Linear Algebra from Scratch: Matrices and Systems

Instructor Anthony
GSU Alumni

Udemy Open Course

M.S. in Mathematics with a Concentration in Bioinformatics

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Example: A System of Two Linear Equations

- Consider the following system of linear equations:

$$\begin{aligned} 2x + 3y &= 5 \\ x - y &= 10 \end{aligned}$$

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$$\begin{aligned} 2x + 3y &= 5 \\ -2x + 2y &= -20 \end{aligned}$$

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- Obtained by multiplying the second row by -2. Adding the first and second equations, will give us:

$$5y = -15 \longrightarrow y = -3$$

  After substituting $y = -3$ into second equation, one obtains $x = 7$. The solution of system of two linear equations is the ordered tuplet: $(7, -3)$.

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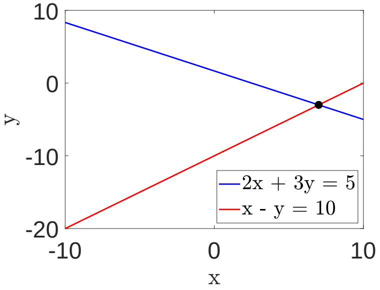
Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:



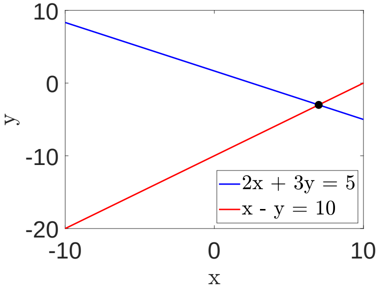
□

Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:
2. The solution exists if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.

□



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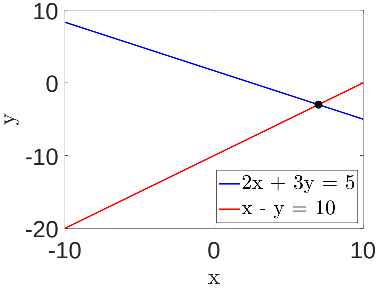
Example: A System of Two Linear Equations

Proof.

1. The solution of the system of the linear equations with two unknowns (x, y) is an ordered pair (x_0, y_0) and is observed when using the graphical method:
2. The solution exists if the lines intersect at least once. We will say that the system is **consistent**. Otherwise, the system is **inconsistent**.
3. The solution is unique if the lines intersect only once.

□



Example: A System of Two Linear Equations

In the previous example, one can classify the different constituents of the system as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \sim \textit{Coefficient Matrix}$$

$$b = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \sim \textit{Column Vector}$$

$$\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \textit{Vector of 2 Unknowns}$$

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General Form of the System of Two Linear Equations

Definition (General form of a System of Two Linear Equations)

A system of two linear equations with two unknowns is expressed as follows:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Where:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \; \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \; and \; b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The system is **linear** if each element belonging to the vector of unknowns x_j has a power of 1 and does not contain nonlinear functions such as trigonometric functions or exponential functions.

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General From of the System of m Linear Equations

Definition (General form of a System of m Linear Equations)

A system of m linear equations with n unknowns has the following form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1j}x_j + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2j}x_j + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ij}x_j + \cdots + a_{in}x_n &= b_i \\
\vdots &= \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mj}x_j + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

We will refer to:

$$a_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} \sim j_{th} \textit{ Column Vector and } a_i{}' = (a_{i1}, \ldots, a_{in}) \sim i_{th} \textit{ Row Vector}$$

General From of the System of m Linear Equations

Definition (Matrix form of System of n Linear Equations)

The matrix form of the set of m linear equations with n unknowns where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ is: $Ax = b$, which is expressed as:

$$Ax = b \leftrightarrow \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$

General From of the System of m Linear Equations

General form of a System of m Linear Equations.

We will refer to the matrix of size mxn where m is the number of rows and n is the number of columns, denoted $A \in \mathbb{R}^{mxn}$, is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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Augmented System of Linear Equations

Definition (Augmented System)

Let: $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$, then:

$$A = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array}\right]$$

is the augmented system of $Ax = b$.

General From of the System of m Linear Equations

General form of a System of m Linear Equations.

The j_{th} column vector of A is denoted $a_j \in \mathbb{R}^{mx1}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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General From of the System of m Linear Equations

General form of a System of m Linear Equations.

