cs208 HW 2

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Question 1

(a) Both (i) and (iv) are (ϵ, 0)-differentially private. I demonstrate that below using two neighboring datasets, $x = [0, 0, ..., 0]$ and $x' = [0, 0, ..., 1]$.

(i)

$$\frac{P([\bar{x} + Z]_0^1 = r)}{P([\bar{x'} + Z]_0^1 = r)}$$

$$\frac{\frac{n}{4} exp(-\frac{n}{2}|r - \bar{x}|)}{\frac{n}{4} exp(-\frac{n}{2}|r - \bar{x'}|)}$$

$$exp(\frac{n}{2}|r - \bar{x'}| - |r - \bar{x}|)$$

By the triangle inequality, this expression is less than or equal to

$$exp(\frac{n}{2}|r - \bar{x'} - r + \bar{x}|) = exp(\frac{n}{2}|\bar{x} - \bar{x'}|)$$

Since $|\bar{x} - \bar{x'}|$ is just the global sensitivity, we get that

$$exp(\frac{n}{2}|\bar{x} - \bar{x'}|) = exp(\frac{n}{2}\frac{1}{n}) = exp(\frac{1}{2})$$

Thus, $M(x)$ is $(\epsilon, 0)$-differentially private for $\epsilon \geq 0.5$.

(iv)

$$\frac{exp(\frac{-n}{10}|y - \bar{x}|)}{exp(\frac{-n}{10}|y - \bar{x'}|)} * \frac{\int_0^1 exp(\frac{-n}{10}|z - \bar{x}|)dz}{\int_0^1 exp(\frac{-n}{10}|z - \bar{x'}|)dz}$$

I'll evaluate this term by term. First, the left term:

$$exp(\frac{n}{10}(-|y - \bar{x}| + |y - \bar{x'}|))$$

$$exp(\frac{n}{10}(-|y| + |y - \frac{1}{n}|))$$

By the triangle inequality, this is less than or equal to

$$exp(\frac{n}{10}(y - \frac{1}{n} - y)) = exp(\frac{1}{10})$$

Now, for the right term:

$$\frac{\int_0^1 exp(\frac{-n}{10}|z-\frac{1}{n}|)dz}{\int_0^1 exp(\frac{-nz}{10})dz}$$

$$\frac{\int_0^{\frac{1}{n}} exp(\frac{-n}{10}(z-\frac{1}{n}))dz + \int_{\frac{1}{n}}^1 exp(\frac{-n}{10}(z-\frac{1}{n}))dz}{\int_0^{\frac{1}{n}} exp(\frac{-nz}{10})dz + \int_{\frac{1}{n}}^1 exp(\frac{-nz}{10})dz}$$

$$\frac{exp(\frac{1}{10})\int_0^{\frac{1}{n}} exp(\frac{-nz}{10})dz + exp(\frac{1}{10})\int_{\frac{1}{n}}^1 exp(\frac{-nz}{10})dz}{\int_0^{\frac{1}{n}} exp(\frac{-nz}{10})dz + \int_{\frac{1}{n}}^1 exp(\frac{-nz}{10})dz}$$

This reduces to $exp(\frac{1}{10})$. Putting the two terms together, we have

$$exp(\frac{1}{10}) * exp(\frac{1}{10}) = exp(\frac{1}{5})$$

Thus, this mechanism is $(\epsilon, 0)$-differentially private for $\epsilon \geq 0.2$.

(b) Mechanisms (ii) and (iii) are not $(\epsilon, 0)$-differentially private. Below I'll provide a counterexample that demonstrates this and find a minimum value of δ for which they are (ϵ, δ)-differentially private.

