Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How perfomant is it ? What about other model such as . . . ?

Using R. R Core Team [2017]

Chapter 1

Analysis

The objective of the current chapter is to measure the hedging performance of the Black-Scholes model when the stock price evolves in a non-log-normal world. The studied models allowing the course of a time-series to go out of the Black-Scholes frame are the Merton jump-diffusion (MJD) and Heston stochastic volatility(HSV) models. In order to give realistic results, they have been adjusted in **??**. For both models, two kinds of parameters have been found, namely, the risk-neutral and risk-averse ones. Therefore, the "no risk" parameters are used to price option, while the risky serve to generate time-series thanks to those models.

The chosen metric to compare the models is the "relative profit and loss" (P&L). Latter is explained in **??**.

The analysis is completed in two steps. The first concerns exclusively the BSM model with all the so related constraints respected. Even though the computation of the options' prices in such an environment does not respect those emerging from reality, the measure of the P&L so computed serve as a benchmark for the other models. The next and last part of the analysis measures the P&L for both models MJD and HSV. Those P&L are computed using the delta-hedging strategy. On one hand by using the delta of Black and Scholes and on the other, by using the appropriate delta of the considered model. Although the Black and Scholes delta is purposely computed to be applied when the underlying asset is driven by a geometric Brownian motion (GBM), the goal is to assess how one can go wrong by using it within the real conditions.

At last, some particular results are commented.

1.1 Measuring performance in a log-normal world

Table 1.1 presents the results of the relative P&Ls got concerning the process of delta-hedging on options with maturities of 3, 6 and 13 months and strikes ranging from 140 to 230. Those outputs are maturities column-wise and grouped by strikes in rows. Furthermore, each row is split into three parts, each of these subsections represents different rebalancing frequency. For instance, the result exhibited in the column "91 dbm" and row "140 > intraday" give the mean relative P&L computed on a series of European call option delta-hedging with a maturity of 91 days (3 months), a strike of 140 and a rebalancing done twice a day.

The samples time-series of the underlying asset are shown in figure 1.1. It means that any path of that series helped forty-five times the analysis because each of it has been involved in the hedge of options with five different strikes having three maturities each,

along with three distinctive rebalancing frequencies. The total number of samples was one hundred, and consequently, Table 1.1 summarizes four thousand and five hundred delta-hedging strategies.

Figure 1.1: Sample geometric Brownian motions
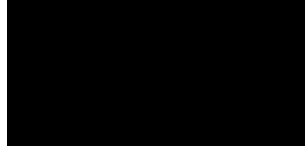
Notes.

From table 1.1 one can observe that the balancing frequency has a positive impact on the quality of the hedge. Indeed, the more frequent balancing, the better hedging. Figure 1.2 shows an extract of the delta-hedging processes with different balancing frequencies. The continuous brown line represents the course of an option price and the green, blue and red dots respectively exhibits the associated delta-neutral portfolios with a rebalancing frequency of twice a day, once a day and once a week.

Figure 1.2: Delta-neutral portfolio with different frequency of balancing
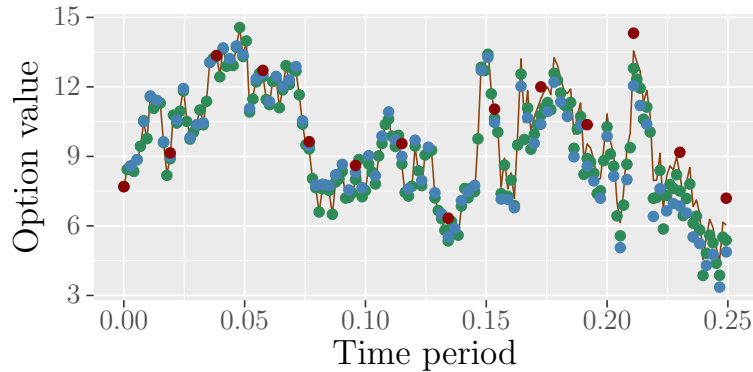
Notes.

