AME 535a - Computational Fluid Dynamics

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BOUNDARY LAYER FLOW

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1 Boundary Layer Flow

Consider a two dimensional, steady, viscous flow over a rigid plate shown in the attached figure. The computational domain extends from $x = 0$ to $x_{max} = L$ in the streamwise direction and from $y = 0$ to $y_{max} = H$ in the vertical direction. The flow enters the domain at $x = 0$ with the uniform velocity u_0. The domain in the vertical direction is large enough so that the flow develops into the boundary layer flow along the plate (in theoretical approaches $H \to \infty$, see fluid dynamics textbooks or online materials for the Blasius boundary solution, e.g., "Incompressible Flow" by Panton).

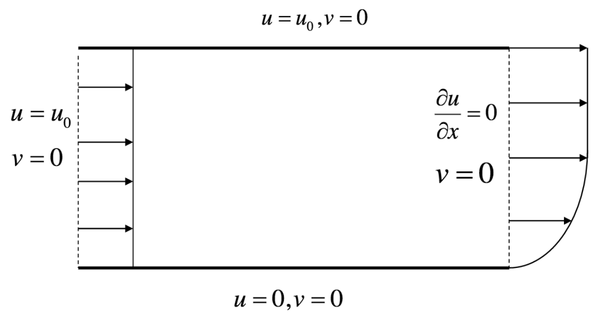

Figure 1: *Boundary Layer Flow*

2 Nondimensionalized 2-D Navier-Stokes Equations

Using L as length scale, u_0 as a velocity scale, and L/u_0 as a time scale it is shown that the nondimensional, 2-D Navier-Stokes equations are:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

where $u = u(x, y)$ and $v = v(x, y)$ are the velocity components, $p(x, y)$ is the pressure, and $Re = u_0 L/\nu$ is the Reynolds number (ν is the kinematic viscosity). The Navier Stokes equation in the x direction (neglecting gravity) is presented,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{\partial p}{\partial x} \tag{4}$$

and in the y direction,

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial p}{\partial y}. \tag{5}$$

The nondimensional variables are $u^* = u/u_0$, $x^* = x/L$, $y^* = y/H$, $t^* = t/(L/u_0)$, $p^* = p/(\rho * u_0^2)$. Substituting the nondimensionalized variables in the equations above, the expression is rewritten as,

$$\rho\frac{u_0^2}{L}\left[\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}\right] = \rho\frac{u_0}{L^2}\left(-\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho}\left[\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right]\right). \tag{6}$$

Dividing the equation by $\rho\frac{u_0^2}{L}$ the equation becomes,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] \tag{7}$$

Similarly for the Navier Stokes equations in the y direction the following expression is obtained after nondimensionalization:

$$\rho \frac{u_0^2}{L}\left[\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right] = \rho\frac{u_0}{L^2}\left(-\frac{\partial p^*}{\partial y^*} + \frac{\mu}{\rho}\left[\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right]\right) \tag{8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] \tag{9}$$

The continuity equation follows as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{10}$$

Substituting nondimensional variables the equation becomes,

$$\frac{u_0}{L}\frac{\partial u^*}{\partial x^*} + \frac{u_0}{L}\frac{\partial v^*}{\partial y^*} = 0 \tag{11}$$

and after dividing the expression by u_0/L, it is rewritten as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{12}$$

3 Solution via Numerical Methods

3.1 Problem Formulation

Numerical code was written to determine the flow in the computational domain for $x > 0$ assuming the boundary conditions shown in the figure. Specifically: no-slip boundary conditions on the upper and the lower wall: $u(x,0)$, $u(x,y_{max}) = u_0$; a uniform velocity at the flow $0 \leq y \leq y_{max} : u(0,y) = u_0$; a Neumann boundary condition for the streamwise velocity at the outflow $\frac{\partial u(x,y)}{\partial x}|_{x=x_{max}} = 0$; and the vertical component of velocity $v = 0$ at all boundaries. A pseudo-transient, fractional step method (time-splitting method) was implemented. The fractional step was split into three sections: nonlinear velocity, pressure and viscosity.

