Asymptotic Hodge Theory for Feynman Integrals

Anthony Massidda

Domodossola 17/07/2024

Work in progress with: Roberta Angius and Sergio L. Cacciatori

amassidda@uninsubria.it

anthony.massidda@mi.infn.it

UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA

INFN Istituto Nazionale di Fisica Nucleare

Outline of the talk

- From multiloop Feynman integrals to periods: Banana integrals and Calabi-Yau
- Moduli space, singularities and variation of the Hodge structure
- Asymptotic Hodge theory and Monodromy: the three punctures case
- Conclusions and Outlooks

Multiloop Feynman integrals

D dimensional l loops Feynman integrals in Feynman parametrization take the form:

$$\boxed{I = \int_\Delta \prod_i x_i^{\nu_i - 1} \frac{\mathcal{U}(x_i)^{\nu-(l+1)\frac{D}{2}}}{\mathcal{F}(x_i)^{\nu - l\frac{D}{2}}}\mu}$$

- *Symanzik polynomials* $\mathcal{U}$ *and* $\mathcal{F}$ *are degree* l *and* $l+1$ *(respectively) homogeneous polynomials in n variables* x_i
- μ *is the* $n-1$ *measure form of* $\mathbb{P}^n$ *:*

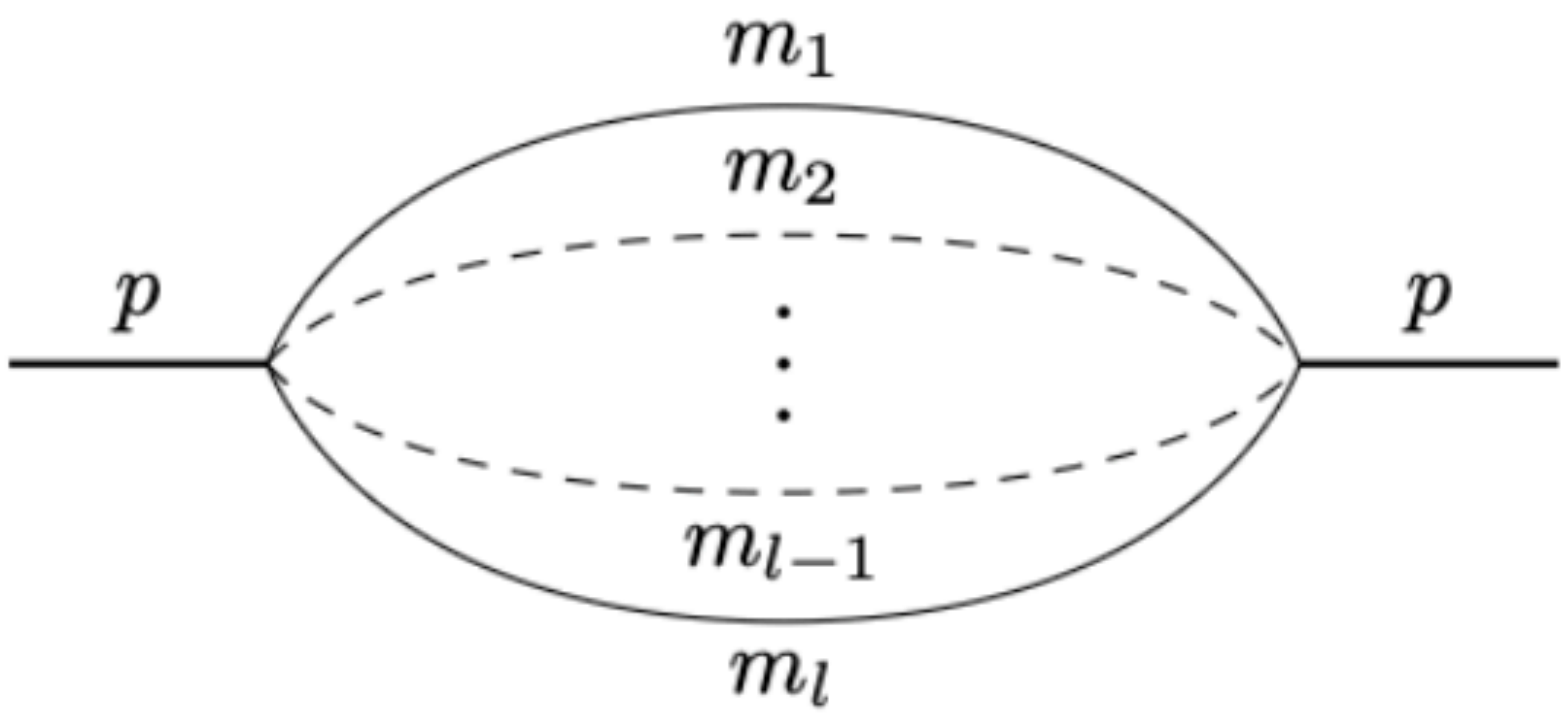
$$\mu = \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge \hat{dx_i} \wedge \ldots \wedge dx_n$$