The i_{th} row vector of A is denoted $a_i{}' \in \mathbb{R}^{1xn}$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

□

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General From of the System of m Linear Equations

General form of a System of m Linear Equations.

The intersection of the i_{th} row of A and j_{th} column of A is the $(i,j)_{th}$ element of A denoted: a_{ij}:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}

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Example

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2. a_1

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Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. a_1

1. 3
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. $\mathbf{a}_1$
3. $\vec{a}'_2$

1. 3
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

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Example

Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

Determine the following:

1. a_{12}
2. $\mathbf{a}_1$
3. $\vec{a}_2'$

1. 3
2. $$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
3. (2, 5)

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Elementary Row Operations

Definition (Elementary Row Operations)

An elementary row operation is an invertible operation applied between two rows of a matrix. Row operations may be classified as:

1. Interchange of two rows AKA row swap
2. Multiply a row by a nonzero real number
3. Replace a row by its sum with a multiple of another row

Elementary Row Operations

Suppose that A_1 is the original matrix and we swap the i_{th} and k_{th} row resulting in the matrix A_2.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{red}{a_{i1}} & \color{red}{a_{i2}} & \dots & \color{red}{a_{ij}} & \dots & \color{red}{a_{in}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{blue}{a_{k1}} & \color{blue}{a_{k2}} & \dots & \color{blue}{a_{kj}} & \dots & \color{blue}{a_{kn}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

After swapping the i_{th} and k_{th} row, one obtains:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kj} & \dots & a_{kn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

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Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

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Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2. After swapping the first and second row of A_1, one obtains:

$$A_2 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

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Elementary Row Operations

Suppose that A_1 is the original matrix and we scale the i_{th} row by a factor of $\alpha \neq 0$ to obtain:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

1. Let:

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Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

2. After multiplying row 2 by a factor of 3 one obtains:

$$A_2 = \begin{bmatrix} 1 & 3 \\ 6 & 12 \end{bmatrix}$$

Elementary Row Operations

Suppose that A_1 is the original matrix and we scale the i_{th} row by a factor of α and add it to the k_{th} row to obtain:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{k1} + \alpha a_{i1} & a_{k2} + \alpha a_{i2} & \dots & a_{kj} + \alpha a_{ij} & \dots & a_{kn} + \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Elementary Row Operations

1. Let:

$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

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Elementary Row Operations

1. Let:
$$A_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$
2. After multiplying row 2 by a factor of 3 and adding it to row 1, one obtains:
$$A_2 = \begin{bmatrix} 7 & 15 \\ 2 & 4 \end{bmatrix}$$

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Equivalent Systems of Linear Equations

Lemma (Equivalence of Systems)

Given two systems of linear equations, the systems are equivalent provided a sequence of elementary row operations can be applied from one matrix to the other.

In other words, if $A_1 \to A_2 \to \cdots \to A_k$ *is an iterative sequence of applying* $k-1$ *row operations, then* $A_1x = b_1$ *and* $A_kx = b_k$ *are equivalent systems.*

Remark*: Two systems are equivalent provided the solution sets of each are identical.*

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Triangular form of a System of Equations

Definition (Triangular Form)

1. Given $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$, the system: $Ax = b$ is strict triangular provided:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \ddots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & a_{mn} \end{bmatrix}$$

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2. In other words, $a_{ij} = 0$ when $i > j$. If $m = n$, then A is referred to as a lower triangular matrix. Alternatively, if in the k_{th} equation, the coefficients of the first $k - 1$ variables are all zero and the coefficient of x_k is nonzero for each $k = 1, 2, \dots, n$.

Example: Triangular Form

1. Consider the system:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 1 \\ x_2 - x_3 &= 2 \\ 2x_3 &= 4 \end{aligned}$$

Which can be expressed as:

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   Which can be expressed as:

2. $$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

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Example: Back Substitution

Given the system:

$$
\begin{aligned}
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$$

We can solve using **Back Substitution**:

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We can solve using **Back Substitution**:

1. $2x_3 = 4 \rightarrow x_3 = 2$
2. $x_2 - x_3 = 2 \rightarrow x_2 = 4$
3. $x_1 = \frac{1-2x+2-x_3}{3} \rightarrow x_1 = -3$

Reference: Leon, 2014 [2]

Example: Back Substitution

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2. Performing **Elementary Row Operations** to get the matrix in a triangular form is the way!

