The Shannon Sampling Theorem—Its Various Extensions and Applications: A Tutorial Review

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Abstract—It has been almost thirty years since Shannon introduced the sampling theorem to communications theory. In this review paper we will attempt to present the various contributions made for the sampling theorems with the necessary mathematical details to make it self-contained.

We will begin by a clear statement of Shannon's sampling theorem followed by its applied interpretation for time-invariant systems. Then we will review its origin as Whittaker's interpolation series. The extensions will include sampling for functions of more than one variable, random processes, nonuniform sampling, nonband-limited functions, implicit sampling, generalized functions (distributions), sampling with the function and its derivatives as suggested by Shannon in his original paper, and sampling for general integral transforms. Also the conditions on the functions to be sampled will be summarized. The error analysis of the various sampling expansions, including specific error bounds for the truncation, aliasing, jitter and parts of various other errors will be discussed and summarized. This paper will be concluded by searching the different recent applications of the sampling theorems in other fields, besides communications theory. These include optics, crystallography, time-varying systems, boundary value problems, spline approximation, special functions, and the Fourier and other discrete transforms.

I. Introduction

THE SAMPLING theorem that we shall discuss in detail was introduced by Shannon [1] to information theory. However, the interest of the communications engineer in the sampling theorem may be traced back to Nyquist [2]. As we shall see in Section II this theorem was originated by both E. T. and J. M. Whittaker [3]–[5] and Ferrar [6], even though some attribute it to Cauchy [7, p. 41]. In the Russian literature this theorem was introduced to communications theory by Kotel'nikov [8], and took its name from him as opposed to Shannon, the Whittaker, or popular sampling theorems in the English literature. In what follows we will use either one of the above references or, in brief, we will use WKS sampling theorem after both Whittakers, Kotel'nikov, and Shannon. We will do this with every sampling theorem that involves a band-limited signal, i.e., represented by a finite limit (truncated) inverse Fourier transform. WKSK will stand for Kramers's [9] and Weiss' [10] generalization of the sampling theorem which involves more general integral transforms than the usual Fourier transform. Attention should be given to the minor variations in the definition and/or the alternate use of the Fourier transform and its inverse.

As we shall illustrate in the following sections, the principal impact of the Shannon sampling theorem on information theory is that it allows the replacement of a continuous band-limited signal by a discrete sequence of its samples without the loss of any information. Also it specifies the lowest rate (the Nyquist rate) of such sample values that is necessary to reproduce the original continuous signal.

We may stress here that the Shannon sampling theorem and most of its extensions are stated primarily for band-limited functions instead of random processes which are more relevant to the information theorist. However, and as we shall see in Section I-D-2, most of these sampling expansions can be extended easily to random processes.

It is our intention to include all possible relevant contributions in communications, mathematics, and other fields, a task which we hope to give the justice it deserves. To this end we have attempted to include an exhaustive bibliography to help the specialist and the interested reader of various disciplines (see [205]–[248]). We will attempt, whenever possible, to unite the different notations used, but attention should be given to such differences, especially when we quote certain detailed results such as estimates of various errors.

A. The Shannon Sampling Theorem

Shannon's original statement [1] of the WKS sampling theorem is the following.

Theorem I-A-1: "If a function $f(t)$ contains no frequencies higher than W cps it is completely determined by giving its ordinates at a series of points spaced $(1/2W)$ s apart." Shannon's proof starts by letting

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F(\omega)e^{-j\omega t}\,d\omega \tag{1}$$

since $F(\omega)$, the spectrum of $f(t)$, is assumed to be zero outside the band $(-2\pi W, 2\pi W)$. The Fourier series expansion of $F(\omega)$ on the fundamental period $-2\pi W < \omega < 2\pi W$ is

$$F(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega n/2W} \tag{2}$$

$$c_n = \frac{1}{4\pi W}\int_{-2\pi W}^{2\pi W} F(\omega)e^{-j\omega n/2W}\,d\omega = \frac{1}{2W}f\left(\frac{n}{2W}\right). \tag{3}$$

We note that the Fourier coefficient c_n is proportional to $f(n/2W)$, the sample of the signal $f(t)$. Also, $\{c_n\}$ determines $F(\omega)$, hence, by the uniqueness property of the Fourier transform, $f(t)$ is determined. Shannon then constructed $f(t)$ as

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the sampling series

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \tag{4}$$

This result is easily established when we use the Fourier exponential series (2) for $F(\omega)$ in (1), exchange the integration and summation, and use (3). We will see in Section II-C that the outline of this proof and the method of constructing $f(t)$ as in (4) is parallel to the work of J. M. Whittaker [4]. In fact, Shannon introduced the physics of time and frequency to the second part of Theorem II-C-1, where (4) is Whittaker's cardinal series. This celebrated theorem, with some variations from the above-mentioned Shannon statement, is discussed briefly in a number of texts [11]–[19] in the field of communications with some detailed illustrations. In the Japanese literature, Someya [19] discussed the sampling theorem at about the same time Shannon did [1]. The variations in the proofs center around different methods of manipulation in Fourier analysis, contour integration, and matrices.

Due to the symmetry of the Fourier transform pairs, the sampling theorem is also valid for time-limited functions, i.e., for $F(\omega)$ the Fourier transform of a function $f(t)$ which is zero for $|t| > T$:

$$F(\omega) = \sum_{n=-\infty}^{\infty} F\left(\frac{n\pi}{T}\right) \frac{\sin (T\omega - n\pi)}{(T\omega - n\pi)}. \tag{5}$$

B. System Interpretation–Time-Invariant Systems

Reza [11, p. 305] gave the following physical interpretation to Shannon's (WKS) sampling theorem. Suppose that $f(t)$ represents a continuous band-limited voltage signal. Then $f(t)$ can be sampled at times $\{n/2W\}$, $n = 0, \mp 1, \mp 2, \cdots$. Here $k(t) = (\sin 2\pi Wt)/(\pi t)$ is known to be the impulse response of an ideal low-pass filter with system function $K(\omega)$ and frequency cutoff at $2\pi W$ (Fig. 1). So $f(t)$ of (4) will be the output of such a filter with input taken to be the pulse train defined by the samples $\{f(n/2W)\}$ as shown in Fig. 1.

As we will see in Section IV-I, Papoulis [13], [14] later extended the WKS sampling theorem in such a way that he obtained a physical interpretation with more relaxed conditions on the filter (Fig. 2) and with a recognizable pulse as input rather than the unattainable impulse. The relaxation of the filter's condition will result in an error that can be minimized by sampling at a rate higher than the Nyquist rate of (4) (see Section IV-I).

II. The Origin of Shannon's Sampling Theorem–Interpolatory Functions

In this section we review the theory of interpolatory functions, since this is where the Shannon [1] sampling theorem originated. As such we intend to show that it is also here that the Weiss [10] and Kramer [9] generalization of the above sampling theorem to other integral transforms besides the Fourier one emerged as a natural extension.

A. The Cardinal Series

E. T. Whittaker [3] set out to find an analytic expression for a function when the values of the function are known for equidistant values $a, a + w, \cdots, a + nw$, of its argument and such that this expression is free of periodic components with a period less than $2w$. This function was called the *Cardinal*

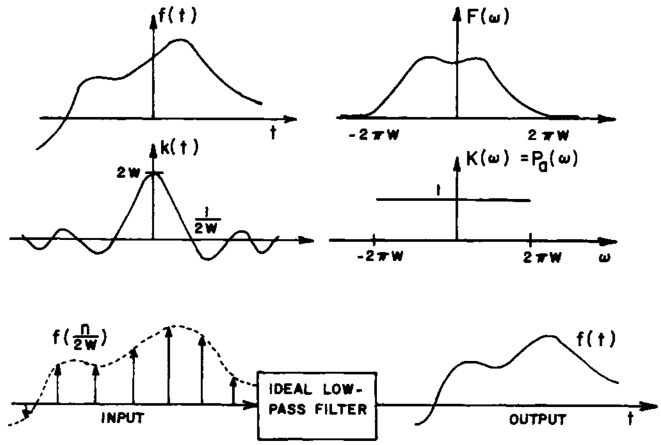

Fig. 1. Physical interpretation of Shannon's sampling expansion (4).

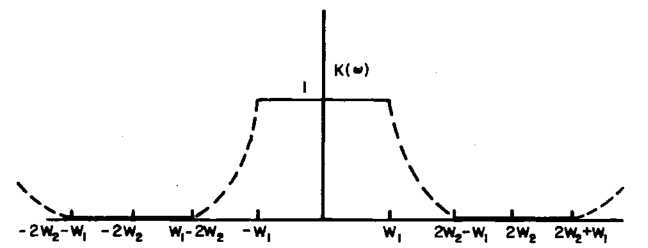

Fig. 2. A more practical system function for a filter of a sampling expansion.

Function. He showed that this analytic expression is not only an interpolatory expression but a representative one as well. At this point, we may say that the sampling theorem of Shannon had its origin. The first thing we notice is that Whittaker's problem is concerned with equally spaced values of the argument so a periodic function is expected. Whittaker considered the tabulated values of the function $f(t)$, i.e., $f(a), f(a + w), \cdots, f(a + nw)$, and derived the final form of the cardinal function as

$$\sum_{n=-\infty}^{\infty} f(a + nw) \frac{\sin \frac{\pi}{w}(t - a - nw)}{\frac{\pi}{w}(t - a - nw)}. \tag{6}$$

We note that this cardinal series is the one Shannon used for his sampling theorem and is what is sometimes called the Whittaker sampling theorem. There are two references here, to E. T. Whittaker [3] and J. M. Whittaker [4], [5]. This may be due to the fact that the final statement of the above sampling theorem in terms of band-limited signals is very close to the more refined statements of J. M. Whittaker [5, p. 68] concerning the relation between the cardinal series (6) and the truncated Fourier integral (1). The most complete recent treatment of Whittaker's cardinal function as a mathematical tool was given by McNamee, Stenger, and Whitney [20]. They linked the cardinal function to the central difference through their similarities, and showed again how the cardinal function provides a link between the Fourier series and Fourier integral. Finally, they showed that the cardinal function can be used for solving integral equations. Very recently Stegner [21] used Whittaker's cardinal function to derive various types of very accurate approximation procedures, along with error bounds, for interpolating, integrating, and evaluating the Fourier and the Hilbert transforms of functions.

B. Suggestions for Other Series

At this point it is not surprising that we raise the question: "Is it possible to consider some expression resembling the cardinal form that deals with the samples of a function at non-equidistant values of its argument, say $\{t_n\}$?" To follow the same procedure we know that $\sin \lambda t$ is the simplest periodic function with period $(2\pi/\lambda)$, so for our case we avoid it and try a more general kernel $K(\lambda, t)$ with a sampling function

$$S_n(t) = S(t, t_n) \tag{7}$$

where $S(t_n, t_m) = \delta_{n,m}$. The explicit expression for such a $S_n(t)$ is given by Kramer [9] for the generalized sampling theorem for any choice of $K(\lambda, t_n)$ as an orthogonal set on $[a, b]$. So we can regard Kramer's generalization as a natural extension of Whittaker's work and the popular sampling theorem.

C. The Cardinal Series and the Fourier Integral

In this section we will discuss J. M. Whittaker's [4] important development toward what we now know as the Shannon sampling theorem. In particular, his explicit theorem involves the cardinal series and Fourier and Fourier–Stieltjes integrals. Hence, he came the closest to the present statement of the sampling theorem as it is given in terms of a band-limited signal (i.e., a truncated inverse Fourier transform). J. M. Whittaker's [4, Theorem 2] theorem is the following.

Theorem II-C-1: "If the series

$$\sum_{n=1}^{\infty} \frac{|a_n| + |a_{-n}|}{n} \tag{8}$$

converges, the cardinal series

$$C(x) = \frac{\sin \pi x}{\pi} \left\{ \frac{a_0}{x} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{a_n}{x-n} + \frac{a_{-n}}{x+n} \right] \right\} \tag{9}$$

is absolutely convergent, and its sum is of the form

$$\int_0^1 [\cos \pi xt \, dF(t) + \sin \pi xt \, dG(t)] \tag{10}$$

where F, G are continuous functions. Given any function $f(x)$ of the form of (10) the series

$$\frac{\sin \pi x}{\pi} \left\{ \frac{f(0)}{x} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{f(n)}{x-n} + \frac{f(-n)}{x+n} \right] \right\} \tag{11}$$

is $(C, 1)$ summable to $f(x)$." The $(C, 1)$ here stands for "Cesáro summability" where, according to a theorem due to Hardy [22], this means that the series (11) converges if $f(n)$ is bounded (see Section V-I).

Previously Ferrar [6] gave the following theorem, which we consider to be even closer to Shannon's original statement of the sampling theorem.

Theorem II-C-2: "If $\sum_{n=-\infty}^{\infty} |a_n|^p$ is convergent, $p > 1$, and $C(x)$ is defined by

$$C(x) = \frac{\sin \pi x}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a_n}{x-n} \tag{12}$$

then

$$C(x) = \frac{\sin \pi(x-b)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n C(b+n)}{x-b-n} \tag{13}$$

where $\{a_n\} \in l_p$ implies that the series in (12) and (13) are convergent." Here $\{a_n\} \in l_p$ means that the series $\sum_{n=-\infty}^{\infty} |a_n|^p$ is convergent. We also note that, by Hardy's theorem, for $C(x)$ as $(C, 1)$ summable to be convergent we need $a_n/(x-n) = 0(1/n)$, i.e., if $C(b+n)$ is bounded. Ferrar called this the *consistency of the cardinal series.* This corresponds to the representation of the sampling theorem as compared to the interpolation only, in the case of interpolatory theory. Again, J. M. Whittaker asserted that, given a sequence $a_0, a_1, \cdots, a_n, \cdots$ of real numbers, then the series (cardinal) of type (11), convergent or $(C, 1)$ summable, affords a means of defining the trigonometric integrals associated with the Fourier and Fourier–Stieltjes series, respectively. For example

$$a(x) = \int_{-\pi}^{\pi} f(x) \cos xt \, dt \tag{14}$$

where $f(x)$ is represented by the Fourier series and $a(x)$ by the cardinal series. Here, we are led to the truncated Fourier cosine integral in (14). At this point we note that the above statement is another, more precise statement of what E. T. Whittaker had started, with almost everything centered around the cardinal series.

Now we may raise a question of a different nature which is still aimed at tying the Kramer generalization of the sampling theorem to a common origin with the Shannon sampling theorem and, hence, is a natural extension of the latter. This question is, "What kind of integral representation would a series other than the cardinal series offer?" As an example, it is sufficient to consider the Bessel function $J_m(xt)$, of the first kind of order m, instead of $\sin xt$. J. M. Whittaker [5, p. 71] came close to touching the question of the generalized sampling expansion when he considered the general partial fraction series [5, p. 64]:

$$H(z) \cdot \left\{ \frac{f(0)}{z} + \sum_{n=1}^{\infty} \left[\frac{f(c_n)}{H'(c_n)(z-c_n)} + \frac{f(-c_n)}{H'(c_n)(z+c_n)} \right] \right\} \tag{15}$$

where the $c_1, c_2, \cdots$, is a strictly increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n^{-2}$ converges and

$$H(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{c_n^2}\right). \tag{16}$$

In addition, he noted that Theorem II-C-1 does not apply to (15) in general, but to the special case $H(z) = \sin \pi z$ and $c_n = n\pi$, $z = cx$, as the cardinal series is in terms of $\{\sin n\pi x\}$, an orthogonal set of functions relative to its zeros in $[0, 1]$. At this point he hinted that a theorem similar to Theorem II-C-1 holds if $c_n = t_n$, the zeros of $J_0(z)$, the Bessel function of the first kind of order zero, and $H(z) = zJ_0(z)$ [in (15)]. So, $H(xc_n)$ is the orthogonal set relative to its zeros with a weight function $\rho(x) = (1/x)$. It is then no surprise to find the Bessel functions among the first examples of the generalized sampling theorem (Section III-A), where we accept the theorem as the natural extension of the work of Ferrar [6] and both Whittakers [3]-[5], and different than their cardinal series.

One advantage of using the finite Bessel (Hankel) transform is that the n-dimensional Fourier transform, with circular symmetry, is reduced to $J_{n-1/2}(x)$- Bessel transform [23] [see (29)].

D. The Sampling Theorem and Interpolation

Jagerman and Fogel [24] considered the WKS sampling theorem as an interpolation formula, then stated and proved a number of interesting extensions. They first considered the Lagrange interpolation polynomial [25]

$$P_n(t) = g_n(t) \sum_{j=0}^{n} \frac{f(t_j)}{(t - t_j) g'(t_j)}, \quad g_n(t_j) = 0 \tag{17}$$

then extended the real variable t to a complex variable z. Note that (17) is the partial fraction expansion of J. M. Whittaker's equation (15). Here $(P_n(z))/(g_n(z))$ is analytic except at the zeros of $g_n(z)$, the sampling points, and $P_n(z)$ is entire, i.e., analytic everywhere. This was generalized to include an infinite number of sampling points. The choice for $g(z)$ was obviously $g(z) = \sin(\pi z/h)$, so

$$P(z) = \sin \frac{\pi z}{h} \sum_{j=-\infty}^{\infty} \frac{(-1)^j f(jh)}{z - jh} \tag{18}$$

is the cardinal series for the entire function $P(z)$. The sample points are uniformly spaced on the complex plane. We remark here that a more general choice for $g(z)$ would be a function such as $J_m(z)$, where the sample point distribution would be asymptotically uniform. For their choice of $g_n(z)$, they stated and proved a number of basic extensions of the WKS sampling theorem, using the method of contour integration and the Paley–Wiener theorem [26, p. 13] which states the equivalence between band-limited functions and square integrable functions of exponential type. Then they extended these sampling theorems to include the samples of the function $f(jh)$ and its derivative $f'(jh)$, an important extension which was remarked on explicitly by Shannon [1], and which we will discuss in detail in Section IV-B.

III. The Generalized Sampling Theorem

In this section we will discuss the generalization of Shannon's sampling theorem to include more general, finite limit (truncated) integral transforms besides the usual Fourier transform. In Section II-C we indicated how Whittaker [5] had suggested a sampling series for a finite limit integral transform with the Bessel function, instead of the exponential function, as its kernel. The first generalization that followed in this direction was considered by Weiss [10] for transforms with kernels which are solutions of the Sturm–Liouville problem associated with second-order differential equations [27]. Kramer [9] followed this by a detailed treatment for nth-order differential equations and illustrated it for the case of the Bessel function as a kernel.

In the following, we will give the statement of the generalized sampling theorem with various illustrations, compare it to Shannon's sampling theorem, present its physical interpretation in terms of time-varying systems, and then discuss its various extensions and applications. As we mentioned in the beginning of Section I, we will refer to this generalized theorem as the WKSK sampling theorem, after both Whittakers [3]-[5], Kotel'nikov [8], Shannon [1], and Kramer [9] as compared to WKS for the Whittakers', Kotel'nikov's, and Shannon's popular sampling theorem.

A. The Sampling Theorem for Hankel (Bessel) and Other Finite Limit Integral Transforms

The final generalization of the sampling theorem was stated by Kramer [9] as the following theorem.