(ii) Consider $x = [0, 0, ..., 0]$ and $x' = [0, 0, ..., 1]$. Now, $P(M(x) = -1) \geq 0$ while $P(M(x') = -1) = 0$. This violates $P(M(x) = -1) \leq exp(\epsilon)P(M(x') = -1)$, so this mechanism is not $(\epsilon, 0)$-differentially private. Now let's consider the minimum value of δ for which it is (ϵ, δ)-differentially private.

$$\delta \geq max_{x\ x'}[\int_y max(P(M(x) = y) - exp(\epsilon)P(M(x') = y), 0)]$$

Since y will be bounded on [-1, 2] here, this is the same as

$$max[\int_{-1}^{2} P(M(x) = y) - \exp(\epsilon)P(M(x') = y), 0]$$

Consider the worst-case scenario I defined above, where $x = [0, ..., 0]$ and $x' = [0, ..., 0, 1]$. The expression inside the integral will only be ≥ 0 for $y \in [-1, -1 + \frac{1}{n}]$ because $P(M(x') = y) = 0$ here.

$$\int_{-1}^{-1+\frac{1}{n}} P(M(x) = y)$$

$$\int_{-1}^{-1+\frac{1}{n}} P(\bar{x} + Z = y)$$

$$\int_{-1}^{-1+\frac{1}{n}} \frac{n}{4} exp(\frac{-n|y-\bar{x}|}{2})$$

$$\int_{-1}^{-1+\frac{1}{n}} \frac{n}{4} exp(\frac{-n(y-\bar{x})}{2})$$

$$\frac{n}{4} exp(\frac{-n\bar{x}}{2}) \int_{-1}^{-1+\frac{1}{n}} exp(\frac{-ny}{2})$$

We know that $\bar{x} = 0$ and after integrating we are left with

$$\frac{1}{2}[exp(\frac{ny}{2})]_{-1}^{-1+\frac{1}{n}}$$

$$\frac{1}{2}[exp(\frac{1}{2})exp(\frac{-n}{2}) - exp(\frac{-n}{2})]$$

$$\frac{1}{2}exp(\frac{-n}{2})[exp(\frac{1}{2}) - 1]$$

Thus, this mechanism is (ϵ, δ)-differentially private for $\delta \geq \frac{1}{2}exp(\frac{-n}{2})[exp(\frac{1}{2}) - 1]$.

(iii) Consider $x = [0, 0, ..., 1]$ and $x' = [0, 0, ..., 0]$. Now, $P(M(x) = 1) = \frac{1}{n}$ while $P(M(x') = 1) = 0$. This clearly violates $P(M(x) = 1) \leq exp(\epsilon)P(M(x') = 1)$, so this mechanism is not $(\epsilon, 0)$-differentially private. Now let's consider the minimum value of δ for which it is (ϵ, δ)-differentially private.

$$\delta \geq max_{x\ x'}[\Sigma_y max(P(M(x) = y) - exp(\epsilon)P(M(x') = y), 0)]$$

$$\Sigma_{y\in[0,1]} max[(P(M(x) = y) - exp(\epsilon)P(M(x') = y), 0]$$
$$max[P(M(x) = 1) - exp(\epsilon)P(M(x') = 1), 0] + max[P(M(x) = 0) - exp(\epsilon)P(M(x') = 0), 0]$$

$$max[\frac{1}{n} - exp(\epsilon) * 0, 0] + max[1 - \frac{1}{n} - exp(\epsilon) * 1, 0] = \frac{1}{n} + 0 = \frac{1}{n}$$

Thus, this mechanism is (ϵ, δ)-differentially-private for $\delta \geq \frac{1}{n}$.

(c)

(d) I guess (i) -we understand laplace mechanism well -clamping to protect outliers?