3.2 Nonlinear Velocity

$$\frac{\partial u}{\partial t} = F_u \tag{13}$$

$$\frac{\partial v}{\partial t} = F_v \tag{14}$$

(F_u, F_v) are nonlinear terms and are discretized:

$$F_u = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} \tag{15}$$

$$F_u = -u_{i,j}^n\left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2\Delta x}\right) - v_{i,j}^n\left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{2\Delta y}\right) \tag{16}$$

$$F_v = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} \tag{17}$$

$$F_v = -u_{i,j}^n\left(\frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta x}\right) - v_{i,j}^n\left(\frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta y}\right) \tag{18}$$

Time advancement using Adams-Bashforth method, i.e.

$$\frac{u_{i,j}^* - u_{i,j}^n}{\Delta t} = \frac{3}{2}F_u^n - \frac{1}{2}F_u^{n-1}, \tag{19}$$

$$\frac{v^*_{i,j} - v^n_{i,j}}{\Delta t} = \frac{3}{2}F^n_v - \frac{1}{2}F^{n-1}_v \tag{20}$$

At time = 0, the expression is reduced

$$\frac{u^*_{i,j} - u^n_{i,j}}{\Delta t} = F^n_u, \tag{21}$$

$$\frac{v^*_{i,j} - v^n_{i,j}}{\Delta t} = F^n_v \tag{22}$$

3.3 Pressure

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \tag{23}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}, \tag{24}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{25}$$

Time discretization

$$\frac{\mathbf{u}^{**} - \mathbf{u}^*}{\Delta t} = -\nabla p \tag{26}$$

Taking divergence of the above equation and assuming that $\mathbf{u}^{**}$ is incompressible, i.e. $\frac{\partial u^{**}}{\partial x} + \frac{\partial v^{**}}{\partial y} = 0$, we get a Poisson equation p

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \frac{1}{\Delta t}\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) \tag{27}$$

which is represented in discretized form below,

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = \frac{1}{\Delta t}\left(\frac{u^*_{i+1,j} - u^*_{i-1,j}}{2\Delta x} + \frac{v^*_{i,j+1} - v^*_{i,j-1}}{2\Delta y}\right) \tag{28}$$

The above equation for p is first solved with the Neumann boundary conditions on all boundaries

$$\frac{\partial p}{\partial x} = 0, at\ x = 0, x_{max}, \tag{29}$$

and

$$\frac{\partial p}{\partial y} = 0, at\ y = 0, y_{max}, \tag{30}$$

and then $\mathbf{u}^{**}$ is found from the previous equation

$$u^{**} = u^* - \Delta t\frac{\partial p}{\partial x}, \tag{31}$$

$$u^{**} = u^* - \Delta t\frac{p^n_{i+1,j} - p^n_{i-1,j}}{2\Delta x} \tag{32}$$

$$v^{**} = v^* - \Delta t\frac{\partial p}{\partial y}. \tag{33}$$

$$v^{**}_{i,j} = v^*_{i,j} - \Delta t\frac{p^n_{i,j+1} - p^n_{i,j-1}}{2\Delta y} \tag{34}$$

3.4 Viscosity

The first viscous step

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial x^2} \tag{35}$$

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\frac{\partial^2 v}{\partial x^2}. \tag{36}$$

Discretization using Crank-Nicolson scheme, i.e. for u

$$\frac{u^{***} - u^{**}}{\Delta t} = \frac{1}{Re}\left(\frac{1}{2}\frac{\partial^2 u^{***}}{\partial x^2} + \frac{1}{2}\frac{\partial^2 u^{**}}{\partial x^2}\right), \tag{37}$$

$$\frac{(-s)u^{***}_{i+1,j} + (1-2s)u^{***}_{i,j} + (-s)u^{***}_{i-1,j}}{\Delta x^2} = \frac{(s)u^{**}_{i+1,j} + (1+2s)u^{**}_{i,j} + (s)u^{**}_{i-1,j}}{\Delta x^2} \tag{38}$$

$$\frac{v^{***} - v^{**}}{\Delta t} = \frac{1}{Re}\left(\frac{1}{2}\frac{\partial^2 v^{***}}{\partial x^2} + \frac{1}{2}\frac{\partial^2 v^{**}}{\partial x^2}\right). \tag{39}$$

$$\frac{(-s)v^{***}_{i,j+1} + (1-2s)v^{***}_{i,j} + (-s)v^{***}_{i,j-1}}{\Delta x^2} = \frac{(s)v^{**}_{i,j+1} + (1+2s)v^{**}_{i,j} + (s)v^{**}_{i,j-1}}{\Delta x^2} \tag{40}$$

The second viscous step similarly solves the following equations

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \tag{41}$$

$$\frac{\partial v}{\partial t} = \frac{1}{Re}\frac{\partial^2 v}{\partial y^2} \tag{42}$$

using the Crank-Nicolson method. For u the discretized formula is shown below,

$$\frac{u^{n+1} - u^{***}}{\Delta t} = \frac{1}{Re}\left(\frac{1}{2}\frac{\partial^2 u^{n+1}}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u^{***}}{\partial y^2}\right) \tag{43}$$

$$\frac{(-s)u^{n+1}_{i+1,j} + (1-2s)u^{n+1}_{i,j} + (-s)^{n+1}_{i-1,j}}{\Delta y^2} = \frac{(s)u^{***}_{i+1,j} + (1+2s)u^{***}_{i,j} + (s)u^{***}_{i-1,j}}{\Delta y^2} \tag{44}$$

$$\frac{v^{n+1} - v^{***}}{\Delta t} = \frac{1}{Re}\left(\frac{1}{2}\frac{\partial^2 v^{n+1}}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u^{***}}{\partial y^2}\right). \tag{45}$$

$$\frac{(-s)v^{n+1}_{i,j+1} + (1-2s)v^{n+1}_{i,j} + (-s)v^{n+1}_{i,j-1}}{\Delta y^2} = \frac{(s)v^{***}_{i,j+1} + (1+2s)v^{***}_{i,j} + (s)v^{***}_{i,j-1}}{\Delta y^2} \tag{46}$$

