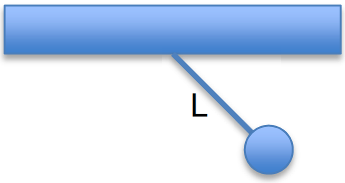
CP409 Problem Solving Block A - Week 2 – Dimensional Analysis

Dimensional Analysis provides a powerful and versatile way to approach new problems. As we will see, in some sense it allows us to cheat, so that we can make good calculations without having to do deep theoretical analyses.

The best way to introduce Dimensional Analysis is through an example. Consider a simple pendulum: how does the period of oscillation T depend on the system parameters of length L, mass M and acceleration due to gravity g?



Clearly we can dust off our Higher Physics, resolve forces and solve the ensuing differential equations. However, an easier approach is to realize that whatever formula we end up with, it must have units of time, and this must severely restrict what the combination of parameters must be in our solution.

We therefore start by writing:

$$T \sim L^{\alpha} \times M^{\beta} \times g^{\gamma} \qquad (1)$$

In other words, we are asking how T will scale with length L etc, and the answer depends on the exponent α (eg, if $\alpha = 2$, the period will vary as the square of the length L).

So how do we determine the exponents α, β, γ? We simply look at the dimensions (or if you prefer units) of the quantities involved, and insist that T ends up with units of time, which I write here as [t]:

$$\begin{aligned}[t] &= [l]^{\alpha} \times [m]^{\beta} \times [g]^{\gamma} \\ &= [l]^{\alpha} \times [m]^{\beta} \times \frac{[l]^{\gamma}}{[t]^{2\gamma}}\end{aligned} \qquad (2)$$

(Here I have used that [g] is length over time-squared, $[g] = [l] \times [t]^{-2}$)

Equation (2) gives us the information we require. Clearly, $\gamma = -1/2$ for the time dimension [t], $\beta = 0$ for mass dimension [m], and $\alpha = -\gamma = 1/2$ for length dimension [l]; there is no dependence on [m] or [l] on the left hand side of (2).

Putting these into equation (1) we have found

$$T = k \times \sqrt{\frac{L}{g}}$$

where the constant k could be found by experiment, or a more detailed theoretical analysis. Hopefully you recognize the result, and particularly the independence of T from the mass of the bob on the pendulum.

Task A2.1

(a) Derive a formula for the power that a wind turbine can generate using Dimensional Analysis. Use the length L of the turbine blades, the density of air ρ, and the speed of the wind v. Estimate the generating capacity of a typical turbine with L=40m.

(b) Estimate the free-fall velocity of some-one falling out of a plane. What is it if they have a parachute?

(c) When a guitar string is plucked, it vibrates at a certain fundamental frequency. Use Dimensional Analysis to find out how this relates to the physical properties of the string.

(d) Derive a formula for the pressure drop per unit length along a pipe with diameter D carrying a high-viscosity fluid with viscosity μ at volumetric flow rate Q.

Task A2.2

Consider a Falling Sphere Viscometer, which is used to measure the viscosity μ of a fluid by observing the terminal velocity of a heavy sphere (density ρ_s and diameter D) falling under gravity in a column of the fluid (density ρ_f).

(a) Use Dimensional Analysis to derive a formula for the drag force exerted on the sphere by the viscous fluid when it is moving at speed v through the fluid.

(b) How is the terminal velocity of the sphere related to the fluid viscosity?

(c) If the sphere starts from rest, use Dimensional Analysis to predict the timescale over which the sphere will reach its terminal velocity.

(d) Derive an appropriate first-order differential equation for the speed of the sphere as a function of time.

(e) Solve your equation from (d) when the falling sphere starts from rest. How does the timescale of your solution compare to that you found above in (c)?

(f) Solve your equation from (d) using the Euler Method in Matlab, and compare your results with those from (e).