

# Linear Algebra from Scratch: Matrices and Systems

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"It does not matter how slowly you go as long as you do not stop." - Confucius

Udemy Open Course



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# Example: A System of Two Linear Equations

- Consider the following system of linear equations:

$$2x + 3y = 5$$

$$x - y = 10$$

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$$-2x + 2y = -20$$

- Obtained by multiplying the second row by  $-2$ . Adding the first and second equations, will give us:

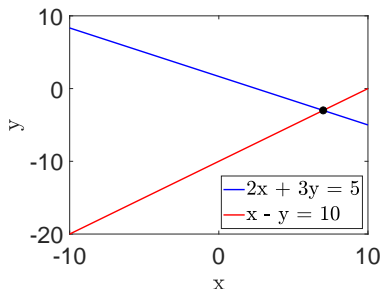
$$5y = -15 \longrightarrow y = -3$$

After substituting  $y = -3$  into second equation, one obtains  $x = 7$ . The solution of system of two linear equations is the ordered tuple:  $(7, -3)$ .

# Example: A System of Two Linear Equations

## Proof.

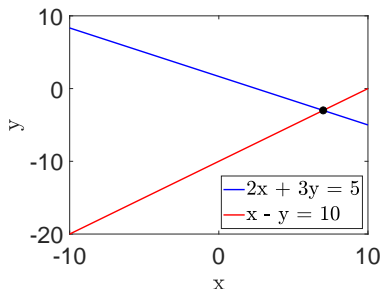
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- 2 The solution **exists** if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.

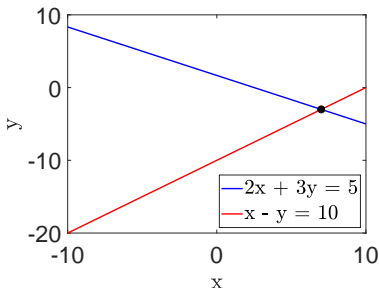




# Example: A System of Two Linear Equations

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- 2 The solution **exists** if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.
- 3 The solution is **unique** if the lines intersect only once.



## Example: A System of Two Linear Equations

In the previous example, one can classify the different constituents of the system as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \sim \textit{Coefficient Matrix}$$

$$b = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \sim \textit{Column Vector}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \textit{Vector of 2 Unknowns}$$

# General Form of the System of Two Linear Equations

## Definition (General form of a System of Two Linear Equations)

A system of two linear equations with two unknowns is expressed as follows:

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

Where:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The system is **linear** if each element belonging to the vector of unknowns  $x_j$  has a power of 1 and does not contain nonlinear functions such as trigonometric functions or exponential functions.

# General Form of the System of $m$ Linear Equations

## Definition (General form of a System of $m$ Linear Equations)

A system of  $m$  linear equations with  $n$  unknowns has the following form:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n = b_2$$

$$\vdots = \vdots$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n = b_i$$

$$\vdots = \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n = b_m$$

We will refer to:

$$a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \sim j_{th} \text{ Column Vector and } a_i' = (a_{i1}, \dots, a_{in}) \sim i_{th} \text{ Row Vector}$$

# General Form of the System of $m$ Linear Equations

## Definition (Matrix form of System of $n$ Linear Equations)

The matrix form of the set of  $m$  linear equations with  $n$  unknowns where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $b \in \mathbb{R}^m$  is:  $Ax = b$ , which is expressed as:

$$Ax = b \Leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$

# General Form of the System of $m$ Linear Equations

## General form of a System of $m$ Linear Equations.

We will refer to the matrix of size  $m \times n$  where  $m$  is the number of rows and  $n$  is the number of columns, denoted  $A \in \mathbb{R}^{m \times n}$ , is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$



# Augmented System of Linear Equations

## Definition (Augmented System)

Let:  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , and  $x \in \mathbb{R}^n$ , then:

$$A = \left[ \begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array} \right]$$

is the augmented system of  $Ax = b$ .

