Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method.

Chapter 1

The underlying models

1.1 Brownian Motion

Before going further a gentle reminder of what is a brownian motion and how it is computed would probably be appreciate especially since it will become a master piece in subsequent computation.

A brownian motion is the continuous in time and in variable version of a scaled symmetric random walk. Basicaly At any time $t+1$ a random walk can take two variables contingent on an experiment realized at t, such as throwing a two fair coins. The idea that the coin is fair is crutial in the determination of the next step value because of the symmetry of the random walk.

Moreover the symmetry underly another key aspect inherited for random walk to brownian motion, which is the martingale property. A martingale characterize a specific motion involving in value through time with a constant drift rate of zero. It means that the expectation of such of process always is its initial value.

I howerver no waste much time in the precise mathematical definition of what a random is. Instead, modeling the browian motion seems to be crutial.

First of all the brownian motion has as initial value zero. According to what has been stated before about the martingale property of a brownian motion, one should expect to get zero in average by entering in a brownian motion process.

Secondly their increments are normally distributed in the value they could take from a previous time and they also are normally distributed beteen them if nonoverlaped.

$$\Delta W(t) \sim N\left(0, \Delta t\right) \tag{1.1}$$

As shown in (1.1) the more the delta time, the bigger the variance of the brownian motion. It is easily explained by the idea that forcasting more distant random data, following a time process, brings more uncertainty.

The concept of time process is key for a Wiener process and is closely attached to that of filtration. A filtration is a collection of σ algebra $\mathcal{F}(\sqcup)$ indexed by time.

At any time t a brownian motion is said to be $(F(t))$–measurable and independent of futur increment as well. It means that all informations accessible at time t is sufficent to determine the value taken by the brownian motion at that time – it is resolved by the information at t. However at this precise time t there is no way to predict with one hundred percent of certainty the futur value taken by the motion of the Wiener process.

These constrains meet what is call the weak form of market efficency and will be a necessary feature for the modeling of stock price thicker, but I digress.

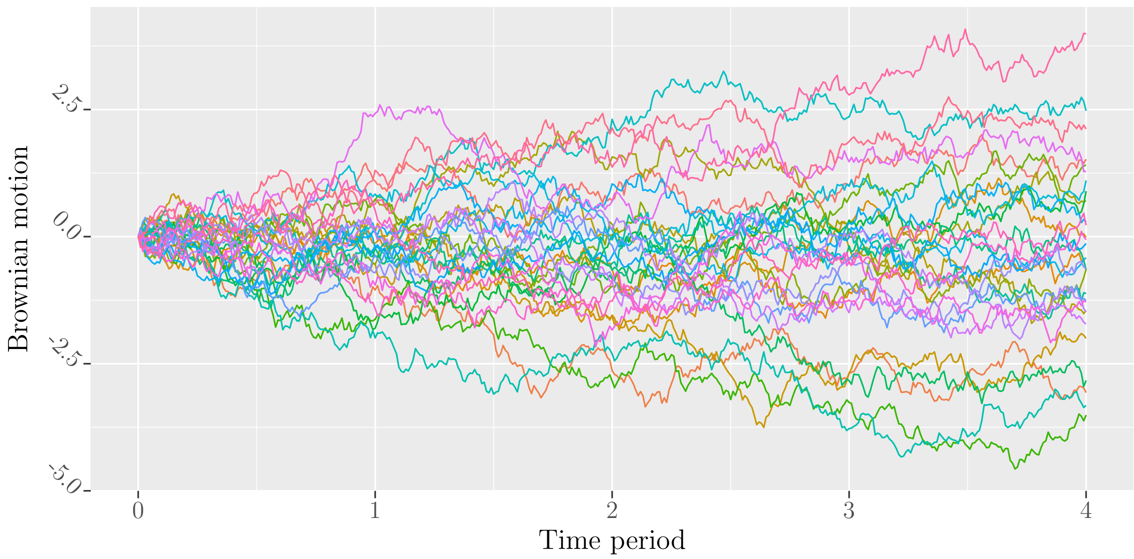

Figure 1.1: Multiple paths of a Brownian Motion

1.2 A log-normally distributed one

1.2.1 Overview

The random model described in this section is used to highlight a behavior that a continuously, in time and in value, stock price motion follows, provided that the distribution of the process used to model the walk of the price is log-normally distributed.

