Linear Algebra from Scratch: Linear Transformations

Instructor Anthony

"It does not matter how slowly you go as long as you do not stop." - Confucius

Udemy Open Course

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Linear Transformation

Definition ($L : V \to W$ is a Map)

Given Vector Spaces: V and W, $L : V \to W$ is a **Linear Transformation** or Linear Mapping provided:

$$(1) : L(\alpha \mathbf{v_1} + \beta \mathbf{v_2}) = \alpha L(\mathbf{v_1}) + \beta L(\mathbf{v_2})$$

For each $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v_1}, \mathbf{v_2} \in V$.

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Linear Transformation

Definition ($L : V \to W$ is a Map)

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$$(2) : L(\mathbf{v_1} + \mathbf{v_2}) = L(\mathbf{v_1}) + L(\mathbf{v_2})$$

$$L(\alpha v_1) = \alpha L(v_1)$$

For each $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v_1}, \mathbf{v_2} \in V$.

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Linear Transformation

Definition ($L : V \to W$ is a Map)

Given Vector Spaces: V and W, $L : V \to W$ is a **Linear Transformation** or Linear Mapping provided:

$$(1): L(\alpha \mathbf{v_1} + \beta \mathbf{v_2}) = \alpha L(\mathbf{v_1}) + \beta L(\mathbf{v_2})$$

$$(2): L(\mathbf{v_1} + \mathbf{v_2}) = L(\mathbf{v_1}) + L(\mathbf{v_2})$$

$$L(\alpha v_1) = \alpha L(v_1)$$

For each $\alpha, \beta \in \mathbb{R}$ and $\mathbf{v_1}, \mathbf{v_2} \in V$.

Remark: If $V = W$, then L is referred to as a **Linear Operator**. Additionally, only (1) or (2) needs to hold.

Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = 3 * \mathbf{x}$
   Show that L is a Linear Operator.

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Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = 3 * \mathbf{x}$

   Show that L is a Linear Operator.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{bmatrix} 3(\alpha x_1 + \beta y_1) \\ 3(\alpha x_2 + \beta y_2) \end{bmatrix} = \begin{bmatrix} 3(\alpha x_1) + 3(\beta y_1) \\ 3(\alpha x_2) + 3(\beta y_2) \end{bmatrix}$$

□

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Example: Linear Operator

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$$= \begin{bmatrix} 3(\alpha x_1) \\ 3(\alpha x_2) \end{bmatrix} + \begin{bmatrix} 3(\beta y_1) \\ 3(\beta y_2) \end{bmatrix}$$

$\square$

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Example: Linear Operator

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Proof.

$$
\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{bmatrix} 3(\alpha x_1 + \beta y_1) \\ 3(\alpha x_2 + \beta y_2) \end{bmatrix} = \begin{bmatrix} 3(\alpha x_1) + 3(\beta y_1) \\ 3(\alpha x_2) + 3(\beta y_2) \end{bmatrix} \\
&= \begin{bmatrix} 3(\alpha x_1) \\ 3(\alpha x_2) \end{bmatrix} + \begin{bmatrix} 3(\beta y_1) \\ 3(\beta y_2) \end{bmatrix} \\
&= \alpha * \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} + \beta * \begin{bmatrix} 3y_1 \\ 3y_2 \end{bmatrix} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}
$$

$\square$

Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = 3 * \mathbf{x}$
   Show that L is a Linear Operator.
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Proof.

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$$= \alpha * \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} + \beta * \begin{bmatrix} 3y_1 \\ 3y_2 \end{bmatrix} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

$\square$

3. **Remark**: The Linear Operator $L(\mathbf{x}) = 3 * \mathbf{x}$ stretches $\mathbf{x}$ by a factor of 3.
   Generally, any constant of greater magnitude than 1 is a **dilation**.

