Linear Algebra from Scratch: Vector Spaces

Instructor Anthony

"It does not matter how slowly you go as long as you do not stop." - Confucius

Udemy Open Course

Table of Contents

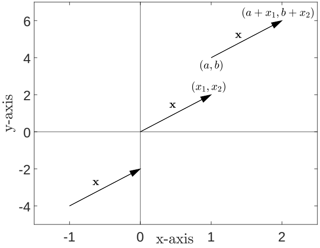
Table of Contents

udemy

Vectors Connect Two Ordered n Tuples

Vectors in $\mathbb{R}^2$.

- Vectors are defined according to their length (magnitude) and direction.

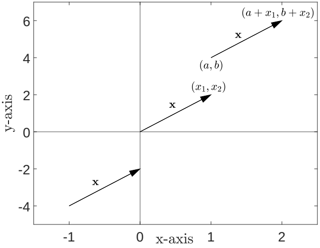


udemy

Vectors Connect Two Ordered n Tuples

Vectors in $\mathbb{R}^2$.

- Vectors are defined according to their length (magnitude) and direction.
- In the special case where $n = 2$, any vector is a connection between tail to tip, shifting the vector does not change the magnitude or direction of the vector.

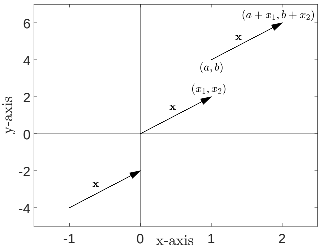


udemy

Vectors Connect Two Ordered n Tuples

Vectors in $\mathbb{R}^2$.

- Vectors are defined according to their length (magnitude) and direction.
- In the special case where $n = 2$, any vector is a connection between tail to tip, shifting the vector does not change the magnitude or direction of the vector.
- Recall a point in the coordinate plane can be represented as: $P : (x_1, y_1)$ *and* $Q : (x_2, y_2)$, the vector connecting from the origin to P is the vector: $\mathbf{x}$ is the same vector connecting points: P to Q.



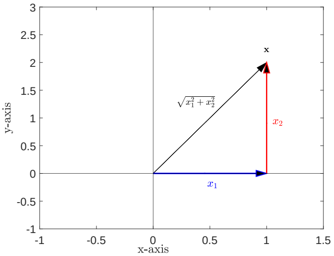
udemy

Vectors have Length and Direction

The Magnitude of a Vector.

- Given vector: $\mathbf{x} = (x_1, x_2)$, the magnitude or $L2$ norm is:

$$||\mathbf{x}||_2 = \sqrt{{x_1}^2 + {x_2}^2}$$



Reference: Leon, 2014 [2]

Vectors have Length and Direction

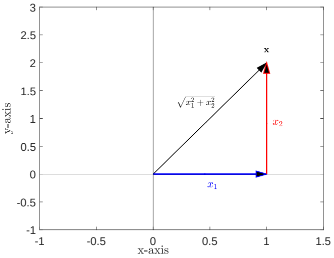
The Magnitude of a Vector.

- Given vector: $\mathbf{x} = (x_1, x_2)$, the magnitude or $L2$ norm is:

$$||\mathbf{x}||_2 = \sqrt{{\color{blue}x_1}^2 + {\color{red}x_2}^2}$$

- Generally, the magnitude of a vector $\mathbf{x} \in \mathbb{R}^n$ is:

$$||\mathbf{x}||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$



Reference: Leon, 2014 [2]

Vectors have Length and Direction

The Magnitude of a Vector.

- Given vector: $\mathbf{x} = (x_1, x_2)$, the magnitude or $L2$ norm is:

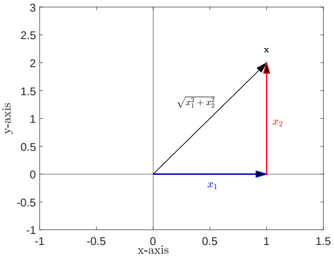
$$||\mathbf{x}||_2 = \sqrt{{x_1}^2 + {x_2}^2}$$

- Generally, the magnitude of a vector $\mathbf{x} \in \mathbb{R}^n$ is:

$$||\mathbf{x}||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Given $\mathbf{x} \in \mathbb{R}^2$, the direction denoted θ, is:

$$\theta = tan^{-1}\left(\frac{x_2}{x_1}\right)$$



udemy

Reference: Leon, 2014 [2]

Vectors have Length and Direction

The Magnitude of a Vector.

- Given vector: $\mathbf{x} = (x_1, x_2)$, the magnitude or $L2$ norm is:

$$||\mathbf{x}||_2 = \sqrt{{x_1}^2 + {x_2}^2}$$

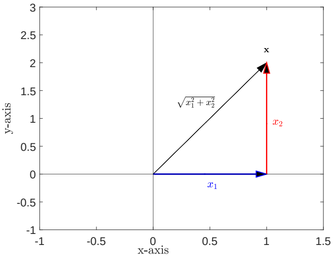
- Generally, the magnitude of a vector $\mathbf{x} \in \mathbb{R}^n$ is:

$$||\mathbf{x}||_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

- Given $\mathbf{x} \in \mathbb{R}^2$, the direction denoted θ, is:

$$\theta = tan^{-1}\left(\frac{x_2}{x_1}\right)$$

- **Remark**: We will generalize this to $\mathbb{R}^n$ and is referred to as **The Pythagorean Law**.



udemy

The distance between two Vectors

Definition (Distance between two Vectors in Euclidean Space)

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then the distance between $\mathbf{x}$ *and* $\mathbf{y}$, denoted $\delta(\mathbf{x}, \mathbf{y})$ is:

$$\delta(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

Generally, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then:

$$\delta(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_j - y_j)^2 + \cdots + (x_n - y_n)^2}$$

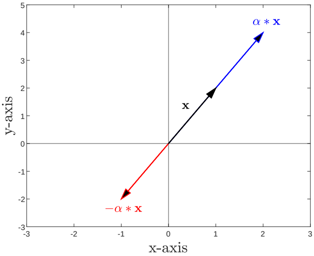
udemy

Vectors can be Scaled by a Real Number

Scaling x by α.

Consider the vector $\mathbf{x} \in \mathbb{R}^2$, then the multiplying by a scalar α has the following properties:

- (1): $\alpha = 0$ **Deletes** the vector.



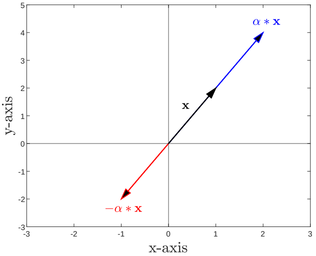
udemy

Vectors can be Scaled by a Real Number

Scaling **x** by α.

Consider the vector $\mathbf{x} \in \mathbb{R}^2$, then the multiplying by a scalar α has the following properties:

- 1: $\alpha = 0$ **Deletes** the vector.
- 2: $||\alpha|| > 1$ **Stretches** the Vector.



udemy

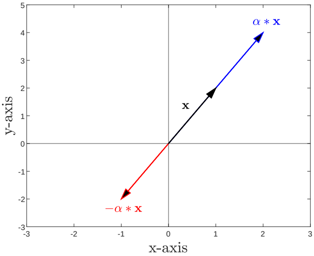
Vectors can be Scaled by a Real Number

Scaling **x** by α.

Consider the vector $\mathbf{x} \in \mathbb{R}^2$, then the multiplying by a scalar α has the following properties:

- (1): $\alpha = 0$ **Deletes** the vector.
- (2): $||\alpha|| > 1$ **Stretches** the Vector.
- (3): $||\alpha|| < 1$ **Shrinks** the Vector.

□



udemy

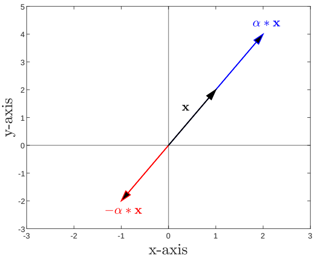
Vectors can be Scaled by a Real Number

Scaling **x** by α.

