Financial Econometrics: A refresher class

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Outline

Consider the following model :

$$Y = X \beta + \varepsilon$$

where X is a non random matrix of full rank, $E(\varepsilon) = 0$ and

$$E(\varepsilon\varepsilon') = \begin{pmatrix} V(\varepsilon_1) & Cov(\varepsilon_1, \varepsilon_2) & \dots & Cov(\varepsilon_1, \varepsilon_T) \\ Cov(\varepsilon_2, \varepsilon_1) & V(\varepsilon_2) & \dots & Cov(\varepsilon_2, \varepsilon_T) \\ \vdots & \vdots & \ddots & \vdots \\ Cov(\varepsilon_T, \varepsilon_1) & Cov(\varepsilon_2, \varepsilon_T) & \dots & V(\varepsilon_T) \end{pmatrix} = \Omega_\varepsilon \qquad (51)$$

If $\Omega_\varepsilon \neq \sigma_\varepsilon^2 \mathbf{I}$ the errors are autocorrelated and/or heteroskedastics.

Consequences on the OLS estimators

In the presence of autocorrelated and/or heteroskedastics errors :

- The OLS estimator is :

$$\hat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$$

  and as a consequence, $\hat{\beta}$ is still an unbiased estimator of β : $E(\hat{\beta}) = \beta$.

- The variance of the estimator is :

$$\begin{aligned} \Omega_{\hat{\beta}} &= E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right] \\ &\neq \sigma_{\varepsilon}^2 (X'X)^{-1} \end{aligned}$$

- The OLS estimator is not the best estimator and the standard deviation are biased upward.

In the presence of autocorrelated and/or heteroskedastics errors, we can correct the errors before applying the OLS.

- Consider the multiple regression model

$$Y = X\beta + \varepsilon \tag{52}$$

- Multiplying by the non singular matrix M of size (T, T), we get :

$$MY = MX\beta + M\varepsilon \tag{53}$$

- The variance-covariance matrix of the error term of this model is

$$E(M\varepsilon\varepsilon' M') =$$

Imposing $\Omega_\varepsilon = \sigma_\varepsilon^2 \Gamma_\varepsilon$, we get

$$E(M\varepsilon\varepsilon' M') =$$

- If $\exists M$ s.t. $M\Gamma_\varepsilon M' = \mathbf{I}$, the OLS estimators of eq. (53) are BLUE.

- The OLS estimator of eq. (53) is :

$$\hat{\beta}_{GLS} = \tag{54}$$

- $\hat{\beta}_{GLS}$ is the Generalized Least Square estimator (GLS) of eq. (52).
- The variance of the estimator is :

$$\Omega_{\hat{\beta}} =$$

$$\Omega_{\hat{\beta}} = (X'\Omega_{\varepsilon}^{-1}X)^{-1} \tag{55}$$

- The GLS estimator is an OLS estimator of eq. (53).

- The estimator of the variance of the errors is

$$\tilde{\sigma}_\varepsilon^2 = \frac{e'e}{T-k-1}$$

where $e = MY - MX\hat{\beta}_{GLS}$

$$\tilde{\sigma}_\varepsilon^2 \quad =$$

GLS estimators

In the presence of autocorrelated and/or heteroskedastics errors, applying the GLS to $Y = X\beta + \varepsilon$, we get

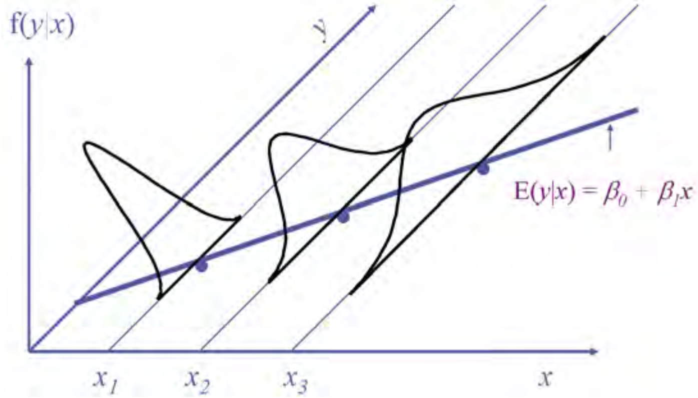
- $\hat{\beta}_{GLS} = (X'\Omega_\varepsilon^{-1}X)^{-1}X'\Omega_\varepsilon^{-1}Y$
- $V(\hat{\beta}_{GLS}) = \Omega_{\hat{\beta}} = (X'\Omega_\varepsilon^{-1}X)^{-1}$
- $\tilde{\sigma}_\varepsilon^2 = \frac{(Y-X\hat{\beta}_{GLS})'\Gamma_\varepsilon^{-\mathbf{1}}(Y-X\hat{\beta}_{GLS})}{T-k-\mathbf{1}}$

The errors are heteroskedastics when

$$V(\varepsilon_t) = \sigma^2_{\varepsilon_t} \neq \sigma^2_{\varepsilon}$$

- Heteroskedasticity is mainly a problem in modelling in cross-section data.
- But can also exist in times series.
- Both theoretical and empirical reasons.
- Sources of Heteroskedasticity :
  - Sample heterogeneity : Mixing bluechips stocks with unfrequently traded stocks ;
  - Omitted explanatory variable ;
  - Asymmetry of the distribution of some explanatory variables ;
  - Wrong data preliminary treatments ;
  - Wrong model (random coefficient model) ;
  - Nature of the data.

