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$$S = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}, \quad A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

1

(a)

$F(\underline{x}) = \frac{1}{2}\underline{x}^T S \underline{x}$, minimise subject to $A^T\underline{x} = b$

$$\Rightarrow L(\underline{x}, \lambda) = \frac{1}{2}\underline{x}^T S \underline{x} + \lambda(b - A^T\underline{x})$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow L(\underline{x}, \lambda) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \lambda\left(b - \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$$

$$\Rightarrow L(\underline{x}, \lambda) = \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5x_1 - 3x_2 \\ -3x_1 + 5x_2 \end{bmatrix} + \lambda(b - x_1 - 3x_2)$$

$$\Rightarrow L(\underline{x}, \lambda) = \frac{1}{2}(5x_1^2 + 5x_2^2 - 6x_1x_2) + \lambda(b - x_1 - 3x_2)$$

(b)

$$\frac{\partial L}{\partial x_1} = 5x_1 - 3x_2 - \lambda = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_2} = 5x_2 - 3x_1 - 3\lambda = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = b - x_1 - 3x_2 = 0 \quad (3)$$

(c)

from (1), we have $\lambda = 5x_1 - 3x_2$, putting this into (2),

$$5x_2 - 3x_1 - 3\lambda = 5x_2 - 3x_1 - 3(5x_1 - 3x_2) = 0 \Rightarrow 14x_2 = 18x_1 \Rightarrow x_2 = \frac{9}{7}x_1$$

putting this into 3,

$$b - x_1 - 3x_2 = b - x_1 - \frac{27}{7}x_1 = 0 \Rightarrow x_1 = \frac{7}{34}b$$

$x_2 = \frac{9}{7}x_1 \Rightarrow \boxed{x_2 = \frac{9}{34}b}$, now, use (1) to find $\lambda$,

$$\lambda = 5x_1 - 3x_2 = \frac{35}{34}b - \frac{27}{34}b = \boxed{\frac{4b}{17} = \lambda}$$

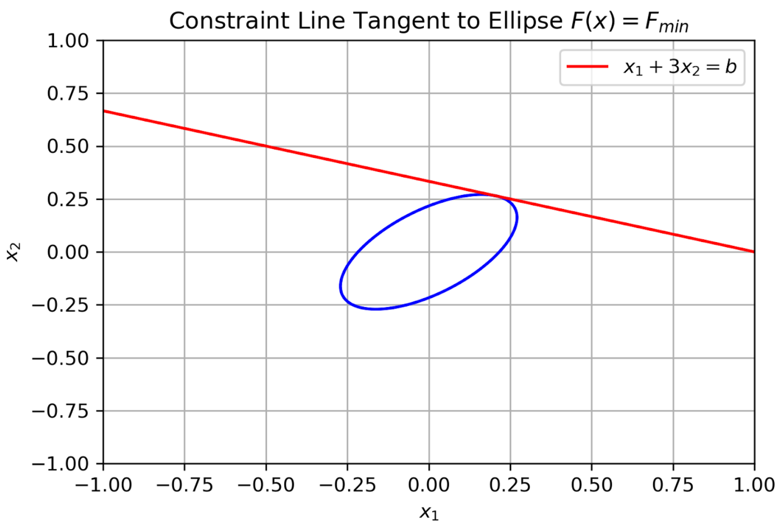
$$\Rightarrow \boxed{\underline{x}^* = b\begin{bmatrix}7/34\\9/34\end{bmatrix}, \quad \lambda^* = \frac{4b}{17}}$$

from (a)

$$f_{min} = f(\underline{x}^*) = \frac{1}{2}\underline{x}^{*T}S\underline{x}^* = \frac{1}{2}\left(5x_1^{*2} + 5x_2^{*2} - 6x_1^*x_2^*\right) = \frac{b^2}{2}\left(\frac{245 + 405 - 378}{1156}\right) = \frac{2b^2}{17}$$

$$\Rightarrow \boxed{f_{min} = \frac{2b^2}{17}}$$

(d)