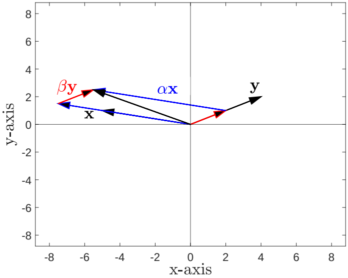
Linear Operator: Schematic Representation

$L : \mathbb{R}^2 \to \mathbb{R}^2$.

1. $|\alpha| > 1 \sim$ *Stretch* (*Contraction*)
2. $|\beta| < 1 \sim$ *Shrink* (*Dilation*)
3. Scaling a Vector ($\mathbf{x}$) in $\mathbb{R}^2$ by a factor of α is a Linear Operator because:

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

$\square$



Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_2, -x_1)$
   Show that L is a Linear Operator.

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Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_2, -x_1)$
   Show that L is a Linear Operator.
2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{bmatrix} \alpha x_2 + \beta y_2 \\ -\alpha x_1 - \beta y_1 \end{bmatrix}$$

$\square$

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Example: Linear Operator

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$\square$

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Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_2, -x_1)$

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2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

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\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{bmatrix} \alpha x_2 + \beta y_2 \\ -\alpha x_1 - \beta y_1 \end{bmatrix} \\
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&= \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}
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$\square$

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Example: Linear Operator

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_2, -x_1)$

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L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{bmatrix} \alpha x_2 + \beta y_2 \\ -\alpha x_1 - \beta y_1 \end{bmatrix} \\
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&= \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}
$$

□

3. **Remark**: The Linear Operator $L(\mathbf{x}) = (x_2, -x_1)$ rotates $\mathbf{x}$ 90° Clockwise.

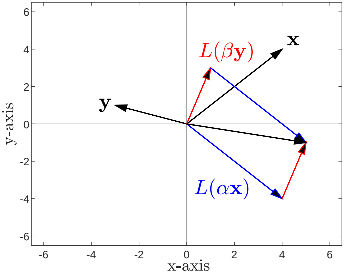
Linear Operator: Schematic Representation

$L : \mathbb{R}^2 \to \mathbb{R}^2$.

(1) The Linear Operator $L(\mathbf{x}) = (x_2, -x_1)$ rotates $\mathbf{x}$ 90° Clockwise. (2) Rotation of a Vector ($\mathbf{x}$) in $\mathbb{R}^2$ is a Linear Operator because:

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

□



Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1, -x_2)$
   Show that L is a Linear Operator.

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1, -x_2)$
   Show that L is a Linear Operator.
2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ -\alpha x_2 - \beta y_2 \end{bmatrix}$$

□

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1, -x_2)$
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\end{aligned}
$$

$\square$

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Example: Linear Transformation

(1) Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1, -x_2)$

Show that L is a Linear Operator.

(2) Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= \begin{bmatrix} \alpha x_1 + \beta y_1 \\ -\alpha x_2 - \beta y_2 \end{bmatrix} \\
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&= \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}$$

$\square$

Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1, -x_2)$

Show that L is a Linear Operator.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

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L(\alpha \mathbf{x} + \beta \mathbf{y}) &= \begin{bmatrix} \alpha x_1 + \beta y_1 \\ -\alpha x_2 - \beta y_2 \end{bmatrix} \\
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&= \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}
$$

$\square$

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3. **Remark**: The Linear Operator $L(\mathbf{x}) = (x_1, -x_2)$ reflects $\mathbf{x}$ across the x-axis.

Reference: Leon, 2014 [2]

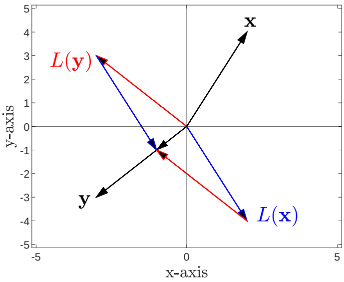
Linear Operator: Schematic Representation

$L : \mathbb{R}^2 \to \mathbb{R}^2$.