Theorem III-A-1: "Let I be an interval and $L_2(I)$ the class of functions $\phi(x)$ for which $\int_I |\phi(x)|^2 \, dx < \infty$. Suppose that for each real t

$$f(t) = \int_I K(x, t) g(x) \, dx \tag{19}$$

where $g(x) \in L_2(I)$. Suppose that for each real t, $K(x, t) \in L_2(I)$, and that there exists a countable set $E = \{t_n\}$ such that $\{K(x, t_n)\}$ is a complete orthogonal set on I. Then

$$f(t) = \lim_{N \to \infty} \sum_{|n| \le N} f(t_n) S_n(t) \tag{20}$$

where

$$S_n(t) = S(t, t_n) = \frac{\int_I K(x, t) \overline{K(x, t_n)} \, dx}{\int_I |K(x, t_n)|^2 \, dx}.\text{"} \tag{21}$$

Here $g(x) \in L_2(I)$ means that $g(x)$ is Lebesque measurable and that $\int_I |g(x)|^2 \, dx < \infty$. Also $\overline{K(x, t)}$ is the complex conjugate of $K(x, t)$. The simplest proof is readily established when we write the orthogonal expansion for $g(x)$ in (19) in terms of $\overline{K(x, t_n)}$

$$g(x) = \sum_{n=1}^{\infty} c_n \overline{K(x, t_n)} \tag{22}$$

$$c_n = \frac{\int_I g(x) K(x, t_n) \, dx}{\int_I |K(x, t_n)|^2 \, dx} = \frac{f(t_n)}{\int_I |K(x, t_n)|^2 \, dx}. \tag{23}$$

Then multiply both sides of (22) by $K(x, t)$ and formally integrate term by term to obtain

$$\int_I K(x, t) g(x) \, dx = f(t) = \sum_{n=1}^{\infty} \frac{f(t_n) \int_I K(x, t) \overline{K(x, t_n)} \, dx}{\int_I |K(x, t_n)|^2 \, dx}$$

$$= \sum_{n=1}^{\infty} f(t_n) S_n(t) \qquad [(19)\text{-}(20)]$$

after using (19) for $f(t)$ and (21) for the sampling function $S_n(t)$, or what we sometimes write as $S(t, t_n)$. We may mention here that a weighting function $\rho(x)$ may be introduced [27], [28] in the integrals of (19) instead of having it implicit in the product $K(x, t) g(x)$. Also we indicate that the same proof can be followed when $K(x, t)$ of (19) is expanded in terms of the same orthogonal functions $\overline{K(x, t_n)}$. However, the shortest proof is to use Parseval's equation [29] for the integral in (19) with the Fourier coefficients c_n of (23) and

$S_n(t)$ of (21) for $g(x)$ and $K(x, t)$, respectively (see Section V-A). Kramer [9] showed that the conditions for this theorem on the kernel $K(x, t)$ in (19) are exhibited by the solutions of nth-order self-adjoint differential equations [27, p. 188, p. 284]. He illustrated it for the cases when $K(x, t) = e^{ixt}$ and when $K(x, t) = J_m(xt)$, where $J_m(xt)$ is the Bessel function of the first kind of order m. Campbell [28] illustrated the case when $K(x, t) = P_t(x)$, where $P_t(x)$ is the Legendre function. Other illustrations including $K(x, t)$ as the associated Legendre, the Gegenbauer, the Chebyshev, and the prolate spheroidal functions [30] were done in detail in [31] with suggestions for their use in scattering problems in physics. A recent illustration for $K(x, t)$ as the associate Laguerre function $L_t^\alpha(x)$, but with an integral defined on the semi-infinite interval $(0, \infty)$ instead of the usual finite interval, is found in [32]. As an illustration of Theorem III-A-1, we present the case of the finite limit J_m-Hankel, or Bessel, transform:

$$f(t) = \int_0^1 xJ_m(xt)F(x)\,dx. \tag{24}$$

The sampling function $S_n(t)$ of (21) is derived as

$$S_n(t) = S(t, t_{m,n}) = \frac{\int_0^1 xJ_m(xt)J_m(xt_{m,n})\,dx}{\int_0^1 x[J_m(xt)]^2\,dx} = \frac{2t_{m,n}J_m(t)}{(t_{m,n}^2 - t^2)J_{m+1}(t_{m,n})} \tag{25}$$

where the $\{t_{m,n}\}$ are the zeros of the Bessel function J_m, i.e., $J_m(t_{m,n}) = 0, n = 1, 2, \cdots$. Here the familiar properties of the Bessel functions were used [33] to evaluate the integrals of (25). The final sampling series (21) for the finite limit Hankel transform becomes

$$f(t) = \sum_{n=1}^{\infty} f(t_{m,n}) \frac{2t_{m,n}J_m(t)}{(t_{m,n}^2 - t^2)J_{m+1}(t_{m,n})} \tag{26}$$

$$J_m(t_{m,n}) = 0, n = 1, 2, \cdots.$$

We note here that the weighting function $\rho(x) = x$ has been introduced explicitly in (24) instead of having it implicit in the product $K(x, t)g(x)$ of (19).

B. On the Equivalence of the Generalized (WKSK) and Shannon (WKS) Sampling Theorems

The first question related to the generalized sampling theorem was raised by Campbell [28] concerning the possibility of applying Shannon's sampling theorem to functions that can be sampled by the generalized sampling theorem. He considered as kernels in (19) the solutions of regular first-order and regular second-order differential equations with separated boundary conditions, and the solutions of the singular Bessel and Legendre equations. For these cases Campbell showed that if a function with such kernels can be expanded by the use of the WKSK sampling theorem, then it can also be expanded by the use of the WKS sampling theorem. These results were extended [34] to include integral transforms with kernels such as the following: $P_t^m(x)$, the associated Legendre function; $C_t^\nu(x)$, the Gegenbauer function; $U_t^\nu(x)$, the Chebyshev function of the second kind; and other functions. For example, in the case of the finite Gegenbauer transform:

$$f(t) = \int_{-1}^{1} C_t^\nu(x)F(x)\,dx \tag{27}$$

we can use the integral representation of $C_t^\nu(x)$ [33, p. 159, equation (27)] as a truncated Fourier transform in (27), then interchange the order of integration and define $H(u)$ in a simple way to obtain

$$f(t) = \frac{\Gamma(t + 2\nu)}{\Gamma(t + 1)} \int_{-\pi}^{\pi} e^{itu} H(u)\,du \tag{28}$$

so the function $(\Gamma(t + 1))/(\Gamma(t + 2\nu))\, f(t)$ and hence $f(t)$ may be sampled by the WKS sampling theorem. To compare the two sampling theorems in a more precise way, some definitions were presented and conditions were found [34] under which the two sampling theorems, namely, the Shannon and the generalized one, are equivalent. In summary, the theorems presented in [34] simply tell us that there is no advantage in using the WKSK sampling theorem when the function is represented by a double inverse Fourier transform with finite limits. This, however, is the case only when we assume that the communications engineer is interested in working with no integral transform other than the Fourier one. So the advantage of the WKSK sampling theorem may become clear when we consider other integral transforms [35], [36] and especially for time-varying systems [37] which we shall discuss in the following section. One other obvious advantage is the use of the Hankel transform in optics [14] where, with circular symmetry, a J_0-Hankel transform is equivalent to a double Fourier transform and, in general, a $J_{(m/2)-1}$-Hankel transform is equivalent to an m-dimensional Fourier transform [23, p. 82]:

$$\rho^{m/2-1}F(\rho) = \int_0^\infty r^{m/2-1} rf(r)J_{m/2-1}(\rho r)\,dr. \tag{29}$$

Here, $F(\vec{\xi}) = F(|\vec{\xi}|) = F(\rho)$ is the m-dimensional Fourier transform of $f(\vec{x}) = f(|\vec{x}|) = f(r)$ with circular symmetry. Hence, in two dimensions we may replace a double WKS sampling series by a single *WKSK* sampling series associated with the Bessel kernel $J_0(x)$.

C. System Interpretation—Time-Varying Systems

As we have presented in Section I-B, the applied interpretation [11] for the special case of $K(\omega, t) = e^{-j\omega t}$, i.e., the Shannon sampling expansion (4), is that $f(t)$ is the output of an ideal low-pass filter with impulse response $h(t, t_n) = 2WS(t, t_n) = [\sin 2\pi W(t - (n\pi/2W))]/[\pi(t - (n\pi/2W))]$ and with the input taken to be the pulse train $f(t_n) = f(n/2W)$. The applied interpretation of the generalized sampling expansion (20) can be given [36], where $f(t)$ is considered as the output of a band- (or transform-) limited [38] and a low-pass filter in the sense of these general integral transforms, with a time-varying impulse response that is related directly to the sampling function in (21) and with the pulse train $\{f(t_n)\}$ as its input. This was done for a transform-limited function:

$$f_I(t) = \int_I \rho(\omega)K(\omega, t)F(\omega)\,d\omega \tag{30}$$

with the Fourier-type inverse:

$$F(\omega) = \int \rho(t)\overline{K(\omega, t)} f(t)\, dt. \tag{31}$$

where $\rho(\omega)$ is a weighting function. Here $\overline{K(\omega, t)}$ stands for the complex conjugate of $K(\omega, t)$. Also, the limits of integration in (31) are not finite and will be specified for the particular integral transform. The complete details of this analyysis are found in [36] with the main definitions being in agreement with those in D'Angelo [37], Zadeh [39], and Zemanian [40].

Some basic properties of such transforms including the Parseval's equality, the orthogonality of the sampling functions on the interval of the integral (31), and the convolution product were derived and the Hankel transform was presented as an example [35], [36].

D. Some Applications (Also Sections VII-B–VII-C)

The first reference to the possible application of the generalized sampling theorem in communications [35] was for time-varying systems analysis [36] which we discussed in the last section. Also in the case of circular symmetry, for example in optics, it is known [14], [23] that the needed double Fourier transform can be replaced by a single J_0-Hankel (Bessel) transform. Hence, it is advantageous to replace a double Shannon sampling series by a single Bessel one. In general, with circular symmetry, an m-dimensional Fourier transform reduces to a one-dimensional $J_{(m/2)-1}$-Hankel transform [see (29)]. The next application was in the field of nuclear scattering [31]. In particular the sampling functions (21) for the generalized sampling theorem are necessary for evaluating the lth eigenvalue of the unitary S-matrix [41] due to the nth Regge pole of the S-matrix. This is especially true when a more general orthogonal expansion is needed rather than the usual Legendre one.

In the field of heat transfer, the generalized sampling theorem was used [42] to facilitate the solution of a conjugated boundary value problem. The analysis is applied to determine the effect of the axial conduction on the temperature field for a fluid with laminar flow in a tube. In this problem the finite J_0-Hankel transform was used to algebraize the radial part of the partial differential equation. To satisfy the boundary condition at the interface of the fluid, the coefficients of the two infinite series solutions are matched to obtain the final solution. However, since the generalized sampling series is applicable to the finite Hankel transforms it was possible [42] to recognize the infinite series in both solutions as the sampling series and hence assign it the transform function value. This resulted in eliminating the infinite series on both sides, thereby eliminating the need for approximations and numerical matching procedures.

The most recent attempt to use the generalized sampling expansion is in the field of general discrete transforms [43], [44]. This is in parallel to the discrete Fourier transform [45], [46] which lead to the fast Fourier transform (FFT) algorithm [46]–[48]. In attempting to develop a discrete Hankel transform [44], [49] we are guided by its corresponding sampling expansion which dictates the sample spacing. This recent investigation indicates that for the discrete J_0-Hankel transform of N terms, the samples are taken at $\{(j_{0,n})/b\}$ and $\{(j_{0,n})/c\}$ in the two t and ω spaces, respectively, with $j_{0,n}$ being the nth zero of $J_0(x)$ and $j_{0,N} < bc < j_{0,N+1}$.

E. Sampling with the Value of the Function and Its Derivatives

When Shannon [1] introduced the sampling theorem to communications he also remarked that the value of the function $f(t)$ can be constructed from the knowledge of the function and its derivative at every other sample point, then extended his remark to higher derivatives. In Section IV-B we will discuss the different methods [14], [24], [50]–[53] of arriving at this result with illustrations and physical interpretation. The truncation error bounds [54], [55] for such series are presented in Section VI-A. This result [24], [52] has been extended [56], [57] to other integral transforms associated with the generalized (WKSK) sampling theorem, which we will discuss at the end of Section IV-B and illustrate for the case of Hankel (Bessel) transforms.

F. Other Extensions–Sampling for an Infinite Limit Laguerre-$L_\nu^\alpha(x)$ Transform

Until recently [32], all direct illustrations of the sampling theorems have been associated with functions represented by finite limit (truncated) integral transforms whose kernels are orthogonal on the same finite interval. The first example of a sampling expansion for functions represented by an integral with infinite limits is that of the associate Laguerre-$L_\nu^\alpha(x)$ transform [32]. Here the associate Laguerre polynomials $L_n^\alpha(x)$ are used, which are orthogonal on the semi-infinite interval $(0, \infty)$ with respect to the weighting function $\rho(x) = e^{-x} x^\alpha$. This result is summarized in the following theorem.

Theorem III-F-1: If the function $F(x)$ is such that $\int_0^\infty e^{-x} \cdot x^\alpha |F(x)|^2\, dx < \infty$, or in brief $F(x) \in L_2\ (I, \rho)$ with $\rho = e^{-x} \cdot x^\alpha$ and I as $(0, \infty)$, then its Laguerre-L_ν^α transform

$$f(\nu) = \int_0^\infty e^{-x}\, x^\alpha\, L_\nu^\alpha\left(\frac{\lambda x}{\lambda - 1}\right) F(x)\, dx, \quad \alpha > -1,\ \nu \geq 0,\ 0 < |\lambda| < 1 \tag{32}$$

has the sampling expansion

$$f(\nu) = \frac{1}{(1-\lambda)^\nu \Gamma(\nu+1)} \cdot \left[f(0) + \lim_{N\to\infty} \sum_{n=1}^{N} (-\nu)_n \sum_{m=0}^{n} \frac{(\lambda-1)^{n-m} f(n-m)}{m!} \right] \tag{33}$$

where

$$(\nu)_n \equiv \frac{\Gamma(\nu+n)}{\Gamma(\nu)}.$$

We note here that in contrast to the other sampling expansions which involve the nth sample $f(n)$ in the nth term of the sampling series, the sampling expansion in (33) involves a combination of the first $n+1$ samples of the function in the nth term of the sampling series. However, for $t = k$, a nonnegative integer, the sampling expansion (33) gives the sample values $f(k)$. To verify this we note that the summation over n in (33) stops at $n = k$ and all the coefficients of $(\lambda - 1)^{n-m}$ in the double series cancel out, except that of $(\lambda - 1)^k$ which reduces (33) to $f(k)$. The rigorous proof, which involves writing the $L_n^\alpha(x)$-Laguerre polynomials orthogonal expansion of $F(x)$, integrating term by term as in (32), and using some special integrals [33] is given in [32]. We may note that the so-called Laguerre function is defined as

$$L_\nu^\alpha(x) = \frac{\Gamma(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\alpha+1)} M(-\nu, \alpha+1; x) \tag{34}$$

where $M(a, b; x)$ is the confluent hypergeometric function. We should point out here that the Laguerre function in (24) is defined differently from that of [33, p. 268, equation (37)],

$$L_\nu^\alpha(x) = \frac{1}{\Gamma(\nu+1)} M(-\nu, \alpha+1; x) \tag{35}$$

but it reduces to the Laguerre polynomial $L_n^\alpha(x)$ when $\nu = n$.

IV. Various Extensions of the Sampling Theorems

In this section we will present most extensions of Shannon's (WKS) and the generalized (WKSK) sampling theorems. This includes sampling in n dimensions, with derivatives, for random processes, with nonuniformly spaced samples, bandpass, implicit sampling, for distributions (generalized functions), for signals with time-varying bands and others.

A. The Sampling Theorems in n Dimensions

Shannon's sampling theorem was extended by Parzen [58] to include sampling for band-limited functions of n variables. The following is the statement given in Reza [11] where the proof follows the same method as used for the one-dimensional (WKS) sampling theorem (Section I-A).

Theorem IV-A-1: "Let $f(t_1, t_2, \cdots, t_n)$ be a function of n real variables, whose n-dimensional Fourier integral exists and is identically zero outside an n-dimensional rectangle and is symmetrical about the origin; that is,

$$g(y_1, y_2, \cdots, y_n) = 0, \quad |y_k| > |\omega_k|, \quad k = 1, 2, \cdots, n. \tag{36}$$

Then

$$f(t_1, t_2, \cdots, t_n) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} f\left(\frac{\pi m_1}{\omega_1}, \cdots, \frac{\pi m_n}{\omega_n}\right) \frac{\sin(\omega_1 t_1 - m_1\pi)}{\omega_1 t_1 - m_1\pi} \cdots \frac{\sin(\omega_n t_n - m_n)}{\omega_n t_n - m_n\pi}.\text{"} \tag{37}$$

Miyakawa [59] presented a sampling theorem for stationary stochastic variables in n dimensions. A very interesting historical review of the sampling theorems with reference to many relevant applications was presented by Petersen [60]. We may remark here that the above Theorem IV-A-1 can also be proved easily by using the Parseval's equation in n dimensions [29]. Also, we can extend this result to include higher dimensional general integral transforms of the type (19) used for the generalized sampling theorem. The proof we give follows the same method used in Section III-A for proving the generalized sampling theorem and in particular the simple use of the general Parseval's equations for such higher dimensional transforms. We may point out again the advantage of the generalized sampling theorem where a $J_{(n/2)-1}$-Hankel transform is equivalent to the above n-dimensional Fourier transform when $g(\vec{y})$ in (36) and $f(\vec{t})$ in (37) possess circular symmetry [23]. Hence, the n-dimensional sampling series (37) may be replaced by a one-dimensional Bessel sampling series (26) with $m = (n/2) - 1$.

Petersen and Middleton [53], [61] presented a very detailed treatment of the sampling theorem in n dimensions [61] that involved the samples of the amplitude and the gradient [53] of an n-dimensional stochastic field (see Sections IV-B–IV-C)

$$\begin{aligned}\hat{f}(\vec{x}) &= \sum_{[k]} [f(\vec{x}_{[k]})g(\vec{x}, \vec{x}_{[k]}) + \vec{\nabla} f(\vec{x}_{[k]}) \cdot \vec{h}(\vec{x}, \vec{x}_{[k]})] \\ &= \sum_{[k]} \left[f(\vec{x}_{[k]})g(\vec{x}, \vec{x}_{[k]}) + \sum_{l=1}^{N} \frac{\partial f(\vec{x}_{[k]})}{\partial x_l} h_l(\vec{x}, \vec{x}_{[k]})\right]\end{aligned} \tag{38}$$

where $\hat{f}(\vec{x})$ is an estimate of the value of the random field $f(\vec{x})$ at every point $\vec{x}$ in the N-dimensional Euclidean space. $g(\vec{x}, \vec{x}_{[k]})$ and $h_l(\vec{x}, \vec{x}_{[k]})$, $l = 1, 2, \cdots, N$ are functions applying, respectively, to the values of amplitude and each component of the gradient measured at the sampling point $\vec{x}_{[k]}$, for reconstruction of the random field at any point $\vec{x}$. $\Sigma_{[k]}$ stands for the N-dimensional summation $\Sigma_{k_1} \Sigma_{k_2} \cdots \Sigma_{k_N}$. In addition to suggesting various applications (see Section VII-E) they [61] concluded that, for deterministic functions, the most efficient lattice is not in general rectangular, nor is a unique reconstruction function associated with a given sampling lattice. In addition, such optimal weighting functions were derived [53] for least mean-square reconstruction of the above (38) N-dimensional stochastic fields from discrete measurements of amplitude and gradient. Montgomery [62] utilized the sampling expansion with a function and its gradient then extended the result of Petersen and Middleton (38) to include the samples of the function and its higher partial derivatives [63] up to order $K \leqslant 1$

$$f(\vec{t}) = \sum_{\vec{n}} \left\{\sum_{j=0}^{K} \frac{1}{j!} [(\vec{t} - \vec{a}_{\vec{n}})\vec{\nabla}]^j \vec{f}(\vec{a}_{\vec{n}})\right\} g(\vec{t} - \vec{a}_{\vec{n}}) \tag{39}$$

where $f(\vec{t})$ is a square integrable complex valued function in the N-dimensional space, $\vec{t}$ is a vector in this space, $\vec{a}_{\vec{n}}$ are points of the sampling lattice and $g(\vec{t} - \vec{a}_{\vec{n}})$ is the weighting function applied in the construction of $f(\vec{t})$. $\vec{a}_{\vec{n}}$ is an integral linear combination of the vectors $\vec{a}_i$, where $\vec{n} = (n_1, n_2, \cdots, n_N)$ gives the integral coefficients used.

Gaarder [64] extended the n-dimensional sampling expansion to allow nonuniform but periodic sampling, a subject which we shall discuss in Section IV-D. Sharma and Mehta [65] extended the generalized (WKSK) sampling theorem, with kernels besides the Fourier one, to higher dimensions for bandpass functions instead of the usual low-pass ones (see Section IV-E).

B. Sampling with the Values of the Function and its Derivatives

1) The Shannon (WKS) Sampling Theorem: As we mentioned in Section III-E, when Shannon introduced the sampling theorem he also remarked that the value of $f(t)$ can be reconstructed from the knowledge of the function and its derivative at every other sample point, and then extended his remarks to higher derivatives. Fogel [50] considered this question without reference to the above remark, and stated and proved the following theorem.

Theorem IV-B-1: "If a function $f(t)$ contains no frequency higher than $W(Hz)$ it is determined by giving M function derivative values at each of a series of points extending throughout the time domain, sampling interval $T = (M/2W)$ being the time interval between instantaneous observations."

Later, Jagerman and Fogel [24] incorporated the above theorem and a theorem dealing with exponential order to give a number of very useful theorems including an explicit form that involves the samples of the function and its derivative. The method of proving their results relies on Lagrange interpolation polynomial and contour integration.

The importance of this result lies in its application. For example, for an aircraft the estimated velocity as well as position are used to determine a continuous course plot of the path with half the sampling rate.

As a generalization to the above results and as an explicit answer to Shannon's remark [1] concerning the reconstruction of a function $f(t)$ when the value of the function and its first R derivatives are given at equidistant sampling points $(R+1)/2W$ seconds apart, Linden [51] and then Linden and Abramson [52] gave the following final result after a minor correction.

Theorem IV-B-2: "Let $f(t)$ be a continuous function with finite Fourier transform $F(\omega)[F(\omega)=0$ for $|\omega| > 2\pi W]$. Then

$$f(t) = \sum_{k=-\infty}^{\infty} \left[\xi(kh) + (t-kh)\xi'(kh) + \cdots + \frac{(t-kh)^R}{R!}\xi^{(R)}(kh)\right]\left[\frac{\sin\frac{\pi}{h}(t-kh)}{\frac{\pi}{h}(t-kh)}\right]^{R+1} \tag{40}$$

where $h = (R+1)/(2W)$."