4 Results

4.1 Velocity Profile

Velocity profiles $u(y)$ for $x = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ in the near wall region $y_{limit} \leq 0.1 * y_{max}$ is presented Figure 2.

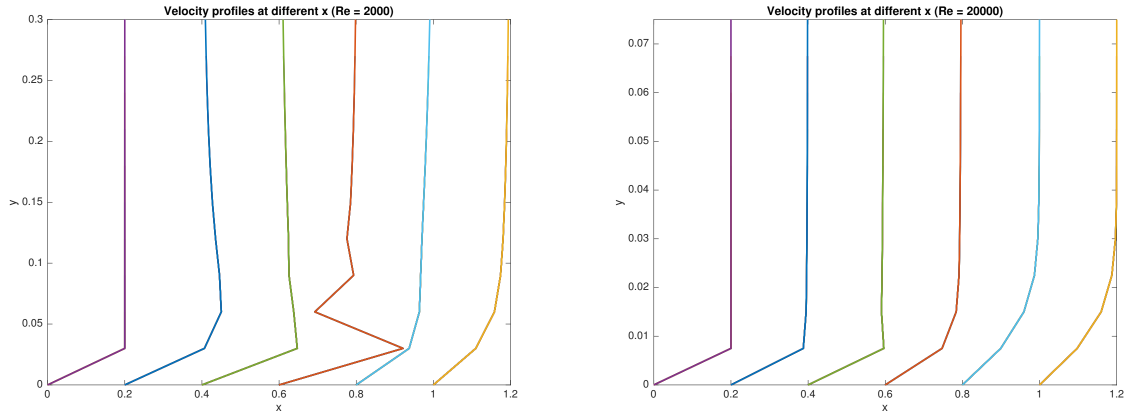

Figure 2: *Velocity profiles for two Reynolds numbers.*

4.2 Blasius Boundary Layer Solution

4.3 Skin Friction Coefficient

The skin friction coefficient $C_f = \frac{\tau_0}{\frac{1}{2}\rho u_0^2}$ was computed, where $\tau_0 = \rho\nu\frac{\partial u}{\partial y}|_{y=0}$ is the viscous shear stress at the wall. Plots of the skin friction coefficient and the Blasius similarity solution $C_{Bf} = 0.664 Re_x^{-0.5}$, where $Re_x = u_0 x/\nu$ are presented below along with the rms error.

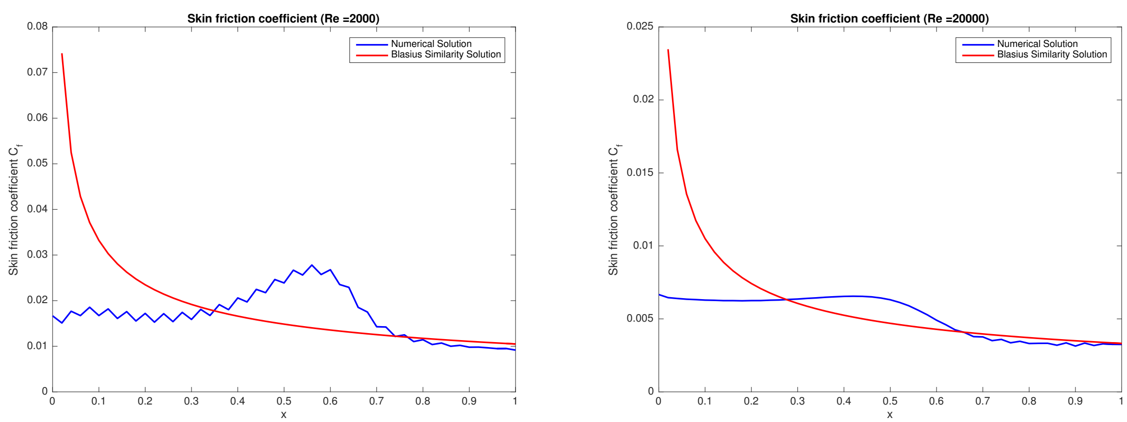

Figure 3: *The left figure has a Reynolds number of 2000 with an RMS error of 0.018 for skin friction coefficient. The right figure has a Reynolds number equal to 20000 with skin friction coefficient RMS error of .005*

