Linear Algebra from Scratch: Eigenvalues

Instructor Anthony

"It does not matter how slowly you go as long as you do not stop." - Confucius

Udemy Open Course

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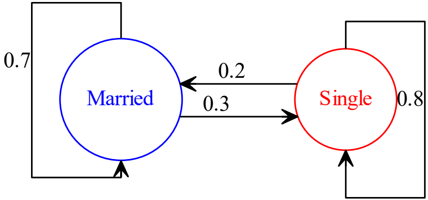
Introduction

Suppose: $\mathbf{w} \in \mathbb{R}^2$ and $A \in \mathbb{R}^{2x2}$ where $w_1 \sim$ number of married women and $w_2 \sim$ number of single women, where $a_{ij} \in [0, 1]$ is the Probability of transition between i_{th} state to the j_{th} state in a year. We will consider the following:

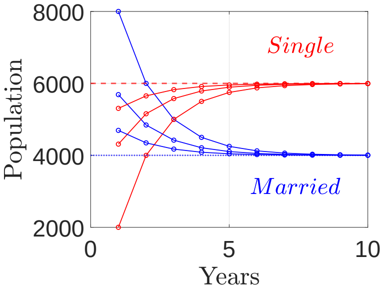
$$A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \ \& \ \mathbf{w}^{(0)} = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$$



Introduction

Since $a_{ij} \sim$ Probability of transition from the i_{th} state to the j_{th} state, then $(i)_{th}$ element of $A^k \mathbf{w}^{(0)} = \mathbf{w}^{(k)}$ is the number of Married (for $i = 1$) or Single (for $i = 2$) on the k_{th} day. As long as the initial distribution of Married and Single add up to 10000, then one may check that $\mathbf{w}^{(0)} \longrightarrow \mathbf{w}^{(ss)}$ as $k \longrightarrow \infty$ where:

$$\mathbf{w}^{(ss)} = (4000, 6000)^T = 2000 * (2, 3)^T.$$



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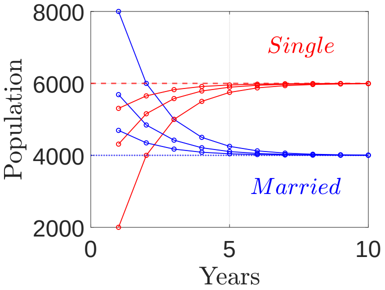
$\mathbf{w}^{(ss)} = (4000, 6000)^T = 2000 * (2, 3)^T$.

We will refer to $\mathbf{w}^{(ss)} \sim$ Steady State (Convergent) Vector. We will denote $\mathbf{x_1} = (2, 3)^T$ and one may check that $A\mathbf{x_1} = 1 * \mathbf{x_1}$. We will refer to

1 $\sim$ **Eigenvalue** and $\mathbf{x_1} \sim$ *Eigenvector*.



The Eigenvalue Problem

Definition (The Eigenvalue Problem)

Suppose $A \in \mathbb{R}^{nxn}$, then $\lambda \in \mathbb{C}$ is an Eigenvalue / Characteristic Value provided $A\mathbf{x} = \lambda\mathbf{x}$. Such $\mathbf{x} \neq \mathbf{0}$ corresponding to λ is the **Eigenvector / Characteristic Vector** corresponding to λ.

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Lemma (Characteristic Equation: $A\mathbf{x} = \lambda\mathbf{x}$)

Suppose $A \in \mathbb{R}^{nxn}$ and take $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \in \mathbb{R}^{n}$, then The Characteristic Equation $(p(\lambda))$ is:

$$A\mathbf{x} = \lambda\mathbf{x} \longleftrightarrow A\mathbf{x} - \lambda\mathbf{x} = 0 \longleftrightarrow (A - \lambda I)\mathbf{x} = 0 \longleftrightarrow Det(A^T A - \lambda I_n) = 0$$

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Remark: $A \in \mathbb{R}^{nxn} \longleftrightarrow p(\lambda) \in P_{n+1} \longrightarrow$ *there are n Complex Roots not including Algebraic multiplicity.*

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Equivalent Conditions of The Eigenvalue: λ

Theorem (The Eigenvalue: λ where $A\mathbf{x} = \lambda\mathbf{x}$)

Suppose $A \in R^{nxn}$ and $\lambda \in \mathbb{C}$, then the following are equivalent:

- *1: $\lambda \sim$ Eigenvalue of A*

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- 5: $Det(A - \lambda I) = 0 \longleftrightarrow p(\lambda) = 0$

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- 2: $(A - \lambda I)\mathbf{x} = 0$ *has a nontrivial solution* $\mathbf{x} \neq \mathbf{0}$.
- 3: $Nul(A - \lambda I) \neq \{\mathbf{0}\}$
- 4: $A - \lambda I \sim$ *Singular.*
- 5: $Det(A - \lambda I) = 0 \longleftrightarrow p(\lambda) = 0$

Remark: $Nul(A - \lambda I) = \{\mathbf{x} \in \mathbb{C}^n \mid A\mathbf{x} = \lambda\mathbf{x}\} \longleftrightarrow$ *Eigenspace of A corresponding to λ*

The Eigenspace

Theorem (Eigenspace: $Nul(A - \lambda I_n)$)

Suppose $A \in \mathbb{R}^{nxn}$ *and* $\mathbf{x} \in \mathbb{C}^n$ *where* $\mathbf{x} \neq \mathbf{0}$. *The Eigenspace is the set of all* $\mathbf{x} \in \mathbb{C}^n$ *such that:* $A\mathbf{x} = \lambda\mathbf{x}$ *for a given* λ *is a Subspace of* $\mathbb{C}^n$.

