Intersection Theory and String Amplitudes

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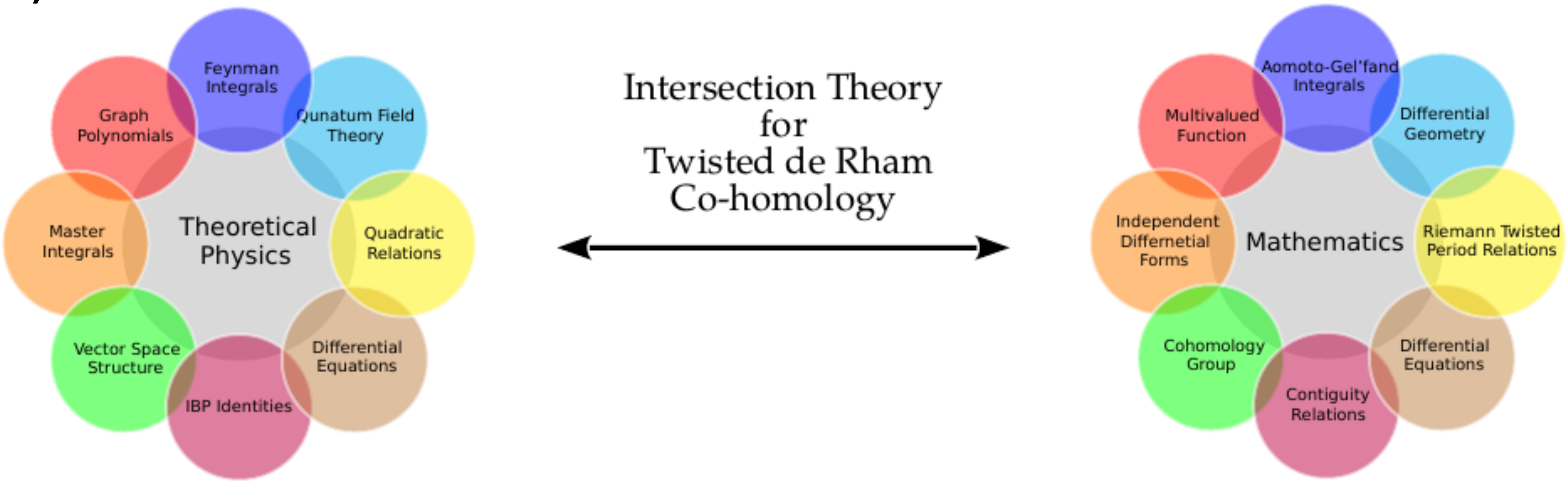
The main task: study of scattering amplitudes structure and new methods for their interpretation and evaluation by means of a modern mathematical theory: Intersection Theory.



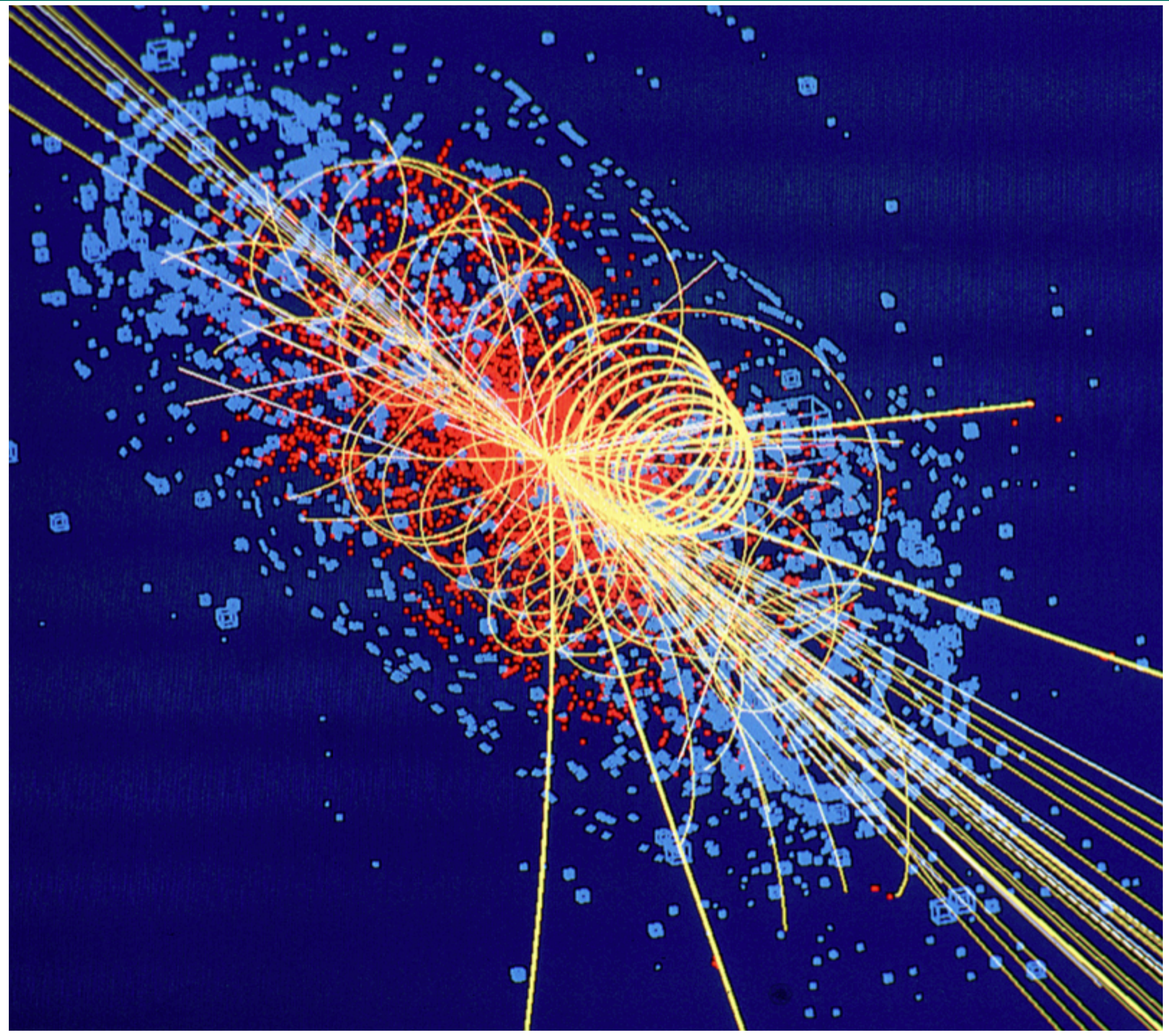
Talk based on:

- Master thesis: A Modern approach to String Amplitudes and Intersection Theory.

  Supervisor: Prof. Pierpaolo Mastrolia (Unipd), Prof. Co-Supervisor: Sergio Luigi Cacciatori (Uninsubria)

- PhD lectures: Advanced Methods for Scattering Amplitudes

  Prof. Sergio Luigi Cacciatori (Uninsubria), Prof. Yoshiaki Goto (Kobe University), Prof. Pierpaolo Mastrolia (Unipd), *Scattering amplitudes encode crucial information about particle collision, that, at the moment, represents the most efficient tool to investigate matter constituents and forces of Nature. Scattering Amplitudes are complex numbers related to the transition probability from a given asymptotic incoming state configuration and a given outgoing one.*



Lagrangian → Feynman rules

Particle scattering → Feynman diagram

Feynman rules + Feynman diagram → Integral representation of the Amplitude

$$I = \int_C (P)^\gamma \frac{dz_1 \wedge dz_2 \wedge \ldots \wedge dz_N}{z_1^{n_1} z_2^{n_2} \ldots z_N^{n_N}}$$

QFT Amplitudes
[Mastrolia, Mizera 2018]
[Padova group 2019/22]

$$I = \int_C (P)^\gamma \frac{dz_1 \wedge dz_2 \wedge \ldots \wedge dz_N}{z_1^{n_1} z_2^{n_2} \ldots z_N^{n_N}}$$

Intersection theory
[Aomoto, Cho, Matsumoto, Kita, Yoshida]

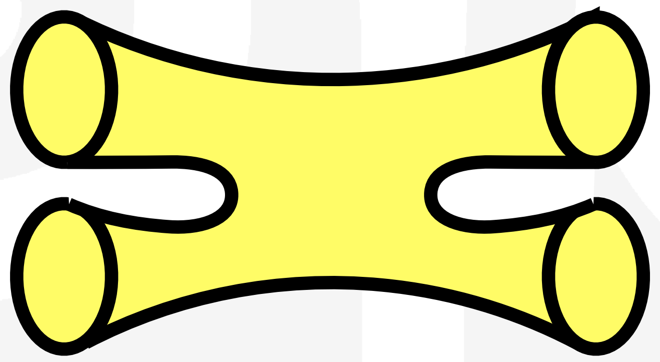
Mathematics

Topology, Algebra, Differential Geometry, Number theory

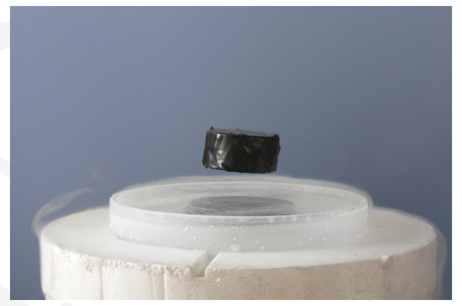
Data Science

Machine learning, Topological computing

String Amplitudes
[Mizera 2016/17]

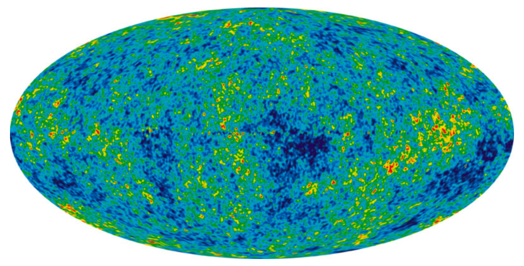


Condensed matter



Correlation functions give us information about phase transitions

Cosmology



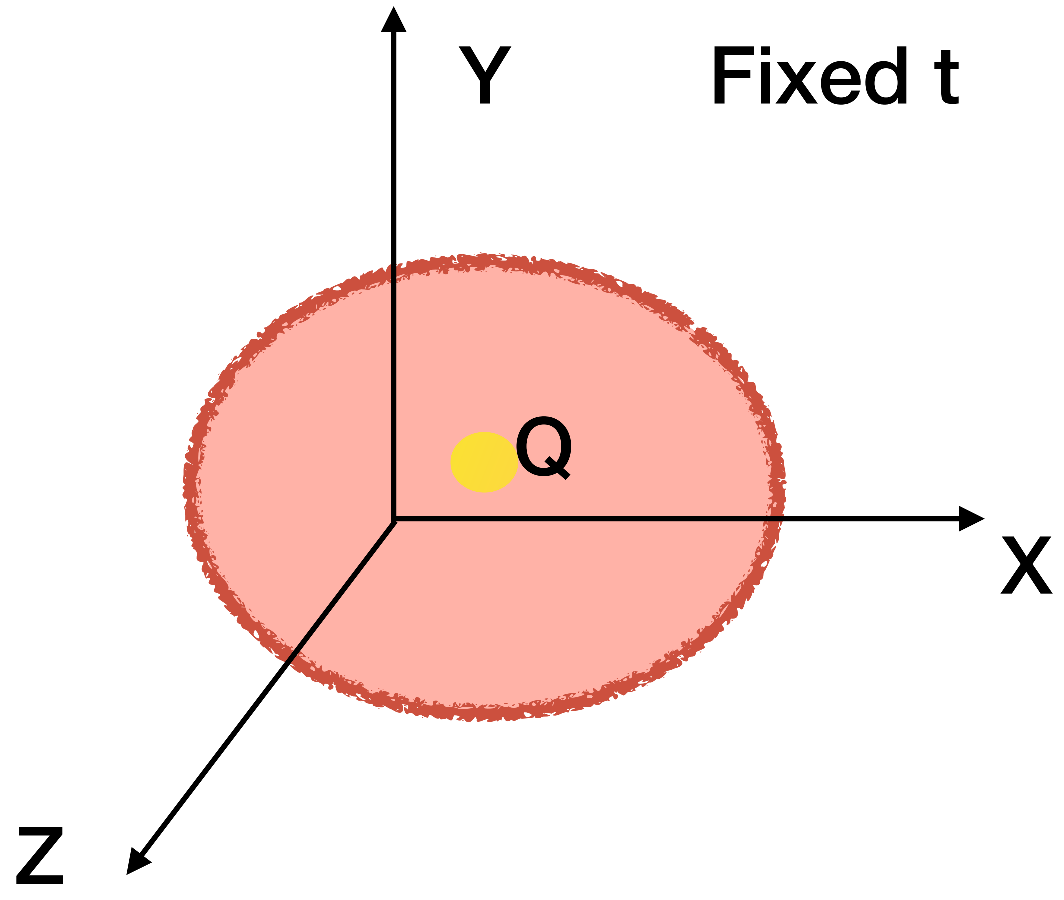
Correlation functions evaluation is necessary to interpret CMB fluctuations and obtain information about early universe.

Stokes' Theorem

$$\int_{\partial C} \varphi = \int_C d\varphi$$

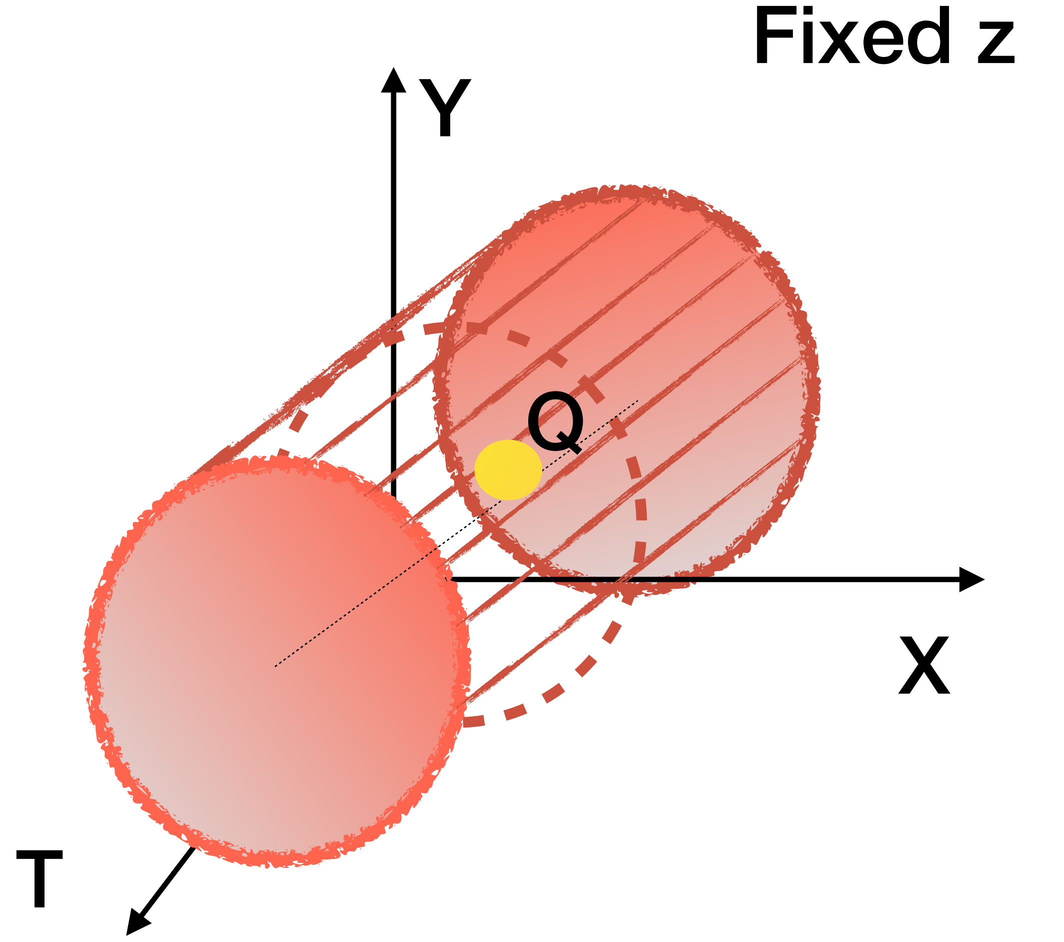
Gauss' law

The closed surface surrounding a charge is the boundary of a 3 dimensional Manifold.



Ampere's law

The closed curve surrounding a wire is actually a (1+1) dimensional surface around the charge, the boundary of a (2+1) dimensional Manifold



Let $\varphi \in \Omega^k$ be a k-form and $\Delta_{k+1} \subset \mathbb{R}^{k+1}$ a (k+1)-simplex

$$\int_{\partial \Delta_{k+1}} \varphi = \int_{\Delta_{k+1}} d\varphi$$

$$\int_{\Delta_k} \varphi = \int_{\Delta_k} \varphi + \lambda, \quad d\lambda = 0$$

λ *is a closed form*

$$\int_{\Delta_k} \varphi = \int_{\Delta_k + \tilde{\Delta}_k} \varphi, \quad \partial \tilde{\Delta}_k = 0$$

$\tilde{\Delta}_k$ *is a closed contour*

$$\int_C \varphi = \langle \varphi \,|\, C]$$

Equivalence classes of forms and contours.

