CP409 Problem Solving Block A Week 3 – Dimensional Analysis 2

Last week we used Dimensional Analysis to consider how various problems can be tackled without the need for deep theoretical analyses. We do this by simply considering how the dimensions (or units) of key physical parameters must combine to give the correct dimensions in the result.

However, as we saw, identifying the key physical parameters for the problem is crucial to the success of this approach. For example, we considered how the drag force depends on air density (but not viscosity) for someone falling out of a plane, and then in a different problem how the drag force depends on fluid viscosity for a heavy sphere falling through a viscous fluid. Surely these situations are related; how can the two approaches be reconciled?

Let us return to the first approach where we ignored viscosity. You can verify for yourself that the drag force F on a sphere with diameter D (or cross-sectional area $A = \frac{\pi}{4}D^2$) falling at speed v through a fluid with density ρ is

$$F = \frac{1}{2}c_d \times \rho \times A \times v^2 \qquad (1)$$

In this equation, the drag coefficient c_d is a dimensionless number that is a constant at high speeds v; you will see what is meant by 'high' later. Note the factor of ½ that has been included in equation (1). This could, of course, be incorporated into c_d, but it is usually explicitly shown since it occurs if you start the analysis from the Bernoulli Equation.

The essential condition is that c_d is dimensionless - but it does not have to be a constant. Therefore it can depend on the fluid viscosity μ (or indeed any other relevant parameter) provided that μ is combined with the other parameters ρ, D, v to make a *dimensionless group*. We can therefore explicitly write c_d as a function of this group:

$$c_d\left(\frac{\mu}{\rho^\alpha \times D^\beta \times v^\gamma}\right) \qquad (2)$$

The exponents α, β, γ are to be determined by Dimensional Analysis so that the drag coefficient has no dimensions.

Task A4.1

(a) Perform the Dimensional Analysis required in equation (2). The result should look familiar – what is this dimensionless group usually called?

(b) Consult a reliable source to see how c_d varies with the numerical value of your dimensionless group, and sketch its form.

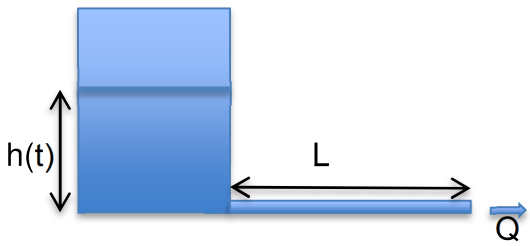
(c) Under which conditions is c_d approximately constant; how does this correspond to the problems you have considered last week?

(d) For very viscous fluids, we anticipate that F in equation (1) will linearly depend on viscosity μ. What is the functional form of c_d in this case, and how does it correspond to the problems you considered last week?

(e) How do your answers to parts (c) and (d) relate to your sketch in (b)?

Task A4.2

In this problem we will consider how long it takes to drain a tank of fluid through a horizontal pipe at its base, as in the figure:



The length of the pipe is L, its diameter is D. The depth of the fluid in the tank is h(t), and starts at h_0. The tank is of regular cross-sectional area A, and contains a fluid with density ρ and viscosity μ.

(a) Referring to last week's tasks, derive the relationship between pressure drop ΔP along the pipe and the volumetric flow rate Q if the fluid is viscous.

(b) How is the pressure at the base of the tank related to the depth of the fluid? Use this to derive a first order differential equation for h(t), and then solve.

(c) What is the timescale for the tank drainage in terms of μ, L etc. What happens to this in the limit

$$(\mu \times L) \rightarrow 0,$$

and is this sensible?

(d) Repeat the Dimensional Analysis of the pressure drop across the pipe, now including dimensionless functions of ρ and of L in your analysis. Sketch the pressure drop as a function of $(\mu \times L)$. What is the expression for ΔP when it is independent from μ and L?

(e) Derive the first order differential equation for h(t) in this limit, and solve. How long does it take the tank to drain when viscosity is negligible?

(f) Solve the differential equations from parts (b) and (e) using the Euler Method in Matlab, and compare to your analytical solutions.