(e)

$$\frac{\partial f_{min}}{\partial b} \overset{!}{=} \lambda^*$$

$$f_{min} = \frac{2b^2}{17}, \quad \lambda^* = \frac{4b}{17}$$

$$\Rightarrow \boxed{\frac{\partial f_{min}}{\partial b} = \frac{4b}{17} = \lambda^*, \text{ as desired}}$$

Q2

(a)

We have $x_{k+1} = x_k - \alpha \nabla f(x_k)$ from 1(a)

$f(\underline{x}) = \frac{1}{2}\underline{x}^T S \underline{x} = \frac{1}{2}(5x_1^2 + 5x_2^2 - 6x_1x_2)$, $\nabla f(\underline{x}) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix}$

$\Rightarrow \nabla f(x) = \begin{bmatrix} 5x_1 - 3x_2 \\ -3x_1 + 5x_2 \end{bmatrix} = S\underline{x}$

$\Rightarrow x_{k+1} = x_k - \alpha S \underline{x}$

want min of $\varphi(\alpha) = f(x_{k+1})$

$f(x_{k+1}) = \frac{1}{2} x_{k+1}^T S x_{k+1} = \frac{1}{2}(x_k - \alpha S x_k)^T S (x_k - \alpha S x_k)$, $S = S^T$ (by inspection of $S$)

$\Rightarrow f(x_{k+1}) = \frac{1}{2}(x_k^T S - \alpha x_k^T S^2)(x_k - \alpha S x_k) = \frac{1}{2}(x_k^T S x_k - \alpha x_k^T S^2 x_k - \alpha x_k^T S^2 x_k + \alpha^2 x_k^T S^3 x_k)$

$\Rightarrow f(x_{k+1}) = \frac{1}{2}(x_k^T S x_k - 2\alpha x_k^T S^2 x_k + \alpha^2 x_k^T S^3 x_k)$

$\varphi(\alpha)' = \frac{d}{d\alpha}(f(x_{k+1})) = -x_k^T S^2 x_k + \alpha x_k^T S^3 x_k = 0$ to find minimum

$\Rightarrow \alpha = \dfrac{x_k^T S^2 x_k}{x_k^T S^3 x_k}$

(b)

optimal point $x^*$ is point s.t. $\nabla f(x) = 0$

$\nabla f(x) = Sx = 0 \Rightarrow x^* = 0$

$\Rightarrow e_k = ||x_k - x^*|| = ||x_k||$

$e_{k+1} = ||x_{k+1} - x^*|| = ||x_k - \alpha S x_k||$

$S$ symmetric $\Rightarrow S = PDP^T$, question asks "in terms of eigenvalues of $S$" so will not calculate explicitly.

We have $x_k = \mathbb{1} x_k = PP^T x_k = P y_k$, where $y_k = P^T x_k$

We have $x_{k+1} = x_k - \alpha S \underline{x} \Rightarrow P^T x_{k+1} = P^T x_k - \alpha P^T S \underline{x}$

$\Rightarrow y_{k+1} = y_k - \alpha \underbrace{P^T P}_{\mathbb{1}} D \underbrace{P^T x_k}_{y_k} \Rightarrow y_{k+1} = y_k - \alpha D y_k$

$\Rightarrow e_{k+1} = \| y_{k+1} - y^* \| = \| y_{k+1} \| = \| y_k - \alpha D y_k \|$

for each component $y_{k,i}$ we have

$y_{k+1,i} = (1 - \alpha \lambda_i) y_{k,i}$ $\Rightarrow$ slowest convergence associated with largest eigenvalue, $\lambda_{max}$

for optimal convergence $|1 - \alpha \lambda_i| = 0$

$\Rightarrow 1 - \alpha \lambda_i = 0 \Rightarrow \alpha = \frac{1}{\lambda_i}$

$\Rightarrow \alpha = \frac{1}{\lambda_i}$, largest eigenvalue corresponds to slowest convergence