The $\xi^{(j)}(kh)$ in (40) are linear combinations of the $f^{(j)}(kh)$:

$$\xi^{(j)}(kh) = \sum_{i=0}^{j} \binom{j}{i}\left(\frac{\pi}{h}\right)^{j-i} \Gamma_{R+1}^{(j-i)} f^{(i)}(kh). \tag{41}$$

Here

$$\Gamma_\alpha^{(\beta)} = \frac{d^\beta}{dt^\beta}\left[(t/\text{sin}t)^\alpha\right]\Big|_{t=0}. \tag{42}$$

The $\Gamma_\alpha^{(\beta)}$ may be expressed in terms of the generalized Bernoulli numbers. Some of these values are

$$\Gamma_\alpha^{(0)} = 1 \quad \Gamma_\alpha^{(2)} = \frac{\alpha}{3} \quad \Gamma_\alpha^{(4)} = \frac{\alpha(5\alpha+2)}{15}$$

$$\Gamma_\alpha^{(6)} = \frac{\alpha(35\alpha^2+42\alpha+16)}{63} \qquad \Gamma_\alpha^{(\beta)} = 0, \quad \text{for odd } \beta.$$

Equation (41) may be obtained by multiplying both sides of (40) by $\{(\pi \sin \pi(t-kh))/(t-kh)\}^{-(R+1)}$ and equating their jth derivatives at $t = (kh)/\pi$. Such expansion makes clear the advantage of sampling with the function and its R derivative since the sample spacing here is $h = (R+1)/2W$, which is $(R+1)$ times that of $h = 1/(2W)$ for the case involving the samples of the function only.

Rearrangement of terms in (40) yields an alternate form which emphasizes the derivatives $f^{(j)}(kh)$ rather than their linear combinations (41). In addition, this alternate form relates the limit of the R-derivative sampling expansion as $R \to \infty$ to Taylor-type series weighed by a Gaussian density function centered about each sample point. An interesting question would be whether two-point (Lindstone interpolation [25, p. 28]) and then N-point Taylor-series-type expansion would reduce to a sampling-type expansion as $N \to \infty$? The proof of Theorem IV-B-2 relies on somewhat involved matrix methods [52]. However, the method of using contour integration can be employed [56], [57] to derive (40) in a very simple fashion.

Among other very interesting results concerning the sampling theorems, Papoulis [14, p. 132] presented a very useful decomposition theorem and utilized it to arrive at a simple method for deriving the sampling expansion with $R = N - 1$ derivatives. The explicit form for $R = 1 (N = 2)$ was easily obtained by this method

$$f(t+\tau) = \frac{4\sin^2(\omega_0\tau/2)}{\omega_0^2} \sum_{n=-\infty}^{\infty} \left[\frac{f(t+2nT)}{(\tau-2nT)^2} + \frac{f'(t+2nT)}{(\tau-2nT)}\right] \tag{43}$$

where ω_0 is the band-limit and $T = \pi/(\omega_0)$. However, this does not seem to be the case when $N > 2$. Note that $R = 0$ or $N = 1$ corresponds to sampling with the function only.

As we mentioned in the last section, Petersen and Middleton [53] gave the sampling expansion for stochastic fields represented by an n-dimensional band-limited Fourier transform that involved only the samples of the function and its gradient (38) (see Section IV-C). They also suggested many applications including crystallography and meteorology where the samples of the function and the gradient were sufficient for their analysis. As such they did not include any higher partial derivatives. Montgomery [63] extended these results to involve higher order partial derivatives (39). Later another method was devised [56], [57] for such extension and was illustrated for the double Fourier transform. Such a method uses a generalization to two dimensions of Linden and Abramson's important lemma [52] that was used for deriving (40). We suggest here that contour integration methods, similar to that used in [24], [56], [57] for functions of several variables [66] may be used to establish the sampling expansions with R derivatives for higher dimension band-limited Fourier and other integral transforms.

As we have shown, sampling with derivatives increases the sample spacing required, or in other words it allows the reconstruction of the band-limited signal with a sampling rate less than the Nyquist rate. Another approach aiming at the same goal was established by Kahn and Liu [67]. They treated the problem of the representation and construction of wide-sense stationary stochastic signals, not from one set of data $\{f(n\pi/a)\}$ but from several sets of sampled values obtained by using a multiple channel sampling scheme. They showed that with the optimum combination of prefilters and post-filters, in the case where two sets of sample values are taken, the frequency range of the input signal is limited by the prefilters to a total width of $4a$. This is instead of the usual total width of $2a$ when a single channel is used, which makes it stand as a natural extension of the latter case. Todd [68] used multiple channels to reconstruct deterministic band-limited signals with a sampling rate less than the Nyquist rate. The sample rate needed is inversely related to the number of the channels used and directly proportional to the Nyquist rate.

2) The Generalized (WKSK) Sampling Theorem: Recently, the sampling with R derivatives (40) was extended [56] to include the finite limit Hankel and other transforms besides the Fourier transform of (40). The method used in such general results employs contour integration which is a generalization of the method used by Jagerman and Fogel [24]. It is shown that, in parallel to the known special case of the truncated Fourier transform, the advantage of sampling with R derivatives is to increase by $(R+1)$-fold the asymptotic spacing between the sampling points. The importance of such an advantage for the Hankel transform, for example, is of course realized in time-varying [36] or spatial-varying systems.

As an example, the generalized sampling expansion with one derivative for the finite limit J_0-Hankel transform

$$f(t) = \int_0^a x J_0(xt) F(x)\, dx \tag{44}$$

is

$$f(t) = \sum_{k=1}^{\infty} \left[\frac{t^2}{t_{0,k}^2} f(t_{0,k}) + \frac{t^2 - t_{0,k}^2}{2t_{0,k}} f'(t_{0,k}) \right] S_k^2(t) \tag{45}$$

where $\{t_{0,k}\}$ are the zeros of the J_0-Bessel function of the first kind of order zero and $S_k(t)$ is the same sampling function as in (25)

$$S_k(t) = \frac{2t_{0,k} J_0(at)}{a(t_{0,k}^2 - t^2) J_1(at_{0,k})}. \tag{46}$$

The general procedure for deriving (45) for the J_0-Hankel transform, and other finite transforms including the Legendre transform is outlined in [56] and presented in detail in [57]. The derivation of the sampling expansion with derivatives for double finite Fourier transform is presented in [57].

C. Sampling Theorems for Random Processes

Another extension of the WKS sampling theorem was considered by Balakrishnan [69] where he showed that the WKS sampling theorem can be used to represent a process of a continuous time parameter. One of his theorems in this direction is the following.

Theorem IV-C-1: "Let $x(t), -\infty < t < \infty$, be a real or complex valued stochastic process, stationary in the "wide sense" (or second-order stationary), possessing a spectral density which vanishes outside the interval $[-2\pi W, 2\pi W]$. Then $x(t)$ has the representation

$$x(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{-N}^{N} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)} \tag{47}$$

for every t, where l.i.m. stands for limit in the mean square." The proof consists of using the WKS sampling theorem for the covariance function of the process, since it is assumed to have a truncated Fourier transform. Then $x^*(t)$, the optimal estimate of $x(t)$, was constructed by using the sampling series to show that the mean-square error is zero.

Middleton [12, chap. 20] also treated random sampling and presented a comparison of random and periodic data sampling. Peterson [60] gave a very detailed treatment for sampling of space–time stochastic processes with application to information and decision systems and a very interesting review. Many applications and extensions of the subject of optimal reconstruction of multidimensional random fields were presented by Petersen and Middleton [53], [70] and Petersen [71]. The complete treatment of this subject will be found in [72].

Among other generalizations of the sampling theorem, Parzen [58] presented simple proofs using Fourier series for the Fourier kernel $e^{i\omega t}$ to establish the above result (47) for random variables. More general theorems that include the above result as a special case were presented by Lloyd [73]. He first presented conditions under which the above random variables $x(t)$ of a stationary (wide-sense) stochastic process $\{x(t), -\infty < t < \infty\}$ are determined linearly by the "sample" random variables $\{x(nh), -\infty < n < \infty\}$. This may be summarized as follows: "the process x is determined linearly by its samples if and only if some set of frequencies Λ containing all the power of the process is disjoint from each of its translates $\Lambda - (r/h)$, $r = \pm 1, \pm 2, \cdots$ (that is no two frequencies in Λ differ by a multiple of $(1/h)$." Then Lloyd showed that such a linear dependence has the form of the sampling series (47) and discussed its convergence properties. Of the many theorems presented in this direction, we give the following theorem [73] and its corollary which is a generalization of the above Theorem IV-C-1. It is noted that most of the results for stochastic processes are based on their corresponding ones for deterministic signals.

Theorem IV-C-2: "If the spectral distribution of process x has an open support Λ whose translates $\{\Lambda - (n/h), -\infty < n < \infty\}$ are mutually disjoint then the sampling series is $(C, 1)$ summable in norm to $x(t)$; i.e.,

$$x(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) x(nh)\, K(t - nh)\text{"} \tag{48}$$

where

$$K(t) = h \int_{\Lambda} e^{2\pi i \lambda t}\, d\lambda, \quad -\infty < t < \infty. \tag{49}$$

A series Σa_j is said to be Césaro summable (or $(C, 1)$) if the mean $\sigma_N = (s_1 + s_2 + \cdots s_N/N)$ of its partial sums $s_1, \cdots, s_N$ converges (see Section V-A).

We may note that if Λ in (49) is the one interval $(-(1/2h), (1/2h))$ then $K(t - nh)$ is the familiar sampling function of (47). The following corollary considers the special case when Λ is a finite union of mutually disjoint open intervals $\{(\lambda_\alpha', \lambda_\alpha''), \alpha = 1, 2, \cdots, n\}$ where, according to (49):

Corollary: "if the set of frequencies Λ is a finite union of intervals, or more generally, if $\text{l.u.b.}_{-\infty < t < \infty} |tK(t)| < \infty$, then the sampling series converges in norm to $x(t)$; i.e.,

$$x(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=-N}^{N} x(nh)\, K(t - nh).\text{"} \tag{50}$$

Here l.u.b. stands for least upper bound which means, as it sounds, that for the set of real numbers A, if x is an upper bound for A and if y is any upper bound for A then $x \leqslant y$, then x is called the least upper bound of A or $x = \text{l.u.b.}\, A$. We may remark here that (50) is a generalization of (47) towards bandpass or multipass systems and away from the usual band-limited ones. The extension to multidimensional space of the sampling theorem of stationary stochastic variables was treated by Miyakawa [59] which combines Theorem IV-C-1 and Parzen's results [58] for the n-dimensional sampling. Miyakawa also considered the application of his extension to crystallography. Petersen and Middleton [53] derived optimum weighting, or sampling, functions for reconstructing the n-dimensional random field $f(\vec{x})$ using the sample measurement of its amplitude and gradient (38). Their criterion for the optimum reconstruction, of the estimate $\hat{f}(\vec{x})$ for $f(\vec{x})$, is to minimize the (statistical) mean-square error $E\ \{[f(\vec{x}) - \hat{f}(\vec{x})]^2\}$ at every point $\vec{x}$.

In a later paper, Balakrishnan [74] considered the question, "that a stationary stochastic process is not physically realizable." As an answer he spoke of "essentially band-limited stochastic processes."

For sampling over a finite interval $(0, T)$ instead of the usual infinite one $(-\infty, \infty)$ Lichtenberger [75] showed that sampling infinitely often over any finite interval $(0, T)$ taken at arbitrary discrete times $\{t_i\}$ leads to perfect reconstruction of an analytic random process $f(t)$. He considered a separable Gaussian random process $f(t)$, with its samples $\{f(t_i)\}$ taken at the arbitrary times $\{t_i\}$ over the fixed interval $(0, T)$, then constructed an estimate $\hat{f}_N(t)$ such that

$$\underset{N\to\infty}{\text{l.i.m.}}\ E\{|f(t) - \hat{f}_N(t)|^2\} = 0, \qquad -\infty \leqslant t \leqslant \infty \tag{51}$$

where

$$\hat{f}_N(t) = \sum_{i=1}^{N} a_i(t) f(t_i). \tag{52}$$

The proof utilized the Lagrange interpolation formula for representing $f(t)$ with its Nth partial sum for $\hat{f}_N(t)$, in (51), then letting $N \to \infty$. An error bound was derived as

$$\mathcal{E}_N = E\{|f(t) - f_N(t)|^2\} < \frac{\tau^{2N} R^{2N}}{[(N+1)!]^2} \tag{53}$$

where $\tau = \max(|t - t_1|, |t - t_N|)$ and R is some finite number defined as $|D^{(N)}R(t, s)| < R^N$. Here $R(t, s)$ is the covariance of $f(t)$ and $D^{(N)}f(t) \equiv (d^N f)/(dt^N)$. More general results in this direction were presented by Beutler [76].

Instead of the usual sampling at equidistant instants or the above arbitrary instants, Beutler and Leneman [77] considered random selection of the sampling points. Leneman [78] and then Leneman and Lewis [79]–[81] considered some specific related results. Barakat [82] used the sampling expansion in one [12], [84] and higher [59], [61] dimensions, in connection with nonlinear transformation of stochastic processes associated with Fourier transforms of band-limited positive functions.

1) Sampling Theorems for Nonstationary Random Processes: The sampling theorems presented so far in this section deal with wide-sense stationary random processes, while the rest of the paper deals mainly with deterministic signals. For nonstationary random processes, Zakai [83] was the first to present a sampling theorem followed by Piranashvili [84] then Gardner [85], who presented the following theorem which required a relatively simple proof and which was motivated toward applications.

Theorem IV-C-3: "Let x be a random process with autocorrelation function $k_x(t, s)$. If the double Fourier transform $K_x(f, \nu)$ of $k_x(t, s)$ satisfies the band-limiting constraint

$$K_x(f, \nu) \triangleq \iint_{-\infty}^{\infty} k_x(t, s)\, e^{-2\pi i(ft - \nu s)}\, dt\, ds = 0$$

for $|f| \geqslant (1/2T)$ and $|\nu| \geqslant (1/2T)$ (for some nonzero T), then

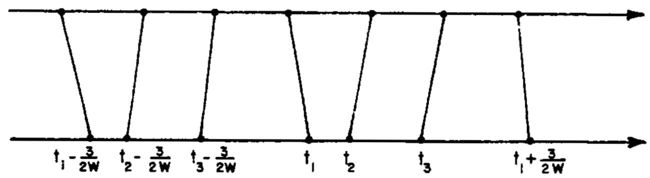

Fig. 3. Recurrent nonuniform sampling, $N = 3$.

x admits the mean-square equivalent sampling representation:"

$$E\left\{\left[x(t) - \sum_{n=-\infty}^{\infty} x(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}\right]^2\right\} = 0 \tag{54}$$

for all $t \in (-\infty, \infty)$."

The proof here is a formal one in the sense that (54) was expanded and the expectation was allowed to be exchanged with the infinite summation to yield the above result of (54). We may note how this theorem is related to two-dimensional deterministic function sampling. Sharma and Mehta [86] presented a generalized sampling theorem for nonstationary processes.

D. Sampling with Nonuniformly Spaced Sampling Points

For the case of a band-limited function $f(t)$ with all the sampling points outside the interval $(-T, T)$ being exactly zero, Shannon [1] remarked, as did others before him, that only then can $f(t)$ be specificed by $2WT$ sampling points where W is the bandwidth. He also remarked that these $2WT$ sampling points need not be equally spaced, an idea that obviously cannot be covered by his version of the WKS sampling theorem and its cardinal series. We review here some of the work which was done in this direction. The first is a statement which was attributed to Cauchy by Black [7, p. 41]:

> If a signal is a magnitude-time function, and if time divided into equal intervals such that each subdivision comprises an interval T seconds (sic) long, where T is less than half the period of the highest significant frequency component of the signal, and if one instantaneous sample *is taken from each sub-interval (sic) in any manner*, then a knowledge of the instantaneous magnitude of each sample plus a knowledge of the instant within each subinterval at which the sample is taken, contains all the information of the original signal.

Yen [87] considered the case where a finite number of uniform sample points migrate in a uniform distribution to new distinct positions. He proved that the band-limited signal $f(t)$ remains uniquely defined, then reconstructed $f(t)$. When the number of migrated points increases without limit he called it a gap and proved a similar theorem. Yen also considered the case of a "recurrent, nonuniform sampling." That is, when the sampling points are divided into groups of N points each, and the groups have a recurrent period of $N/2W$ s, as shown in Fig. 3 where W is the maximum frequency of the band-limited function $f(t)$. He determined $f(t)$ uniquely and reconstructed it in terms of its values at $t = t_p + (mN/2W)$, $p = 1, 2, \cdots, N$, and $m = \cdots, -1, 0, 1, \cdots$ as follows.

Theorem IV-D-1: "A bandwidth-limited signal is uniquely determined by its values at a set of recurrent sample points $t = \tau_{pm} = t_p + (mN/2W)$, $p = 1, 2, \cdots, N$; $m = \cdots, -1, 0, 1, \cdots$. The reconstruction is

$$f(t) = \sum_{m=-\infty}^{\infty} \sum_{p=1}^{N} f(\tau_{pm})\, \Psi_{pm}(t) \tag{55}$$

where

$$\Psi_{pm}(t) = \frac{\prod_{q=1}^{N} \sin \frac{2\pi W}{N}(t - t_q)}{\prod_{q=1 \neq p}^{N} \sin \frac{2\pi W}{N}(t_p - t_q)} \cdot \frac{(-1)^{mN}}{\frac{2\pi W}{N}\left(t - t_p - \frac{2mN}{2W}\right)}." \tag{56}$$

Recently, Sankur and Gerhardt [88] considered various methods for reconstructing a continuous signal from its nonuniform samples. They employed and compared a number of techniques including low-pass filtering, spline interpolation, and Yen's [87] interpolation. The spline- or hill-function-type interpolation is a special polynomial expansion which is relatively new, with best approximation properties (see Section VII-C). Their observation, from the simulation experiments with these and other techniques, was that even though Yen's method was impractical to realize, it still proved superior to the other methods. This is in the sense that it is insensitive to sample migration and signal-to-noise ratio (SNR)

$$\text{SNR} = \frac{\sum_{i=1}^{N} s_i^2}{\sum_{i=1}^{N} (s_i - \hat{s}_i)^2} \tag{57}$$

where the s_i are the signal samples and $\hat{s}_i$ are the samples from the reconstructed signal. Recently Marvasti and Gerhardt [89] presented a practical treatment for signal transmission using nonuniform sampling. The special case of Yen's nonuniform but periodic sampling was extended to higher dimensions by Gaarder [64] with explicit sampling series which he then applied to nonrectangular lattices.

Yao and Thomas [90] derived sampling representation for band-limited functions when the sampling instants are not necessarily spaced uniformly but each deviate less than $(1/\pi)$ ln $2 \simeq 0.22$ from its corresponding Nyquist instant, as required by the WKS sampling theorem. They termed such representation as "semiuniform" and used nonharmonic Fourier series for its derivation. Finally they remarked that a sample representation is not possible when all the sampling instants are allowed to deviate nonuniformly by (1/4) unit from their corresponding Nyquist instants, or if an arbitrary finite number of the sampling instants are placed arbitrarily, or if additional sample points are added. Prior to this, Beutler [91, p. 111], in his unified approach to sampling theorems (see Section V-C), treated the same "perturbation" question and concluded that the sampling times need not be periodic, but may vary from the true periodicity by over 20 percent without sacrificing capability of restoring the signal $f(t)$. Also, Leneman [92] presented error bounds for jittered sampling. In a later paper, Yao and Thomas [93] considered the question of the stability of the WKS sampling expansion in the sense that a small change in the amplitude of sample values should lead to small changes in the reconstructed function. This subject will be discussed in Section VI-D. They showed that the uniform Lagrange interpolation sampling expansion preserves some stable properties while a general nonuniform sampling expansion need not possess these stability properties. For their "semiuniform sampling expansion" [93] where $|t_n - (n\pi/a)| \leqslant d < (1/4)$ they showed that it is stable while for the nonuniform sampling, i.e., when $d > (1/4)$ the Lagrange interpolation sampling expansion is not stable [93, Theorem 1]. They also gave a simple example with uniform sampling that is not stable (see Section VI-D). Beutler [94] considered and proved what is called the "folk theorem" in the sense that a signal $f(t)$ may be represented by any linear combination of irregularly spaced samples $f(t_n)$, provided that the average sampling rate exceeds the Nyquist rate, i.e., that the number of samples per unit time exceed (on the average) twice the highest frequency present in the signal. Also he showed that only the past need to be sampled at an average rate greater than the Nyquist rate to assure error-free recovery. Even more, the recovery is sometimes feasible if the average rate is less than the Nyquist rate, e.g., if sampling is concentrated in rare bursts of higher than the Nyquist rate sampling. Like Yao and Thomas [93], his proofs utilized nonharmonic series expansion, but within a more general mathematical setting. Furthermore, Beutler [94] applied his results to deterministic as well as wide-sense stationary stochastic processes.