Question 2

(a)

```
poissonDGP <- function(n){ return(rpois(n, lambda=10)) }
```

(b) I use the first mechanism from Question 1 to answer this question.

```
sgn <- function(x) {      # function borrowed from class
  return(ifelse(x < 0, -1, 1))
}

rlap = function(mu=0, b=1, size=1) {     # function borrowed from class
  p <- runif(size) - 0.5
  draws <- mu - b * sgn(p) * log(1 - 2 * abs(p))
  return(draws)
}

clip <- function(x, lower, upper){     # function borrowed from class
  x.clipped <- x
  x.clipped[x.clipped<lower] <- lower
  x.clipped[x.clipped>upper] <- upper
```

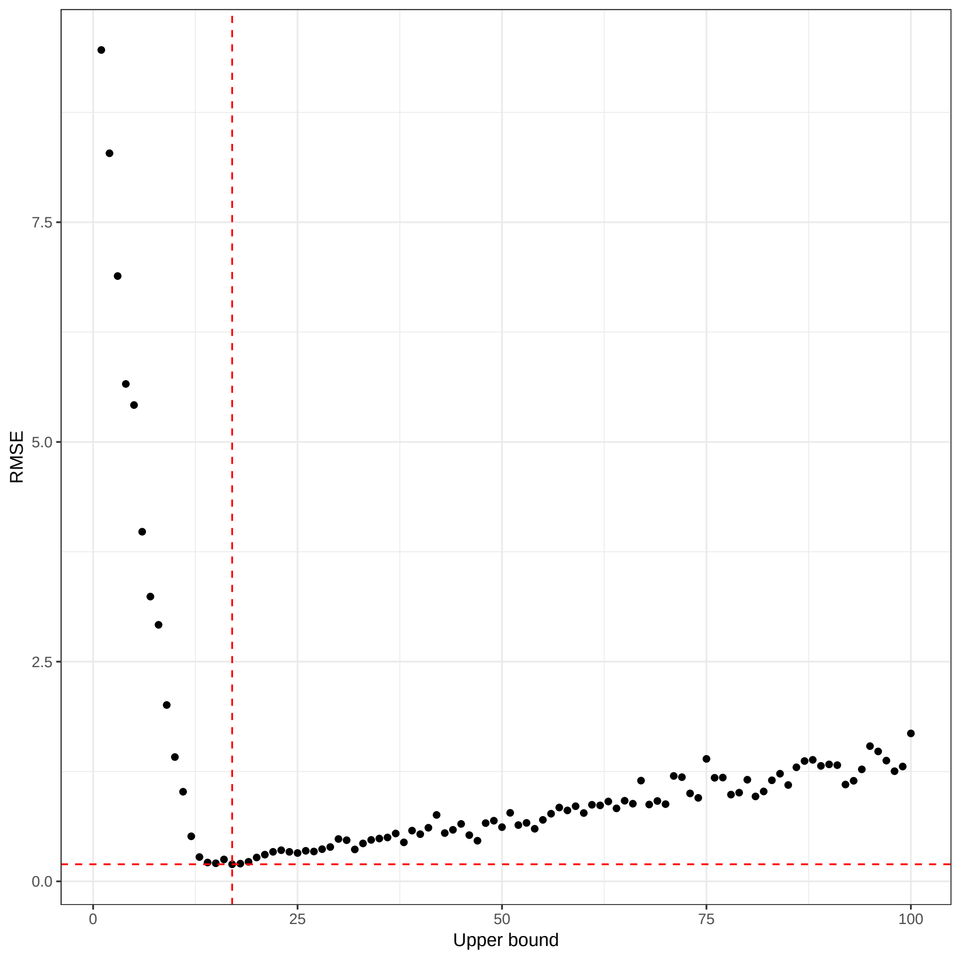
```
  return(x.clipped)
}

laplaceClampMeanRelease <- function(x, epsilon, a=0, b=1){
  n <- length(x)
  sensitivity <- (a - b)/n
  scale <- sensitivity / epsilon

  x.clipped <- clip(x, a, b)
  clipped.mean <- mean(x.clipped)
  noisy.mean <- clipped.mean + rlap(mu=0, b=scale, size=1)
  release.mean <- clip(noisy.mean, a, b)
  true.mean <- mean(x)

  return(list(release=release.mean, true=true.mean))
}
```

(c) Find the code for this portion of the problem in the Appendix. Based on my analysis, the optimal value for the upper bound b - i.e., the one that minimizes RMSE - is 19. I will use that value for the remiander of this problem and for Question 3.