Figure : Heteroskedasticity



Looking/searching for heteroskedasticity in the errors, we can do

- **Graphical inspection :** OLS estimation of the model + scatter plot of e_t^2 against $\hat{y}_t$.
- **Statistical tests :** Goldfeld and Quandt, Breusch-Pagan, White, ARCH. Based on that hypothesis of homoskedasticity $H_0 : \sigma^2_{\varepsilon_1} = \sigma^2_{\varepsilon_2} = \ldots = \sigma^2_{\varepsilon_T} = \sigma^2_{\varepsilon}$

Looking for heteroskedasticity II

Figure : Heteroskedasticity

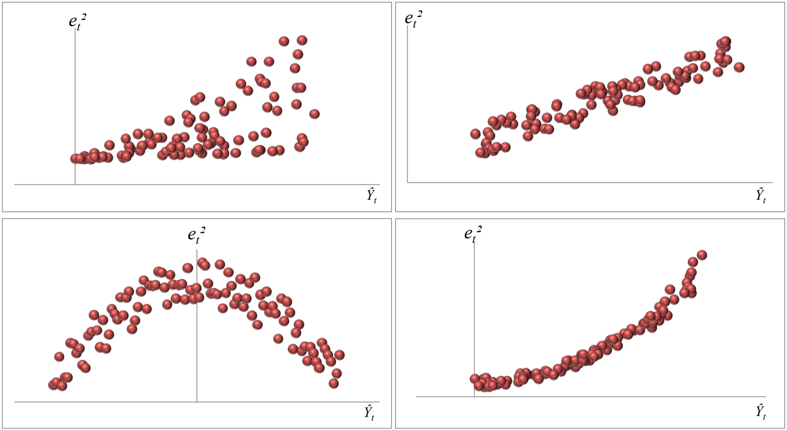


Figure : Homoskedasticity

Figure : Heteroskedasticity/Homoskedasticity

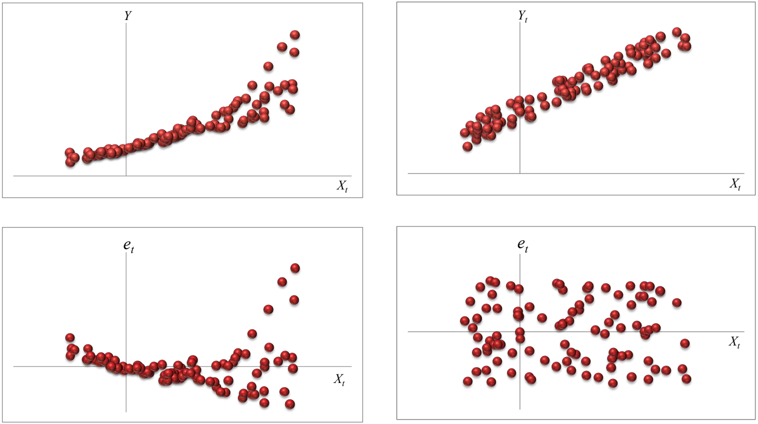


Figure : Heteroskedasticity/Homoskedasticity

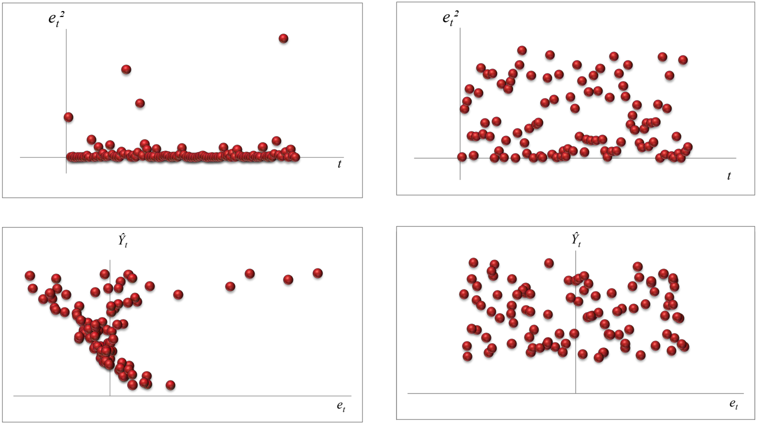
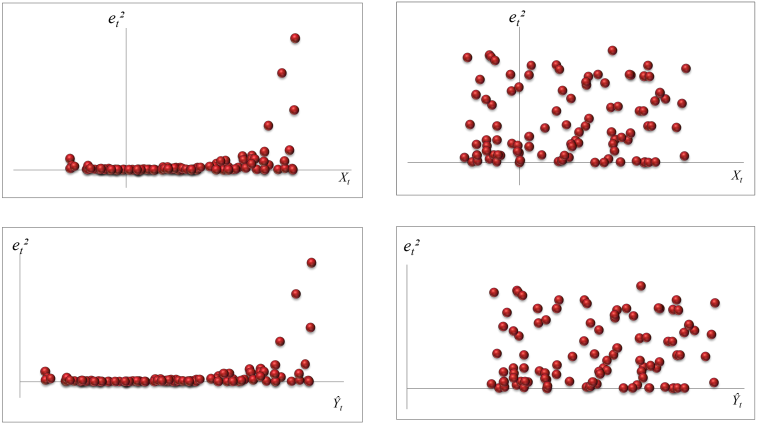


Figure : Heteroskedasticity/Homoskedasticity



This test is used when one of the regressors is considered the proportionality factor of heteroskedasticity. It is based on the hypothesis that the variance of the errors is related to a regressor, say X.

1. The observations on Y and X are sorted in ascending order of the regressor X.
2. The total sample is divided into three sub-samples. Omitting the central one with m observations, we are left with two sub-samples of length $T_1 = (T-m)/2$ and $T_2 = (T-m)/2$.
3. Running the regression on the two sub-samples, we get the two residuals variances as
$$S_1^2 = \frac{e_1' e_1}{T_1 - k - 1}, \ S_2^2 = \frac{e_2' e_2}{T_2 - k - 1}.$$
4. The null hypothesis is that the two variances of the disturbances are equal $H_0 : \sigma_1^2 = \sigma_2^2$ against the alternative hypothesis $H_a : \sigma_1^2 \neq \sigma_2^2$.
5. The statistics
$$\begin{aligned} GQ = \frac{S_2^2}{S_1^2} = \frac{ESS_2}{ESS_1} \quad &\sim_{H_0} \quad F(T_1 - k - 1, T_2 - k - 1) \\ &= \quad F\left(\frac{T - m - 2(k+1)}{2}, \frac{T - m - 2(k+1)}{2}\right) \end{aligned}$$

6. The decision rule is :
    - If GQ is greater than the critical value, at a chosen significance level, we reject the null hypothesis of homoskedasticity.
    - If GQ is lower than the critical value, at a chosen significance level, we accept the null hypothesis of homoskedasticity.