In summary, for a band-limited function on $(-a, a)$ the freedom of having irregular sampling, or allowing the sampling instants t_n to deviate from those of the Nyquist instants $(n\pi/a)$, stems from some theorems due to Levinson [95]. These theorems give conditions on a set of real numbers $\{t_n\}$ which assure that

$$\int_{-a}^{a} e^{i\omega t_n} g(\omega)\, d\omega = 0, \qquad \text{for all } n,\ g \in L_p(-a, a) \tag{58}$$

implies that $g = 0$ almost everywhere. Here $g \in L_p(-a, a)$ means that $\int_{-a}^{a} |g(\omega)|^p \, d\omega < \infty$ where the usual case of $p = 2$ defines finite energy signals. Also the condition (58) defines the set $\{e^{i\omega t_n}\}$ as a *closed* set.

Brown [96] treated the nonuniform sampling for band-limited and finite energy signals using a finite energy and band-limited Lagrange interpolating function. He gave conditions on the nonuniform sampling instants $\{t_n\}$ such that $f(t)$ is uniquely determined by the sample values $\{f(t_n)\}$ and possesses a uniformly convergent representation

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)\, \Psi_n(t), \quad -\infty < t < \infty. \tag{59}$$

The function $\Psi_n(t)$ is a Lagrange interpolating function which is band-limited to the same band as $f(t)$ and $\Psi_n(t_k) = \delta_{n,k}$ for integers n and k. The conditions on the sequence $\{t_n\}$ are that it is both stable, as defined by Yao and Thomas [93], and exact. By "exact" or "minimal" set it is meant that the closure, as defined by (58) of the set $\{e^{i\omega t_n}\}$ on the interval $(-a, a)$, is destroyed by the deletion of any single term from it.

A detailed treatment of the sampling theorem including nonequidistant sampling points is presented in Churgin and Iakovlev [97].

E. Sampling for Bandpass Functions

Kohlenberg [98] was the first to consider sampling expansions for a bandpass function which lies in the frequency band $(W_0,\ W_0 + W)$ instead of the usual low-pass function with band limits $(-W,\ W)$. This is to be distinguished from the bandpass function which vanishes outside the intervals $[W_0, W_0 + W] \cup [-W_0 - W, -W_0]$. Because of the possible nonuniqueness of the equispaced sampling expansion, he introduced what he termed "second-order sampling" which guaranteed a unique representation. Second-order sampling involves two interleaved sequences of equispaced sampling points. In general a pth-order sampling is defined as

$$g(t) = \sum_{i=1}^{p} g_i(t) = \sum_{i=1}^{p} \sum_{n} f(a_i n + k_i) S_i(t - a_i n - k_i) \quad (60)$$

in which the ith sampling series $g_i(t)$ has particular "sample spacing" a_i, "phase" k_i, and "sampling function" $S_i(t)$. He first considered a first-order sampling with $a_1 = a = (1/2W)$, $k_1 = 0$ for band-limited function $f(t)$ in the frequency range $(0, W)$ and used Fourier analysis to obtain the WKS sampling series

$$f(t) = \sum_{n} f\left(\frac{n}{2W}\right) \frac{\sin \pi (2Wt - n)}{\pi(2Wt - n)} \quad (61)$$

where the samples are independent, since $S_1(0) = 1$ and $S_1(n/2W) = 0$ for $n = +1, \pm 2, \cdots$, and the sampling rate of $2W$ per second is minimum. For the case of first-order sampling with $a_1 = a < (1/2W)$

$$f(t) = \sum_{n} f(an) \frac{\sin 2\pi W(t - an)}{2\pi W(t - an)}. \quad (62)$$

It is clear that the samples are not independent since $S_1(an) \neq 0$ for $n \neq 0$. To treat this problem for the general case of bandpass function $f(t)$ on $(W_0, W_0 + W)$, a necessary and sufficient condition of $W_0 = cW$, $c = 0, 1, 2, \cdots$, was found to permit the exact construction of $f(t)$ from its samples at the minimum rate of $2W$ per second. In contrast, the following second-order sampling expansion ($a_1 = a_2 = (1/W)$, $k_1 = 0$, $k_2 = k \neq 0$ in (60)), permits the use of $2W$ samples per second for any W_0 and W [98], [12, p. 215].

Theorem IV-E-1: For a function $f(t)$ in a band $(W_0, W_0 + W)$, the exact interpolation formula is

$$f(t) = \sum_{n} \left[f\left(\frac{n}{W}\right) S\left(t - \frac{n}{W}\right) + f\left(\frac{n}{W} + k\right) S\left(\frac{n}{W} + k - t\right) \right] \quad (63)$$

$$S(t) = \frac{\cos [2\pi(W_0 + W) t - (r + 1) \pi Wk] - \cos [2\pi(rW - W_0) t - (r + 1) \pi Wk]}{2\pi Wt \sin (r + 1) \pi Wk} + \frac{\cos [2\pi(rW - W_0) t - r\pi Wk] - \cos [2\pi W_0 t - r\pi Wk]}{2\pi Wt \sin r\pi Wk}. \quad (64)$$

In (63), we have two groups of samples each at a rate of W per second with spacing $(1/W)$ shifted by a phase k from each other. k in (64) is a constant such that $kWr, kW(r + 1) \neq 0, 1, \cdots$, and where r is an integer such that $(2W_0/W) < r < 2W_0/W + 1$. Such development of sampling for bandpass functions is discussed in Middleton [12, p. 215], and was also derived by Linden [51] and Parzen [58, Theorem 4] using somewhat similar but more direct and simpler methods of Fourier analysis. Linden [51] relied on the convolution theorem of Fourier analysis and very clear graphical illustrations to derive (63) and (64) and also gave second-order sampling expansion for the usual band-limited function. The result (63), (64) can be derived easily with the help of the Hilbert transform [17, p. 76].

1) The Generalized (WKSK) *Sampling Theorem for Bandpass Functions:* Sharma and Mehta [65] derived the sampling expansion for bandpass functions represented by more general integral transforms than the Fourier transform. This is a generalization of the WKSK sampling theorem (Theorem III-A-1) where the frequency spectrum vanishes outside the bandpass region $R_{BP} = [W_0 - \pi W, W_0 + \pi W] \cup [-W_0 - \pi W, -W_0 + \pi W]$ instead of the interval I in (19)–(21). Their main result [65, Theorem 2.3] is the following.

Theorem IV-E-2: "Let $g(\omega)$ be a complex valued function on $-\infty < \omega < \infty$ with $g(\omega) \in L_1$, i.e., $\int_{-\infty}^{\infty} |g(\omega)| \, d\omega < \infty$, and let $K(t, \omega)$ be a complex function of time such that $|K(t, \omega)| = |K(t, -\omega)|$. Consider a bandpass signal $f(t)$ which is real valued and band-limited to the bandpass region $R_{BP} = [-W_0 - \pi W, -W_0 + \pi W] \cup [W_0 - \pi W, W_0 + \pi W]$. If

$$f(t) = \int_{R_{BP}} g(\omega) K(t, \omega) \, d\omega \quad (65)$$

then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{W}\right) L_n(t) \quad (66)$$

where

$$L_n(t) = \frac{2 \int_{R_{BP}} K(t, \omega) \overline{K\left(\frac{n}{W}, \omega\right)} \, d\omega}{\int_{R_{BP}} \left| K\left(\frac{n}{W}, \omega\right) \right|^2 d\omega}." \quad (67)$$

The explicit expression for $L_n(t)$ and an example of $K(t, \omega) = J_0(\omega, t)$, the Bessel function of the first kind of zeroth order, were also presented. We may remark that the method of proving this general result is a simple and straightforward one which parallels the proof for the generalized (WKSK) sampling theorem (Theorem III-A-1). But in contrast to the remark made by Kohlenberg [98] and others [12], [51], [58], concerning the required sampling rate in order to guarantee the algebraic independence of the samples, no such remark was mentioned for the above generalization or its special case. However the above Theorem IV-E-2 was extended to higher dimensions [65, Theorem 3.1] which is a generalization of Theorem IV-A-1 for bandpass functions with more general kernels.

F. Implicit Sampling

All the sampling expansions that we have discussed up till now may be termed "explicit" samplings in the sense that a band-limited function $f(t)$ is represented in terms of its samples $f(t_n)$ at preselected instants $\{t_n\}$ which are independent of $f(t)$. "Implicit" sampling may refer to the case when the function is represented in terms of the instants $\{t_n\}$ in which the function assumed a predetermined value, for example its zero crossings $\{t_n : f(t_n) = 0\}$ or its crossings with a cosine function $\{t_n : f(t_n) = C \cos 2\pi w t_n\}$. The first "implicit" sampling expansion was considered by Bond and Cahn [99] as they extended the WKS sampling theorem when the sampling instants $\{t_n\}$ are not independent of the sampled signal $f(t)$. Their justification was that such a procedure had proved valuable in minimizing the error caused by infinite clipping, which means that we can transmit a continuous signal over a discrete channel if the zero crossings of $f(t)$ are preserved. For $f(t)$, a band-limited function on $(0, W)$, they extended t to a complex variable z and used Titchmarsh's [100] result that

$$F(z) = \int_{-W}^{W} e^{2\pi i f z}\, V(f)\, df \tag{68}$$

is a real, entire function, described by the location of its zeros which are either real or occur as complex conjugate pairs. In general, the zeros tend to cluster near the real axis. Furthermore, the aggregate of the zeros occur at the Nyquist rate. Thus

$$f(z) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \tag{69}$$

where $f(0) \neq 0$, $z_n = R_n e^{i\theta_n}$, $R_n \leqslant R_{n+1}$, and $\lim_{n\to\infty} (2WR_n/n) = 1$. Note that the formula (69) needs all the past and future zeros, both real and complex, which makes it impracticable. Instead, they suggested another, more practicable problem with specified interval $(-(T/2), (T/2))$ and zeros inside this interval occurring at slightly less than the Nyquist rate, zeros outside are real and occurring at the Nyquist rate. Let N be the largest integer not exceeding WT; then there are a maximum of $2N$ real or complex zeros, $z_n = t_n + iu_n$, $|t_n| < T/2$. Outside this interval the zeros occur at $t_n = \pm(n/2W)$, for $n = N+1, N+2, \cdots$. Using this in (69) and referring to the infinite product representation of the sine function, they obtained

$$f(t) = \sum_{n=-N}^{N} (-1)^n A_n \frac{\sin(2\pi W t - n\pi)}{2\pi W t - n\pi} \tag{70}$$

where A_n is expressed in terms of the values of the $2N$ zeros z_m inside the interval

$$A_n = f(0) \frac{\prod\limits_{m} \frac{1}{2Wz_m} \prod\limits_{m} (2Wz_m - n)}{\prod\limits_{m=1}^{N} - \left(\frac{1}{m}\right)^2 \prod\limits_{\substack{m=-N\\ m\neq n}}^{N} (m - n)} \tag{71}$$

where m in the numerator of (71) as the index of the zeros within the interval $(-(T/2), (T/2))$. Later Bond, Cahn, and Hancock [101] found a relation between the above "implicit sampling" and the Fourier coefficients that allows a Fourier series representation of a band-limited function in terms of its zero crossings. More work on the subject of implicit sampling was done by Voelker [102] which was simulated on a computer by Sekey [103]. Bar-David [104] considered the important case of implicit sampling in terms of real variables alone. For example, the instants $\{t_n\}$ at which a bounded band-limited function $f(t)$ crosses a cosine function $\{t_n : f(t_n) = \cos 2\pi w t_n\}$, where $f(0)$ and $\{t_n\}$ determine $f(t)$ uniquely. He considered bounded functions which are band-limited in the usual sense or as extended by Zakai [83], to give the following implicit sampling theorem.

Theorem IV-F-1: "Let $f(t)$ be a *bounded* band-limited function of bandwidth W_0 such that the sampling expansion

$$f(z) = \lim_{n\to\infty} \sum_{-n}^{n} f\left(\frac{k}{2w}\right) \frac{\sin(2\pi w z - k\pi)}{(2\pi w z - k\pi)} \tag{72}$$

converges uniformly, for $w > W_0$, in any bounded region of the z-plane. Let $C > B \geqslant |f(t)|$ and let

$$\{t_k\} = \{t : f(t) = C \cos 2\pi w t\}, \quad k = , \cdots, -2, -1, 0, 1, 2, \cdots. \tag{73}$$

Then the following infinite product also converges uniformly, though conditionally, in the same region:

$$f(z) = [f(0) - C] \lim_{n\to\infty} \prod_{-n}^{n} \left(1 - \frac{z}{t_k}\right) + C \cos \pi w z. \tag{74}$$

A sufficient condition for convergence is that $t_{\pm k}$ should indicate the kth zero to the right (left) of the origin."

G. Sampling for Generalized Functions (Distributions)

The extension of the WKS sampling theorem (1), (2) to band-limited generalized functions was first considered by Campbell [105]. He noted that the WKS sampling expansion

$$f(t) = \sum_{-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} \tag{75}$$

of the band-limited function

$$f(t) = \int_{-\Omega}^{\Omega} e^{-j\omega t} g(\omega)\, d\omega \tag{76}$$

where $g(\omega)$ is integrable, is valid when $g(\omega)$ is replaced by the Dirac delta function $\delta(\omega - a)$, which is a special case of a generalized function. In this case it is obvious that (75) reduces to

$$e^{-jat} = \sum_{n=-\infty}^{\infty} \exp - \frac{jn\pi a}{\Omega} \left[\frac{\sin \Omega(t - n\pi)}{\Omega(t - n\pi)}\right] \tag{77}$$

as a Fourier series expansion of the function $f(a) = e^{-jat}$. However when $g(\omega) = \delta'(\omega - a)$, the derivative of $\delta(\omega - a)$, the Fourier transform (76) in this case is $jt\, e^{-jat}$ and $f(n\pi/\Omega) = (jn\pi/\Omega)\, e^{-(jn\pi/\Omega)} = O(n)$, which makes the series (75) diverge. Here we let O and o have their usual meaning, i.e., $F(x) = O(g(x))$ means that there exists an M such that $F(x) \leqslant Mg(x)$ and $F(x) = o(g)$ means that $\lim_{x\to x_0} (F(x)/g(x)) = 0$. Thus Campbell concluded that the WKS sampling theorem (75) does not extend to the Fourier transform of an *arbitrary* distribution with bounded support. He then investigated functions which are Fourier transforms of distributions with bounded support and showed that these band-limited distribution functions are still entire and are completely determined by their sample values at $n\pi/\Omega$. Here Ω serves as a bandwidth for the support $(-\Omega, \Omega)$, which we shall present next as Theorem IV-G-1. The statements of Campbell's theorems need a few definitions and the usual notation, as given in Zemanian [106]. One of his main results in this direction is the following.

Theorem IV-G-1: "Let $g(\omega)$ be a distribution with support contained in the open interval $\{\omega : |\omega| < (1 - q)\,\Omega\}$ where $0 < q < 1$. Let $f(t)$ be the Fourier transform of $g(\omega)$. Then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{(\Omega t - n\pi)} S(q[\Omega t - n\pi]) \tag{78}$$

where $S(y)$ is the additional factor defined as

$$S(y) = \frac{\int_{-1}^{1} \exp\,[1/(x^2 - 1) - jxy]\, dx}{\int_{-1}^{1} \exp\,[1/(x^2 - 1)]\, dx}.\text{"} \tag{79}$$

Campbell [105] also derived an expression for the truncation error (see (181)) of (78) that reduced to previously derived errors of the usual WKS sampling series, which we shall present at the end of Section VI-A.

Pfaffelhuber [107] presented additional results for the band-limited generalized functions which include that, for a suitably restricted space of test functions, the WKS sampling theorem expansion is valid in its classical form, and that a band-limited generalized function $f(t)$ can be represented by a series of delta functions

$$f(t) = \frac{\pi}{\Omega} \sum_{n=-\infty}^{\infty} f(t_n)\, \delta(t - t_n) \tag{80}$$

concentrated at the sampling points $\{t_n\}$ with weights equal to the sampling values $f(t_n)$. This means that, as it is the case for ordinary band-limited functions, the information contained in the whole signal is equal to the information provided by the sample values at $\{t_n\}$.

As we have indicated [29] in Section IV-A for sampling in n dimensions and other extensions of the sampling theorem, the proper extension of Parseval's equation [108, p. 64] offers the simplest method of proof.

Pfaffelhuber [107] also gave a representation for band-limited distribution that looked like a combination of a Taylor series and a conventional sampling series [107, Theorem 2].

H. Sampling for Time-Varying Systems with Time-Varying Bands

Horiuchi [109] was the first to extend the WKS sampling theorem for the analysis of continuous signals specified by time-varying spectra with time-varying bands. This is in line with what we had presented in Section III-C for using the generalized WKSK sampling theorem for time-varying systems. In both cases the analysis is justified only when the fluctuations of the time-varying parameters of the system are predicted in advance.

Consider the class of continuous signals

$$f(t) = \frac{1}{2\pi} \int_{2\pi w_1(t)}^{2\pi w_2(t)} F(\omega, t)\, e^{i\omega t}\, d\omega \tag{81}$$

with time-varying spectrum $F(\omega, t)$ and time-varying bands, $2\pi w_2(t)$, $2\pi w_1(t)$. Here $w_1(t)$ and $w_2(t)$ are bounded real-valued piecewise-continuous functions such that $w_2(t) > w_1(t)$, $w_2(t) > 0$.

Signal $f(t)$ of (81) can be specified by a time-varying signal

$$f(t, \tau) = \frac{1}{2\pi} \int_{2\pi w_1(\tau)}^{2\pi w_2(\tau)} F(\omega, \tau)\, e^{i\omega\tau}\, d\tau \tag{82}$$

and its Fourier transform

$$F(\omega, \tau) = \int_{-\infty}^{\infty} f(t, \tau)\, e^{-i\omega t}\, dt \tag{83}$$

where $f(t) \equiv f(t, t)$.

Since the function $F(\omega, t)$ vanishes outside the interval $B: 2\pi w_1(t) \leqslant w \leqslant 2\pi w_2(t)$, the signal $f(t)$ is said to have the time-varying spectrum $F(\omega, t)$ and the time-varying band B. Also for constant w_1 and w_2 in (81) this represents the band-pass signal which we discussed in Section IV-E.

The expansion for the signal $f(t)$ of (81) in terms of the samples of $f(t, \tau)$ in (82) is

$$f(t) = \sum_{k=-\infty}^{\infty} f\left\{\frac{k}{2w(t)}, t\right\} \phi_k(t) \tag{84}$$

where

$$w(t) = \tfrac{1}{2}\,[w_2(t) - w_1(t)] \tag{85}$$

$$\phi_k(t) = \frac{\sin \pi\{2tw(t) - k\}}{\pi\{2tw(t) - k\}} \exp\left\{2\pi i w_0(t)\left[t - \frac{k}{2w(t)}\right]\right\} \tag{86}$$

$$w_0(t) = \tfrac{1}{2}\,\{w_1(t) + w_2(t)\}. \tag{87}$$

The derivation of the sampling expansion (84) is obtained simply by writing the Fourier series of $F(\omega, t)$ in terms of $\exp\{-(ik/2w(t))\,\omega\}$

$$F(\omega, t) = \frac{1}{2w(t)} \sum_{k=-\infty}^{\infty} f\left\{\frac{k}{2w(t)}, t\right\} \exp\left\{-\frac{ik}{2w(t)}\,\omega\right\} \tag{88}$$

then substituting in (81).

We note that the coefficients $f((k/2w(t)), t)$ of (84), as specified by (82), are not the same as the samples $\{f(t_k)\}$ of the signal $f(t)$ except at the zeros $\{t_k\}$ of the equation

$$2tw(t) - k = 0. \tag{89}$$

In this case, $\phi_k(t)$ may be called a pseudo-sampling function since it plays the role of the usual sampling function

$$\phi_k(t_n) = \delta_{k,n}, \qquad k, n = 0, \mp 1, \cdots. \tag{90}$$

Horiuchi [109] then considered some special cases and showed that only for restricted cases the expansion (84) can be realized as a sampling expansion with the usual physical interpretation of the WKS sampling theorem. We may remark here that the expansion (84) may be realized when we consider the generalized WKSK sampling theorem and its physical interpretation in terms of time-varying systems (see Section III-C). Applications of both the WKS and the generalized WKSK sampling theorems to time-varying systems are discussed in Section VII-B.

I. Other Extensions

Papoulis [13], [14], [110], [111] presented and discussed in detail various extensions of the WKS sampling theorem. Some of these extensions were utilized to give a more practical physical interpretation of the sampling theorem than the one associated with an ideal low-pass filter (Section I-B). He then presented different error bounds for the sampling expansion which we shall discuss in Sections VI-A and VI-B. We will present here the main extensions which lead to such relaxed physical interpretation of the sampling series to which we hinted in Section I-B (see Fig. 1 and Fig. 2, Section I-B). Papoulis considered a band-limited signal $f(t)$

$$f(t) = \frac{1}{2\pi} \int_{-w_1}^{w_1} F(\omega)\, e^{j\omega t}\, d\omega \tag{91}$$

but constructed it in a more general way than that of the WKS sampling theorem (4) to give

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin w_0(t - nT)}{w_2(t - nT)} \tag{92}$$

where $w_2 \equiv (\pi/T) \geqslant w_1$, $w_1 \leqslant w_0 \leqslant 2w_2 - w_1$. We note here that with the band limit w_1, the sampling spacing $T \equiv (\pi/w_2) \leqslant (\pi/w_1)$, which means that such a relaxed extension (92) requires higher sampling rates. The proof of (92) is considered to be a particularly elegant version [14, p. 120] of the proof of the WKS sampling theorem. Papoulis also proved a converse to the above result, i.e., "Given an arbitrary sequence of numbers $\{a_n\}$, if we form the sum

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin w_0(t - nT)}{w_2(t - nT)} \tag{93}$$

then $x(t)$ is band-limited by w_0." The proof assumes the Fourier series expansion of $F(\omega)$ then multiplication by $e^{-jnT} p_{w_0}(\omega)$ and integration term by term. Of course, we must have a condition on the coefficients $\{a_n\}$ that allows such term by term integration. A sufficient condition is that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. A number of theorems in this direction for the WKS and the WKSK sampling theorems are presented in [34]. Ericson and Johansson [112] also derived necessary and sufficient conditions for some variations of the WKS sampling series.