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1. But how do we get the matrix into triangular form in the first place?
2. Performing **Elementary Row Operations** to get the matrix in a triangular form is the way!
3. Formally, we will refer to this process as **Gaussian Elimination**.

Example: Row Operations to obtain a triangular matrix

1. Given the matrix:

$$A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 3 & 3 & 0 & 3 \\ -3 & -2 & 1 & 3 \end{array}\right]$$

By subtracting the first row from the second row, one obtains:

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Example: Row Operations to obtain a triangular matrix

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   And adding the first row to the third row, one has:

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   And adding the first row to the third row, one has:

3. $$A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

   Which can be solved using **Back Substitution**

Row Echelon Form

We will refer to the nonzero element of a_{ij} where $a_{i1} = a_{i2} = \cdots = a_{i,j-1} = 0$ as a pivot corresponding to the i_{th} row. Given the matrix:

$$A = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

The pivot of the third row is 2, the pivot the second row is 1, and the pivot of the first row is 3. The matrix can be reduced into **Row Echelon Form (REF)** provided the leading nonzero entry (pivot) have a value of 1. Diving each row by its corresponding pivot value gives:

$$A = \left[\begin{array}{ccc|c} 1 & 2/3 & 1/3 & 1/3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Row Echelon Form

Definition (Row Echelon Form)

The system of equations $Ax = b$ is in **Row Echelon Form (REF)** provided the following conditions hold:

- The first nonzero entry in each row is 1.

-
$$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Row Echelon Form

Definition (Row Echelon Form)

The system of equations $Ax = b$ is in **Row Echelon Form (REF)** provided the following conditions hold:

- The first nonzero entry in each row is 1.
- If the k_{th} row does not consist entirely of zeros, then the number of leading zeros in the next row $(k+1)$ is greater than the number of rows in the k_{th} row.

- $$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Row Echelon Form

Definition (Row Echelon Form)

The system of equations $Ax = b$ is in **Row Echelon Form (REF)** provided the following conditions hold:

- The first nonzero entry in each row is 1.
- If the k_{th} row does not consist entirely of zeros, then the number of leading zeros in the next row $(k+1)$ is greater than the number of rows in the k_{th} row.
- If a row has entries all zero, then it must be directly below the rows having nonzero entries.

- $$A = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Reduced Row Echelon Form

Lemma (Reduced Row Echelon Form)

The system of equations $Ax = b$ is in **Reduced Row Echelon Form (RREF)** *provided the following conditions hold:*

- *The Matrix is in Row Echelon Form*

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

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Reduced Row Echelon Form

Lemma (Reduced Row Echelon Form)

The system of equations $Ax = b$ *is in* **Reduced Row Echelon Form (RREF)** *provided the following conditions hold:*

- *The Matrix is in Row Echelon Form*
- *The first nonzero entry in each row is the only nonzero entry in its column*

$$A = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

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Gaussian Elimination

Definition (Gaussian Elimination)

Given the system of linear equations $Ax = b$, where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$ whose augmented system is:

$$A = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} & b_i \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} & b_m \end{array}\right]$$

The process of utilizing the 3 elementary row operations to transform the linear system from an augmented matrix to Row Echelon Form is referred to as **Gaussian Elimination**.

Gauss-Jordan Elimination

Lemma (Gauss-Jordan Elimination)

The process of utilizing the 3 elementary row operations to transform the linear system from an augmented matrix to Reduced Row Echelon Form is referred to as **Gauss-Jordan Elimination**.

Over-determined Systems

1. **Definition (Over-determined Systems)**

   An **over-determined** system of linear equations is a system of linear equations whose number of equations exceeds the number of unknowns. If $Ax = b$ is the system of linear equations where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$, then: $m > n$.

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Over-determined Systems

1.

Definition (Over-determined Systems)

An **over-determined** system of linear equations is a system of linear equations whose number of equations exceeds the number of unknowns. If $Ax = b$ is the system of linear equations where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$, then: $m > n$.

