5.1 HW 6 (n! on 151), 8, 14, 18, 28, 38, 40

6) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ whenever n is a positive integer.

Basis Step: n x n! = (n+1)! – 1
P(1): 1x1 = 1+1 – 1 = 1, so 1 = 1 for P(1).
Because P(1) is true, this shows the basis step.

Inductive Step: States P(k), where P(k) is $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ is true.
We must show that if P(k) is true, then P(k+1) is also true.
To do this, add (k +1) x (k + 1)! to each side.

(K x k)! + (k+1)x(k+1)! = (k+1)! -1 + (k+1) x (k+1)!
= (k+1)![1 + k + 1] – 1
= (k+1)!(k + 2) -1
= (k+2)! - 1

This shows the inductive hypothesis is true and completes the inductive argument. By showing the basis and inductive argument to be true, we can conclude that P(n) is also true.

8) Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1}) / 4$ whenever n is a nonnegative integer.

Basis Step: $2(-7)^n = (1 - (-7)^{n+1}) / 4$
P(0) gives us 2 = 2, proving that the basis step is true

Inductive Step: We must show that if P(k) is true, then P(k+1) is also true.
Add $2(-7)^{n+1}$ to both sides:
$2(-7)^n + 2(-7)^{n+1} = ((1 - (-7)^{n+1}) / 4) + 2(-7)^{n+1}$
$= 1 - (-7)^{n+1} + 8(-7)^{n+1} / 4$
$= 1 + 7(-7)^{n+1}/4$
$= 1 + (-7)^{n+2}/4$

This shows the inductive hypothesis to be true where if P(k), then P(k+1). Because we have shown the basis and inductive steps to be true, we can conclude that the statement is true.

14) Prove that for every positive integer n, $\sum_{k=1}^{n} n2^\wedge n = (n - 1)2^{n+1} + 2$.

Basis Step:
P(1): 2 = 2
Because 2 = 2, we have shown that the basis step, P(1) is true

Inductive Step:
Show that if P(k) is true, then P(k+1) is also true

Add $(k+1)2^{n+1}$ to each side, making the right side:
$(k-1)2^{k-1} + 2 + (k+1)2^{k+1} = 2^{k+1}(k-1+k+1) + 2$
$=2^{k+1}(2k) + 2$
$=2^{k+2}(k) + 2$
This shows that the inductive hypothesis is true for P(k), and that P(k+1) must also be true. Because we have shown the basis and inductive steps to be true, we can conclude that the statement is true.

18) Let *P(n)* be the statement that $n! < n^n$, where *n* is an integer greater than 1.

a) What is the statement *P(*2*)*?
$2 < 4$ or just $2! < 2^2$

b) Show that *P(*2*)* is true, completing the basis step of the proof.
Because $(1 \text{ x } 2) < (2)^{(2)}$, we have shown the basis step that P(2) is true.

c) What is the inductive hypothesis?
For all positive integers n, where n is greater than 1, and P(n) is true, then $P^{(n+1)}$ is true. $k+1! < k + 1^{k+1}$

d) What do you need to prove in the inductive step?
I need to prove that for every n, if P(n) is true, then P(n+1) is also true.

e) Complete the inductive step.
$K!(k + 1) < k^2(k+1) < (k + 1)^k (k+1) = (k+1)^{k+1}$

f) Explain why these steps show that this inequality is true whenever *n* is an integer greater than
These steps show both the basis step and inductive step to be true, showing that the statement is true.

28) Prove that $n^2 - 7n + 12$ is nonnegative whenever *n* is an integer with $n \geq 3$.

Basis Case:
P(3): 9 – 21 + 12 = 0, which is a nonnegative integer.
Because P(3) is true, we can say that the base case is true.

Inductive Case:
If P(k) is true, show that P(k+1) is also true.
Insert (k+1) to the equation
$(k+ 1)^2 – 7(k + 1)+12 = k^2 + 2k +1 – 7k + -7 + 12$
$= (k^2 -7k + 12) + (2k – 6)$
$= (k^2 -7k + 12) + 2(k – 3)$
2(k-3) will be non-neg whenever k > or = to 3, and $(k^2 -7k + 12)$ has already been proven for P(k).
We have shown that the inductive hypothesis, if true for P(k), is also true for P(k+1). By proving the basis case and the inductive case true, we can conclude that the statement is true.

38) Prove that if *A*1*,A*2*, . . . , An* and *B*1*,B*2*, . . . , Bn* are sets such that *Aj* ⊆ *Bj* for *j* = 1*,* 2*, . . . , n,*

then Un j = 1 Aj ⊆ Un j=1 Bj

Basis Step: P(1) shows $A_1 \subseteq B_1$

Inductive Step: P(k) is T where $A_1 \subseteq B_1$, then Uk+1, j = 1, Aj ⊆ U k+1 j =1, Bj.
X is an arbitry are element of (Uk+1, j=1, Aj) U Ak + 1, where x will be an element of the first or second.
If it is of the first, then we can clearly conclude x is an element of U k+1 j =1, Bj. If it is of the second, we can say that x is an element of Bk + 1 because $A_{k+1} \subseteq B_{k+1}$.

We have thus shown the inductive hypothesis P(k) is true, and that then P(k+1) must also be true. Because we have shown the basis step and the inductive step to be true, we can conclude that the original statement is true.

40) Prove if A1, A2, … An, and B1, B2, … Bn are sets such that Aj ⊆ Bj for j = 1,2, …n then $(A_1 \cap A_2 \cap \ldots \cap A_n) \cup B = (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_n \cup B)$

Basis Step: P(1): $A_1 \cup B = A_1 \cup B$ is true, and proves the Basis step.
Inductive Step: If we assume P(k) is true, then we must show that P(k+1) is also true.
This gives us $(A_1 \cap A_2 \cap \ldots \cap A_k \cap A_{k+1}) \cup B$
$= [(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}] \cup B$ - Associative Law
$= [(A_1 \cap A_2 \cap \ldots \cap A_k) \cup B] \cap (A_{k+1} \cup B)$ - Distributive Law
$= (A_1 \cup B) \cap (A_2 \cup B) \cap \ldots \cap (A_k \cup B) \cap (A_{k+1} \cup B)$
We have thus shown that the inductive hypo P(k) is true, and that P(K+1) then must also be true. We have then shown both the basis case and the inductive case, showing that the original statement is true.