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Proof.

$$1 : \textit{Assume } \mathbf{x}, \mathbf{y} \in Nul(A - \lambda I_n) \ \& \ \textit{Take } \alpha, \beta \in \mathbb{R}$$

$\square$

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$$1 : \textit{Assume } \mathbf{x}, \mathbf{y} \in Nul(A - \lambda I_n) \ \& \ \textit{Take } \alpha, \beta \in \mathbb{R}$$

$$2 : A(\alpha \mathbf{x} + \beta \mathbf{y}) = A(\alpha \mathbf{x}) + A(\beta \mathbf{y})$$

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Proof.

1 : *Assume* $\mathbf{x}, \mathbf{y} \in Nul(A - \lambda I_n)$ & *Take* $\alpha, \beta \in \mathbb{R}$

2 : $A(\alpha\mathbf{x} + \beta\mathbf{y}) = A(\alpha\mathbf{x}) + A(\beta\mathbf{y})$

3 : $\alpha A\mathbf{x} + \beta A\mathbf{y}$

□

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$$\text{3} : \alpha A\mathbf{x} + \beta A\mathbf{y}$$

$$\text{4} : \alpha\lambda\mathbf{x} + \beta\lambda\mathbf{y} = \lambda(\alpha\mathbf{x} + \beta\mathbf{y})$$

$\square$

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Example: Computing the Eigenvalues and Eigenvectors

Find the Eigenvalues and Eigenvectors of $A \in \mathbb{R}^{2x2}$

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \longrightarrow Det(A - \lambda I) = 0 \longrightarrow \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix}$$

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One can also check that:

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \longrightarrow A\mathbf{x} = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 * \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3 * \mathbf{x}$$

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The Eigenvectors are:

$$A - 2I = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \longrightarrow Nul(A - 2I) = Span\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \longrightarrow \mathbf{x}_{\lambda_1} = (1, 1)^T$$

$$A - 3I = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \longrightarrow Nul(A - 3I) = Span\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \longrightarrow \mathbf{x}_{\lambda_2} = (2, 1)^T$$

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The Sum and Product of Eigenvalues

Theorem (Eigenvalues of $A \in \mathbb{R}^{nxn}$)

Suppose $A \in \mathbb{R}^{nxn}$ and given the Eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$, then:

$$\sum_{j=1}^{n} a_{jj} = \sum_{j=1}^{n} \lambda_j$$

$$Det(A) = \prod_{j=1}^{n} \lambda_j$$

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Lemma (Characteristic Polynomial of $A \in \mathbb{R}^{2x2}$)

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ *where* $\mathbf{x} \neq \mathbf{0}$, *then:*

$$p(\lambda) = Det(A - \lambda I_2) = \lambda^2 - Trace(A)\lambda + Det(A)$$

$$Where: \; Trace(A) = a_{11} + a_{22}$$

Similar Matrices and Eigenvalues

Theorem (Similar Matrices have identical Eigenvalues)

Suppose $A \sim Similar \sim B$ where $A, B \in \mathbb{R}^{nxn}$, then $p_A(\lambda) = p_B(\lambda)$.
That is, A and B have the same set of Eigenvalues.
Remark: *The set of all eigenvalues corresponding to a matrix, A, and is referred to as the spectrum of A.*

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A System of Differential Equations

Definition (System of Differential Equations)

Suppose $y_i(t) \in \mathcal{C}^1[a,b]$ for each $i = 1, 2, \ldots, n$ and $a_{ij} \in \mathbb{R}$ for $i, j = 1, 2, \ldots, n$ then: a **System of Linear Differential Equations** has the form:

$$
\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots a_{1j}y_j + \cdots + a_{1n}y_n \,\Big|\, y_1(a) = y_1^{(0)} \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2j}y_j + \cdots + a_{2n}y_n \,\Big|\, y_2(a) = y_2^{(0)} \\
\vdots &= \vdots \\
y_i' &= a_{i1}y_1 + a_{i2}y_2 + \cdots + a_{ij}y_j + \cdots + a_{in}y_n \,\Big|\, y_i(a) = y_i^{(0)} \\
\vdots &= \vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nj}y_j + \cdots + a_{nn}y_n \,\Big|\, y_n(a) = y_n^{(0)}
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\end{aligned}
$$

Remark: One can take: $A = [a_{ij}] \in \mathbb{R}^{nxn}$, $Y = [y_1, y_2, \ldots, y_j, \ldots, y_n]$, and $Y' = [y_1', y_2', \ldots, y_j', \ldots, y_n']$ as a vector Function, then the matrix form is: $Y' = AY$

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Systems of Differential Equations

Since $n = 1 \longrightarrow y' = ay \longrightarrow y(t) = ce^{at}$, then it is natural to make the educated guess: $\mathbf{Y}(t) = [y_1(t), y_2(t) \ldots, y_j(t), \ldots, y_n(t)]$ and $\mathbf{Y}(t) = \mathbf{x}e^{\lambda t}$ for some $\mathbf{x} \in \mathbb{R}^n$.

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It follows that:

$$\mathbf{Y}' = \lambda \left(\mathbf{x}e^{\lambda t}\right) = \lambda Y$$

$$Since: \ \mathbf{Y}' = A\mathbf{Y} \longrightarrow A\mathbf{Y} = \lambda \mathbf{Y}$$

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If $(\mathbf{x}, \lambda)$ is an Eigenpair associated to A, then $\mathbf{x}e^{\lambda t}$ is a solution to $\mathbf{Y}' = A\mathbf{Y}$ since:

$$\mathbf{Y}' = A\mathbf{Y} = A\left(\mathbf{x}e^{\lambda t}\right) = e^{\lambda t}(A\mathbf{x}) = e^{\lambda t}(\lambda\mathbf{x}) = \lambda\mathbf{x}e^{\lambda t}$$

The Initial Value Problem (IVP)

Lemma (IVP's are Subspaces)