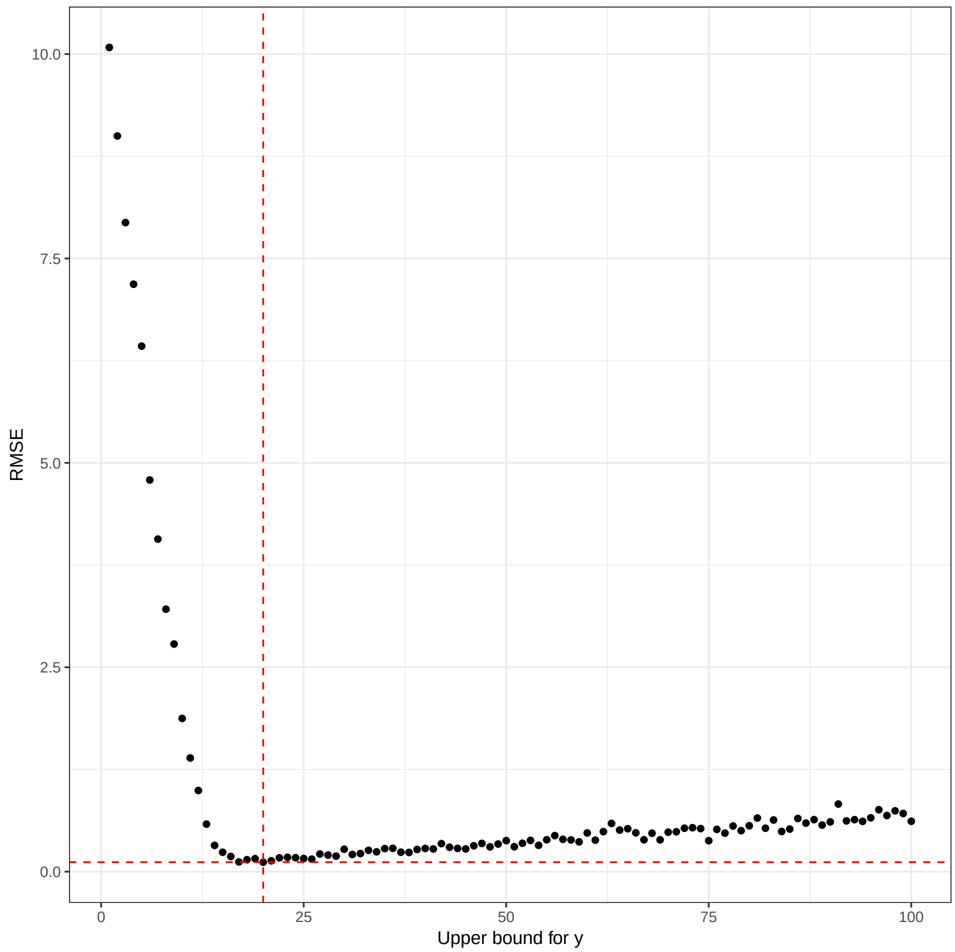


(d) -protecting outliers -why doesn't bootstrap do this

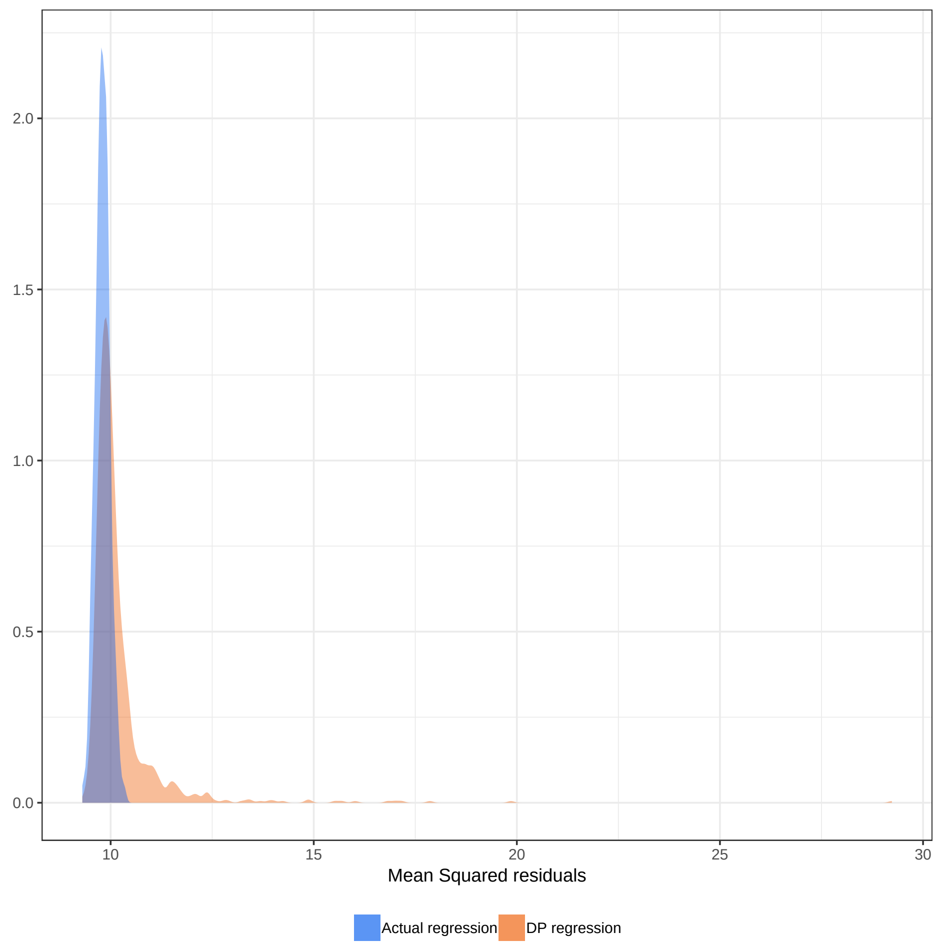
(e) -propose new mechanism -local sensitivity –> lower sensitivity in general, means thay we draw less noisy means -BOOM -or a diffeent statistic, different mechanism -see section notes

Question 3

(a) -4 DP subroutines: S_{xy}, S_{xx}, $\bar{x}$, and $\bar{y}$ and so we split epsilon over these (2 mean release and one regression coefcciicient release) -how to clamp y? –> run same code as in 2, but generate poisson, plug that into noisy linear, and then find lowest rmse for some values of an upper bound for y —— 17 is the value I find



(b) -should true parameters be computed from clipped data?



Question 4

First we can write out

$$\frac{\Sigma_{i=1}^{n} P[A(M(X))_i = X_i]}{n}$$

$$\frac{\Sigma_{i=1}^{n} P[A(M(X_i, X_{-i})) = X_i]}{n}$$

By the definition of (ϵ, δ)-differential privacy, we have

$$\frac{\Sigma_{i=1}^{n} P[A(M(X_i, X_{-i})) = X_i]}{n} \leq \frac{\Sigma_{i=1}^{n} exp(\epsilon) P[A(M(*, X_{-i})) = X_i]}{n} + \delta$$

where $*$ is either 0 or 1, depending on whether 0 or 1 is the most common value in the dataset.

- average of indicator functions?

Appendix

I put the code for all of my analyses here. You can also find it on Github.