The limits of this test are :

White test

Based on the comparison of the variances of the estimated residuals when the model is estimated under the null of homoskedasticity and under the alternative hypothesis of heteroskedasticity (the form of the heteroskedasticity is not specified). This test does not need the error to be normally distributed.

1. Consider the model
$$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_{1t}$$
and get the estimated residuals $e_{1t} = \hat{\varepsilon}_{1t}$.
2. Run the auxiliary regression
$$e_{1t}^2 = \gamma_0 + \gamma_1 x_{1t} + \gamma_2 x_{2t} + \gamma_3 x_{1t}^2 + \gamma_4 x_{2t}^2 + \gamma_5 x_{1t} x_{2t} + \varepsilon_{2t}$$
3. We test the null hypothesis of homoskedasticity : $H_0 : \gamma_1 = \ldots = \gamma_5 = 0$
4. From R^2 of the above regression, we get that under the null, the statistics TR^2 is asymptotically distributed as a chi-square $\chi^2(p)$, with p being the number of tested parameters. Here $p = 5$.
5. If $TR^2 > \chi^2_\alpha(p)$ the homoskedasticity hypothesis is rejected at the confidence level α.

The limits of this test are :

General test that cover a lot of different forms of heteroskedasticity. It is an asymptotic test and, as such, can be run only on large samples.

- Consider the model

$$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + ... + \beta_k x_{kt} + \varepsilon_t \tag{56}$$

  where the residuals are normally distributed with variance $\sigma_{\varepsilon_t} = f(a_0 + a_1 Z_{1t} + \ldots + a_p Z_{pt})$. f can be any function, the coefficients a_i are not linked to the regression coefficients and $Z_{1t}, \ldots, Z_{pt}$ are possible sources of heteroskedasticity.

- The null hypothesis of homoskedasticity is $H_0 : a_1 = a_2 = \ldots = a_p = 0$ since in that case, $\sigma_{\varepsilon_t} = f(a_0)$ is constant.
- The test procedure is :
  1. Estimate the regression (56) using OLS and get the estimated residuals, $e_t, t = 1, \ldots, T$
  2. Compute the (ML) estimated variance of disturbances $\tilde{\sigma}^2_{ML} = \frac{1}{T}\sum_{t=1}^{T} e_t^2$.
  3. Regress the variable $h_t = \frac{e_t^2}{\tilde{\sigma}^2_{MV}}$ on $Z_{1t}, \ldots, Z_{pt}$ by OLS and compute the RSS.
  4. Under the null hypothesis of homoskedasticity,
  $$BP = \frac{1}{2} RSS \sim \chi^2_p.$$
  5. If $BP > \chi^2_p$, reject the null hypothesis of homoskedasticity.

The limits of this test are :

Exercise 12 : Heteroskedasticity tests comparison

From what you have seen in class and based on your own research, which test will you favor and when ?

1. Heteroskedasticity-consistent standard errors have been popularized by White (HCCME [2])

- The OLS estimator is : $\hat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$ and, $\hat{\beta}$ is still an unbiased estimator of β : $E(\hat{\beta}) = \beta$.
- The variance of this estimator is :$\Omega_{\hat{\beta}}(X'X)^{-1}X'\Omega_\varepsilon X(X'X)^{-1} \neq \sigma^2_\varepsilon(X'X)^{-1}$
- White shows that an estimator of that matrix can be optained by replacing Ω_ε by a diagonal matrix of squared errors :

$$\hat{\Omega}_{\hat{\beta}} = (X'X)^{-1}X'\hat{\Omega}_d X(X'X)^{-1},$$

$$\text{where} \quad \hat{\Omega}_d = \begin{pmatrix} \hat{\varepsilon}_1^2 & 0 & \dots & 0 \\ 0 & \hat{\varepsilon}_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{\varepsilon}_n^2 \end{pmatrix} = diag\{\hat{\varepsilon}_1^2, \hat{\varepsilon}_2^2, \dots, \hat{\varepsilon}_n^2\}$$

This estimator is robust to heteroscedasticity but is not BLUE anymore. The best unbiased estimator is that of the Generalized Least Squares (GLS).

2. The Generalized Least Squares (GLS)

- We do have a regression model (52) as before $Y = X\beta + \varepsilon$ where the variance of the error is $V(\varepsilon) = \Omega_\varepsilon$.
- The estimator of the coefficients is $\hat{\beta}_{GLS} = (X'\Omega_\varepsilon^{-1}X)^{-1}X'\Omega_\varepsilon^{-1}Y$
- The estimator of the variance of the coefficient is $V(\hat{\beta}) = (X'\Omega_\varepsilon^{-1}X)^{-1}$
- The GLS estimator is BLUE and if $\Omega_\varepsilon = \sigma^2\mathbf{I}$, we get back the OLS estimator.
- The GLS are a generalized version of the OLS.
- The criterium to minimize is : $\underset{\beta}{Min}(y - X\beta)'\Omega_\varepsilon^{-1}(y - X\beta)'$
- We associate the larger weights to the lower variances of y.
  - If $\Omega_\varepsilon = \sigma_\varepsilon^2\mathbf{I}$, we minimize the ESS, and we get back to the OLS minimization problem.
  - If $\Omega_\varepsilon = \Omega_d = \begin{pmatrix} \sigma_{\varepsilon_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\varepsilon_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{\varepsilon_T}^2 \end{pmatrix}$, we minimize the sum of the squared residuals weighted by Ω_d^{-1}, i.e. by $\frac{1}{\sigma_{u_i}^2}$ and the GLS are called weighted least squares (WLS).
    This is equivalent to applying the OLS on the following tranformed equation :

$$\frac{y_i}{\sigma_{\varepsilon_i}} = \frac{X_i}{\sigma_{\varepsilon_i}}\beta + \frac{\varepsilon_i}{\sigma_{\varepsilon_i}}, \quad \frac{\varepsilon_i}{\sigma_{\varepsilon_i}} \sim IID(0,1) \qquad (57)$$

Exercise 13 : Dealing with heteroskedasticity

In the general model such that

$$Y = X\beta + \varepsilon,$$

the GLS method consists in applying the OLS method to a modified model. In each of the following cases, give the M matrix (of the matrix representation) and the linear modified model where to apply the OLS method to.