$\Rightarrow \alpha = \frac{1}{\lambda_{max}}$

(c)

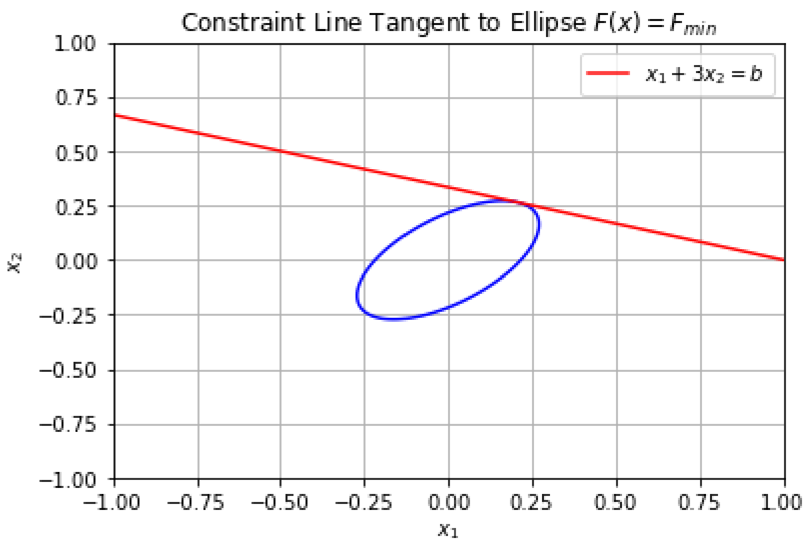
for $\underline{x_i}$ to be in the worst-case scenario, it has to align with the eigenvector, $v_{max}$, corresponding to $\lambda_{max}$. This causes slowest convergence because $|1 - \alpha \lambda_i|$ is closest to 1 (slowest) when $\lambda_i = \lambda_{max}$

$\Rightarrow x_i = c v_{max}$ .

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

1d

In [9]:
```
#arbitrary b
b=1
S=np.array([[5, -3], [-3, 5]])

F_min = 2*b**2 /17

x1_vals=np.linspace(-1,1, 200)
x2_vals=np.linspace(-1,1, 200)
X1, X2=np.meshgrid(x1_vals, x2_vals)


F_x=0.5*(5*X1**2+5*X2**2-6*X1*X2)

#contour of F_x=F_min
plt.contour(X1,X2,F_x,levels=[F_min], colors='b')
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.title('Constraint Line Tangent to Ellipse $F(x) = F_{min}$')

#constraint
plt.plot(x1_vals,(b -x1_vals)/3,color='r',label=r'$x_1 + 3x_2 = b$')

plt.legend()
plt.grid(True)
plt.savefig("1d.png",dpi=300)
plt.show()
```




2d

In [122…
```
S=np.array([[5,-3], [-3,5]])
x0=np.array([3,5])

eigenvalues, eigenvectors=np.linalg.eig(S)
lam_max=max(eigenvalues)
lam_min=min(eigenvalues)

alpha=1/(lam_max)
```

