MATMEK-4270 Mandatory 1

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I. 1.2.3. EXACT SOLUTION

Wave equation in 2D is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

, show that $u(t, x, y) = \exp[i(k_x x + k_y y - \omega t)]$ satisfies the wave equation.
First, find $\frac{\partial^2 u}{\partial t^2}$

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 \exp[i(k_x x + k_y y - \omega t)],$$

where $\omega^2 = c^2(k_x x + k_y y)$.
Second, find the right-hand side.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\exp[i(k_x x + k_y y - \omega t)](k_x x + k_y y) = -\exp[i(k_x x + k_y y - \omega t)]\omega^2/c^2$$

Now combining everything

$$-\omega^2 \exp[i(k_x x + k_y y - \omega t)] = c^2(-\exp[i(k_x x + k_y y - \omega t)])\omega^2/c^2$$

$$1 = 1 \text{ Q.E.D.}$$

II. 1.2.4. DISPERSION COEFFICIENT

Assuming that $m_x = m_y$, such that $k_x = k_y = k$ discrete u reads

$$u_{ij}^n = e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)} \tag{2}$$

, where $\tilde{\omega}$ is a numerical dispersion coefficient, i.e., the numerical approximation of the exact ω.
Show that the discrete equation (eq. 2) with CFL number $C = 1/\sqrt{2}$ plugged in

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right) \tag{3}$$

gives $\tilde{\omega} = \omega$.
First calculate the terms u_{ij}^{n+1}, u_{ij}^n, and u_{ij}^{n-1} from the discrete equation

$$\begin{aligned}
u_{ij}^{n+1} &= e^{\imath(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} = u_{ij}^n e^{-\imath \tilde{\omega} \Delta t}, \\
u_{ij}^n &= e^{\imath(kh(i+j) - \tilde{\omega} n \Delta t)}, \\
u_{ij}^{n-1} &= e^{\imath(kh(i+j) - \tilde{\omega}(n-1)\Delta t)} = u_{ij}^n e^{\imath \tilde{\omega} \Delta t}.
\end{aligned}$$

Substituting these into the left-hand side of the wave equation

$$\frac{u_{ij}^{n+1} - 2u_{ij}^{n} + u_{ij}^{n-1}}{\Delta t^2} = \frac{u_{ij}^{n}(e^{-\imath\tilde{\omega}\Delta t} - 2 + e^{\imath\tilde{\omega}\Delta t})}{\Delta t^2}.$$

Using the identity $e^{\imath\theta} + e^{-\imath\theta} = 2\cos(\theta)$, this simplifies to

$$\frac{u_{ij}^{n} \cdot 2(\cos(\tilde{\omega}\Delta t) - 1)}{\Delta t^2}.$$

Next, compute the spatial differences for $i+1$, $i-1$, $j+1$, and $j-1$

$$\begin{aligned}
u_{i+1,j}^{n} &= e^{\imath(kh(i+1+j) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{\imath kh},\\
u_{i-1,j}^{n} &= e^{\imath(kh(i-1+j) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{-\imath kh},\\
u_{i,j+1}^{n} &= e^{\imath(kh(i+j+1) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{\imath kh},\\
u_{i,j-1}^{n} &= e^{\imath(kh(i+j-1) - \tilde{\omega} n \Delta t)} = u_{ij}^{n} e^{-\imath kh}.
\end{aligned}$$

The spatial difference on the right-hand side becomes

$$\frac{u_{i+1,j}^{n} - 2u_{ij}^{n} + u_{i-1,j}^{n}}{h^2} + \frac{u_{i,j+1}^{n} - 2u_{ij}^{n} + u_{i,j-1}^{n}}{h^2}.$$

Substituting the values for $u_{i+1,j}^{n}$, $u_{i-1,j}^{n}$, etc.

$$\frac{u_{ij}^{n}(e^{\imath kh} - 2 + e^{-\imath kh})}{h^2} + \frac{u_{ij}^{n}(e^{\imath kh} - 2 + e^{-\imath kh})}{h^2}.$$

Using $e^{\imath\theta} + e^{-\imath\theta} = 2\cos(\theta)$, this simplifies to

$$\frac{u_{ij}^{n} \cdot 4(\cos(kh) - 1)}{h^2}.$$

For the discrete wave equation to hold, it needs

$$\frac{u_{ij}^{n} \cdot 2(\cos(\tilde{\omega}\Delta t) - 1)}{\Delta t^2} = c^2 \cdot \frac{u_{ij}^{n} \cdot 4(\cos(kh) - 1)}{h^2}.$$

Cancelling u_{ij}^{n} on both sides and simplifying, obtain the numerical dispersion relation

$$\frac{\cos(\tilde{\omega}\Delta t) - 1}{\Delta t^2} = 2c^2 \frac{\cos(kh) - 1}{h^2}.$$

Given that the CFL number C is defined as

$$C = \frac{c\Delta t}{h}$$

and given $C = \frac{1}{\sqrt{2}}$, one can express the time step as

$$\frac{c\Delta t}{h} = \frac{1}{\sqrt{2}} \implies \Delta t = \frac{h}{c\sqrt{2}}.$$

Now substitute this expression for Δt into the numerical dispersion relation

$$\frac{\cos\left(\tilde{\omega}\frac{h}{c\sqrt{2}}\right) - 1}{\left(\frac{h}{c\sqrt{2}}\right)^2} = 2c^2 \frac{\cos(kh) - 1}{h^2}.$$

Simplifying both sides

$$\frac{\cos\left(\tilde{\omega}\frac{h}{c\sqrt{2}}\right) - 1}{\frac{h^2}{2c^2}} = 2c^2 \frac{\cos(kh) - 1}{h^2}.$$

Multiplying both sides by $\frac{h^2}{2c^2}$ gives

$$\cos\left(\tilde{\omega}\frac{h}{c\sqrt{2}}\right) - 1 = \cos(kh) - 1.$$

Thus

$$\cos\left(\tilde{\omega}\frac{h}{c\sqrt{2}}\right) = \cos(kh).$$

Since $\omega = \sqrt{2}ck$, one can substitute $kh = \omega\frac{h}{\sqrt{2}c}$, leading to

$$\tilde{\omega}\frac{h}{c\sqrt{2}} = kh = \omega\frac{h}{\sqrt{2}c}.$$

Finally, dividing both sides by $\frac{h}{\sqrt{2}c}$, one finds

$$\tilde{\omega} = \omega.$$