Consider the vector $\mathbf{x} \in \mathbb{R}^2$, then the multiplying by a scalar α has the following properties:

- (1): $\alpha = 0$ **Deletes** the vector.
- (2): $||\alpha|| > 1$ **Stretches** the Vector.
- (3): $||\alpha|| < 1$ **Shrinks** the Vector.
- (4): $\alpha < 0$ **Reflects** the vector.

□



udemy

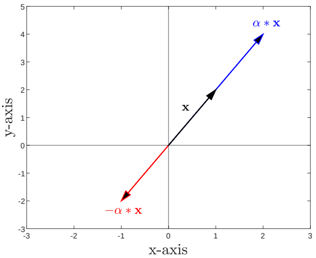
Vectors can be Scaled by a Real Number

Scaling **x** by α.

Consider the vector $\mathbf{x} \in \mathbb{R}^2$, then the multiplying by a scalar α has the following properties:

- (1): $\alpha = 0$ **Deletes** the vector.
- (2): $||\alpha|| > 1$ **Stretches** the Vector.
- (3): $||\alpha|| < 1$ **Shrinks** the Vector.
- (4): $\alpha < 0$ **Reflects** the vector.
- **Remark**: The results hold if $\mathbf{x} \in \mathbb{R}^n$

$\square$



udemy

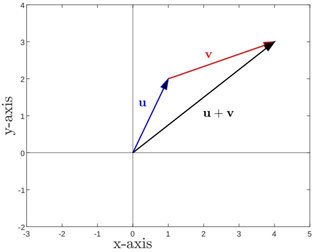
Vector Addition

Vector Addition in the Plane.

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, then:

- Geometrically, adding two vectors corresponds to placing one vector's Tail at the Tip of the other vector.

$\square$



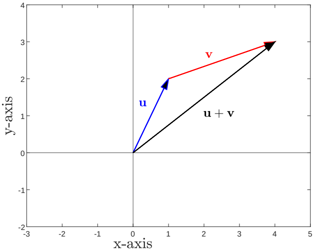
udemy

Vector Addition

Vector Addition in the Plane.

Suppose $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, then:

- Geometrically, adding two vectors corresponds to placing one vector's Tail at the Tip of the other vector.
- Algebraically, $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$



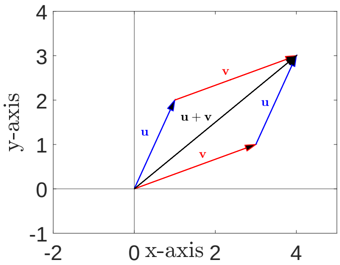
udemy

Vectors Addition Commutes

Vector Addition Commutes.

- Geometrically, vector addition commutes as either vector's Tail can be placed on the other vector's Tip.

□



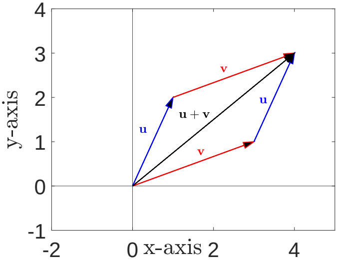
udemy

Vectors Addition Commutes

Vector Addition Commutes.

- Geometrically, vector addition commutes as either vector's Tail can be placed on the other vector's Tip.
- Algebraically, one has:
  $\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) = \mathbf{v} + \mathbf{u}$

$\square$



udemy

Vectors in $\mathbb{R}^n$

Definition (Vector Addition and Scalar Multiplication)

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, then:

$$\alpha \mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_j \\ \vdots \\ \alpha x_n \end{bmatrix} \quad \mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_j + y_j \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- 1: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*

udemy

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha \mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- 1: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- 2: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x}+\mathbf{y} \in V$ when $\mathbf{x}, \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- 1: $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- 2: $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*
- 3: There is $\mathbf{0}$ such that: $\mathbf{0}+\mathbf{v} = \mathbf{v} = \mathbf{v}+\mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*

udemy

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x}, \ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- (1): $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- (2): $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*
- (3): There is $\mathbf{0}$ such that: $\mathbf{0} + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*
- (4): For every $\mathbf{v}$, there is $-\mathbf{v}$ such that: $\mathbf{v} + -\mathbf{v} = \mathbf{0} = -\mathbf{v} + \mathbf{v} \sim$ *Additive Inverse*

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- (1): $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- (2): $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*
- (3): There is $\mathbf{0}$ such that: $\mathbf{0} + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*
- (4): For every $\mathbf{v}$, there is $-\mathbf{v}$ such that: $\mathbf{v} + -\mathbf{v} = \mathbf{0} = -\mathbf{v} + \mathbf{v} \sim$ *Additive Inverse*
- (5): For any $\alpha, \mathbf{x}, \mathbf{y} \in V$, one has: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} \sim$ *Scalar Distribution*

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x}+\mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- 1: $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- 2: $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*
- 3: There is $\mathbf{0}$ such that: $\mathbf{0}+\mathbf{v} = \mathbf{v} = \mathbf{v}+\mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*
- 4: For every $\mathbf{v}$, there is $-\mathbf{v}$ such that: $\mathbf{v}+-\mathbf{v} = \mathbf{0} = -\mathbf{v}+\mathbf{v} \sim$ *Additive Inverse*
- 5: For any $\alpha, \mathbf{x}, \mathbf{y} \in V$, one has: $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y} \sim$ *Scalar Distribution*
- 6: For any $\alpha, \beta, \mathbf{x} \in V$, one has: $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x} \sim$ *Vector Distribution*

udemy

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x}+\mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- 1: $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ for each $\mathbf{x},\mathbf{y} \in V \sim$ *Commutative Additive Property*
- 2: $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ for each $\mathbf{x},\mathbf{y},\mathbf{z} \in V \sim$ *Additive Associative Property*
- 3: There is $\mathbf{0}$ such that: $\mathbf{0}+\mathbf{v} = \mathbf{v} = \mathbf{v}+\mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*
- 4: For every $\mathbf{v}$, there is $-\mathbf{v}$ such that: $\mathbf{v} + -\mathbf{v} = \mathbf{0} = -\mathbf{v}+\mathbf{v} \sim$ *Additive Inverse*
- 5: For any $\alpha,\mathbf{x},\mathbf{y} \in V$, one has: $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y} \sim$ *Scalar Distribution*
- 6: For any $\alpha,\beta,\mathbf{x} \in V$, one has: $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x} \sim$ *Vector Distribution*
- 7: For any $\alpha,\beta,\mathbf{x} \in V$, one has: $(\alpha\beta)*\mathbf{x} = \alpha*(\beta\mathbf{x})$ *Scalar Associative Property*

Vector Space Axioms

Definition (Vector Space Axioms)

Suppose V is a set closed under scalar multiplication and vector addition, i.e., $\alpha\mathbf{x} \in V$ when $\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$ when $\mathbf{x},\ \mathbf{y} \in V$, then $V \sim$ **Vector Space** if:

- (1): $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for each $\mathbf{x}, \mathbf{y} \in V \sim$ *Commutative Additive Property*
- (2): $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V \sim$ *Additive Associative Property*
- (3): There is $\mathbf{0}$ such that: $\mathbf{0} + \mathbf{v} = \mathbf{v} = \mathbf{v} + \mathbf{0}$ for each $\mathbf{v} \in V \sim$ *Additive Identity*
- (4): For every $\mathbf{v}$, there is $-\mathbf{v}$ such that: $\mathbf{v} + -\mathbf{v} = \mathbf{0} = -\mathbf{v} + \mathbf{v} \sim$ *Additive Inverse*
- (5): For any $\alpha, \mathbf{x}, \mathbf{y} \in V$, one has: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y} \sim$ *Scalar Distribution*
- (6): For any $\alpha, \beta, \mathbf{x} \in V$, one has: $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x} \sim$ *Vector Distribution*
- (7): For any $\alpha, \beta, \mathbf{x} \in V$, one has: $(\alpha\beta) * \mathbf{x} = \alpha * (\beta\mathbf{x})$ *Scalar Associative Property*
- (8): For any $\mathbf{x} \in V$, one has: $1 * \mathbf{x} = \mathbf{x} = \mathbf{x} * 1 \sim$ *Multiplicative Identity*