Suppose $\mathbf{Y_1}, \mathbf{Y_2}, \ldots, \mathbf{Y_n}$ *are solutions to* $\mathbf{Y}' = A\mathbf{Y}$*, then:*
$c_1\mathbf{Y_1} + c_2\mathbf{Y_2} + \cdots + c_j\mathbf{Y_j} + \cdots + c_n\mathbf{Y_n}$ *is also a solution to* $\mathbf{Y}' = A\mathbf{Y}$*.*
We refer $S = \left\{\mathbf{Y_1}, \mathbf{Y_2}, \ldots, \mathbf{Y_n} \,\middle|\, \mathbf{Y}_j' = A\mathbf{Y}_j \text{ and } \mathbf{Y}_j \in \mathcal{C}^1[a,b]\right\} \subset C^1[a,b]$ *and is in fact, a subspace of* $C^1[a,b]$*.*
If each function is prescribed to an initial value $y_i(a) = y_i^{(0)}$ *then:*

$$\mathbf{Y}' = A\mathbf{Y} \ \&\ \mathbf{Y}(a) = \mathbf{Y}_0 \text{ where } \mathbf{Y}_0 = [y_1^{(0)}, y_2^{(0)}, \ldots, y_j^{(0)}, \ldots, y_n^{(0)}]$$

Is the **Initial Value Problem***.*

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Is the **Initial Value Problem***.*

Lemma (General and Particular Solution to IVP's)

Given the n_{th} *dimensional IVP:* $\mathbf{Y}' = A\mathbf{Y}$ *subject to:* $\mathbf{Y}(a) = \mathbf{Y}_0$ *the* **general solution** *is:*

$$\mathbf{Y} = \sum_{j=1}^{n} c_j \mathbf{x} e^{\lambda_j t} \text{ for some } c_j \in \mathbb{R}$$

The **particular solution** *is determined by using the initial conditions and solving for* c_j*.*

Complex Vectors are vectors with complex entries

Definition (Complex Vector Space: $\mathbb{C}^n$)

The set of all complex vectors with n components is denoted:

$$\mathbb{C}^n = \left\{ \mathbf{z} \;\middle|\; \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_j \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} a_1 + i * b_1 \\ a_2 + i * b_2 \\ \vdots \\ a_j + i * b_j \\ \vdots \\ a_n + i * b_j \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_j \\ \vdots \\ a_n \end{bmatrix} + i * \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \\ \vdots \\ b_n \end{bmatrix} \right\}$$

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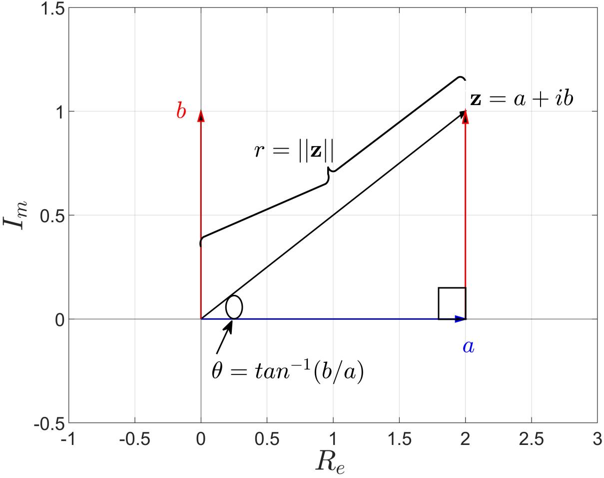
Complex Vectors are vectors with complex entries



Complex Eigenvalues

Lemma

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ *for* $A \in \mathbb{R}^{nxn}$, $\mathbf{x}$ *is an* $n \ x\ 1$ *vector where* $\mathbf{x} \neq \mathbf{0}$, *then*

$$Det(A - \lambda I_n) = 0$$

$$If: \ \lambda \in \mathbb{C} \longrightarrow \mathbf{x}_\lambda \in \mathbb{C}^n$$

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$$Det(A - \lambda I_n) = 0$$

$$If: \ \lambda \in \mathbb{C} \longrightarrow \mathbf{x}_\lambda \in \mathbb{C}^n$$

Remark: *by* **The Fundamental Theorem of Algebra**, *the roots of the characteristic equation* $(p(\lambda) = Det(A - \lambda I_n))$ *admit up to n Complex Roots including multiplicity and the eigenvectors corresponding to the eigenvalues will be complex if there is at least one pair of complex eigenvalues.*

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Example: IVP

Solve the following Initial Value Problem:

$$y_1' = -2y_1 + 2y_2 \mid y_1(0) = 1$$
$$y_2' = -y_1 - 2y_2 \mid y_2(0) = -1$$

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Observe the IVP can be transformed into its vector function equivalence:

$$\mathbf{Y}' = A\mathbf{Y} \,\Big|\, A = \begin{bmatrix} -2 & 2 \\ -1 & -2 \end{bmatrix} \; \mathbf{y}^{(0)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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By previous theory, the solution is of the form $\mathbf{C} * \mathbf{x}e^{\lambda t}$ for $\mathbf{x}, \mathbf{C} \in \mathbb{R}^2$.
Solving the characteristic equation to find the eigenvalues of A:
$Det(A - \lambda I_2) = 0 \longrightarrow \lambda^2 - Trace(A)\lambda + Det(A) = 0 \longrightarrow \lambda^2 - (-4)\lambda + 6 = 0 \longrightarrow$
$\lambda_{1,2} = -2 \pm i\sqrt{2}$ and the eigenvectors associated to the eigenvalues are:

$$\mathbf{x}_1 = (0.82, 0.58i) \;\&\; \mathbf{x}_2 = (0.82, -0.58i)$$

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The **general solution** of the IVP is:

$$y_1 = c_1\left(0.82e^{(-2+i\sqrt{2})t}\right) + c_2\left(0.58ie^{(-2-i\sqrt{2})t}\right)$$
$$y_2 = c_1\left(0.82e^{(-2+i\sqrt{2})t}\right) - c_2\left(0.58ie^{(-2-i\sqrt{2})t}\right)$$

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Example: IVP Cont'd

After substituting $y_1(0) = 1$ and $y_2(0) = -1$ one obtains:

$$\begin{aligned} 1 &= 0.82c_1 + 0.58ic_2 \\ -1 &= 0.82c_1 - 0.58ic_2 \end{aligned}$$