# General Form of the System of $m$ Linear Equations

## General form of a System of $m$ Linear Equations.

The  $j_{th}$  column vector of  $A$  is denoted  $\mathbf{a}_j \in \mathbb{R}^{m \times 1}$ :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & \mathbf{a}_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \mathbf{a}_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & \mathbf{a}_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \mathbf{a}_{mj} & \dots & a_{mn} \end{bmatrix}$$





# General Form of the System of $m$ Linear Equations

## General form of a System of $m$ Linear Equations.

The  $i_{th}$  row vector of  $A$  is denoted  $\mathbf{a}_i' \in \mathbb{R}^{1 \times n}$ :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{a}_{i1} & \mathbf{a}_{i2} & \dots & \mathbf{a}_{ij} & \dots & \mathbf{a}_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$



# General Form of the System of $m$ Linear Equations

## General form of a System of $m$ Linear Equations.

The intersection of the  $i_{th}$  row of  $A$  and  $j_{th}$  column of  $A$  is the  $(i,j)_{th}$  element of  $A$  denoted:  $a_{ij}$ :

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$



## Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

①  $a_{12}$

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1 3

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Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

1 3

Determine the following:

1  $a_{12}$

2  $a_1$

# Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

1 3  
2

Determine the following:

1  $a_{12}$

2  $a_1$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

# Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

① 3  
②

Determine the following:

①  $a_{12}$

②  $a_1$

③  $a'_2$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

# Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

①  $a_{12}$

②  $a_1$

③  $a'_2$

① 3

②

③ (2, 5)

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



# Elementary Row Operations

## Definition (Elementary Row Operations)

An elementary row operation is an invertible operation applied between two rows of a matrix. Row operations may be classified as:

- 1 Interchange of two rows AKA row swap
- 2 Multiply a row by a nonzero real number
- 3 Replace a row by its sum with a multiple of another row

# Elementary Row Operations

Suppose that  $A_1$  is the original matrix and we swap the  $i_{th}$  and  $k_{th}$  row resulting in the matrix  $A_2$ .

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

# Elementary Row Operations

After swapping the  $i_{th}$  and  $k_{th}$  row, one obtains:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

# Elementary Row Operations

1 Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

# Elementary Row Operations

① Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

② After swapping the first and second row of  $A_1$ , one obtains:

$$A_2 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

# Elementary Row Operations

Suppose that  $A_1$  is the original matrix and we scale the  $i_{th}$  row by a factor of  $\alpha \neq 0$  to obtain:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

# Elementary Row Operations

① Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

② After multiplying row 2 by a factor of 3 one obtains:

$$A_2 = \begin{bmatrix} 1 & 3 \\ 6 & 12 \end{bmatrix}$$



# Elementary Row Operations

Suppose that  $A_1$  is the original matrix and we scale the  $i_{th}$  row by a factor of  $\alpha$  and add it to the  $k_{th}$  row to obtain:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} + \alpha a_{i1} & a_{k2} + \alpha a_{i2} & \dots & a_{kj} + \alpha a_{ij} & \dots & a_{kn} + \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

# Elementary Row Operations

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# Elementary Row Operations

① Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

② After multiplying row 2 by a factor of 3 and adding it to row 1, one obtains:

$$A_2 = \begin{bmatrix} 7 & 15 \\ 2 & 4 \end{bmatrix}$$

# Equivalent Systems of Linear Equations

## Lemma (Equivalence of Systems)

*Given two systems of linear equations, the systems are equivalent provided a sequence of elementary row operations can be applied from one matrix to the other.*

*In other words, if  $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_k$  is an iterative sequence of applying  $k - 1$  row operations, then  $A_1x = b_1$  and  $A_kx = b_k$  are equivalent systems.*

**Remark:** *Two systems are equivalent provided the solution sets of each are identical.*

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# Triangular form of a System of Equations

## Definition (Triangular Form)

- ① Given  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $b \in \mathbb{R}^m$ , the system:  $Ax = b$  is strict triangular provided:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & a_{mn} \end{bmatrix}$$

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- ② In other words,  $a_{ij} = 0$  when  $i > j$ . If  $m = n$ , then  $A$  is referred to as a lower triangular matrix. Alternatively, if in the  $k_{th}$  equation, the coefficients of the first  $k - 1$  variables are all zero and the coefficient of  $x_k$  is nonzero for each  $k = 1, 2, \dots, n$ .

