MAA Metro NY Section
Problem of the Month

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1 Problem

The index of the palindrome p is the positive integer i such that p is the ith palindrome. For example, the index of the palindrome 11 is 10. A palindromic-index palindrome is a palindrome whose index is also a palindrome. For example, 22 is a palindromic-index palindrome, since its index 11 is a palindrome. Moreover, 22 is the 10th palindromic-index palindrome. **What is the 111th palindromic-index palindrome?**

2 Solution

The 111th palindromic-index palindrome is 222222. Let $P(x)$ be a well-defined function that maps integer x to the x^{th} palindromic-index palindrome. Let $p(x)$ be a well-defined function that maps integer x to the x^{th} palindrome. $P(x)$ can be expressed as a composition of $p(x)$, namely $P(x) = p(p(x))$. Function $p(x)$ can be defined as

$$
\begin{aligned}
&p(x) = \left(10^{q(x)-1}\right)(C(x)+10)\,x \\
&-G(x)\left((A(x)-Z(x)-1)+Y(x)-(C(x)10^{2(q(x)-1)}-1)-B(x)\left(\left\lfloor \frac{x+1}{10^{q}} \right\rfloor - 2\right)\right)
\end{aligned}
$$

The subfuctions are defined as follows:

$$
\begin{aligned}
w(x) &= x - (10^{\lfloor \log_{10}(x)) \rfloor - 1} - 1) \\
q(x) &= \lfloor \log_{10}(w(x)) \rfloor \\
v(x) &= w(x) - 9 \left(10^{\lfloor \log_{10}(w(x)) - 1 \rfloor} \right) \\
m(x) &= \lfloor \log_{10}(v(x)) \rfloor \\
C(x) &= (q(x) - m(x)) \mod 2 \\
B(x) &= (q(x) - m(x) + 1) \mod 2 \\
A(x) &= \frac{10^{q(x)} - 1}{9} \left(10^{q(x)+1} \right) \\
Z(x) &= \frac{10^{q(x)-C(x)} - 1}{9} \left(10^{q(x)-C(x)} \right) \\
Y(x) &= \sum_{n=1-C(x)}^{q(x)-C(x)-1} \left(\left(C(x) 10^{2(q(x)-1)-n} - 10^{n} \right) \left\lfloor \frac{(x+1) \mod \left(10^{q(x)-n+B(x)} \right)}{10^{q(x)-n-1-B(x)}} \right\rfloor \right) \\
G(x) &= \min(\left\lfloor \frac{x}{10} \right\rfloor, 1)
\end{aligned}
$$

Let x = 111.

$$
\begin{aligned}
P(111) &= p(p(111)) \\
&= p(1221) \\
&= 222222
\end{aligned}
$$

3 Python Code Snippet

The following code was derived from the functions above. Functions such as q(x) were replaced with variables to save computation time. Y(x), a summation, remains a function of x.

```
from math import * #to use floor and log base 10 operations

def Y(x): #the only summation term, represented with a for loop
    global C,B,q,m,w,v
    t = 0
    for n in range(1-C, q-C): #bounds of summation, proportional to log(x)
        p1 = C*(10**(2*(q-1)-n)) - (10**n)
        p2 = ((x+1) % (10**(q-n+B)))//(10**(q-n-1+B))
        t+= p1*p2 #each term in the series is this product
    return t #result of summation
```

```python
def p(x): #function for finding the xth palindrome
    global C,B,q,m,w,v
    w = x-(10**(floor(log10(x))-1)-1)
    q = (floor(log10(w)))
    v = w-9*10**(q-1)
    m = floor(log10(v))
    G = min(x//10,1) #returns 0 if x is between 1 and 9 inclusive, returns 1 otherwise
    C = ((q-m) % 2) #returns 1 if the xth palindrome has even number of digits
    B = ((q-m+1) % 2)  #returns 1 if the xth palindrome has odd number of digits
    A =  ((10**(q)-1)//9) * 10**(q+1)
    Z =  ((10**(q-C)-1)//9) * 10**(q-C)

    #Dividing the expression into components
    p1 = 10**(q-1) * (C+10)
    p2 = A-Z-1
    p3 = Y(x)
    p4 = C*(10**(2*(q-1))-1)
    p5 = B*(((x+1)//(10**(q)))-2)
    ans = p1*x-G*(p2+p3-p4-p5)

    #The xth palindrome
    return int(ans)

def P(x): #function for finding the xth palindromic-index palindrome
    return(p(p(x)))

#Calling the function to find the 111th palindromic-index palindrome
print(P(111)) #p(p(111)) = p(1221) = 222222
```