Finite Difference Methods

Kaiza Amouh

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Outline

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Principle of Finite Difference

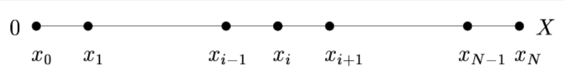
- A Partial Differential Equation (PDE)for a multivariate function $u(x1, ..., x_n)$ is an equation of the form

$$f\left(x_1, ..., x_N; u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}; \frac{\partial^2 u}{\partial x_1 \partial x_1}, ..., \frac{\partial^2 u}{\partial x_1 \partial x_n}; ...\right) = 0$$

- Derivatives of u are approximated by linear combinations of the values of u at some points on a discretization grid

Example of a $1D$ grid on a domain $\Omega = (0, X)$

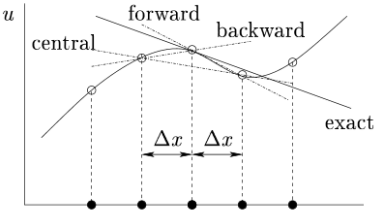
$i = 0, 1, ..., N$ Mesh size $= \Delta x := \frac{X}{N}$ $x_i := i\Delta x$ $u_i := u\left(x_i\right)$



Approximation of 1^{st} order derivatives

The theoretical 1^{st} order derivative is defined by three equivalent expressions

$$\frac{\partial u}{\partial x}(x_i) = \begin{cases} \lim\limits_{\Delta x \to 0} \frac{u(x_i+\Delta x)-u(x_i)}{\Delta x} \\ \lim\limits_{\Delta x \to 0} \frac{u(x_i)-u(x_i-\Delta x)}{\Delta x} \\ \lim\limits_{\Delta x \to 0} \frac{u(x_i+\Delta x)-u(x_i-\Delta x)}{2\Delta x} \end{cases} \tag{1}$$



$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_i}{\Delta x} \quad \textit{forward difference}$$

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_i - u_{i-1}}{\Delta x} \quad \textit{backward difference}$$

$$\left(\frac{\partial u}{\partial x}\right)_i \approx \frac{u_{i+1} - u_{i-1}}{2\Delta x} \quad \textit{forward difference}$$

Approximation of 2^{nd} order derivatives

The second order discrete approximation can be obtained by two consecutive applications of 1^{st} order discretization.

$$
\begin{aligned}
\left(\frac{\partial^2 u}{\partial x^2}\right)_i &= \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]_i \\
&= \lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{i+1/2} - \left(\frac{\partial u}{\partial x}\right)_{i-1/2}}{\Delta x} \\
&\approx \frac{\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} \\
&= \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}
\end{aligned}
$$

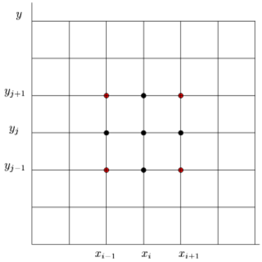
Mixed 2^{nd} order derivatives in 2D

Let $u : (x, y) \longmapsto u(x, y)$. We have :

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{\left(\frac{\partial u}{\partial y}\right)_{i+1,j} - \left(\frac{\partial u}{\partial y}\right)_{i-1,j}}{2\Delta x} + \mathcal{O}\left(\Delta x\right)^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{i+1,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2\Delta y} + \mathcal{O}\left(\Delta y\right)^2$$

$$\left(\frac{\partial u}{\partial y}\right)_{i-1,j} = \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2\Delta y} + \mathcal{O}\left(\Delta y\right)^2$$



2^{nd} order difference approximation

$$\left(\frac{\partial^2 u}{\partial x \partial y}\right)_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4\Delta x \Delta y} + \mathcal{O}\left[(\Delta x)^2, (\Delta y)^2\right]$$

PDEs arising in Finance

- Most one dimensional PDEs in Finance can be expressed in the following generic format:

$$\begin{cases} V_t + b(x,t)V_x + \frac{1}{2}a(x,t)V_{xx} - r(x,t)V + f(x,t) = 0 \\ V(x,T) = g(x), \qquad (x,t) \in \mathbb{R}x[0,T] \end{cases} \tag{2}$$

- f is called the source term, and may represent some revenue (dividends)
- x is the *space variable* and t the *time variable*
- We assume that the functions a, b and r are *continuous* and *bounded*, and that a is *nonnegative*