In the light of the above figure 1.2, and as confirmed by the table 1.1, a portfolio more regularly balanced will give a better result in the BSM delta-hedging strategy, within a log-normal world.

The vast majority of these results shows that with a less rebalancing frequency, the average value of P&Ls is negative, meaning that the delta-neutral portfolio underperform with respect to the option itself. The reason is due to the option's gamma. As a reminder, gamma gives the acceleration of the call function with respect to the stock price. With a positive gamma, and gamma is always positive for a vanilla stock option within the BSM model, the function $c(S(t), t)$, with t constant, is concave up. As Shreve [2004] shows, the delta-neutral portfolio is tangent below the curve of the call function and due to the convexity of the latter, meaning that an instantaneous change of the asset price, either by increasing or decreasing, always make the related delta-neutral portfolio suffer. That kind of portfolio is called short gamma.

On other note, the average worst result comes from the deep-out-in-the-money options weekly rebalanced. However, by opting for a more frequent rebalancing strategy, the mean

Table 1.1: Hedging with BSM: Relative P&L

		91 dbm[a]	182 dbm	399 dbm
140	intraday	0	0	0
	daily	0	0	0
	weekly	0	0	0
160	intraday	0	0	0
	daily	0	0	0
	weekly	-0.001	-0.001	-0.003
186	intraday	0	0.001	-0.001
	daily	-0.005	-0.002	-0.004
	weekly	-0.01	-0.019	-0.021
200	intraday	0.022	0.008	-0.001
	daily	-0.002	-0.005	-0.006
	weekly	-0.007	-0.052	-0.037
230	intraday	0.02	0.042	-0.007
	daily	0.022	-0.063	-0.022
	weekly	0.317	-0.285	-0.136

[a]dbm: days before maturity

relative P&L reduces. Indeed, it goes from 31.7% to 2% by choosing a rhythm of twice a day instead of once a week. Figure 1.3 shows the distribution of the above-mentioned P&Ls with a portfolio balancing occurring either daily (blue curve), twice (green curve) a day or once a week (red curve). As expected, the more frequently rebalanced portfolios show less variance and more value near zero for the associated P&Ls.

Figure 1.3: Sample geometric Brownian motions
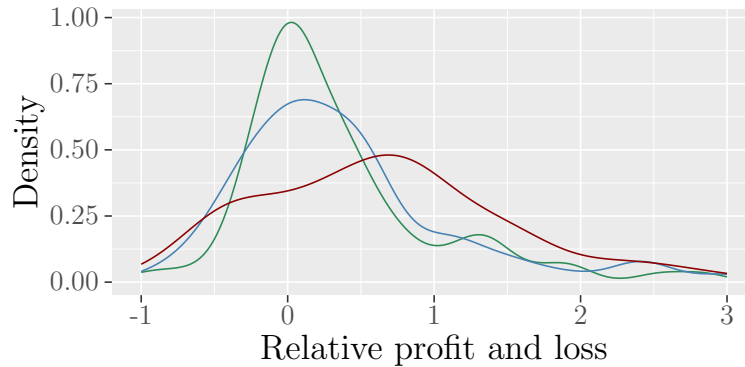
Notes.

The more disruptive observation is given by the average P&L of the deep-out-of-the-money, weekly rebalanced options with 91 days before maturity, namely, 0.317. Unlike the other P&Ls, this one is positive, meaning that the delta-neutral portfolio that replicates a position in a long European call won on average. The underpinned reason is explained by a high value of option's theta i.e., the first derivative of the call function with respect to time, all other things staying unchanged. Table 1.3 lists all the P&Ls of that hedging scenario.