Even though, as it were previously mentioned, the stock model is defined in a way that not only its independent variable (time component) is continuous but is also its depent variable (stock motion component), it is however convenient in practice to not consider the move between two random value $dS(t)$ as fully continuous, provided that these moves are normally distributed. The reason is only matter of computation. Because a component of the stock price motion function is random, i.e. a Brownian Motion, the value it would take in a subsequent time is by definition of its randomness unpredictable. Needless to say that less the jump between two values of $S(t)$ is and less this approximation against the theoretical model brings inacuracy. It is why the following subsection introduce first the stock price motion model with continuity in its delta value and the next one features the same model exept that the move between the value the stock take is discrete.

$$S(t) = S(0)\, e^{\sigma W(t) + \left(\alpha - \frac{1}{2\sigma^2}\right)t} \tag{1.2}$$

For both continuous or discrete value process, the equation (1.2) can be use to model it. It is only when one want to deal with delta that the equations discribing the continuous or the descrite value process differs.

In (1.2) distinction are made in a sense to separate what variables are not random from the one that is random. Stricly speaking and because the model deals with process trhough time, a continuous random variable that evolves while time passes is called stochastic. The only stochastic variable involved in the model is the Brownian Motion , $W(t)$. Roughly a Brownian Motion , synonym with Wiener process (which is a standard normally distributed Markov process), is a stochastic process that evolves through time with independent increments normally distributed – if not overlapped. Because this variable in the only one that is stochastic in the model, is is the one that bring uncertainty to the stock price motion described following (1.2) Finally the two other ingredients cannot be called variables but rater contant, because they keep the same value as time passes. They are σ and μ, respectively for the stock's standard deviation and expectation.

1.2.2 Continuous through time

As previously stated, the delta – which mean the evolution between two specified given time – of a stock price motion that is continuous in the value it could take between two time increments are here considered.

The equation (1.2) could be derived using Itô's formula (1.3), in order to get a differential based equation. The provided differential formula will be next used to find out the first and second moments of the distribution underpinned by the stock price random process.

$$df\left(t, W\left(t\right)\right) = \left[\frac{\partial f\left(t, W\left(t\right)\right)}{\partial t} + \frac{1}{2}\frac{\partial^2 f\left(t, W\left(t\right)\right)}{\partial x^2}\right] dt + \frac{\partial f\left(t, W\left(t\right)\right)}{\partial x} dW\left(t\right) \tag{1.3}$$

The term $df\left(t, W\left(t\right)\right)$ in (1.3) represents any changes in the value of a function $f\left(t, W\left(t\right)\right)$ occuring over a infinitesimally move in time dt. The related function is any one involving time and a Brownian motion. For that matter in the present case highlighted in this section it is considered that $f\left(t, W\left(t\right)\right) = S\left(t\right)$. By applying the tranformation inccured by Itô, the two following equations (1.4a, 1.4b) emerge:

$$dS\left(t\right) = \alpha S\left(t\right) dt + \sigma S\left(t\right) dW\left(t\right) \tag{1.4a}$$

$$\frac{dS\left(t\right)}{S\left(t\right)} = \alpha dt + \sigma dW\left(t\right) \tag{1.4b}$$

Although (1.4a) imparts a first building block of the model to consider as a proxy for strock prive movement, the equation (1.4b) gives more clues on the distribution of the XXX RETURN (LOG)?. Because the increments of the Brownian Motion are are normally distributed with mean 0 and variance dt, the distribution of (1.4b) also be normally distributed with mean α and variance σ^2, as stated in (1.5).

$$\frac{dS\left(t\right)}{S\left(t\right)} \sim N(\mu dt, \sigma^2 t) \tag{1.5}$$

The Itô formula is so widely used and so well establish that it could be trustworthy applyed in any transformation of borel–measurable function $f\left(t, W\left(t\right)\right)$ implying time and Brownian Motion – even more broadly any stochastic process with a slightly arrangement of the Itô's formula previously introduced –.

1.2.3 Discrete through time

In the present subsection, the stochastic price motion process (1.2) come in a slightly different flavor and led to another representation than the one highlighted by equation 1.4a.

$$
\begin{aligned}
dS\left(t\right) &= \alpha S\left(t\right)\Delta t + \sigma S\left(t\right)dW\left(t\right) && \text{(1.6a)}\\
\frac{\Delta S\left(t\right)}{S\left(t\right)} &= \alpha \Delta t + \sigma \delta W\left(t\right) && \text{(1.6b)}
\end{aligned}
$$

The unique difference between respectively (1.4a) – (1.6a) and (1.4b) – (1.6b) is the frequence at which data can be recorded. In the realm of real world, the model described by (1.6a) fairly matches the data pickup requirements in the sense that only discrete measurement can be performed from the bunch of market available data. Moreover as particaly than theoreticaly the process (1.6a) could be as precise as one would by moving the delta time cursor down ($\Delta t \downarrow 0$). A somehow key warning can therefore be raised in the accuracy of such a model that is indeed closely related to the choice of Δt in conjucture with the stock variance(SURE THAT IS THE STOCK VARIANCE OR THE STOCK PRICE EVOLUTION VARIANCE ?).