Determining the coefficient vector $C = (c_1, c_2)^T$ by solving the augmented system:

$$\left[\begin{array}{cc|c} 0.82 & 0.58i & 1 \\ 0.82 & -0.58i & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0.61 + 0.87i \\ 0 & 1 & 0.61 - 0.87i \end{array}\right] \longrightarrow \mathbf{C} = (c_1, c_2)^T$$

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Example: IVP Cont'd

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Substituting the coefficient vector into the general solution gives:

$$y_1 = (0.61 + 0.87i) * 0.82e^{(-2+i\sqrt{2})t} + (0.61 - 0.87i) * 0.58ie^{(-2-i\sqrt{2})t}$$
$$y_2 = (0.61 + 0.87i) * 0.82e^{(-2+i\sqrt{2})t} - (0.61 - 0.87i) * 0.58ie^{(-2-i\sqrt{2})t}$$

$$e^{a+ib} = e^a * (Cos(b) + i * Sin(b)) \ \& \ \tau \stackrel{\triangle}{\equiv} \sqrt{2}t \rightarrow$$

$$y_1 = e^{-\sqrt{2}\tau} * [(0.50 + 0.71i)(Cos(\tau) + iSin(\tau)) + (0.50 + 0.36i)(Cos(\tau) + iSin(\tau))]$$
$$y_2 = e^{-\sqrt{2}\tau} * [(0.50 + 0.71i)(Cos(\tau) + iSin(\tau)) - (0.50 + 0.36i)(Cos(\tau) + iSin(\tau))]$$

Example: IVP Cont'd

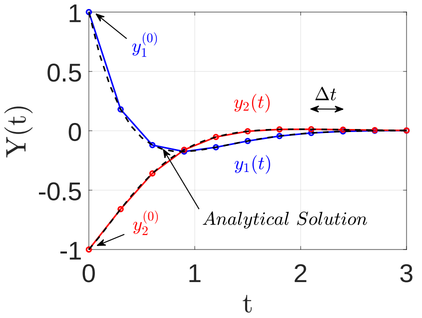
1: The analytical Solution is of the form:

$$\mathbf{Y} = c_1 \mathbf{x}_{\lambda_1} e^{\lambda_1 t} + c_2 \mathbf{x}_{\lambda_2} e^{\lambda_2 t}$$

2: The **numerical solution** is solved using an RK45 integrator in MATLAB.

3: Δt is the integration step.

4: Generally, numerical solutions approximate the "true" (**analytical**) **solution**.



Properties of Complex Values

Theorem

Suppose $z, w \in \mathbb{C}$ *where* $z = a + ib$ *and* $w = c + id$ *then:*

- 1: $\overline{z} = a - ib \sim$ *conjugate of* z.

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Suppose $z = 3 + 2 * i$

$$\overline{z} = 3 - 2 * i$$
$$r = \sqrt{9 + 4} = \sqrt{13}$$
$$\theta = tan^{-1}(-2/3) \approx 326^\circ$$

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Complex Matrices

Definition ($\mathbb{C}^{mxn}$)

Complex matrices are matrices where each entry is a complex value. The set of all complex matrices is of the form:

$$\mathbb{C}^{mxn} = \{Z \mid z_{kj} = a_{kj} + i * b_{kj} \ \& \ Z = A + i * B \ \ni: \ A, B \in \mathbb{R}^{mxn}\}$$

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Example:

$$Z = \begin{bmatrix} 2+3i & 2-3i \\ 3-i & 2i \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} + i * \begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix} = A + i * B$$

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The Complex Inner Product

Definition ($\langle \cdot \rangle$ over $\mathbb{C}^n$)

Suppose $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ then the Complex Inner Product between $\mathbf{z} = (a_1 + i * b_1, a_2 + i * b_2, \dots, a_n + i * b_n)$ and $\mathbf{w} = (c_1 + i * d_1, c_2 + i * d_2, \dots, c_n + i * d_n)$ is:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H * \mathbf{z} = \overline{\mathbf{w}}^T * \mathbf{z}$$

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$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H * \mathbf{z} = \overline{\mathbf{w}}^T * \mathbf{z}$$

Suppose $\mathbf{z} = (5 + i, 1 - 3i)$ and $\mathbf{w} = (2 + i, -2 + 3i)$ then:

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H * \mathbf{z} = (2 - i, -2 - 3i) * \begin{bmatrix} 5 + i \\ 1 - 3i \end{bmatrix} = (11 - 3i) + (-11 + 3i) = 0$$

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Thus $\langle \mathbf{z}, \mathbf{w} \rangle = 0$ are orthogonal. Diving each vector by its magnitude: $\mathbf{u_z} = \frac{\mathbf{z}}{\sqrt{\langle \mathbf{z}, \mathbf{z} \rangle}}$ and $\mathbf{u_w} = \frac{\mathbf{w}}{\sqrt{\langle \mathbf{w}, \mathbf{w} \rangle}}$ will generate an Orthonormal set on $\mathbb{C}^2$:

$$S = \{\mathbf{u_z}, \mathbf{u_w}\}$$

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Properties of Inner Product

$\mathbb{R}^n$	$\mathbb{C}^n$
$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y}$	$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{z} = \overline{\mathbf{w}}^T * \mathbf{z}$
$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$	$\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$
$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$	$\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$
$\langle \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z} \rangle = \alpha \langle \mathbf{x}, \mathbf{z} \rangle + \beta \langle \mathbf{y}, \mathbf{z} \rangle$	$\langle \alpha \mathbf{z} + \beta \mathbf{w}, \mathbf{u} \rangle = \alpha \langle \mathbf{z}, \mathbf{u} \rangle + \beta \langle \mathbf{w}, \mathbf{u} \rangle$
$\lvert\lvert \mathbf{x} \rvert\rvert^2 = \mathbf{x}^T \mathbf{x}$	$\lvert\lvert \mathbf{z} \rvert\rvert^2 = \mathbf{z}^H \mathbf{z}$