Pure diffusion equation

- The equation (2) becomes a pure diffusion equation when b and r are identically zero

$$\begin{cases} V_t + \frac{1}{2} a(x,t) V_{xx} + f(x,t) & = 0, \qquad (x,t) \in \mathbb{R}x[0,T] \\ V(x,T) & = g(x), \quad x \in \mathbb{R} \end{cases}$$

- Let the time step be $h_0 = T/N$, where N is a positive integer, and the space step be $h_1 > 0$
- For $j \in \mathbb{Z}$ and $k = 0$ to N, we denote

$$v_j^k := V\left(jh_1, kh_0\right), \quad a_j^k := a\left(jh_1, kh_0\right)$$

$$f_j^k := f\left(jh_1, kh_0\right), \quad g_j := g\left(jh_1\right)$$

Explicit Scheme for pure diffusion

The *Explicit* scheme of the pure diffusion equation is

$$\begin{cases} \frac{v_j^k - v_j^{k-1}}{h_0} + \frac{1}{2} a_j^k \frac{v_{j+1}^k + v_{j-1}^k - 2v_j^k}{h_1^2} + f_j^k = 0, & j \in \mathbb{Z},\ k = 1, ..., N \\ v_j^N = g_j, & j \in \mathbb{Z} \end{cases}$$

The *ordered form* (w.r.t. to the components of v^k) of the scheme is

$$v_j^{k-1} = \left(1 - \frac{h_0}{h_1^2} a_j^k\right) v_j^k + \frac{1}{2} \frac{h_0}{h_1^2} a_j^k \left(v_{j-1}^k + v_{j+1}^k\right) + h_0 f_j^k$$

- The r.h.s. is, apart from the source term $h_0 f_j^k$, a linear combination of the values at the previous step
- The coefficients have a sum equal to 1. They are *nonnegative* if the following *monotonicity condition* holds:

$$\frac{h_0}{h_1^2} ||a||_\infty \leq 1$$

Implicit Scheme for pure diffusion

The standard *Implicit Scheme* for the pure diffusion equation is obtained by replacing the increment $(v_j^k - v_j^{k-1})$ in the *Explicit Scheme* by : $(v_j^{k+1} - v_j^k)$

$$\begin{cases} \frac{v_j^{k+1} - v_j^k}{h_0} + \frac{1}{2} a_j^k \frac{v_{j+1}^k + v_{j-1}^k - 2v_j^k}{h_1^2} + f_j^k & = 0, \;\; j \in \mathbb{Z}, \;\; k = 1, \ldots, N \\ v_j^N & = g_j, \;\; j \in \mathbb{Z} \end{cases}$$

The first relation may be written in the *fixed-point form*

$$v_j^k = \left(1 + \frac{h_0}{h_1^2} a_j^k\right)^{-1} \left[\frac{1}{2} \frac{h_0}{h_1^2} a_j^k \left(v_{j-1}^k + v_{j+1}^k\right) + v_j^{k+1} + h_0 f_j^k\right]$$

Implicit Scheme - Stability

- In order to assess the stability, we set:

$$\gamma_j^k := \frac{h_0}{h_1^2} a_j^k \left(1 + \frac{h_0}{h_1^2} a_j^k\right)^{-1}, \quad \gamma^k := \sup_j \gamma_j^k$$

- Since $s \mapsto s/(1+s)$ is increasing with image in $(-1, 1)$, we have that

$$\gamma^k \leq \gamma(a,h) := \frac{h_0}{h_1^2} ||a||_\infty \left(1 + \frac{h_0}{h_1^2} ||a||_\infty\right)^{-1} < 1$$

- The contraction factor $\gamma(a,h) \leq 1 \implies$ *existence* and *uniqueness* of the *fixed-point*

Diffusion with actualization

- When r is not identically zero, the equation (2) becomes :

$$\begin{cases} V_t + \frac{1}{2} a(x,t) V_{xx} - r(x,t) v + f(x,t) & = 0, \qquad (x,t) \in \mathbb{R}x[0,T] \\ V(x,T) & = g(x), \quad x \in \mathbb{R} \end{cases}$$

- It is convenient to write the contribution of the actualization term as *implicit* (rather than *explicit*)

We set :

$$r_j^k := r\left(jh_1, kh_0\right)$$

Scheme with actualization

For $j \in \mathbb{Z}$ and $k = 1, ..., N$, we have

$$\begin{cases} \frac{v_j^k - v_j^{k-1}}{h_0} + \frac{1}{2} a_j^k \frac{v_{j+1}^k + v_{j-1}^k - 2v_j^k}{h_1^2} - r_j^{k-1} v_j^{k-1} + & f_j^k = 0 \\ & v_j^N = g_j \end{cases}$$