Papoulis [14] then presented the sampling expansion for $f^2(t)$, instead of $f(t)$, of (92) as

$$f^2(t) = \sum_{n=-\infty}^{\infty} f^2(nT) \frac{\sin w_0(t - nT)}{w_2(t - nT)} \tag{94}$$

where $w_2 = (\pi/T)$, $w_2 \geqslant 2w_1$ (instead of $w_2 \geqslant w_1$ for $f(t)$ in (92)), and w_0 is such that $2w_1 \leqslant w_0 \leqslant 2w_2 - 2w_1$. We note here how (94) with $T \leqslant (\pi/2w_1)$ requires more than double the usual sampling rate. Papoulis then used this result (94) for deriving the round-off error of the sampling series which we shall discuss in Section VI-C.

More significance should be assigned to the sampling expansion (94) since in applying the sampling theorem to scattering problems or crystallography it is the intensity $|f(t)|^2$, and not the wave function $f(t)$, that is to be constructed from its measured samples $|f(nT)|^2$, as we shall see in Section VII-A.

Papoulis' most recent generalization [111] of the Shannon sampling theorem is to express the band-limited signal

$$f(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} F(\omega)\, e^{i\omega t}\, d\omega \tag{95}$$

in terms of the sample values $g(nT)$ of the output

$$g(t) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} F(\omega)\, H(\omega)\, e^{i\omega t}\, d\omega \tag{96}$$

of a system $H(\omega)$ driven by $f(t)$. The sampling expansion is

$$f(t) = \sum_{n=-\infty}^{\infty} g(nT)\, y(t - nT) \tag{97}$$

where

$$y(t) = \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \frac{e^{i\omega t}}{H(\omega)}\, d\omega \tag{98}$$

and the proof is a straightforward one after writing the Fourier series expansion for $(e^{i\omega t}/H(\omega))$ on $(-\sigma, \sigma)$.

Another extension of the sampling theorem is the prediction of band-limited processes from past samples. Brown [113] considered $f(t)$ as either a deterministic or a stochastic signal which is band-limited to the frequency interval $|\omega| \leqslant \pi$

$$f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\omega)\, e^{i\omega t}\, d\omega, \quad -\infty < t < \infty \tag{99}$$

with $|F|^2$ integrable on $[-\pi, \pi]$. He showed that for any constant sampling spacing T satisfying $0 < T < (1/2)$, $f(t)$ may be approximated arbitrarily well by a linear combination of past samples $f(t - kT)$ taken at any constant rate that exceeds twice the associated Nyquist rate

$$\lim_{n\to\infty} |f(t) - \sum_{k=1}^{n} a_{kn} f(t - kT)| = 0 \tag{100}$$

uniformly for $-\infty < t < \infty$. Here the coefficients $a_{kn} = (-1)^{k+1} (\cos \pi T)^k \binom{n}{k}$ are independent of the detailed structure of the signal $f(t)$. This result provides a sharpening of a previous result by Wainstein and Zubakov [114] which requires a sampling rate in excess of three times the Nyquist rate. Beutler [94] showed that there exist coefficients for recovering the signals with sampling rates required to exceed only the Nyquist rate. However, the explicit form for such coefficients are not given and in general they are not independent of the structure of the predicted function $f(t)$.

Maeda [115] treated the sampling theorem for band-limited periodic signals [17] with nonuniformly spaced points. These results were extended by Isomichi [116] to band-limited signals with finite energy.

Among other extensions of the WKS sampling theorem are the integral sampling [117] in which the sample is taken over a whole sampling period T and the sampling for signals as solutions of nth-order linear differential equations with constant coefficients [118]. Holt, Hill, and Linggard [117] developed a network that allows integrating the input signal $f(t)$ over the whole sampling period. The sampled output signal $f^*(t)$ of such a circuit is expressed as

$$f^*(t) = \sum_{n=0}^{\infty} \left\{ \int_{nT-T}^{nT} f(\tau)\, d\tau - \int_{nT-2T}^{nT-T} f(\tau)\, d\tau \right\} u(t - nT) \tag{101}$$

where $u(t)$ is the unit step function. In addition to his detailed presentation and applications of the sampling theorem, Lathi [18] considered what is termed "natural" sampling where very narrow pulses of finite width are considered instead of the instantaneous impulses of the WKS sampling theorem. Kishi and Maeda [118] considered nth-order linear differential equations with constant coefficients and showed that their solutions obey sampling theorems similar to those of band-limited functions. Their main result is that a waveform $f(t)$ which is a finite linear combination of $t^m e^{\alpha t}$ or $t^n \sin(\omega t + \theta)$ can be constructed from a finite set of its sample values at equal intervals. In addition they showed that $f(t)$ can also be constructed uniquely in terms of the samples of the derivative $f'(t)$ as well as $f(t)$. Maeda [119] showed the relations between such signals [118] and band-limited signals, and then gave some theorems [120] for the interpolatory functions. Kishi and Maeda [121] followed this treatment by an application to waveform approximation.

For integral transforms with infinite limits, like the Hilbert transform, Linden [51] gave a sampling expansion in terms of the samples of a bandpass function and its Hilbert transform. Papoulis [14, p. 130] presented a sampling expansion for a signal represented by the infinite limit Hilbert transform. The sampling expansion for the infinite limit Laguerre-$L_\nu^\alpha(x)$ transfrom (33) was presented in Section III-F.

Wunsch [122] and Kioustelidis [123] considered sampling of duration- (or time-) limited functions, instead of band-limited functions. Recently Butzer and Splittstosser [124] presented a more detailed treatment of such sampling expansion for continuous duration-limited functions whose spectrum is absolutely integratable. They also gave a bound on the truncation error of such series. Slepian and Pollak [30] treated the problem of reconstructing a finite-duration-finite-energy (FDFE) signal that is observed through an ideal low-pass filter. They showed that such construction could be performed without error by expanding the time-limited signal as a series of prolate spheroidal functions. Stuller [125] used certain time domain sampling arguments, along with the series of prolate spheroidal functions [30], to derive an interpolation formula for the FDFE signal from equally spaced samples of the observed waveform. He showed that in the noiseless case, perfect reconstruction of the FDFE signal can be obtained when the sampling rate exceeds one half the minimum sampling rate specified by the WKS sampling theorem. The limitations imposed by measurement noise were also described. Kramer [126] developed a very useful property of the band-limited functions in the sense that in digital computations "continuous" operations are replaced by "discrete" ones. In particular he gave explicit relations between the samples of the higher derivative of a band-limited function $f(t)$ and the samples of $f(t)$. This was also done for bandpass functions.

V. Different Methods, Conditions, and Representation of the Sampling Series

In this section we will outline most of the methods used in deriving the WKS and the WKSK sampling series. The emphasis here is to relax the conditions on the sampled signals. This includes sampling for not necessarily finite energy signals, i.e., signals whose transforms are not necessarily square integrable but may be absolutely integrable. Besides the usual finite limit integral representation of the signal, a triple integral representation, that allows different physical interpretations will also be presented. This will be concluded by a summary of the various attempts to unify the different aspects of the sampling expansion in the sense of a general mathematical setting which we will not pursue here in much detail. We refer the interested reader to the original papers.

A. Different Methods and Conditions for Deriving the Sampling Series

In Sections II and III, we presented the WKS and the generalized WKSK sampling theorems and offered the usual proofs that included contour integration for the sampled function $f(t)$ and orthogonal expansion for the kernel $K(x, t)$ or the transformed function $F(x)$ in

$$f(t) = \int_I K(x, t)\, F(x)\, dx \tag{19}$$

where both $F(x)$ and $K(x, -)$ are assumed to be square integrable, i.e., $F(x), K(x, -) \in L_2(I)$ and where I is a finite interval. In the case of the WKS sampling theorem $K(x, t) = e^{ixt} \in L_2(I)$ and so it is the purpose of this section to investigate relaxing the condition $F(x) \in L_2(I)$ on the transformed function $F(x)$.

A simple mode of proof was offered by Brown [127] for the WKS sampling theorem which employed the prototype Parseval equation; that is, when $g, h \in L_2(I)$ and c_n, d_n are their respective Fourier coefficients, for the orthogonal expansion in terms of $\{K(x, t_n)\}$, then

$$\int_I g(x)\, \overline{h(x)}\, dx = \lim_{N \to \infty} \sum_{|n| \leq N} c_n \overline{d_n}\, \|K(x, t_n)\|_2^2 \tag{102}$$

where

$$\|K(x, t_n)\|_2^2 = \int_I |K(x, t_n)|^2\, dx.$$

Brown considered the band-limited function

$$f(t) = \int_{-a}^{a} e^{ixt}\, F(x)\, dx \tag{103}$$

and employed (102) with $g(x) = e^{ixt}$, $h(x) = F(x); K(x, t_n) = e^{i(n\pi x/a)}$. Clearly, $d_n = (1/2a) f(n\pi/a)$, $\|K(x, t_n)\|_2^2 = 2a$, and

$$c_n = S_n(t) = \frac{\sin (at - n\pi)}{(at - n\pi)}. \tag{104}$$

When (104) is used in (102) we obtain

$$f(t) = \int_{-a}^{a} e^{ixt}\, F(x)\, dx = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{a}\right) \frac{\sin (at - n\pi)}{(at - n\pi)} \tag{105}$$

as the WKS sampling series.

Unless otherwise indicated, a summation like $\Sigma\, c_n$ will assume limits as in (102) while $\Sigma^n c_n$ signifies the nth partial sum. It is clear that this method of proof is valid for the WKSK sampling theorem when c_n and d_n are given in (23) and (21) as Fourier coefficients for $F(x)$ and $K(x, t)$ of (61), respectively. The WKSK theorem can also be proved by using the Schwarz inequality [9] on $f(t)$ in (19)–(20):

$$\left| f(t) - \sum^n f(t_n)\, S_n(t) \right|^2$$

$$= \left| \int_I \left[K(x, tn) - \sum^n K(x, t_n)\, S_n(t) \right] F(x)\, dx \right|^2$$

$$\leq \int_I \left| K(x, t) - \sum^n K(x, t_n)\, S_n(t) \right|^2 dx \int_I |F(x)|^2\, dx \tag{106}$$

and noting that the orthogonal series inside the last integral converges in the mean to the square integrable kernel $K(x, t)$ as

$$K(x, t) = \underset{N \to \infty}{\text{l.i.m.}} \sum_{|n| \leq N} S_n(t)\, K(x, t_n). \tag{107}$$

In attempting to move away from the condition of square integrability, or finite energy, band-limited signals, Brown [127] raised a question concerning the necessity for a "new mode of proof" when $F(x) \notin L_2(I)$. He gave the example of Bessel function of zeroth order

$$J_0(\pi t) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\exp(ixt)\, dx}{\sqrt{\pi^2 - x^2}} \tag{108}$$

which is band-limited to $[-\pi, \pi]$ and with a Fourier transform $F(x) = (2/\sqrt{\pi^2 - x^2})$ which is in $L_1(-\pi, \pi)$, i.e., $\int_{-\pi}^{\pi} |2/\sqrt{\pi^2 - x^2}|\, dx < \infty$ but not in $L_2(-\pi, \pi)$. Hence, his version of using the above prototype Parseval equation or the other method of using the Schwarz inequality [9] cannot be used for they both require $F(x) \in L_2(-\pi, \pi)$. To answer such questions, we will make use of the *Hölder inequality*, as opposed to its special case, the Schwarz inequality, and the many extensions of the Parseval equation [29]. Among the various methods for relaxing the conditions on $F(x)$, both the Hölder inequality and an extension of the Parseval equation allow the validity of the WKS sampling expansion (105) when $F(x) \in L_1(-a, a)$, which answers Brown's question of (108) as a special case.

Before we introduce the Hölder inequality and the proofs of the sampling theorems, we will present here a few definitions and state some basic and very clear results [22].

Let $f(x) \in L_p(a, b)$ mean that $f(x)$ is Lebesque measurable and that $\int_a^b |f(x)|^p\, dx < \infty$, where we have already used its two usual special cases L_2 and L_1 for $p = 2$ and 1, respectively. The *norm* $\|f\|_p$ of f is defined by

$$\|f\|_p = \left[\int_a^b |f(x)|^p\, dx\right]^{1/p}. \tag{109}$$

Let $f(x) \in C^k$ mean that $f(x)$ is k times continuously differentiable. Also $f(x) \in BV$ means that the function is of bounded variation, which is equivalent to saying that f is the difference between two monotonic functions. The following inclusion relation may prove valuable. For $k \geqslant 0$, $0 < q < p < \infty$,

$$C^\infty \subset \cdots \subset C^{k+1} \subset C^k \subset \cdots C^0 \subset L_\infty \subset L_p \subset Lq. \tag{110}$$

As we had considered in Section II-C (Theorem II-C-1), the nth partial sum $S_N(x)$,

$$S_N(x) = \sum_{n=-N}^{N} c_n(x) \tag{111}$$

is said to be $(C, 1)$ Ćesaro summable if the arithmetic mean

$$\sigma_N(x) = \frac{\sum_{j=0}^{N} S_j(x)}{N+1} \tag{112}$$

converges. *Hardy's theorem* states that if $c_n = o(1/n)$ then the $(C, 1)$ Ćesaro summability implies convergence. $S_N(x)$ is said to converge, *in the mean of order p*, to $f(x)$ if

$$\lim_{N\to\infty} \int_a^b |f(x) - S_N(x)|^p\, dx = 0. \tag{113}$$

1) Hölder Inequality: For a finite or infinite interval let $g \in L_p$ and $h \in L_{p'}$, where $1 \leqslant p \leqslant \infty$, $(1/p) + (1/p') = 1$, then

$$\int |gh|\, dx \leqslant \left[\int |g|^p\, dx\right]^{1/p} \left[\int |h|^{p'}\, dx\right]^{1/p'} \tag{114}$$

It is clear that when $p = p' = 2$ this reduces to Schwarz's inequality.

2) Parseval's Equation: Consider the Fourier series expansion for $g(x)$ and $h(x)$ in terms of the orthogonal functions $\psi_n(x) = e^{inx}$ on the interval $[-\pi, \pi]$ with the Fourier coefficients c_n and d_n, respectively. The following are some relaxed versions of the Parseval equation:

If $1 < p < \infty$ and $g \in L_p$, $h \in L_{p'}$, $1/p + 1/p' = 1$, then

$$\sum c_n d_n \text{ is } (C, 1) \text{ Ćesaro summable to } \tfrac{1}{2} \int_{-\pi}^{\pi} g(x) h(x)\, dx. \tag{115}$$

The case $p = p' = 2$ is the prototype Parseval equation used by Brown [127]. The case of interest here, $p = 1$ is taken to correspond to $p' = \infty$ and (115) is still valid. For $f \in L_1$, the series converges absolutely and uniformly when $g \in C^2$. A very brief proof for Brown's example of $J_0(\pi t)$ in (108) will make use of the Parseval equation (115). Since $e^{ixt} \in L_\infty(-\pi, \pi)$ and $g(x) \in L_1(-\pi, \pi)$, its sampling series is $(C, 1)$ summable and hence convergent when we appeal to Hardy's theorem which requires that $f(n)\ (\sin \pi(t-n))/(\pi(t-n)) = o(1/n)$. This is the case since $f(n) = J_0(\pi n)$ is bounded, which can be shown by using the Hölder inequality (114) on (108).

The first proof we give here for the WKS sampling theorem makes immediate use of the Hölder inequality (114). Let $S_N(t)$ and $D_N(x)$ be the partial sums of the series expansion for $f(t)$ and $K(x, t) = e^{ixt}$ in (105) and (107), respectively, and consider

$$f(t) - S_N(t) = \int_{-a}^{a} [e^{ixt} - D_N(x)] F(x)\, dx. \tag{116}$$

If we use the Hölder inequality (114) we obtain

$$|f(t) - S_N(t)| \leqslant \left[\int_{-a}^{a} |e^{ixt} - D_N(x)|^p\, dx\right]^{1/p} \cdot \left[\int_{-a}^{a} |F(x)|^{p'}\, dx\right]^{1/p'}. \tag{117}$$

Hence the series in (105) converges to $f(t)$ when we know that $D_N(x)$ converges in the mean of order p and $F(x) \in L_{p'}(-a, a)$. When $p = \infty$, which is the case, then (105) is valid when $F(x) \in L_1(-a, a)$. By employing the Parseval equation (115), we can find other conditions on $F(x)$ for the WKS sampling theorem. Also the Hölder inequality may be used to relax the condition on $F(x)$ in (19) for the generalized sampling theorem. This of course depends on our close examination of the convergence, in the mean of order p, of D_N to the particular kernel $K(x, t)$. These extensions and others including generalized functions and sampling in n dimensions for the WKS and the WKSK sampling theorems are presented in [29].

Boas [128] presented a simple proof for the Shannon sampling theorem using well-known summation formulas. He illustrated this method by using Poisson's summation formula to derive the sampling expansion including the case when the Fourier transform is integrable. Also he used such expansions to derive an estimate for the aliasing error which results when the sampling series is applied to a function which is not band-limited (see Section VI-B).

B. Different Representations of the Sampling Series

As we state any sampling theorem for a band-limited function $f(t)$ in terms of its discrete values $f(t_n)$, we involve two different representations. The first one is the integral representation for the general band-limited signal

$$f(t) = \int_I K(x, t) g(x)\, dx \tag{19}$$

which in our various proofs implies a series representation, i.e., the sampling expansion

$$f(t) = \lim_{N \to \infty} \sum_{|n| \leq N} f(t_n) S_n(t) \tag{20}$$

where $S_n(t) = S(t, t_n)$ is the sampling function (21). The converse of the sampling theorem is to have (20) imply (19). Theorems in both such directions and in the direction of WKS versus WKSK sampling representations are given in [34] and [34], [28], respectively.

Another familiar and very useful representation is the contour integral one, which was used in deriving various extensions of the sampling theorems [24], [54], [56]. This method employs the residue theorem, to establish a series expansion, which states the following.

Theorem V-B-1: "Any function $h(z)$ which is meromorphic (analytic except at a finite number of points) inside C_R for every R, where C_R is a circular contour of radius R centered at the origin, may be represented by an expansion of the form

$$h(z) = - \sum_j R_{z_j} \left\{ \frac{h(\xi)}{z - \xi} \right\} \tag{118}$$

if the integral $1/(2\pi i) \int_{C_R} (h(\xi))/(\xi - z)\, d\xi$ along C_R approaches zero as $R \to \infty$." In (118), R_{z_j} denotes the residue at $\{z_j\}$ and Σ_j stands for the summation over the poles of $h(\xi)$. This residue theorem can be used to produce a variety of sampling expansions. All we need to do is to let $h(z) = (f(z))/(g(z))$ and choose the proper function $g(z)$ that has zeros at the sampling points of $f(z)$. This was used [24] to derive the sampling expansion (115) by letting $f(z)$ have a finite Fourier transform representation and $g(z) = \sin z$. The corresponding expansion with the function and its first derivative (42) required $g(z) = \sin^2 z$. The same method was used [56], [57] to derive the sampling expansion with N derivatives for the general band-limited integral transform (19) by letting $h(z) = f(z)/g^{N+1}(z)$. In the specific case of $f(z)$ as a band-limited Hankel transform associated with the Bessel function $J_n(z)$, the choice is $h(z) = f(z)/J_n^{N+1}(z)$. Here the sampling points are $j_{n,m}$, $m = 1, 2, \cdots$, the zeros of $J_n(z)$. The cases of $N = 1$ and $N = 2$ for sampling with the function alone and the function and its first derivative are presented in (26) and (45), (46), respectively.

A somewhat novel method for deriving the sampling expansion of the WKSK sampling theorem was introduced by Haddad and Thomas [129] then by Haddad, Yao, and Thomas [130]. This method represents the sampled function $f(t)$ in (19) as a triple integral where two of its possible six permutations correspond to the two most used methods of derivation, the orthogonal expansion and contour integration. The significance of this representation lies in the fact that each of the six possible permutations represents an interesting physical interpretation. The derivation of the triple integral representation of

$$f(t) = \frac{1}{2\pi i} \cdot \oint_C \int_a^b \int_a^b [-G(x, \xi, z)] F(\xi) \rho(\xi) \rho(x) \bar{\psi}(x, \lambda)\, d\xi\, dx\, dz \tag{119}$$

assumes $f(t)$ as a finite limit transform of $F(x)$. The kernel $\psi(x, \lambda)$ of this transform is a solution of the nth-order self-adjoint boundary value problem and $F(x)$ in (119) is the nonhomogeneous term of its associated nonhomogeneous problem with $G(x, \xi, z)$ as its Green's function [130].