For each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $\mathbf{z}, \mathbf{w}, \mathbf{u} \in \mathbb{C}^n$, and $\alpha, \beta \in \mathbb{C}$

Types of Complex Matrices

Definition (Conjugate Transpose)

Suppose $Z \in \mathbb{C}^{mxn}$ and $Z = A + i * B$ for $A, B \in R^{mxn}$ then the **Conjugate Transpose** of Z is $Z^H = \left(\overline{A + i * B}\right)^T = A^T - i * B^T$

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Definition (Hermitian Matrices)

$Z \in \mathbb{C}^{nxn}$ is a **Hermitian Matrix** $\longleftrightarrow Z = Z^H$

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Reference: Leon, 2014 [2]

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Definition (Normal Matrices)

$A \in \mathbb{C}^{nxn}$ is a **Normal Matrix** $\longleftrightarrow A^H A = AA^H$

Properties of the Conjugate Transpose

$\mathbb{R}^{mxn}$	$\mathbb{C}^{mxn}$
$(A^T)^T = A$	$(Z^H)^H = Z$
$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T$	$(\alpha Z + \beta W)^H = \overline{\alpha} Z^H + \overline{\beta} W^H$
$(AB)^T = B^T A^T$	$(ZW)^H = W^H Z^H$

For each $\alpha, \beta \in \mathbb{C}$ compatible (or conformal) A, B, Z, W

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Euler's Formula

Theorem (Euler's Formula)

Suppose $z \in \mathbb{C} \longrightarrow z = a + ib$ *and* $a, b, r \in \mathbb{R}$ *then the following will hold:*

$$z = re^{i\theta} = r * (Cos(\theta) + iSin(\theta)) = r * Cos(\theta) + i * r * Sin(\theta)$$

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$$z = re^{i\theta} = r * (Cos(\theta) + iSin(\theta)) = r * Cos(\theta) + i * r * Sin(\theta)$$

Remark: *Any complex number has an associated magnitude (radius:* $r = \sqrt{a^2 + b^2}$*) and angle (*$\theta = Tan^{-1}(b/a)$*)*

Example: The Saddle

Solve the following system:

$$y_1' = 3y_1 + 4y_2$$
$$y_2' = 3y_1 + 2y_2$$

Taking: $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$

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Taking: $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$ The eigenvalues are $Det(A - \lambda I_2) = 0 \longrightarrow$ $\lambda_{1,2} = 6, -1$ with corresponding eigenvectors: $\mathbf{x}_{\lambda_1} = (4, 3)^T$ and $\mathbf{x}_{\lambda_2} = (1, -1)^T$

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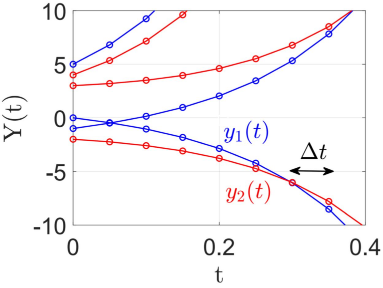
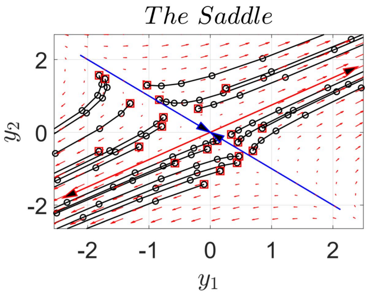
The general solution is:

$$\mathbf{Y}' = c_1 \mathbf{x}_{\lambda_1} e^{\lambda_1 t} + c_2 \mathbf{x}_{\lambda_2 t} e^{\lambda_2 t} = \begin{bmatrix} 4c_1 e^{6t} + c_2 e^{-t} \\ 3c_1 e^{6t} - c_2 e^{-t} \end{bmatrix}$$

Where choice of $c_1, c_2 \in \mathbb{R}$ depend on initial conditions.

Example: The Saddle





The Saddle.

Saddle nodes often separate a stationary (fixed) point between stable and unstable manifolds. Depending on the initial conditions, y_1 and y_2 converge to different extrema in the ω-limit as $t \longrightarrow \infty$

Remark: The *intersection* of the *eigenvectors* corresponds to a stationary point.

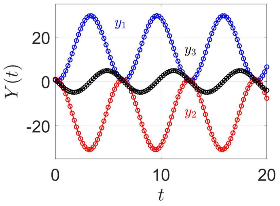
Generally, **stationary points** satisfy: $\mathbf{Y}' = 0$ for some $\mathbf{y}_{ss} = [y_1^*, y_2^*, \ldots, y_j^*, \ldots, y_n^*]$

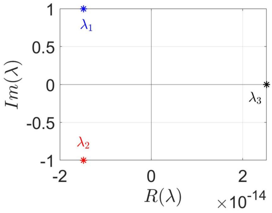
Reference: Leon, 2014 [2]

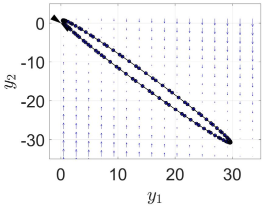
Example: The Andronov-Hopf gives rise to Periodic Behavior

Periodic behavior can occur when there is at least one pair of purely imaginary eigenvalues and the dominant real eigenvalue is "close" to zero ($\max(R(\lambda)) \approx 0$).

Periodic behavior is often found in conservative systems such as Simple Harmonic Motion.