1 The Linear Operator $L(\mathbf{x}) = (x_1, -x_2)$ reflects across the *x-axis*. 2 Reflecting a Vector $(\mathbf{x})$ in $\mathbb{R}^2$ is a Linear Operator because:

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y})$$

$\square$



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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}$ where $L(\mathbf{x}) = x_1 + x_2 + x_3$
   Show that L is a Linear Transformation.

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}$ where $L(\mathbf{x}) = x_1 + x_2 + x_3$

Show that L is a Linear Transformation.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then:

Proof.

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3)$$

$\square$

Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}$ where $L(\mathbf{x}) = x_1 + x_2 + x_3$

Show that L is a Linear Transformation.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then:

Proof.

$$\begin{aligned} L(\alpha \mathbf{x} + \beta \mathbf{y}) &= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\ &= (\alpha x_1 + \alpha x_2 + \alpha x_3) + (\beta y_1 + \beta y_2 + \beta y_3) \end{aligned}$$

□

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}$ where $L(\mathbf{x}) = x_1 + x_2 + x_3$

Show that L is a Linear Transformation.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then:

Proof.

$$
\begin{aligned}
L(\alpha \mathbf{x} + \beta \mathbf{y}) &= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\
&= (\alpha x_1 + \alpha x_2 + \alpha x_3) + (\beta y_1 + \beta y_2 + \beta y_3) \\
&= \alpha * (x_1 + x_2 + x_3) + \beta * (y_1 + y_2 + y_3)
\end{aligned}
$$

□

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}$ where $L(\mathbf{x}) = x_1 + x_2 + x_3$

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2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, then:

Proof.

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\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_1 + \beta y_1) + (\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) \\
&= (\alpha x_1 + \alpha x_2 + \alpha x_3) + (\beta y_1 + \beta y_2 + \beta y_3) \\
&= \alpha * (x_1 + x_2 + x_3) + \beta * (y_1 + y_2 + y_3) \\
&= \alpha L(\mathbf{x}) + \beta L(\mathbf{y})
\end{aligned}
$$

□

Example: Not a Linear Transformation

1. Consider the mapping: $M : \mathbb{R}^n \to \mathbb{R}$ where $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
   Show that M is not a Linear Transformation.

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Example: Not a Linear Transformation

1. Consider the mapping: $M : \mathbb{R}^n \to \mathbb{R}$ where $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
   Show that M is not a Linear Transformation.

2. Take $\alpha \in \mathbb{R}$, then:

Proof.

$$M(\alpha\mathbf{x}) = \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + \cdots + (\alpha x_n)^2}$$

$\square$

Example: Not a Linear Transformation

1. Consider the mapping: $M : \mathbb{R}^n \to \mathbb{R}$ where $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Show that M is not a Linear Transformation.

2. Take $\alpha \in \mathbb{R}$, then:

Proof.

$$\begin{aligned} M(\alpha \mathbf{x}) &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + \cdots + (\alpha x_n)^2} \\ &= |\alpha| \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \end{aligned}$$

□

Example: Not a Linear Transformation

1. Consider the mapping: $M : \mathbb{R}^n \to \mathbb{R}$ where $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
   Show that M is not a Linear Transformation.
2. Take $\alpha \in \mathbb{R}$, then:

Proof.

$$
\begin{aligned}
M(\alpha \mathbf{x}) &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + \cdots + (\alpha x_n)^2} \\
&= |\alpha| \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\
&\neq \alpha * M(\mathbf{x})
\end{aligned}
$$

□

Example: Not a Linear Transformation

1. Consider the mapping: $M : \mathbb{R}^n \to \mathbb{R}$ where $M(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Show that M is not a Linear Transformation.

2. Take $\alpha \in \mathbb{R}$, then:

Proof.