C. A Unified Approach to the Different Aspects of the Sampling Theorems

In this section, we will only summarize what was done in the direction of a unified and rigorous approach to the sampling expansion, since such development needs more mathematical background than this review paper is intended to deal with. Beutler [91] presented a unified approach to the sampling theorems for (wide-sense) stationary random processes as it rests upon the concept of Hilbert space. His treatment included, as he had done in a more recent paper [94], the recovery of the process $x(t)$ from nonperiodic samples, or when any finite number of the samples are missing or deleted. He also gave conditions for obtaining $x(t)$ when only the past is sampled and a criterion for restoring $x(t)$ from a finite number of consecutive samples.

Yao [131] considered a number of cases for the WKSK sampling theorem as it is represented by the band-limited integral transforms with kernels including the Bessel, exponential, sine and cosine functions. He considered such general finite energy transforms as a realization of the abstract:

1) Reproducing Kernel Hilbert Space H (RKHS): This is a Hilbert space of functions defined on a set T such that there exists a unique function or kernel $\mathcal{K}(x, t)$ defined on the cross product $T \times T$ such that $\mathcal{K}(-, t) \in H$, for all $t \in T$ and that

$$x(t) = (x, \mathcal{K}(-, t)) \equiv \int x(\omega) \overline{\mathcal{K}(\omega, t)}\, d\omega \tag{120}$$

for all $t \in T$ and for all $x \in H$. The function $\mathcal{K}(\omega, t)$ is the *reproducing kernel* of the RKHS. As one of Yao's examples, he proved that the class of finite energy, Fourier-transformed band-limited signals is a realization of the abstract RKHS. He also made the same statement for the Hankel transform, indicating the same method of proof. This is not so clear since his proof in the case of the Fourier transform uses the convolution theorem which is not as feasible or simple in the case of Hankel transforms [44], [49].

We may remark here that the reproducing kernel $\mathcal{K}(t, \tau)$ is different from $K(x, t)$, the kernel of the integral transform (19), but it is very much related to $S(t, \tau)$

$$S(t, \tau) \int_I \rho(\omega) |K(\omega, \tau)|^2\, d\tau = \int p^I(\omega) \rho(\omega) K(\omega, t) \overline{K(\omega, \tau)}\, d\omega \tag{121}$$

as it represents, aside from the norm factor, the impulse response of a time-varying system when a general integral transform with symmetric kernel $K(x, t)$ (30)–(31) is used. Indeed we have shown [36] that in this general case

$$\int f_I(t)\rho(t)\overline{S(t,t_m)}\,dt = \frac{f_I(t_m)}{\|K(-,t_m)\|_2^2} \tag{122}$$

and even closer to (120) we have

$$\int f(t)\rho(t)\overline{S(t,\tau)}\,dt = \frac{f_I(\tau)}{\|K(-,\tau)\|_2^2} \tag{123}$$

where $S(t, \tau)$ is defined in (121).

Yao [131] also discussed the relevance of the RKHS for extremum problems for general integral transforms and finally gave an upper bound for the truncation error (173) of the generalized WKSK sampling series, which we shall present in Section VI-A.

Among other information theoretic results, Jagerman [132] presented the approximation of band-limited functions in an abstract setting and derived an upper bound for the truncation error of the sampling series (see Section VI-A).

As we mentioned in Section II-A, a general treatment of the cardinal or sampling functions was presented by McNamee, Stenger, and Whitney [20]. They showed that the cardinal functions provide a link between the Fourier series and Fourier transform. They also linked the cardinal functions to the central difference in numerical analysis. A subject similar to this is Schoenberg's work [133] in extending the cardinal series expansion to splines, which we will present in Section VII-C.

A more abstract generalization of the sampling theorem was established and proved by Kluvánek [134] in terms of abstract harmonic analysis. In this analysis, the role of the real line, in the case of the band-limited integral, is replaced by an arbitrary locally compact Abelian group and the role of the sample instants $(n\pi/a)$ in (105) by its discrete subgroups.

VI. Error Analysis in Sampling Representation

In this chapter, we will present a review of the various errors that may arise in the practical implementation of the sampling theorems. This includes the *truncation error* which results when only a finite number of samples are used instead of the infinite samples needed for the sampling representation, the *aliasing error* which is caused by violating the band-limitedness of the signal, the *jitter error* which is caused by sampling at instants different from the sampling points, the *round-off error*, and the *amplitude error* which is the result of the uncertainty in measuring the amplitude of the sample values. A comprehensive treatment of some of these errors with their upper bounds were presented by Thomas and Liu [135] and Papoulis [14], [110].

As we mentioned in the Introduction, attention should be given to the different notations used especially for the signal representation as a truncated inverse Fourier transform [see (1) and (124)].

A. The Truncation Error and Its Bounds

For the band-limited signal

$$f(t) = \int_{-a}^{a} e^{i\omega t}F(\omega)\,d\omega \tag{124}$$

and its WKS sampling representation

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{a}\right)\frac{\sin(at-n\pi)}{(at-n\pi)} \tag{125}$$

the truncation error ϵ_T is the result of considering the partial sum $f_N(t)$ with only $2N+1$ terms of the infinite series (125),

$$\mathcal{E}_T = f(t) - f_N(t) = \sum_{|n|>N} f\left(\frac{n\pi}{a}\right)\frac{\sin(at-n\pi)}{(at-n\pi)}. \tag{126}$$

Unless otherwise indicated we will use $\mathcal{E}_T$ for all the different truncation errors.

Tsybakov and Iakovlev [136] gave the first truncation error bound as

$$|\mathcal{E}_T(t)| \leqslant \frac{\sqrt{2}}{\pi}E\left|\sin\frac{\pi t}{\Delta t}\right|\sqrt{\frac{T\Delta t}{(T^2-t^2)}} \tag{127}$$

for $-T \leqslant t \leqslant T$ and where $\Delta t < (1/W)$, W is the highest frequency of $f(t)$, and E is the total finite energy which is carried by the signal $f(t)$:

$$E = \int_{-W}^{W} |F(\omega)|^2\,d\omega. \tag{128}$$

Helms and Thomas [54] considered the truncation error when $f(t)$ is approximated by the following finite sum

$$f_N(t) = \sum_{n=K-N}^{K+N} f\left(\frac{n}{2W}\right)\frac{\sin(2W\pi t - n\pi)}{(2W\pi t - n\pi)}, \quad 0<N<\infty \tag{129}$$

with $a = 2\pi W$ in (125); K is an integer which is assumed to be a function of t such that $2Wt - (1/2) \leqslant K(t) \leqslant 2Wt + (1/2)$, and N is a fixed integer. Thus the truncation error $\mathcal{E}_T(t) = f(t) - f_N(t)$ will approach zero as N approaches infinity provided that $f(t)$ is band-limited to $a = 2\pi W$. In their treatment they considered a band limit $rW < W$ with $0<r<1$ which, as we shall see, improved the bound on the truncation error:

$$|\mathcal{E}_T(t)| \leqq \frac{4M}{\pi^2 N(1-r)} = \frac{4M}{\pi^2 Nq}, \quad -\infty<t<\infty. \tag{130}$$

where $M \equiv \max|f(t)|$ for all t, $q = 1-r$. As an example, when $N = 24$, $W = 1000$ Hz, $f(t)$ has the highest frequency of 750, the truncation error $\mathcal{E}_T$ of (130) is bounded by 0.068.

A similar bound was given by Jordan [137]

$$\mathcal{E}_T \leqq \frac{4M|\sin 2\pi Wt|}{\pi Nq}. \tag{131}$$

When $f(t)$ is approximated by an asymmetrical partial sum,

$$f_{N_1,N_2}(t) = \sum_{n=K-N_1}^{K+N_2} f\left(\frac{n}{2W}\right)\frac{\sin(2W\pi - n\pi)}{(2W\pi - n\pi)} \tag{132}$$

the truncation error $\mathcal{E}_T(t) = f(t) - f_{N_1,N_2}$ is also shown to be bounded as

$$|\mathcal{E}_T(t)| \leqslant \frac{2M}{\pi^2(1-r)}\left[\frac{1}{N_1}+\frac{1}{N_2}\right], \quad -\infty<t<\infty. \tag{133}$$

To obtain an upper bound that decreases faster with N than that of (130), they [54] considered a "self truncating" sampling expansion

$$f(t)=\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\left\{\frac{\sin\frac{2qW\pi}{m}\left(t-\frac{n}{2W}\right)}{\frac{2qW\pi}{m}\left(t-\frac{n}{2W}\right)}\right\}^m \cdot \frac{\sin 2W\pi\left(t-\frac{n}{2W}\right)}{2W\pi\left(t-\frac{n}{2W}\right)} \quad (134)$$

with its Nth partial sum f_N and the corresponding truncation error $\mathcal{E}_T(t)=f(t)-f_N(t)$. With the choice of m in (134) to equal approximately the optimum value $(Nq\pi)/e$ they gave a new upper bound:

$$|\mathcal{E}_T(t)| \leqslant 1.48M(qN-1.74)^{-1/2}(3.2)^{-qN}. \quad (135)$$

To compare this upper bound to that in (130) they considered the same values of the last example to find that for (135) $|\mathcal{E}_T(t)|<6.8\times 10^{-4}\,M$, which is 1/100 as large as the bound of (130).

When they considered the truncation error for the sampling expansion with the function and its first derivative the bound was the same as that of $\mathcal{E}_T(t)$ in (130). However, when they used the same series with the self-truncating factor as in (134), the bound on the truncation error was derived as

$$|\mathcal{E}_T(t)| \leqslant 2.1M(qN-0.87)^{-1/2}(3.2)^{-2qN} \quad (136)$$

where M and q are defined in (130) and m in the "self-truncating series," of the function and its derivative, is an integer which is set equal to approximately the optimum value of $m=(2Nq\pi/e)$. For the same example of $\mathcal{E}_T$ in (130) the error $|\mathcal{E}_T|$ of (136) is bounded by $8\times 10^{-7}M$. The detailed proofs for (130), (133), (135), and (136) are presented in [54] where contour integration was effectively and elegantly used as a powerful tool. The convolution theorem was also employed for deriving the sampling series (134), since its self-truncating factor is the mth repeated convolution of a gate function $p_b(\omega)$ with $b=(2qW/m)$. Petersen [60] also presented a treatment for the truncation error.

An upper bound on the truncation error was given by deFrancesco [138]. Razyner and Bason [139] used the error formula for Lagrange interpolation [25] to derive an expression for the truncation error bound in terms of the sampling rate and the Nyquist frequency for regular sampling and central interpolation. The usual error formula [25] for interpolation over N samples of $g(t)$ is

$$\mathcal{E}_T(t)=\frac{g^N(\xi)}{N!}\prod_{p=1}^{N}(t-t_p) \quad (137)$$

where $g^N(\xi)$ denotes the Nth derivative at some undetermined ξ in the sample range, t is the interpolation point, and t_p are the arbitrary sample locations. In terms of the total energy E and the cutoff frequency W, they derived a bound as

$$\mathcal{E}_T \leqslant \left|\frac{1}{N}\left(\frac{2EW}{\pi}\right)^{1/2}\cdot(\pi hW)^N\right| \quad (138)$$

for samples at regular equal intervals h in equal numbers on either side of the interpolation point. In order that this interpolation converges, it is sufficient to have $h<(1/\pi W)$ which requires approximately a 50 percent faster sampling rate than that required by the sampling theorem with $h<(1/2W)$. A tighter bound than that of (133) was given by Yao and Thomas [55] with $|f(t)|\leqslant M$ for all t, $0\leqslant r<1$:

$$|\mathcal{E}_T(t)| \leqslant \frac{M|\sin 2\pi Wt|}{2\pi\cos\frac{r\pi}{2}}\left[\frac{1}{N_1}+\frac{1}{N_2}\right] \quad (139)$$

which they extended to the case of sampling signals $f(t_1, t_2, \cdots, t_m)$ in m dimensions (37),

$$|\mathcal{E}_T(t_1,\cdots,t_m)| \leqslant \frac{M|\sin 2\pi W_1 t_1|\cdots|\sin 2\pi W_m t_m|}{(2\pi)^m\left[\cos\frac{\pi r_1}{2}\right]\cdots\left[\cos\frac{\pi r_m}{2}\right]}\cdot\left\{\frac{1}{N_1^1}+\frac{1}{N_2^1}\right\}\cdots\left\{\frac{1}{N_1^m}+\frac{1}{N_2^m}\right\} \quad (140)$$

and to sampling with the function and one derivative (42),

$$\mathcal{E}_T(t) \leqslant \frac{M\sin^2\pi Wt}{\pi^2(\sin\pi r)/\pi r}\left\{\frac{1}{N_1}+\frac{1}{N_2}\right\}. \quad (141)$$

When the restriction $|f(t)|\leqslant M$ is replaced by the condition that $f(t)$ has a finite energy E (128), i.e., $f(t)\in L_2(-\infty,\infty)$ and that $f(t)$ is band-limited to rW Hz, the bound (139) would be

$$|\mathcal{E}_T(t)| \leqslant \frac{2[2ErW]^{1/2}|\sin 2\pi Wt|}{\pi^2(1-r)}\cdot\left\{\frac{1}{N_1}+\frac{1}{N_2}\right\}. \quad (142)$$

Brown [140] used real analysis methods to obtain bounds that have the same asymptotic behavior as that of (142) as $N_1, N_2\to\infty$ for $F(\omega)\in L_2(-\pi r,\pi r)$ and $F(\omega)\in L_1(-\pi r,\pi r)$. For the first case of finite energy the error bound was given as

$$|\mathcal{E}_T(t)| \leqslant \frac{2\sqrt{2}}{\pi^{3/2}}|\sin\pi t|\sqrt{E}\left(\tan\frac{\pi r}{2}\right)^{1/2}\left[\frac{1}{N_1}+\frac{1}{N_2}\right],\quad |t|\leqslant\frac{1}{2}. \quad (143)$$

In comparing (143) to (142), note that $W=(1/2)$ and that (143) represents an improvement over (142) when r is close to 1. This is clear from the ratio R of the upper bounds of (143) and (142),

$$R=\frac{\text{upper bound of (143)}}{\text{upper bound of (142)}}=\frac{\sqrt{2}\pi\left(\tan\frac{\pi r}{2}\right)^{1/2}(1-r)}{\sqrt{r}} \quad (144)$$

and Fig. 4.

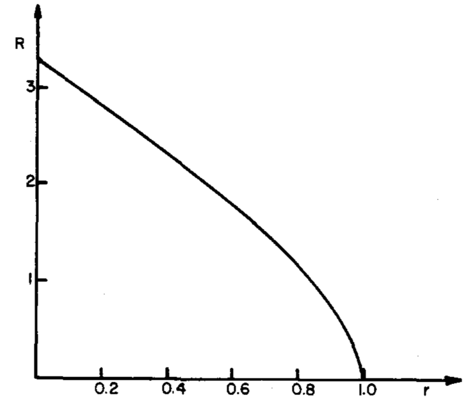

Fig. 4. R: Ratio of Brown's truncation bound to Yao and Thomas' bound.

For the second case of $F(\omega) \in L_1(-\pi r, \pi r)$, i.e.,

$$\int_{-\pi r}^{\pi r} |F(\omega)| d\omega < \infty, \quad M = \max_{-\infty < t < \infty} |f(t)| \tag{145}$$

$$|\mathcal{E}_T(t)| \leq \frac{2M}{\pi} |\sin \pi t| \left\{ c_0 + \left(\frac{\pi r}{18}\right)^{1/2} \tan^{3/2}\left(\frac{\pi r}{2}\right)\left[\frac{1}{N_1} + \frac{1}{N_2}\right]\right\} \tag{146}$$

where $|t| \leq \frac{1}{2}$ and

$$c_0 = \frac{1}{\pi r} \ln \left[\frac{1 + \sin \frac{\pi r}{2}}{1 - \sin \frac{\pi r}{2}}\right] \tag{147}$$

Piper [141] used real analysis methods to derive a truncation error bound for finite energy signals that are band-limited to $(-\pi r, \pi r)$:

$$|\mathcal{E}_T(t)| < \frac{1}{\pi^{3/2}} \left[E \tan\left(\frac{\pi r}{2}\right)\right]^{1/2} [1 + 2^{1/2}] |\sin \pi t| [(N+m)^{-1} + (N-m)^{-1}] \tag{148}$$

where m is the nearest integer to t and $|t| < N$. This represents an improvement over the bound (142) of Yao and Thomas [55] for $0.73 < r < 1$ and that (143) of Brown [140] for all values of r. A tighter error bound than that of Helms and Thomas [54] was derived by Hagenauer [142]. He considered the truncated sampling series

$$f_{N_1,N_2}(t) = \sum_{n=K(t)-N_2}^{K(t)+N_1} f\left(\frac{n\pi}{\omega_0}\right) \frac{\sin(\omega_0 t - n\pi)}{(\omega_0 t - n\pi)} \tag{149}$$

with $\omega_0 t/\pi - \frac{1}{2} < K(t) < \omega_0 t/\pi + \frac{1}{2}$, then used a self-truncating factor

$$h_0(t) = \left(\frac{\sin(\delta\omega_0 t/n)}{\delta\omega_0 t/n}\right)^n = \frac{(2n+1)!!}{(\delta\omega_0 t)^{n+1/2}} J_{n+1/2}(\delta\omega_0 t) \tag{150}$$

with the sampling series

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\omega_0}\right) h_0\left(t - \frac{k\pi}{\omega_0}\right) \frac{\sin(\omega_0 t - k\pi)}{(\omega_0 t - k\pi)} \tag{151}$$

where $0 < \delta < 1$, $(2n+1)!! = (2n+1)(2n-1)(2n-3) \cdots 3 \cdot 1$ and $J_{n+1/2}$ is the Bessel function of the first kind of order $n + (1/2)$. We note that this self-truncating factor is similar to that used in (134) with $\delta = q$ and $\omega_0 = 2W$. The bound on the truncation error $\mathcal{E}_T = f(t) - f_{N_1,N_2}$ was given as

$$|\mathcal{E}_T| \leq M(2n+1)!! \left[\frac{1}{(\delta\pi N_1)^{n+1}} \sum_{k=0}^{n} \frac{c_{n,k}}{n+k+1} \cdot \frac{1}{(2\delta\pi N_1)^k} + \frac{1}{(\delta\pi N_2)^{n+1}} \sum_{k=0}^{n} \frac{c_{n,k}}{n+k+1} \cdot \frac{1}{(2\delta\pi N_2)^k}\right] \tag{152}$$

where $c_{n,k} = \binom{n}{k}(k+n)!/n!$.

Papoulis [110] presented and proved the following bound for the truncation error of (126)

$$|\mathcal{E}_T(t)| \leq \frac{|\sin at|}{a} \sum_{|n|>N} \frac{f(n\pi/a)}{|t - (n\pi/a)|}. \tag{153}$$

For $|t| < (N\pi/a)$, he presented the following bound and attributed it to Jagerman [143, Theorem 1.]:

$$|\mathcal{E}_T(t)| \leq \frac{|\sin at|}{\sqrt{\pi a}} \left\{\left[\frac{1}{(N\pi/a)+t} \sum_{n=-\infty}^{-(N+1)} f^2\left(\frac{n\pi}{a}\right)\right]^{1/2} + \left[\frac{1}{(N\pi/a)-t} \sum_{n=N+1}^{\infty} f^2\left(\frac{n\pi}{a}\right)\right]^{1/2}\right\}, \quad |t| < \frac{N\pi}{a}, \quad N \geqq 1. \tag{154}$$

Later Papoulis [144] used some of his results to show that for a signal with finite energy E the maximum of the truncation error is bounded by the mean-square value of the error $\eta(\omega)$ resulting in approximately $e^{j\omega t}$ by a truncated Fourier series,

$$|\mathcal{E}_T(t+\tau)|^2 \leq \frac{E}{2\pi} \int_{-a}^{a} |\eta(\omega)|^2 d\omega = \frac{E}{2\pi} \int_{-a}^{a} \left| e^{j\omega\tau} - \sum_{n=-N}^{N} e^{j(n\pi\omega/a)} \frac{\sin(a\tau - n\pi)}{(a\tau - n\pi)}\right|^2 d\omega \tag{155}$$

for any t. Then when we expand and use Parseval's equation,

$$|\mathcal{E}_T(t+\tau)|^2 \leq \frac{aE}{\pi} \sum_{|n|>N} \left[\frac{\sin(a\tau - n\pi)}{(a\tau - n\pi)}\right]^2. \tag{156}$$

In terms of the energy of such error, $E_g = \int_{-\infty}^{\infty} |\mathcal{E}_T(t)|^2 dt$, the bound is

$$|\mathcal{E}_T(\tau)|^2 \leq \frac{aE_g}{\pi} \sum_{|n|>N} \frac{\sin^2 a(\tau - nT)}{a^2(\tau - nT)^2}. \tag{157}$$

For signals with *finite power*

$$\overline{|f(t)|^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt, \quad T = \frac{\pi}{a} \tag{158}$$

the mean square value of $\mathcal{E}_T$ is bounded by the maximum $\eta(\omega_M)$ of $|\eta(\omega)|$ in (155):

$$\overline{|\mathcal{E}_T(t+\tau)|^2} \leq |\eta(\omega_M)|^2 \, \overline{|f(t)|^2} \tag{159}$$

where the average is with respect to t. In the derivation of (159), Papoulis commented that the sampling expansion

$$f(t+\tau) = \sum_{n=-\infty}^{\infty} f\left(t + \frac{n\pi}{a}\right) \frac{\sin(a\tau - n\pi)}{(a\tau - n\pi)} \tag{160}$$

used is not valid in general for finite power signals; however, he showed that it holds in the following mean square sense

$$\underset{T\to\infty}{\text{l.i.m.}} \frac{1}{2T} \int_{-T}^{T} \left| f(t+\tau) - \sum_{n=-\infty}^{\infty} f(t+nT) \frac{\sin a(\tau - nT)}{a(\tau - nT)}\right|^2 dt = 0, \quad T = \frac{\pi}{a} \tag{161}$$

where l.i.m. stands for "limit in the mean." He then remarked that such results [144] can be extended to stochastic processes by formally replacing time averages by expected values [69], [91]. He also extended similar results to two dimensions and Hankel transforms and then related such results to the uncertainty principle in one and two dimensions [145].