Multivalued function

$$I = \int_{\Delta_k} u\varphi$$

- $$\int_{\Delta_k \otimes u} u\varphi \equiv \int_{\Delta_k} u_\Delta \varphi$$

$$\omega \equiv d \log u_\Delta$$

- $$\int_{\partial \Delta_{k+1}} u_\Delta \varphi = \int_{\Delta_{k+1}} d(u_\Delta \varphi)$$

- $$d(u_\Delta \varphi) = du_\Delta \wedge \varphi + u_\Delta \wedge d\varphi = u_\Delta \left(\frac{du_\Delta}{u_\Delta} \wedge \varphi + d\varphi \right)$$
$$= u_\Delta (d\varphi + \omega \wedge \varphi) \equiv u_\Delta \nabla_\omega \varphi$$

Twisted Stokes' Theorem

$$\int_{\Delta_{k+1} \otimes u} u \nabla_\omega \varphi = \int_{\partial_\omega \Delta_{k+1}} u_\Delta \varphi$$

[Aomoto, Kita, Yoshida, Cho, Matsumoto…]

Given a set $\{G_p(X)\}$ of algebraic objects (groups) defined on X and a set of morphisms $\{\Delta_p : G_p \to G_{p-1}\}$, we call a **sequence** on X the structure:

$$(G_p, \Delta_p) : \qquad 0 \xrightarrow{\Delta_{p+1}} G_p \xrightarrow{\Delta_p} G_{p-1} \xrightarrow{\Delta_{p-1}} \dots \xrightarrow{\Delta_1} G_0 \xrightarrow{\Delta_0} 0$$

Similarly, the dual sequence is defined as

$$(G^p, \Delta^p) : \qquad 0 \xleftarrow{\Delta^{p+1}} G^p \xleftarrow{\Delta^p} G^{p-1} \xleftarrow{\Delta^{p-1}} \dots \xleftarrow{\Delta^1} G^0 \xleftarrow{\Delta^0} 0$$

If $\{G_p(X)\}$ are abelian groups on a topological space X the sequence is called **Chain Complex** and group homomorphisms are called **Boundary Operators.**

If the image of one morphism equals the kernel of the next, the sequence is said to be **exact**

$$Im[\Delta_p] = Ker[\Delta_{p-1}]$$

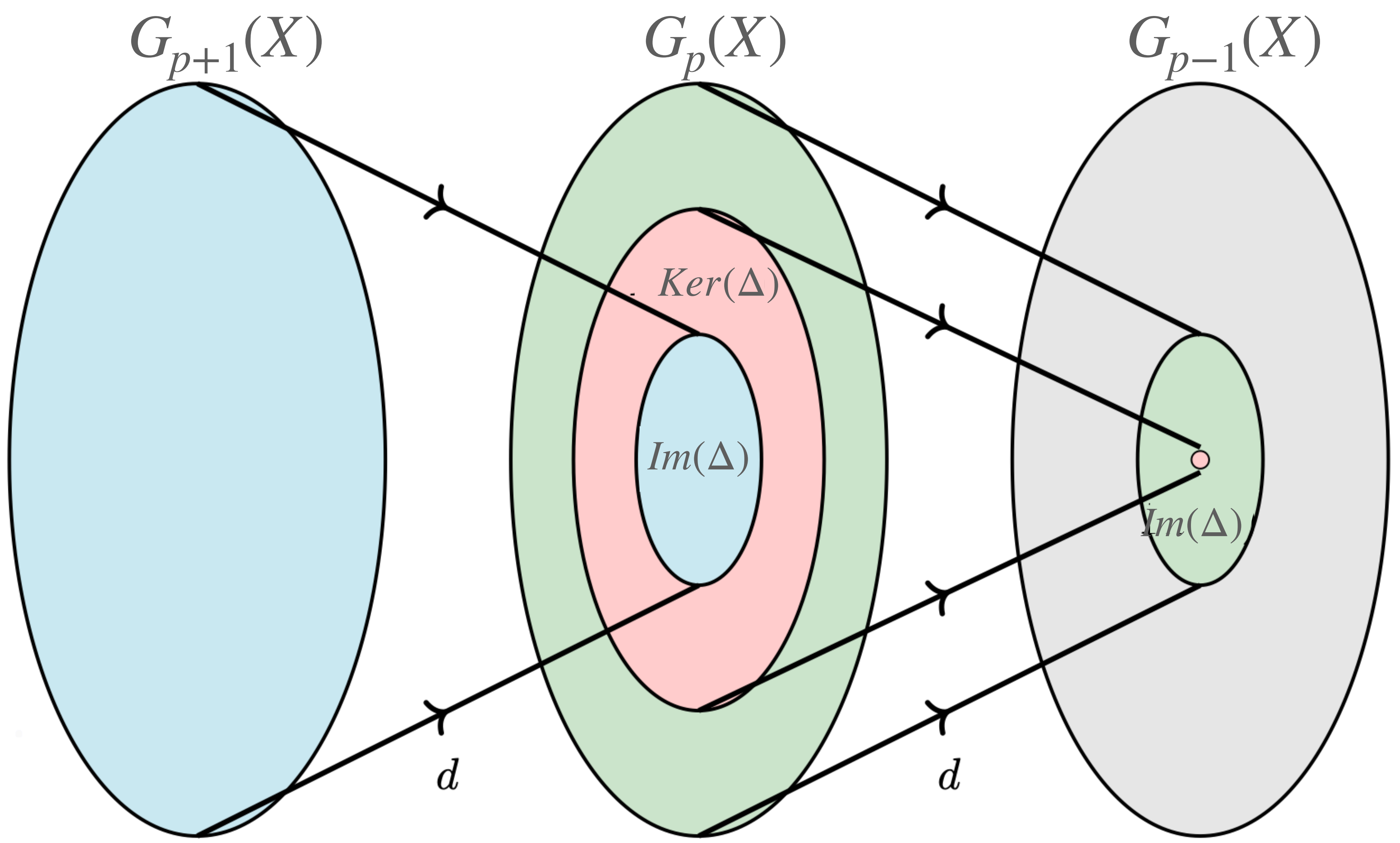
If the sequence is not exact is useful to define the groups:

The n-th homology group

$$H_p(X) = \frac{Ker[\Delta_p]}{Im[\Delta_{p+1}]}$$

The n-th Cohomology group

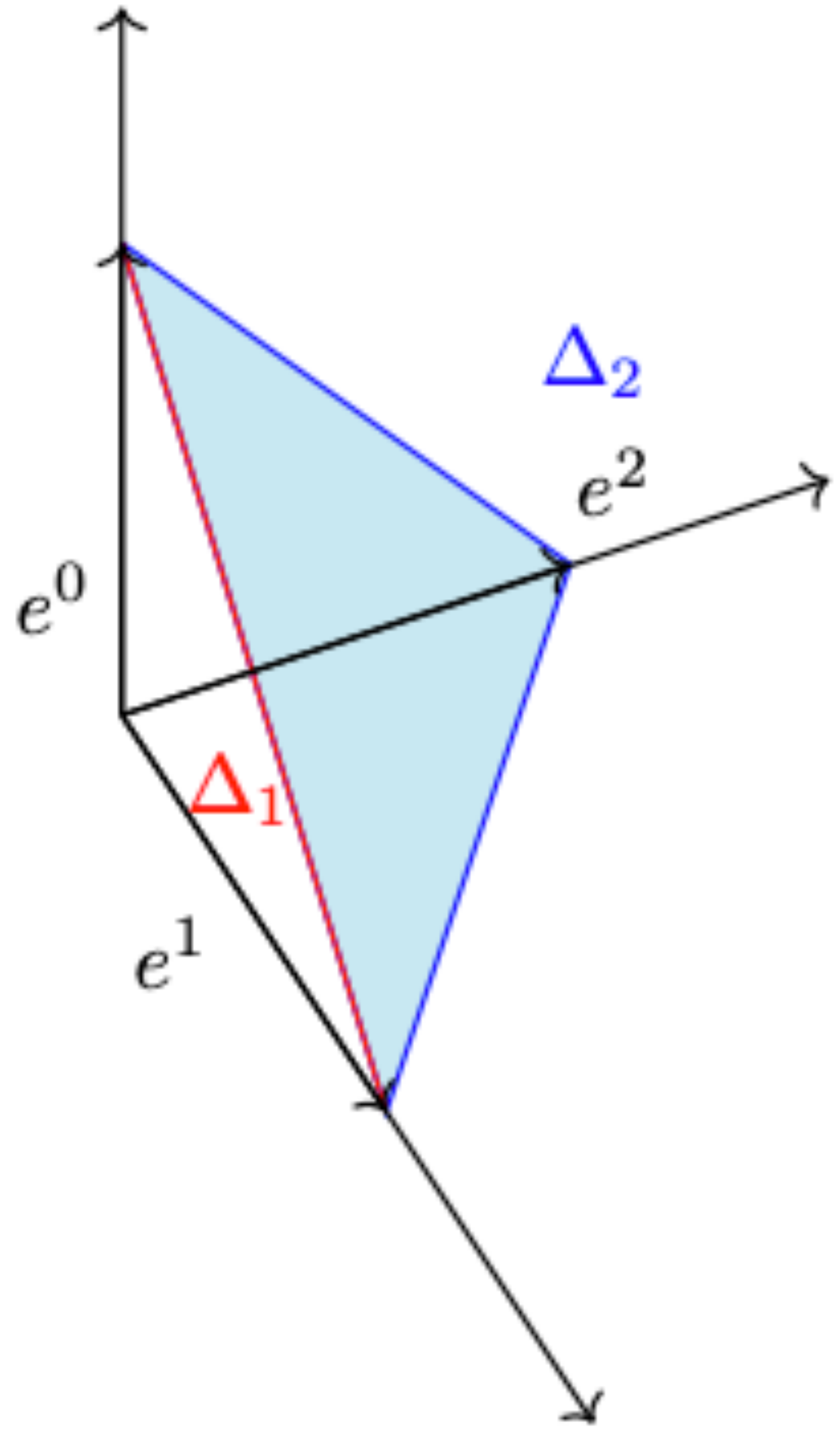
$$H^p(X) = \frac{Ker[\Delta^p]}{Im[\Delta^{p-1}]}$$



They describe how images are included in kernels

A standard p-simplex is a planar p dimensional subset of $\mathbb{R}^{p+1}$

$$\Delta_p = \left\{ \sum_{i=0}^{p} \alpha_i e^i \mid \sum \alpha_i = 1, \alpha_i \geq 0 \right\}$$



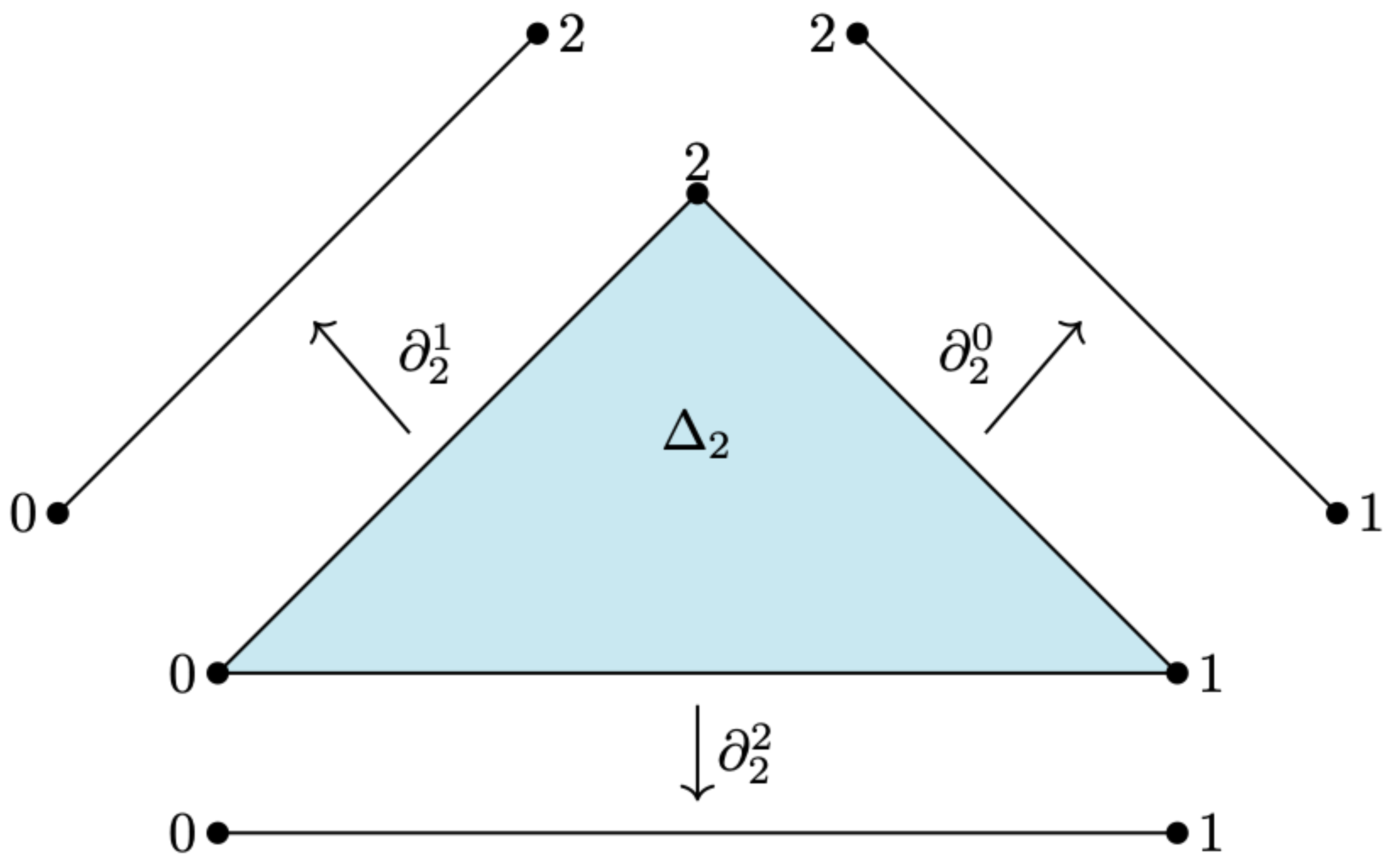
Order p-simplex (fixed vertices) $\Delta_p = \langle 012 \ldots p \rangle$

p-chain $c_p = \sum_i a_i \Delta_p^i \in C_p$

Consider the face map: $\partial_p^i\langle 012..i..p\rangle = \langle 012..\hat{i}..p\rangle$

Boundary Operator $\partial_p : C_p \to C_{p-1}$

$$\boxed{\partial_p = \sum_{i=0}^{p} (-1)^i \partial_p^i}$$



Example: $\partial_p\langle 012\rangle = \langle 12\rangle - \langle 02\rangle + \langle 01\rangle = \langle 12\rangle + \langle 20\rangle + \langle 01\rangle$

A p-chain having no boundary is called p-cycle.

The chain complex of chain groups with the boundary operators define the sequence

$$0 \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \dots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

p-th simplicial homology group

$$\boxed{H_p^{simpl} = \frac{Ker[\partial_p]}{Im[\partial_{p+1}]}}$$

Defining

$$Hom[\,-\,, K] : \begin{cases} Hom[C_p, K] = Hom(C_n, K) \\ Hom[\partial_p, k] = \delta_p : Hom(C_{p-1}, K) \to Hom(C_p, K) \end{cases}$$

The dual sequence $$0 \xleftarrow{\delta_{p+1}} Hom(C_p, K) \xleftarrow{\delta_p} Hom(C_{p-1}, k) \xleftarrow{\delta_{p-1}} \dots \xleftarrow{\delta_1} Hom(C_0, K) \xleftarrow{\delta_0} 0$$

p-th simplicial cohomology group

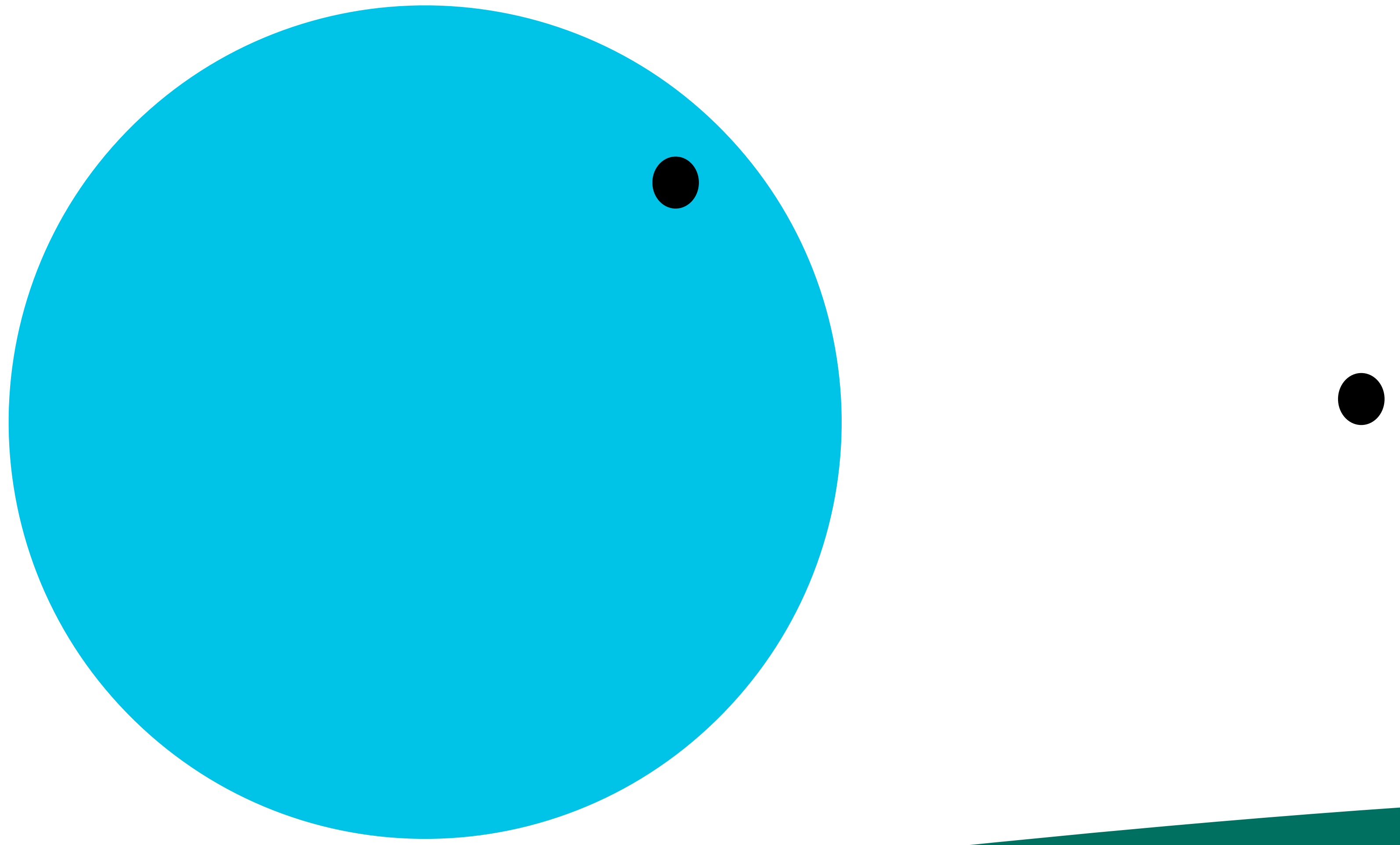
$$\boxed{H^p_{simpl} = \frac{Ker[\delta_p]}{Im[\delta_{p-1}]}}$$