- *The domain of integration is* $\mathbb{P}^n \supset \Delta_n = \{[x_1 : x_2 : \ldots : x_n] \in \mathbb{P}^n \,|\, x_i \geq 0\}$
- ν_i *are the exponent of propagators in momentum space and* $\nu = \sum \nu_i$

Banana Diagrams

l-loops banana integrals:

$$I^{(B)} = \int_{\Delta} \frac{\mathcal{U}(x_i)^{\frac{1}{2}(l+1)(2-D)}}{\mathcal{F}(x_i)^{\frac{1}{2}l(2-D)+1}} \mu$$



Where

$$\mathcal{U}(x_i) = \left(\prod_{i=1}^{l+1} x_i\right)\left(\sum_{i=1}^{l+1} \frac{1}{x_i}\right) \qquad \mathcal{F}(x_i) = -p^2\left(\prod_{i=1}^{l+1} x_i\right) + \left(\sum_{i=1}^{l+1} m_i^2 x_i\right)\mathcal{U}(x_i)$$

In D=2

$$I^{(B)}_{D=2} = \int_{\Delta} \frac{\mu_n}{\mathcal{F}(x_i)}$$

[Bloch,Vanhove 2014]

The residue Map

Let $f : \mathbb{CP}^n \to \mathbb{C}$ *and* $\omega \in \mathscr{H}^{n-1}(\mathbb{CP}^n)$.

We call residue of the meromorphic n-form $\mathscr{H}^n(\mathbb{CP}^n \backslash \mathscr{Z}) \ni \Omega = \omega \wedge d\, log f$

$$Res(\Omega) = Res\left(\omega \wedge \frac{df}{f}\right) = \omega|_{\mathscr{Z}}$$

with $\mathscr{Z} \equiv Ker(f)$

Having the propriety:

$$\int_{\Gamma_{n-1} \times S^1} \Omega = \int_{\Gamma_{n-1}} \omega|_{\mathscr{Z}}$$

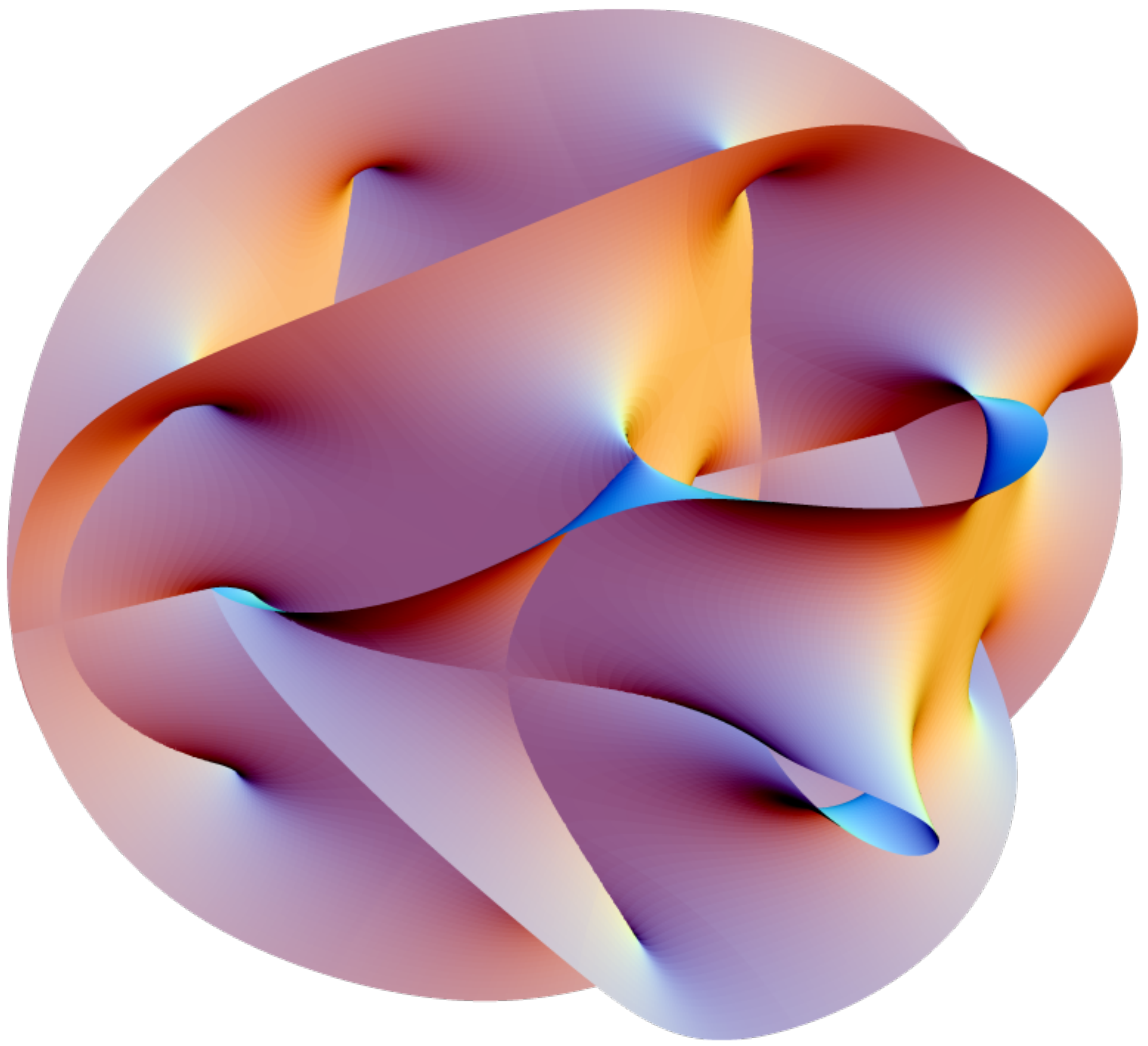
$$\boxed{Res : \mathscr{H}^n(\mathbb{CP}^n \backslash \mathscr{Z}) \to \mathscr{H}^{n-1}(\mathscr{Z})}$$

