Midterm Study Guide Combinatorics

December 6, 2017

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2.1 Unordered Selections 2.3

THM 2.1. $\binom{n}{r} = \binom{n}{k-r}$. $(0 \leqslant r \leqslant n)$.

THM 2.2. Let n be a positive integer. Then, if $(1+x)^n$ is expanded as a sum of powers of x, the coefficient of x^r is $\binom{n}{r}$.

ie. $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{0}x^2 + ... + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$.

THM 2.3. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

THM 2.4. $(a+b)^n = \binom{n}{0}a^{n-1}b + \binom{n}{1}a^{n-2}b^2 + ... + \binom{n}{n}b^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$.

THM 2.5. If n is any positive integer, then

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - ... + (-1)^n\binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r x^r.$$

THM 2.6. If n is any positive integer, then

$$(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + ... = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r.$$

Choose n from k	Number of ordered selections	Number of unordered selections
Repetitions not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
Repetitions allowed	n^k	$\binom{n+k-1}{k}$

3 CH3

3.1 Pairing within a set

Properties
(3.1) The total number of different pairings of 2n objects is

$$\frac{(2n)!}{(2!)^n n!}$$

(3.2) If distinct representatives do exist, then, for every value of k; any k sets contain between them at least k elements
(3.3) In general, a sequence $(a_1, a_2, ..., a_n), a_1 \geqslant a_2 \geqslant ... \geqslant a_n$, of non-negative integers can be a score sequence only if
(3.4)

$$a_1 + a_2 + ... + a_n = \binom{n}{2}$$

and (3.5)

$$a_{n-r+1} + a_{n-r+2} + ... + a_n \geqslant \binom{r}{2}$$

THM 3.1. Let S be a set of mn objects. Then S can be split up (partitioned) into n sets of m elements in

$$\frac{(mn)!}{(m!)^n n!}$$

THM 3.2. If a graph has 2n verticies, each of degree $\geqslant$ n, then the graph has a perfect matching.

3.2 Pairings between sets

THM 3.3.(Philip Hall's theorem on distinct representatives). The sets $A_1, ..., A_n$ possess a system of distinct representatives if and only if, for all k=$1, ..., n$, any k $A_i s$ contain at least k elements in their union.
THM Assignment. This assignment problem has a solution if and only if there is no value of k for which there are k jobs with fewer than k suitable applicants between them.
THM Marriage. Given a set of men and a set of women, each man makes a list of the women he is willing to marry. Then each man can be married off to a woman on his list if and only if,

$$(*) \begin{cases} \text{for every value of k, any k lists contain in their union at least k names.} \end{cases}$$

THM 3.4. If r < n, any r x n Latin rectangle can be extended to an (r+1) x n Latin rectangle.
THM 3.5.(Landau's theorem). the non-negative integers $a_1 \geqslant ... \geqslant a_n$ form the score sequence of a tournament if and only if conditions (3.4) and (3.5) are satisfied.
THM 3.6.(The Harem Theorem). Let $w_1, ..., w_n$ be non-negative integers, and suppose that men $M_1, ..., M_n$ each makes a list of the women he is willing to marry. Then each M_i can be married to w_i women on his list if and only if, for any subset $\{i_1, ..., i_r\}$ of $\{1, ..., n\}$, the lists of men $M_{i_1}, ..., M_{i_r}$ contain in their union at least $w_{i_1}, ..., w_{i_r}$ names.

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6 CH6

6.1 Block Designs 6.1

DEF: $(b, v, r, k, \lambda) - design$
b - subsets (blocks),
v - elements (varieties),
r - each element is in exactly r blocks,
k - each subset has k elements,
λ - each pair of elements appears in λ subsets

THM 6.1: In a block design each element lies in exactly r blocks, where

$$r(k-1) = \lambda(v-1)$$

THM 6.2: For a $(b, v, r, k, \lambda) - design$,

$$b \geq v$$

6.2 Square Block Designs 6.2

Properties $(v, k, \lambda) - design's$ incidence matrix has the following properties:

1. Any row contains k 1's.
2. Any column contains k 1's
3. Any pair of columns both have 1's in exactly λ rows
4. Any pair of rows both have 1's in exactly λ columns.

THM 6.3: If A is a square (0, 1)-matrix (i.e. a matrix all of whose entries are 0 or 1) and if A satisfies

$$A^T A = (k - \lambda)I - \lambda J$$

with $k > \lambda$, then

$$AA^T = (k - \lambda)I - \lambda J$$

also holds.

DEF: A finite projective plane of order q is defined to be a $(v, k, \lambda) - configuration$ with the properties:

1. $v = q^2 + q + 1$
2. $k = q + 1$

3. $\lambda = 1$

fpp properties: A fpp of order q has the following properties

1. Any line contains $q + 1$ points
2. Any point lies on $q + 1$ lines
3. Any pair of points are joined on exactly one line
4. Any pair of lines intersect in exactly one point

other knowledge about ffp's

a. A plane of order q definitely exists if $q \geq 2$ is a prime or a power of a prime.

b. No plane of any other order is known to exist.

c. There is definitely no plane of order 6, or in general of any order n, where n is of the form $(4k + 1)$ or $(4k + 2)$, and is divisible an odd number of times by a prime of the form $(4h + 3)$.

FACT: There is no finite projective plane of order 6.

6.3 Hadamard configurations

DEF: A $(v, k, \lambda) - configuration$ is called a Hadamard configuration when $v = 4m - 1$, $k = 2m - 1$, $\lambda = m - 1$ for some integer $m \geq 2$.
idea: Hadamard Matrix is formed from taking the incidence matrix of a Hadamard configuration and changing the 0's to 1's.
DEF: A nxn matrix is a Hadamard matrix of order n if:

1. $a_{ij} = \pm 1, \forall i, j$
2. $AA^T = nI$

FACT: Given a Hadamard matrix, it is permissible to interchange any two rows or any two columns, or to multiply any row or column by -1, for these operations do not effect the properties required by the definition.

THM 6.5: If A is an $n \times n$ Hadamard matrix with $n > 2$, then $n = 4m$ for some positive integer m. Further, each row has exactly $2m$ +1s and $2m$ -1s, and, for any two chosen rows, there are exactly m columns in which both rows have +1.

THM 6.6: Each normalized Hadamard matrix A of order $4m \geq 8$ yields a $(4m - 1, 2m - 1, m - 1) - configuration$.

Process: A $mn \times mn$ Hadamard matrix can be formed by taking 2 Hadamard matrices, A and B, of order m and n respectively, and place A at every 1 in B and $-A$ for every -1 in B.

6.4 Error-correcting codes

THM 6.7: A code will detect all sets of h or fewer errors if any two words differ in at least $(h+1)$ places.

THM 6.8: A code will correct all sets of h or fewer errors if any two words differ in at least $(2h+1)$ places.

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