A closed k-cell e^k is a space homeomorphic to the ball $B^k = \{x \in \mathbb{R}^k : |x| \leq 1\}$

A CW complex X is a topological space obtained from a 0-cell (point) by gluing k-cells via an attaching map

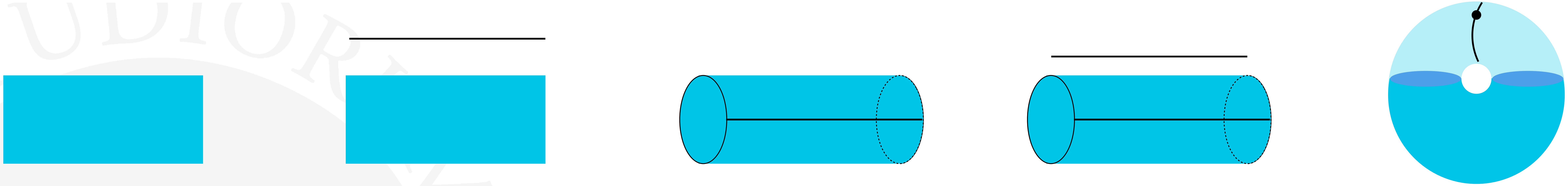
If X is a manifold its **handlebody decomposition** is a CW complex

The n-Sphere
$S^n = e^0 \cup e^n$



The Torus

$T^2 = e^0 \cup e^1 \cup e^1 \cup e^2$



A function $f : M \to \mathbb{R}$ is said to be a Morse function its critical points p_i are non degenerate ($\det \partial_i \partial_j f|_{p_i} \neq 0$)

Let $M_s(t) = \{p \in M \,|\, s \leq f(p) \leq t\}$

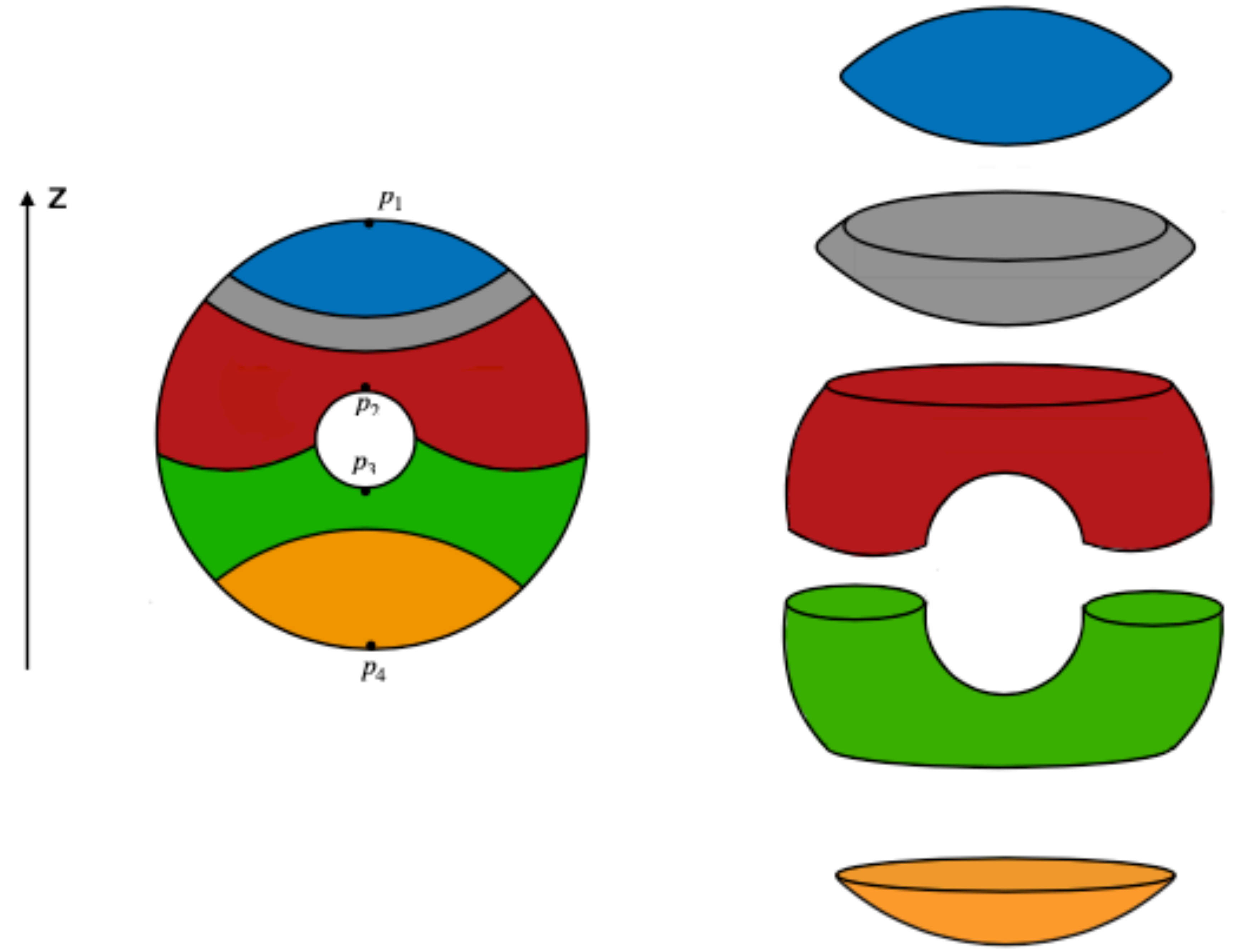
- If $[a, b]$ contains no critical values, then

$$M_s(a) \cong M_s(b)$$

- If $[a, b]$ contains a critical values $c_i = f(p_i)$ then

$$M_s(b) \cong M_s(a) \sqcup (B^m \times B^{m-\lambda})$$

The integer λ is called index of the critical point



Euler-Poincarè $\quad \chi = \sum (-1)^\lambda n_\lambda = \sum (-1)^\lambda b_\lambda$

Morse Inequality $\quad n_\lambda \geq b_\lambda$

Let $X^{(2)} = e^0 \cup_n e^1 \cup_m e^2$ be the two skeleton of a CW complex X with one 0-cell.
The fundamental group of X is generated by n generators with m relations:

$$\pi_1(X) = \pi_1(X^{(2)}) = \langle g_1, g_2 \,..\, g_n \,|\, r_1 = 1, r_2 = 1, \ldots r_m = 1 \rangle$$

Circle $\qquad \pi_1(S^1) = \pi_1(e^0 \cup e^1)\langle g \,|\, 0 \rangle = \mathbb{Z}$

n-Sphere $\qquad \pi_1(S^n) = \pi_1(e^0 \cup e^2 \delta_{n2})\langle 0 \,|\, r = 1 \rangle = 0$

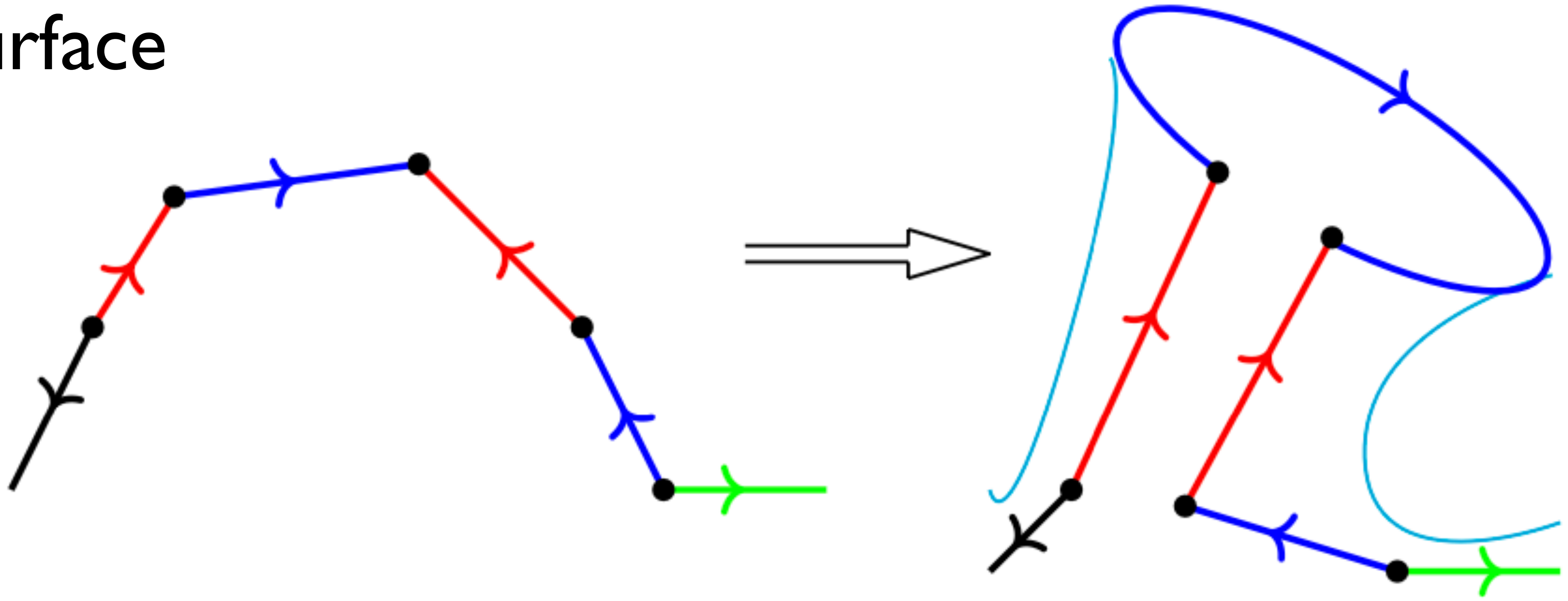
Torus $\qquad \pi_1(T^2) = \pi_1(e^0 \cup e^1 \cup e^1 \cup e^2) = \langle g_1, g_2 \,|\, r = 1 \rangle = ?$

Klein Bottle $\qquad \pi_1(\mathbb{K}) = \pi_1(e^0 \cup_i e^1 \cup_i e^1 \cup e^2) = \langle g_1, g_2 \,|\, r = 1 \rangle = ?$

Projective plane $\qquad \pi_1(\mathbb{R}P^n) = \pi_1(e^0 \cup e^1 \cup e^2) = \langle g \,|\, r = 1 \rangle = ?$

The g-gone construction

Any Riemann surface of genus g can be realized from a planar surface with 4g edges (4g-gone): $a_i, a_i^{-1}, b_i, b_i^{-1}$ $(i = 1,\ldots,g)$



Where a_i and b_i are clockwise oriented and a_i^{-1} and b_i^{-1} counterclockwise oriented

The fundamental parallelogram gives genus zero surfaces

$Sphere$	RP^1	$Klein\,bottle$	$Torus$
$abb^{-1}a^{-1}$	$abab$	$abab^{-1}$	$aba^{-1}b^{-1}$

Torus $\pi_1(T^2) = \pi_1(e^0 \cup e^1 \cup e^1 \cup e^2) = \langle g_1, g_2 \,|\, \underline{g_1 g_2 g_1^{-1} g_2^{-1}} = 1 \rangle = \mathbb{Z} \times \mathbb{Z}$

Klein Bottle $\pi_1(\mathbb{K}) = \pi_1(e^0 \cup_i e^1 \cup_i e^1 \cup e^2) = \langle g_1, g_2 \,|\, \underline{g_1 g_2 g_1 g_2^{-1}} = 1 \rangle = \mathbb{Z} \rtimes \mathbb{Z}$

Projective plane $\pi_1(\mathbb{R}P^n) = \pi_1(e^0 \cup e^1 \cup e^2) = \langle g \,|\, \underline{g^2} = 1 \rangle = \mathbb{Z}_2$

Let M be a differentiable manifold and TM its tangent bundle.
The cotangent bundle is

$$T^*M : TM \to \mathbb{R}$$

Elements of the cotangent bundle are called algebraic one forms, sections of the cotangent bundle are called differential one forms.

Exterior product

$$\omega_1 \wedge \gamma_1 = -\gamma_1 \wedge \omega_1 = \omega\gamma dx_1 \wedge dx_2$$

A k-form (differential) is a smooth section of the k-th external algebra of the cotangent bundle

$$\omega_k(x) : M \to \bigwedge_k (T^*M)$$

Let $\Omega^p(M)$ be the group of p-forms of the manifold M, and $d : \Omega^p(M) \to \Omega^{p+1}(M)$ the external derivative

$$0 \xrightarrow{d} \Omega^0(\mathcal{M}) \xrightarrow{d} \Omega^1(\mathcal{M}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(\mathcal{M}) \xrightarrow{d} 0,$$

Closed forms

De Rham Cohomology

$$\boxed{H^p_{dR}(\mathcal{M}) = \frac{Ker[d : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})]}{Im[d : \Omega^{p-1}(\mathcal{M}) \to \Omega^p(\mathcal{M})]}}$$

Exact forms

It turns $H^p_{dR} \approx H^p_{simpl}$, then

Non degenerate pairing: $H_p^{simpl} \times H^p_{dR} \to \mathbb{C}$ $\qquad \langle \omega_p | c_p] = \int_{c_p} \omega_p$

Let $x, y \in X$ be two points of a topological space and γ^i_{xy} the classes of homotopically inequivalent paths having extrema in x and y .

Let $\{G_x\}$ be a set of groups and $\phi[\gamma^i_{xy}] : G_x \to G_y$ a group isomorphism.
The couple $\mathscr{L} = (\{G_x\}, \phi)$ is called a **local system** if

$$\phi[\gamma^i_{xy}] \circ \phi[\gamma^j_{yz}] = \phi[\gamma^i_{xy}\gamma^j_{yz}]$$

- $\phi[i] = I$
- $\phi[\gamma^i_{xy}] = \phi[\gamma^i_{yx}]^{-1}$
- $\phi[\gamma^i_{xx}] \in Aut(G_x) \quad \to \quad \pi_x(X) \cong Aut(G_x)$
- If G_x is abelian, then $\phi[\gamma^i_{xy}] = \phi(x, y)$

It is a fiber bundle with trivial transition function

[Steenrod, 1943]
[Zein, Snoussi, 2000]

Consider a multivalued function

$$u(z) = \prod_{j=1}^{m} P_j(z)^{\alpha_j}, \quad \alpha_j \in \mathbb{C} \setminus \mathbb{Z}$$

The solutions of the differential equation

$$\nabla_\omega h = dh + \sum_{j=1}^{m} \boxed{\alpha_j \frac{dP_j}{P_j}} h = 0$$

define Local System .