Notice for n=1 it reduces to the well known residue

$$Res\left(\frac{dz}{z - z_0}\omega(z)\right) = \omega(z_0)$$

Applying the residue map on the 2D banana (complexified) integral

$$I^{(B)*}_{D=2} = \int_{\Delta} \frac{\mu_n^*}{\mathscr{F}^*(x_i)} = \int_{\Gamma_{n-1}} \omega \,\big|_{\mathscr{F}=0}$$



The integral of an algebraic form on an algebraic domain: a ***Period***

[Kontsevich,Zagier 2001]

Fact: The locus of zeros of a homogeneous degree $(l+1)$ polynomial in $\mathbb{CP}^l$ is Calabi-Yau $(l-1)$ fold !

Example: Periods of the torus and Elliptic integrals

Consider the integral in $\mathbb{CP}^2$:

$$I_{\mathscr{E}} = \int_{\Gamma_n} \frac{\mu_n}{f(x,y,z)} = \int_{\Gamma_n} \frac{x dy \wedge dz - y dx \wedge dz + z dx \wedge dy}{y^2 z - x^3 + g_2 x z^2 + g_3 z^3}$$

The integrand can be written as

$$\frac{\mu_n}{f} = \frac{x dz + z dx}{\partial_y f} \wedge \frac{df}{f} - \frac{3 dx \wedge dz}{\partial_y f}$$

Thus:

$$I_{\mathscr{E}} = \int_{\Gamma_{n-1}} Res\left(\frac{\mu_n}{f}\right) = \int \left(\frac{x dz + z dx}{2yz}\right)_{f=0} = \frac{1}{2}\int \left(\frac{dx}{y}\right)_{f=0} = \frac{1}{2}\int \frac{dx}{\sqrt{x^3 - g_2 x - g_3}}$$

Periods of a Torus $\mathbb{T}^2$

Example: Periods of the torus and Elliptic integrals

If the parameters g_i are fixed and such that the discriminant

$$\Delta = 4g_2^3 + 27g_3^2 \neq 0$$

The torus is smooth and everything is fine!

But $g_i = g_i(m_i, p^2)$ and for some "critical value" of physical parameters $\Delta = 0$, the elliptic curve degenerates and the elliptic surface is not smooth!

The original integral is related to the periods of a family of elliptic surfaces depending on physical parameters and some of the surfaces of the family are singular

How do we handle singularities?

Simplest answer: we avoid them!