Question 2

```
# c
rmse <- function(pred, true){ return(sqrt(mean((pred-true)^2))) }

n = 200
epsilon = 0.5
b_vals = seq(from=1, to=100, by=1)
n_sims <- 100

results <- matrix(NA, nrow=(length(b_vals)*n_sims), ncol=4)
i = 1
for (b in b_vals){
  dat <- poissonDGP(n)
  for (j in 1:n_sims){
    DPrelease <- laplaceClampMeanRelease(dat, epsilon, a=0, b=b)
    results[i,1] <- b
    results[i,2] <- j
    results[i,3] <- DPrelease$release
    results[i,4] <- DPrelease$true
    i = i + 1
  }
}
results_df <- data.frame(results)
names(results_df) <- c("b", "sim", "release", "true")
avg_results_df <- results_df %>% group_by(b) %>% summarise(rmse = rmse(release, true))

q2_plot <- ggplot(data=avg_results_df, aes(x=b, y=rmse)) +
  geom_point() + geom_hline(yintercept = min(avg_results_df$rmse), col="red", lty=2) +
  geom_vline(xintercept = avg_results_df[which.min(avg_results_df$rmse), ]$b, col="red", lty=2) +
  labs(x="Upper bound", y="RMSE") + theme_bw()
```

Question 3

```
# a
poissonDGP <- function(n){ return(rpois(n, lambda=10)) }
noisyLinearDGP <- function(x, n, alpha, beta, mu=0, sd=1) { return(beta*x + alpha + rnorm(n, mu, sd)) }

sgn <- function(x) {     # function borrowed from class
  return(ifelse(x < 0, -1, 1))
}

rlap = function(mu=0, b=1, size=1) {     # function borrowed from class
  p <- runif(size) - 0.5
  draws <- mu - b * sgn(p) * log(1 - 2 * abs(p))
  return(draws)
}

clip <- function(x, lower, upper){     # function borrowed from class
  x.clipped <- x
  x.clipped[x.clipped<lower] <- lower
  x.clipped[x.clipped>upper] <- upper
  return(x.clipped)
```

```
}

rmse <- function(pred, true){ return(sqrt(mean((pred-true)^2))) }

laplaceClampMeanRelease <- function(x, epsilon, a=0, b=1){     # from q2
  n <- length(x)
  sensitivity <- (a - b)/n
  scale <- sensitivity / epsilon

  x.clipped <- clip(x, a, b)
  clipped.mean <- mean(x.clipped)
  noisy.mean <- clipped.mean + rlap(mu=0, b=scale, size=1)
  release.mean <- clip(noisy.mean, a, b)
  true.mean <- mean(x)

  return(list(release=release.mean, true=true.mean))
}

regressionRelease <- function(y, x, ylower=0, yupper=17, xlower=0, xupper=19, eplsilon, partition){
  x <- clip(x, xlower, xupper)
  y <- clip(y, ylower, yupper)

  n <- length(x)
  sens.Sxy <- ((xupper-xlower)*(yupper-ylower))
  sens.Sxx  <- ((xupper-xlower)^2)

  scale.Sxy <- sens.Sxy / (epsilon*partition$Sxy)
  scale.Sxx <- sens.Sxx / (epsilon*partition$Sxx)

  true.beta <- sum((x - mean(x))*(y - mean(y))) / sum((x - mean(x))^2)
  true.alpha <- mean(y) - true.beta*mean(x)

  release.Sxy <- sum((x - mean(x))*(y - mean(y)))  + rlap(mu=0, b=scale.Sxy, size=1)
  release.Sxx <- sum((x - mean(x))^2) + rlap(mu=0, b=scale.Sxx, size=1)
  release.beta <- release.Sxy/release.Sxx

  release.x.bar <- laplaceClampMeanRelease(x, epsilon*partition$x.bar, a=xlower, b=xupper)$release
  release.y.bar <- laplaceClampMeanRelease(y, epsilon*partition$y.bar, a=ylower, b=yupper)$release