2. Consider the system:

$$
\begin{aligned}
x_1 + x_2 &= 1 \\
x_1 - x_2 &= 3 \\
-x_1 + 2x_2 &= -2
\end{aligned}
$$

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Over-determined Systems

- Which has an augmented system of:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right]$$

After subtracting the first row from the second row and adding the first row to the third row:

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Over-determined Systems

- Which has an augmented system of:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right]$$

  After subtracting the first row from the second row and adding the first row to the third row:

- $$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array}\right]$$

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Over-determined Systems

- Which has an augmented system of:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right]$$

After subtracting the first row from the second row and adding the first row to the third row:

- $$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array}\right]$$

- After dividing the second row by 2 and adding 3 times the result to the third row:

$$A = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

Over-determined Systems

The last row of the reduced form implies that $0x_1 + 0x_2 = 2$, which is not possible and a solution does not exist!
Therefore, the system of linear equations is **inconsistent**!
Remark: Generally, over-determined systems are more often than not inconsistent, but not necessarily.

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Under-determined Systems

-

Definition (Under-determined Systems)

An **under-determined** system of linear equations is a system of linear equations whose number of equations is less than the number of rows. If $Ax = b$ is the system of linear equations where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$, then: $m < n$.

Under-determined Systems

-

Definition (Under-determined Systems)

An **under-determined** system of linear equations is a system of linear equations whose number of equations is less than the number of rows. If $Ax = b$ is the system of linear equations where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^m$, then: $m < n$.

- Take the following system:

$$
\begin{aligned}
x_1 + 2x_2 + x_3 &= 1 \\
2x_1 + 4x_2 + 2x_3 &= 3
\end{aligned}
$$

Under-determined Systems

- The augmented form is:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{array}\right]$$

After multiplying the first row by -2 and adding it to the second row one obtains:

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Under-determined Systems

- The augmented form is:

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \end{array}\right]$$

  After multiplying the first row by -2 and adding it to the second row one obtains:

- $$A = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Which implies that: $x_1 = 1 - 2x_2 - x_3$ where x_2 and x_3 are free variables. For simplicity, we will take: $x_2 = \alpha$ and $x_3 = \beta$, then:

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Under-determined Systems

- The augmented form is:

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Which implies that: $x_1 = 1 - 2x_2 - x_3$ where x_2 and x_3 are free variables. For simplicity, we will take: $x_2 = \alpha$ and $x_3 = \beta$, then:

- $$\mathbf{x} = \begin{bmatrix} 1 - 2\alpha - \beta \\ \alpha \\ \beta \end{bmatrix}$$

As our solution. Generally, under-determined systems will not have unique solutions (α, β are free).

The Homogeneous Equation

-

Definition (The Homogeneous Equation)

The **homogeneous equation** is expressed as: $\boldsymbol{Ax} = \mathbf{0}$, where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$

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The Homogeneous Equation

-

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-

Lemma (Solution to Under-determined Systems)

Given the under-determined system of the homogeneous equation: $Ax = 0$, where where $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$ and $m < n$, there exists a non-trivial solution: $\boldsymbol{x} \neq 0$.

Remark: *The trivial solution $\boldsymbol{x} = 0$, is always a solution to the homogeneous equation: $Ax = 0$*

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Matrix Addition I

Definition (Matrix Addition)

Suppose $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{mxn}$ where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & b_{ij} & \dots & b_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mj} & \dots & b_{mn} \end{bmatrix}$$

If A and B have the same shape, then we will say that A and B are **conformal** and matrix addition is well-defined.

Matrix Addition II

Definition (Matrix Addition)

We will define the addition of matrices $C = A + B$ where the $(i, j)_{th}$ element of C is the sum of the corresponding elements of A and B, e.g., $c_{ij} = a_{ij} + b_{ij}$:

$$C = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) & \dots & (a_{1j} + b_{1j}) & \dots & (a_{1n} + b_{1n}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) & \dots & (a_{2j} + b_{2j}) & \dots & (a_{2n} + b_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{i1} + b_{i1}) & (a_{i2} + b_{i2}) & \dots & (a_{ij} + b_{ij}) & \dots & (a_{in} + b_{in}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (a_{m1} + b_{m1}) & (a_{m2} + b_{m2}) & \dots & (a_{mj} + b_{mj}) & \dots & (a_{mn} + b_{mn}) \end{bmatrix}$$

A simplified notation is: $[c_{ij}] = [a_{ij} + b_{ij}]$ for each $i = \overline{1, m}$ and $j = \overline{1, n}$.