## Example: Triangular Form

① Consider the system:

$$3x_1 + 2x_2 + x_3 = 1$$

$$x_2 - x_3 = 2$$

$$2x_3 = 4$$

Which can be expressed as:



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Which can be expressed as:

$$\left[ \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

# Example: Back Substitution

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①  $2x_3 = 4 \rightarrow x_3 = 2$

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③  $x_1 = \frac{1-2x_2+2-x_3}{3} \rightarrow x_1 = -3$

## Example: Back Substitution

Given the system:

$$3x_1 + 2x_2 + x_3 = 1$$

$$x_2 - x_3 = 2$$

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① But how do we get the matrix into triangular form in the first place?

We can solve using **Back Substitution**:

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- ② Performing **Elementary Row Operations** to get the matrix in a triangular form is the way!

## Example: Back Substitution

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We can solve using **Back Substitution**:

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$$\textcircled{2} \quad x_2 - x_3 = 2 \rightarrow x_2 = 4$$

$$\textcircled{3} \quad x_1 = \frac{1 - 2x_2 + 2 - x_3}{3} \rightarrow x_1 = -3$$

- $\textcircled{1}$  But how do we get the matrix into triangular form in the first place?
- $\textcircled{2}$  Performing **Elementary Row Operations** to get the matrix in a triangular form is the way!
- $\textcircled{3}$  Formally, we will refer to this process as **Gaussian Elimination**.



## Example: Row Operations to obtain a triangular matrix

- ① Given the matrix:

$$A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 3 & 3 & 0 & 3 \\ -3 & -2 & 1 & 3 \end{array} \right]$$

By subtracting the first row from the second row, one obtains:

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And adding the first row to the third row, one has:

③

$$A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

Which can be solved using **Back Substitution**

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# Row Echelon Form

We will refer to the nonzero element of  $a_{ij}$  where

$a_{i1} = a_{i2} = \dots = a_{i,j-1} = 0$  as a pivot corresponding to the  $i_{th}$  row. Given the matrix:

$$A = \left[ \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

The pivot of the third row is 2, the pivot the second row is 1, and the pivot of the first row is 3. The matrix can be reduced into **Row Echelon Form (REF)** provided the leading nonzero entry (pivot) have a value of 1. Diving each row by its corresponding pivot value gives:

$$A = \left[ \begin{array}{ccc|c} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

# Row Echelon Form

## Definition (Row Echelon Form)

The system of equations  $Ax = b$  is in **Row Echelon Form (REF)** provided the following conditions hold:

- The first nonzero entry in each row is 1.

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Row Echelon Form

## Definition (Row Echelon Form)

The system of equations  $Ax = b$  is in **Row Echelon Form (REF)** provided the following conditions hold:

- The first nonzero entry in each row is 1.
- If the  $k_{th}$  row does not consist entirely of zeros, then the number of leading zeros in the next row ( $k + 1$ ) is greater than the number of rows in the  $k_{th}$  row.

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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The system of equations  $Ax = b$  is in **Row Echelon Form (REF)** provided the following conditions hold:

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- If the  $k_{th}$  row does not consist entirely of zeros, then the number of leading zeros in the next row ( $k + 1$ ) is greater than the number of rows in the  $k_{th}$  row.
- If a row has entries all zero, then it must be directly below the rows having nonzero entries.

$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Reduced Row Echelon Form

## Lemma (Reduced Row Echelon Form)

*The system of equations  $Ax = b$  is in*  
**Reduced Row Echelon Form (RREF)**  
*provided the following conditions hold:*

- *The Matrix is in Row Echelon Form*

$$A = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$



# Reduced Row Echelon Form

## Lemma (Reduced Row Echelon Form)

The system of equations  $Ax = b$  is in **Reduced Row Echelon Form (RREF)** provided the following conditions hold:

- The Matrix is in Row Echelon Form
- The first nonzero entry in each row is the only nonzero entry in its column

$$A = \left[ \begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array} \right]$$

# Gaussian Elimination

## Definition (Gaussian Elimination)

Given the system of linear equations  $Ax = b$ , where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $b \in \mathbb{R}^m$  whose augmented system is:

$$A = \left[ \begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array} \right]$$

The process of utilizing the 3 elementary row operations to transform the linear system from an augmented matrix to **Row Echelon Form** is referred to as **Gaussian Elimination**.