Twisted chains are chains with coefficients in the Local System:

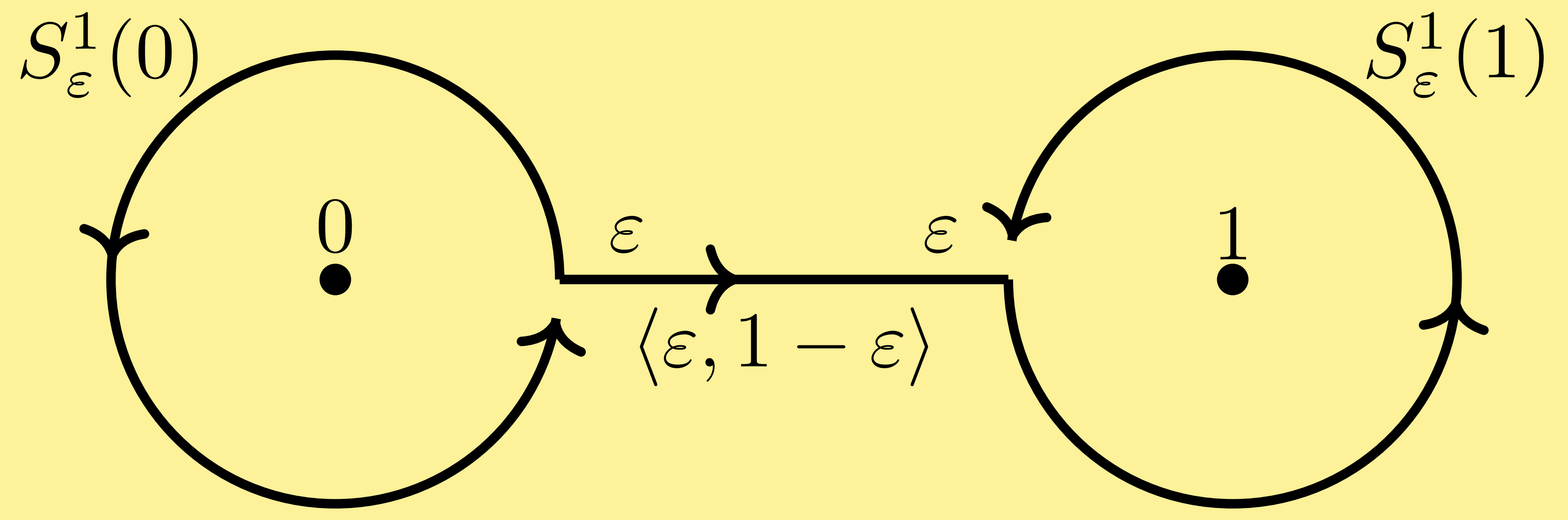
$$C_p(\mathcal{M}, \mathcal{L}_\omega) \ni c_p = \sum_i \Delta_p^i \otimes \mathcal{L}_\omega$$

We introduce a Boundary operator:

$$\partial_\omega(\Delta \otimes u_\Delta) \equiv \sum_{j=0}^{p} (-1)^j \langle 01..\hat{j}...p\rangle \otimes u_{\langle 01..\hat{j}...p\rangle}$$

$$u(z) = z^{\alpha_1}(z-1)^{\alpha_2}$$

$$\Delta(\omega) = \frac{1}{e^{2\pi\alpha_1}-1} S_\varepsilon^1(0) \otimes u_{S_\varepsilon^1(0)} + \langle \varepsilon, 1-\varepsilon\rangle \otimes u_{(0,1)} - \frac{1}{e^{2\pi\alpha_2}-1} S_\varepsilon^1(1) \otimes u_{S_\varepsilon^1(1)}$$



$$\partial_\omega \Delta(\omega) = 0$$

Twisted Homology is defined as:

$$\boxed{H_p(\mathcal{M}, \mathcal{L}_\omega) \equiv \frac{Ker\, \partial_\omega}{Im \partial_\omega}}$$

[Aomoto,Kita,Yoshida,Cho,Matsumoto,Goto…]

We analogously introduce k-forms with values in the Local System

$$\Omega^k(\mathcal{M},\mathcal{L}_\omega) = \Omega^k(\mathcal{M},\mathbb{C}) \otimes \mathcal{L}_\omega$$

And a boundary operator

$$\nabla_\omega : \Omega^k(\mathcal{M},\mathcal{L}_\omega) \to \Omega^{k+1}(\mathcal{M},\mathcal{L}_\omega)$$

Twisted Cohomology is the cohomology of the sequence

$$0 \xrightarrow{\nabla_\omega^0} \Omega^0(\mathcal{M},\mathcal{L}_\omega) \xrightarrow{\nabla_\omega^1} \Omega^1(\mathcal{M},\mathcal{L}_\omega) \xrightarrow{\nabla_\omega^2} \dots \xrightarrow{\nabla_\omega^n} \Omega^n(\mathcal{M},\mathcal{L}_\omega) \xrightarrow{\nabla_\omega} 0$$

$$\boxed{H^k(\mathcal{M},\nabla_\omega) \equiv \frac{ker[\nabla_\omega^k]}{Im[\nabla_\omega^{k+1}]}}$$

[Aomoto,Kita,Yoshida,Cho,Matsumoto…]

Cycle-cocycle pairing

$$H^k(\mathcal{M},\nabla_\omega)\times H_k(\mathcal{M},\mathcal{L}_\omega^\vee)\to\mathbb{C}\qquad \langle\varphi_L|C_R] = \int_{C_R} u\varphi_L$$

Cocycle-cocycle pairing

$$H^k_c(\mathcal{M},\nabla_\omega)\times H^k(\mathcal{M},\nabla_{-\omega})\to\mathbb{C}$$

$$\langle\varphi_L|\varphi_R\rangle = \int reg_c(\varphi_L)\wedge\varphi_R. \tag{2}$$

Cycle-cycle pairing

$$H^{lf}_k(\mathcal{M},\mathcal{L}_\omega)\times H_k(\mathcal{M},\mathcal{L}_\omega^\vee)\to\mathbb{C}$$

$$[C_L|C_R] = reg_h(C_L)\cdot C_R\otimes u_L u_R. \tag{3}$$

oto,Kita,Yoshida,Cho,Matsumoto…]

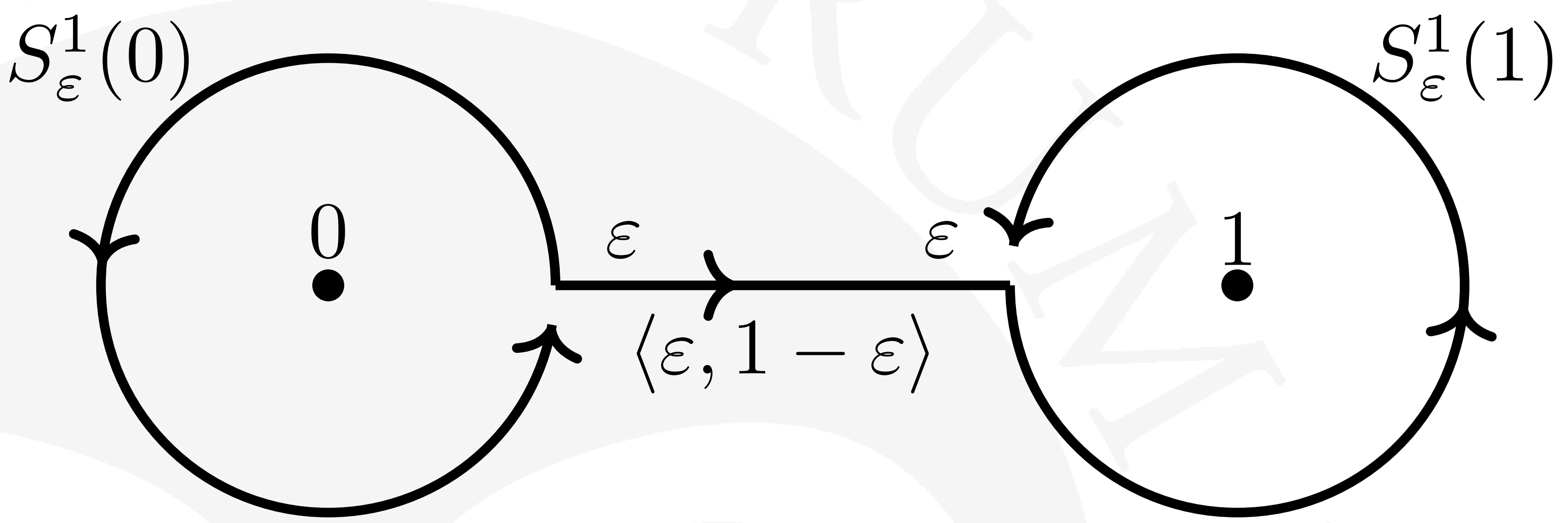
Intersection among contours: $[C_1 \,|\, C_2]$

[Mimachi, Yoshida, 2003]

[Mizera, 2019]

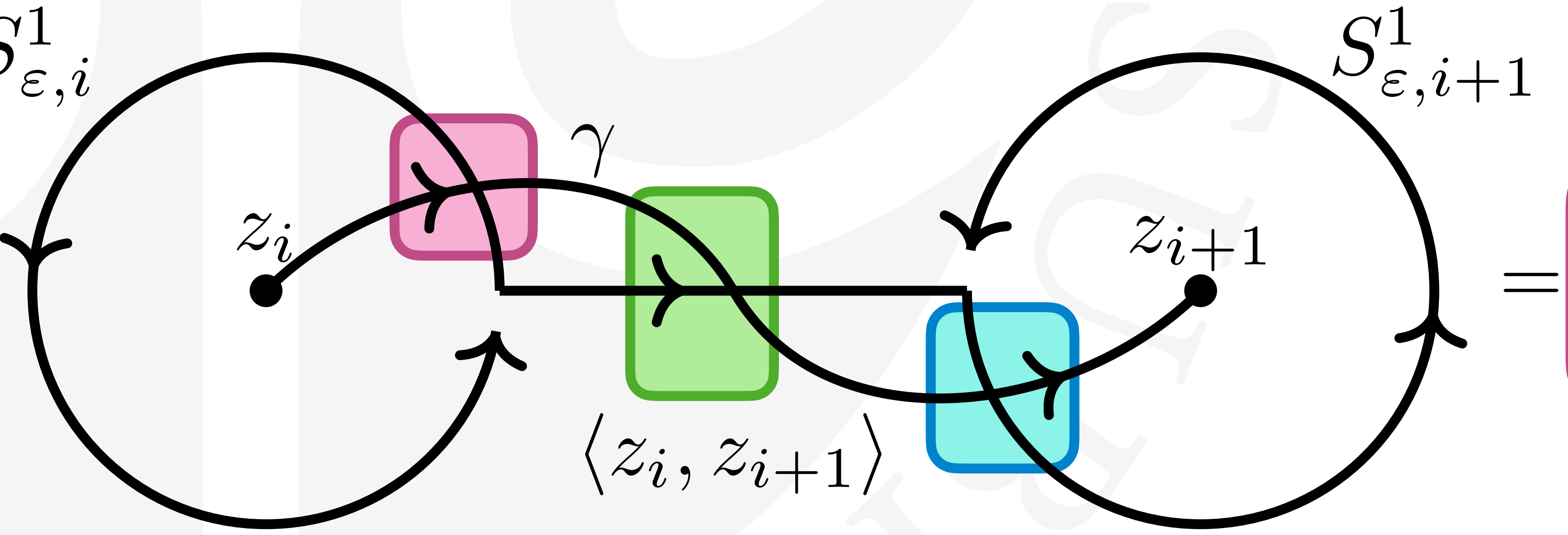
Diagrammatical Techniques

$\Delta_i =$

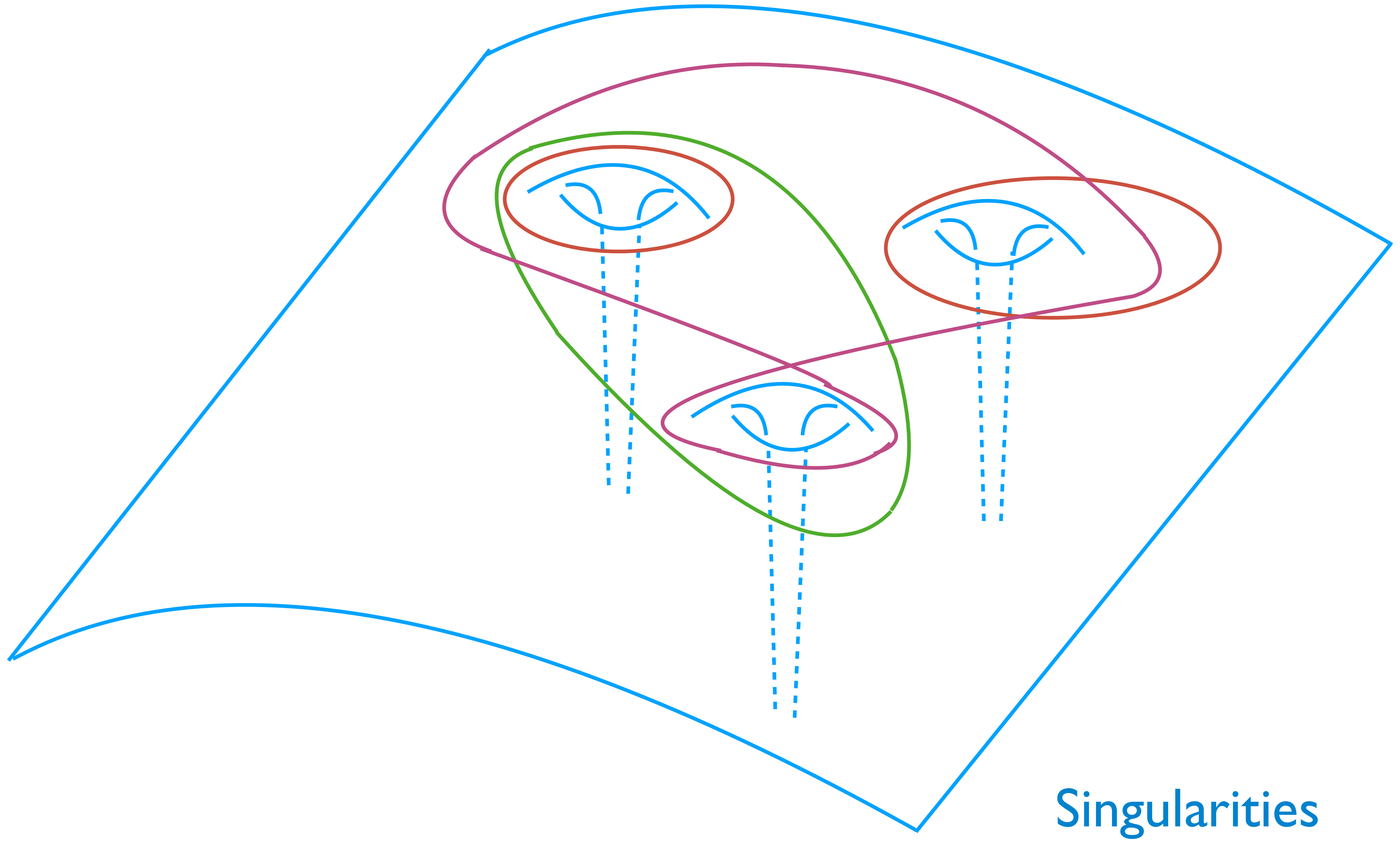


$\Delta_i(\omega) =$

$[\Delta_i(\omega) \,|\, \Delta_i] =$

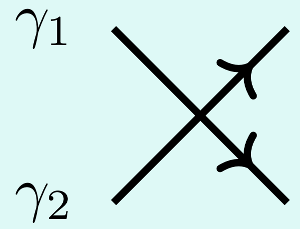


$$= -\frac{1}{e^{2\pi i \alpha_i}-1} - 1 - \frac{1}{e^{2\pi i \alpha_{i+1}}-1}$$



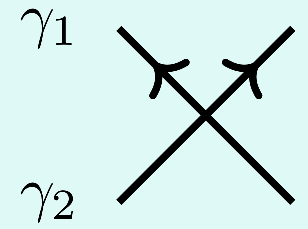
Topological intersection

$\gamma_1 \cdot \gamma_2 =$



$= +1$

$\gamma_1 \cdot \gamma_2 =$



$= -1$

Intersection among forms: $\langle \varphi_1 | \varphi_2 \rangle$

[Matsumoto, 1998]

[Padova group 2019/2022]

[Weinzierl, 2021]

$\mathscr{P}_\omega$ = poles of ω $\quad$ $\mathscr{Z}_\omega$ = zeros of ω $\quad$ P_ω = poles of $\Omega^{(n)}$

- *Univariate forms* *[Cho, Matsumoto (1995)] [Matsumoto (1998)]*

$$\langle \varphi_L | \varphi_R \rangle \equiv \frac{1}{2\pi i} \int_{\mathcal{M}} reg_c(\varphi_L) \wedge \varphi_R \; = \sum_j Res_{z=z_j} (\psi_j \varphi_R).$$

With $\psi_j(z)$ the local solution of

$$\nabla_\omega \psi_j(z) = \varphi_L(z)$$

- *(Multivariate) dlog forms* *[Mizera (2017)]*

$$\langle \varphi_L | \varphi_R \rangle = \int \prod_{i=1}^{n} dz_i \delta[\omega_i] \varphi_L \varphi_R = \sum_{z^*_{ij}} |\partial_i \omega_j|^{-1} \varphi_L \varphi_R \Big|_{z_i = z^*_{ij}}.$$

- *Multivariate generic forms* *[Mizera (2019)][Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera, (2019)]*

$$\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{((\mathbf{n}))} \rangle = \sum_{p \in P_\omega} \operatorname*{Res}_{z_n = p} \left(\psi_i^{(n)} \varphi_{Rj}^{(n)} \right)$$

Let $\langle e_i|, |h_i\rangle, [C_{Li}|, |C_{Ri}]$ be basis for (co-homology spaces). Intersection matrices are defined as:

$$C_{ij} \equiv \langle e_i|h_j\rangle \quad H_{ij} = [C_{Li}|C_{Rj}].$$

The identity operator can be written as:

$$\sum_{ij} |h_i\rangle (C^{-1})_{ij} \langle e_j| = \mathbf{1}_c \qquad \sum_{ij} |C_{Ri}] (H^{-1})_{ij} [C_{Lj}| = \mathbf{1}_h$$

Master decomposition formula (linear)

$$\boxed{\langle \varphi_L|C_R] = \sum_{ij} \langle \varphi_L|h_i\rangle (C^{-1})_{ij} \langle e_j|C_R]}$$

[Mastrolia,Mizera (2019)]

[Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera, (2019-20)]

Twisted Riemann period relations (quadratic)

$$\boxed{\langle \varphi_L|\varphi_R\rangle = \sum_{ij} \langle \varphi_L|C_{Ri}] (H^{-1})_{ij} [C_{Lj}|\varphi_R\rangle}$$

[Cho, Matsumoto (1995)]

See for review [Cacciatori, Conti, Trevisan (2021)]

$$\boxed{d = d(\chi)}$$

$$I = \sum_{i=1}^{d} \boxed{c_i}\,\boxed{M_i}$$

Master integrals

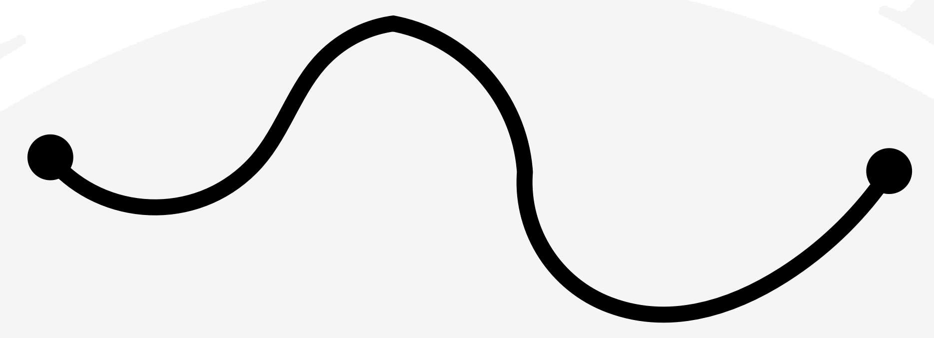
Intersection numbers

- Any Feynman integral can be decomposed into a finite set of Master integrals
- The number of Master integral depends on the integration domain topology
- The coefficient of the expansion are intersection number.