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

□

udemy

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

- 1: $\mathbb{R}^n \sim n_{th}$ *Dimensional Euclidean Space*

udemy

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

- 1: $\mathbb{R}^n \sim n_{th}$ *Dimensional Euclidean Space*
- 2: $\mathbb{R}^{mxn}$ *Collection of mxn Matrices with Real Entries*

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

- 1: $\mathbb{R}^n \sim n_{th}$ *Dimensional Euclidean Space*
- 2: $\mathbb{R}^{mxn}$ *Collection of mxn Matrices with Real Entries*
- 3: $\mathcal{C}[a, b] \sim$ the set of all continuous functions with real values over the Interval: $I = [a, b]$

□

udemy

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

- 1: $\mathbb{R}^n \sim n_{th}$ *Dimensional Euclidean Space*
- 2: $\mathbb{R}^{mxn}$ *Collection of mxn Matrices with Real Entries*
- 3: $\mathcal{C}[a,b] \sim$ the set of all continuous functions with real values over the Interval: $I = [a,b]$
- 4: $P_n \sim$ The Polynomials of degree: $n-1$

udemy

Examples of Vector Spaces

Examples of Vector Spaces.

The following are examples of Sets that Satisfy the Vector Space Axioms:

- (1): $\mathbb{R}^n \sim n_{th}$ *Dimensional Euclidean Space*
- (2): $\mathbb{R}^{mxn}$ *Collection of mxn Matrices with Real Entries*
- (3): $\mathcal{C}[a,b] \sim$ the set of all continuous functions with real values over the Interval: $I = [a, b]$
- (4): $P_n \sim$ The Polynomials of degree: $n-1$
- **Remark**: Any Polynomial in P_n is the form of:

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_j x^j + \cdots + \alpha_{n-1} x^{n-1}$$

Additional Properties of Vector Spaces

Theorem (Properties of Vector Spaces)

Suppose $V \sim$ *Vector Space, then:*

udemy

Additional Properties of Vector Spaces

Theorem (Properties of Vector Spaces)

Suppose $V \sim$ *Vector Space, then:*

- $0 * \mathbf{x} = \mathbf{0} = \mathbf{x} * 0$ *for any* $\mathbf{x} \in V$

udemy

Additional Properties of Vector Spaces

Theorem (Properties of Vector Spaces)

Suppose $V \sim$ *Vector Space, then:*

- $0 * \mathbf{x} = 0 = \mathbf{x} * 0$ *for any* $\mathbf{x} \in V$
- $\mathbf{x} + \mathbf{y} = 0 \rightarrow \mathbf{y} = -\mathbf{x}$ *for any* $\mathbf{x}, \mathbf{y} \in V \sim$ *Uniqueness of Additive Inverse*

udemy

Additional Properties of Vector Spaces

Theorem (Properties of Vector Spaces)

Suppose $V \sim$ Vector Space, then:

- $0 * \mathbf{x} = 0 = \mathbf{x} * 0$ *for any* $\mathbf{x} \in V$
- $\mathbf{x} + \mathbf{y} = 0 \rightarrow \mathbf{y} = -\mathbf{x}$ *for any* $\mathbf{x}, \mathbf{y} \in V \sim$ *Uniqueness of Additive Inverse*
- $(-1)\mathbf{x} = -\mathbf{x} = \mathbf{x}(-1)$

udemy

Table of Contents

udemy

The Subspace

Definition (The Subspace)

Suppose $V \sim$ *Vector Space* satisfying the vector space axioms. If $S \subset V$, then $S \sim$ **Subspace** of V. For any $\alpha \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in S$. It is sufficient to show that S is a subspace of V provided:

$$1 : \mathbf{x} + \mathbf{y} \in S \quad 2 : \alpha \mathbf{x} \in S$$

Remark: $\emptyset$ and V are the **Trivial Subspaces**, e.g., $\emptyset \subset V$ and $V \subset V$. We refer to $S \subset V$ such that: $S \neq \emptyset$ and $S \neq V$ as a **Proper Subspace**.
Remark: To show $S \neq \emptyset$, one must show that: $\mathbf{0} \in S$.

udemy

Example

1. Consider the following set:

$$S = \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, x_2 = 2 * x_1 \right\}$$

Then, $S \subset \mathbb{R}^2$, let $c = y_1, k = x_1 \in \mathbb{R}$, then:

Example

1. Consider the following set:

$$S = \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, x_2 = 2 * x_1 \right\}$$

   Then, $S \subset \mathbb{R}^2$, let $c = y_1, k = x_1 \in \mathbb{R}$, then:

2. For any $\mathbf{x}, \mathbf{y} \in S$:

$$1 : \mathbf{x} + \mathbf{y} = \begin{bmatrix} k + c \\ 2(k + c) \end{bmatrix} \; 2 : \alpha * \mathbf{x} = \begin{bmatrix} \alpha k \\ \alpha * 2k \end{bmatrix}$$

udemy

Example

1. Consider the following set:

$$S = \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, x_2 = 2 * x_1 \right\}$$

   Then, $S \subset \mathbb{R}^2$, let $c = y_1, k = x_1 \in \mathbb{R}$, then:

2. For any $\mathbf{x}, \mathbf{y} \in S$:

$$(1): \mathbf{x} + \mathbf{y} = \begin{bmatrix} k + c \\ 2(k + c) \end{bmatrix} (2): \alpha * \mathbf{x} = \begin{bmatrix} \alpha k \\ \alpha * 2k \end{bmatrix}$$

3. Therefore, $S \sim \textit{Subspace}$ of $\mathbb{R}^2$.

udemy

Example

- Let $S = \{p(x) \in P_n \mid p(0) = 0\}$, show that $S \sim Subspace\ of\ P_n$.

Example

- Let $S = \{p(x) \in P_n \mid p(0) = 0\}$, show that $S \sim \textit{Subspace of } P_n$.
- Recall that: $p(x) \in P_n$ is of the form:

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$$

If $p(0) = 0 \rightarrow \alpha_0 = 0$

Example

- Let $S = \{p(x) \in P_n \mid p(0) = 0\}$, show that $S \sim Subspace\ of\ P_n$.
- Recall that: $p(x) \in P_n$ is of the form:

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$$

If $p(0) = 0 \to \alpha_0 = 0$

- Note that one has the following:

$$1 : p(\beta * 0) = \beta * 0 = 0 \quad 2 : \ (p+q)(0) = p(0) + q(0) = 0$$

udemy

Example

- Let $S = \{p(x) \in P_n \mid p(0) = 0\}$, show that $S \sim \textit{Subspace of } P_n$.
- Recall that: $p(x) \in P_n$ is of the form:

$$p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}$$

  If $p(0) = 0 \rightarrow \alpha_0 = 0$

- Note that one has the following:

$$1 : p(\beta * 0) = \beta * 0 = 0 \quad 2 : \; (p+q)(0) = p(0) + q(0) = 0$$

- Thus, $S \sim \textit{Subspace } P_n$.

Example

1. Suppose: $S = \{f \in C^2[a, b] \mid f''(x) + f(x) = 0\}$, then $S \sim Subspace\ of\ C^2[a, b]$

Example

1. Suppose: $S = \{f \in C^2[a,b] \mid f''(x) + f(x) = 0\}$, then
   $S \sim \textit{Subspace of } C^2[a,b]$
2. $S \neq \emptyset$ as S contains the zero function. Additionally, one has:

$$1 : (\alpha f)''(x) + \alpha f = \alpha(f'' + f) = 0$$

udemy

Example

1. Suppose: $S = \{f \in C^2[a, b] \mid f''(x) + f(x) = 0\}$, then $S \sim \textit{Subspace of } C^2[a, b]$
2. $S \neq \emptyset$ as S contains the zero function. Additionally, one has:

$$1 : (\alpha f)''(x) + \alpha f = \alpha(f'' + f) = 0$$

3. And:

$$(f + g)''(x) + (f + g)(x) = f''(x) + f(x) + g''(x) + g(x) = 0$$

Example

1. Suppose: $S = \{f \in C^2[a, b] \mid f''(x) + f(x) = 0\}$, then $S \sim \textit{Subspace of } C^2[a, b]$
2. $S \neq \emptyset$ as S contains the zero function. Additionally, one has:

$$1 : (\alpha f)''(x) + \alpha f = \alpha(f'' + f) = 0$$

3. And:

$$(f + g)''(x) + (f + g)(x) = f''(x) + f(x) + g''(x) + g(x) = 0$$

4. Therefore, $S \sim \textit{Subspace of } C^2[a, b]$.

udemy

The Null Space: $Ax = 0$

Definition (The Null Space)

- Let: $Ax = 0$ where $A \in \mathbb{R}^{mxn}$ and $x \in \mathbb{R}^n$, then: The Null Space of A, denoted: $Nul(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$

Proof.