Deformation theory

Singular → Smooth

$$\mathscr{V}[f] \to \mathscr{V}_D[f] \equiv \mathscr{V}[f + D]$$

$$D \in \frac{\mathbb{C}[x_i]}{f + J_f}$$

The simplest deformation is by a constant (respect to x_i) $\quad f \to f - \tau$

$$\mathscr{V}_\tau(f) = \{[x_1 : \ldots : x_{n+1}] \in \mathbb{CP}^n \,|\, f(x_1, \ldots x_{n+1}) = \tau\}$$

Deformations and Milnor Fibration

Consider the (quasi) fibration $f : \mathbb{CP}^n \to \mathbb{CP}^1$ *whose fibers are*

$$f^{-1}(\tau) \equiv \mathscr{F}_\tau \cong \mathscr{V}_\tau[f]$$

For an isolated singularity, fibers are smooth and isomorphic to each other on $\mathbb{CP}^1_* = \mathbb{CP}^1 \setminus \{0\}$ *thus f induces a well defined fibration on* $\mathbb{CP}^1_*$ *called Milnor Fibration*

[Milnor, 1968]

The information on the removed singular fiber is store in the Monodromy of the base space (Moduli Space)

How? *The presence of singularity corresponds to the shrinking of some cycle (Vanishing cycles) i.e. to the change of the homology of the fiber because of:*

$$\pi_1 : \mathbb{CP}^1_* \to \mathbb{CP}^1_* \overset{Induces}{\Rightarrow} \pi_1^* : \mathscr{H}^n(\mathscr{V}_\tau[f]) \to \mathscr{H}^n(\mathscr{V}_\tau[f])$$

Hodge structure

Because regular fibers are complex analytic manifolds, their cohomology admit an Hodge decomposition.
In particular we are interested in the middle cohomology (Remember: our initial integral was real valued!!)

$$\mathscr{H}^n(\mathscr{F}_\tau, \mathbb{C}) = \bigoplus_{p+q=n} \mathscr{H}^{p,q}(\mathscr{F}_\tau, \mathbb{C})$$

Let choose and fix a basis $\{\gamma_i\}$ *(symplectic/orthogonal) for the vector space* $F^0 = \bigoplus_{p+q=n} \mathscr{H}^{p,q}(\mathscr{F}_\tau, \mathbb{C})$.

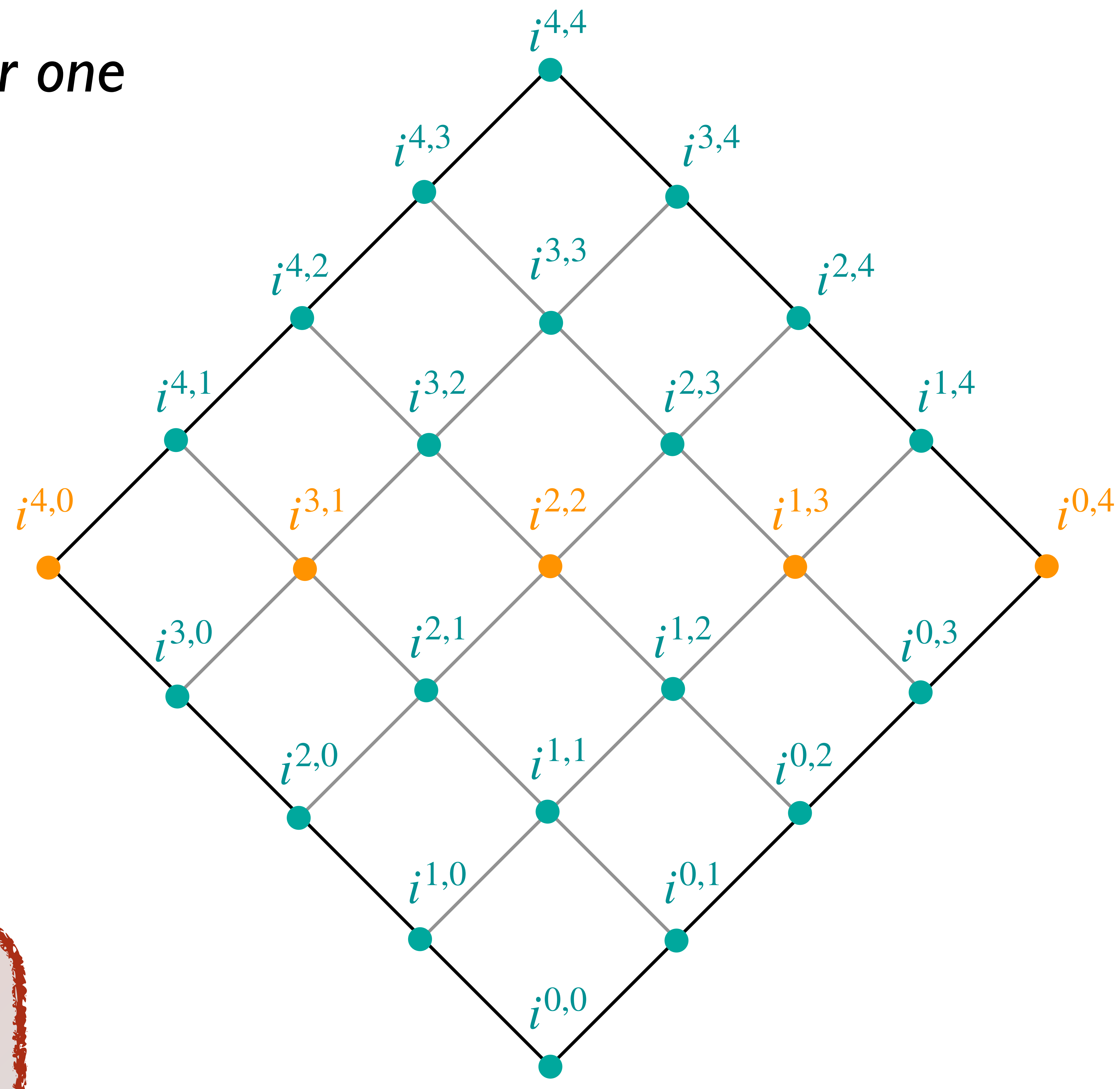
$$\Omega(\tau) = \Pi^i(\tau)\gamma_i \quad \textit{with} \quad \boxed{\Pi^i(\tau) = \int_{\tilde{\gamma}_i} \Omega(\tau)} \quad \textit{for any} \quad \Omega(\tau) \in \mathscr{H}^n(\mathscr{F}_\tau, \mathbb{C})$$