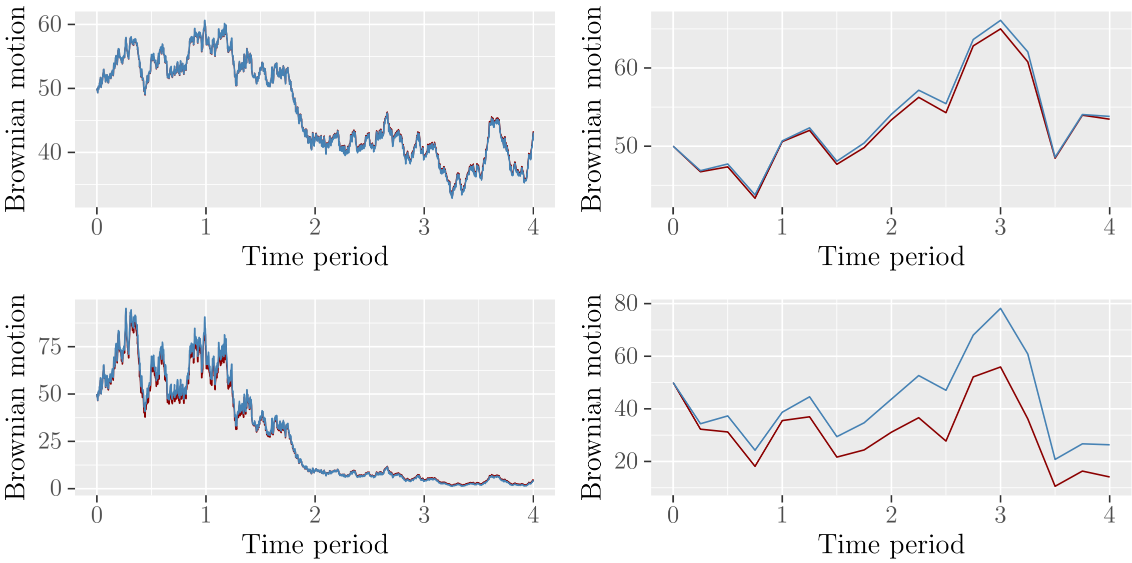

Figure 1.2: Sample output from tikzDevice

The graphs (REFERENCE) makes a point of showing how precise is the estimation made by using the approximation (1.4a) against (1.2) as dt decrease.

Four different simulations of stock price motion are represented throughout charts (REF). They vary by two means, horizontally the stock volatility change whilst the time period between two measures remain identical. While vertically, the volatilities are the same and only the time steps are differents (daily basis for top charts and quarterly measurement for bottom ones). The blue line are the one ones for which computation has been made using (1.2) whereas those with a pink dotted representation have been constructed over the Itô's approximation (1.4a). On one hand, the first – which show the most accurate approximation, more reliable – is constructed with a standard deviation of 0.4 and a duration of one day between two measures. On the other hand, the worst scenario of getting reliable measure by using (1.4a) is represented by the graph in the odd pane, for which the measures are set quarterly and the volatility.

As previously stated $\frac{\Delta S(t)}{S(t)}$ is normally distributed with $\mathbb{E}\,\frac{\Delta S(t)}{S(t)} = \alpha\Delta t$ and $var\frac{\Delta S(t)}{S(t)} = \sigma^2\Delta t$. The figure ([REFERENCE]) shows that the normality expectation holds. The solid line exibits the normal distribution bell curve while the bins represents the samples' distribution of 300 stock processes equally splited into the same Δt with parameters $\alpha = 0$, $\sigma = 40\%$.

