Midterm Study Guide Combinatorics

December 11, 2017

1 CH1

2 CH2

2.1 Unordered Selections 2.3

THM 2.1. $\binom{n}{r} = \binom{n}{k-r}$. $(0 \leqslant r \leqslant n)$.

THM 2.2. Let n be a positive integer. Then, if $(1+x)^n$ is expanded as a sum of powers of x, the coefficient of x^r is $\binom{n}{r}$.

ie. $(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{0}x^2 + ... + \binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}x^r$.

THM 2.3. $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

THM 2.4. $(a+b)^n = \binom{n}{0}a^{n-1}b + \binom{n}{1}a^{n-2}b^2 + ... + \binom{n}{n}b^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$.

THM 2.5. If n is any positive integer, then

$$(1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - ... + (-1)^n\binom{n}{n}x^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r x^r.$$

THM 2.6. If n is any positive integer, then

$$(1-x)^{-n} = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + ... = \sum_{r=0}^{\infty}\binom{n+r-1}{r}x^r.$$

Choose n from k	Number of ordered selections	Number of unordered selections
Repetitions not allowed	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$
Repetitions allowed	n^k	$\binom{n+k-1}{k}$

3 CH3

3.1 Pairing within a set

Properties
(3.1) The total number of different pairings of 2n objects is

$$\frac{(2n)!}{(2!)^n n!}$$

(3.2) If distinct representatives do exist, then, for every value of k; any k sets contain between them at least k elements
(3.3) In general, a sequence $(a_1, a_2, ..., a_n), a_1 \geqslant a_2 \geqslant ... \geqslant a_n$, of non-negative integers can be a score sequence only if
(3.4)

$$a_1 + a_2 + ... + a_n = \binom{n}{2}$$

and (3.5)

$$a_{n-r+1} + a_{n-r+2} + ... + a_n \geqslant \binom{r}{2}$$

THM 3.1. Let S be a set of mn objects. Then S can be split up (partitioned) into n sets of m elements in

$$\frac{(mn)!}{(m!)^n n!}$$

THM 3.2. If a graph has 2n verticies, each of degree $\geqslant$ n, then the graph has a perfect matching.

3.2 Pairings between sets

THM 3.3.(Philip Hall's theorem on distinct representatives). The sets $A_1, ..., A_n$ possess a system of distinct representatives if and only if, for all k=$1, ..., n$, any k $A_i s$ contain at least k elements in their union.
THM Assignment. This assignment problem has a solution if and only if there is no value of k for which there are k jobs with fewer than k suitable applicants between them.
THM Marriage. Given a set of men and a set of women, each man makes a list of the women he is willing to marry. Then each man can be married off to a woman on his list if and only if,

$$(*) \begin{cases} \text{for every value of k, any k lists contain in their union at least k names.} \end{cases}$$

THM 3.4. If r < n, any r x n Latin rectangle can be extended to an (r+1) x n Latin rectangle.
THM 3.5.(Landau's theorem). the non-negative integers $a_1 \geqslant ... \geqslant a_n$ form the score sequence of a tournament if and only if conditions (3.4) and (3.5) are satisfied.
THM 3.6.(The Harem Theorem). Let $w_1, ..., w_n$ be non-negative integers, and suppose that men $M_1, ..., M_n$ each makes a list of the women he is willing to marry. Then each M_i can be married to w_i women on his list if and only if, for any subset $\{i_1, ..., i_r\}$ of $\{1, ..., n\}$, the lists of men $M_{i_1}, ..., M_{i_r}$ contain in their union at least $w_{i_1}, ..., w_{i_r}$ names.

4 CH4 Recurrence

4.1 Misc. problems

Q: The problem of derangements. Suppose that n jobs have been assigned to n people. In how many ways can they be reassigned the following day so that no person is given the same job as before?
A: $a_n = (n-1)a_{n-1} + (n-1)a_{n-2}$, and $a_1 = 0$, $a_2 = 2$, $a_3 = 2$
Explanation: let $n > 2$ and examine a derangement of $1, ..., n$. There are 2 Cases:

1. n switches places with some other element r, so there are $(n-1)$ choices for r, and a_{n-2} derangements of the remaining $n-2$ elements
2. r moves to the n^{th} place, and n does not move to r's place. Relabel n by r. Now n is fixed, with $(n-1)$ elements to derange, which can be done in a_{n-1} ways.