Table 1.2: Worst relative P&L for BSM

	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
[0]	1.01	8.52	-0.52	0.67	-15.42	0.55	3.62	0.35	2.45	2.15
[10]	2.55	0.85	1.61	0.06	-1.68	-4.93	3.16	0.61	0.27	0.97
[20]	-9.32	0.38	3.34	-0.43	-9.27	0.60	-0.37	-0.27	0.27	-0.57
[30]	1.54	0.83	17.43	1.04	-37.44	1.09	1.48	1.73	0.92	0.77
[40]	1.17	0.29	-2.33	0.45	-0.51	8.98	-3.76	-0.55	0.83	0.69
[50]	-8.85	0.18	-0.96	0.92	4.38	0.29	2.95	4.55	5.29	23.50
[60]	-0.11	4.71	3.83	-0.26	3.70	-2.95	1.28	7.00	0.80	0.15
[70]	-0.21	11.00	0.54	0.23	0.71	-38.71	15.04	7.75	-0.49	0.85
[80]	-15.18	1.62	5.40	-9.23	-0.43	1.39	0.65	1.31	-7.34	1.53
[90]	-0.02	3.19	-1.68	13.05	-0.12	-6.70	-0.34	8.68	0.69	2.18

All the value of theta for the option with the biggest P&L (the one highlighted in table 1.3) are listed in **??**.

Table 1.3: Worst relative P&L for BSM

Time	0.00	0.02	0.04	0.06	0.08	0.10	0.12	...
Theta	-1.83	-3.80	-8.50	-5.26	-13.88	-13.84	-20.39	...
Time	...	0.13	0.15	0.17	0.19	0.21	0.23	0.25
Theta	...	-14.89	-19.05	-19.84	-35.54	-47.81	-65.66	Inf

The biggest value of theta appears at the end of the life of the derivative. The reason, as shown through FIGURE X, is due to the fact that at the time near maturity, the values of the option's theta is computed when the option is just in-the-money, letting by consequence the time with a big influence on its value.

By resuming the solution of theta, given in Shreve [2004], for the European call option, one gets the following equation.

$$\Theta = -rKe^{-r(T-t)}N\left(d_-(T-t, S(t))\right) - \frac{\sigma S(t)}{2\sqrt{T-t}}N'\left(d_+(T-t, S(t))\right) \tag{1.1}$$

with

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right] \tag{1.2}$$

With N and N^- respectively standing for the normal cumulative distribution function (CDF) and for the normal probability density function (PDF).

Intuitively, according to equation (1.1) As $\tau \downarrow \implies \sigma S(t)/(2\sqrt{T-t}) \uparrow$ and $N'(d_+(T-t, S(t)))$ gets its higher value when $d_+(T-t, S(t)) \to 0$. In addition, in accordance with equation (1.2), for any given τ sufficiently small, the function d_+ is the nearest to zero as $x \to S(t)$. The **??** shows that it is exactly what happens for the observed option.

1.2   Merton jump-diffusion performance measuring

Table 1.4 is arranged the same way than table 1.1 but except that the columns are subdivided to give the information on the relative P&L obtained by either hedging with

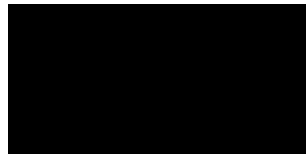Figure 1.4: European call option with higher theta as time goes to maturity
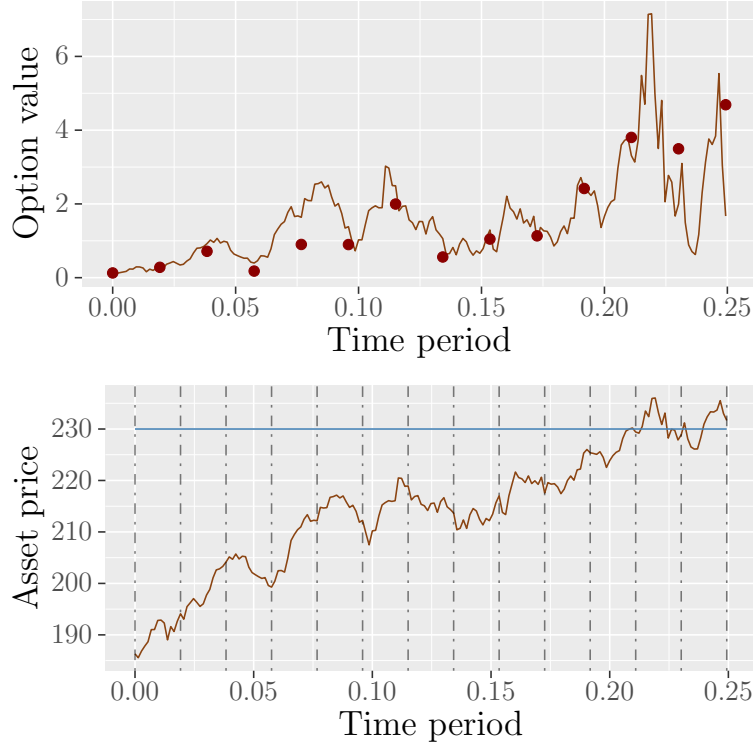
the delta MJD or BSM. For instance, the result exhibited in the column "91 dbm > $delta_{bsm}$" and row "140 > intraday" gives the mean relative P&L computed on a series of delta-hedging on European call options with a maturity of 91 days (3 months), a strike of 140, a rebalancing done twice a day and with $\Delta(t)$ computed using the BSM equation, while the output in the column "91 dbm > $delta_{mjd}$" and row "140 > intraday", give the same information but with $\Delta(t)$ computed by using the Merton equation.

