Linear Algebra from Scratch: Determinants

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Udemy Open Course

M.S. in Mathematics with a Concentration in Bioinformatics

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Row Equivalence

Definition (Elementary Matrix)

Let $A \in \mathbb{R}^{mxn}$, suppose we apply a single Elementary Row Operation (Row Swap, Scale a row by $\alpha \neq 0$, add a scalar multiple of one row to another row), the resulting matrix will be referred to as an **Elementary Matrix**: E

Definition (Row Equivalent Matrices)

$A \in \mathbb{R}^{nxn}$ is row equivalent to $B \in \mathbb{R}^{nxn}$ if there exists a sequence of k Elementary Matrices such that:

$$B = E_k * E_{k-1} * \cdots * E_1 * A$$

The Types of Elementary Matrices I

Lemma (Types of Elementary Matrices)

Let: E be an elementary matrix row equivalent to the identity matrix I_n and $\alpha \neq 0$ There are three types of Elementary Matrices that can be pre/post-multiplied by $A \in \mathbb{R}^{nxn}$ to obtain a row equivalent matrix $B \in \mathbb{R}^{nxn}$. The first elementary matrix has $\alpha \neq 0$ in the $(j, j)_{th}$ element of in place of 1 in I_n:

$$E_1 = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Pre-multiplying A by E_1, i.e. $(E_1 A)$, scales the j_{th} **row** *of A by α, while Post-multiplying A by E_1, i.e. (AE_1), scales the j_{th}* **column** *of A by α.*

The Types of Elementary Matrices II

Lemma (Types of Elementary Matrices)

The second elementary matrix, E_2, is obtained by swapping the i_{th} row with the k_{th} row of the Identity Matrix I_n:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{blue}{0} & \color{blue}{0} & \dots & \color{blue}{1} & \dots & \color{blue}{0} & \dots & \color{blue}{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{red}{0} & \color{red}{0} & \dots & \color{red}{0} & \dots & \color{red}{1} & \dots & \color{red}{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix} \longrightarrow E_2 = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{red}{0} & \color{red}{0} & \dots & \color{red}{0} & \dots & \color{red}{1} & \dots & \color{red}{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{blue}{0} & \color{blue}{0} & \dots & \color{blue}{1} & \dots & \color{blue}{0} & \dots & \color{blue}{0} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Pre-multiplying A by E_2, i.e. (E_2A), interchanges the i_{th} and k_{th} **rows** *of A, while*
Post-multiplying A by E_2, i.e. (AE_2), interchanges the i_{th} and k_{th} **columns** *of A.*

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The Types of Elementary Matrices III

Lemma (Types of Elementary Matrices)

The third elementary matrix has $\alpha \neq 0$ in the $(i,j)_{th}$ element of in place of 0 in I_n:

$$E_3 = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Pre-multiplying A by E_3, i.e. (E_3A), scales the j_{th} **row** *of A by α and adds it to the i_{th} row of A, while Post-Multiplying A by E_3, i.e. (AE_3), scales the j_{th}* **column** *of A by α and adds it to the i_{th} column of A.*

Example: Elementary Matrices

Suppose that:

1. $$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad E_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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2. Then:

$$AE_1 = \begin{bmatrix} 3 & 2 & 3 \\ 12 & 5 & 6 \\ 21 & 8 & 9 \end{bmatrix} \quad E_1 A = \begin{bmatrix} 3 & 6 & 9 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

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3. And:
$$AE_2 = \begin{bmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{bmatrix} \quad E_2 A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$$

Example: Elementary Matrices

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3. And:
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4. Lastly:
$$AE_3 = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 5 & 18 \\ 7 & 8 & 30 \end{bmatrix} \quad E_3A = \begin{bmatrix} 22 & 26 & 30 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Equivalent Conditions of a Nonsingular (Invertible) Matrix

Lemma (Conditions for a Nonsingular Matrix)