$\square$

udemy

The Null Space: $Ax = 0$

Definition (The Null Space)

- Let: $Ax = 0$ where $A \in \mathbb{R}^{mxn}$ and $x \in \mathbb{R}^n$, then: The Null Space of A, denoted: $Nul(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
- **Remark**: It is clear that $Nul(A) \subset \mathbb{R}^n$, and $A \sim Invertible \longleftrightarrow Nul(A) = \{0\}$.

Proof.

$\square$

udemy

The Null Space: $Ax = 0$

Definition (The Null Space)

- Let: $Ax = 0$ where $A \in \mathbb{R}^{mxn}$ and $x \in \mathbb{R}^n$, then: The Null Space of A, denoted: $Nul(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
- **Remark**: It is clear that $Nul(A) \subset \mathbb{R}^n$, and $A \sim \textit{Invertible} \longleftrightarrow Nul(A) = \{0\}$.

Proof.

- Suppose $A \in \mathbb{R}^{mxn}$ and let $x, y \in \mathbb{R}^n, \alpha \in \mathbb{R}$.

□

udemy

The Null Space: $Ax = 0$

Definition (The Null Space)

- Let: $Ax = 0$ where $A \in \mathbb{R}^{mxn}$ and $x \in \mathbb{R}^n$, then: The Null Space of A, denoted: $Nul(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
- **Remark**: It is clear that $Nul(A) \subset \mathbb{R}^n$, and $A \sim Invertible \longleftrightarrow Nul(A) = \{0\}$.

Proof.

- Suppose $A \in \mathbb{R}^{mxn}$ and let $x, y \in \mathbb{R}^n, \alpha \in \mathbb{R}$.
- Then:

$$1 : A(x+y) = Ax + Ay = 0 \quad 2 : \; A(\alpha x) = \alpha Ax = 0$$

$\square$

udemy

The Null Space: $Ax = 0$

Definition (The Null Space)

- Let: $Ax = 0$ where $A \in \mathbb{R}^{mxn}$ and $x \in \mathbb{R}^n$, then: The Null Space of A, denoted: $Nul(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$
- **Remark**: It is clear that $Nul(A) \subset \mathbb{R}^n$, and $A \sim \textit{Invertible} \longleftrightarrow Nul(A) = \{0\}$.

Proof.

- Suppose $A \in \mathbb{R}^{mxn}$ and let $x, y \in \mathbb{R}^n, \alpha \in \mathbb{R}$.
- Then:

$$1 : A(x + y) = Ax + Ay = 0 \quad 2 : \ A(\alpha x) = \alpha Ax = 0$$

- Clearly, $Nul(A) \neq \emptyset$ as $\mathbf{0} \in \mathbb{R}^n$.

$\square$

Example

1. Suppose we wish to solve the following system:

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned}$$

Example

1. Suppose we wish to solve the following system:

$$\begin{aligned}\alpha_1 + 2\alpha_2 + 4\alpha_3 &= a\\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b\\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c\end{aligned}$$

2. Noting that the coefficient matrix is singular:

$$A = \begin{bmatrix} 1 & 2 & 4\\ 2 & 1 & -1\\ 4 & 3 & 1\end{bmatrix} \quad Det(A) = 0$$

Example

1. Suppose we wish to solve the following system:

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned}$$

2. Noting that the coefficient matrix is singular:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix} \quad Det(A) = 0$$

3. From previous hypothesis, we know that $A \not\sim I_3$, putting A in RREF:

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

1. Suppose we wish to solve the following system:

$$\begin{aligned} \alpha_1 + 2\alpha_2 + 4\alpha_3 &= a \\ 2\alpha_1 + \alpha_2 - \alpha_3 &= b \\ 4\alpha_1 + 3\alpha_2 + \alpha_3 &= c \end{aligned}$$

2. Noting that the coefficient matrix is singular:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & -1 \\ 4 & 3 & 1 \end{bmatrix} \quad Det(A) = 0$$

3. From previous hypothesis, we know that $A \not\sim I_3$, putting A in RREF:

$$A \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Taking: $x_3 = \alpha$, the general solution is: $\mathbf{x} = \begin{bmatrix} 2\alpha \\ -3\alpha \\ \alpha \end{bmatrix}$

The Span of a Collection of Vectors

Definition (The Linear Combination of a Set of Vectors)

Suppose that: $S = \{v_1, v_2, \ldots, v_n\}$ such that $v_j \in V$ where $V \sim$ *Vector Space* and take $\alpha_j \in \mathbb{R}$, then a **Linear Combination** of the vectors belonging to S is:

$$\sum_{j=1}^{n} \alpha_j v_j = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_j v_j + \cdots + \alpha_n v_n$$

Definition (The Span of a Collection of Vectors)

Consider the set: $S = \{v_1, v_2, \ldots, v_j, \ldots v_n\}$ where $v_j \in V$ where $V \sim$ *Vector Space*, then the **Span** of S is:

$$Span(v_1, v_2, \ldots, v_n) = \left\{ \sum_{j=1}^{n} \alpha_j v_j \;\middle|\; \alpha_j \in \mathbb{R} \right\}$$

The Span of a Collection of Vectors is a Subspace

Theorem (The Span of a Collection of Vectors is a Subspace)

Consider the following set: $S = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$ *where* $v_j \in V$ *such that* $V \sim$ *Vector Space, then:*

$$Span(v_1, v_2, \ldots, v_j, \ldots, v_n) \sim Subspace\ of\ V$$

udemy

The Spanning Set of a Vector Space

Definition (The Spanning Set of a Vector Space)

1. Let $V \sim \textit{Vector Space}$ and take $S = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$, then S **Spans** V provided every vector can be expressed as a Linear Combination of vectors from V. If $z \in V$, S spans V if:

udemy

The Spanning Set of a Vector Space

Definition (The Spanning Set of a Vector Space)

1. Let $V \sim \textit{Vector Space}$ and take $S = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$, then S **Spans** V provided every vector can be expressed as a Linear Combination of vectors from V. If $z \in V$, S spans V if:

2. $$z = \sum_{j=1}^{n} \alpha_j v_j \sim \textit{For Some } \alpha_j \in \mathbb{R}$$

udemy

The Spanning Set of a Vector Space

Definition (The Spanning Set of a Vector Space)

1. Let $V \sim$ *Vector Space* and take $S = \{v_1, v_2, \ldots, v_j, \ldots, v_n\}$, then S **Spans** V provided every vector can be expressed as a Linear Combination of vectors from V. If $z \in V$, S spans V if:
2. $$z = \sum_{j=1}^{n} \alpha_j v_j \sim \textit{For Some } \alpha_j \in \mathbb{R}$$
3. **Remark**: Equivalently, one can say S is a **spanning set** of V, $Span(v_1, v_2, \ldots, v_j, \ldots, v_n) = V$, the vectors: $v_1, v_2, \ldots v_j, \ldots, v_n$ **span** V, or S **Spans** V.