Example

- Let:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

  and:

$$B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

Example

- Let:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$

  and:

$$B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix}$$

- Then: $C = A + B$ is:

$$C = \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$$

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Scalar Multiplication I

Definition (Scalar Multiplication)

Suppose that $A \in \mathbb{R}^{mxn}$ and $\alpha \in \mathbb{R}$, then one has the following:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \alpha A = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \dots & \alpha a_{1j} & \dots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots & \alpha a_{2j} & \dots & \alpha a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{i1} & \alpha a_{i2} & \dots & \alpha a_{ij} & \dots & \alpha a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \dots & \alpha a_{mj} & \dots & \alpha a_{mn} \end{bmatrix}$$

Alternative notation is: $\alpha[a_{ij}] = [\alpha a_{ij}]$

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Scalar Multiplication II

Suppose that:

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

After multiplying $B = 3 * A$, one obtains:

$$B = \begin{bmatrix} 3 & 9 \\ 15 & 21 \end{bmatrix}$$

Matrix Transposition

Definition (Matrix Transposition)

Let $A \in \mathbb{R}^{mxn}$, then the transposition of A denoted A' or A^T is a matrix whose $(i,j)_{th}$ element of A is the $(j,i)_{th}$ element of A'

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix} \quad A' = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

Remark: Generally, if $A \in \mathbb{R}^{mxn}$, then: $A' \in \mathbb{R}^{nxm}$

Example

- Consider the following matrix:

$$A = \begin{bmatrix} 1 & 3 & 5 & 6 \\ 11 & 21 & 34 & 100 \\ 23 & 50 & 10 & 25 \end{bmatrix}$$

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Example

- Consider the following matrix:

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- The transposition of A is:

$$A' = \begin{bmatrix} 1 & 11 & 23 \\ 3 & 21 & 50 \\ 5 & 34 & 10 \\ 6 & 100 & 25 \end{bmatrix}$$

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- Consider the following matrix:

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- The transposition of A is:

$$A' = \begin{bmatrix} 1 & 11 & 23 \\ 3 & 21 & 50 \\ 5 & 34 & 10 \\ 6 & 100 & 25 \end{bmatrix}$$

- **Remark**: Note that $A' \in \mathbb{R}^{4x3}$, yet $A \in \mathbb{R}^{3x4}$. Generally, if $A \in \mathbb{R}^{mxn}$, then $A' \in \mathbb{R}^{nxm}$.

Matrix Multiplication I

Definition (Matrix Multiplication)

Suppose $A \in \mathbb{R}^{mxp}$, $B \in \mathbb{R}^{qxn}$, then: $C = AB$ is a well defined matrix product provided $p = q$. The resulting matrix $C \in \mathbb{R}^{mxn}$, whose $(i, j)_{th}$ element is obtained by taking the weighted sum of the i_{th} row of A and j_{th} column of B

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \color{red}{a_{i1}} & \color{red}{a_{i2}} & \dots & \color{red}{a_{ij}} & \dots & \color{red}{a_{ip}} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mp} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \dots & \color{blue}{b_{1j}} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & \color{blue}{b_{2j}} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i1} & b_{i2} & \dots & \color{blue}{b_{ij}} & \dots & b_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \dots & \color{blue}{b_{pj}} & \dots & b_{pn} \end{bmatrix}$$

When the number of columns of A is the the same as the number of rows of B, in which matrix product AB is well-defined, we will say that A and B are **compatible**.

Matrix Multiplication II

Definition (Matrix Multiplication)

Recalling that the $(i,j)_{th}$ element of C is the product between the i_{th} row of A and the j_{th} column of B one has:

$$
\begin{aligned}
c_{ij} &= {a_i}' * b_j \\
&= (a_{i1}, a_{i2} \ldots a_{ik} \ldots a_{ip})' * \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{kj} \\ \vdots \\ b_{pn} \end{bmatrix} \\
&= \sum_{k=1}^{p} a_{ik} * b_{kj}
\end{aligned}
$$

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Example

Take:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$$

Then:

$$\begin{aligned} C = AB &= \begin{bmatrix} 1*2+3*3 & 1*4+3*5 & 1*6+3*7 \\ 2*2+4*3 & 2*4+4*5 & 2*6+4*7 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 19 & 27 \\ 16 & 28 & 40 \end{bmatrix} \end{aligned}$$

Remark: Generally, the product $C = AB$ can be expressed as: $[Ab_1, A_{b2} \ldots Ab_j \ldots Ab_n]$, meaning the j_{th} column of C is Ab_j.