Mendelovicz and Sherman [146] used a generalization of Papoulis's approach [145] to give a least upper bound (l.u.b.) on the truncation error for energy bounded band-limited functions. This was done for the WKS sampling expansion (125) and the self-truncating series (134). They also treated the problem of finding an optimum sampling function that minimizes the truncation error. These results compared very favorably with those of Yao and Thomas [55] and Brown [140] especially when sampling above the Nyquist rate. They concluded that a few percent oversampling gives a significant improvement in error performance. In comparison with the cardinal series which converges slowly but is best at the Nyquist rate, they suggested other series for over sampling that may converge faster. Later Mendelovicz, Sherman, and Murphy [147] presented a more detailed treatment where they considered both stochastic and deterministic signals.

Jagerman [143] presented various estimates for the truncation error under some appropriate constraints on the signal $f(t)$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\omega t} F(\omega)\, d\omega. \tag{162}$$

The estimate in (154) is one of his first theorems for $f(t) \in L_2(-\infty, \infty)$. A corollary to this theorem is the special case when $|t| \leqq \pi/2a$

$$|\mathcal{E}_T(t)| \leqq \frac{|\sin at|}{\pi} \frac{\left[\sum_{n<-N} f^2\left(\frac{n\pi}{a}\right)\right]^{1/2} + \left[\sum_{n>N} f^2\left(\frac{n\pi}{a}\right)\right]^{1/2}}{\sqrt{N - \frac{1}{2}}}. \tag{163}$$

With the different condition $t^k f(t) \in L_2(-\infty, \infty)$, k positive integer, $N \geqq 1$, he gave the following estimate

$$\mathcal{E}_T(t) \leqq \frac{|\sin at| E_k}{\pi h^k \sqrt{1 - 4^{-k}}} \left[\frac{1}{\sqrt{Nh - t}} + \frac{1}{\sqrt{Nh + t}}\right] \frac{1}{(N+1)^k}, \quad |t| < \frac{N\pi}{a} \tag{164}$$

where $h = (\pi/a)$ and

$$E_k = \left[\int_{-\infty}^{\infty} t^{2k} |f(t)|^2\, dt\right]^{1/2}. \tag{165}$$

An immediate consequence of (164) is the special case for $|t| \leqq (\pi/2a)$:

$$\mathcal{E}_T(t) \leqq \frac{2}{\pi} \frac{|\sin at|}{\sqrt{1 - 4^{-k}}} \cdot \frac{E_k}{h^{k+1/2}} \cdot \frac{1}{(N+1)^k \sqrt{N - 1/2}}, \quad |t| < \frac{\pi}{2a}. \tag{166}$$

For the truncation error $\mathcal{E}_T(t)$ (135) of the Helms-Thomas "self truncating series" (134) (with $a = 2\pi W$) the estimate for $|t| \leqq (\pi/2a)$ is

$$|\mathcal{E}_T(t)| < \frac{2S}{\pi m} e^{-m[1-(1/\nu)]}, \quad |t| \leqq \frac{\pi}{2a} \tag{167}$$

where $0 < q < 1$, $\nu = (\pi q/e)\,(N - (1/2))$, $N \geqq 1$, $m = [\![\nu]\!] + 1$; $f(x)$ has radian band width $(1 - q)a$ and $S = \text{l.u.b.}_{-\infty<n<\infty}\, f(n\pi/a)$. Here $[\![\nu]\!]$ is the integral part of ν, i.e., it is the unique integer satisfying $\nu - 1 < [\![\nu]\!] \leqq \nu$. Jagerman [143] also gave an estimate of the truncation error in terms of the l.u.b. of the sampling series error

$$q_N = \underset{j\geqq N}{\text{l.u.b.}} \left| \sum_{k>j} f\left(\frac{k\pi}{a}\right) \frac{\sin(at - k\pi)}{(at - k\pi)} \right| \tag{168}$$

$$s_N = \underset{j\geqq N}{\text{l.u.b.}} \left| \sum_{k>j} f\left(-\frac{k\pi}{a}\right) \frac{\sin(at + k\pi)}{(at + k\pi)} \right|. \tag{169}$$

His estimate for $0 < q < 1$, $\nu = (\pi q/e)(N + (1/2))$, $N \geqq 1$, $m = [\![\nu]\!] + 2$, $|t| \leqq (\pi/2a)$, and $f(x)$ having radian bandwidth $(1 - \delta)a$ is

$$|\mathcal{E}_T(t)| \leqq (e+2)(q_N + s_N) e^{-m(1-(2/\nu))}, \quad |t| \leqq \frac{\pi}{2a}. \tag{170}$$

Other estimates including when the conditions of both (164) and (170) are met were also presented.

The most recent treatment of the truncation error, for deterministic functions as well as wide-sense stationary processes, was presented by Beutler [148]. In contrast to other methods, his method depends on the use of the Dirichlet kernel representation for the truncated series and on properties of functions of bounded variation. Other integral kernels were also employed. It is known that the truncated series for functions $f(t)$ with absolutely integrable Fourier transform is slowly convergent [141]. As we indicated earlier, bounds for such functions were found [55], [140], [143] provided that there is a guard band δ, i.e., provided that the Fourier transform $F(\omega)$ of $f(t)$ is supported on $[-\pi + \delta, \pi - \delta]$. Beutler [148] showed that a similar upper bound can be obtained without the guard requirement which he replaced by requiring that the Fourier transform of $f(t)$ be of bounded variation in the neighborhood of $-\pi$ and π.

1) Truncation Error for the Generalized WKSW Sampling Expansion: As we mentioned in Section V-C, Yao [131] was the first to give an upper bound for the truncation error of the generalized WKSK sampling expansion. He considered this and other expansions as a realization of his abstract RKHS of functions $f(t)$ and the reproducing kernel $K(s, t)$ defined on a set T (see Section V-C),

$$f(t) = \sum_{n\in I} f(t_n) \psi_n(t, t_n). \tag{171}$$

Here $\psi_n(t, t_n)$ is a sampling function where $\psi_i(t_i, t_j) = \delta_{i,j}$ and the series (171) is uniformly convergent for all $t \in T$. He considered the partial sum $f_0(t)$ of (171) with a finite number of terms I', as a proper subset of I, and showed that the truncation error

$$\mathcal{E}_T(t) = f(t) - f_0(t) = \sum_{n\in(I-I')} f(t_n) \psi_n(t, t_n) \tag{172}$$

is bounded as

$$|\mathcal{E}_T(t)| \leqq \left[E - \sum_{n\in I'} c_n^2 f^2(t_n)\right]^{1/2} \cdot \left[\sum_{n\in(I-I')} c_n^2 K^2(t, t_n)\right]^{1/2}, \quad f \in H'' \tag{173}$$

for any $f \in H''$ where $H'' = \{f \in H : \|f\|_2^2 \leqq E\}$, i.e., finite energy signals. Here the constants c_n are defined as $c_n f(t_n) = (f, \phi_n)$, $n \in I'$, where f_0 is an element of smallest norm satisfying

$$(f_0, \phi_n) = b_n, \quad n \in I' \tag{174}$$

$\{b_n, n \in I'\}$ are fixed constants, $\{\phi_n, n \in I\}$ is a complete orthonormal set for the finite energy functions, and $\psi_n(t, t_n) = c_n\phi_n(t, t_n) = c_n^2 K(t, t_n)$. In the case of the Shannon (WKS) sampling expansion,

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \tag{174}$$

he considered only $I' = \{n_0(t) - M \leqslant n \leqslant n_0(t) + N\}$ terms where $n_0(t)$ is the nearest integer to t and M and N are positive integers. The truncation error is

$$\mathcal{E}_T(t) = \sum_{n=-\infty, n_0(t)-M-1}^{n_0(t)+N+1, \infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)} \tag{175}$$

and the upper bound was given for finite energy functions f as

$$|\mathcal{E}_T(t)| < \begin{cases} \dfrac{E_0^{1/2}}{\pi} \left| \dfrac{1}{M} + \dfrac{2}{2N-1} \right|^{1/2}, & t \in \left(n_0(t), \ n_0(t) + \dfrac{1}{2}\right] \\ \dfrac{E_0^{1/2}}{\pi} \left| \dfrac{2}{2M-1} + \dfrac{1}{N} \right|^{1/2}, & t \in \ n_0(t) - \dfrac{1}{2}, n_0(t)\Big] \end{cases} \tag{176}$$

where

$$E_0 = \sum_{n=-\infty, n_0(t)+M-1}^{n_0(t)+N+1, \infty} f^2(n) < E < \infty. \tag{177}$$

The limits of the summations (175) and (177) are those of $I - I'$.

2) Truncation Error for Band-Limited Distributions (Generalized Functions: Campbell [105] established a bound for the truncation error

$$\mathcal{E}_T(t) = f(t) - \sum_{n=-N}^{N} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} \cdot S(q[\Omega t - n\pi]) \tag{178}$$

for the sampling expansion (78) of band-limited generalized functions (distributions). For f, g, S, q, Ω as in Theorem IV-G-1, let r be an integer and b a number such that $|f(t)| \leqq b|t|^r$ for $|t| > (N\pi/\Omega)$. Also let j be an integer greater than r and let

$$c_j = 4(9)^{j-1} [(j-1)!]^2 (2j)! j^{-1/2} \pi^{-3/2} bK \tag{179}$$

where

$$K^{-1} = \int_{-1}^{1} \exp(x^2 - 1)^{-1} \, dx. \tag{180}$$

Let $0 \leqq |\Omega t| < N\pi$. Then the estimate for the bound of the truncation error $\mathcal{E}_T(t)$ of (178) is

$$|\mathcal{E}_T(t)| \leqq \frac{c_j |\sin \Omega t|}{q^j (N\pi - |\Omega t|)^j} \left(\frac{N\pi}{\Omega}\right)^r. \tag{181}$$

B. The Aliasing Error and its Bounds

In practice, the signals that we deal with are not necessarily band-limited in the sense required by the Shannon sampling expansion. The aliasing error $\mathcal{E}_A(t) = f(t) - f_s(t)$ is the result from applying the sampling theorem representation $f_s(t)$ to signals $f(t)$ with samples $f(n\pi/a)$ even when they are not band-limited or band-limited to different limits than those used in sampling expansion. In this section we will present estimates for the aliasing error for band-limited bandpass functions and for the generalized WKSK sampling theorem.

In their review paper for error problems in sampling representation, Thomas, and Liu [135] showed that the mean-square aliasing error is equal to twice the spectral power outside the Nyquist range $|\omega| > 2\pi W$:

$$E[(f - f_s)^2] = \frac{1}{\pi} \int_{|\omega| > 2\pi W} \Phi_{ff}(\omega) \, d\omega \tag{182}$$

where E stands for expectation value and Φ_{ff} is the power spectral density of $f(t)$. However when an optimum prefilter is used then the mean-square aliasing error is reduced to one half of the error without prefiltering (182). In the deviation of (182), the sample for $f_s(t)$ was taken at $(n/2W) - \alpha$ where α is uniformly distributed in the interval $[0, (1/2W)]$. The phase averaging process resulted in a wide-sense stationary process. Brown [149] considered the samples at $(n/2W)$ and showed that the mean-square error is less than or equal to twice that of (182). He also added that his result cannot be improved without additional processing such as the above random phase averaging or prefiltering. It is clear that unless the signal is band-limited to $(-2\pi W, 2\pi W)$ there will always be an aliasing error when we sample at the required Nyquist rate. So if there is any alias free sampling it must be based on a rate different from that of the Nyquist rate or in other words sampling at unequally spaced instants of time. To develop an alias free sampling, Shapiro and Silverman [150] found conditions on the sampling instants. They showed that various schemes with randomly chosen sampling instants satisfy these conditions. This problem was later treated by Beutler [151].

Weiss [152] considered the aliasing error resulting from applying the sampling theorem to a function even when it is not band-limited, i.e., when $F(\omega)$ does differ from zero for $|\omega| > a$. For $F(\omega) \in L_1(-\infty, \infty)$, $F(\omega) = \bar{F}(-\omega)$, of bounded variation and $2F(\omega) = F(\omega + 0) + F(\omega - 0)$, let $f_s(t)$ be the sampling series (125) with samples $f(n\pi/a)$ of the nonband-limited function $f(t)$. Weiss gave an upper bound for the aliasing error as

$$|\mathcal{E}_A(t)| = |f(t) - f_s(t)| \leqq \frac{2}{\pi} \int_a^{\infty} |F(\omega)| \, d\omega. \tag{183}$$

Papoulis [110] derived an upper bound for the aliasing error in terms of the area of the spectrum $E_A(\omega)$ of such an error

$$B = \int_{-\infty}^{\infty} |E_A(\omega)| \, d\omega \tag{184}$$

as

$$|\mathcal{E}_A(t)| \leqslant \frac{B}{2\pi} |\sin at| \tag{185}$$

where the upper bound can be attained. Brown [153] showed that the first of the four Weiss conditions, i.e., $F(\omega) \in L_1(-\infty, \infty)$, is sufficient for the validity of (183) with the estimate as

$$|\mathcal{E}_A(t)| = |f(t) - f_s(t)| = \frac{1}{\pi} \int_{|\omega| > a} |F(\omega)| \, d\omega. \tag{186}$$

As we have indicated in Section V-A, Boas [128] used the Poisson summation formula to derive the sampling expansion and the above two estimates of Weiss [152] and Brown [153] for the aliasing error. Brown then considered the same aliasing error for the sampling expansion of bandpass functions (63) [12, p. 216], where again $F(\omega)$ does not necessarily vanish outside the bandpass interval I_{BP} as in (63). The bound on this error is

$$|f(t) - f_s(t)| \leqslant 2 \int_{\omega \notin I_{BP}} |F(\omega)| \, d\omega \qquad (187)$$

where f_s is the sampling series (63) with samples of $f(t)$. He concluded and proved [153], [154] that the constant multiple in (186) and (187) cannot be reduced further and so the crude estimate (187) is a very good estimate of aliasing error for almost all values of t except those for which t is a sampling point of the form $(n\pi/a)$. Standish [155] showed that a bound of the form

$$|\mathcal{E}_A(t)| \leqslant \alpha(t) \int_{|\omega|>a} |F(\omega)|^2 \, d\omega, \quad -\infty < t < \infty \qquad (188)$$

with $\alpha(t)$ independent of f and bounded *cannot* hold for all signals of finite energy ($F(\omega) \in L_2(-\infty, \infty)$). Stickler [156] reported the same bound as Weiss (183) without reference to Weiss's result and then derived a tighter but cumbersome bound

$$|\mathcal{E}_A(t)| \leqslant \frac{2}{\pi} \int_{w=a}^{\infty} |F(\omega)| \sin \frac{\lambda(\omega)t}{2} \, d\omega \qquad (189)$$

where

$$\lambda(\omega) = \begin{cases} 0, & |\omega| < a \\ 2ah \left[\dfrac{|\omega| - a}{2a} \right] \operatorname{sgn} \omega, & |\omega| > a \end{cases} \qquad (190)$$

$$h(x) = \text{smallest integer} \geqslant |x| \text{ and } \operatorname{sgn} x = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Then he gave a bound for the aliasing error when $f(t)$ is band-limited but with a larger band ω_0 than a, i.e., $\omega_0 > a$:

$$|\mathcal{E}_A(t)| \leqslant \frac{2}{\pi} (\omega_0 - a) \max_{a \leqslant \omega \leqslant \omega_0} |F(\omega)| \qquad (191)$$

which he compared to that given by Papoulis (185)

$$|\mathcal{E}_A(t)| \leqslant \frac{|\sin at|}{2\pi} \int_{\omega=-\infty}^{\infty} |E_a(\omega)| \, d\omega \qquad (185)$$

with the comment that (189) or (191) is more useful than (185) for some cases.

The aliasing error bounds (183) and (187) were extended by Mehta [157] to the case of the generalized WKSK sampling theorem for its deviation from "the generalized" band-limited and bandpass functions, respectively,

$$|\mathcal{E}_A(t)| \leqslant 2\alpha \int_{\omega \notin I} |F(\omega)| \, d\omega \qquad (192)$$

$$|\mathcal{E}_A(t)| \leqslant 2\alpha \int_{\omega \notin R_{BP}} |F(\omega)| \, d\omega \qquad (193)$$

where I is the interval on which $\{K(\omega, t_n)\}$ are orthogonal, as in (19), R_{BP} is the bandpass region as in (65). Mehta took $\alpha = |K(\omega, t)|$ for all real ω but it would be more practical to take $\alpha = \max |K(\omega, t)|$, which does not affect his derivation.

C. The Jitter, Round-Off, and Other Errors

The amplitude error is caused by the uncertainty in the sample values due to either quantization or to some fluctuation where the round-off error may be considered as a special case.

The jitter error results from sampling at instants $t_n = nT + \gamma_n$ which differ in a random fashion by γ_n from the required Nyquist sampling instants nT. As it turned out [135] the jitter and amplitude errors are related and require very similar theoretical treatments. Thomas and Liu [135] gave a thorough review of both subjects with a summary of the original work by Franklin [158], Lloyd and MacMillan [159], Stewart [160], Spilker [161], Chang [162], Brown [163], Middleton and Petersen [164], and Ruchkin [165], on the amplitude error, and by Shapiro and Silverman [150], Balakrishnan [69], [166], Brown [167], and Brown and Palermo [168] on the jitter problem. Middleton [12, ch. 4] treated various sampling procedures including the "jittered" samples. Papoulis [110] also gave a simple treatment of the jitter and the round-off error where he utilized some extensions of the sampling theorem which we presented in Section IV-I. In this section, we will present the bounds of both errors that were developed by Papoulis and refer the reader to the above references and in particular [135].

In his study of the error analysis for the sampling theorem Papoulis [110] applied (94)

$$f^2(t) = \sum_{n=-\infty}^{\infty} f^2(nT) \frac{\sin w_0(t - nT)}{w_2(t - nT)} \qquad (94)$$

where $w_2 \equiv (\pi/T)$ and $w_2 \geqslant 2w_1$ (instead of $w_2 \geqslant w_1$ in the case for $f(t)$) and w_0 is such that $2w_1 \leqslant w_0 \leqslant 2w_2 - 2w_1$, to the *round-off error*

$$\epsilon_n = f(nT) - \bar{f}(nT) \qquad (194)$$

where $\bar{f}(nT)$ is the recorded or tabulated sampled values which differ from the exact sampled values by ϵ_n. Using the cardinal series (92) with $w_0 = w_1$ and sampled values $\bar{f}(nT)$ he constructed the function $f_r(t)$, which differs from $f(t)$ by the *total round-off error* $\mathcal{E}_r(t)$. Combined with the above results in (94) he showed that this error $\mathcal{E}_r(t)$ is bounded by its own total energy E_r; that is,

$$|\mathcal{E}_r(t)| \leqslant \left(\frac{w_1 E_r}{\pi} \right)^{1/2} \qquad (195)$$

$$E_r = \int_{-\infty}^{\infty} \mathcal{E}_r^2(t) \, dt. \qquad (196)$$

Papoulis then considered the jitter problem, which arises when the sample values are not exactly at the sampling points nT but are at some other instants $nT - u_n$, where $\{u_n\}$ is the set of deviations of the sampling points from nT. He considered

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} u_n \frac{\sin \omega_2(\tau - nT)}{\omega_2(\tau - nT)} \qquad (197)$$

to be band-limited, using (93). Higgins [169] presented two series representations for an error free reconstruction of a band-limited finite energy signal from its irregularly spaced or "jittered" samples. The basic theoretical treatment for this problem was developed by Beutler [91], [94].