The variation of the periods along a compact path $\lambda(\tau)$ on the moduli space encompasses the variation of the Hodge structure, but for paths approaching a singularity periods diverge and the information on the Hodge structure is lost

That it, because singular fibers do not admit Hodge-decomposition, but a finer one called ***Hodge-Deligne splitting***

$$F^0 = \bigoplus_{p+q=n} \mathscr{H}^{p,q}(\mathscr{F}_\tau, \mathbb{C}) \to W^{2n} = \bigoplus_{p+q=0}^{2n} \mathscr{I}^{p,q}(\mathscr{F}_\tau, \mathbb{C})$$

Setting $h^{p,q} \equiv dim\mathscr{I}^{p,q}$*, the splitting can be represented by the so called Hodge-Deligne diamond.*



[Hodge-Deligne diamond for CY_4]

The type of the singularity uniquely determines the corresponding Hodge-Deligne splitting, (and viceversa), and the latter is completely fixed by the local monodromy

We got to the point!

Is it possible to determine the periods of a family of varieties from the monodromy of the moduli space?

FINITENESS THEOREM (Deligne '87)

For a given moduli space with fixed singularity structure there are only finitely many monodromy groups Γ supporting that structure.

RIGIDITY THEOREM (Schmid '73)

Given a monodromy group Γ, the period vector that gives rise to this monodromy behavior is determined uniquely.

Work plan

1. *Assuming a specific singularities structure, enumerate the possible monodromy groups*
2. *Compute for those groups the corresponding period vectors.*

Example: Elliptic curve in $\mathscr{M} = \mathbf{CP}^1 \backslash \{0, \mu^{-1}, \infty\}$

Turning around a boundary point induces a monodromy:

$$\mathbf{\Pi}(z) \mapsto \mathbf{\Pi}(e^{2\pi i} z) = M \mathbf{\Pi}(z) \qquad \textit{with} \qquad M \in SL(2,\mathbb{Z})$$

such that:

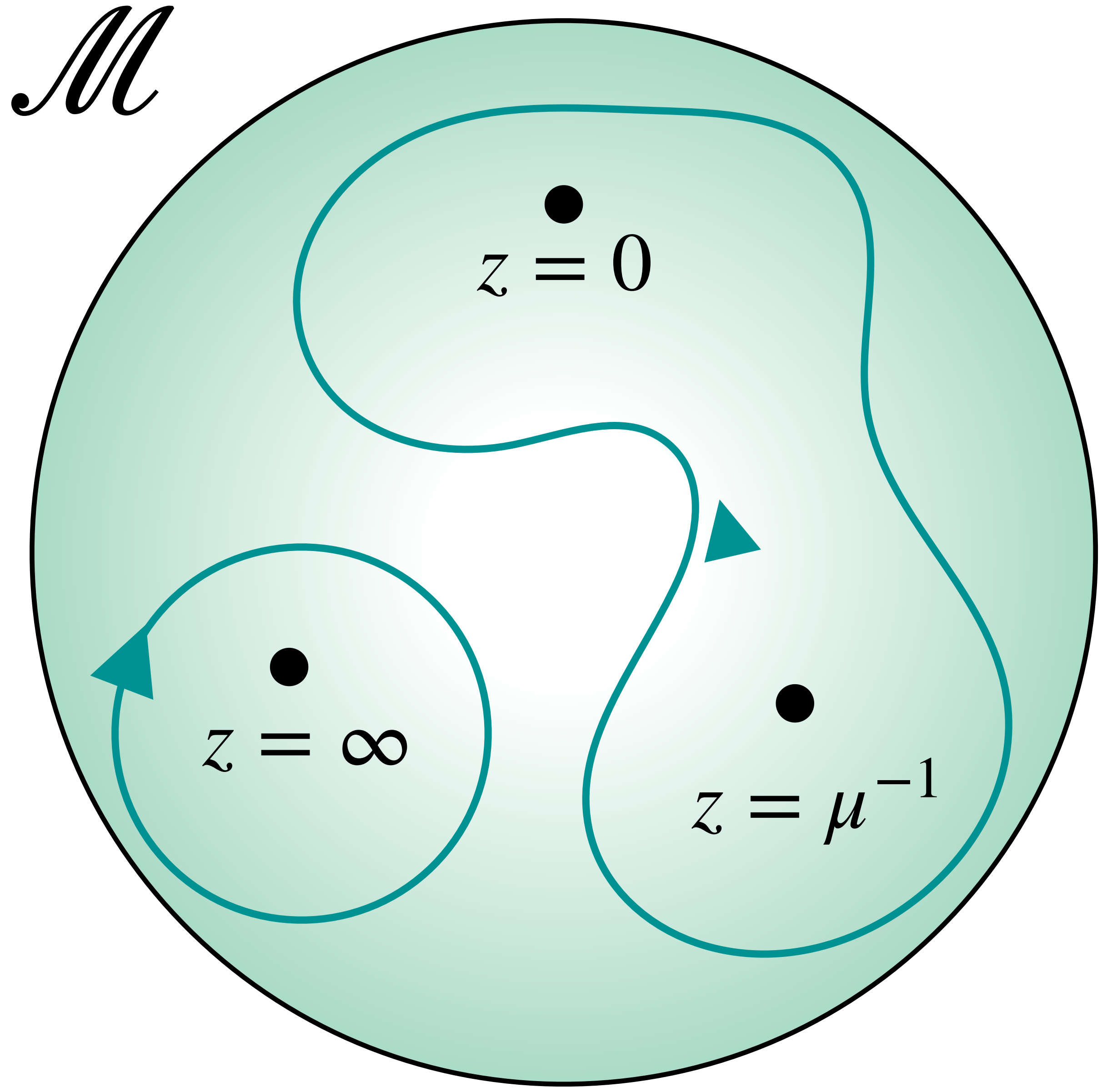
1. *The monodromy matrix* M *is* ***quasi-unipotent***:

$$\exists\ l, d \in \mathbb{Z} \mid \left(M^l - \mathbb{I}\right)^d \neq 0 \qquad \left(M^l - \mathbb{I}\right)^{d+1} = 0$$

$$M = \underbrace{M_u}_{\text{Unipotent of degree } d} \underbrace{M_s}_{\text{Semi-simple}}$$

2. *Equivalent loops have the same monodromy:*

$$M_0 M_{\mu^{-1}} = M_\infty^{-1}$$



Assumptions:

- MUM point for $CY_n, n > 1$

$$M_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

- Conifold point for $CY_n, n > 1$

$$M_{\mu^{-1}} = \begin{pmatrix} 1 & -\kappa \\ 0 & 1 \end{pmatrix}$$

Example: Elliptic curve in $\mathscr{M} = \mathbf{CP}^1 \backslash \{0, \mu^{-1}, \infty\}$

The monodromy M_∞ around $z = \infty$:

1. *Has to be fixed as the inverse of the product of the other monodromies:*

$$M_\infty = \left(M_0 M_{\mu^{-1}}\right)^{-1} = \begin{pmatrix} 1-\kappa & \kappa \\ -1 & 1 \end{pmatrix}$$