Recall that $A \in \mathbb{R}^{nxn}$ *is a* **Nonsingular** *matrix,* A*, provided* A *has a Multiplicative Inverse,* A^{-1}*, such that:*

$$AA^{-1} = I_n = A^{-1}A$$

Given Matrix A*, the following are equivalent:*

1. *1: A is Nonsingular*

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3. (3): *A is row equivalent to I_n*

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Given Matrix A, the following are equivalent:

1. *(1): A is Nonsingular*
2. *(2): $Ax = 0$ has* **only** *the trivial solution: $x = 0$*
3. *(3): A is row equivalent to I_n*
4. **Remark**: *One can solve the augmented system $[A|I_n]$ to obtain the inverse: $[I_n|A^{-1}]$*

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Example

1. Consider the matrix:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

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2. One can augment the system with the Identity Matrix: I_2 to obtain:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{array}\right]$$

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2. One can augment the system with the Identity Matrix: I_2 to obtain:

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3. After row reducing to RREF on the left hand side (left as an exercise to its reader) one obtains:

$$\left[\begin{array}{cc|cc} 1 & 0 & -0.1 & 0.3 \\ 0 & 1 & 0.4 & -0.2 \end{array}\right]$$

One can check the right hand side is the inverse of A.

The Principal Minor

Definition (Principal Minor)

Let $A \in \mathbb{R}^{nxn}$, then the $(i,j)_{th}$ **Principal Minor**, denoted A_{ij} is obtained by crossing out the i_{th} row of A and j_{th} column of A:

$$A_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,j-1} & \cancel{a_{1j}} & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,j-1} & \cancel{a_{2j}} & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & \cancel{a_{i-1,j}} & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ \cancel{a_{i,1}} & \cancel{a_{i,2}} & \cdots & \cancel{a_{i,j-1}} & \cancel{a_{ij}} & \cancel{a_{i,j+1}} & \cdots & \cancel{a_{i,n}} \\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & \cancel{a_{i+1,j}} & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{m,j-1} & \cancel{a_{mj}} & a_{m,j+1} & \cdots & a_{mn} \end{bmatrix}$$

Remark: $A_{ij} \in \mathbb{R}^{n-1,n-1}$

Reference: Leon, 2014 [2]

Example

1. Consider the matrix:

$$A = \begin{bmatrix} 1 & \cancel{2} & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & \cancel{8} & 9 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & \cancel{2} & 3 \\ \cancel{4} & \cancel{5} & \cancel{6} \\ 7 & \cancel{8} & 9 \end{bmatrix}$$

2. Then, the A_{22} **Principal Minor** is obtained by crossing out the second row and column of A:

$$A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$$

The Determinant

Definition (The Determinant of a Square Matrix)

- Let $A \in \mathbb{R}^{nxn}$ be a square matrix.
  Case I: Suppose that $n = 1$ then $Det(A) = a_{11}$

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- **Case II**: Suppose that $n = 2$, then:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Then $Det(A) = a_{11}a_{22} - a_{12}a_{21}$

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  Then $Det(A) = a_{11}a_{22} - a_{12}a_{21}$
- **Case III**: Suppose $n = 3$, then:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  Then $Det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

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  Then $Det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$
- **Remark**: $Det(A) = a_{11}Det(A_{11}) - a_{12}Det(A_{12}) + a_{13}Det(A_{13})$

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Example

- Suppose that:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix}$$

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Example

- Suppose that:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix}$$

- Then $Det(A) = 2 * (3 - 0) - 4(1 - 20) + 6(0 - 12) = 10$

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Determinants and Singular Matrices

Lemma (Determinant of a Singular Matrix)

Let $A \in \mathbb{R}^{nxn}$, then the following are equivalent:

- *1: $A \sim$ Singular*

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Let $A \in \mathbb{R}^{nxn}$, then the following are equivalent:

- 1: $A \sim Singular$
- 2: $Det(A) = 0$

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Lemma (Determinant of a Singular Matrix)

Let $A \in \mathbb{R}^{nxn}$*, then the following are equivalent:*

- 1: $A \sim$ *Singular*
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- 3: $Ax = 0$ *has a nontrivial solution* $x \neq 0$