The paths of the samples time-series of the underlying asset are all displayed in figure 1.10. Any of them was used ninety times in the analysis because all have been involved in the hedge of options with five different strikes having three maturities each, along with three distinctive rebalancing frequencies and depending on two deltas as well. The total number of samples was one hundred, and consequently, Table 1.4 summarizes nine thousand delta-hedging strategies.

Figure 1.5: Sample geometric Brownian motions

From table 1.1, the effect of the rebalancing frequency is not as clear as it was with the hedging of the BSM model in the log-normal world. At first sight, however, one can observe that with the increase of the balancing rhythm comes a rise in the value of the

delta-neutral portfolio, on average, which even tends to systematically outperform the European call. Nonetheless, the information given by table 1.1 is only the mean of all the computed P&Ls and therefore lacks in exhaustiveness.

Table 1.4: Hedging with MJD: Relative P&L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}
	intraday	0.004	0.006	0.011	0.012	0.01	0.021
140	daily	0.002	0.006	0.008	0.012	0.016	0.021
	weekly	0.004	0.006	0.006	0.011	0.007	0.021
	intraday	0.011	0.018	0.021	0.029	0.025	0.042
160	daily	0.016	0.018	0.022	0.029	0.019	0.042
	weekly	0.013	0.016	0.018	0.026	0.018	0.04
	intraday	0.036	0.021	0.078	0.055	0.079	0.074
186	daily	0.039	0.022	0.072	0.055	0.068	0.074
	weekly	0.014	-0.008	0.055	0.037	0.057	0.061
	intraday	0.072	-0.002	0.139	0.061	0.13	0.086
200	daily	0.06	-0.013	0.131	0.057	0.115	0.085
	weekly	-0.02	-0.1	0.083	0.005	0.085	0.053
	intraday	0.955	0.331	0.444	-0.061	0.301	0.063
230	daily	1.098	0.466	0.409	-0.091	0.261	0.054
	weekly	-0.741	-1.335	0.085	-0.438	0.174	-0.088

In order to get more comprehension of that result, let us focus on the distribution of the P&Ls. Latters are displayed in figure 1.6 where the green, blue and red-filled densities respectively denote the P&Ls' distributions when the delta-neutral portfolios are balanced twice a day, daily or weekly.

In the light of figure 1.6, one can observe that as the rhythm of balancing increased, the variance of the averaged relative profits and losses decrease with the most of the distribution so concentrated around the mean.

Likewise, the time to maturity and the strike price both affect the distributions of the P&Ls.

On the one hand, concerning the effect of the maturity on the shape of the distributions, the observations are split into two categories. The first concentrates the computed measures when $0 < K/S(t) < 1$ at time zero, that is, all the [deep]-in-the-money options, while the second gathers all the others, namely, [deep]-out-of|at-the-money. To observe how such results affect the distribution, figure 1.6 is zoomed-in for the interesting P&Ls' density curves of the first considered category, so giving figure 1.7.