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Higher Order Systems

Theorem

Suppose $y \in C^2[a,b]$, $\alpha, \beta, \gamma \in \mathbb{R}$ then the second order differential equation:

$$\gamma y''(t) + \alpha y'(t) + \beta y = 0 \,\Big|\, y(a) = y^{(0)} \;\&\; y'(a) = y'^{(0)}$$

can be equivalently solved by substituting $y_1 = y$, $y_2 = y'$ and evaluating the following coupled system:

$$\begin{aligned} y_1' &= y_2 \mid y_1(a) = y(a) \\ y_2' &= \frac{1}{\gamma}(-\alpha y_2 - \beta y_1) \,\Big|\, y_2(a) = y'(a) \end{aligned}$$

Remark: *One can convert any n_{th} order differential equation into a system of n coupled linear equations.*

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Example: Simple Harmonic Motion

Solve the following conservative system:

$$m * y''(t) = -k * y \mid y(0) = 1, y'(0) = 1$$

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After substituting $y_1 = y$ and $y_2 = y_1'$ one obtains:

$$\begin{aligned} y_1' &= 0y_1 + y_2 \mid y_1(0) = 1 \\ y_2' &= -\frac{k}{m}y_1 + 0y_2 \,\Big|\, y_2(0) = 1 \end{aligned}$$

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One can express the linear system as:e

$$\mathbf{Y}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} * \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

A close observation one sees: $\lambda_{1,2} = \pm i * \sqrt{k/m}$. Actually, the angular frequency of the oscillation is $\omega^2 = k/m$ and is inversely proportional to the mass as expected.

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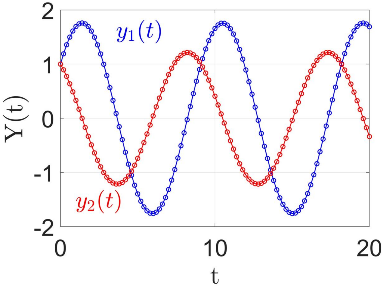
Example: Simple Harmonic Motion

An object at rest stays at rest unless acted upon by an external force. In this context, because each of the eigenvalues is purely imaginary, the energy of the system is conserved.



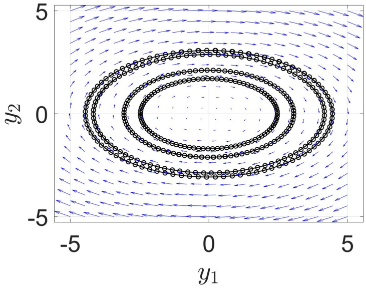


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Table of Contents

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Fundamental Theorem of Algebra Revisited

Lemma

Suppose one solves the eigenvalue problem $A\mathbf{x} = \lambda\mathbf{x}$ for $A \in \mathbb{R}^{nxn}$, $\lambda \in \mathbb{C}$, and $\mathbf{x} \neq 0$ or equivalently solves for the n roots of the characteristic equation $p_A(\lambda) = Det(A - \lambda I_n) = 0$ with respect to λ, then A has n complex roots (including repetition).

In fact, the characteristic equation of A can be expressed as follows:

$$p_A(\lambda) = (\lambda - \lambda_1)^{k_1} * (\lambda - \lambda_2)^{k_2} * \cdots * (\lambda - \lambda_r)^{k_r}$$

Where r is the number of distinct (unique) eigenvalues and k_j is the Algebraic Multiplicity (number of duplicates) corresponding to the j_{th} eigenvalue for $j = 1, 2, \ldots, r$.

Fundamental Theorem of Algebra Revisited

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Where r is the number of distinct (unique) eigenvalues and k_j is the Algebraic Multiplicity (number of duplicates) corresponding to the j_{th} eigenvalue for $j = 1, 2, \ldots, r$.

Remark: $k_1 + k_2 + \cdots + k_r = n$

If A has unique eigenvalues, then: $r = n$

The Geometric Multiplicity of λ_j is

$$dim(Nul(A - \lambda_j I_n)) = Nullity(A - \lambda_j I_n)$$

Linear Independence of Eigenvectors

Definition (Spectrum of a Matrix A)

The **spectrum** of an $n \times n$ matrix A is a multi-set (some elements may be repeated) of the eigenvalues corresponding to the solution $A\mathbf{x} = \lambda\mathbf{x}$ for some $n \times 1$ vector $\mathbf{x}$.

That is, each element in the multi-set must satisfy the eigenvalue problem.

Linear Independence of Eigenvectors

Definition (Spectrum of a Matrix A)

The **spectrum** of an $n \times n$ matrix A is a multi-set (some elements may be repeated) of the eigenvalues corresponding to the solution $A\mathbf{x} = \lambda\mathbf{x}$ for some $n \times 1$ vector $\mathbf{x}$.

That is, each element in the multi-set must satisfy the eigenvalue problem.

Theorem (Distinct Eigenvalues)

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct eigenvalues of the characteristic equation: $p_A(\lambda) = 0 = Det(A - \lambda I_n)$ where $A \in \mathbb{R}^{nxn}$, then the corresponding eigenvectors $\mathbf{x}_{\lambda_1}, \mathbf{x}_{\lambda_2}, \ldots, \mathbf{x}_{\lambda_n}$ form a **Linearly Independent** *set.*

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Remark: *Any collection of $k \leq n$ distinct eigenvalues with associated k eigenvectors is also Linearly Independent.*

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Diagonalizable Matrices

Definition (Diagonalization or Spectral Decomposition of $A \in \mathbb{R}^{nxn}$)

Suppose $A \in \mathbb{R}^{nxn}$, then A is diagonalizable provided:

$$A = XDX^{-1} \longleftrightarrow X^{-1}AX = D \longleftrightarrow AX = XD$$

For some nonsingular $X \in \mathbb{C}^{nxn}$ and some diagonal matrix $D \in \mathbb{C}^{nxn}$.
Remark: If A is diagonalizable, then X diagonalizes A.
Additionally, one refers to $A = XDX^{-1}$ as the **Diagonalization** of A by X or **Spectral Decomposition** of A.

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Theorem

Given $A \in \mathbb{R}^{nxn}$, we say A is ***diagonalizable*** $\longleftrightarrow$ *A has n* ***Linearly Independent*** *eigenvectors*

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A is diagonalizable $\longleftrightarrow$ A has n Linearly Independent Eigenvalues.

