Linear Algebra from Scratch: Matrices and Systems

Instructor Anthony
GSU Alumni

Udemy Open Course

M.S. in Mathematics with a Concentration in Bioinformatics

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Example: A System of Two Linear Equations

- Consider the following system of linear equations:

$$\begin{aligned} 2x + 3y &= 5 \\ x - y &= 10 \end{aligned}$$

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$$\begin{aligned} 2x + 3y &= 5 \\ -2x + 2y &= -20 \end{aligned}$$

- Obtained by multiplying the second row by -2. Adding the first and second equations, will give us:

$$5y = -15 \longrightarrow y = -3$$

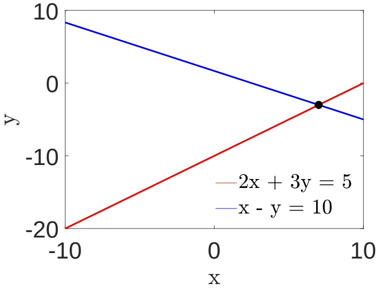
  After substituting $y = -3$ into second equation, one obtains $x = 7$. The solution of system of two linear equations is the ordered tuplet: $(7, -3)$.

Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:

□



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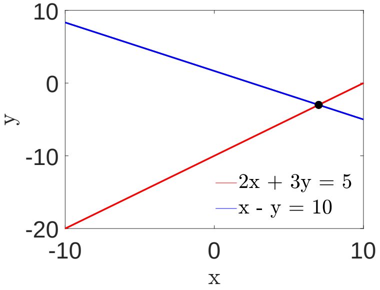
Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:
2. The solution exists if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.

□



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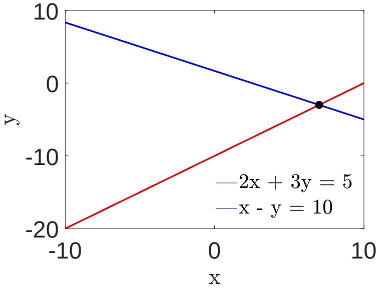
Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:
2. The solution exists if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.
3. The solution is unique if the lines intersect only once.

□



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Example: A System of Two Linear Equations

In the previous example, one can classify the different constituents of the system as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \sim \mathit{Coefficient\ Matrix}$$

$$b = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \sim \mathit{Column\ Vector}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \mathit{Vector\ of\ 2\ Unknowns}$$

General Form of the System of Two Linear Equations

Definition (General form of a System of Two Linear Equations)

A system of two linear equations with two unknowns is expressed as follows:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Where:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \; \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; and \; b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The system is **linear** if each element belonging to the vector of unknowns x_j has a power of 1 and does not contain nonlinear functions such as trigonometric functions or exponential functions.

General From of the System of n Linear Equations

Definition (General form of a System of n Linear Equations)

A system of n linear equations with n unknowns has the following form:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n &= b_i \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

We will refer to:

$$a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \sim j_{th} \textit{ Column Vector and } a_i{}' = (a_{i1}, \ldots, a_{in}) \sim i_{th} \textit{ Row Vector}$$

General From of the System of n Linear Equations

Definition (Matrix form of System of n Linear Equations)

The matrix form of the set of m linear equations with n where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ is: $Ax = b$, which is expressed as:

$$Ax = b \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$

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General From of the System of n Linear Equations

General Form of a System of n Linear Equations.

We will refer to the matrix of size mxn where m is the number of rows and n is the number of columns, denoted $A \in \mathbb{R}^{mxn}$, is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

□

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Augmented System of Linear Equations

Definition (Augmented System)

Let: $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^{m}$, and $x \in \mathbb{R}^{n}$, then:

$$A = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array}\right]$$

is the augmented system of $Ax = b$.

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General From of the System of n Linear Equations

General Form of a System of n Linear Equations.

The j_{th} column vector of A is denoted $a_j \in \mathbb{R}^{mx1}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

□

General From of the System of n Linear Equations

General Form of a System of n Linear Equations.

The i_{th} row vector of A is denoted $a_i{}' \in \mathbb{R}^{1xn}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

General From of the System of n Linear Equations

General Form of a System of n Linear Equations.

The intersection of the i_{th} row of A and j_{th} column of A is the $(i,j)_{th}$ element of A denoted: a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

□

Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}

Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

1. 3

Determine the following:

1. a_{12}

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Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

1. 3

Determine the following:

1. a_{12}
2. a_1

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Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. $\mathbf{a}_1$

1. 3
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. $\mathbf{a}_1$
3. $\vec{a}'_2$

1. 3
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. a_1
3. a_2'

1. 3
2. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
3. (2, 5)

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Elementary Row Operations

Definition (Elementary Row Operations)

An elementary row operation is an invertible operation applied between two rows of a matrix. Row operations may be classified as:

1. Interchange of two rows AKA row swap
2. Multiply a row by a nonzero real number
3. Replace a row by its sum with a multiple of another row

Elementary Row Operations

Suppose that A_1 is the original matrix and we swap the i_{th} and k_{th} row resulting in the matrix A_2.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

After swapping the i_{th} and k_{th} row, one obtains:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2. After swapping the first and second row of A_1, one obtains:

$$A_2 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Elementary Row Operations

Suppose that A_1 is the original matrix and we scale the i_{th} row by a factor of $\alpha \neq 0$ to obtain:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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Elementary Row Operations

1. Let:

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Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2. After multiplying row 2 by a factor of 3 one obtains:

$$A_2 = \begin{bmatrix} 1 & 3 \\ 6 & 12 \end{bmatrix}$$

Elementary Row Operations

Suppose that A_1 is the original matrix and we scale the i_{th} row by a factor of α and add it to the k_{th} row to obtain:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} + \alpha a_{i1} & a_{k2} + \alpha a_{i2} & \dots & a_{kj} + \alpha a_{ij} & \dots & a_{kn} + \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

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Elementary Row Operations

1. Let:
$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$
2. After multiplying row 2 by a factor of 3 and adding it to row 1, one obtains:
$$A_2 = \begin{bmatrix} 7 & 15 \\ 2 & 4 \end{bmatrix}$$

Equivalent Systems of Linear Equations

Lemma (Equivalence of Systems)

Given two systems of linear equations, the systems are equivalent provided a sequence of elementary row operations can be applied from one matrix to the other.
In other words, if $A_1 \to A_2 \to \cdots \to A_k$ *is an iterative sequence of applying* $k-1$ *row operations, then* $A_1x = b_1$ *and* $A_kx = b_k$ *are equivalent systems.*
Remark*: Two systems are equivalent provided the solution sets of each are identical.*

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