1. *If* $var(\varepsilon_t) = \sigma^2_{\varepsilon,t}$ *;*
2. *If* $var(\varepsilon_t) = \sigma^2_{\varepsilon} x^2_{2,t}$ *;*

Exercise 14 : GLS estimators

Rewrite model (57) by setting $Y^* = X^*\beta + \varepsilon^*$.

1. *What do we have in* Y^*, X^*, ε^* *? Why do we get the same* β *as in equation 52.*
2. *Show that the OLS estimators of 57 are the same as the GLS estimators of 52.*

In practice, we do not know Ω and we can only apply the GLS if we manage to get a consistent estimator of Ω.
Dealing with heteroskedasticity under Gretl or R.

- The White correction method is available on any econometric package. Under Gretl, in the `Tools` Menu, choose the `Preferences` menu and pick `General`. Under the spreadsheet `HCCME`, check the box `"Use robust covariance matrix by default"` and choose your correction method. Example : HC0 for the White correction. There are several possibility to perform a White correction. See Gretl's manual for details.
  Under R, you can use the `car` and `lmtest` or `sandwich` libraries and specify in the coeftest command the type of correction in `vcov`, like hccm.
- Others alternatives : Weighted least square - Choose `Model`, `Other linear models` and then `Weighted least square` or Choose `Other linear models` and `Heteroskedasticity corrected` under Gretl or use lm in R and specify the weights with the option `weight=`.

2. HCCME : Heteroskedasticity Consistent Covariance Matrix Estimator

The errors are autocorrelated when

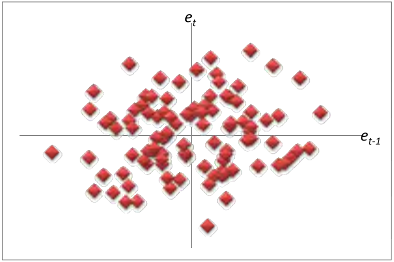
$$E(\varepsilon_{t'}\varepsilon_t) \neq 0.$$

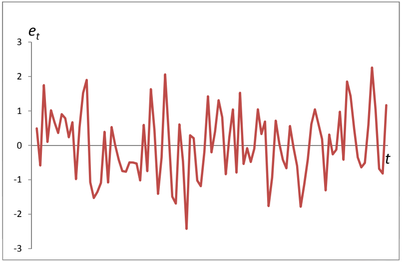
- Times series variables are rarely time independent.
- OLS estimators are unbiased but non efficient and the OLS estimator of the variance-covariance matrix of the parameters is biased $\Longrightarrow$ Tests and confidence intervals are wrong. The errors are non independent and contain information that could improve predictions.
- Sources of autocorrelation :
  - Omitted explanatory variable ;
  - Model misspecification (non linear model with a linear specification) ;
  - Asymmetry of the distribution of some explanatory variables ;
  - Wrong data preliminary treatments (smoothing, sampling ...) ;
  - Nature of the data.
  - Misspecification of the true random error (change in regime, link with some "long lasting" effect factor.)

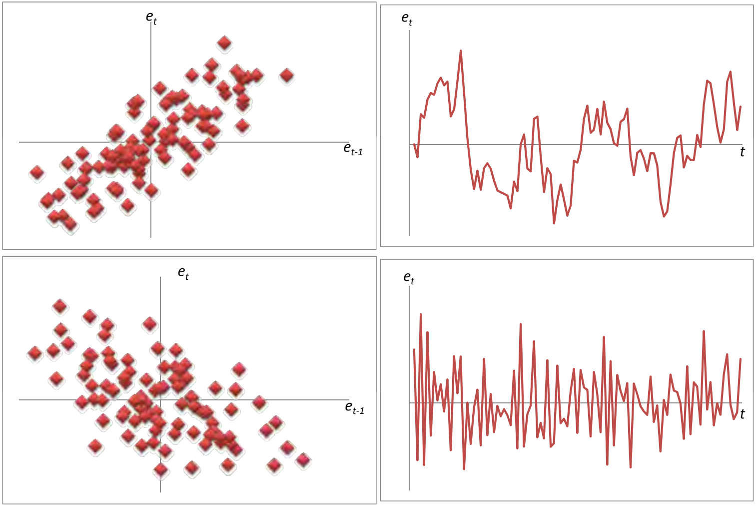
Looking/searching for autocorrelation in the errors, you can do

- **Graphical inspection :** OLS estimation of the model + plot over time of e_t and scatter plot of e_t against e_{t-1}.
- **Statistical tests :** Durbin-Watson test, Breusch-Godfrey test. Based on that hypothesis of autocorrelation $H_0 : Cov(\varepsilon_t, \varepsilon_{t-1}) = Cov(\varepsilon_t, \varepsilon_{t-2}) = \ldots = Cov(\varepsilon_t, \varepsilon_{t-r}) = 0$

Figure : No serial correlation





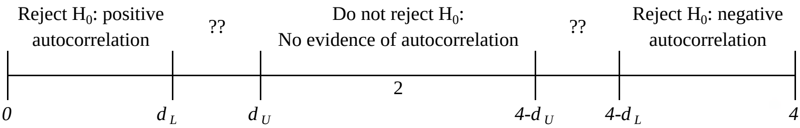

e_t
e_{t-1}
e_t
t
e_t
e_{t-1}
e_t
t

Durbin Watson test I

- Durbin-Watson (DW) is a test of the first order autocorrelation.
  1. Consider the error $\varepsilon_t = \rho\varepsilon_{t-1} + \nu_t$, where $\nu \sim N(0, \sigma_\nu^2)$.
  2. The hypotheses of this test are : $H_0 : \rho = 0$ and $H_a : \rho \neq 0$.
  3. The Durbin-Watson of autocorrelation of order 1, is :

$$DW = \frac{\sum_{t=2}^{T} (e_t - e_{t-1})^2}{\sum_{t=2}^{T} e_t^2} \approx 2 - 2\frac{\sum_{t=2}^{T} e_t e_{t-1}}{\sum_{t=2}^{T} e_t^2} = 2\,(1 - \hat{\rho}) \tag{58}$$

Figure : Rejection regions for DW test



Durbin Watson test II

DW test validity conditions

- There must be a constant term in the regression ;
- The regressors must be non stochastic ;
- There must be no lags of the dependent variable.