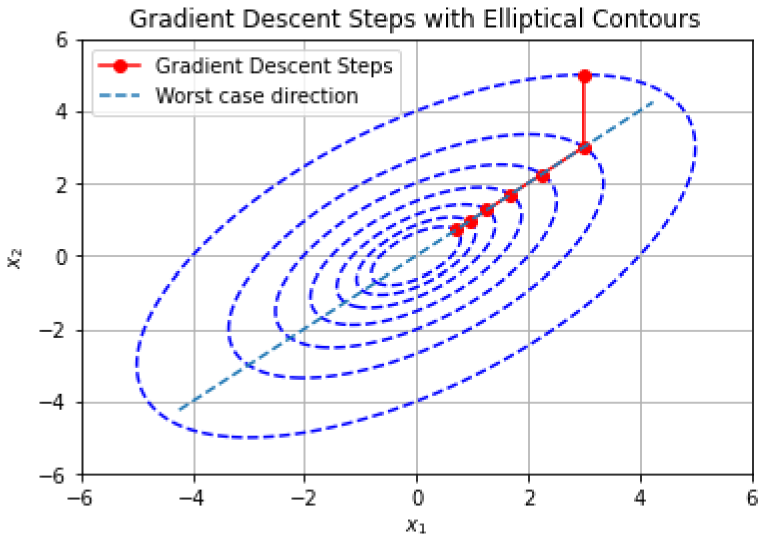
```
x_k= x0
vals=[x_k]

for i in range(6):
    x_k=x_k-alpha*S@x_k
    vals.append(x_k)

for i, x in enumerate(vals):
    print(f"Step {i}: x = {x}")
```

```
Step 0: x = [3 5]
Step 1: x = [3. 3.]
Step 2: x = [2.25 2.25]
Step 3: x = [1.6875 1.6875]
Step 4: x = [1.265625 1.265625]
Step 5: x = [0.94921875 0.94921875]
Step 6: x = [0.71191406 0.71191406]
```

e

In [123…

```
def F(x):
    return 0.5 *x.T@S@ x

#find and sort contour levels (have to be strictly increasing for plt.contour)
contour_levels=sorted([F(x) for x in vals])

x_vals=np.linspace(-6,6,400)
y_vals=np.linspace(-6,6,400)
X,Y=np.meshgrid(x_vals,y_vals)
Z=0.5*(S[0, 0]*X**2+2*S[0,1]*X*Y+S[1, 1]*Y**2)

plt.contour(X,Y,Z,levels=contour_levels,colors='b',linestyles='dashed')

vals=np.array(vals)
plt.plot(vals[:,0],vals[:,1],'o-',color='r',label='Gradient Descent Steps')

lam_max_idx=np.argmax(eigenvalues)
vec_max=eigenvectors[:,lam_max_idx]

line = np.array([vec_max * t for t in np.linspace(-6, 6, 2)])
#-1 const
plt.plot(line[:, 0],-line[:, 1],'--',label='Worst case direction')


plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.title("Gradient Descent Steps with Elliptical Contours")
plt.legend()
plt.grid(True)

plt.show()
```




f

In [124…

```
errors=[np.linalg.norm(x) for x in vals]
for i,e_k in enumerate(errors):
    print(f"Step{i}:Error e_{i}={e_k}")

print("\nConvergence over the first three steps:")
error_ratios=[errors[i+1]/errors[i] for i in range(len(errors)-1)]
for i in range(3):
    print(f"Error ratio e_{i+1}/e_{i}={error_ratios[i]}")
```

```
Step0:Error e_0=5.830951894845301
Step1:Error e_1=4.242640687119285
Step2:Error e_2=3.181980515339464
Step3:Error e_3=2.386485386504598
Step4:Error e_4=1.7898640398784484
Step5:Error e_5=1.3423980299088363
Step6:Error e_6=1.0067985224316272

Convergence over the first three steps:
Error ratio e_1/e_0=0.7276068751089988
Error ratio e_2/e_1=0.75
Error ratio e_3/e_2=0.7500000000000001
```