# Over-determined Systems

1

## Definition (Over-determined Systems)

An **over-determined** system of linear equations is a system of linear equations whose number of equations exceeds the number of unknowns. If  $Ax = b$  is the system of linear equations where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $b \in \mathbb{R}^m$ , then:  $m > n$ .

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2 Consider the system:

$$x_1 + x_2 = 1$$

$$x_1 - x_2 = 3$$

$$-x_1 + 2x_2 = -2$$

## Over-determined Systems

- Which has an augmented system of:

$$A = \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array} \right]$$

After subtracting the first row from the second row and adding the first row to the third row:

# Over-determined Systems

- Which has an augmented system of:

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$$A = \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array} \right]$$

# Over-determined Systems

- Which has an augmented system of:

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After subtracting the first row from the second row and adding the first row to the third row:



$$A = \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array} \right]$$

- After dividing the second row by 2 and adding 3 times the result to the third row:

$$A = \left[ \begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array} \right]$$



# Over-determined Systems

The last row of the reduced form implies that  $0x_1 + 0x_2 = 2$ , which is not possible and a solution does not exist!

Therefore, the system of linear equations is **inconsistent**!

**Remark:** Generally, over-determined systems are more often than not inconsistent, but not necessarily.

# Under-determined Systems



## Definition (Under-determined Systems)

An **under-determined** system of linear equations is a system of linear equations whose number of equations is less than the number of rows. If  $Ax = b$  is the system of linear equations where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $b \in \mathbb{R}^m$ , then:  $m < n$ .

# Under-determined Systems



## Definition (Under-determined Systems)

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- Take the following system:

$$x_1 + 2x_2 + x_3 = 1$$

$$2x_1 + 4x_2 + 2x_3 = 3$$

# Under-determined Systems

- The augmented form is:

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{array} \right]$$

After multiplying the first row by  $-2$  and adding it to the second row one obtains:

# Under-determined Systems

- The augmented form is:

$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{array} \right]$$

After multiplying the first row by  $-2$  and adding it to the second row one obtains:



$$A = \left[ \begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Which implies that:  $x_1 = 1 - 2x_2 - x_3$  where  $x_2$  and  $x_3$  are free variables.  
For simplicity, we will take:  $x_2 = \alpha$  and  $x_3 = \beta$ , then:

# Under-determined Systems

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For simplicity, we will take:  $x_2 = \alpha$  and  $x_3 = \beta$ , then:

- $$\mathbf{x} = \begin{bmatrix} 1 - 2\alpha - \beta \\ \alpha \\ \beta \end{bmatrix}$$

As our solution. Generally, under-determined systems will not have unique solutions ( $\alpha$ ,  $\beta$  are free).

# The Homogeneous Equation



## Definition (The Homogeneous Equation)

The **homogeneous equation** is expressed as:  $\mathbf{Ax} = \mathbf{0}$ , where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$

# The Homogeneous Equation



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The **homogeneous equation** is expressed as:  $\mathbf{Ax} = \mathbf{0}$ , where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$



## Lemma (Solution to Under-determined Systems)

*Given the under-determined system of the homogeneous equation:  $Ax = 0$ , where where  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$  and  $m < n$ , there exists a **non-trivial** solution:  $x \neq 0$ .*

**Remark:** *The trivial solution  $x = 0$ , is always a solution to the homogeneous equation:  $Ax = 0$*



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# Matrix Addition I

## Definition (Matrix Addition)

Suppose  $A \in \mathbb{R}^{m \times n}$ ,  $B \in \mathbb{R}^{m \times n}$  where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} & \dots & b_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mj} & \dots & b_{mn} \end{bmatrix}$$

If  $A$  and  $B$  have the same shape, then we will say that  $A$  and  $B$  are **conformal** and matrix addition is well-defined.