Systems of Equations: Revisited

Theorem (General Solution to $Ax = b$)

1. *Suppose one solves:* $Ax = b$ *for* $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, $b \in R^m$, *then a solution to the linear system is:*

udemy

Systems of Equations: Revisited

Theorem (General Solution to $Ax = b$)

1. *Suppose one solves:* $Ax = b$ *for* $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, $b \in R^m$, *then a solution to the linear system is:*
2. $$y = x_0 + z \text{ where : } Ax_0 = b \;\&\; Az = 0$$

   F *e.g.,* $z \in Nul(A)$, *and* x_0 *is a particular solution to* $Ax = b$.

udemy

Systems of Equations: Revisited

Theorem (General Solution to $Ax = b$)

1. *Suppose one solves:* $Ax = b$ *for* $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in R^m$, *then a solution to the linear system is:*
2. $$y = x_0 + z \text{ where}: \ Ax_0 = b \ \& \ Az = 0$$

   F e.g., $z \in Nul(A)$, *and* x_0 *is a particular solution to* $Ax = b$.
3. **Remark**: *Additionally,* x_0 *is the unique solution* $\longleftrightarrow Nul(A) = \{0\}$.

udemy

Table of Contents

udemy

Example

1. Consider the following system:

$$\begin{aligned} x1 - 2x_2 - x_3 &= 0 \\ -x_1 + 3x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 8x_3 &= 0 \end{aligned}$$

Example

1. Consider the following system:

$$\begin{aligned} x1 - 2x_2 - x_3 &= 0 \\ -x_1 + 3x_2 + 3x_3 &= 0 \\ 2x_1 + x_2 + 8x_3 &= 0 \end{aligned}$$

2. First, observing that the coefficient matrix is singular:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \\ 2 & 1 & 8 \end{bmatrix} \quad Det(A) = 0$$

   Additionally, observing that the third column can be expressed as a linear combination of the preceding columns:

Example

1. Consider the following system:

$$
\begin{aligned}
x1 - 2x_2 - x_3 &= 0 \\
-x_1 + 3x_2 + 3x_3 &= 0 \\
2x_1 + x_2 + 8x_3 &= 0
\end{aligned}
$$

2. First, observing that the coefficient matrix is singular:

$$
A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \\ 2 & 1 & 8 \end{bmatrix} \quad Det(A) = 0
$$

Additionally, observing that the third column can be expressed as a linear combination of the preceding columns:

3.

$$
\mathbf{x_3} = {\color{blue}3}x_1 + {\color{red}2}x_2
$$

Remark: Actually, $A \in \mathbb{R}^{nxn}$ and $Det(A) = 0 \longleftrightarrow$ At least one $\mathbf{a_j}$ can be expressed as a linear combination of the other columns.

Lastly, $Span(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = Span(\mathbf{x_1}, \mathbf{x_2}) = Span(\mathbf{x_1}, \mathbf{x_3}) = Span(\mathbf{x_2}, \mathbf{x_3})$.

udemy

Linear Combination and Spanning Set of Vectors

Theorem (Linear Combination and a Spanning Set of Vectors)

Suppose that: $S = \{v_1, v_2, \ldots, v_n\}$ *are a collection of vectors in* V, *then the following will hold:*

- 1: *If* S *spans* V *and one of the vectors* $v_j \in S$ *can be expressed as a linear combination of the other vectors, then* V *can be spanned by any of the* $n-1$ *vectors.*

udemy

Linear Combination and Spanning Set of Vectors

Theorem (Linear Combination and a Spanning Set of Vectors)

Suppose that: $S = \{v_1, v_2, \ldots, v_n\}$ *are a collection of vectors in* V*, then the following will hold:*

- 1: *If* S *spans* V *and one of the vectors* $v_j \in S$ *can be expressed as a linear combination of the other vectors, then* V *can be spanned by any of the* $n-1$ *vectors.*
- 2: *Suppose that at least one of the vectors* v_j *can be expressed as a linear combination of the other vectors:* $v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$*, then there are scalars* $c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n \neq 0$ *such that:*

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_j\mathbf{v_j} + \cdots + c_n\mathbf{v_n} = 0$$

udemy

Linear Independence

Definition (Linear Independence)

Consider the following set of Vectors: $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ are **Linearly Independent** iff:

$$\sum_{j=1}^{n} c_j \mathbf{v_j} = c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \cdots + \cdots + c_n \mathbf{v_n} = 0 \longrightarrow c_j = 0$$

udemy

Example

- Consider the vectors: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then:

Example

- Consider the vectors: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then:
- $$c_1\mathbf{v_1} + c_2\mathbf{v_2} = 0 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

udemy

Example

- Consider the vectors: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then:
- $$c_1\mathbf{v_1} + c_2\mathbf{v_2} = 0 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$
- Implying that: $c_1 + c_2 = 0 = c_1 + c_2$, e.g., $c_1 = 0 = c_2$, which implies that $V = \{\mathbf{v_1}, \mathbf{v_2}\}$ is a linearly independent collection of vectors.

udemy

Example

- Consider the vectors: $\mathbf{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ then:
- $$c_1\mathbf{v_1} + c_2\mathbf{v_2} = 0 \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$
- Implying that: $c_1 + c_2 = 0 = c_1 + c_2$, e.g., $c_1 = 0 = c_2$, which implies that $V = \{\mathbf{v_1}, \mathbf{v_2}\}$ is a linearly independent collection of vectors.
- **Remark**: Observe that $Det(C) \neq 0$ and can be generalized to a general case for linearly independent vectors.

Linear Independence and Determinants

Theorem (Linear Independence and Determinants)

Suppose that: $X \in \mathbb{R}^{nxn}$ *and* $X = [\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}]$*, then:*

$$Det(X) \neq 0 \longleftrightarrow Column\ Vectors\ of\ X\ are\ Linearly\ Independent$$

Remark*:* $Det(X) = 0 \longleftrightarrow$ *at least one of the columns of* X *can be expressed as a linear combination of the other columns of* X*.*

udemy

Unique Representation of a Linear Combination

Theorem (Unique Representation of a Linear Combination)

Suppose we are given the spanning set $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ on Vector Space V, then for any $\mathbf{v} \in V$ one has:

$$\mathbf{v} = \sum_{j=1}^{n} c_j \mathbf{v_j} = c_1 \mathbf{v_1} + c_2 \mathbf{v_2} + \cdots + c_n \mathbf{v_n}$$

*For some $c_j \in \mathbb{R}$. Furthermore, the representation is unique provided $S \sim$ **Linearly Independent**.*

udemy

Linear Independence of the Polynomial Space

Linear Independence of Polynomials.

1. Suppose that: $\mathbf{p} \in P_n$, where $\mathbf{p} = [p_1, p_2, \ldots, p_k]$ where each polynomial is of the form:

$$p_j(x) = \alpha_0^{(j)} + \alpha_1^{(j)} x + \cdots + \alpha_{n-1}^{(j)} x^{n-1}$$

□

udemy

Linear Independence of the Polynomial Space

Linear Independence of Polynomials.

1. Suppose that: $\mathbf{p} \in P_n$, where $\mathbf{p} = [p_1, p_2, \ldots, p_k]$ where each polynomial is of the form:

$$p_j(x) = \alpha_0^{(j)} + \alpha_1^{(j)} x + \cdots + \alpha_{n-1}^{(j)} x^{n-1}$$

2. $\mathbf{p}$ will form a **Linearly Independent** set of vectors provided:

$$c_1 p_1(X) + c_2 p_2(x) + \cdots + c_j p_j(x) + \cdots + c_k p_k = 0 \longrightarrow c_j = 0$$

$\square$

udemy

Linear Independence of Functions

Definition (Vector Space on $C^{(n-1)}$)

Let $F = \{f_1, f_2, \dots, f_n\}$ be a collection of functions where $f_j \in C^{(n-1)}[a, b]$, then F is a linear independent collection provided:

$$c_1 f_1 + c_2 f_2 + \cdots + c_j f_j + \cdots + c_n f_n = 0 \rightarrow c_j = 0$$

Which will hold computing up to the i_{th} Derivative:

$$c_1 f_1^{(i)} + c_2 f_2^{(i)} + \cdots + c_j f_j^{(i)} + \cdots + c_n f_n^{(i)} = 0 \longrightarrow c_j = 0$$