Roughly linear, bar first ratio almost completely linear

3a

In [125…

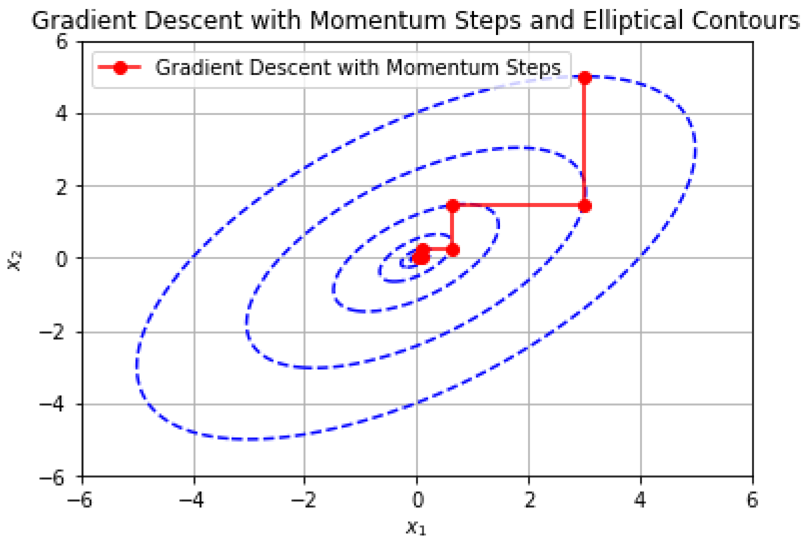
```
alpha=4/(np.sqrt(lam_max)+np.sqrt(lam_min))**2
beta=((np.sqrt(lam_max)-np.sqrt(lam_min))/(np.sqrt(lam_max)+np.sqrt(lam_min)))**2

x_k=x0
v_k=np.array([0,0])
vals=[x_k]

for i in range(6):
    v_k=beta*v_k-alpha*(S@x_k)
    x_k=x_k+v_k
    vals.append(x_k)

for i, x in enumerate(vals):
    print(f"Step {i}: x = {x}")
```

```
Step 0: x = [3 5]
Step 1: x = [3.         1.44444444]
Step 2: x = [0.62962963 1.44444444]
Step 3: x = [0.62962963 0.25925926]
Step 4: x = [0.10288066 0.25925926]
Step 5: x = [0.10288066 0.03978052]
Step 6: x = [0.01508916 0.03978052]
```

b

In [126…

```
#find and sort contour levels (have to be strictly increasing for plt.contour)
contour_levels = sorted([F(x) for x in vals])

plt.contour(X,Y,Z,levels=contour_levels,colors='b',linestyles='dashed')

vals=np.array(vals)
plt.plot(vals[:,0],vals[:,1],'o-',color='r',label='Gradient Descent with Momentum S

plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.title("Gradient Descent with Momentum Steps and Elliptical Contours")
plt.legend()
plt.grid(True)
plt.show()
```




c

In [127…

```
errors=[np.linalg.norm(x) for x in vals]
for i,e_k in enumerate(errors):
    print(f"Step{i}:Error e_{i}={e_k}")

print("\nConvergence over the first three steps:")
error_ratios1=[errors[i+1]/errors[i] for i in range(len(errors)-1)]
for i in range(3):
    print(f"Error ratio e_{i+1}/e_{i}={error_ratios1[i]}")
```

```
Step0:Error e_0=5.830951894845301
Step1:Error e_1=3.329627569727043
Step2:Error e_2=1.5757072137913068
Step3:Error e_3=0.6809176411426004
Step4:Error e_4=0.27892614325646925
Step5:Error e_5=0.11030376127832407
Step6:Error e_6=0.042546124607052815

Convergence over the first three steps:
Error ratio e_1/e_0=0.5710264172596781
Error ratio e_2/e_1=0.47323827689247555
Error ratio e_3/e_2=0.4321346219544464
```

d

In [129…

```
print(error_ratios)
print('\n',error_ratios1)
```

```
[0.7276068751089988, 0.75, 0.7500000000000001, 0.75, 0.75, 0.75]

 [0.5710264172596781, 0.47323827689247555, 0.4321346219544464, 0.4096327167973247
7, 0.3954586687017756, 0.3857178043067658]
```

**Without Momentum**: Gradient descent without momentum reduces the error linearly but relatively slowly, without acceleration, depending directly on the step size and the condition number of S.

**With Momentum**: Momentum accelerates convergence (rather than a linear convergence) by including a "memory" of the previous step. This leads to a faster error reduction, especially in directions associated with smaller eigenvalues. As can be seen in our error ratios for both, with momentum shows a quicker drop in error, achieving faster convergence compared to standard gradient descent.