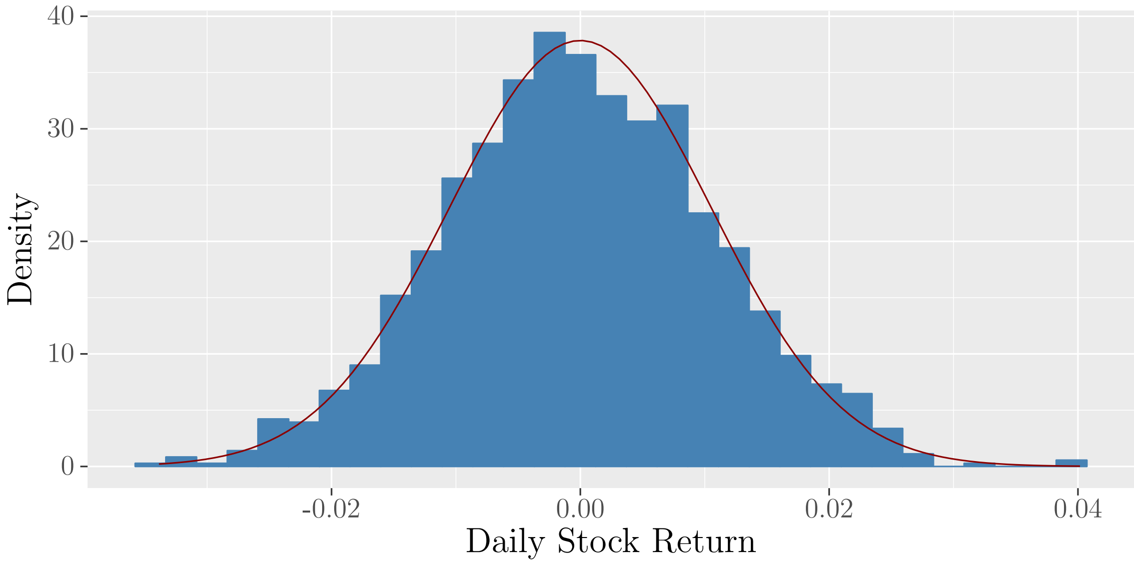

Figure 1.3: Daily Basis Stock Return Density

1.2.4 Distribution of the stock price process

According to Itô's approximation and by setting $f(t, W(t))$ to be $\ln S(t)$ it is showed that the natural logarith of $S(t)$ is normaly distributed with mean $\left(\alpha - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2\Delta t$. Because every process with normally distributed logarithm are log–normally distributed, and following the relationship between the two laws, $S(t)$ is consequently log–normally distributed with mean () and var ().

MEAN VAR

+ FIGURE

1.2.5 Flaws

1.3 Volatility issues

1.4 Jump issues

Chapter 2

The Black–Scholes–Merton model

2.1 OverView

The Black–Schole–Merton equation is meant to provide the non arbitrage price of any kind of derivative assets. In the present document the model derived by the Black–Scholes–Merton formula is focus on the pricing of one of a kind trivial derivative, the call vanilla option.

In order to price this derivative throughout the model highlighted in this present chapter, some contraints are set. The section 2.2 exibits the assumptions made to keep the model between boundaries. In latter chapter however it will be shown how the model is relevant by going on the edge on that constraints and even beyond.

2.2 Assumptions

As previously stated, the Black–Scholes–Merton is a framework to define the non arbitrage price for call option. As a model it is somehow far away from reality in a bunch of way. The list below itemize all the contraint attached to the model.

- The stock price process involving in the computation of Black–Scholes–Merton is lognormally distributed with mean and variance as described in 1.2.4
- Short selling is not forbidden as hands-on market opperations
- The price for bid and ask quote are identical. It means that there is no bid–ask spread to be considerered
- The underlying does not provide dividend at any time
- No arbitrage opportunity are implied
- Are constant, the risk-free intererst rate (r), the stock volatility (σ) and, even though no considered in the Black–Scholes–Merton solution for the purpose of vanilla option pricing, also is the stock rate of return (α)

[TODO: Send reader to METHODOLOGY chapter where is explain how the analysis will go away from these constraints]

2.3 The Black–Sholes–Merton equation

The Black–Scholes–Merton model provides a backward parabolic differential equation (2.1) with terminal condition () and boundary contirion () , () with the purpose of finding out the fair price of an option provided that assumptions (2.2) are respected.

Because the dummy variable x and t are dummies, the equation (2.1) involved no uncertainty. As it is shown through chapter (GREEKS REF), the uncertainty underlying to the model is removed thank to the delta hedging rule.

$$rc\left(t,x\right)=\frac{\partial c\left(t,x\right)}{\partial t}+rx\frac{\partial c\left(t,x\right)}{\partial x}+\frac{1}{2}\sigma^{2}x^{2}\frac{\partial^{2}c\left(t,x\right)}{\partial x^{2}} \tag{2.1}$$

The terminal condition (2.2) is intended to support that the value of a call option at maturity equals its payoff.

$$c\left(T,x\right)=\left(x-K\right)^{+} \tag{2.2}$$

Whilst the terminal condition focuses on the value taken at maturity, the bondaries condition fix some constraints on the extreme values likely to be taken by the shares of stock at any time during the optio like. In that regard, the boundary condition (2.3) shows that an option with an worthless underlying is itself valueless while whenever the option is deep in the money, simulated with $x=\infty$ (2.4), the value of the derivative is equal to the value of a forward contract involving the same underlying and with the same maturity date.

$$c\left(t,0\right)=0 \tag{2.3}$$

$$\lim_{x\to\infty}\left[c\left(t,x\right)-\left(x-e^{-r(T-t)}\right)\right]=0 \tag{2.4}$$

2.4 Solution for vanilla option pricing method

Chapter 3

The Greeks

3.1 The purpose of the Greeks

3.2 Delta

3.3 Theta

3.4 Gamma

3.5 Vega

3.6 Rho

3.7 An hedging strategy involving the Greeks and Black–Scholes–Merton

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