2. *Has to be quasi-unipotent. This is possible for the following values of (l, d, κ):*

$$\left(\begin{pmatrix} 1-\kappa & \kappa \\ -1 & 1 \end{pmatrix}^l - \mathbb{I}_{2\times 2}\right)^{d+1} = 0 \longrightarrow$$

(l, d, κ)	(a_1, a_2)
$(6, 0, 1)$	$\left(\frac{1}{6}, \frac{5}{6}\right)$
$(4, 0, 2)$	$\left(\frac{1}{4}, \frac{3}{4}\right)$
$(3, 0, 3)$	$\left(\frac{1}{3}, \frac{2}{3}\right)$
$(2, 1, 4)$	$\left(\frac{1}{2}, \frac{1}{2}\right)$

Once we known the monodromy matrix M_∞, we can compute the corresponding eigenvalues and construct the Picard-Fuchs equations

$$arg\left[\lambda_i\right] = 2\pi a_i$$

Picard-Fuchs equations

The periods obey a set of differential equations constructed through the Picard-Fuchs operator
In the case of one modulus we have only a single equation:

$$L\omega(z) = \left[\theta^{n+1} - \mu z \prod_{i=1}^{n+1} (\theta + a_i)\right] \omega = 0 \qquad \textit{with} \qquad \theta = z\frac{d}{dz}$$

and $n = 1$ *for elliptic curves or* $n > 1$ *for* CY_n.

- *The Riemann P-symbol for* $\mathbb{P}^1 \backslash \{0, \mu^{-1}, \infty\}$ *is fully determined by monodromies:*

$$\mathscr{P}\begin{Bmatrix} 0 & \mu^{-1} & \infty \\ 0 & 0 & a_1 \\ 0 & 0 & a_2 \end{Bmatrix} \qquad \textit{and} \qquad \mathscr{P}\begin{Bmatrix} 0 & \mu^{-1} & \infty \\ 0 & 0 & a_1 \\ 0 & 1 & a_2 \\ \dots & \dots & \dots \\ 0 & n-1 & a_n \\ 0 & (n-1)/2 & a_{n+1} \end{Bmatrix}$$

for elliptic curves *for Calabi-Yau n-folds*

[Doran, Morgan 2005]

[Van de Heisteeg, 2024]

Frobenius basis

Solving the Picard-Fuchs equations around the three singular points provides the following Frobenius solutions:

- $z = 0, \mu^{-1}$

$$\left.\begin{aligned} \varpi_0 &= f_0(z) \\ 2\pi i \varpi_1 &= f_0 \log[z] + f_1(z) \end{aligned}\right\} \quad \textit{with} \quad f_i(z) = \delta_{0,i} + \sum_{k=1}^{\infty} c_{i,k} z^k, \quad \textit{and} \quad c_{i,k} \in \mathbb{Q}$$

In particular we have:

$$\varpi_0 = {}_2F_1(a_1, a_2; 1; \mu z) \qquad \varpi_1 = \frac{i}{\sqrt{k}}\, {}_2F_1(a_1, a_2; 1; 1 - \mu z)$$

- $z = \infty \quad (a_1; a_2)$

$$\left.\begin{aligned} \varpi_0 &= z^{a_1} \tilde{f}_0(z) \\ \varpi_1 &= z^{a_2} \tilde{f}_1(z) \end{aligned}\right\} \quad \textit{with} \quad \tilde{f}_i(z) = 1 + \sum_{k=1}^{\infty} \tilde{c}_{i,k} z^k$$