The first relation may be written in the ordered form

$$v_j^{k-1} = \left(1 + h_0 r_j^{k-1}\right)^{-1} \left[\left(1 - \frac{h_0}{h_1^2} a_j^k\right) v_j^k + \frac{1}{2} \frac{h_0}{h_1^2} a_j^k \left(v_{j-1}^k + v_{j+1}^k\right) + h_0 f_j^k \right]$$

We can see that the expression within the square brackets is the *explicit* scheme when $r = 0$

The family of θ schemes

For $\theta \in [0,1]$, consider the following scheme for $j \in \mathbb{Z}$ and $k = 0, ..., (N-1)$

$$\begin{cases} \frac{v_j^{k+1} - v_j^k}{h_0} + \frac{1}{2}\theta a_j^k \frac{v_{j+1}^k + v_{j-1}^k - 2v_j^k}{h_1^2} + \frac{1}{2}(1-\theta) a_j^{k+1} \frac{v_{j+1}^{k+1} + v_{j-1}^{k+1} - 2v_j^{k+1}}{h_1^2} + f_j^k = 0 \\ v_j^N = g_j \end{cases}$$

- The scheme is *explicit* for $\theta = 0$, and *implicit* for $\theta = 1$
- It can be seen as the combination of an explicit step followed by an implicit step, both of length h_0
- $\theta = \frac{1}{2} \Longrightarrow$ *Cranck-Nicholson* scheme

The condition for a θ scheme to be simply monotone is

$$(1-\theta)\frac{h_0}{h_1^2}||a||_\infty \leq 1$$

Outline

Stochastic Differential Equations (SDEs)

- A *Stochastic Differential Equation* is a differential equation in which one or more of the terms is a Stochastic Process
- SDEs contain a variable which represents a random *White Noise*
- The *White Noise* is calculated as the derivative of a *Brownian motion* (or *Wiener Process*)

Definition

A Brownian Motion is a stochastic process $(B_t)_{t\geq 0}$ with continuous trajectories such that $B_0 = 0$ and for $0 \leq s \leq t$,

- $(B_t - B_s) \sim \mathcal{N}(0, t-s)$
- $(B_t - B_s)$ is independent of B_r, $\forall r \leq s$

Simulation of the Brownian Motion

The simulation relies on the probability distribution of increments

Simulation

Given times $t_1, t_2, ..., t_k, ...$, $(B_t)_{t\geq 0}$ is simulated by the following recurrent scheme

$$\begin{cases} B_{t_{k+1}} = B_{t_k} + \sqrt{t_{k+1} - t_k} G_{k+1} \\ B_0 = 0 \end{cases}$$

Where $(G_k)_{k\geq 1}$ is an *independent* and *identically distributed* sequence and

$$\forall k \geq 1, \quad G_k \sim \mathcal{N}(0,1)$$

Solution of an SDE

Consider the following SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 \in \mathbb{R}^d$$

Existence and Uniqueness of the solution

If b and σ are Regular, locally Lipschitz, and verify the condition

$$\langle b(x), x\rangle + \frac{1}{2} Tr\left(\sigma\sigma'\right)(x) \leq \lambda\left(1 + |x|^2\right)$$

Then the SDE has a unique solution : the *Markovian Process* $(X_t)_{t\geq 0}$ defined for all $t \geq 0$ by :

$$X_t = X_0 + \int_0^t b\left(X_s\right)ds + \int_0^t \sigma\left(X_s\right)dB_s$$

Euler Scheme

Let $t_0, ..., t_n$ be a partition of $[0, T]$. We have

$$\begin{aligned} X_{t_{k+1}} &= X_{t_k} + \int_{t_k}^{t_{k+1}} b\left(X_s\right) ds + \int_{t_k}^{t_{k+1}} \sigma\left(X_s\right) dB_s \\ &\approx X_{t_k} + b\left(X_{t_k}\right)\left(t_{k+1} - t_k\right) + \sigma\left(X_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right) \end{aligned}$$

The *Euler* scheme is therefore defined by the recurrence

$$\begin{cases} \overline{X}_{t_{k+1}} = \overline{X}_{t_k} + b\left(\overline{X}_{t_k}\right)\left(t_{k+1} - t_k\right) + \sigma\left(\overline{X}_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right) \\ \\ \overline{X}_0 = X_0 \end{cases}$$