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\begin{aligned}
M(\alpha\mathbf{x}) &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2 + \cdots + (\alpha x_n)^2} \\
&= |\alpha|\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \\
&\neq \alpha * M(\mathbf{x})
\end{aligned}
$$

For $\alpha < 0$. Therefore, M is not a Linear Operator. $\square$

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)$
   Show that L is a Linear Transformation.

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Example: Linear Transformation

(1) Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)$

Show that L is a Linear Transformation.

(2) Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1, \alpha(x_1 + x_2) + \beta(x_2 + y_2))^T$$

$\square$

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)$
   Show that L is a Linear Transformation.
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Proof.

$$
\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1, \alpha(x_1 + x_2) + \beta(x_2 + y_2))^T \\
&= \begin{bmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha(x_1 + x_2) \end{bmatrix} + \begin{bmatrix} \beta y_2 \\ \beta y_1 \\ \beta(y_1 + y_2) \end{bmatrix}
\end{aligned}
$$

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)$
Show that L is a Linear Transformation.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$
\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1, \alpha(x_1 + x_2) + \beta(x_2 + y_2))^T \\
&= \begin{bmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha(x_1 + x_2) \end{bmatrix} + \begin{bmatrix} \beta y_2 \\ \beta y_1 \\ \beta(y_1 + y_2) \end{bmatrix} \\
&= \alpha * L(\mathbf{x}) + \beta * L(\mathbf{y})
\end{aligned}
$$

$\square$

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Example: Linear Transformation

1. Consider the mapping: $L : \mathbb{R}^2 \to \mathbb{R}^3$ where $L(\mathbf{x}) = (x_2, x_1, x_1 + x_2)$
   Show that L is a Linear Transformation.

2. Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then:

Proof.

$$
\begin{aligned}
L(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1, \alpha(x_1 + x_2) + \beta(x_2 + y_2))^T \\
&= \begin{bmatrix} \alpha x_2 \\ \alpha x_1 \\ \alpha(x_1 + x_2) \end{bmatrix} + \begin{bmatrix} \beta y_2 \\ \beta y_1 \\ \beta(y_1 + y_2) \end{bmatrix} \\
&= \alpha * L(\mathbf{x}) + \beta * L(\mathbf{y})
\end{aligned}
$$

□

Remark: If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \to A\mathbf{x} = L(\mathbf{x})$

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Reference: Leon, 2014 [2]

The Matrix Representation

Lemma (General Representation)

Consider $\mathbf{x} \in \mathbb{R}^n$ *and the Linear Mapping* $L : \mathbb{R}^n \to \mathbb{R}^m$, *then the Matrix Representation given* $A \in \mathbb{R}^{mxn}$ *is:*

$$L_A(\mathbf{x}) = A\mathbf{x}$$

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The Matrix Representation

Lemma (General Representation)

Consider $\mathbf{x} \in \mathbb{R}^n$ *and the Linear Mapping* $L : \mathbb{R}^n \to \mathbb{R}^m$*, then the Matrix Representation given* $A \in \mathbb{R}^{mxn}$ *is:*

$$L_A(\mathbf{x}) = A\mathbf{x}$$

Proof.

Take $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then:

$$L_A(\alpha\mathbf{x} + \beta\mathbf{y}) = A(\alpha\mathbf{x} + \beta\mathbf{y}) = A(\alpha\mathbf{x}) + A(\beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y} = \alpha L_A(\mathbf{x}) + \beta L_A(\mathbf{y})$$

$\square$

Linear Transformation Properties

Definition (Properties of the Linear Transformation)

Take $V, W \sim$ *Vector Spaces* and $L : V \to W$ a Linear Transformation, then the following holds:

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Example: Linear Transformation

1. Consider the mapping: $L : C[a, b] \to \mathbb{R}$ where $L(\mathbf{f}) = \int_a^b f(x)dx$
   Show that L is a Linear Transformation.