Figure : Durbin-Watson significance table. Testing $\rho = 0$ against $\rho > 0$, at 5 %

TABLE de DURBIN-WATSON : Test unilatéral de ρ = 0 contre ρ > 0, au seuil de 5% (test bilatéral seuil α = 10%)

	k' = 1		k' = 2		k' = 3		k' = 4		k' = 5		k' = 6		k' = 7		k' = 8		k' = 9		k' = 10	
n	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u	d_L	d_u
15	1,08	1,36	0,95	1,54	0,82	1,75	0,69	1,97	0,56	2,21	0,45	2,47	0,34	2,73	0,25	2,98	0,17	3,22	0,11	3,44
16	1,10	1,37	0,98	1,54	0,86	1,73	0,74	1,93	0,62	2,15	0,50	2,40	0,40	2,62	0,30	2,86	0,22	3,09	0,15	3,30
17	1,13	1,38	1,02	1,54	0,90	1,71	0,78	1,90	0,67	2,10	0,55	2,32	0,45	2,54	0,36	2,76	0,27	2,97	0,20	3,20
18	1,16	1,39	1,05	1,53	0,93	1,69	0,82	1,87	0,71	2,06	0,60	2,26	0,50	2,46	0,41	2,67	0,32	2,87	0,24	3,07
19	1,18	1,40	1,08	1,53	0,97	1,68	0,86	1,85	0,75	2,02	0,65	2,21	0,46	2,40	0,46	2,59	0,37	2,78	0,29	2,97
20	1,20	1,41	1,10	1,54	1,00	1,68	0,90	1,83	0,79	1,99	0,69	2,16	0,60	2,34	0,50	2,52	0,42	2,70	0,34	2,88
21	1,22	1,42	1,13	1,54	1,03	1,67	0,93	1,81	0,83	1,96	0,73	2,12	0,64	2,29	0,55	2,46	0,46	2,63	0,38	2,81
22	1,24	1,43	1,15	1,54	1,05	1,66	0,96	1,80	0,86	1,94	0,77	2,09	0,68	2,25	0,59	2,41	0,50	2,57	0,42	2,73
23	1,26	1,44	1,17	1,54	1,08	1,66	0,99	1,79	0,90	1,92	0,80	2,06	0,71	2,21	0,63	2,36	0,54	2,51	0,46	2,67
24	1,27	1,45	1,19	1,55	1,10	1,66	1,01	1,78	0,93	1,90	0,84	2,03	0,75	2,17	0,67	2,32	0,58	2,46	0,51	2,61
25	1,29	1,45	1,21	1,55	1,12	1,66	1,04	1,77	0,95	1,89	0,87	2,01	0,78	2,14	0,70	2,28	0,62	2,42	0,54	2,56
26	1,30	1,46	1,22	1,55	1,14	1,65	1,06	1,76	0,98	1,88	0,90	1,99	0,82	2,12	0,73	2,25	0,66	2,38	0,58	2,51
27	1,32	1,47	1,24	1,56	1,16	1,65	1,08	1,76	1,01	1,86	0,92	1,97	0,84	2,09	0,77	2,22	0,69	2,34	0,62	2,47
28	1,33	1,48	1,26	1,56	1,18	1,65	1,10	1,75	1,03	1,85	0,95	1,96	0,87	2,07	0,80	2,19	0,72	2,31	0,65	2,43
29	1,34	1,48	1,27	1,56	1,20	1,65	1,12	1,74	1,05	1,84	0,97	1,94	0,90	2,05	0,83	2,16	0,75	2,28	0,68	2,40
30	1,35	1,49	1,28	1,57	1,21	1,65	1,14	1,74	1,07	1,83	1,00	1,93	0,93	2,03	0,85	2,14	0,78	2,25	0,71	2,36
31	1,36	1,50	1,30	1,57	1,23	1,65	1,16	1,74	1,09	1,83	1,02	1,92	0,95	2,02	0,88	2,12	0,81	2,23	0,74	2,33
32	1,37	1,50	1,31	1,57	1,24	1,65	1,18	1,73	1,11	1,82	1,04	1,91	0,97	2,00	0,90	2,10	0,84	2,20	0,77	2,31
33	1,38	1,51	1,32	1,58	1,26	1,65	1,19	1,73	1,13	1,81	1,06	1,90	0,99	1,99	0,93	2,08	0,86	2,18	0,79	2,28
34	1,39	1,51	1,33	1,58	1,27	1,65	1,21	1,73	1,15	1,81	1,08	1,89	1,01	1,98	0,95	2,07	0,88	2,16	0,82	2,26
35	1,40	1,52	1,34	1,58	1,28	1,65	1,22	1,73	1,16	1,80	1,10	1,88	1,03	1,97	0,97	2,05	0,91	2,14	0,84	2,24
36	1,41	1,52	1,35	1,59	1,29	1,65	1,24	1,73	1,18	1,80	1,11	1,88	1,05	1,96	0,99	2,04	0,93	2,13	0,87	2,22
37	1,42	1,53	1,36	1,59	1,31	1,66	1,25	1,72	1,19	1,80	1,13	1,87	1,07	1,95	1,01	2,03	0,95	2,11	0,89	2,20
38	1,43	1,54	1,37	1,59	1,32	1,66	1,26	1,72	1,21	1,79	1,15	1,86	1,09	1,94	1,03	2,02	0,97	2,10	0,91	2,18
39	1,43	1,54	1,38	1,60	1,33	1,66	1,27	1,72	1,22	1,79	1,16	1,86	1,10	1,93	1,05	2,01	0,99	2,08	0,93	2,16
40	1,44	1,54	1,39	1,60	1,34	1,66	1,29	1,72	1,23	1,79	1,17	1,85	1,12	1,92	1,06	2,00	1,01	2,07	0,95	2,14
45	1,48	1,57	1,43	1,62	1,38	1,67	1,34	1,72	1,29	1,78	1,24	1,84	1,19	1,90	1,14	1,96	1,09	2,00	1,04	2,09
50	1,50	1,59	1,46	1,63	1,42	1,67	1,38	1,72	1,34	1,77	1,29	1,82	1,25	1,87	1,20	1,93	1,16	1,99	1,11	2,04
55	1,53	1,60	1,49	1,64	1,45	1,68	1,41	1,72	1,38	1,77	1,33	1,81	1,29	1,86	1,25	1,91	1,21	1,96	1,17	2,01
60	1,55	1,62	1,51	1,65	1,48	1,69	1,44	1,73	1,41	1,77	1,37	1,81	1,33	1,85	1,30	1,89	1,26	1,94	1,22	1,98
65	1,57	1,63	1,54	1,66	1,50	1,70	1,47	1,73	1,44	1,77	1,40	1,80	1,37	1,84	1,34	1,88	1,30	1,92	1,27	1,96
70	1,58	1,64	1,55	1,67	1,52	1,70	1,49	1,74	1,46	1,77	1,43	1,80	1,40	1,84	1,37	1,87	1,34	1,91	1,30	1,95
75	1,60	1,65	1,57	1,68	1,54	1,71	1,51	1,74	1,49	1,77	1,46	1,80	1,43	1,83	1,40	1,87	1,37	1,90	1,34	1,94
80	1,61	1,66	1,59	1,69	1,56	1,72	1,53	1,74	1,51	1,77	1,48	1,80	1,45	1,83	1,42	1,86	1,40	1,89	1,37	1,92
85	1,62	1,67	1,60	1,70	1,57	1,72	1,55	1,75	1,52	1,77	1,50	1,80	1,47	1,83	1,45	1,86	1,42	1,89	1,40	1,92
90	1,63	1,68	1,61	1,70	1,59	1,73	1,57	1,75	1,54	1,78	1,52	1,80	1,49	1,83	1,47	1,85	1,44	1,88	1,42	1,91
95	1,64	1,69	1,62	1,71	1,60	1,73	1,58	1,75	1,56	1,78	1,54	1,80	1,51	1,83	1,49	1,85	1,46	1,88	1,44	1,90
100	1,65	1,69	1,63	1,72	1,61	1,74	1,59	1,76	1,57	1,78	1,55	1,80	1,53	1,83	1,51	1,85	1,48	1,87	1,46	1,90
150	1,72	1,75	1,71	1,76	1,69	1,77	1,68	1,79	1,66	1,80	1,65	1,82	1,64	1,83	1,62	1,85	1,60	1,86	1,59	1,88
200	1,73	1,78	1,75	1,79	1,73	1,80	1,73	1,81	1,72	1,82	1,71	1,83	1,70	1,84	1,69	1,85	1,68	1,86	1,66	1,87