Milstein Scheme

The *Milstein* scheme is obtained by an enhancement of the Stochastic integral

$$\begin{aligned}\int_{t_k}^{t_{k+1}} \sigma\left(X_s\right) dB_s \quad &\approx \sigma\left(X_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right) \\ &+ \tfrac{1}{2}\sigma\sigma'\left(X_{t_k}\right)\left[\left(B_{t_{k+1}} - B_{t_k}\right)^2 - \left(t_{k+1} - t_k\right)\right]\end{aligned}$$

The *Milstein* scheme is therefore defined by the following recurrence

$$\begin{aligned}\overline{X}_{t_{k+1}} = \overline{X}_{t_k} \quad &+ \left(b - \tfrac{1}{2}\sigma\sigma'\right)\left(\overline{X}_{t_k}\right)\left(t_{k+1} - t_k\right) + \sigma\left(\overline{X}_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right) \\ &+ \tfrac{1}{2}\sigma\sigma'\left(\overline{X}_{t_k}\right)\left(B_{t_{k+1}} - B_{t_k}\right)^2\end{aligned}$$

Strong error for Euler and Milstein schemes

We define the L^2 strong error of the scheme $\overline{X}$ by

$$\mathcal{E}_2^{strong} := \sqrt{\mathbb{E}\left[\left(\sup_{t_k}|X_{t_k} - \overline{X}_{t_k}|\right)^2\right]}$$

Theorem

- For the *Euler* scheme with time step h, the strong error is of order $\frac{1}{2}$
$$\mathcal{E}_2^{strong} = \mathcal{O}\left(\sqrt{h}\right)$$
- For the *Milstein* scheme with time step h, the strong error is of order 1
$$\mathcal{E}_2^{strong} = \mathcal{O}\left(h\right)$$

Outline

Feynmann-Kac results

- The expectation of a (function of) random variable can be obtained as a solution of an associated PDE
- Let $X_t = (X_t^1, ..., X_t^d)$ be a stochastic process, solution of the following system of equations

$$dX_t^i = \mu_i\left(t, X_t\right) dt + \sigma_i\left(t, X_t\right) dB_t^i$$

  Where $B_t^1, ..., B_t^d$ are Brownian motions with correlation

$$d\langle B^i, B^j\rangle_t = \rho_{ij} dt$$

- Let $H(x) = H(x_1, ..., x_d)$ be some payoff

Feynmann-Kac Formula without interest rate

Link between PDEs and SDEs

The function

$$u(t,x) := \mathbb{E}\left[H(X_T)/X_t = x\right]$$

is solution of the PDE

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \mu(t,x)\frac{\partial u}{\partial x_i} + \frac{1}{2}\sum_{i,j=1} \rho_{ij}\sigma_i(t,x)\sigma_j(t,x)\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$$

with final condition

$$u(T,x) = H(x)$$

Feynmann-Kac Formula with interest rate involved

- Let $\mathcal{A}$ be the *generator* of the SDE, defined for every $f : \mathbb{R}^+ x\mathbb{R}^d \to \mathbb{R}$ by

$$(\mathcal{A}f)(t,x) = \sum_{i=1}^{d} \mu_i(t,x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d}\left(\sigma\sigma^T\right)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

- Let $X_t^{t,x} = x$ be the initial condition of the SDE
- Let $u \in \mathcal{C}^{1,2}\left(\mathbb{R}^+ x\mathbb{R}^d\right)$ satisfy the equation

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}u - ru = 0, \quad \text{for all } (t,x) \in [0,T]x\mathbb{R}^d \\ u(T,x) = f(x) \end{cases}$$

Then for all $(t,x) \in [0,T]x\mathbb{R}^d$, the solution $u(t,x)$ is

$$u(t,x) = \mathbb{E}\left[e^{-\int_t^T r(s,X_s)ds} f(X_T) \Big/ X_t = x\right] = \mathbb{E}\left[e^{-\int_t^T r(s,X_s^{t,x})ds} f(X_T^{t,x})\right]$$