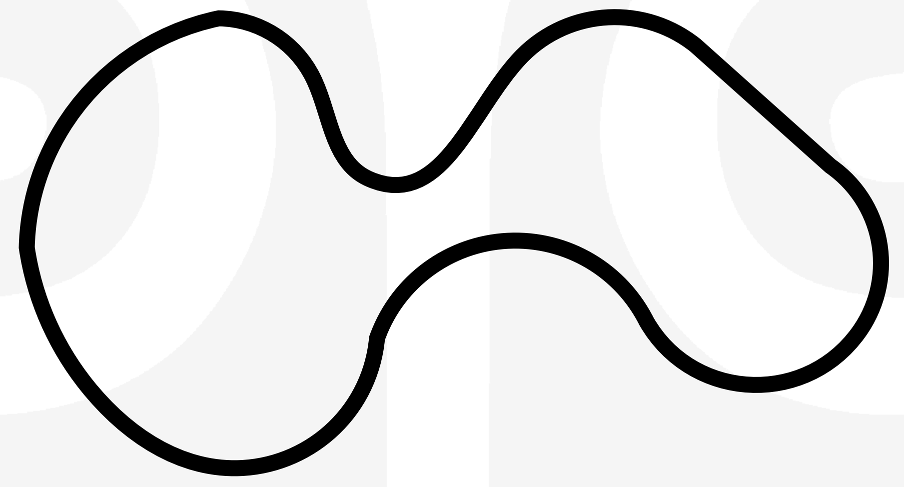
Intersection numbers control relations among Scattering Amplitudes

[Frellesvig, Gasparotto, Mandal, Mastrolia, Matiazzi, Mizera, 2019]

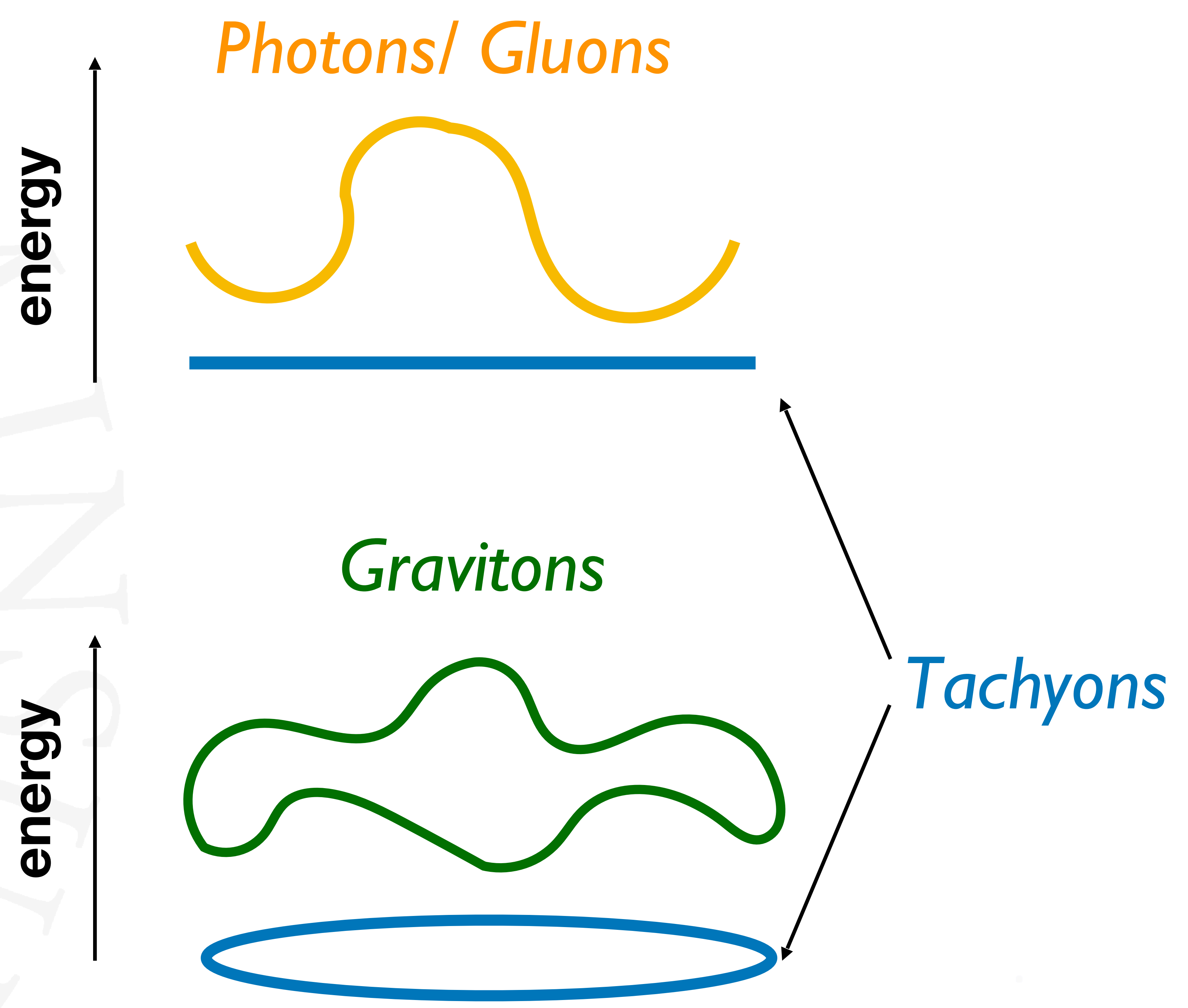
Open



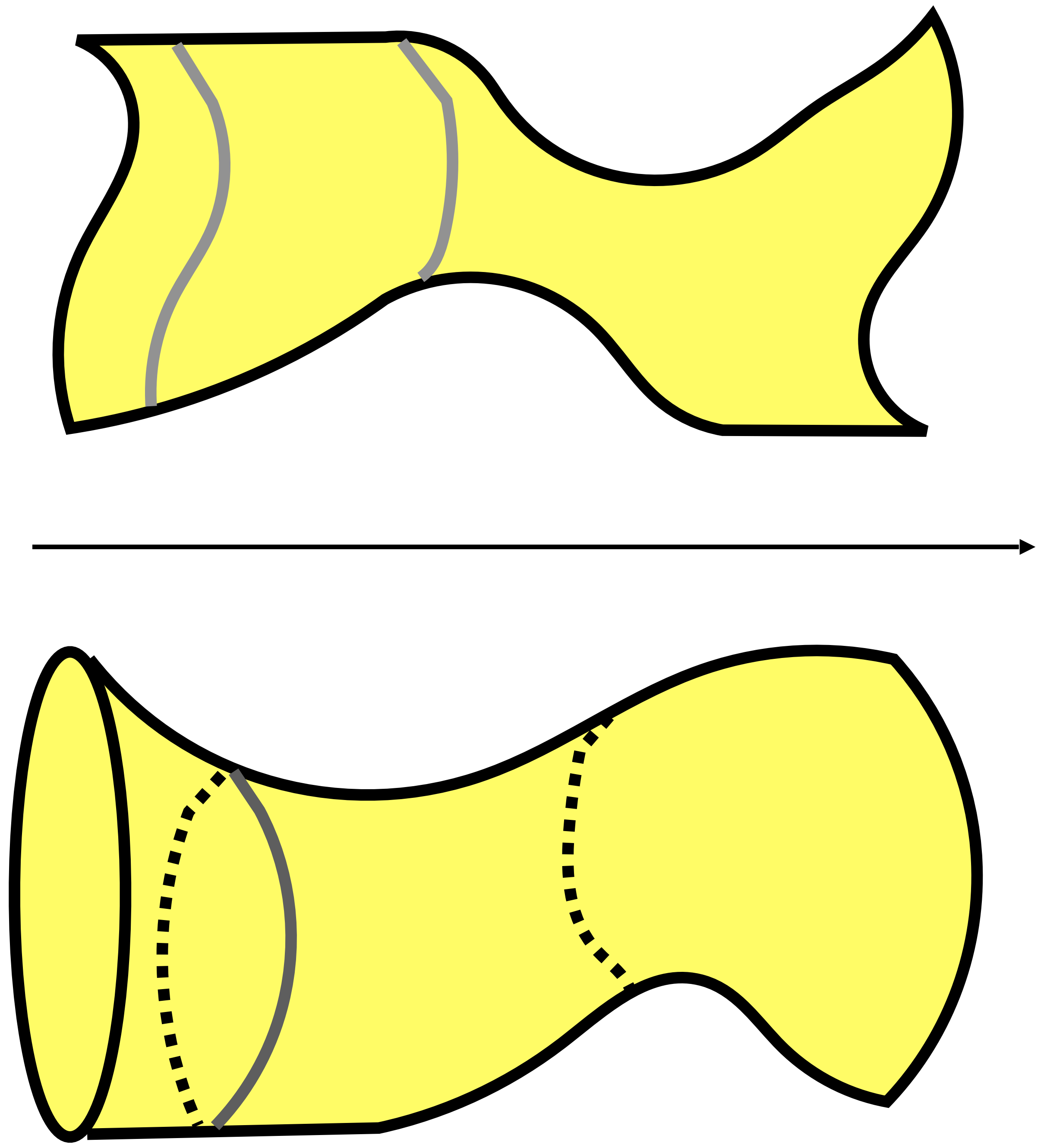
Closed



String states



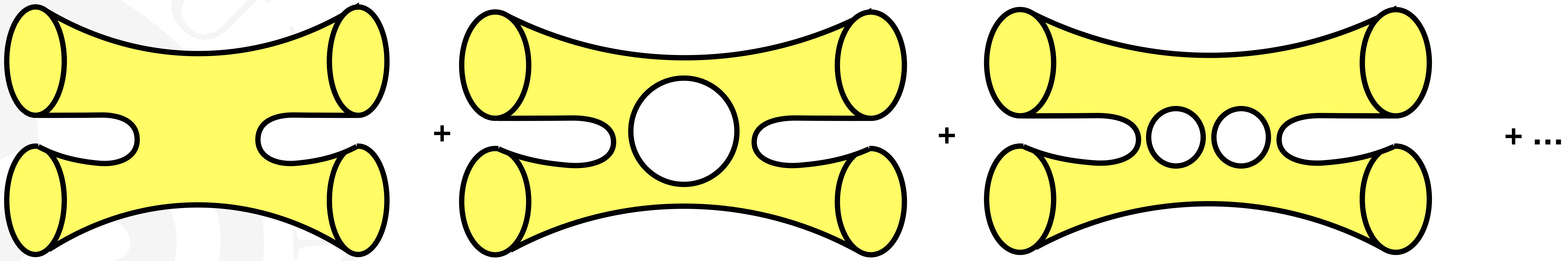
Worldsheets



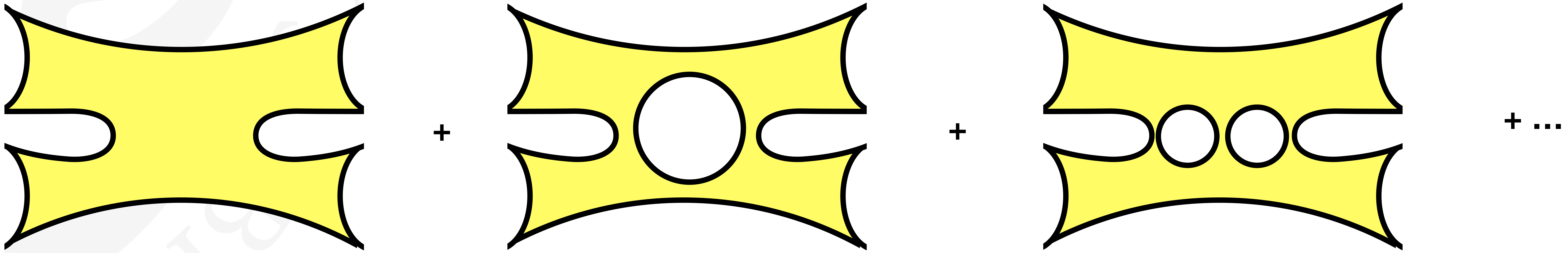
The n string amplitude is defined as:

$$\mathcal{A}_n = \int_W DXDh\, e^{-S_p[X,h]} \prod_{i=1}^{n} V_i = \sum_g e^{-\lambda\chi(g)} \int_{W_g} DXDh\, e^{-S_p[X,h]} \prod_{i=1}^{n} V_i .$$