- Durbin-Watson (DW) is a test of the first order autocorrelation under some circumstances. The Breusch-Godfrey test is a more general test for autocorrelation up to order r.
- Consider the error $\varepsilon_t = \rho_1 \varepsilon_{t-1} + \rho_2 \varepsilon_{t-2} + \ldots + \rho_r \varepsilon_{t-r} + \nu_t$, where $\nu \sim N(0, \sigma^2_\nu)$.
- The hypotheses of this test are : H_0 : $\rho_1 = 0$ and $\rho_2 = 0$ and ... and $\rho_r = 0$ against H_a : $\rho_1 \neq 0$ or $\rho_2 \neq 0$ or ... or $\rho_r \neq 0$.
- To conduct a Breusch-Godfrey test :
  1. Consider the model
  $$y_t = \beta_0 + \beta_1 x_{1t} + \beta_2 x_{2t} + \varepsilon_t \tag{59}$$
  and obtain the residuals, $e_t = \hat{\varepsilon_t}$.
  2. Run the auxiliary regression
  $$e_t = \gamma_0 + \gamma_1 x_{1t} + \gamma_2 x_{2t} + \rho_1 e_{t-1} + \rho_2 e_{t-2} + \ldots + \rho_r e_{t-r} + \nu_t,$$
  $\nu_t \sim N(0, \sigma^2_\nu)$.
  3. From R^2 of the above regression, we get that under the null, the statistics $(T-r)R^2$ is asymptotically distributed as a chi-square $\chi^2(r)$.
  4. If $(T-r)R^2 > \chi^2_\alpha(r)$ the no serial correlation hypothesis is rejected at the confidence level α.

Exercise 15 : Autocorrelation tests comparison

From what you have seen in class and based on your own research, which test will you favor and when ?

1. Data pre-treatments

- Prewhitening the data.
- Respecifying the model and treating seasonality : using lagged depend variable, differencing the dependent variable, using seasonal explanatory variables.
- Correct autocorrelation : Variance correction (Robust covariance matrix : heteroskedastic and autocorrelation consistent - HAC), General Least Squares, ARIMA (Autoregressive Integrated Moving Average).

2. Cochrane–Orcutt method

Consider the model

$$y_t = \beta_0 + X_t\beta + \varepsilon_t, \tag{60}$$

where X_t is a matrix of explanatory variables and ε_t the error term is serially correlated over time :

$$\varepsilon_t = \rho\varepsilon_{t-1} + \nu_t, |\rho| < 1. \tag{61}$$

The Cochrane-Orcutt procedure is based on :

$$y_t - \rho y_{t-1} = \beta_0(1-\rho) + (X_t - \rho X_{t-1})\beta + \nu_t.$$

This equation can be rewritten in

$$y_t^* = \beta_0^* + X_t^*\beta + \nu_t, \tag{62}$$

with $y_t^* = y_t - \rho y_{t-1}$, $\beta_0^* = \beta_0(1-\rho)$, and $X_t^* = X_t - \rho X_{t-1}$.

This transformation is known as *Quasi-differencing*.