For each $i = \overline{0, n-1}$. It follows that one must have...

udemy

The Wronskian

Definition (The Wronskian of $f_1, f_2, \ldots, f_n$)

Suppose $f_j \in C^{(n-1)}[a, b]$, for $j = 0, 1, \ldots, n-1$; then $F = [f_1, f_2, \ldots f_n]$ forms a **Linearly Independent** set provided:

$$\begin{pmatrix} f_1 & f_2 & \ldots & f_n \\ f^{(1)} & f_2^{(1)} & \ldots & f_n^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ f_1^{(i)} & f_2^{(i)} & \ldots & f_n^{(i)} \\ \vdots & \ddots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_i \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Which implies that: $\alpha_j = 0$

udemy

The Wronskian Cont'd

Definition (The Wronskian Cont'd)

Continuing where we left off, the set is **Linearly Dependent** provided the Wronskian is 0:

$$W = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f^{(1)} & f_2^{(1)} & \dots & f_n^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ f_1^{(i)} & f_2^{(i)} & \dots & f_n^{(i)} \\ \vdots & \ddots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

Remark: Generally, if F forms a linearly independent set of functions, then we will refer to F as a **Fundamental Set**.

udemy

Example

1. Show that e^x and e^{-x} are **linearly independent** on $C(-\infty, \infty)$:

udemy

Example

1. Show that e^x and e^{-x} are **linearly independent** on $C(-\infty, \infty)$:
2. $$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$

udemy

Example

1. Show that e^x and e^{-x} are **linearly independent** on $C(-\infty, \infty)$:
2. $$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0$$
3. Since, $W \neq 0$, then it follows that: e^x and e^{-x} are **Linearly Independent**.

udemy

Table of Contents

The Basis

Definition (The Basis)

Consider the following collection: $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots, \mathbf{v_n}\}$ such that: $\mathbf{v_j} \in V$, where $V \sim$ *Vector Space*, then S is a **Basis** on V iff:

1. ①: The vectors in S form a **Linearly Independent** set.
2. ②: S is a **Spanning** Set on V.

udemy

The Standard Basis on $\mathbb{R}^n$

Definition (The Standard Basis)

Given the Vector Space: $\mathbb{R}^n$ and the *nxn* **Identity** Matrix: I_n, then the columns of I_n denoted $\mathbf{e_j}$ span $\mathbf{R}^n$:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ 0 & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

udemy

The Standard Basis on $\mathbb{R}^n$

Proof.

Take $\mathbf{v} \in \mathbb{R}^n$, then:

$$\mathbf{v} = v_1\mathbf{e_1} + v_2\mathbf{e_2} + \cdots + v_j\mathbf{e_j} + \cdots + v_n\mathbf{e_n}$$

This concludes the proof!

Remark: The basis need not be unique but the representation of the vector in terms of the basis elements is Unique!

The following collection of vectors is also a basis on $\mathbb{R}^3$:

$$S' = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\square$

The Natural Basis on $\mathbb{R}^{mxn}$

Definition (The Natural Basis)

Suppose that: $A \in \mathbb{R}^{mxn}$, then A can be spanned by a collection of $U_{ij} \in \mathbb{R}^{mxn}$ such that:

$$U_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

udemy

Spanning Set of Vectors and Linear Independence

-

Corollary

Consider: $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots \mathbf{v_n}\}$ *a collection of vectors such that:* $\mathbf{v_j} \in V$ *where* S *spans* V*, then any collection of* m *vectors:* $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}, \ldots \mathbf{w_m}\}$ *such that* $m > n$ *is* **Linearly Dependent**.

Spanning Set of Vectors and Linear Independence

-

Corollary

Consider: $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots \mathbf{v_n}\}$ *a collection of vectors such that:* $\mathbf{v_j} \in V$ *where* S *spans* V*, then any collection of* m *vectors:* $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}, \ldots \mathbf{w_m}\}$ *such that* $m > n$ *is* **Linearly Dependent**.

-

Corollary

Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_j}, \ldots \mathbf{v_n}\}$ *and* $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}, \ldots \mathbf{w_m}\}$ *are collection of vectors that are a* **Basis** *on* V*, then it follows that* $n = m$.

The Dimension of a Vector Space

Definition (The Dimension of a Vector Space)

Suppose that $V \sim \textit{Vector Space}$.

1: If V is spanned by a finite collection of vectors, then we say $V \sim \textit{Finite} - \textit{Dimensional Vector Space}$.

2: If V is spanned by a infinite collection of vectors, then $V \sim \textit{Infinite} - \textit{Dimensional Vector Space}$.

3: If $V = \{0\}$, then the dimension of the Vector Space is $\mathbf{0}$.

Generally, the **Dimension** refers to the number of vectors in the Basis.

udemy

Properties of the Dimension

-

Theorem

Suppose that $V \sim Finite - Dimension: \; n > 0$, *then:*

1. (1): *Any collection of n linearly independent vectors* **Spans** V.
2. (2): *Any collection of n vectors that Span* V *are* **Linearly Independent**.

udemy

Properties of the Dimension

-

Theorem

Suppose that $V \sim Finite - Dimension: \; n > 0$, *then:*

1. (1): *Any collection of* n *linearly independent vectors* **Spans** V.
2. (2): *Any collection of* n *vectors that Span* V *are* **Linearly Independent**.

Theorem

Suppose that $V \sim Finite - Dimension: n > 0$, *then:*

1. (1): *Any collection of fewer than* n *vectors in* V **cannot Span** V.
2. (2): *Any sub-collection of less than* n *Linearly Independent vectors can be extended to form a Basis on* V.
3. (3): *Any collection of greater than* n *vectors that spans* V *can be parred down to obtain a basis on* V.

Table of Contents

udemy

Change of Basis in $\mathbb{R}^2$

Motivation: why study the change of basis?

1. (1): Recall $S = \{e_1, e_2\}$ is the Standard Basis on $\mathbb{R}^2$ and for any $\mathbf{x}$:

$$\mathbf{x} = x_1 e_1 + x_2 e_2$$

udemy

Change of Basis in $\mathbb{R}^2$

Motivation: why study the change of basis?

1. (1): Recall $S = \{e_1, e_2\}$ is the Standard Basis on $\mathbb{R}^2$ and for any $\mathbf{x}$:

$$\mathbf{x} = x_1 e_1 + x_2 e_2$$

2. (2): Given a basis, the representation is unique and $x_S = (x_1, x_2)$ is the coordinate vector of $\mathbf{x}$ with respect to the ordered basis S.

Change of Basis in $\mathbb{R}^2$

Motivation: why study the change of basis?

1. (1): Recall $S = \{e_1, e_2\}$ is the Standard Basis on $\mathbb{R}^2$ and for any $\mathbf{x}$:

$$\mathbf{x} = x_1 e_1 + x_2 e_2$$

2. (2): Given a basis, the representation is unique and $x_S = (x_1, x_2)$ is the coordinate vector of $\mathbf{x}$ with respect to the ordered basis S.
3. (3): Recalling that the Basis need not be unique, consider a distinct Ordered Basis $S' = \{y, z\}$, we wish to find the constants: c_1, c_2 such that for each $\mathbf{x} \in \mathbb{R}^2$:

$$\mathbf{x} = c_1 y + c_2 z$$

Where $\mathbf{c} = (c_1, c_2)$ is the coefficient vector of $\mathbf{x}$ with respect to S'.

$\square$

Example

Suppose that: $u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and take: $U = [u_1, u_2]$, one can show that $U \sim I_2$ implying that $S = \{u_1, u_2\}$ is an ordered basis on $\mathbb{R}^2$. We are interested in answering the two following questions:

WNTS.

$\square$

udemy

Example

Suppose that: $u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and take: $U = [u_1, u_2]$, one can show that $U \sim I_2$ implying that $S = \{u_1, u_2\}$ is an ordered basis on $\mathbb{R}^2$.
We are interested in answering the two following questions:

WNTS.

- 1: For each $\mathbf{x} \in \mathbb{R}^2$, we wish to find the coordinate vector with respect to S.