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□

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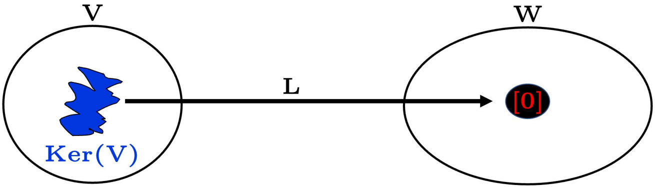
The Kernel

Definition (Kernel of a Linear Transformation)

Suppose V and W are Linear Spaces and $L : V \to W$ is a Linear Transformation, then the Kernel of L are all the elements in V that map to the null vector in W:

$$Ker(V) = \{\mathbf{v} \in V \mid L(v) = \mathbf{0}_w\}$$



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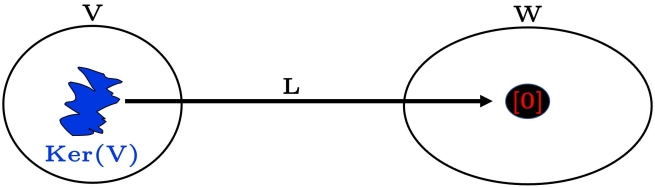
Remark: $Ker(V) \subset V$



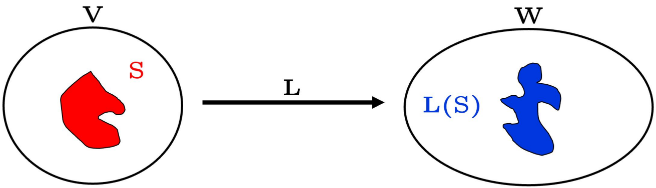
The Image

Definition (The Image of a Linear Transformation)

Suppose V and W are Linear Spaces and $S \subset V$. Consider $L : V \to W$ as a Linear Transformation, then the Image of S under L are all the elements mapped to W from an element in S:

$$L(S) = \{\mathbf{w} \in W \mid \mathbf{w} = L(v) \text{ for some } \mathbf{v} \in S\}$$

Remark: The **Range** of L is the Image of V under L denoted: $\mathbf{L(V)}$.



The Image

Definition (The Image of a Linear Transformation)

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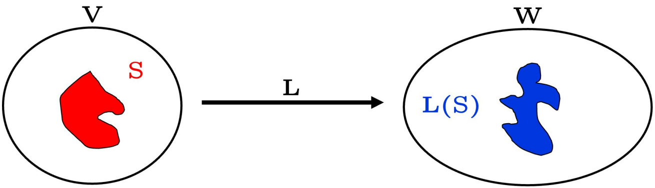

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1. Consider the mapping: $D : P_3 \to P_2$ where $D(\mathbf{p}(x)) = p'(x)$
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Proof.

$$\begin{aligned}
\mathbf{p}(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 \\
D(\mathbf{p}) &= \alpha_1 + 2\alpha_2 x = 0 \longleftrightarrow \mathbf{p}(x) = \alpha_0 \\
Ker(D) &= \{\mathbf{p} \in P_3 \mid \mathbf{p} = \alpha_0\} \\
D(P_3) &= P_2
\end{aligned}$$

□

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Table of Contents

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The Matrix Representation of L

Theorem (Matrix Representation)

Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a Linear Transformation, then the Matrix Representation of a Linear Transformation is $A \in \mathbb{R}^{mxn}$ such that:

$$Ax = L(x)$$

For each $\mathbf{x} \in \mathbb{R}^n$, where $a_j = L(e_j)$.

Proof.

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$$L(\mathbf{x}) = L(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n)$$

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$$\begin{aligned} L(\mathbf{x}) &= L(x_1 e_1 + x_2 e_2 + \cdots + x_n e_n) \\ &= x_1 L(e_1) + x_2 L(e_2) + \cdots + x_n L(e_n) \end{aligned}$$

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For each $\mathbf{x} \in \mathbb{R}^n$, where $a_j = L(e_j)$.

Proof.