Knab and Schwartz [170] considered the truncation error $\mathcal{E}_T$ combined with channel error $\mathcal{E}_c$ which is caused by uncorrelated noise samples. Let the band-limited signal $f(t)$, with band limit w_0, bounded by M and with a Fourier transform of bounded variation be corrupted with zero-mean additive noise samples ϵ_k of variance σ^2 that are uncorrelated. They used a truncated sampling series to approximate $f(t)$ in the interval $|t| \leqslant (T/2)$ as $\hat{f}(t)$:

$$\hat{f}(t) = \sum_{k=-N}^{N} \hat{f}(kT)\theta(t - kT) \tag{198}$$

where $\theta(t)$ is a "self-truncating" sampling function as in (134) with $q = 1 - \delta$ and $a = w_0$. The reconstruction mean-square error $E|\&(t)|^2$ is found as

$$E|\&(t)|^2 = E|\&_c(t)|^2 + [\&_T(t)]^2 = \sigma^2 \sum_{k=-N}^{N} \theta^2(t - kT) + [\&_T(t)]^2. \tag{199}$$

The error bound for such combined errors was found for all $t \leqslant T/2$ as

$$E|\&(t)|^2 \leqslant M^2 \left(\frac{\sin \pi t}{T}\right)^2 \frac{4}{m^2\pi^2} \left[\frac{M}{\pi N\delta}\right]^{2m} + \sigma^2 \left[\left\{1 - \frac{1}{3}\delta m^{-0.515}\right\}\left\{1 - \cos\frac{2\pi}{T}t\right\} + \cos\frac{2\pi}{T}t\right] \tag{200}$$

where m is as in (134).

Knab [171] derived an error bound using Lagrange polynomials for interpolation and extrapolation of finite power band-limited signals. He compared this bound to that derived by Helms and Thomas [54] for the self-truncating sampling series (134). He also noted that extrapolation is possible with his series while it is not the case with (134). However, the Lagrange extrapolation method is not numerically stable as the channel error becomes large very fast with N, the number of terms in the series.

Ericson [172] presented the sampling series for signals which are not necessarily band-limited and where the samples are subject to distortion before being used for signal reconstruction. He considered a stationary time-continuous process $x(t)$ whose spectral density (which is assumed to be integrable) is zero outside a more general frequency set than the band-limit interval.

D. Conditions for a Stable Sampling Expansion

As it is the case of the solutions of many problems in applied mathematics, we always seek a stable solution in the sense (of Hadamard) that a small error in the input produces a correspondingly small error in the output or in other words the output depends continuously on the input. Yao and Thomas [93] considered the stability of the sampling expansion in the sense that a small error in reading the sample values produces only a correspondingly small error in the recovered signal. They gave a condition for a stable sampling that "for a band-limited function $g(t)$ to possess a stable sampling expansion with respect to a class of sampling sequence $\{t_n\}$ there must exist a positive finite absolute constant C (C is independent of $f(t)$ and $\{t_n\}$), such that

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt \leqslant C \sum |f(t_n)|^2." \tag{201}$$

For an example of an unstable sample expansion, they considered the simple constructed function $f(t) = 0$ with uncorrupted samples $\{f(n) = 0, n = 1, 2, \cdots\}$. When these sample values $f(n)$ were corrupted by the specific noise samples $h(n)$ of

$$h(t) = \frac{b^{1/4} \sin[(\pi - \epsilon)(b + t)]}{\pi(b + t)}, \qquad b > 0 \tag{202}$$

they became $\{f(n) + h(n) = h(n)\}$. For this noise (202) it is easy to show that

$$\int_{-\infty}^{\infty} |h(t)|^2\, dt = \frac{\sqrt{b}\,(\pi - \epsilon)}{\pi} \tag{203}$$

which becomes unbounded as the arbitrary b approaches infinity which makes such sampling unstable.

Landau [173] considered the WKS sampling theorem and used the Parseval equation (102) on (4) whereby

$$\frac{1}{2W}\sum |a_n|^2 = \frac{1}{2W}\sum \left|f\left(\frac{n}{2W}\right)\right|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt \tag{204}$$

and gave the following interesting interpretation as it relates the Nyquist rate with the stability of the sampling expansion.

1) Every signal $f(t)$ of finite energy, i.e., $\int_{-\infty}^{\infty} f^2(t)\, dt < \infty$, and bandwidth W(Hz) may be completely recovered in a simple way, from the knowledge of its samples taken at the Nyquist rate of $2W$ per second. Moreover, the recovery is stable, in the sense of Yao and Thomas (or Hadamard), such that a small error in reading sample values produces only a correspondingly small error in the recovered signal.

2) Every square-summable sequence of numbers may be transmitted at the rate of $2W$ per second over an ideal channel of bandwidth W(Hz) by being represented as the samples of an easily constructed band-limited signal of finite energy.

In relation to the required Nyquist rate for the transmitted sequence of samples or the recovered ones, Landau considered other configurations, besides the band-limited finite energy signals, in the hope of improving such rates. This included moving to differently chosen sampling instants or to bandpass or multiband (rather than band-limited) signals. Emphasizing that only stable sampling is meaningful in practice, he proved the following two very sharp and useful results.

1) Stable sampling cannot be performed at a lower rate than the Nyquist rate.

2) Data cannot be transmitted as samples at a rate higher than the Nyquist rate regardless of the location of sampling instants, the nature of the set of frequencies which the signal occupy, or the method of construction.

These results also apply to bounded signals besides finite energy signals.

VII. Other Applications of the Sampling Theorems

In this chapter we will discuss a number of applications of the WKS and the generalized WKSK sampling theorems in other fields besides the usual communications theory. The latter applications of the sampling theorems are found in most texts [11]-[17], research papers in information theory, and, in particular [13], [14].

A. Optics and Crystallography

Barakat [174] presented a direct application of the sampling theorem to optical diffraction theory as a computational tool and credited Gabor [175] and others for pioneering the introduction of the sampling theorem concept in optics. He developed formulas, in terms of sampled values of the point spread function for the transform function, total illuminance, line spread function, and cumulative line spread function. Then he presented a theory for general point spread functions for slit and square aperture, where the WKS sampling theorems in one and two dimensions are used, respectively. For circular apertures with rotationally symmetric point spread functions, the one-dimensional generalized WKSK sampling theorem, associated with the J_0-Bessel function, was used instead of the two-dimensional WKS sampling theorem. This, of course, is an advantage of the generalized WKSK, as we pointed out in Section III-B, in general, with circular symmetry, a $J_{(m/2)-1}$-Hankel transform is equivalent to an m-dimensional Fourier transform [see (29)].

As an example, the transfer function $T(w)$ and the point spread function $t(v)$ for a slit aperture are related by a band-limited Fourier transform

$$t(v) = \frac{1}{2} \int_{-2}^{2} T(w) e^{ivw} \, dw \tag{205}$$

where the factor (1/2) enters in order that $t(0)$ be unity for a perfect system. Using the WKS sampling theorem, the point spread function $t(v)$ can be written in terms of its discrete measured values $t(n\pi/2)$ as

$$t(v) = \sum_{n=-\infty}^{\infty} t\left(\frac{n\pi}{2}\right) \frac{\sin(2v - n\pi)}{(2v - n\pi)}. \tag{206}$$

Som [176] used the two-dimensional WKS sampling theorem in the frequency domain for a coherent optical processor to obtain multiple reproduction of spaced-limited functions in two dimensions. This approach was backed by an experiment where it was found that the relative separation and the relative brightness of the multiple reproductions of a given input function can be quantitatively controlled by simply choosing an appropriate sampling function.

An early application of the sampling theorem in opticis is due to diFrancia [177] where he used the WKS sampling expansion to compute the number of degrees of freedom of an image. He then extended the analysis to antenna theory [178]. Gori and Guattari [179]–[181] used nonuniform sampling in the analysis of holographic restoration and optical processing. Various applications of the sampling theorem in optics were made by Lohmann [182]. Another related application in the general field of Fourier spectroscopy is due to Vanasse and Sakai [183].

Hopper [184] utilized the N-dimensional sampling theorem for wavenumber-limited functions with examples of the two-dimensional case for coherent optical systems. In particular, in the case of computer generation and construction of holograms, sampling must be done in space and, hence, it is of advantage to use the most efficient sampling in two dimensions as presented by Petersen and Middleton [61] and Miakawa [59]. Instead of the two-dimensional sampling in Cartesian coordinates (x, y), Blažek [185], [186] considered the sampling theorem in polar coordinates (ρ, θ) to find the number of spatial degrees of freedom of optical wave fields at the output of optical systems with circular symmetric apertures of various shapes. The difference between this sampling and that of the WKS sampling is that the former is based on sampling circles instead of sampling points. This implies that an integration of a function over a given circle is used instead of the value of the function at a given point. We may remark that such polar coordinate sampling is a combination of the WKS sampling in θ with exponential kernel and the WKSK sampling in ρ with kernel as Bessel function. Blažek mentioned as an example a radiotelescope with circular symmetric transfer function. This makes use of the sampling theorem as a tool to solve system analysis problems from the point of view of transmission of pictorial information.

Marks, Walkup, and Hazler [187] developed a sampling expansion which is applicable to the class of linear space-variant systems characterized by sufficiently slowly varying line-spread functions. They showed that the desired sampling rate is determined by both the system and the input and that the corresponding output is band-limited. McDonnell [188] introduced the "line-segment-limited" function for image restoration where the emphasis is on the samples themselves and the continuously restored image obtained by the usual sampling expansion. This is in contrast with the usual sampling series where these samples are convolved with the sampling function to reconstruct the restored image. In avoiding the usual sampling function, he showed that sampling can be performed at a rate lower than the Nyquist rate. These results were extended to two dimensions.

1) Crystallography: Brillouin [189, p. 105] presented the WKS sampling series and Fourier methods for analyzing periodic crystal structures. He considered the electron density $F(\vec{r})$ as periodic in the three dimensions with the corresponding translations $\vec{d}_1$, $\vec{d}_2$, and $\vec{d}_3$:

$$F(\vec{r} + \rho_1 \vec{d}_1 + \rho_2 \vec{d}_2 + \rho_3 \vec{d}_3) = F(\vec{r}) \tag{207}$$

where $\vec{r}$ is the vector (x_1, x_2, x_3); ρ_1, ρ_2, and ρ_3 are positive or negative integers. The Fourier series for $F(\vec{r})$ is

$$F(x_1, x_2, x_3) = \sum_{h_1, h_2, h_3} \sum \sum C_{h_1 h_2 h_3} \cdot e^{2\pi i (h_1 b_1 x_1 + h_2 b_2 x_2 + h_3 b_3 x_3)} \tag{208}$$

$$C_{h_1 h_2 h_3} = \frac{1}{V_d} \int_0^{d_1} \int_0^{d_2} \int_0^{d_3} F(x_1, x_2, x_3) \cdot e^{-2\pi i (h_1 b_1 x_1 + h_2 b_2 x_2 + h_3 b_3 x_3)} \, dx_1 \, dx_2 \, dx_3 \tag{209}$$

where $V_d \doteq 1/b_1 b_2 b_3$, the volume of the fundamental lattice. Brillouin then turned to the Fourier series of the autocorrelation of $F(\vec{r})$ or what is called the *Patterson function* [189],

$$P(u_1, u_2, u_3) = \frac{1}{V_d} \int_0^{d_1} \int_0^{d_2} \int_0^{d_3} F(x_1 + u_1, x_2 + u_2, x_3 + u_3) F(x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3 \tag{210}$$

whose Fourier coefficients are the intensities

$$|C_{h_1 h_2 h_3}|^2 = C_{h_1 h_2 h_3} \overline{C_{h_1 h_2 h_3}}$$

and

$$P(u_1, u_2, u_3) = \sum_{h_1, h_2, h_3} \sum \sum |C_{h_1 h_2 h_3}|^2 \cdot e^{2\pi i (h_1 b_1 u_1 + h_2 b_2 u_2 + h_3 b_3 u_3)}. \tag{211}$$

This is a Fourier series analysis whereby the autocorrelation of the electron density is determined in terms of the measured samples of the intensity $|C_{h_1 h_2 h_3}|^2$ where we don't seem to find the explicit use of the sampling theorem except in Brillouin's other information theoretic discussions. To apply the sampling theorem it is tempting to recognize the intensity as the square of a band-limited function in three dimensions (209) where an extension of Papoulis's [14] sampling expansion (94) for $f^2(t)$ to three dimensions may be used to interpolate the discrete values of the intensity $|C_{h_1 h_2 h_3}|^2$. Since (h_1, h_2, h_3) determines the direction in which the intensity is measured, it seems possible, for some physical reasons, that measurements cannot be done at some angles and, hence, there is a gap or nonuniform samples which can be treated by the methods of Section IV-D.

Some related applications of the sampling theorems were made by Frank [190] then Frank and Ali [191] in the area of radiation damage caused by the electrons used for imaging in the electron microscope. Earlier, Frank [192] presented a detailed review for computer processing of electron micrographs.

B. Time-Varying Systems, Boundary Value Problems, and Discrete Fourier and Other Transforms

1) Time-Varying Systems: The Shannon sampling theorem with its band-limited Fourier transform and the simple convolution translation is a natural tool for the analysis of time-invarying systems. Time-varying systems are also analyzed using Fourier transform as evident in the work of Zadeh [39] and others, however, we emphasize here a time-varying system with a generalized convolution translation [36] which is associated with the generalized integral transforms of the WKSK sampling theorem. In Section III-C we attempted to give an applied interpretation of this generalized WKSK theorem in terms of a time-varying impulse response. In Section IV-H we discussed the WKS sampling theorem, using the Fourier transform, but with time-varying bands [109]. Šťastný [193] considered the reconstruction of time-varying signals with particular emphasis on the distortion which arises during such restoration. He employed the sampling expansion for bandpass functions and derived a dependence of the distortion on the sampling frequency which is valid for stationary Gaussian random signals and for deterministic signals with continuous spectrum having a low frequency character. In Section III-D we discussed the possibility of replacing the m-dimensional Fourier transform of functions with circular symmetry by a one-dimensional $J_{(m/2)-1}$-Hankel transform where the WKSK sampling theorem can be used. This may relate to spatial-varying problems where, in the case of polar coordinates, the sampling is done on circles instead of at points.

2) Boundary Value Problems: In Section III-D we discussed how the generalized WKSK sampling theorem was used [42] to facilitate the study of the effect of the axial heat conduction on the temperature field for a fluid with laminar flow in a tube. This development can still be extended to problems which require matching boundary conditions and uses general orthogonal expansion. For example, in the case of a spherical geometry, associated Legendre polynomials or spherical Bessel function expansions may be used. In the simple case of laminar flow between plates, the Cartisian coordinates are used and, hence, a finite Fourier transform and its Shannon's sampling theorem may be employed.

3) Discrete Fourier and Other Transforms: The role of the WKS sampling theorem is very evident [45], [46] in determining the required spacing of the discrete Fourier transform which led to the powerful tool of the fast Fourier transform (FFT) algorithm. Petersen [194] employed the WKS sampling theorem [53] for the discrete transform and FFT for N-dimensional lattices. As we mentioned in Section III-D, it is in this direction that we have attempted to employ the generalized WKSK sampling theorem for determining the spacing of general discrete transforms associated with classical orthogonal polynomials [43] and the Bessel functions [44].

C. The Sampling Series and the Hill Functions (B-Splines)

As we have briefly indicated in Section VI-A, the hill function $\phi_{R+1}(a(R+1), \omega)$ of order $R+1$ is defined as the Rth-fold Fourier convolution of the gate function $p_a(\omega) \equiv \phi_1(a, \omega)$. Hence, it is the Fourier transform of $[(2 \sin at)/(t)]\ R+1$ which can be recognized as related to the sampling function for sampling with R derivatives (43) or the factor for the self-truncating series (134) which improved the error bound for the truncation error (135). This function was also used [195] as a self-truncating Fourier coefficient for efficient evaluation of the hill function of higher order

$$\phi_{R+1}(\omega) = \frac{a_0}{2} + \frac{1}{a(R+1)} \sum_{n=1}^{\infty} \left[\frac{2 \sin (n\pi)/(R+1)}{(n\pi)/(a(R+1))}\right]^{R+1} \cdot \cos \frac{n\pi}{a(R+1)}, -a(R+1) < \omega < a(R+1) \quad (212)$$

where $a_0 = ((2a)^{R+1})/(a(R+1)) = 2a\,\bar{\phi}_{R+1}$ and $\bar{\phi}_{R+1}$ is the average value of $\phi_{R+1}(\omega)$ over the interval $[-a(R+1), a(R+1)$. In (212) we note the simple form of the coefficients and their advantage in making the series a self-truncating one for large R. A very thorough treatment of the sampling expansion (cardinal interpolation) and spline functions is given by Schoenberg [133]. The above discussion is exclusive for the WKS sampling theorem and its very familiar tool the Fourier transform. As of yet no mention is made of some "valuable function" like the spline function which can be defined as the Rth-fold general transform convolution [36] and which may play the role of improving the error bound for the truncation error of the generalized WKSK sampling theorem.

D. Special Functions

The subject of this section is varied as we can see that the generalized WKSK sampling theorem is an extension of the simple exponential function, as the kernel of Fourier transform, to other functions; as solutions of nth-order self-adjoint differential equations, for the kernel of more general integral transforms. In Section III-B we have already shown the conditions for the equivalence between the Fourier and other transforms and, hence, the two sampling theorems. In particular, it is known that the sampling functions $\{(\sin (at - n\pi))/(at - n\pi)\}$ which are band-limited to $(-a, a)$ are orthogonal on $(-\infty, \infty)$ and we have shown [36] that the generalized sampling functions $\{S(t, t_n)\}$ of (21) are also orthogonal on the interval used for the integral transform inverse (31). This shows that the general Fourier-type transforms (30), (31) preserve orthogonality since $\{\overline{K(x, t_n)}\}$ in (21) is an orthogonal set and so is its transform $\{S(t, t_n)\}$. Some of the following results relating band-limited functions to the transform of some special functions are closely related to the above treatments.

Yao and Thomas [196] noted that $(\sin t)/(t)$ is a band-limited function of the gate function $p_a(\omega)$ which is a polynomial of order zero on $(-1, 1)$. This is the same for the prolate spheroidal functions [27], which are also band-limited and have been used extensively for approximating band-limited signals. They showed that a more general frequency function, instead of the gate function, band-limited to $(-1, 1)$ can be obtained by orthogonalizing the polynomials $1, \omega, \omega^2, \cdots$ with respect to the weighting function $\rho(\omega)$ using the Gram-Schmidt process. Such functions are the weighted Jacobi polynomials which are complete in $L_2(-1, 1)$ and are also solutions of second-order self-adjoint differential equations [27]. The time functions as the inverse Fourier transform of these functions are well-known special functions which are also orthogonal [26] on $(-\infty, \infty)$. Some of the special cases that they exhibited are the Gegenbauer, the Legendre, and the Tchebyschev polynomials. Mehta [197] considered the same problem for other examples like the associated Legendre polynomials and the prolate spheroidal functions.

One of the simplest applications of the WKS sampling theorem is in establishing a relation between the continuous special function and its discrete or sample values. For example, from Titchmarsh [198, p. 186] we have the following band-limited function representation for a certain combination of gamma functions of x

$$\frac{\pi\Gamma(a-1)}{2^{a-2}\Gamma((a+x)/2)\Gamma((a-x)/2)} = \int_{-(\pi/2)}^{(\pi/2)} [\cos u]^{a-2}\, e^{ixu}\, du, \quad a > 1. \tag{213}$$

We recognize that this is a band-limited function of x, for which we can immediately write the Shannon sampling series

$$\frac{1}{\Gamma((a+x)/2)\Gamma((a-x)/2)} = \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma((a+2n)/2)\Gamma((a-2n)/2)} \cdot \frac{\sin((\pi/2)x - n\pi)}{((\pi/2)x - n\pi)}, \quad a > 1 \tag{214}$$

a result which apparently is not easily accessible in the literature. Setting $a = 2$ in (214) leads to the well-known special case

$$\frac{1}{\Gamma(x/2)\Gamma(1-(x/2))} = \frac{\sin(\pi/2)x}{\pi x}$$

which is the finite limit Fourier transform of the gate function $p_{(\pi/2)}(t)$ as in (213) with $a = 2$.

In parallel to the WKS sampling theorem or the Whittaker cardinal series [1], [5], Higgins [199] considered an interpolatory series associated with the WKSK sampling theorem [9] and in particular the one associated with the Hankel (Bessel) transform. He proved a number of theorems that included basic properties of the WKSK sampling functions [36]. Then he utilized particular cases of his results for special functions expansion which is in parallel to the above expansion (214) of the gamma functions as a WKS sampling series.

E. Other Applications

Among other applications of the Shannon sampling theorem, Petersen and Middleton utilized the multidimensional sampling theorem [53], [61] for the analysis of meteorological data [200]. Also this sampling expansion [53], [61] was referred to by Belyayev [201] for oceanographic applications. Radzyner [202] employed the nonuniform sampling expansion and developed error bounds resulting from the simplification of a mathematical model for the cardiac pacemaker.

A seemingly far removed application is that of sampling the fractional derivative $(d^\alpha)/(dt^\alpha)\, f(t)g(t)$ (Leibnitz rule—α not necessarily an integer) [203] of the product of two functions $f(t)g(t)$ in terms of its samples, the usual nth derivatives $(d^n)/(dt^n)\, f(t)g(t)$. We remark here that such sampling may also be attempted for the fractional integrals [204].

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