$\square$

udemy

Example

Suppose that: $u_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and take: $U = [u_1, u_2]$, one can show that $U \sim I_2$ implying that $S = \{u_1, u_2\}$ is an ordered basis on $\mathbb{R}^2$. We are interested in answering the two following questions:

WNTS.

- 1: For each $\mathbf{x} \in \mathbb{R}^2$, we wish to find the coordinate vector with respect to S.
- 2: Given an arbitrary $\mathbf{x} = c_1 u_1 + c_2 u_2$, we wish to find the coordinate vector with respect to The Standard Basis $\{e_1, e_2\}$.

$\square$

udemy

Example Cont'd

- 2: Observe that:

$$u_1 = 3e_1 + 2e_2$$
$$u_2 = e_1 + e_2$$

It follows that: $\mathbf{x} = c_1 u_1 + c_2 u_2 = (3c_1 + c_2)e_1 + (2c_1 + c_2)e_2$.
Thus, the coordinate vector of x with respect to The Standard Basis is: $\begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{bmatrix}$, which can be expressed as: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = U * c$

Example Cont'd

- (2): Observe that:

$$u_1 = 3e_1 + 2e_2$$
$$u_2 = e_1 + e_2$$

It follows that: $\mathbf{x} = c_1 u_1 + c_2 u_2 = (3c_1 + c_2)e_1 + (2c_1 + c_2)e_2$. Thus, the coordinate vector of x with respect to The Standard Basis is: $\begin{bmatrix} 3c_1 + c_2 \\ 2c_1 + c_2 \end{bmatrix}$, which can be expressed as: $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = U * c$

- (1): Since $U \sim I_2$, then $Det(U) \neq 0$ and $c = U^{-1}\mathbf{x}$. Since $\mathbf{x} = x_1 e_1 + x_2 e_2$, one only needs to pre-multiply $\mathbf{x}$ by U^{-1} to obtain the coordinate vector of $\mathbf{x}$ with respect to S.

The Change of Basis in $\mathbb{R}^2$

Definition (General Case in $\mathbb{R}^2$)

- 1 Suppose $F = \{u_1, u_2\}$ and $E = \{v_1, v_2\}$ are two ordered basis on $\mathbb{R}^2$, take $\mathbf{x} \in \mathbb{R}^2$, c and d the Coordinate vectors of $\mathbf{x}$ with respect to U and V respectively. Assume c is known and we wish to find d.

The Change of Basis in $\mathbb{R}^2$

Definition (General Case in $\mathbb{R}^2$)

- (1) Suppose $F = \{u_1, u_2\}$ and $E = \{v_1, v_2\}$ are two ordered basis on $\mathbb{R}^2$, take $\mathbf{x} \in \mathbb{R}^2$, c and d the Coordinate vectors of $\mathbf{x}$ with respect to U and V respectively. Assume c is known and we wish to find d.
- (2) Then:

$$Uc = \mathbf{x} = Vd \longrightarrow d = V^{-1}Uc$$

Remark: V^{-1} exists as E is an ordered basis on $\mathbb{R}^2$.

udemy

The Change of Basis in $\mathbb{R}^2$

Definition (General Case in $\mathbb{R}^2$)

- 1 Suppose $F = \{u_1, u_2\}$ and $E = \{v_1, v_2\}$ are two ordered basis on $\mathbb{R}^2$, take $\mathbf{x} \in \mathbb{R}^2$, c and d the Coordinate vectors of $\mathbf{x}$ with respect to U and V respectively. Assume c is known and we wish to find d.
- 2 Then:

$$Uc = \mathbf{x} = Vd \longrightarrow d = V^{-1}Uc$$

  Remark: V^{-1} exists as E is an ordered basis on $\mathbb{R}^2$.
- 3 Additionally, $S = V^{-1}U$ is referred to as the Transition Matrix, which transforms the coordinates of $\mathbf{x}$ with respect to F to the coordinates with respect to E.
  Lastly, the Coordinate vector of $\mathbf{x}$ with respect to the ordered basis E, and d is often denoted: $[\mathbf{x}]_E$,

udemy

Example

- (1) Suppose that one has: $V = [v_1, v_2] = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $U = [u_1, u_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

Example

- (1) Suppose that one has: $V = [v_1, v_2] = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $U = [u_1, u_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$
- (2) If $S = \{u_1, u_2\}$ and $S' = \{v_1, v_2\}$ are two ordered basis on $\mathbb{R}^2$ where: $Uc = \mathbf{x} = Vd$.

udemy

Example

- (1) Suppose that one has: $V = [v_1, v_2] = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $U = [u_1, u_2] = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$
- (2) If $S = \{u_1, u_2\}$ and $S' = \{v_1, v_2\}$ are two ordered basis on $\mathbb{R}^2$ where: $Uc = \mathbf{x} = Vd$.
- (3) Then, the transition matrix from coordinate vector c to coordinate vector d is:

$$S = V^{-1}U = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$$

Example

1. (1) Suppose $E = \{v_1, v_2, v_3\}$ and $F = \{u_1, u_2, u_3\}$ are two ordered basis where:

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix},\ V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix},\ [\mathbf{x}]_E = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Example

1. (1) Suppose $E = \{v_1, v_2, v_3\}$ and $F = \{u_1, u_2, u_3\}$ are two ordered basis where:

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{x}]_E = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

2. (2) It follows that any $\mathbf{x} \in \mathbb{R}^3$ can be expressed as:

$$\mathbf{x} = 3v_1 + 2v_2 + (-1)v_3$$

If we are interested in expressing $\mathbf{x}$ in terms of ordered basis of F.

udemy

Example

1. ① Suppose $E = \{v_1, v_2, v_3\}$ and $F = \{u_1, u_2, u_3\}$ are two ordered basis where:

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{x}]_E = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

2. ② It follows that any $\mathbf{x} \in \mathbb{R}^3$ can be expressed as:

$$\mathbf{x} = 3v_1 + 2v_2 + (-1)v_3$$

   If we are interested in expressing $\mathbf{x}$ in terms of ordered basis of F.

3. ③ Recalling that $[\mathbf{x}]_F = S[\mathbf{x}]_E = U^{-1}V[\mathbf{x}]_E$, then:

$$[\mathbf{x}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}$$

udemy

Example

1. ① Suppose $E = \{v_1, v_2, v_3\}$ and $F = \{u_1, u_2, u_3\}$ are two ordered basis where:

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{x}]_E = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

2. ② It follows that any $\mathbf{x} \in \mathbb{R}^3$ can be expressed as:

$$\mathbf{x} = 3v_1 + 2v_2 + (-1)v_3$$

If we are interested in expressing $\mathbf{x}$ in terms of ordered basis of F.

3. ③ Recalling that $[\mathbf{x}]_F = S[\mathbf{x}]_E = U^{-1}V[\mathbf{x}]_E$, then:

$$[\mathbf{x}]_F = \begin{bmatrix} 1 & 1 & -3 \\ -1 & -1 & 0 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 3 \end{bmatrix}$$

4. ④ The last equation implies that: $\mathbf{x} = 8u_1 + (-5)u_2 + 3u_3$

udemy

Table of Contents

udemy

$R(A)$ and $C(A)$ are Vector Subspaces

Definition (Row and Column Spaces of $A \in \mathbb{R}^{mxn}$)

1. (1) Suppose $A \in \mathbb{R}^{mxn}$, recall that $A = [a_1, a_2, \ldots, a_j, \ldots, a_n]$ where $a_j \in \mathbb{R}^{mx1}$ is the j_{th} column vector of A.
   Then the **Column Space** of A is:

$$C(A) = \left\{ \sum_{j=1}^{n} c_j a_j \,\middle|\, c_j \in \mathbb{R} \right\} \quad \textbf{Remark} : C(A) \subset \mathbb{R}^m$$

$R(A)$ and $C(A)$ are Vector Subspaces

Definition (Row and Column Spaces of $A \in \mathbb{R}^{mxn}$)