Suppose $A \in \mathbb{R}^{nxn}$ and suppose $\mathbf{x}_{\lambda_j}$ be the eigenvector associated to eigenvalue λ_j for $j = 1, 2, \ldots, n$ (not necessarily distinct). Then:

$$
\begin{aligned}
AX &= A[\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] \\
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By the **Invertible Matrix Theorem**, X is nonsingular $\longleftrightarrow$ $Det(X) \neq 0 \longleftrightarrow X$ has n Linearly Independent columns.

Since the j_{th} column of X is $\mathbf{x}_{\lambda_j}$ it follows that X is nonsingular $\longleftrightarrow$ X has n Linearly Independent columns. Then: $A = XDX^{-1}$

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- (5): *If A is* **diagonalizable**, *then* $A^k = XD^kX^{-1}$ *for any* $k \in \mathbb{N}$

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Example: Diagonalization

Determine the Spectral Decomposition of $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

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Example: Diagonalization with a Repeated Root

Determine if $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$ is diagonalizable.

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Then: $\lambda_{1,2} = 2$ and $\lambda_3 = 4$. Recall that A is diagonalization $\longleftrightarrow$ the eigenvectors of A are Linearly Independent.

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$$A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \\ -3 & 6 & 0 \end{bmatrix} \longrightarrow \mathbf{x}_{\lambda_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \; \& \; \mathbf{x}_{\lambda_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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Likewise, one can show that:

$$\mathbf{x}_{\lambda_3} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Taking $X = [\mathbf{x}_{\lambda_1}, \mathbf{x}_{\lambda_2}, \mathbf{x}_{\lambda_3}]$ one can show $Det(X) \neq 0 \longleftrightarrow A \sim$ Diagonalizable.

Stochastic Processes

Definition (Stochastic Process)

A **Stochastic Process** is a sequence of trials for which each trial depends only on the current state and is probabilistic.

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- (1): There are a finite number of possible outcomes.
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- (3): The probabilities do not depend nor change in time.

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Markov Chain

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- Typically, one defines a probability vector: $\mathbf{x}^{(0)} = (x_1, x_2, \ldots, x_n)^T$ where $\sum_{j=1}^{n} x_j = 1$ then:

$$\mathbf{x}^{(k)} = P^k \mathbf{x}^{(0)} \sim k_{th} \text{ } state$$

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Convergence & Steady State of a Markov Chain

Definition (Steady State of Markov Chain)

Suppose P is the Stochastic Matrix corresponding to the Markov Chain where p_{ij} is the probability of transition between i_{th} state and j_{th} state and $\mathbf{x}^{(0)}$ is the initialized probability vector.

The Markov Chain is **convergent** provided a **Steady State** vector: $\mathbf{x}^*$ exists if the j_{th} entry of $\mathbf{x}^*$ satisfies:

$$\lim_{k \longrightarrow \infty} x_j^{(k)} = x_j^* \; for \; each \; j = 1, 2, \ldots, n$$

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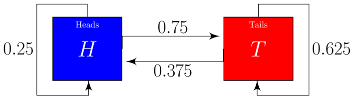
Theorem (Convergence of a Markov Chain)

If P is the transition matrix corresponding to the Markov Chain and $\mathbf{x}^*$ is a **Steady State** *vector, then:*

$$\text{1} : \sum_{j=1}^{n} x_j^* = 1 \longleftrightarrow \mathbf{x}^* \sim \textit{Probability vector}$$

$$\text{2} : \max|\lambda_j| = \lambda_1 = 1 \;\&\; \mathbf{x}^* \sim \textit{Eigenvector of } \lambda_1 = 1$$

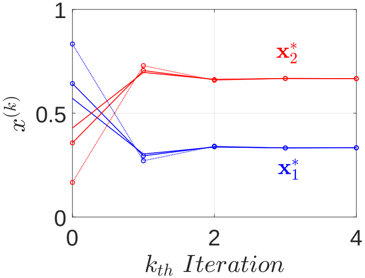
Example: Markov Chain



The Transition Matrix is:

$$P = \begin{bmatrix} 0.25 & 0.375 \\ 0.75 & 0.625 \end{bmatrix}$$

Remark: The steady state vector is unique if $\lambda_1 = 1$



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$$A = U\Sigma V^T$$

For some m x n matrix Σ such that $\sigma_{jj} = \sqrt{\lambda_j}$ where $\lambda_j \geq \lambda_{j+1}$ where λ_j is an eigenvalue to $A^T A$ for each $j = 1, 2, \ldots, n-1$ and U is an orthogonal m x m matrix, V is an orthogonal n x n matrix.

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Lemma (Eigenvalues and Symmetric Matrices)

The eigenvalues of any Symmetric Matrix n x n matrix A are all real.
Remark: *The eigenvalues of* $A^T A$ *are non-negative (*$\lambda_j \geq 0$ *for each* $j = 1, 2, \ldots, n$*).*

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SVD Properties

Theorem (Equivalent Conditions of the SVD)

Given an m x n matrix A with $Rank(A) = r$ that has a Singular Value Decomposition $A = U\Sigma V^T$ then the following are equivalent:

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- (7): *$S_3 = \{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ form an* **Orthonormal** *Basis on $R(A)$.*
- (8): *$S_4 = \{\mathbf{u_{r+1}}, \mathbf{u_{r+2}}, \ldots, \mathbf{u_m}\}$ form an* **Orthonormal** *Basis on $Nul(A^T)$.*

Reference: Leon, 2014 [2]

Compact Singular Value Decomposition

Lemma

$Rank(A) = r$ is the number of nonzero singular values. Suppose $r < n$ and let $U_1 = [\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}]$ and $V_1 = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}]$ and $\Sigma_1 = Diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$ then:

$$A = U_1 \Sigma_1 V_1^T$$

Is the Compact Singular Value Decomposition.