4.2 Fibonacci-type relations

THM: Suppose a_1 and a_2 are given, and that

$$a_n = Aa_{n-1} + Ba_{n-2} \quad (n \geq 3)$$

Then :

1. if the roots α, β of the equation $x^2 = Ax + B$ are distinct, then

$$a_n = K_1\alpha^n + K_2\beta^n$$

   where K_1 and K_2 are determined uniquely by a_1 and a_2.

2. if $x^2 = Ax + B$ has a repeated root, then

$$a_n = (K_1 + K_2 n)\alpha^n$$

4.3 Using Generating Functions

Example: Partitions of an integer. i.e.

$$5 = 4+1 = 3+2 = 3+1+1 = 2+2+1 = 2+1+1+1 = 1+1+1+1+1.$$

Note that the partitions are always in decreasing order. Let $p(n)$ be the number of partitions of n, so $p(5) = 7$. Let $f(x)$ be the generating function:

$$f(x) = p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + ...$$

Now consider the expression

$$(1-x)^{-1}(1-x^2)^{-2}(1-x^3)^{-3}... = (1+x+x^2+x^3+...)(1+x^2+x^4+x^6+...)(1+x^3+x^6+x^9+...)$$

The coeficient of x^n in this expression is equal to $p(n)$

4.4 Misc. Methods

Example: Binomial recurrence relation:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

with

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1$$

We will take the recurrence function:

$$f(n,k) = f(n-1,k) + f(n,k-1)$$

with the conditions:

$$f(1,k) = 1, \quad f(n,1) = n$$

and modify it so that it fits the pattern defined by the binomial recurrence. Define g by

$$f(n,k) = g(n+k,k)$$

Then we have

$$g(n+k,k) = g(n+k-1,k) + g(n+k-1,k-1)$$

with the boundary conditions

$$g(k+1,k) = 1, g(n+1,1) = n$$

If we let $m = n+k$, then

$$g(m,k) = g(m-1,k) + g(m-1,k-1)$$

however the boundary conditions are not correct. So we will try $u = n+k-1$. Then let h be defined as,

$$f(n,k) = h(n+k-1,k) = h(u,k)$$

Thus

$$h(u,k) = h(u-1,k) + h(u-1,k-1)$$

with conditions

$$h(k,k) = 1, \quad h(n,1) = n$$

Now it follows that $h(u,k) = \binom{u}{k}$ and finally,

$$f(n,k) = \binom{n+k-1}{k}$$

5 CH5 Principle of Inclusion-Exclusion

5.1 General Idea

Lets say we want to count the number of items that satisfy constraint C_1 and constraint C_2. We can just add $|C_1|$ and $|C_2|$, but (and this is easiest to imagine if you draw a Venn Diagram) then we need to subtract the value that is in $|C_1 \cup C_2|$ because when we add the two, we have double counted the overlap. This is formalized for r constraints in the next section.

5.2 Formula

$$\sum_{i=0}^{r}(-1)^{r-1}C(1,2,3,...,r)$$

Where $C(1,2,3,...,r)$ represents all combinations of r constraints joined: $|C_1 \cup C_2 \cup C_3 \cup ... \cup C_r|$.

5.3 Basic Applications of PIE

[1] **P 5.1:**
Determine the number of numbers less that n that are divisibly by 4, 3, and 7.
P 5.2:
How many solutions are there to $x_1 + x_2 + x_3 = 17$ where $x_i \leq 7 \forall i$
P 5.3
How many derangements of **DISCRETEMATHROCKS** have no consecutive pairs.
P 5.4
How many ways are there to arrange 15 flowers on five shelves such that each shelf has between 1 and 5 flowers?
P 5.5
How many ways are there to arrange the word **ARRANGEMENT** such that there are exactly two pairs of consecutive letters (i.e. RR, EE, etc)? At least three pairs of consecutive letters?

5.4 Rook Polinomials

6 CH6

6.1 Block Designs 6.1

DEF: $(b, v, r, k, \lambda) - design$
b - subsets (blocks),
v - elements (varieties),
r - each element is in exactly r blocks,
k - each subset has k elements,
λ - each pair of elements appears in λ subsets

THM 6.1: In a block design each element lies in exactly r blocks, where

$$r(k-1) = \lambda(v-1)$$

THM 6.2: For a $(b, v, r, k, \lambda) - design$,

$$b \geq v$$

[1]Solutions to these exercises are nicely explained: https://www.youtube.com/watch?v=Y0CYHMqomgI&t=473s

6.2 Square Block Designs 6.2

Properties $(v, k, \lambda) - design's$ incidence matrix has the following properties:

1. Any row contains k 1's.
2. Any column contains k 1's
3. Any pair of columns both have 1's in exactly λ rows
4. Any pair of rows both have 1's in exactly λ columns.