4.4 Drag Coefficient

The drag coefficient C_D was obtained by integrating C_f over the first and the second half of the wall, $0 < x < 0.5$ and $0.5 < x < 1$, respectively. When $Re = 2000$ the Blasius drag coefficient is 0.64 for the first half of the plate and 0.32 on the second half. The corresponding numerical drag coefficient is 0.46 for the left half of the plate and 0.42 on the right half. When $Re = 20000$ the Blasius drag coefficient for the left half is 0.20 and it is 0.10 on the right half. The numerical drag coefficient is 0.16 on the left half and 0.10 on the right half. The numerical and Blasius solutions converge toward the end of the plate and there is greater agreement at higher Reynolds numbers.

4.5 Boundary Layer Thickness

The boundary layer thickness δ_{99}, i.e., distance from the wall at which the velocity $u(\delta_{99})$, is plotted as a function of x. It is compared with $\delta_{99} \approx 4.9\sqrt{\nu x/u_0}$ given by the Blasius solution.

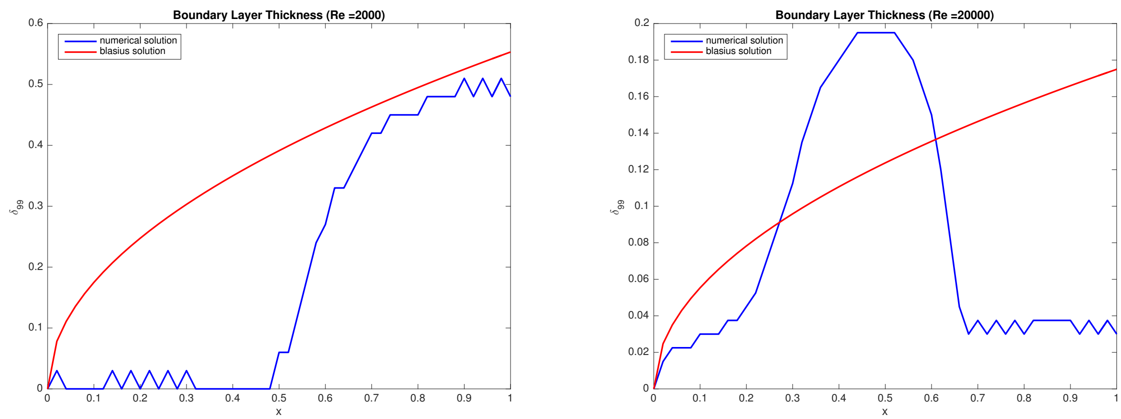

Figure 4: *When Re = 2000 there is greater overlap with the Blasius solution toward the end of the plate, whereas when Re = 20000 the numerical and blasius solution show greater correlation near the front of the plate.*

4.6 Flow Rate

The flow rate at each distance x on the plate is plotted, along with the entrance flow rate. RMS errors are provided.

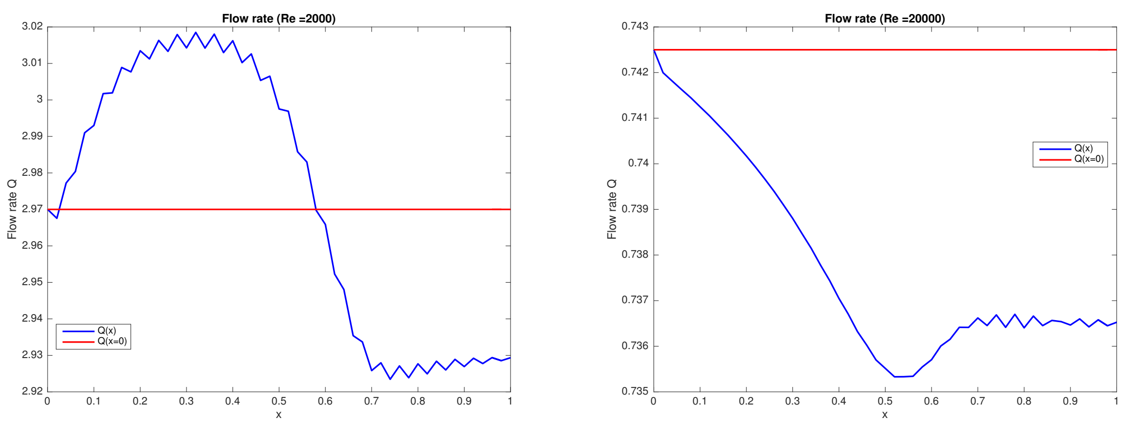

Figure 5: *At Re = 2000 the rms error is 2.97. At Re = 20000 the rms error is 0.74*