From figure 1.7, it can be observed that as the time to maturity increase for options with a strike below the underlying asset price at original date, the volatility of the hedging portfolios performance also increase. Whereas, according to figure 1.6, the inverse behavior is observed for the volatility of the performance of options with the higher strike price, i.e., the further the maturity, le lower the uncertainty about the associated P&Ls.

On the other hand, as shown by figures 1.6 and 1.7 and table 1.4, the performances of poorly rebalanced portfolios are better for options the more in-the-money. Such a behavior is explain by a weaker gamma for the deep-in-the-money options. Indeed, as time passes, the prices of the underlying assets tend to get more valuable in the vast

Figure 1.6: Sample geometric Brownian motions
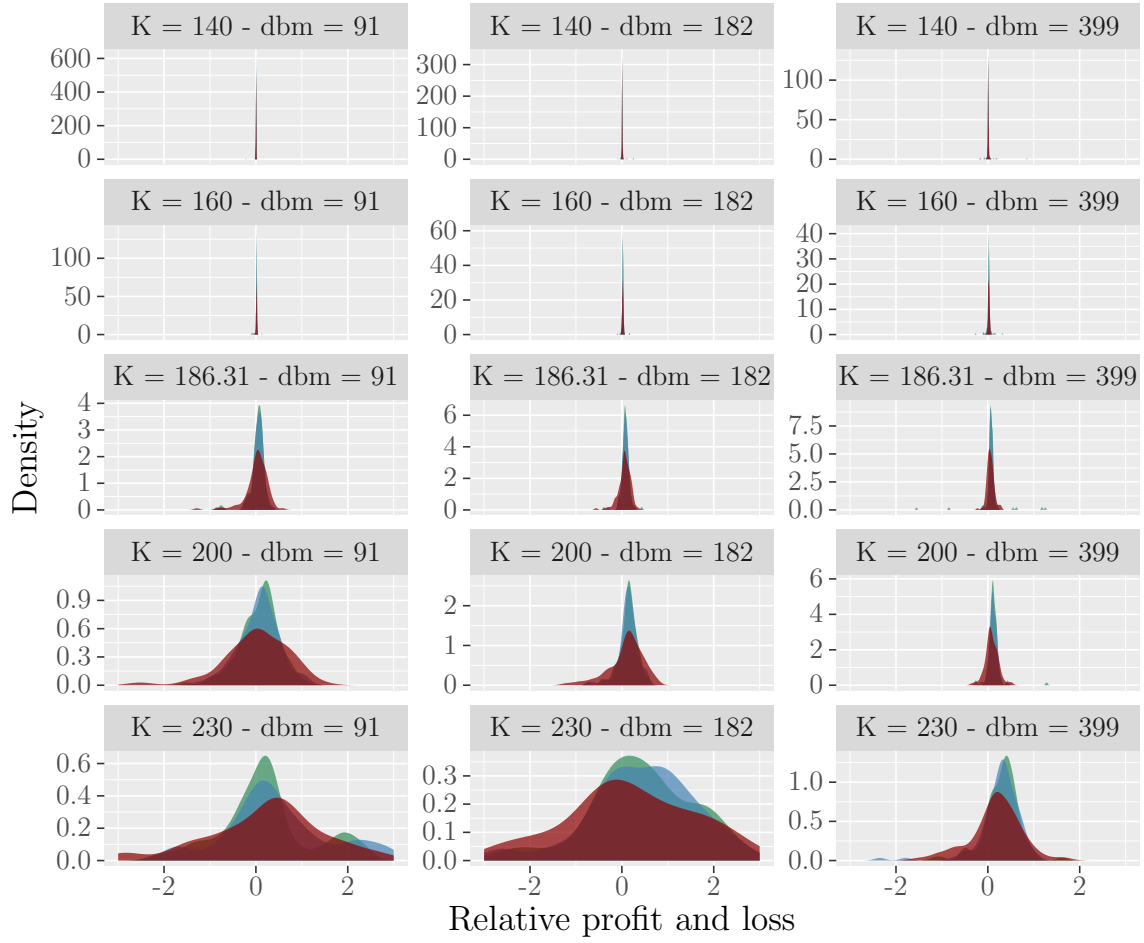
Notes.