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Example: Matrix Representation

Consider the mapping: $L : \mathbb{R}^3 \to \mathbb{R}^2$ where $L(\mathbf{x}) = (x_1 + x_2, x_2 + x_3)$

Find the Matrix Representation of the Linear Transformation.

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$$A = [a_1, a_2, a_3] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\square$

Example: Matrix Representation

Consider the Linear Operator $L(\mathbf{x}) = (-x_2, x_1)$ that rotates a vector $\mathbf{x} \in \mathbb{R}^2$ by 90° Counter-Clockwise.

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Suppose that the first column of A is:

$$a_1 = L(e_1) = \begin{bmatrix} cos(\theta) \\ sin(\theta) \end{bmatrix} \quad a_2 = L(e_2) = \begin{bmatrix} -sin(\theta) \\ cos(\theta) \end{bmatrix}$$

Where $\theta = tan^{-1}(x_2/x_1) \in \mathbb{R}$ is the angle rotated Counter-Clockwise.

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$\square$

Matrix Representation Theorem

Theorem (Matrix Representation of a $L : V \to W$)

Suppose: $E = \{v_1, v_2, \ldots, v_n\}$ *and* $F = \{w_1, w_2, \ldots, w_n\}$ *are ordered basis on Vector Space* V *and* W *respectively. Consider* $L : V \to W$ *as a Linear Transformation and take* $[v]_E$ *as the coordinate vector with respect to the ordered basis* E*, then there is* $A \in \mathbb{R}^{mxn}$ *such that:*

$$[L(v)]_F = A[v]_E \textit{ For each } \mathbf{v} \in V$$

We will refer to A *as the* **Matrix Representing** L **relative to the ordered basis***:* E *and* F*.*

Ordered Basis of P_n

Lemma (An Ordered Basis of P_n)

Suppose that P_n is the vector space consisting of all polynomials with degree $n-1$, then $E = \{1, x, x^2, \ldots, x^j, \ldots, x^{n-1}\}$ is an **Ordered Basis** *on P_n.*

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Proof.

For any $\mathbf{p}(x) \in P_n$, one has:

$$\mathbf{p}(x) = \sum_{j=0}^{n-1} \alpha_j x^j$$

One can take $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})' \stackrel{\triangle}{\equiv} [\mathbf{p}]_E$ as the **coordinate vector** of $\mathbf{p}(x)$ with respect to P_n. $\square$

Example: Matrix Representation

(1) Consider the mapping: $D : P_3 \to P_2$ where $D(\mathbf{p}) = \mathbf{p}'$. Suppose that $E = \{1, x, x^2\}$ and $F = \{1, x\}$ are ordered basis on P_3 and P_2 respectively.

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One obtains the Matrix Representation of the Linear Transformation by applying D to each basis element of E and expressing in terms of elements of the ordered basis on F:

$$D(1) = 0 * 1 + 0 * x \ \& \ D(x) = 1 * 1 + 0 * x \ \& \ D(x^2) = 0 * 1 + 2 * x$$

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$$a_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ a_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ a_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \to A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

□

Representation of Geometric Operations in $2D$

(1) We can create a **Triangle**, by expressing the **ordered tuples** of the vertices as columns of $T \in \mathbb{R}^{2x4}$: $T = \begin{bmatrix} x_1 & y_1 & z_1 & x_1 \\ x_2 & y_2 & z_2 & x_2 \end{bmatrix}$ The 4_{th} column is the same as the 1_{st} column to guarantee formation of a closed polygon.

Geometric Transformations in $\mathbb{R}^2$.