# Matrix Addition II

## Definition (Matrix Addition)

We will define the addition of matrices  $C = A + B$  where the  $(i, j)_{th}$  element of  $C$  is the sum of the corresponding elements of  $A$  and  $B$ , e.g.,  $c_{ij} = a_{ij} + b_{ij}$ :

$$C = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) & \dots & (a_{1j} + b_{1j}) & \dots & (a_{1n} + b_{1n}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) & \dots & (a_{2j} + b_{2j}) & \dots & (a_{2n} + b_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{i1} + b_{i1}) & (a_{i2} + b_{i2}) & \dots & (a_{ij} + b_{ij}) & \dots & (a_{in} + b_{in}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{m1} + b_{m1}) & (a_{m2} + b_{m2}) & \dots & (a_{mj} + b_{mj}) & \dots & (a_{mn} + b_{mn}) \end{bmatrix}$$

A simplified notation is:  $[c_{ij}] = [a_{ij} + b_{ij}]$  for each  $i = \overline{1, m}$  and  $j = \overline{1, n}$ .

# Example

- Let:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

and:

$$B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

# Example

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$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

and:

$$B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

- Then:  $C = A + B$  is:

$$C = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

# Scalar Multiplication I

## Definition (Scalar Multiplication)

Suppose that  $A \in \mathbb{R}^{m \times n}$  and  $\alpha \in \mathbb{R}$ , then one has the following:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1j} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2j} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mj} & \dots & \alpha a_{mn} \end{bmatrix}$$

Alternative notation is:  $\alpha[a_{ij}] = [\alpha a_{ij}]$

# Scalar Multiplication II

Suppose that:

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

After multiplying  $B = 3 * A$ , one obtains:

$$B = \begin{bmatrix} 3 & 9 \\ 15 & 21 \end{bmatrix}$$

# Matrix Transposition

## Definition (Matrix Transposition)

Let  $A \in \mathbb{R}^{m \times n}$ , then the transposition of  $A$  denoted  $A'$  or  $A^T$  is a matrix whose  $(i, j)_{th}$  element of  $A$  is the  $(j, i)_{th}$  element of  $A'$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad A' = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{i1} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{i2} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1j} & a_{2j} & \dots & a_{ij} & \dots & a_{mj} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{in} & \dots & a_{mn} \end{bmatrix}$$

**Remark:** Generally, if  $A \in \mathbb{R}^{m \times n}$ , then:  $A' \in \mathbb{R}^{n \times m}$



## Example

- Consider the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 6 \\ 11 & 21 & 34 & 100 \\ 23 & 50 & 10 & 25 \end{bmatrix}$$

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- The transposition of  $A$  is:

$$A' = \begin{bmatrix} 1 & 11 & 23 \\ 3 & 21 & 50 \\ 5 & 34 & 10 \\ 6 & 100 & 25 \end{bmatrix}$$

- Remark:** Note that  $A' \in \mathbb{R}^{4 \times 3}$ , yet  $A \in \mathbb{R}^{3 \times 4}$ . Generally, if  $A \in \mathbb{R}^{m \times n}$ , then  $A' \in \mathbb{R}^{n \times m}$ .

# Matrix Multiplication I

## Definition (Matrix Multiplication)

Suppose  $A \in \mathbb{R}^{m \times p}$ ,  $B \in \mathbb{R}^{q \times n}$ , then:  $C = AB$  is a well defined matrix product provided  $p = q$ . The resulting matrix  $C \in \mathbb{R}^{m \times n}$ , whose  $(i, j)_{th}$  element is obtained by taking the weighted sum of the  $i_{th}$  row of  $A$  and  $j_{th}$  column of  $B$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \textcolor{red}{a_{i1}} & \textcolor{red}{a_{i2}} & \dots & \textcolor{red}{a_{ij}} & \dots & \textcolor{red}{a_{ip}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mp} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & \textcolor{blue}{b_{1j}} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \textcolor{blue}{b_{2j}} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & \textcolor{blue}{b_{ij}} & \dots & b_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & \textcolor{blue}{b_{pj}} & \dots & b_{pn} \end{bmatrix}$$

When the number of columns of  $A$  is the the same as the number of rows of  $B$ , in which matrix product  $AB$  is well-defined, we will say that  $A$  and  $B$  are **compatible**.