Linear Combination

1.

Definition (Linear Combination)

Let $A \in \mathbb{R}^{mxn}$, such that $A = [a_1, a_2 \ldots a_j \ldots a_n]$ and $\alpha_1, \alpha_2, \ldots \alpha_j \ldots \alpha_n$ be arbitrary scalars, then the a linear combination of the columns of A is:

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_j a_j + \cdots + \alpha_n a_n = \sum_{j=1}^{n} \alpha_j a_j$$

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$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_j a_j + \cdots + \alpha_n a_n = \sum_{j=1}^{n} \alpha_j a_j$$

2.

Lemma (Existence of a Solution to System of Equations)

Given the system of linear equations: $Ax = b$ where $A \in \mathbb{R}^{mxn}$, $b \in \mathbb{R}^m$, and $x \in \mathbb{R}^n$, the system is **consistent** *provided b can be expressed as a linear combination of the columns of A, i.e.,:*

$$b = \sum_{j=1}^{n} x_j a_j$$

for some $x_j \in \mathbb{R}$.

Reference: Leon, 2014 [2]

Special Types of Matrices I

1.

Definition (Square Matrix)

A **square matrix** is a matrix whose number of rows are the same as the number of columns: if $A \in \mathbb{R}^{mxn}$ then: $m = n$

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Special Types of Matrices I

1. **Definition (Square Matrix)**

A **square matrix** is a matrix whose number of rows are the same as the number of columns: if $A \in \mathbb{R}^{mxn}$ then: $m = n$

2. **Definition (Symmetric Matrix)**

A square matrix $A \in \mathbb{R}^{nxn}$ is **symmetric** provided $A' = A$, which implies that the $(i, j)_{th}$ element of A is the $(i, j)_{th}$ of A'.

Special Types of Matrices I

1. **Definition (Square Matrix)**

   A **square matrix** is a matrix whose number of rows are the same as the number of columns: if $A \in \mathbb{R}^{mxn}$ then: $m = n$

2. **Definition (Symmetric Matrix)**

   A square matrix $A \in \mathbb{R}^{nxn}$ is **symmetric** provided $A' = A$, which implies that the $(i,j)_{th}$ element of A is the $(i,j)_{th}$ of A'.

3. **Definition (Equality of Matrices)**

   Given two conformal matrices $A, B \in \mathbb{R}^{mxn}$, **two matrices are equal** $A = B$ iff:

$$a_{ij} = b_{ij} \text{ } for \text{ } each \text{ } i,j = \overline{1,m}; \text{ } \overline{1,n}$$

Special Types of Matrices II

Definition (Singular Matrix)

Suppose we are given the homogeneous system of equations: $Ax = 0$ $A \in \mathbb{R}^{nxn}$, $x \in \mathbb{R}^n$, we will say that $A \sim \boldsymbol{Singular}$ provided a nontrivial solution $\boldsymbol{x} \neq 0$ exists. Otherwise $A \sim Non - Singular$. **Remark**: A Non-Singular matrix is also referred to as an invertible matrix.

1

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Special Types of Matrices II

Definition (Singular Matrix)

Suppose we are given the homogeneous system of equations: $Ax = 0$ $A \in \mathbb{R}^{nxn}$, $x \in \mathbb{R}^n$, we will say that $A \sim$ ***Singular*** provided a nontrivial solution $\boldsymbol{x} \neq 0$ exists. Otherwise $A \sim Non - Singular$. **Remark**: A Non-Singular matrix is also referred to as an invertible matrix.

1.

Definition (Identity Matrix)

The **Identity Matrix**, denoted I_n or I, is a special square matrix whose diagonal elements are 1, and off-diagonal elements are 0:

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ 0 & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

Special Types of Matrices III

1.