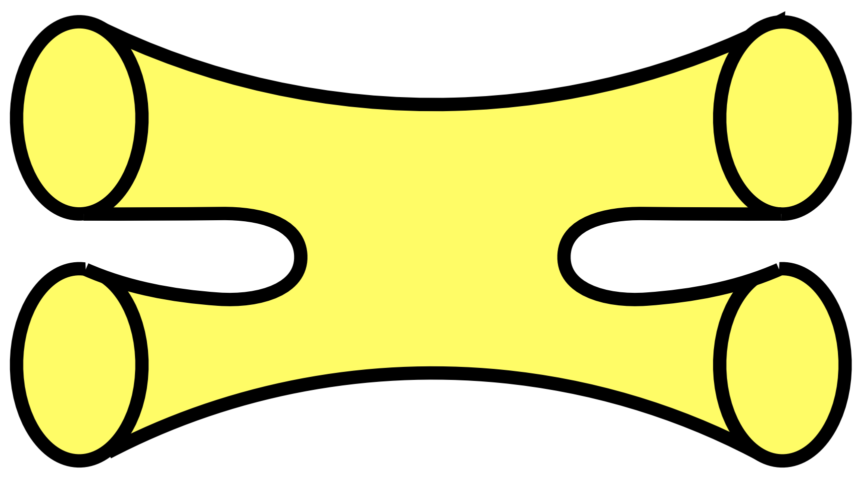
$\mathcal{A}_4^{closed} =$



$\mathcal{A}_4^{open} =$

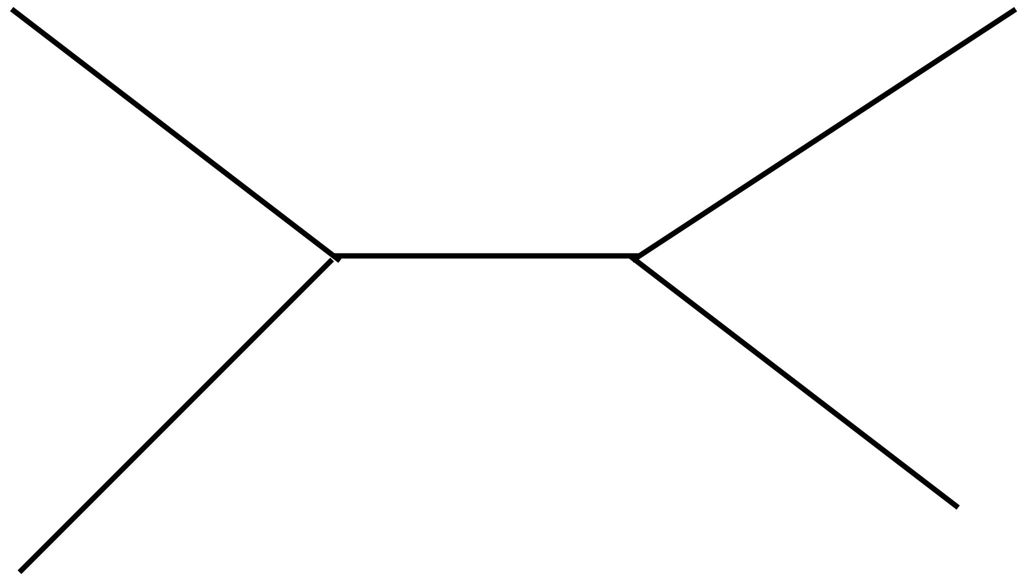


String Amplitudes

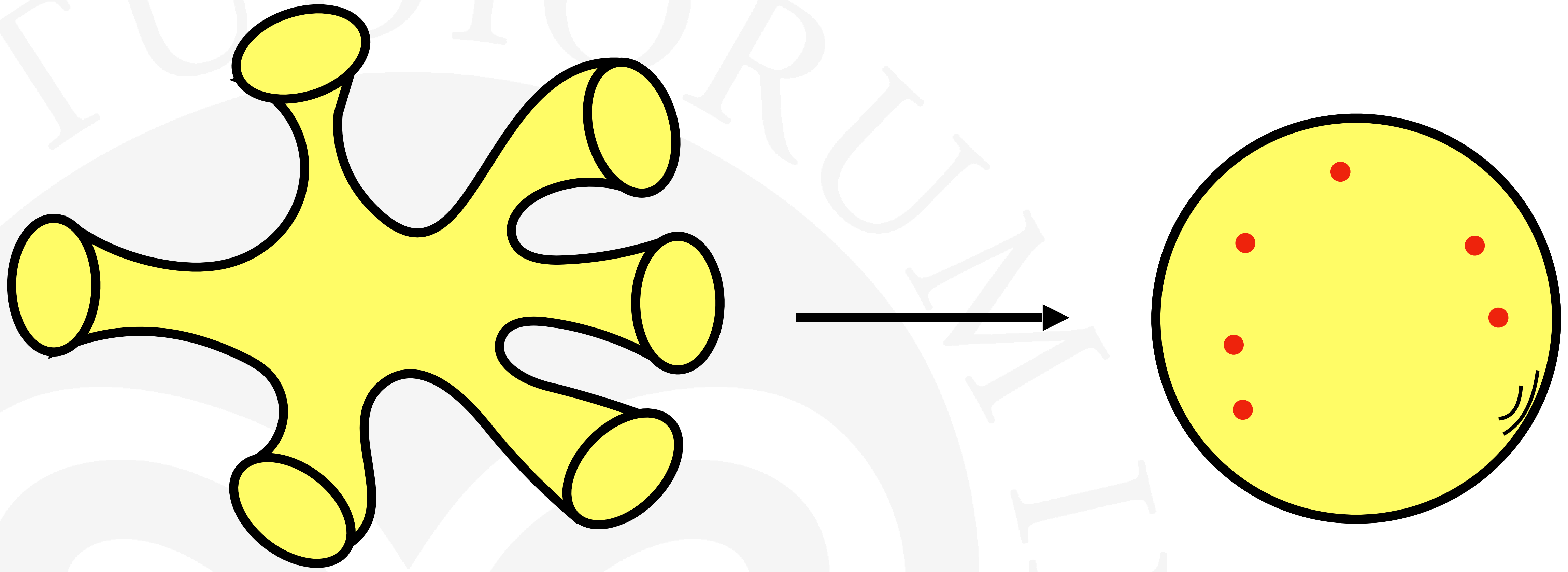


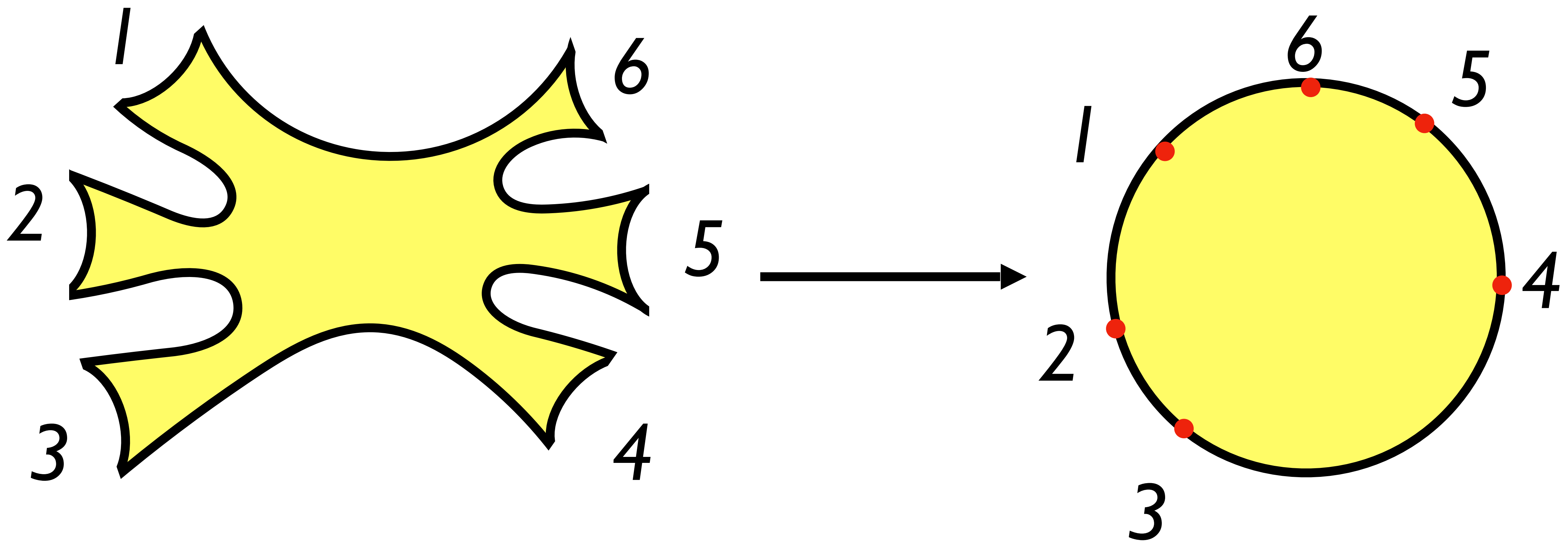
$\alpha' \to 0$

QFT Amplitudes



String tree level Amplitudes



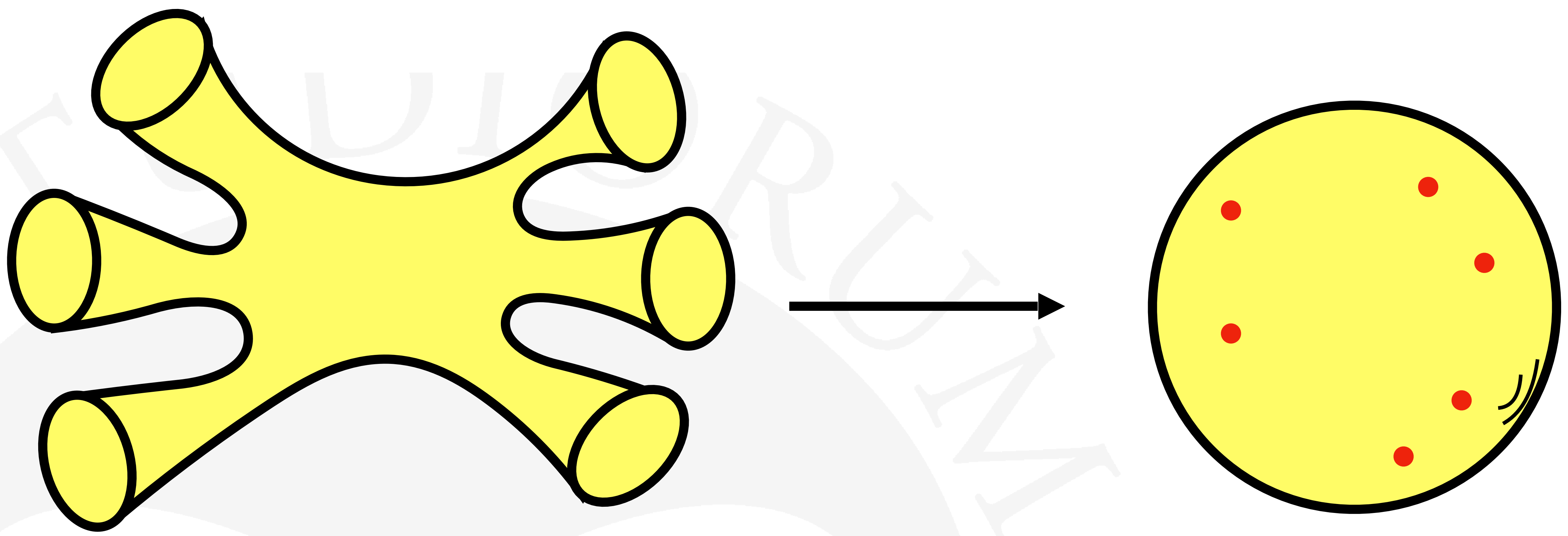


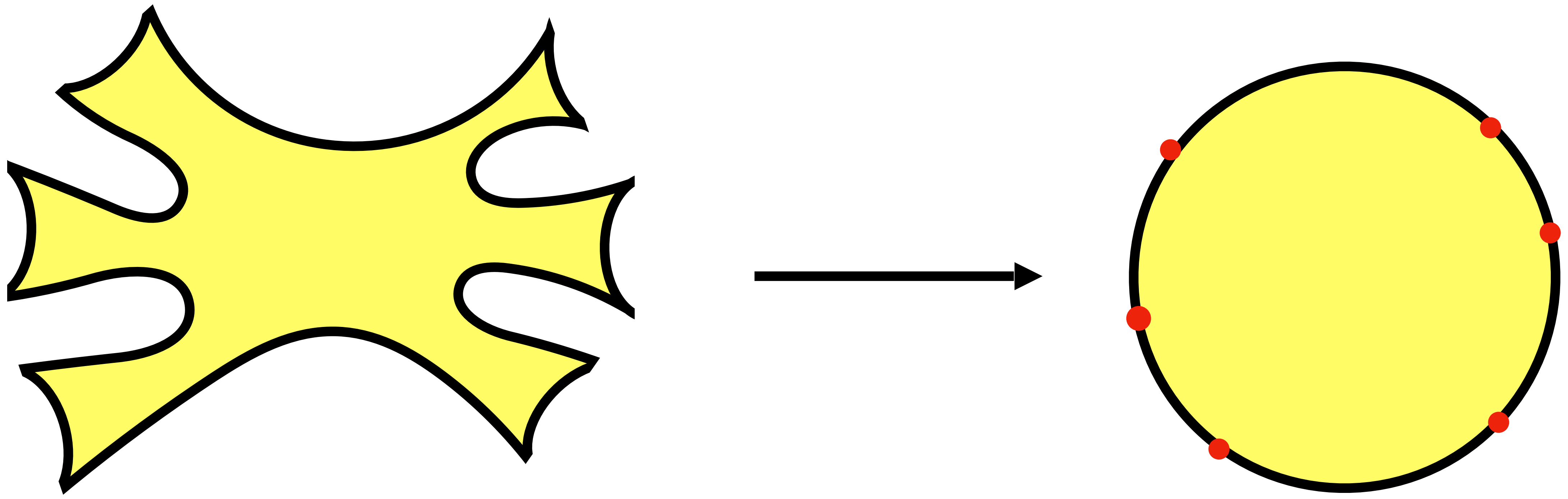
$$\mathcal{A}_n^{closed} = \int_{\mathcal{M}_{0,n}} \frac{1}{SL(2,\mathbb{C})} \prod_{i=1}^{n} d^2 z_i \prod_{i<j=1}^{n} |z_{ij}|^{2\alpha k_i k_j} F(z) F(\bar{z})$$

$$\mathcal{A}_n^{open} = \int_{\mathcal{D}(1,2..n)} \frac{1}{SL(2,\mathbb{R})} \prod_{i=1}^{n} dx_i \prod_{i<j=1}^{n} |x_{ij}|^{\alpha k_i k_j} F(x)$$

Where $z_{ij} = z_i - z_j$ *and* $F(z)$ *is a rational function depending on the state polarization*

For tachyons $F(z) = 1$

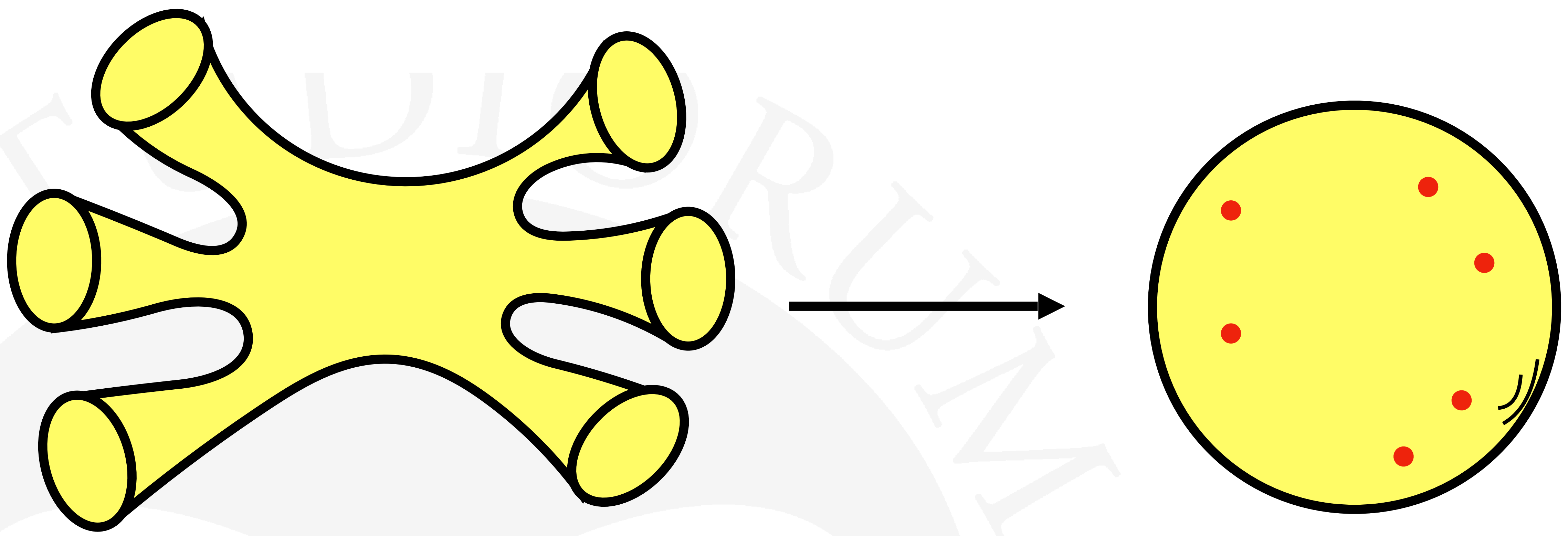


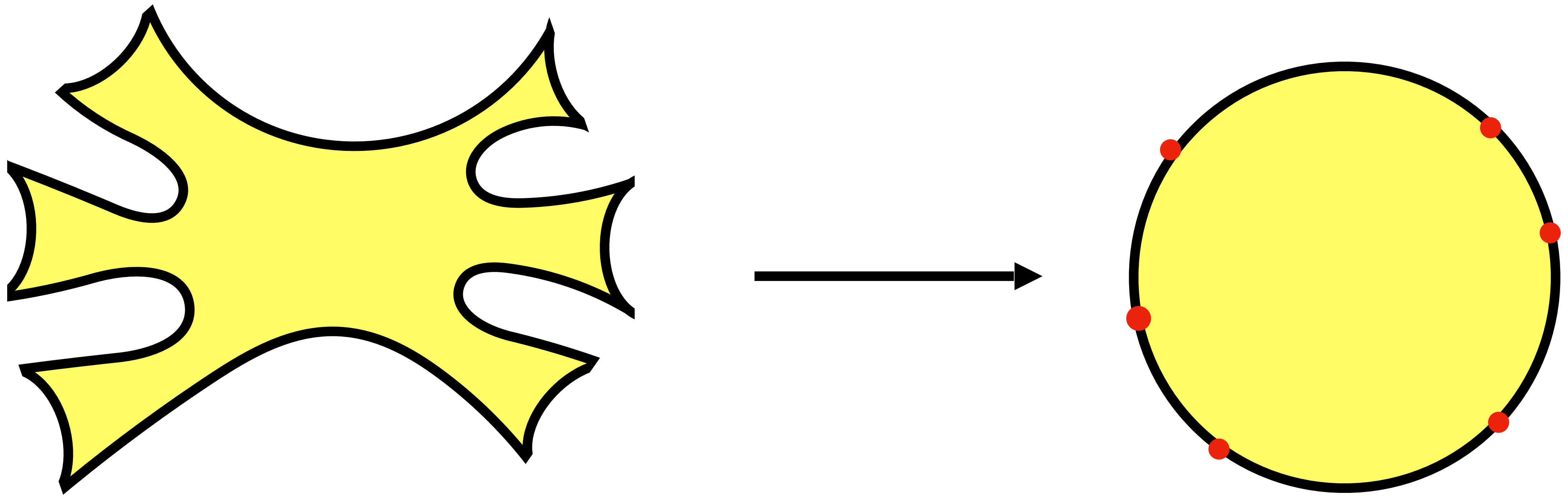


$$\mathcal{A}_n^{closed} = \int_{\mathcal{M}_{0,n}} \boxed{\frac{1}{SL(2,\mathbb{C})}} \prod_{i=1}^{n} d^2 z_i \prod_{i<j=1}^{n} |z_{ij}|^{2\alpha k_i k_j} F(z) F(\bar{z})$$

$$\mathcal{A}_n^{open} = \int_{\mathcal{D}(1,2..n)} \boxed{\frac{1}{SL(2,\mathbb{R})}} \prod_{i=1}^{n} dx_i \prod_{i<j=1}^{n} |x_{ij}|^{\alpha k_i k_j} F(x)$$