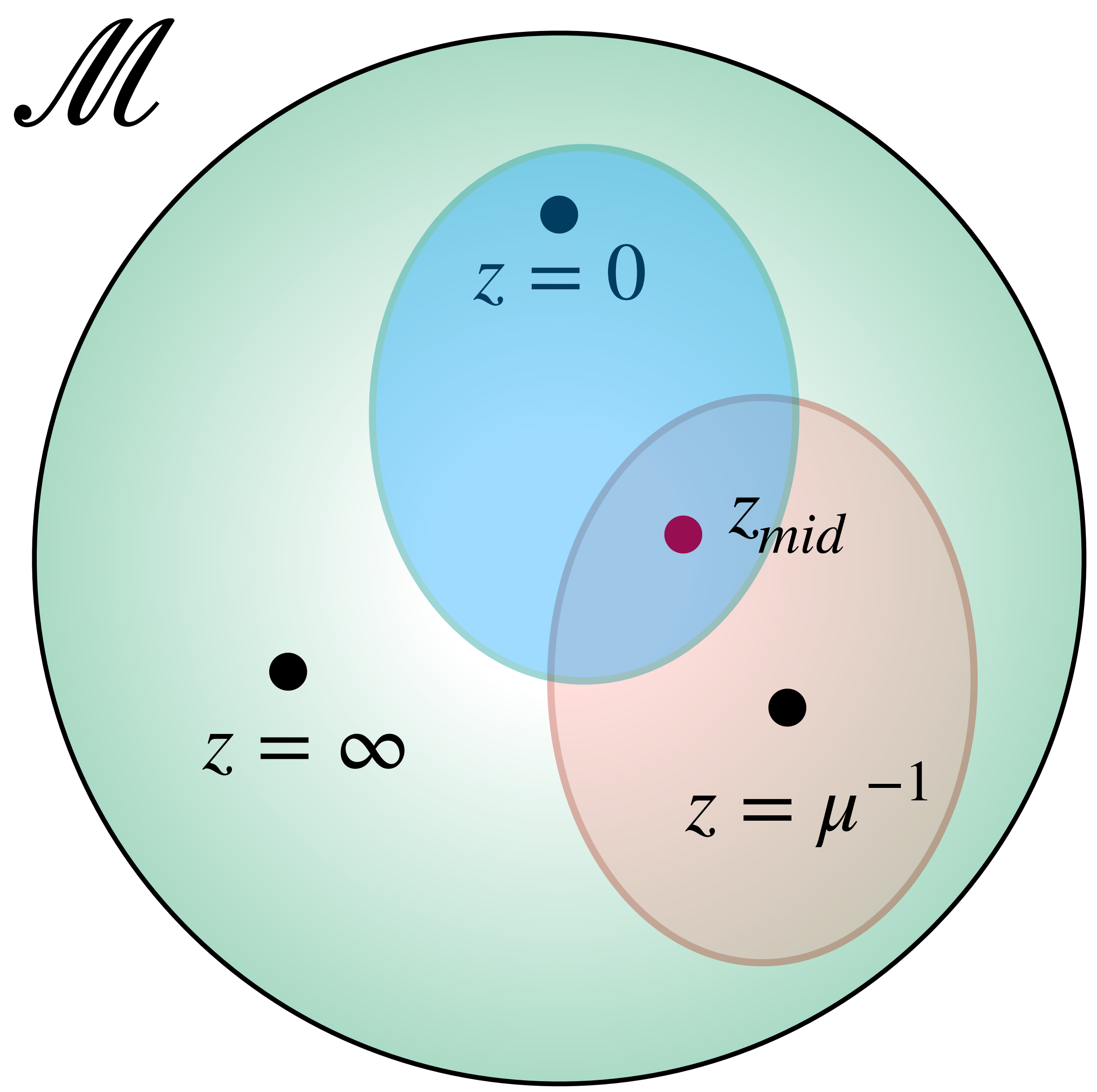
$$\left.\begin{aligned} \varpi_0 &= \sqrt{z}\, f_0(z) \\ 2\pi i \varpi_1 &= \sqrt{z}\, f_0 \log[z] + \sqrt{z}\, f_1(z) \end{aligned}\right\} \quad \textit{for} \quad (a_1; a_2) = \left(\frac{1}{2}; \frac{1}{2}\right)$$

Frobenius basis are local and complex (around each singularity)! We want a global integral basis valid in the whole moduli space.

- *Analytic continuation of the Frobenius basis solutions around* $z = 0$
- *Analytic continuation of the Frobenius basis solutions around* $z = \mu^{-1}$
- *Match the two solutions in a point* $z_{mid} \in A_0 \cap A_{\mu^{-1}}$

$$T_0^{\mu^{-1}} = \frac{i}{2\sqrt{k}} \begin{pmatrix} \kappa & -2\kappa \\ 2 & 0 \end{pmatrix}$$

Transition Matrix



- *Now we have all periods in terms of the Frobenius basis around the MUM*
- *We know the transition* T_{MUM} *from Frobenius to Integral around the MUM (in terms of the mirror CY)*
- *We have a global integral basis for the middle cohomology! Whose dual cycle basis are vanishing cycles!*

The structure of multiloop Feynman integrals is encoded in the monodromy of the moduli space

Some of the questions whose answers we are looking for are:

- What about Moduli spaces with more punctures? (Es. 2D Sunrise with different masses)
- What about higher dimensional Moduli spaces?

Asymptotic hodge theory give us the rules to classify all the singularity structure in terms of local monodromies

Thank you

Backup slides