Definition (Null Matrix)

The **Null Matrix** is the matrix consisting entirely of zeros and is denoted 0_{mxn} or 0.

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Special Types of Matrices III

1. **Definition (Null Matrix)**

   The **Null Matrix** is the matrix consisting entirely of zeros and is denoted 0_{mxn} or 0.

2. **Definition (Invertible Matrix)**

   We will say that $A \in \mathbb{R}^{nxn}$ is **invertible** if there exists a matrix B such that: $AB = I_n$ and $BA = I_n$.

Special Types of Matrices III

1. **Definition (Null Matrix)**

   The **Null Matrix** is the matrix consisting entirely of zeros and is denoted 0_{mxn} or 0.

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   We will say that $A \in \mathbb{R}^{nxn}$ is **invertible** if there exists a matrix B such that: $AB = I_n$ and $BA = I_n$.

3. **Definition (Orthogonal Matrix)**

   The **Orthogonal Matrix** $Q \in \mathbb{R}^{nxn}$ is a matrix that satisfies:

$$Q'Q = I_n$$

Which implies that the Orthogonal Matrix's inverse is its transpose.

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Algebraic Rules of Matrix Operations

Theorem (Algebraic Rules)

Suppose A, B, C *are matrices of sizes so that the operations are well-defined and take:* α, β *as scalars, then the following will hold:*

1. *Commutative Property of Addition:* $A + B = B + A$

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4. *Left (or Right) Distributive Property of Matrix Multiplication I:* $A(B + C) = AB + AC$

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Algebraic Rules of Matrix Operations

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6. *Associative Property of Scalar Multiplication I:* $(\alpha\beta)A = \alpha(\beta A)$

Algebraic Rules of Matrix Operations

Theorem (Algebraic Rules)

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Algebraic Rules of Matrix Operations

Theorem (Algebraic Rules)

Suppose A, B, C *are matrices of sizes so that the operations are well-defined and take:* α, β *as scalars, then the following will hold:*

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2. *Associative Property of Addition:* $(A + B) + C = A + (B + C)$
3. *Associative Property of Matrix Multiplication:* $(AB)C = A(BC)$
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Proof of Left Distributive Property

Proof of 5: $(A + B)C = AC + BC$.

1: Suppose $A, B \in \mathbb{R}^{mxn}$, $C \in \mathbb{R}^{nxp}$.

2: Let $D = (A + B)C$ and $E = AC + BC$, then:

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1. Observe that the $(i, j)_{th}$ element of D is:

$$
\begin{aligned}
d_{ij} &= \sum_{k=1}^{n} (a_{ik} + b_{ik}) c_{kj} \\
&= \sum_{k=1}^{n} [a_{ik} c_{kj} + b_{ik} c_{kj}] \\
&= \sum_{k=1}^{n} a_{ik} c_{kj} + \sum_{k=1}^{n} b_{ik} c_{kj}
\end{aligned}
$$

Which is the $(i, j)_{th}$ element of E!

$\square$

Lemma

The Invertible Matrix] Suppose $A \sim$ *nonsingular (or invertible nxn matrix), let* $A^{-1} \sim$ *Inverse of A, then:*

$$AA^{-1} = I_n$$

Theorem (Algebraic Rules of the Inverse Operation)

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- 5: **Remark**: *By Induction, we can generalize* 3 *to:*

$$(A_1 * A_2 * \cdots * A_{k-1} * A_k)^{-1} = A_k^{-1} * A_{k-1}^{-1} * \cdots * A_2^{-1} * A_1^{-1}$$

Proof that the product of non-singular matrices is non-singular

Proof of 3: $(AB)^{-1} = B^{-1}A^{-1}$.

- 1: Let $A, B \in \mathbb{R}^{nxn}$ and invertible matrices.