The iterative estimation procedure is the following :

1. Estimate equation (60) by OLS and compute the residuals $e_t = \hat{\varepsilon}_t$.
2. Estimate the first order autocorrelation coefficient $\hat{\rho}$ from equation (61).
3. Take the quasi-difference of the variables.
4. Estimate equation (62) by OLS.
5. Use the estimated coefficients to compute the new ε_t from equation (60) and go back to step 2.
6. Iterate until two consecutive estimated values of ρ remain the same.

3. Robust variance estimator of Newey and West (HAC)

The Newey and West estimator is an extention of the heteroscedasticity consistent estimator of White. Their estimator is not only consistent in the presence of heteroscedasticity but also in the presence of autocorrelation, hence its name of HAC for Heteroskedasticity and Autocorrelation Consistent.

- The OLS estimator is : $\hat{\beta} = \beta + (X'X)^{-1}X'\varepsilon$ and as a consequence, $\hat{\beta}$ is still an unbiased estimator of β : $E(\hat{\beta}) = \beta$.
- The variance of the estimator is :

$$\begin{aligned} \Omega_{\hat{\beta}} &= E\left[(\hat{\beta}-\beta)(\hat{\beta}-\beta)'\right] \\ &= E\left[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}\right] \\ &= (X'X)^{-1}X'\Omega_\varepsilon X(X'X)^{-1} \neq \sigma_\varepsilon^2(X'X)^{-1} \end{aligned}$$

- Newey and West show that we can get an estimator of that matrix replacing Ω_ε by a matrix which components are some functions of the residuals. This heteroskedasticity and autocorrelation consistent estimator is not BLUE and the best unbiased estimator is the GLS estimator.

4. The return of the Generalized Least Squares (GLS)

- We consider the regression model

$$Y = X\beta + \varepsilon \tag{63}$$

  where the variance is $V(\varepsilon) = \Omega_\varepsilon$.

- The vector of the estimated parameters is $\hat{\beta}_{GLS} = (X'\Omega_\varepsilon^{-1}X)^{-1}X'\Omega_\varepsilon^{-1}Y$
- The estimator of the variance-covariance matrix of the parameter is $V(\hat{\beta}) = (X'\Omega_\varepsilon^{-1}X)^{-1}$
- The GLS estimator is BLUE and if $\Omega_\varepsilon = \sigma^2\mathbf{I}$, we get back the OLS estimator.

- The GLS are a generalized version of the OLS.
- The criterium to minimize is : $\underset{\beta}{Min}(y - X\beta)'\Omega_{\varepsilon}^{-1}(y - X\beta)'$
- We associate the larger weights to the lower variances of y.
    - If $\Omega_{\varepsilon} = \sigma_{\varepsilon}^{2}\mathbf{I}$, we minimize the ESS, and we get back to the OLS minimization problem.
    - If $\Omega_{\varepsilon} = \Omega_{d} = \begin{pmatrix} \sigma^{2} & \sigma_{\varepsilon_{1},\varepsilon_{2}} & \cdots & \sigma_{\varepsilon_{1},\varepsilon_{T}} \\ \sigma_{\varepsilon_{1},\varepsilon_{2}} & \sigma^{2} & \cdots & \sigma_{\varepsilon_{2},\varepsilon_{T}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{\varepsilon_{1},\varepsilon_{T}} & \sigma_{\varepsilon_{2},\varepsilon_{T}} & \cdots & \sigma^{2} \end{pmatrix}$.

Once again, Ω is unknown and we can only apply the GLS only if we have a convergent estimator of Ω. The Cochrane−Orcutt procedure is a way to get a consistent estimator of Ω (feasible GLS).

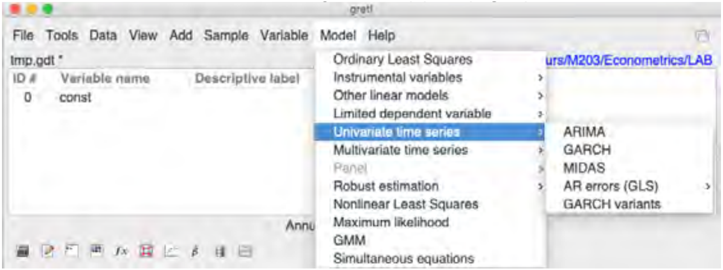
Exercise 16 : Cocharne-Orcutt method

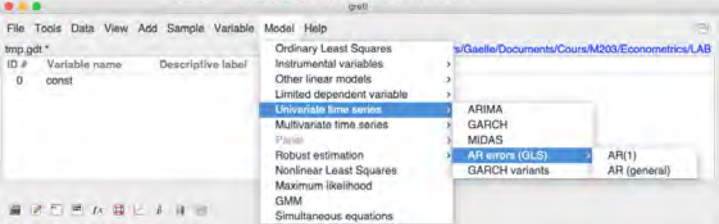
What should be Ω in the particular case of the Cochrane−Orcutt method ? Follow the steps to answer that question :

1. *Write the AR(1) model of equation 61 as a function of lag values of ν_t.*
2. *Use the properties of ν_t to get $V(\varepsilon_t)$.*
3. *Compute $Cov(\varepsilon_t, \varepsilon_{t-1})$.*
4. *Deduce $Cov(\varepsilon_t, \varepsilon_{t-j})$.*
5. *Finally, deduce Ω.*

Several possibilities with Gretl and R.

- To get robust standard errors under Gretl, in the `Tools` Menu, choose the `Preferences` menu and select `General`. Under the spreadsheet `HCCME`, check the box `"Use robust covariance matrix by default"` and choose HAC for heteroskedasticity and autocorrelation consistent standard errors. See Gretl's manual for details.
  Under R, you can use the `car` and `lmtest` or `sandwich` libraries and specify in the coeftest command the type of correction in `vcov`, like HAC1.
- To use time series models, choose "Model", "Univariate Time Series" and the appropriate model, under Gretl. The iterated Cochrane-Orcutt method is in the `AR1 menu` > `AR errors (GLS)` or in command ar1 y const x In R, the "orcutt" package is available then specify `cochrane.orcutt(modelname)`. The `garch` command is avaible in the ts library...