majority, as illustrated by figure 1.10. Therefore, the associated options with smaller strikes continue to gain value but at a lower rate because the incidences of the stock price changes become less impacting. The inverse reasoning is to be applied to deep-out-of-the-money options. Effectively, the initial price of such options is near zero, so are the associated deltas. However, with high growth time-series as described in table 1.4, the collateral effect is that they rapidly affect the option price, just as the values of the delta are influenced. Consequently, according to and for the measured options, gamma is lower for in-the-money and higher for out-of-the-money options.

figure 1.8 shows the distribution of the P&Ls when the hedging is built with BSM or MJD deltas. The red densities concern the data related to BSM deltas while those in green represent the distributions of the performance metrics for the MJD delta-hedging.

According to the results of table 1.4 and figure 1.8, hedging in the real life with delta BSM seems not so bad with respect to the gotten measures. Let us dive more into deeply the delta-hedging by observing what happens behind the hood.

Figure 1.9 illustrates how the hedgings work depending on the strike prices and the deltas computed. The small circles represent the replicated portfolios at each rebalancing time, whereas the underpinned black lines are the course of the options' values as time passes. The blue and red circles respectively stand for the MJD and BSM delta-hedging

Figure 1.7: Sample geometric Brownian motions
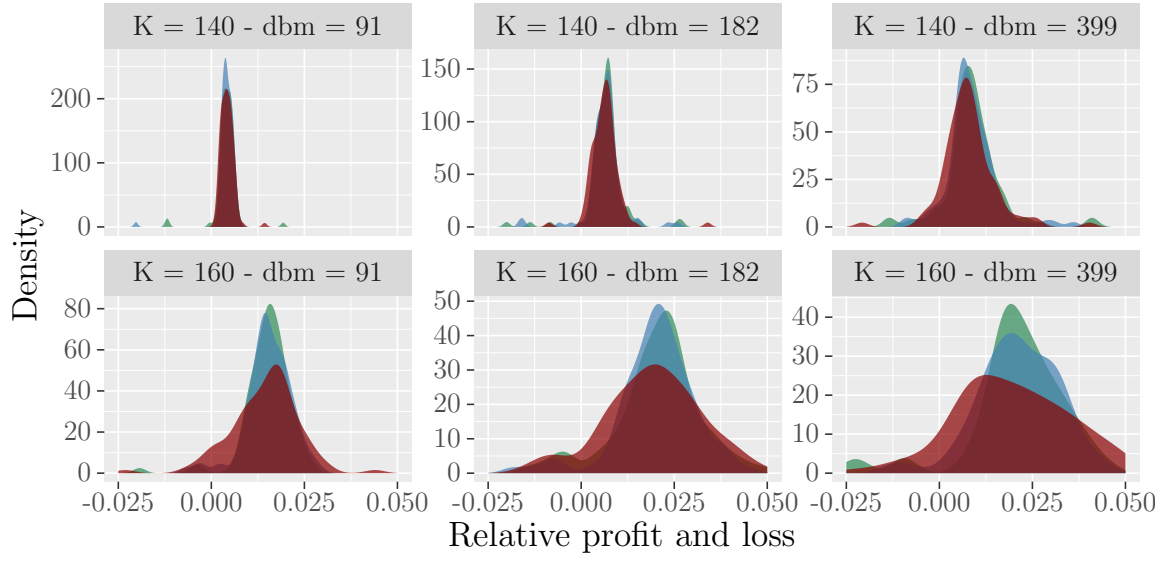
Notes.

portfolios' values.

1.3 Heston stochastic volatility performance measuring

In Table 1.5, the inforamtion are completely organized the same way that in table 1.5.