Proof of the Feynmann-Kac formula

- Lets apply the Ito lemma to the function $F_t := e^{-\int_0^t r(s,X_s)ds} u(t, X_t)$
- Since $e^{-\int_0^s r(v,X_v)dv}$ is differentiable and therefore of bounded variation, we have $\langle\, e^{-\int_0^s r(v,X_v)dv},\ u(s, X_s)\,\rangle = 0$

$$
\begin{aligned}
F_t = \quad F_0 \quad & + \int_0^t \left[d\left(e^{-\int_0^s r(v,X_v)dv} \right) u(s, X_s) + e^{-\int_0^s r(v,X_v)dv} du(s, X_s) + 0 \right] \\
= u(0, X_0) \quad & + \int_0^t \left[\left(e^{-\int_0^s r(v,X_v)dv} \right) u(s, X_s)\left(-r(s, X_s)\right) ds + e^{-\int_0^s r(v,X_v)dv} \frac{\partial u}{\partial t} ds \right] \\
& + \int_0^t e^{-\int_0^s r(v,X_v)dv} \left(\sum_{i=1}^d \frac{\partial u}{\partial x_i} dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j} dX_s^i dX_s^j \right) \\
= u(0, X_0) \quad & + \int_0^t e^{-\int_0^s r(v,X_v)dv} \left[-ru + \frac{\partial u}{\partial t} + \sum_{i=1}^d \frac{\partial u}{\partial x_i} \mu_i(s, X_s) ds \right. \\
& \left. + \sum_{i=1}^d \frac{\partial u}{\partial x_i} \sum_k \sigma_{i,k}(s, X_s) dB_s^k + \sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j} \sum_{k,l} \sigma_{i,k}\sigma_{j,l} \underbrace{dB_s^k dB_s^l}_{=\delta_{k,l} ds} \right]
\end{aligned}
$$

Proof of the Feynmann-Kac formula

$$F_t = u(0,X_0) + \int_0^t e^{-\int_0^s r(v,X_v)dv}\left(-ru + \frac{\partial u}{\partial t} + \mathcal{A}u\right)ds + \int_0^t e^{-\int_0^s r(v,X_v)dv}\nabla u.\sigma.dB_s$$

Since the process $u(0,X_0) + \int_0^t e^{-\int_0^s r(v,X_v)dv}\nabla u.\sigma.dB_s$ is a martingale, then the process

$$M_t := e^{-\int_0^t r(s,X_s)ds}u(t,X_t) - \int_0^t e^{-\int_0^s r(v,X_v)dv}\left(\frac{\partial u}{\partial t} + \mathcal{A}u - ru\right)(s,X_s)ds$$

is a martingale. Hence the process $(M_{t'})_{t \le t' \le T}$ given by

$$\begin{aligned} M_{t'} : \quad &= e^{-\int_t^{t'} r(s,X_s^{t,x})ds}u(t',X_{t'}^{t,x}) - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv}\underbrace{\left(\frac{\partial u}{\partial t} + \mathcal{A}u - ru\right)}_{=0}(s,X_s^{t,x})ds \\ &= e^{-\int_t^{t'} r(s,X_s^{t,x})ds}u(t',X_{t'}^{t,x}) \quad \textit{is also a martingale.} \end{aligned}$$

Proof of the Feynmann-Kac formula

$$
\begin{aligned}
u(t,x) &= M_t \\
&= \mathbb{E}\left[M_t\right] \\
&= \mathbb{E}\left[M_T\right] \\
&= \mathbb{E}\left[e^{-\int_t^T r(s,X_s^{t,x})ds} u\left(T, X_T^{t,x}\right)\right] \\
&= \mathbb{E}\left[e^{-\int_t^T r(s,X_s^{t,x})ds} f\left(X_T^{t,x}\right)\right]
\end{aligned}
$$

Feynmann-Kac with a source term

For some function g, by rewriting $M_{t'}$ as

$$M_{t'} := e^{-\int_t^{t'} r(s,X_s^{t,x})ds} u(t', X_{t'}^{t,x}) \quad - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv} \left(\frac{\partial u}{\partial t} + \mathcal{A}u - ru\right)(s, X_s^{t,x})ds \\ - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv} g\left(s, X_s^{t,x}\right) ds$$

The solution of the PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}u - ru = g \\ \\ v(T,x) = f(x) \end{cases}$$

is obtained as

$$v(t,x) := \mathbb{E}\left[e^{-\int_t^T r(s,X_s^{t,x})ds} f\left(X_T^{t,x}\right) - \int_t^T e^{-\int_t^s r(v,X_v^{t,x})dv} g\left(s, X_s^{t,x}\right) ds\right]$$