# Matrix Multiplication II

## Definition (Matrix Multiplication)

Recalling that the  $(i, j)_{th}$  element of  $C$  is the product between the  $i_{th}$  row of  $A$  and the  $j_{th}$  column of  $B$  one has:

$$\begin{aligned}c_{ij} &= \mathbf{a}_i' * \mathbf{b}_j \\&= (\mathbf{a}_{i1}, \mathbf{a}_{i2} \dots \mathbf{a}_{ik} \dots \mathbf{a}_{ip})' * \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \\ \vdots \\ b_{pn} \end{bmatrix} \\&= \sum_{k=1}^p \mathbf{a}_{ik} * \mathbf{b}_{kj}\end{aligned}$$

## Example

Take:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

Then:

$$\begin{aligned} C = AB &= \begin{bmatrix} 1 * 2 + 3 * 3 & 1 * 4 + 3 * 5 & 1 * 6 + 3 * 7 \\ 2 * 2 + 4 * 3 & 2 * 4 + 4 * 5 & 2 * 6 + 4 * 7 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 19 & 27 \\ 16 & 28 & 40 \end{bmatrix} \end{aligned}$$

**Remark:** Generally, the product  $C = AB$  can be expressed as:  $[Ab_1, Ab_2 \dots Ab_j \dots Ab_n]$ , meaning the  $j_{th}$  column of  $C$  is  $Ab_j$ .

# Linear Combination

1

## Definition (Linear Combination)

Let  $A \in \mathbb{R}^{m \times n}$ , such that  $A = [a_1, a_2 \dots a_j \dots a_n]$  and  $\alpha_1, \alpha_2, \dots \alpha_j \dots \alpha_n$  be arbitrary scalars, then the a linear combination of the columns of  $A$  is:

$$\alpha_1 a_1 + \alpha_2 a_2 + \dots + \alpha_j a_j + \dots + \alpha_n a_n = \sum_{j=1}^n \alpha_j a_j$$

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2

## Lemma (Existence of a Solution to System of Equations)

Given the system of linear equations:  $Ax = b$  where  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ , and  $x \in \mathbb{R}^n$ , the system is **consistent** provided  $b$  can be expressed as a linear combination of the columns of  $A$ , i.e.,:

$$b = \sum_{j=1}^n x_j a_j$$

for some  $x_j \in \mathbb{R}$ .





# Special Types of Matrices I

1

## Definition (Square Matrix)

A **square matrix** is a matrix whose number of rows are the same as the number of columns: if  $A \in \mathbb{R}^{m \times n}$  then:  $m = n$

2

## Definition (Symmetric Matrix)

A square matrix  $A \in \mathbb{R}^{n \times n}$  is **symmetric** provided  $A' = A$ , which implies that the  $(i, j)_{th}$  element of  $A$  is the  $(i, j)_{th}$  of  $A'$ .

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3

## Definition (Equality of Matrices)

Given two conformal matrices  $A, B \in \mathbb{R}^{m \times n}$ , **two matrices are equal**  $A = B$  iff:

$$a_{ij} = b_{ij} \text{ for each } i, j = \overline{1, m}; \overline{1, n}$$

# Special Types of Matrices II

## Definition (Singular Matrix)

Suppose we are given the homogeneous system of equations:  $Ax = 0$   
 $A \in \mathbb{R}^{n \times n}$ ,  $x \in \mathbb{R}^n$ , we will say that  $A \sim$  **Singular** provided a nontrivial solution  $x \neq 0$  exists. Otherwise  $A \sim$  *Non – Singular*. **Remark:** A **Non-Singular** matrix is also referred to as an **invertible** matrix.

1

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1

## Definition (Identity Matrix)

The **Identity Matrix**, denoted  $I_n$  or  $I$ , is a special square matrix whose diagonal elements are 1, and off-diagonal elements are 0:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ 0 & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

# Special Types of Matrices III

1

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The **Null Matrix** is the matrix consisting entirely of zeros and is denoted  $0_{m \times n}$  or  $0$ .

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2

## Definition (Invertible Matrix)

We will say that  $A \in \mathbb{R}^{n \times n}$  is **invertible** if there exists a matrix  $B$  such that:  $AB = I_n$  and  $BA = I_n$ .

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3

## Definition (Orthogonal Matrix)

The **Orthogonal Matrix**  $Q \in \mathbb{R}^{n \times n}$  is a matrix that satisfies:

$$Q'Q = I_n$$

Which implies that the Orthogonal Matrix's inverse is its transpose.