THM 6.3: If A is a square (0, 1)-matrix (i.e. a matrix all of whose entries are 0 or 1) and if A satisfies

$$A^T A = (k - \lambda)I - \lambda J$$

with $k > \lambda$, then

$$AA^T = (k - \lambda)I - \lambda J$$

also holds.

DEF: A finite projective plane of order q is defined to be a $(v, k, \lambda) - configuration$ with the properties:

1. $v = q^2 + q + 1$
2. $k = q + 1$
3. $\lambda = 1$

fpp properties: A fpp of order q has the following properties

1. Any line contains $q + 1$ points
2. Any point lies on $q + 1$ lines
3. Any pair of points are joined on exactly one line
4. Any pair of lines intersect in exactly one point

other knowledge about ffp's

a. A plane of order q definitely exists if $q \geq 2$ is a prime or a power of a prime.

b. No plane of any other order is known to exist.

c. There is definitely no plane of order 6, or in general of any order n, where n is of the form $(4k + 1)$ or $(4k + 2)$, and is divisible an odd number of times by a prime of the form $(4h + 3)$.

FACT: There is no finite projective plane of order 6.

6.3 Hadamard configurations

DEF: A $(v, k, \lambda) - configuration$ is called a Hadamard configuration when $v = 4m-1$, $k = 2m-1$, $\lambda = m-1$ for some integer $m \geq 2$.
idea: Hadamard Matrix is formed from taking the incidence matrix of a Hadamard configuration and changing the 0's to 1's.
DEF: A nxn matrix is a Hadamard matrix of order n if:

1. $a_{ij} = \pm 1, \forall i, j$
2. $AA^T = nI$

FACT: Given a Hadamard matrix, it is permissible to interchange any two rows or any two columns, or to multiply any row or column by -1, for these operations do not effect the properties required by the definition.

THM 6.5: If A is an $n \times n$ Hadamard matrix with $n > 2$, then $n = 4m$ for some positive integer m. Further, each row has exactly $2m$ +1s and $2m$ -1s, and, for any two chosen rows, there are exactly m columns in which both rows have +1.

THM 6.6: Each normalized Hadamard matrix A of order $4m \geq 8$ yields a $(4m-1, 2m-1, m-1) - configuration$.

Process: A $mn \times mn$ Hadamard matrix can be formed by taking 2 Hadamard matrices, A and B, of order m and n respectively, and place A at every 1 in B and $-A$ for every -1 in B.

6.4 Error-correcting codes

THM 6.7: A code will detect all sets of h or fewer errors if any two words differ in at least $(h+1)$ places.

THM 6.8: A code will correct all sets of h or fewer errors if any two words differ in at least $(2h+1)$ places.

7 CH7

7.1 Introductory Remarks

Sphere packing, skipped

7.2 Steiner Systems

Steiner Triple System: $(b, v, r, k, \lambda) - configuration$ in which $k = 3, \lambda = 1$ and $b = v(v-1)/6$.
Example with $v = 9$

$\{1,2,3\}$ $\{4,5,6\}$ $\{7,8,9\}$ $\{1,4,7\}$ $\{1,5,8\}$ $\{1,6,9\}$ $\{2,4,8\}$ $\{2,5,9\}$ $\{2,6,7\}$ $\{3,4,9\}$ $\{3,5,7\}$

DEF: Steiner System $S(l, m, n)$ is a collection of m-element subsets of an n-element set B such that every l-element subset of B lies in exactly one of the m-element sets. B is called the base set.
A Steiner Triple system is a $S(1, 3, n)$
The seven point plane is a $S(2, 3, 7)$

Formula: the number of m-element sets in an $S(l, m, n)$ is

$$\binom{n}{l} / \binom{m}{l}$$

THM: If a system $S(l, m, n)$ exists, so does a system $S(l-1, m-1, n-1)$. Consequently, so does a system $S(l-2, m-2, n-2)$ and in general a system $S(l-u, m-u, n-u)$ for each $u < l$.