$\square$

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- **Remark**: Generalization to 5 using Induction is left as an exercise to its reader

$\square$

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Algebraic Rules for Transpose

Theorem (Rules of the Transpose)

Suppose that $\alpha \in \mathbb{R}$ *and* $A, B \in \mathbb{R}^{mxn}$ *then the following will hold:*

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- *4*: $(AB)' = B'A'$
- *5*: **Remark**: *We can generalize 3 using induction:*
  $(A_1 + A_2 + \cdots + A_k)' = {A_1}' + {A_2}' + \cdots + {A_k}'$

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- 6: **Remark**: *We can generalize* 4 *using induction:* $(A_1 * A_2 * \ldots A_{k-1} * A_k)' = A_k{}' * A_{k-1}{}' * \cdots * A_2' * A_1'$

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Proof of the Transposition of the Product of Matrices

Proof of 6: $(A_1 * A_2 * \cdots * A_k)' = A_k' * \cdots * A_2' * A_1'$.

- 1: Let $A_{j-1},\ A_j$ be compatible matrices for $j = 1, 2, \ldots, k-1$

Proof of the Transposition of the Product of Matrices

Proof of (6): $(A_1 * A_2 * \cdots * A_k)' = A_k{}' * \cdots * A_2' * A_1'$.

- (1): Let $A_{j-1}, \ A_j$ be compatible matrices for $j = 1, 2, \ldots, k-1$
- (2): **Basis Step**: Let $k = 2$, WLOG: take $A_1 = A \in \mathbb{R}^{mxn}$ and $A_2 = B \in \mathbb{R}^{nxp}$, and denote $C = AB$

□

Proof of the Transposition of the Product of Matrices

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- Note that the $(i,j)_{th}$ element of C is the $(j,i)_{th}$ element of C'

$$c_{ji} = \sum_{k=1}^{n} a_{ki} b_{jk} = \sum_{k=1}^{n} b_{jk} a_{ki}$$

Which is the $(j,i)_{th}$ element of $B'A'$

$\square$

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  Which is the $(j, i)_{th}$ element of $B'A'$
- 4: **Inductive Step**: Suppose that claim is true for $k-1$ and denote $Z = A_1 * A_2 * \cdots * A_{k-1}$ such that: $Z' = A_{k-1}' * \cdots * A_2' * A_1'$, then:

□

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- $(A_1 * A_2 * \cdots * A_{k-1} * A_k)' = (Z * A_k)' = {A_k}' Z' = {A_k}' * {A_{k-1}}' * \cdots * A_2' * A_1'$

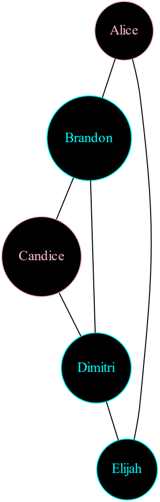
$\square$

Who will sell the most in a village?

Networks.

- 1: Let $a_1' = [Alice, \; Brandon, \; Candice, \; Dimitri, \; Elijah]$ be a row vector of states the and the $(i,j)_{th}$ element of A: a_{ij} be the sales connection from one person to the next. Who will have the greatest sales potential?

$\square$



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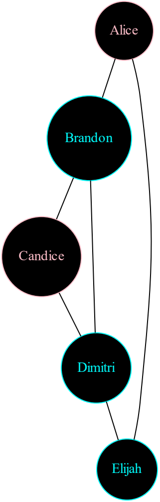
- We can denote the connections in the village by an Adjacency Matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

□

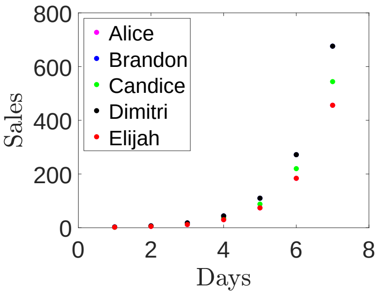


Evaluating the total number of connections in time

Lemma (Connectivity over k steps)

Let $A \in \mathbb{R}^{nxn}$ be a **Connectivity Matrix** *or* **Adjacency Matrix** *whose $(i, j)_{th}$ element is a connection from the i_{th} node to the j_{th} node (and $A' = A$). Then: $a_{ij}^{(k)}$ Is the number of connections from node i to j in k steps. Additionally, summing over the any row (or column) is the total number of connections in k steps.*

$$A^4 = \begin{bmatrix} 9 & 3 & 6 & 11 & 1 \\ 3 & 15 & 8 & 7 & 11 \\ 6 & 8 & 8 & 8 & 6 \\ 11 & 7 & 8 & 15 & 3 \\ 1 & 11 & 6 & 3 & 9 \end{bmatrix}$$



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