  release.alpha <- release.y.bar - release.beta*release.x.bar

  release.mean.sq.residuals <- mean((y - release.beta*x - release.alpha)^2)
  true.mean.sq.residuals <- mean((y - true.beta*x - true.alpha)^2)

  return(list(release.beta=release.beta,
              release.alpha=release.alpha,
              true.beta=true.beta,
              true.alpha=true.alpha,
              release.mean.sq.residuals=release.mean.sq.residuals,
              true.mean.sq.residuals=true.mean.sq.residuals))
}

# get optimal upper bound for y
n = 200
epsilon = 1
```

```
b_vals = seq(from=1, to=100, by=1)
n_sims <- 100

results_y <- matrix(NA, nrow=(length(b_vals)*n_sims), ncol=4)
i = 1
for (b in b_vals){
  dat <- poissonDGP(n)
  y <- noisyLinearDGP(dat, n, alpha=1, beta=1, mu=0, sd=1)
  for (j in 1:n_sims){
    DPrelease <- laplaceClampMeanRelease(y, epsilon, a=0, b=b)
    results_y[i,1] <- b
    results_y[i,2] <- j
    results_y[i,3] <- DPrelease$release
    results_y[i,4] <- DPrelease$true
    i = i + 1
  }
}
results_y_df <- data.frame(results_y)
names(results_y_df) <- c("b", "sim", "release", "true")
avg_results_y_df <- results_y_df %>% group_by(b) %>% summarise(rmse = rmse(release, true))

q3_plot1 <- ggplot(data=avg_results_y_df, aes(x=b, y=rmse)) +
  geom_point() + geom_hline(yintercept = min(avg_results_y_df$rmse), col="red", lty=2) +
  geom_vline(xintercept = avg_results_y_df[which.min(avg_results_y_df$rmse), ]$b, col="red", lty=2) +
  labs(x="Upper bound for y", y="RMSE") + theme_bw()
pdf("plots/q3_plot1.pdf", width=8, height=8)
q3_plot1
dev.off()

# b
equal_partition <- list(Sxy=0.25, Sxx=0.25, x.bar=0.25, y.bar=0.25)
n = 1000
alpha = beta = eplison = sd = 1
n_sims = 1000

results_reg <- matrix(NA, nrow=n_sims, ncol=6)
for (i in 1:n_sims){
  x <- poissonDGP(n)
  y <- noisyLinearDGP(dat, n, alpha=1, beta=1, mu=0, sd=1)
  DPrelease <- regressionRelease(y, x, ylower=0, yupper=17, xlower=0, xupper=19, eplsilon, equal_partit
  results_reg[i,1] <- DPrelease$release.beta
  results_reg[i,2] <- DPrelease$release.alpha
  results_reg[i,3] <- DPrelease$true.beta
  results_reg[i,4] <- DPrelease$true.alpha
  results_reg[i,5] <- DPrelease$release.mean.sq.residuals
  results_reg[i,6] <- DPrelease$true.mean.sq.residuals
}
results_reg_df <- data.frame(results_reg)
names(results_reg_df) <- c("release.beta", "release.alpha", "true.beta",
                           "true.alpha", "release.mean.sq.residuals", "true.mean.sq.residuals")

semi.blue <- rgb(0,90,239,50,maxColorValue=255)
semi.red  <- rgb(239,90,0,200,maxColorValue=255)
q3_plot2 <- ggplot(data=results_reg_df) +
```

```
  geom_density(aes(x=release.mean.sq.residuals, fill="DP regression"), alpha=0.4, colour=NA) +
  geom_density(aes(x=true.mean.sq.residuals, fill="Actual regression"), alpha=0.4, colour=NA) +
  labs(x="Mean Squared residuals", y="") + theme_bw() +
  scale_fill_manual(values=c(semi.blue, semi.red)) +
  theme(legend.position="bottom", legend.title=element_blank())

# d
(more code to come)
```