

Midterm Study Guide Combinatorics

December 6, 2017

1 CH1

2 CH2

THM 2.1. $\binom{n}{r} = \binom{n}{n-r}$. ($0 \leq r \leq n$).

THM 2.2. Let n be a positive integer. Then, if $(1+x)^n$ is expanded as a sum of powers of x , the coefficient of x^r is $\binom{n}{r}$.

ie. $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{n}x^n = \sum_{r=0}^n \binom{n}{r}x^r$.

THM 2.3. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

THM 2.4. $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n}b^n = \sum_{r=0}^n \binom{n}{r}a^{n-r}b^r$.

THM 2.5. If n is any positive integer, then

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \dots + (-1)^n \binom{n}{n}x^n = \sum_{r=0}^n \binom{n}{r}(-1)^r x^r.$$

THM 2.6. If n is any positive integer, then

$$(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \dots = \sum_{r=0}^{\infty} \binom{n+r-1}{r}x^r.$$

3 CH3

THM 3.1. Let S be a set of mn objects. Then S can be split up (partitioned) into n sets of m elements in

$$\frac{(mn)!}{(m!)^n n!}$$

4 CH4

5 CH5

6 CH6

6.1 Block Designs 6.1

DEF: $(b, v, r, k, \lambda) - design$

b - subsets (blocks),

v - elements (varieties),

r - each element is in exactly r blocks,

k - each subset has k elements,

λ - each pair of elements appears in λ subsets

THM 6.1: In a block design each element lies in exactly r blocks, where

$$r(k-1) = \lambda(v-1)$$

THM 6.2: For a $(b, v, r, k, \lambda) - design$,

$$b \geq v$$

6.2 Square Block Designs 6.2

Properties $(v, k, \lambda) - design's$ incidence matrix has the following properties:

1. Any row contains k 1's.
2. Any column contains k 1's
3. Any pair of columns both have 1's in exactly λ rows
4. Any pair of rows both have 1's in exactly λ columns.

THM 6.3: If A is a square $(0, 1)$ -matrix (i.e. a matrix all of whose entries are 0 or 1) and if A satisfies

$$A^T A = (k - \lambda)I - \lambda J$$

with $k > \lambda$, then

$$A A^T = (k - \lambda)I - \lambda J$$

also holds.

DEF: A finite projective plane of order q is defined to be a $(v, k, \lambda) - configuration$ with the properties:

1. $v = q^2 + q + 1$
2. $k = q + 1$

3. $\lambda = 1$

fpp properties: A fpp of order q has the following properties

1. Any line contains $q + 1$ points
2. Any point lies on $q + 1$ lines
3. Any pair of points are joined on exactly one line
4. Any pair of lines intersect in exactly one point

other knowledge about fpp's

- a. A plane of order q definitely exists if $q \geq 2$ is a prime or a power of a prime.
- b. No plane of any other order is known to exist.
- c. There is definitely no plane of order 6, or in general of any order n , where n is of the form $(4k + 1)$ or $(4k + 2)$, and is divisible an odd number of times by a prime of the form $(4h + 3)$.

FACT: There is no finite projective plane of order 6.

6.3 Hadamard configurations

DEF: A (v, k, λ) - *configuration* is called a Hadamard configuration when $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$ for some integer $m \geq 2$.

idea: Hadamard Matrix is formed from taking the incidence matrix of a Hadamard configuration and changing the 0's to 1's.

DEF: A $n \times n$ matrix is a Hadamard matrix of order n if:

1. $a_{ij} = \pm 1, \forall i, j$
2. $AA^T = nI$

FACT: Given a Hadamard matrix, it is permissible to interchange any two rows or any two columns, or to multiply any row or column by -1, for these operations do not effect the properties required by the definition.

THM 6.5: If A is an $n \times n$ Hadamard matrix with $n > 2$, then $n = 4m$ for some positive integer m . Further, each row has exactly $2m + 1$ s and $2m - 1$ s, and, for any two chosen rows, there are exactly m columns in which both rows have +1.

THM 6.6: Each normalized Hadamard matrix A of order $4m \geq 8$ yields a $(4m - 1, 2m - 1, m - 1)$ - *configuration*.

Process: A $mn \times mn$ Hadamard matrix can be formed by taking 2 Hadamard matrices, A and B , of order m and n respectively, and place A at every 1 in B and $-A$ for every -1 in B .

6.4 Error-correcting codes

THM 6.7: A code will detect all sets of h or fewer errors if any two words differ in at least $(h + 1)$ places.

THM 6.8: A code will correct all sets of h or fewer errors if any two words differ in at least $(2h + 1)$ places.

7 CH7