Conformal Killing group





$$\mathcal{A}_n^{closed} = \int_{\mathcal{M}_{0,n}} \frac{1}{SL(2,\mathbb{C})} \prod_{i=1}^{n} d^2 z_i \boxed{\prod_{i<j=1}^{n} |z_{ij}|^{2\alpha k_i k_j}} F(z) F(\bar{z}) \qquad \mathcal{A}_n^{open} = \int_{\mathcal{D}(1,2..n)} \frac{1}{SL(2,\mathbb{R})} \prod_{i=1}^{n} dx_i \boxed{\prod_{i<j=1}^{n} |x_{ij}|^{\alpha k_i k_j}} F(x)$$

Koba-Nielsen factor

Let

$$\varphi(z) = F(z)\Pi dz_i \qquad \overline{\varphi(z)} = F(\bar{z})\Pi d\bar{z}_i \qquad u = \Pi_{i<j=1}|z_{ij}|^{\alpha k_i k_j} \qquad \mathscr{L}^{\vee}_{\omega} = \mathscr{L}_{\overline{\omega}}$$

The tree level n closed string amplitude can be written as

$$\mathcal{A}_n^{closed} = \int |u(z)|^2 \varphi(z)\overline{\varphi(z)} = \langle \varphi(z) | \overline{\varphi(\bar{z})} \rangle$$

Cocycle-Cocycle pairing

The tree level n open string amplitude can be written as

$$\mathcal{A}_n^{open}(\sigma) = \int_{\Delta(\sigma)} u(z)\varphi(z) = \langle \varphi(z) | \Delta_\sigma]$$

Cycle-Cocycle pairing [Mizera, 2017]

The dimension of the (n-3)-(co)homology is:

$$\nu = (n-3)!.$$

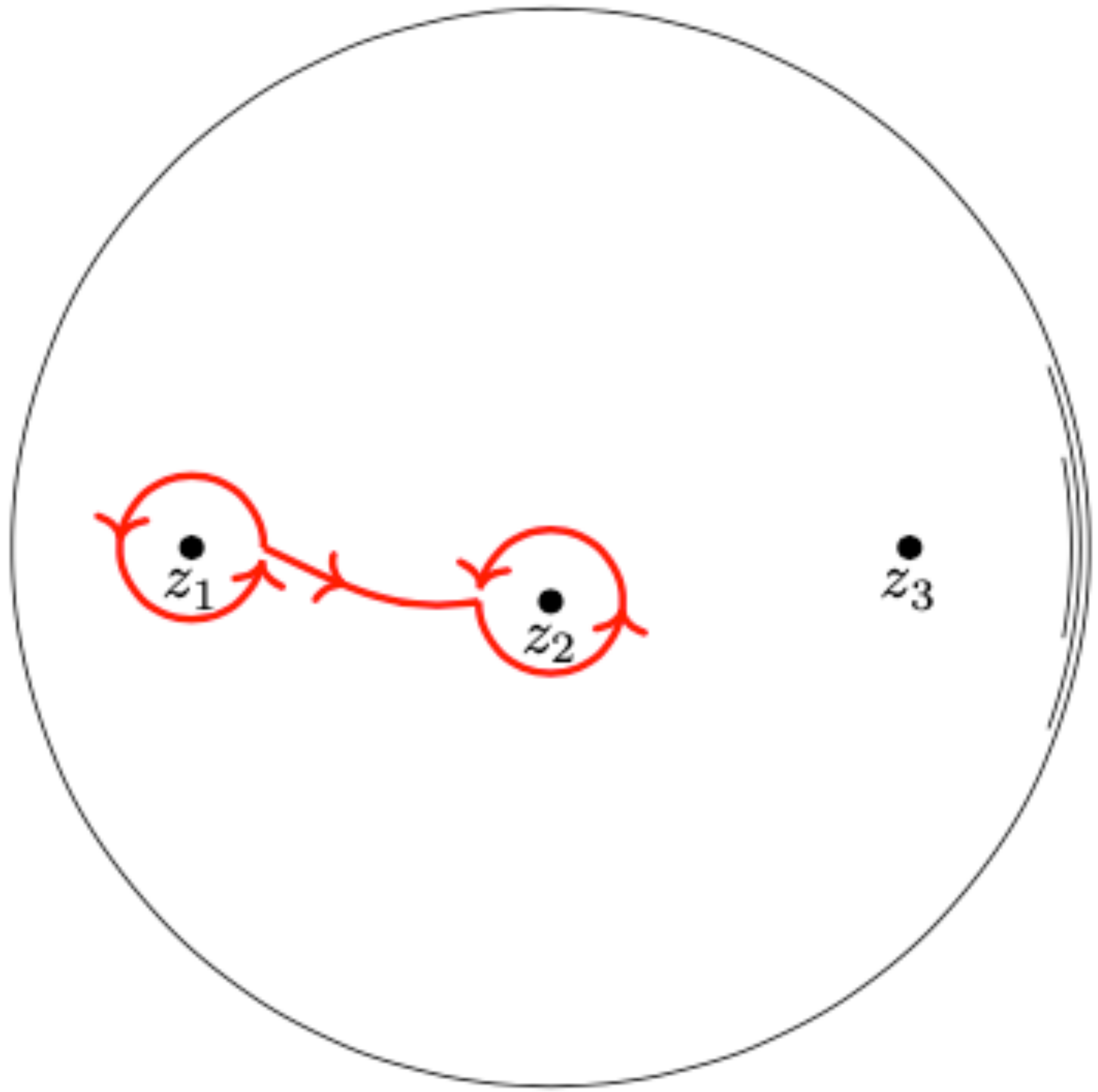
Cycles are defined as

$$\Delta(\beta) \equiv \overline{\{0 < z_{\beta(2)} < z_{\beta(3)} \cdots < z_{\beta(n-2)} < 1\}}.$$

Intersection numbers among twisted cycles is

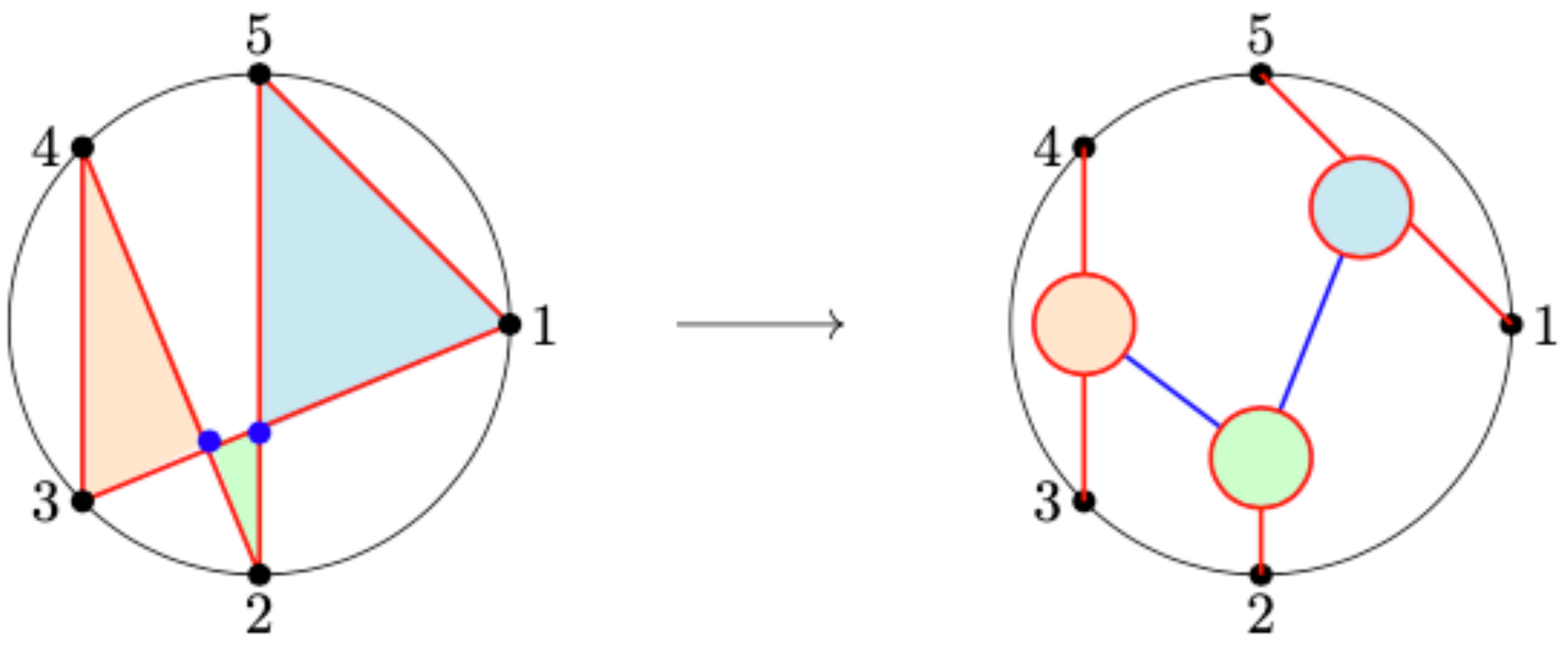
$$[\Delta(\alpha)|\Delta(\beta)] = \left(\frac{i}{2}\right)^{n-3} m(\alpha|\beta)$$

Where $m(\alpha\,|\,\beta)$ can be diagrammatically computed



[Cachazo,He,Yuan(2014)]
[Mizera (2017)]

$$m(\alpha\,|\,\beta) = m(12345\,|\,13425)$$

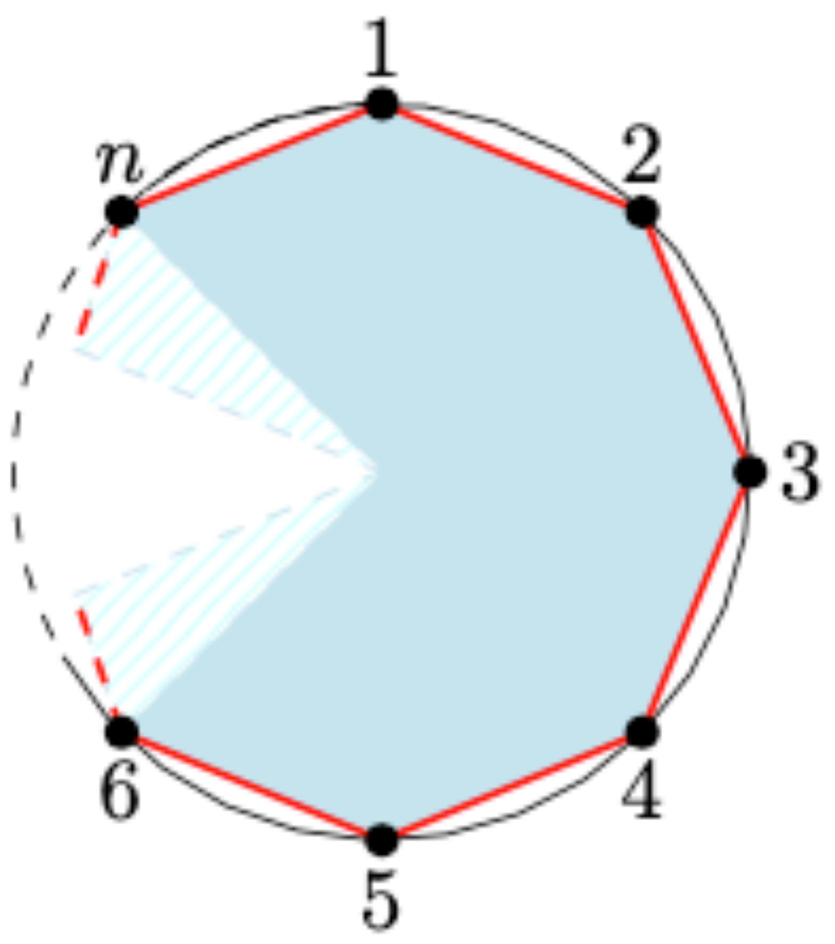


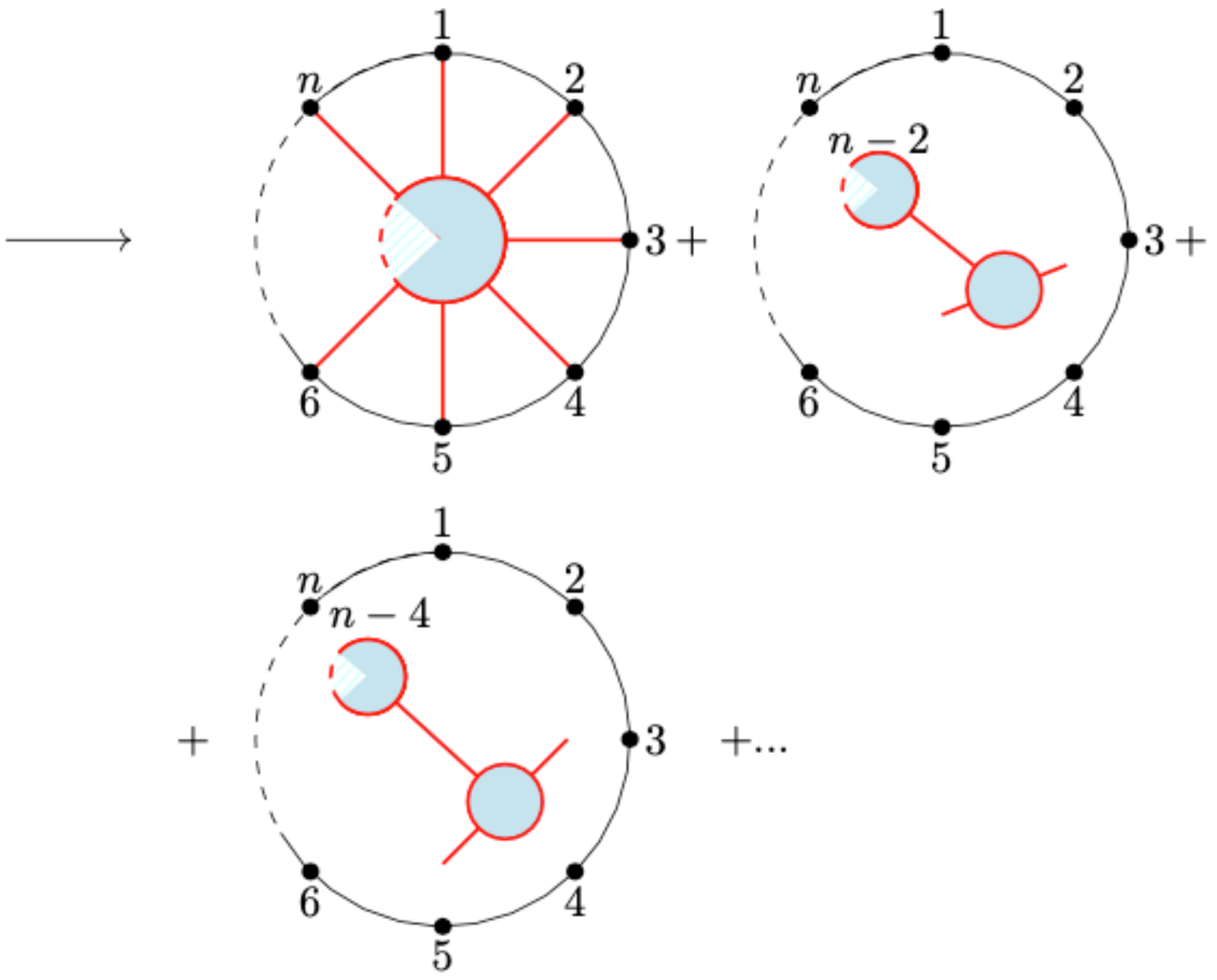
- Draw points along a circle according to α and connect them according to β
- Draw a dual diagram
- Attach to each internal leg a propagator $\frac{1}{\sin \pi s_{ij..}}$ with $s_{ij..} = \alpha' \sum k_i k_j$
- Each bubble with n legs corresponds to a n-3 selfintersection
- The sign is $(-1)^{w(\alpha|\beta)+1}$

$$m(12345\,|\,13425) = \frac{1}{\sin \pi s_{15}} \frac{1}{\sin \pi s_{34}}$$

[Cachazo,He,Yuan(2014)]
[Mizera (2017)]

$m(\alpha\,|\,\beta)$





- Draw all possible diagrams

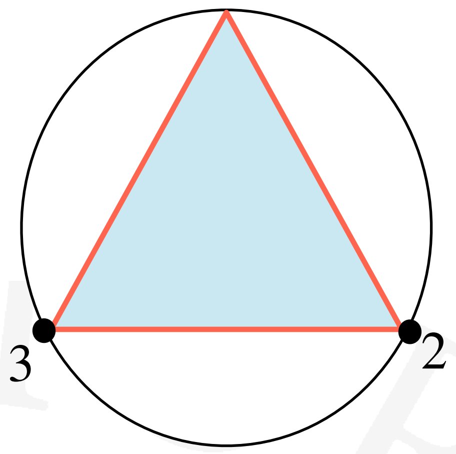
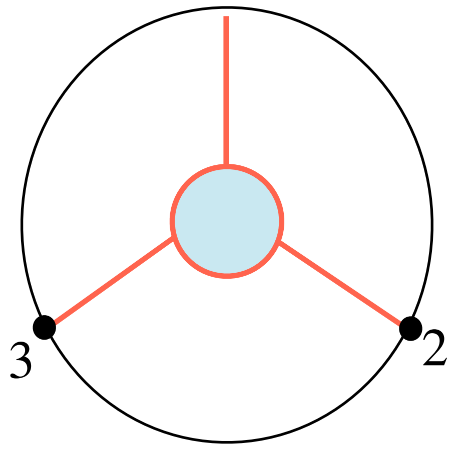
  with n-2,n-4...bubbles