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- 1: $A \sim$ *Singular*
- 2: $Det(A) = 0$
- 3: $Ax = 0$ *has a nontrivial solution* $x \neq 0$
- 4: **Remark**: $Det(A) \neq 0 \longleftrightarrow A \sim$ *Nonsingular*

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The Determinant of a Matrix by Cofactor Expansion

Lemma (Determinant by Cofactor Expansion)

Suppose that $A \in \mathbb{R}^{nxn}$ *and let* A_{ij} *be the* $(i, j)_{th}$ **Principal Minor** *obtained by crossing out the* i_{th} *row and* j_{th} *column of* A*, then:*

$$\begin{aligned} Det(A) &= a_{1j}(-1)^{1+j}Det(A_{1j}) + a_{2j}(-1)^{2+j}Det(A_{2j}) + \cdots + a_{nj}(-1)^{n+j}Det(A_{nj}) \\ &= a_{i1}(-1)^{i+1}Det(A_{i1}) + a_{i2}(-1)^{i+2}Det(A_{i2}) + \cdots + a_{in}(-1)^{i+n}Det(A_{in}) \end{aligned}$$

Generally, we refer to $C_{ij} = (-1)^{i+j}Det(A_{ij})$ *as the* $(i, j)_{th}$ *Co-Factor and the previous equation as the* **Cofactor Expansion** *of* A*,*

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Determinant of the Transpose of a Matrix

Lemma (Determinant of the Transpose of a Matrix)

Suppose $A \in \mathbb{R}^{nxn}$*, then:*

$$Det(A) = Det(A')$$

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The Determinant of a Product of Matrices

Lemma (Determinant of a Product of Matrices)

- *Suppose* $A, B \in \mathbb{R}^{nxn}$, *then:*

$$Det(AB) = Det(A) * Det(B) = Det(BA)$$

Although Matrix Multiplication need not commute, **the determinant of a product of matrices does commute**.
Furthermore, if A and B are both Nonsingular, then AB is Nonsingular and hence invertible.

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Although Matrix Multiplication need not commute, **the determinant of a product of matrices does commute***.*
Furthermore, if A *and* B *are both Nonsingular, then* AB *is Nonsingular and hence invertible.*

- *By Induction, given* $A_1, A_2, \ldots, A_k \in \mathbb{R}^{nxn}$*, one can show:*

$$Det(A_1 * A_2 \cdots * A_k) = Det(A_1) * Det(A_2) * \cdots * Det(A_k)$$

Determinant of Elementary Matrices

Theorem (Determinant of Elementary Matrices)

Let $E \in \mathbb{R}^{nxn}$ Elementary Matrix and $A \in \mathbb{R}^{nxn}$, then:

- 1: *If $E = E_1$ where the j_{th} column and row is scaled by $\alpha \neq 0$, then:*

$$Det(E_1 A) = \alpha Det(A)$$

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$$Det(E_1 A) = \alpha Det(A)$$

- 2: *If* $E = E_2$ *where the* i_{th} *and* k_{th} *rows are interchanged, then:*

$$Det(E_2 A) = -1 * Det(A)$$

Generally, interchanging rows k *times scales the original determinant of* A *by:* $(-1)^k$.

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Generally, interchanging rows k *times scales the original determinant of* A *by:* $(-1)^k$.

- 3: *If* $E = E_3$ *where the* $(i,j)_{th}$ *element of* E *is* $\alpha \neq 0$ *and* $i \neq j$*, then:*

$$Det(E_3 A) = Det(A)$$

That is, multiplying one row by $\alpha \neq 0$ *and adding it to another row does not change the determinant of the original matrix.*

Determinant of a Triangular Matrix

Lemma (Determinant of a Triangular Matrix)

Suppose that $A \in \mathbb{R}^{nxn}$ such that $a_{ij} = 0$ when $i > j$ (A is **Upper Triangular**), *i.e.,*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & a_{nn} \end{bmatrix}$$

then:

$$Det(A) = a_{11} * a_{22} * \cdots * a_{jj} * \cdots * a_{nn}$$

Remark: *The results are identical if $a_{ij} = 0$ when $i < j$ (A is Lower Triangular). Additionally, $A \sim$ Singular if at least one $a_{jj} = 0$.*

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Example

- An alternative method of taking the determinant of a matrix is to first transform A to REF form using Gaussian Elimination, then take the determinant of the final matrix.