1. (1) Suppose $A \in \mathbb{R}^{mxn}$, recall that $A = [a_1, a_2, \ldots, a_j, \ldots, a_n]$ where $a_j \in \mathbb{R}^{mx1}$ is the j_{th} column vector of A.
   Then the **Column Space** of A is:

$$C(A) = \left\{ \sum_{j=1}^{n} c_j a_j \;\middle|\; c_j \in \mathbb{R} \right\} \quad \textbf{Remark} : C(A) \subset \mathbb{R}^m$$

2. (2) Additionally, $A = (a'_1, a'_2, \ldots, a'_i, \ldots, a'_m)$ where $a'_i \in \mathbb{R}^{1xn}$ is the i_{th} row vector of A.
   The **Row Space** of A is:

$$R(A) = \left\{ \sum_{i=1}^{m} c_i {a_i}' \;\middle|\; c_i \in \mathbb{R} \right\} \quad \textbf{Remark} : R(A) \subset \mathbb{R}^n$$

Example

- ① Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

udemy

Example

- 1 Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- 2 Then, $R(A) = \alpha a_1' + \beta a_2' = (\alpha, \beta, 0)$ where $\alpha, \beta \in \mathbb{R}$.

udemy

Example

- (1) Consider the matrix:

$$A = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \end{bmatrix}$$

- (2) Then, $R(A) = \alpha a_1' + \beta a_2' = (\alpha, \beta, 0)$ where $\alpha, \beta \in \mathbb{R}$.
- (3) The $C(A) = \alpha a_1 + \beta a_2 + \gamma a_3 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ for $\alpha, \beta \in \mathbb{R}$

udemy

Example

- ① Consider the matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- ② Then, $R(A) = \alpha a_1' + \beta a_2' = (\alpha, \beta, 0)$ where $\alpha, \beta \in \mathbb{R}$.
- ③ The $C(A) = \alpha a_1 + \beta a_2 + \gamma a_3 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ for $\alpha, \beta \in \mathbb{R}$
- **Remark**: There are two pivots of A and the number of independent rows (2) is referred to as the **Row Rank** while to the number of linearly independent columns (2) is referred to as the **Column Rank**. Since the number of rows is the same as the **Row Rank**, then A has **full Row Rank**.

udemy

The Row Rank and Column Rank are the same

- ①:

Definition (The Rank)

Suppose $A \in \mathbb{R}^{mxn}$, then the **Row Rank** $= dim(R(A))$ and the **Column Rank** $= dim(C(A))$.

Furthermore, $dim(R(A)) = dim(C(A)) \triangleq \textbf{Rank}(\textbf{A})$.

udemy

The Row Rank and Column Rank are the same

- ①:

Definition (The Rank)

Suppose $A \in \mathbb{R}^{mxn}$, then the **Row Rank** $= dim(R(A))$ and the **Column Rank** $= dim(C(A))$.

Furthermore, $dim(R(A)) = dim(C(A)) \triangleq \mathbf{Rank}(\mathbf{A})$.

- ②:

Lemma

Given any $A \in \mathbb{R}^{mxn}$*, one has:* $Rank(A) \leq \min\{m, n\}$

udemy

The Row Rank and Column Rank are the same

- ①:

Definition (The Rank)

Suppose $A \in \mathbb{R}^{mxn}$, then the **Row Rank** $= dim(R(A))$ and the **Column Rank** $= dim(C(A))$.

Furthermore, $dim(R(A)) = dim(C(A)) \stackrel{\triangle}{\equiv} \mathbf{Rank(A)}$.

- ②:

Lemma

Given any $A \in \mathbb{R}^{mxn}$, one has: $Rank(A) \leq \min\{m, n\}$

- ③:

Lemma

If B is Row Equivalent to $A \longrightarrow R(B) = R(A)$.
Remark: *If B is row equivalent to A, then there are k Elementary Matrices such that when pre-multiplied on A give B, yet row operations preserve the Row Space.*

Relevance of $C(A)$ to $Ax = b$

Theorem (The Column Space and Consistency)

- (1) *Suppose* $A \in \mathbb{R}^{mxn}$ *and one solves* $Ax = b$ *where* $x \in R^n$*, and* $b \in \mathbb{R}^m$.

Relevance of $C(A)$ to $Ax = b$

Theorem (The Column Space and Consistency)

- 1 *Suppose* $A \in \mathbb{R}^{mxn}$ *and one solves* $Ax = b$ *where* $x \in R^n$, *and* $b \in \mathbb{R}^m$.
- 3 *Observe that:*

$$b = A\mathbf{x} = [a_1, a_2, \ldots, a_j, \ldots, a_n]\mathbf{x}$$

Relevance of $C(A)$ to $Ax = b$

Theorem (The Column Space and Consistency)

- 1 *Suppose* $A \in \mathbb{R}^{mxn}$ *and one solves* $Ax = b$ *where* $x \in R^n$, *and* $b \in \mathbb{R}^m$.
- 3 *Observe that:*

$$b = A\mathbf{x} = [a_1, a_2, \ldots, a_j, \ldots, a_n]\mathbf{x}$$

- 3 *Which implies that:*

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{i1} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{i2} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_j \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{ij} \\ \vdots \\ a_{mj} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{in} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Relevance of $C(A)$ to $Ax = b$

Theorem (The Column Space and Consistency)

The system of equations $Ax = b$ is **consistent** $\longleftrightarrow$ $b_i = \sum_{j='1}^{n} x_j a_j$ *for each* $i = 1, 2, \ldots, j, \ldots, n$. $\longleftrightarrow$ $b \in C(A)$ **The columns of** A **span** $\mathbb{R}^m$. *Furthermore, there is at most one solution provided the* **columns** *of A are* **Linearly Independent**.

Theorem (Rank-Nullity Theorem)

Consider $A \in \mathbb{R}^{mxn}$, *then*
$Rank(A) + Nullity(A) = n \longleftrightarrow Rank(A) = n - Nullity(A)$ *is referred to as the* **Rank-Nullity Theorem** *of Matrix.*

Corollary (Conditions for Non-singularity)

Any $A \in \mathbb{R}^{nxn}$ *is* **non-singular** *provided the columns of A form a basis on* $\mathbb{R}^n$, *i.e.,* $\mathbb{R}^n \subset C(A)$ *or* $C(A) = \mathbb{R}^n$

Example

- (1) Consider the matrix: $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$ and suppose that we are interested in finding a Basis on $N(A)$ and $R(A)$.

Example

- (1) Consider the matrix: $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$ and suppose that we are interested in finding a Basis on $N(A)$ and $R(A)$.
- (2) The matrix can be row reduced to: $U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the Row Space of A is invariant under row operations, one obtains:

$$R(A) = \{\alpha * (1, 2, 0, 3) + \beta * (0, 0, 1, 2) \mid \alpha, \beta \in \mathbb{R}\}$$

Augmenting $(U \mid \mathbf{0})$, one obtains: $x_1 = -2x_2 - 3x_4$, $x_3 = -2x_4$ and taking $x_2 = \alpha$, $x_4 = \beta$ as free variables.

Example

- (1) Consider the matrix: $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$ and suppose that we are interested in finding a Basis on $N(A)$ and $R(A)$.
- (2) The matrix can be row reduced to: $U = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the Row Space of A is invariant under row operations, one obtains:

$$R(A) = \{\alpha * (1, 2, 0, 3) + \beta * (0, 0, 1, 2) \mid \alpha, \beta \in \mathbb{R}\}$$

  Augmenting $(U \mid \mathbf{0})$, one obtains: $x_1 = -2x_2 - 3x_4$, $x_3 = -2x_4$ and taking $x_2 = \alpha$, $x_4 = \beta$ as free variables.
- (3) Then:

$$N(A) = \left\{ \alpha * \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta * \begin{bmatrix} -3 \\ 0 \\ -2 \\ -1 \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

udemy

References I

[1] David Harville. *Matrix Algebra From a Statistician's Perspective*. New York: Springer-Verlag, 1997.

[2] Leon Stephen. *Linear Algebra with Applications (9th Edition) (Featured Titles for Linear Algebra*. London, England: Pearson, 2014.

udemy