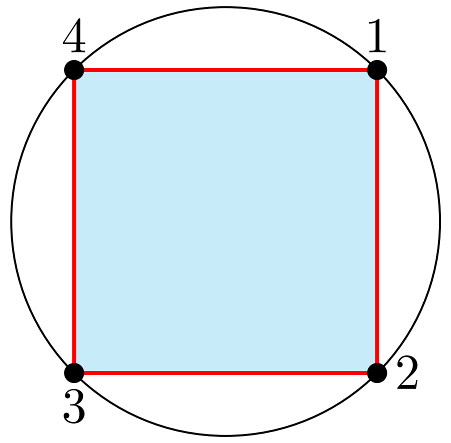
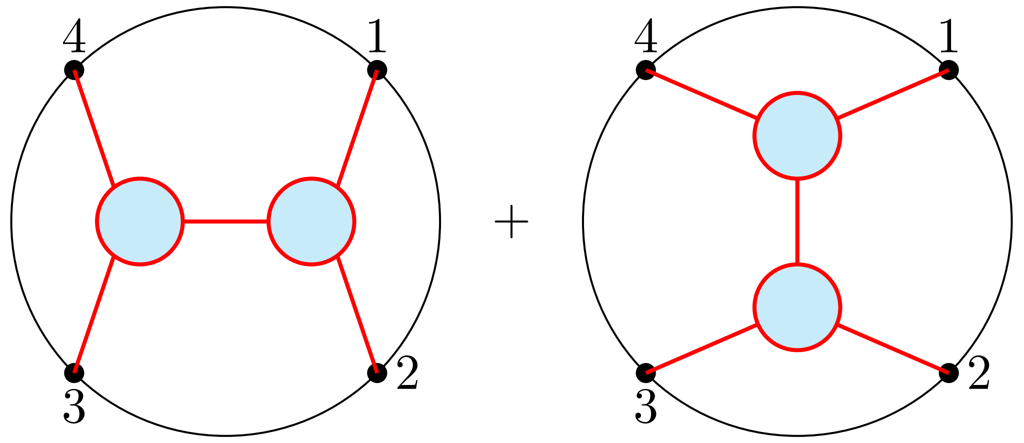
- Attach to each internal leg a propagator $\dfrac{1}{\tan \pi s_{ij..}}$

- One bubble diagrams give a contributes of 1,1,2,5...

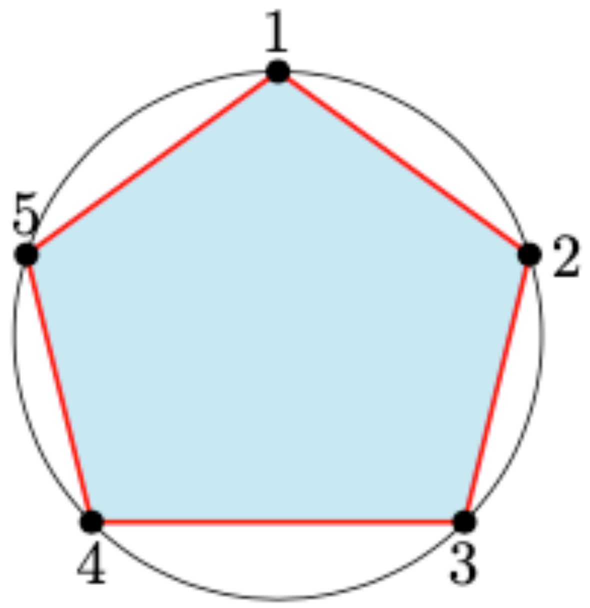
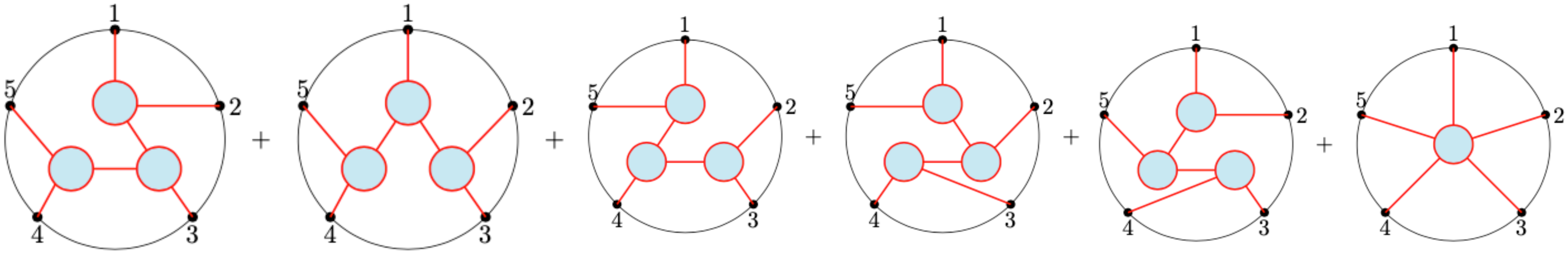
[Cachazo,He,Yuan(2014)]
[Mizera (2017)]

Diagrammatic rule for higher order twisted cycle intersection

$m(123\,|\,123) =$  $\longrightarrow$  $= 1$

$m(1234\,|\,1234) =$  $\longrightarrow$ 

$$[\Delta(1234)|\Delta(1234)] = \frac{i}{2}\left(\frac{1}{\tan\pi s_{12}} + \frac{1}{\tan\pi s_{23}}\right)$$

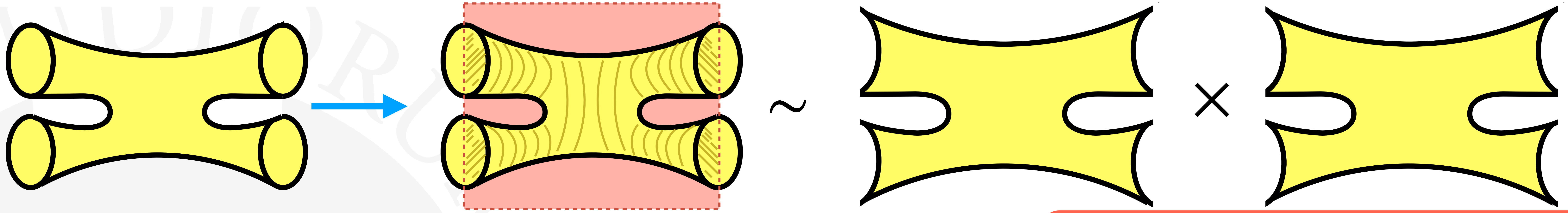
 $\longrightarrow$ 

$$[\Delta(12345)|\Delta(12345)] = -\frac{1}{4}\left(\frac{1}{\tan\pi s_{12}}\frac{1}{\tan\pi s_{45}}\right) + \left(\frac{1}{\tan\pi s_{23}}\frac{1}{\tan\pi s_{45}}\right) + \left(\frac{1}{\tan\pi s_{15}}\frac{1}{\tan\pi s_{23}}\right) +$$
$$+\left(\frac{1}{\tan\pi s_{15}}\frac{1}{\tan\pi s_{34}}\right) + \left(\frac{1}{\tan\pi s_{12}}\frac{1}{\tan\pi s_{34}}\right) + 1,$$

[Cachazo,He,Yuan(2014)]
[Mizera (2017)]

Kawai-Lewellen-Tye relations

Kawai-Lewellen-Tye (KLT) relations encode open/closed string duality: $\mathcal{A}^{\text{closed}} \sim (\mathcal{A}^{\text{open}})^2$



Gravitons $\sim$ (Gluons) $\times$ (Gluons)

KLT as Twisted Riemann period relations

$$\langle \varphi(z) | \overline{\varphi(\bar{z})} \rangle = \sum_{\alpha\beta} \langle \varphi(z) | \Delta_\alpha] (H^{-1})_{\alpha\beta} [\overline{\Delta_\beta} | \overline{\varphi(\bar{z})} \rangle = \sum_{\alpha\beta} \left(\int_{\Delta(\alpha)} u(z) \varphi(z) \right) (H^{-1})_{\alpha\beta} \left(\int_{\overline{\Delta}(\beta)} \bar{u}(z) \overline{\varphi(\bar{z})} \right) . \tag{5.5}$$

KLT kernel

Example: Shapiro-Virasoro and Veneziano amplitudes

$$\mathcal{A}_4^{closed} = \frac{i}{2} \mathcal{A}^{open}(1234) \left(\frac{1}{\tan \pi\alpha' k_1 k_2} + \frac{1}{\tan \pi\alpha' k_1 k_2} \right)^{-1} \mathcal{A}^{open}(1234)$$

[Kawai,Lewellen,Tye (1986)]
[Mizera (2017)]

$$\mathcal{A}_5^{closed} = \int |z_2|^{2\alpha' k_1 k_2} |z_3|^{2\alpha' k_1 k_3} |z_2 - 1|^{2\alpha' k_2 k_4} |z_3 - 1|^{2\alpha' k_3 k_4} |z_2 - z_3|^{2\alpha' k_2 k_3} dz_2 \wedge dz_3 =$$

$$= \int |u(z_2, z_3)|^2 dz_2^2 \wedge dz_3^2 = \langle 1|1 \rangle.$$

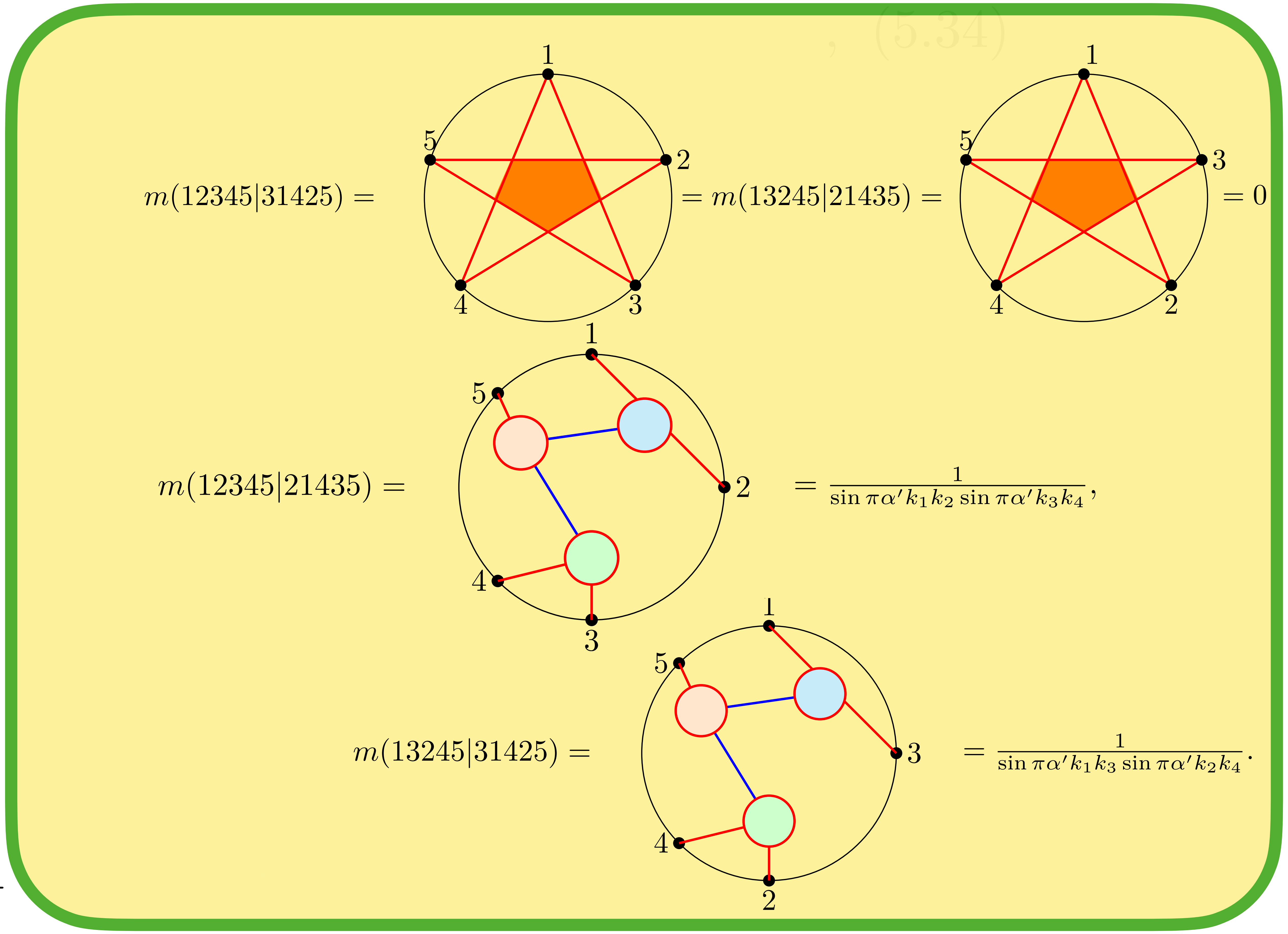
KLT decomposition reads:

$$\langle 1|1\rangle = \sum_{\alpha\beta} \langle 1|\Delta(\alpha)] H^{-1}_{\alpha\beta} [\Delta(\beta)|1\rangle,$$

$$H_{\alpha\beta} = \begin{pmatrix} [\Delta(12345)|\Delta(21435)] & [\Delta(12345)|\Delta(31425)] \\ [\Delta(13245)|\Delta(21435)] & [\Delta(13245)|\Delta(31425)] \end{pmatrix},$$

And we finally obtained

$$\mathcal{A}_5^{closed} = -4\mathcal{A}_5^{open}(12345)\bar{\mathcal{A}}_5^{open}(21435)\sin(\pi\alpha' k_1 k_2)\sin(\pi\alpha' k_3 k_4) +$$
$$- 4\mathcal{A}_5^{open}(13245)\bar{\mathcal{A}}_5^{open}(31425)\sin(\pi\alpha' k_1 k_3)\sin(\pi\alpha' k_2 k_4).$$



[A modern approach to String Amplitudes and Intersection theory (M. Thesis)]

$$\mathcal{A}_5^{open} = \sum_\beta \mathcal{A}_5^{open}(\beta) = \sum_\beta \int_{\Delta(\beta)} z_2^{\alpha' k_1 k_2} z_3^{\alpha' k_1 k_3} (z_2-1)^{\alpha' k_2 k_4} (z_3-1)^{\alpha' k_3 k_4} (z_2-z_3)^{\alpha' k_2 k_3} dz_2 \wedge dz_3$$
$$= \sum_\beta \int_{\Delta(\beta)} u(z_2,z_3) dz_2 \wedge dz_3 = \sum_\beta \langle 1|\Delta(\beta) \otimes u]. \tag{5.39}$$

Master decomposition into Parke-Taylor basis reads:

$$\langle 1|\Delta_\beta] = \sum_{\alpha\sigma} \langle 1|PT(\alpha)\rangle C^{-1}_{\alpha\sigma} \langle PT(\sigma)|\Delta_\beta]$$

$$C_{\alpha\beta} = \begin{pmatrix} \langle PT(12345)|PT(12345)\rangle & \langle PT(12345)|PT(12345)\rangle \\ \langle PT(13245)|PT(12345)\rangle & \langle PT(13245)|PT(13245)\rangle \end{pmatrix}$$

For instance:

$$\mathcal{A}_5^{open}(12345) = \int_0^1 \int_0^1 z_2^{\alpha_1} z_3^{\alpha_2} (z_2-1)^{\alpha_3} (z_3-1)^{\alpha_4} (z_2-z_3)^{\alpha_5} dz_2 \wedge dz_3 =$$
$$\Lambda_1 \int_0^1 \int_0^1 z_2^{\alpha_1 - 1} z_3^{\alpha_2} (z_2-1)^{\alpha_3} (z_3-1)^{\alpha_4 - 1} (z_2-z_3)^{\alpha_5 - 1} dz_2 \wedge dz_3 +$$
$$+ \Lambda_2 \int_0^1 \int_0^1 z_2^{\alpha_1} z_3^{\alpha_2 - 1} (z_2-1)^{\alpha_3 - 1} (z_3-1)^{\alpha_4} (z_2-z_3)^{\alpha_5 - 1} dz_2 \wedge dz_3$$

Parke-Taylor forms

$$PT(12345) = \frac{dz_2 \wedge dz_3}{z_2(z_2-z_3)(z_3-1)},$$
$$PT(13245) = \frac{dz_2 \wedge dz_3}{z_3(z_3-z_2)(z_2-1)}.$$

[A modern approach to String Amplitudes and Intersection theory (M. Thesis)]

Future goal:

- *Investigation of duality relations at higher order and in superstrings;*
- *Beyond Twisted De Rham Theory (Hodge structure, Stratifold, Supermoduli space);*
- *The role of boundaries in QFT and ST (Instantons, Cobordism and Distance Conjecture).*

Thank you