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# Algebraic Rules of Matrix Operations

## Theorem (Algebraic Rules)

*Suppose  $A, B, C$  are matrices of sizes so that the operations are well-defined and take:  $\alpha, \beta$  as scalars, then the following will hold:*

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Proof of ⑤:  $(A + B)C = AC + BC$ .

①: Suppose  $A, B \in \mathbb{R}^{m \times n}$ ,  $C \in \mathbb{R}^{n \times p}$ .

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① Observe that the  $(i, j)_{th}$  element of  $D$  is:

$$\begin{aligned}d_{ij} &= \sum_{k=1}^n (a_{ik} + b_{ik})c_{kj} \\&= \sum_{k=1}^n [a_{ik}c_{kj} + b_{ik}c_{kj}] \\&= \sum_{k=1}^n a_{ik}c_{kj} + \sum_{k=1}^n b_{ik}c_{kj}\end{aligned}$$

Which is the  $(i, j)_{th}$  element of  $E$ !



## Lemma (The Invertible Matrix)

Suppose  $A \sim$  nonsingular (or invertible  $n \times n$  matrix), let  $A^{-1} \sim$  Inverse of  $A$ , then:

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- ⑤: **Remark:** By Induction, we can generalize ③ to:

$$(A_1 * A_2 * \cdots * A_{k-1} * A_k)^{-1} = A_k^{-1} * A_{k-1}^{-1} * \cdots * A_2^{-1} * A_1^{-1}$$

# Proof that the product of non-singular matrices is non-singular

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- **Remark:** Generalization to ⑤ using Induction is left as an exercise to its reader



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## Theorem (Rules of the Transpose)

Suppose that  $\alpha \in \mathbb{R}$  and  $A, B \in \mathbb{R}^{m \times n}$  then the following will hold:

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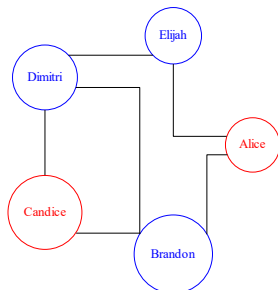
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# Who will sell the most in a village?

## Networks.

- Let  $a'_1 = [Alice, Brandon, Candice, Dimitri, Elijah]$  be a row vector of states the and the  $(i,j)_{th}$  element of  $A$ :  $a_{ij}$  be the sales connection from one person to the next. Who will have the greatest sales potential?

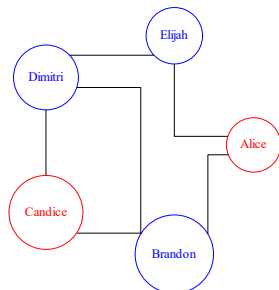


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- We can denote the connections in the village by an Adjacency Matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

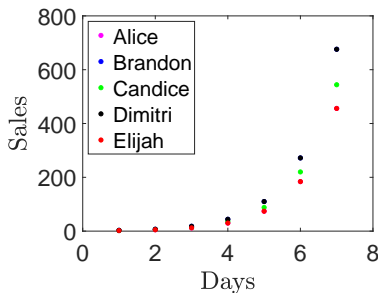


# Evaluating the total number of connections in time

## Lemma (Connectivity over $k$ steps)

Let  $A \in \mathbb{R}^{n \times n}$  be a **Connectivity Matrix** or **Adjacency Matrix** whose  $(i, j)_{th}$  element is a connection from the  $i_{th}$  node to the  $j_{th}$  node (and  $A' = A$ ). Then:  $a_{ij}^{(k)}$  is the number of connections from node  $i$  to  $j$  in  $k$  steps. Additionally, summing over the any row (or column) is the total number of connections in  $k$  steps.

$$A^4 = \begin{bmatrix} 9 & 3 & 6 & 11 & 1 \\ 3 & 15 & 8 & 7 & 11 \\ 6 & 8 & 8 & 8 & 6 \\ 11 & 7 & 8 & 15 & 3 \\ 1 & 11 & 6 & 3 & 9 \end{bmatrix}$$



# References I

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- [2] Leon Stephen. *Linear Algebra with Applications (9th Edition)* (*Featured Titles for Linear Algebra*). London, England: Pearson, 2014.