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Example: SVD

Find the Singular Value Decomposition of: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

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Find the Singular Value Decomposition of: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$

We see that $Rank(A) = 1$ and:

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \ \& \ Det(A - \lambda I_2) = 0 \longrightarrow \lambda = 4, 0$$

The corresponding eigenvectors and Singular Value Matrix are:

$$\mathbf{x}_{\lambda_1} = (1, 1)^T \ \& \ \mathbf{x}_{\lambda_2} = (1, -1)^T \ \& \ \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Normalizing the eigenvectors to obtain:

$$V = [\mathbf{v_1}, \mathbf{v_2}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Since V is an orthogonal matrix, then: $A = U \Sigma V^T \longleftrightarrow AV = U\Sigma \longleftrightarrow A\mathbf{v_j}/\sigma_j = \mathbf{u_j}$ for each $j = 1, 2, \ldots, n$ After solving $\mathbf{v_j} = \mathbf{u_j}/\sigma_j \longrightarrow$

$$U = [\mathbf{u_1}, \mathbf{u_2}] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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A Brief Excursion on the Frobenius Norm

Definition (Frobenius Norm and Matrix Inner Product)

Given $A, B \in \mathbb{R}^{mxn}$ one can take $\langle A, B\rangle = \sum_{i,j} a_{ij} b_{ij}$.
The Frobenius Norm of a matrix $A \in \mathbb{R}^{mxn}$:

$$||A||_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$$

Lemma (Frobenius Norm and Singular Values)

Given any $A \in \mathbb{R}^{mxn}$ the frobenius norm is the square root of the sum of squares of its singular values, e.g., one has:

$$||A||_F = \sqrt{\sum_{j=1}^{n} \sigma_j^2}$$

Where σ_j is the j_{th} singular value of A.

Caveats of the Database Matrix

Given m keywords, which appear in n databases and take $A \in \mathbb{Z}^{mxn}$, where a_{ij} is the number of times the i_{th} keyword appears in the j_{th} database. By Gram-Schmidt Orthogonalization, we can compute a $Q \in \mathbb{R}^{mxn}$ where Q is an Orthogonal matrix.

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Caveats of the Database Matrix

Given m keywords, which appear in n databases and take $A \in \mathbb{Z}^{mxn}$, where a_{ij} is the number of times the i_{th} keyword appears in the j_{th} database. By Gram-Schmidt Orthogonalization, we can compute a $Q \in \mathbb{R}^{mxn}$ where Q is an Orthogonal matrix. Given k keywords of the m total criteria, we found $\mathbf{y} = Q^T\mathbf{x}$ is the optimal search vector in the direction of unit vector $\mathbf{x}$ where $Q \in \mathbb{R}^{mxn}$ with orthonormal columns. This approach did not take into consideration polysemy (words with different meanings and similar spelling) and synonymy (words with similar meanings but different spelling). The database matrix, Q, may be expressed as $Q = P + E$ where P represents the perfect database matrix and E represents the error associated to database Q. We can approximate Q by Q_1 where $Rank(Q_1) \leq Rank(Q)$ and replace E with E_1 such that $||E_1||_F \leq ||E||_F$ (Latent Semantic Indexing):

$$||E||_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} e_{ij}^2}$$

Q_1 is obtained through the Singular Value Decomposition: $Q_1 = U_1\Sigma_1 V_1^T$ where $U_1 = [\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}]$, $\Sigma_1 = Diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$, and $V_1 = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}]$

Principal Component Analysis

Suppose we conduct n *aptitude* tests on m *individuals* and let $X \in \mathbb{R}^{mxn}$ be the **Deviation Matrix**.

Recall $S = \frac{1}{n-1}X^TX$ is the **Covariance Matrix** and $C = U^TU$ where $\mathbf{u_j} = \frac{\mathbf{x_j}}{||\mathbf{x_j}||}$ is the **Correlation Matrix** where c_{ij} is the linear correlation between the i_{th} and j_{th} *aptitude* tests.

We are interested in solving for underlying factors (assumed to be uncorrelated). We require a collection of mutual orthogonal unit vectors $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_r}$ that span $R(X)$.

$$X = U_1\Sigma_1 V_1^T = U_1 W$$

For some $U_1 = [\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}]$, $\Sigma_1 = Diag(\sigma_1, \sigma_2, \ldots, \sigma_r)$, and $V_1 = [\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}]$ where $W = \Sigma_1 V_1$.

It follows that $R(X) = Span(\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\})$ and the underlying Intelligence Factors are driven by $\mathbf{u_j}$ with corresponding strength of $\sigma_j\mathbf{v_j}$.

The left singular vectors of X: $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_r}\}$ are the **Principal Component Vectors**.

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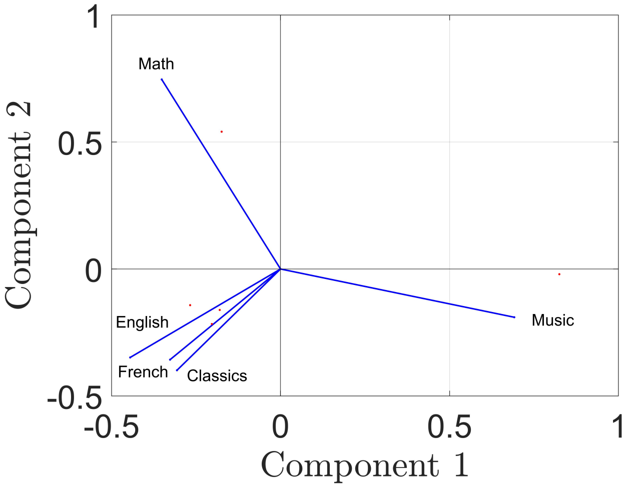
Principal Component Analysis: Dimension Reduction



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Reference: Leon, 2014 [2]

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[1] David Harville. *Matrix Algebra From a Statistician's Perspective*. New York: Springer-Verlag, 1997.

[2] Leon Stephen. *Linear Algebra with Applications (9th Edition) (Featured Titles for Linear Algebra*. London, England: Pearson, 2014.

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