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(4) Reflection: $L(\mathbf{x}) = (x_1, -x_2) \sim \textit{about the x-axis} \to A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ &

$$L(\mathbf{x}) = (-x_1, x_2) \sim \textit{about the y-axis} \to A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(5) Rotation: $L \sim$ Linear Transformation that rotates $\mathbf{x} \in \mathbb{R}^2$ by an angle of θ Counter-Clockwise. $L(\mathbf{x}) = A\mathbf{x}$ where: $A = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$

Linear Operator: Schematic Representation

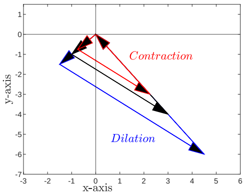

Figure: Scaling

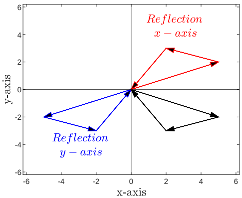

Figure: Reflection

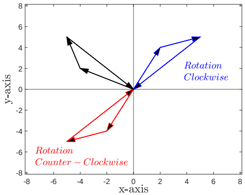

Figure: Rotation

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Similarity between Matrices

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Example.

(2) Let $L(\mathbf{x}) = (2x_1, x_1 + x_2)^T$. We will first find the matrix representation according to the Standard Basis: $E = \{\mathbf{e_1}, \mathbf{e_2}\}$.

$$L(e_1) = a_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$L(e_2) = a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

□

Example Cont'd

1. Consider the ordered basis: $F = \{\mathbf{u_1}, \mathbf{u_2}\}$ where: $U = [\mathbf{u_1}, \mathbf{u_2}] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Example Cont'd

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(2) The Transition Matrix from ordered basis F to E is: $S_1 = I_2^{-1} U = U$.

Likewise, The Transition Matrix from ordered basis E to F is: $S_2 = U^{-1} I_2 = U^{-1}$.

Example Cont'd

1. Consider the ordered basis: $F = \{\mathbf{u_1}, \mathbf{u_2}\}$ where: $U = [\mathbf{u_1}, \mathbf{u_2}] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

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3. It follows that $U^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$.
To obtain the matrix representation of L wrt to Basis F we will pre-multiply by S_2 on Au_j for $j = 1, 2$.

Example Cont'd

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Computation.

$$U^{-1} A u_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad U^{-1} A u_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$B = U^{-1} A U = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

□

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Summary of Previous Example

We obtained a **Similar** Matrix $B = U^{-1}AU$ provided:

Conditions.

$\square$

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1 B is The Matrix Representation of L with respect to the Ordered Basis, $F = \{\mathbf{u_1}, \mathbf{u_2}\}$.

□

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Summary of Previous Example

We obtained a **Similar** Matrix $B = U^{-1}AU$ provided:

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1 B is The Matrix Representation of L with respect to the Ordered Basis, $F = \{\mathbf{u_1}, \mathbf{u_2}\}$.
2 A is The Matrix Representation of L with respect to the Ordered Basis: $E = \{\mathbf{e_1}, \mathbf{e_2}\}$.

$\square$

Summary of Previous Example

We obtained a **Similar** Matrix $B = U^{-1}AU$ provided:

Conditions.

(1) B is The Matrix Representation of L with respect to the Ordered Basis, $F = \{\mathbf{u_1}, \mathbf{u_2}\}$.
(2) A is The Matrix Representation of L with respect to the Ordered Basis: $E = \{\mathbf{e_1}, \mathbf{e_2}\}$.
(3) U is the Transition Matrix corresponding to the change of basis from Fe to E. □

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Similarity Matrices in General

Theorem (Similar Matrices)

Consider $E = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ *and* $F = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}\}$ *as two Ordered Baseis on Vector Space* V*. Consider a Linear Operator:* $L : V \to V$*.*
Take A *as the Matrix Representation of L wrt* E*.*
Take B *as the Matrix Representation of L wrt* F*, then there is nonsingular* S *such that:*

$$B = S^{-1}AS$$

We say that B *is* **Similar** *to* A*.*
Remark*: In fact,* A *is* **Similar** *to* B *since:* $A = SBS^{-1}$*.*
We refer to A *and* B *as* **Similar** *Matrices.*

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[1] David Harville. *Matrix Algebra From a Statistician's Perspective*. New York: Springer-Verlag, 1997.

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