gretl
File Tools Data View Add Sample Variable Model Help
tmp.gdt *
ID # Variable name Descriptive label
0 const
urs/M203/Econometrics/LAB
Ordinary Least Squares
Instrumental variables
Other linear models
Limited dependent variable
Univariate time series
Multivariate time series
Panel
Robust estimation
Nonlinear Least Squares
Maximum likelihood
GMM
Simultaneous equations
ARIMA
GARCH
MIDAS
AR errors (GLS)
GARCH variants


gretl
File Tools Data View Add Sample Variable Model Help
tmp.gdt *
ID # Variable name Descriptive label
0 const
s/Gaelle/Documents/Cours/M203/Econometrics/LAB
Ordinary Least Squares
Instrumental variables
Other linear models
Limited dependent variable
Univariate time series
Multivariate time series
Panel
Robust estimation
Nonlinear Least Squares
Maximum likelihood
GMM
Simultaneous equations
ARIMA
GARCH
MIDAS
AR errors (GLS)
GARCH variants
AR(1)
AR (general)

Outline

- Run the APT regression.
```
Model APT <- ols ermsoft const ersandp dprod dcredit dinflation
dmoney dspread dterm
```
- Obtain the estimated residuals and the explained endogenous variable.
```
genr e = $uhat
genr yhat = $yhat
```
- Calculate e_t^2 and plot e_t^2 against $\hat{y}_t$, e_t^2 against time and $\hat{y}_t$ against e_t

Figure : Graphs with Gretl

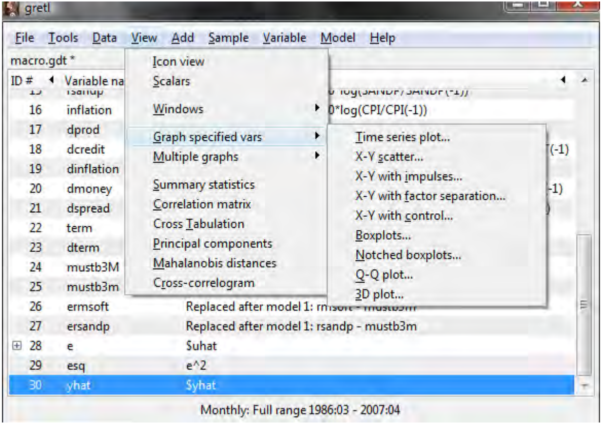


Figure : Visual inspection

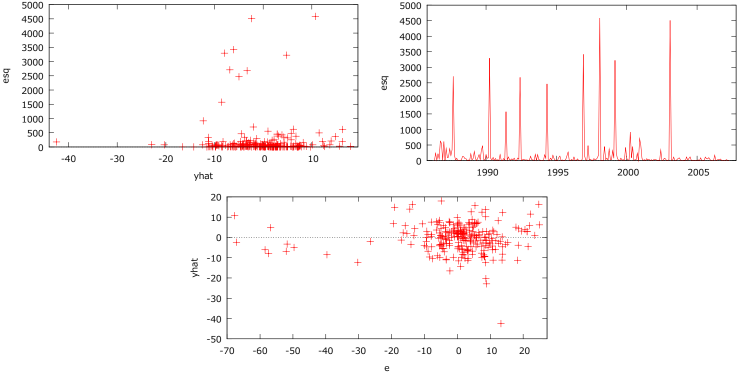


Figure : Heteroskedasticity tests with Gretl

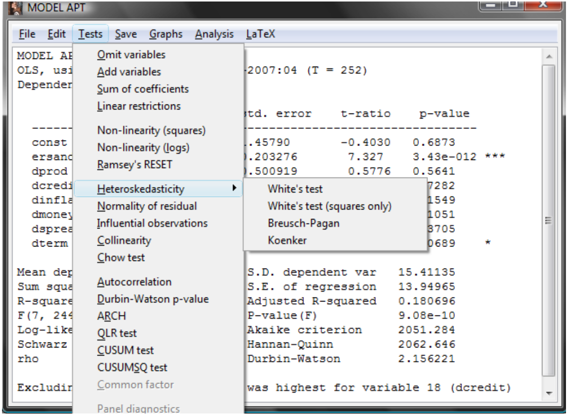


Figure : Heteroskedasticity tests results

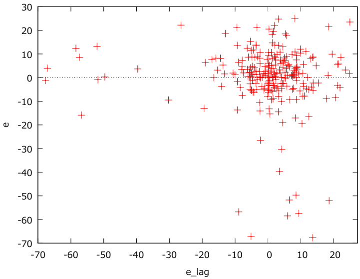
MODEL APT

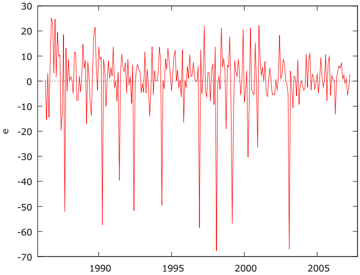
File Edit Tests Save Graphs Analysis LaTeX

```
White's test for heteroskedasticity -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 15.2806
  with p-value = P(Chi-Square(35) > 15.2806) = 0.998477

White's test for heteroskedasticity (squares only) -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 7.41525
  with p-value = P(Chi-Square(14) > 7.41525) = 0.917517

Breusch-Pagan test for heteroskedasticity -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 17.9756
  with p-value = P(Chi-Square(7) > 17.9756) = 0.0120807

Breusch-Pagan test for heteroskedasticity (robust variant) -
  Null hypothesis: heteroskedasticity not present
  Test statistic: LM = 3.41157
  with p-value = P(Chi-Square(7) > 3.41157) = 0.844501
```

- Run the APT regression.
  ```
  Model APT <- ols ermsoft const ersandp dprod dcredit dinflation
  dmoney dspread dterm
  ```
- Obtain the estimated residuals.
  ```
  genr e $uhat
  ```
- Calculate e_{t-1} and plot e_t against e_{t-1}.

Figure : Visual inspection





- The Durbin-Watson's test gives :
```
Durbin-Watson statistic = 2.15622
p-value = 0.887608
```
- The Breusch-Godfrey's test gives :
```
LM test for autocorrelation up to order 12 -
Null hypothesis: no autocorrelation
Test statistic: LMF = 1.53681
with p-value = P(F(12,232) > 1.53681) = 0.111992
```