School of Computer Science
Faculty of Science
National University of Engineering

Entrance Exam

Subject: Computational Mathematics **Period: 2020.1**

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1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(u,v) = (u\cos(v), u\sin(v))$. Let $A = [2,3] \times [\pi/6, \pi/3]$ and let $B := F(A)$. Then, find

$$\int_B \exp\left(x^2 + y^2\right) dxdy.$$

Hint:

Theorem 1. Let $F(u,v) = (x(u,v), y(u,v))$ denote a smooth change of variables that maps a smoothly bounded set A onto a smoothly bounded set B, so that the boundary of A is mapped to the boundary of B. Denote by f a continuous function on B. Then

$$\int_B f(x,y)dxdy = \int_A f\left(x(u,v), y(u,v)\right) |JF(u.v)| dudv.$$

where JF is the Jacobian of the mapping. ♢

Definition 2. A **smoothly bounded** set A in the plane is a closed bounded set whose boundary is the union of a finite number of curves each of which is the graph of a continuously differentiable function, either

$$y = f(x), \quad x \text{ in some closed interval,}$$

or

$$x = f(y), \quad y \text{ in some closed interval.}$$

♢

2.

(a) Show that if A and B are disjoint, then A and B cannot be independent unless $\mathbf{P}(A) = 0$ or $\mathbf{P}(B) = 0$.

(b) Let $\mathbf{P}(C) > 0$. Show that

$$\mathbf{P}((A \cup B) \,|\, C) = \mathbf{P}(A \,|\, C) + \mathbf{P}(B \,|\, C) - \mathbf{P}(A \cap B \,|\, C).$$

(c) Assume that $\mathbf{P}(C) > 0$ and $A_1, \ldots, A_n$ are all pairwise disjoint. Show that

$$\mathbf{P}\left(\bigcup_{j=1}^{n} A_j \,\middle|\, C\right) = \sum_{j=1}^{n} \mathbf{P}(A_j \,|\, C).$$

(d) Let A be any event and $0 < \mathbf{P}(B) < 1$. Show that

$$\mathbf{P}(A) = \mathbf{P}(A \,|\, B)\,\mathbf{P}(B) + \mathbf{P}\left(A \,|\, B^{\complement}\right)\mathbf{P}\left(B^{\complement}\right).$$

3. A professor has equal number of male and female students. In any given year the probability that a male student has a claim in an academic semester is α, independently of other semesters. The analogous probability for females is β. Assume the professor selects a student at random.

(a) What is the probability the selected student will make a claim this semester?

(b) What is the probability the selected student will make a claim in two consecutive semesters?

4. Consider the following function that takes as input a sequence A of integers with `n` elements, `A[1],A[2],..,A[n]` and an integer `k` and returns an integer value. The function `length(S)` returns the length of sequence `S`.

```
function mystery(A, k) {
    n = length(A);
    if (k > n) return A[n];
    v = A[1];
    AL = [ A[j] : 1 <= j <= n, A[j] < v ]; // AL has elements < v in A
    Av = [ A[j] : 1 <= j <= n, A[j] == v ]; // Av has elements = v in A
    AR = [ A[j] : 1 <= j <= n, A[j] > v ]; // AR has elements > v in A
    if (length(AL) >= k) return mystery(AL,k);
    if (length(AL) + length(Av) >= k) return v;
    return mystery(AR, k - (length(AL) + length(Av)));
}
```

(a) Explain what the function computes.

(b) What is the worst-case complexity of this algorithm in terms of the length of the input sequence A?

(c) Give an example of a worst-case input for this algorithm.

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