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- Consider the following example:

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -4 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 3 \\ 0 & 11 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

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- An alternative method of taking the determinant of a matrix is to first transform A to REF form using Gaussian Elimination, then take the determinant of the final matrix.
- Consider the following example:

$$A = \begin{bmatrix} 2 & 5 & 3 \\ -4 & 1 & 2 \\ 0 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 & 3 \\ 0 & 11 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$

- Since the Matrix on the right hand side is Upper Triangular, then the determinant is the product of the diagonal elements:
$Det(A) = 2 * 11 * 9 = 198$

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The Classical Adjoint

Definition (The Classical Adjoint)

Suppose $A \in \mathbb{R}^{nxn}$ and let A_{ij} denote the $(i,j)_{th}$ Principal Minor, then the **Classical Adjoint** of A denoted $\boldsymbol{Adj(A)}$ is:

$$Adj(A) = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1j} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2j} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{i1} & C_{i2} & \dots & C_{ij} & \dots & C_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nj} & \dots & C_{nn} \end{bmatrix}'$$

Where: $C_{ij} = (-1)^{i+j} Det(A_{ij})$ is the $(i,j)_{th}$ Cofactor of A.

The Adjoint and the Inverse of a Matrix

-

Theorem (The Adjoint and the Inverse of a Matrix)

1: *Suppose that:* $A \in \mathbb{R}^{nxn}$ *and* $Det(A) \neq 0$, *then the* **Inverse of a Matrix** *is:*

$$A^{-1} = \frac{1}{Det(A)} Adj(A)$$

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The Adjoint and the Inverse of a Matrix

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$$A^{-1} = \frac{1}{Det(A)} Adj(A)$$

-

Lemma (The Adjoint of a Matrix)

2: *Suppose* $A \in \mathbb{R}^{2x2}$, *such that* $Det(A) \neq 0$, *then:*

$$A^{-1} = \frac{1}{a_{22}a_{11} - a_{12}a_{21}} * \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

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Example

1. Suppose that:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

2. Then, the **Classical Adjoint** Adj(A) is:

$$Adj(A) = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 0 \\ -13 & 8 & -(-2) \\ 5 & -3 & -1 \end{bmatrix}$$

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Example

1. Suppose that:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

2. Then, the **Classical Adjoint** Adj(A) is:

$$Adj(A) = \begin{bmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}' = \begin{bmatrix} 1 & -1 & 0 \\ -13 & 8 & -(-2) \\ 5 & -3 & -1 \end{bmatrix}$$

3. Additionally, the Inverse of A is:

$$A^{-1} = \frac{1}{Det(A)} * Adj(A) = \begin{bmatrix} -1 & 1 & 0 \\ 13 & -8 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

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Cramer's Rule

Theorem (Cramer's Formula for $Ax = b$)

Suppose one solves $Ax = b$ where $A \in \mathbb{R}^{nxn}$, $b \in \mathbb{R}^n$ and $Det(A) \neq 0$, then $A \sim$ Invertible and x is the unique solution can be determined using **Cramer's Formula**

$$x_i = \frac{Det(A_i)}{Det(A)}, \;\; for \;\; i = 1, 2, \ldots, n$$

Where A_i is the matrix obtained by replacing the i_{th} column with b.

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Example

- Consider the System:

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 0x_1 + 1x_1 &= 5 \end{aligned} \longrightarrow A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \; b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Example

- Consider the System:

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 0x_1 + 1x_1 &= 5 \end{aligned} \longrightarrow A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

- Then:

$$x_1 = \frac{Det([b, a_2])}{Det(A)} = \frac{1}{2} * \begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix} = \frac{-12}{2} = -6$$

$$x_2 = \frac{Det([a_1, b])}{Det(A)} = \frac{1}{2} * \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} = \frac{10}{2} = 5$$

References I

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