Figure 1.10 illustrates all the dummy time-series that will be considered for the analysis of delta-hedging using HSV as a stochastic process for the underlying asset.
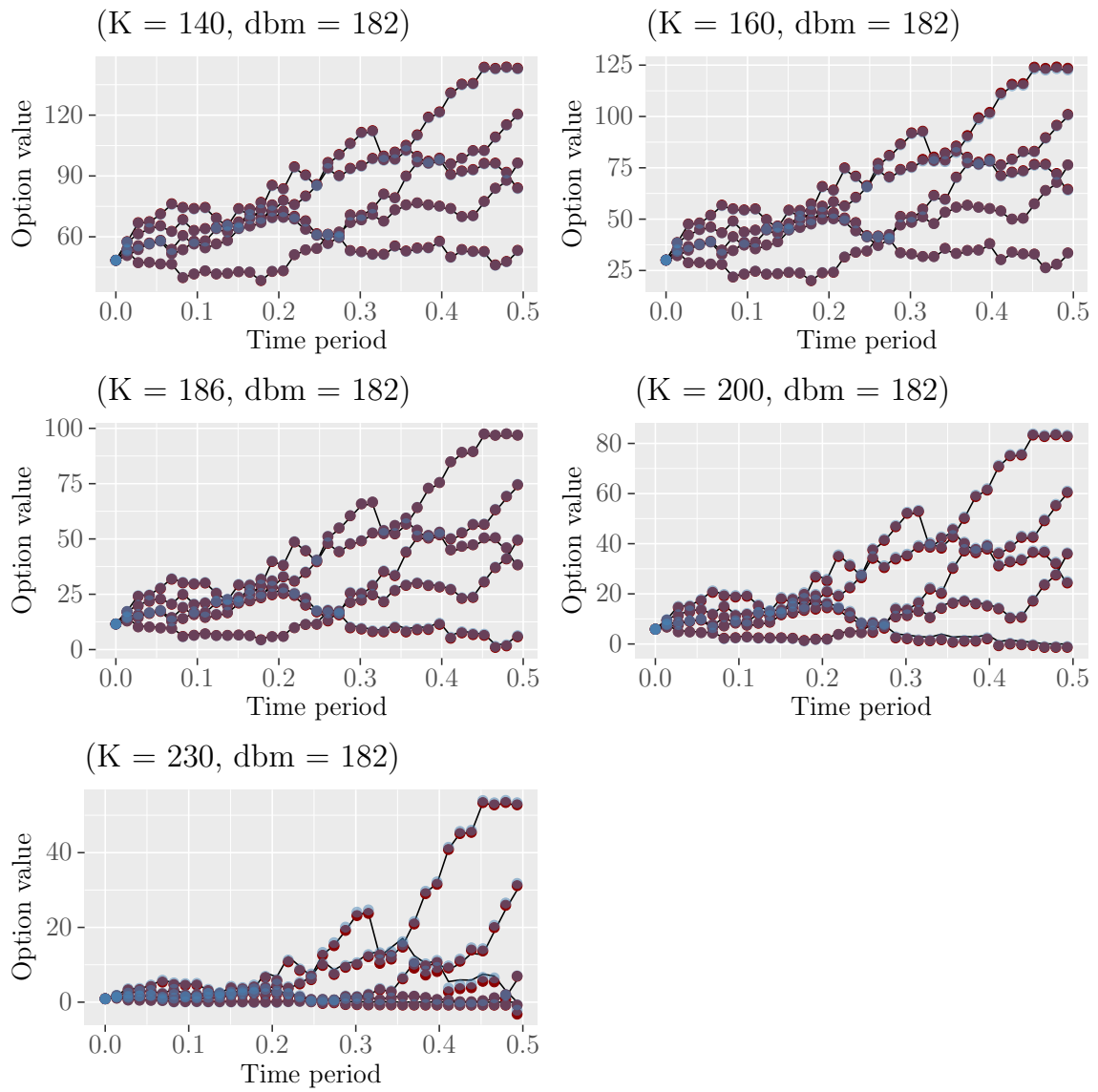
Figure 1.8: Sample geometric Brownian motions
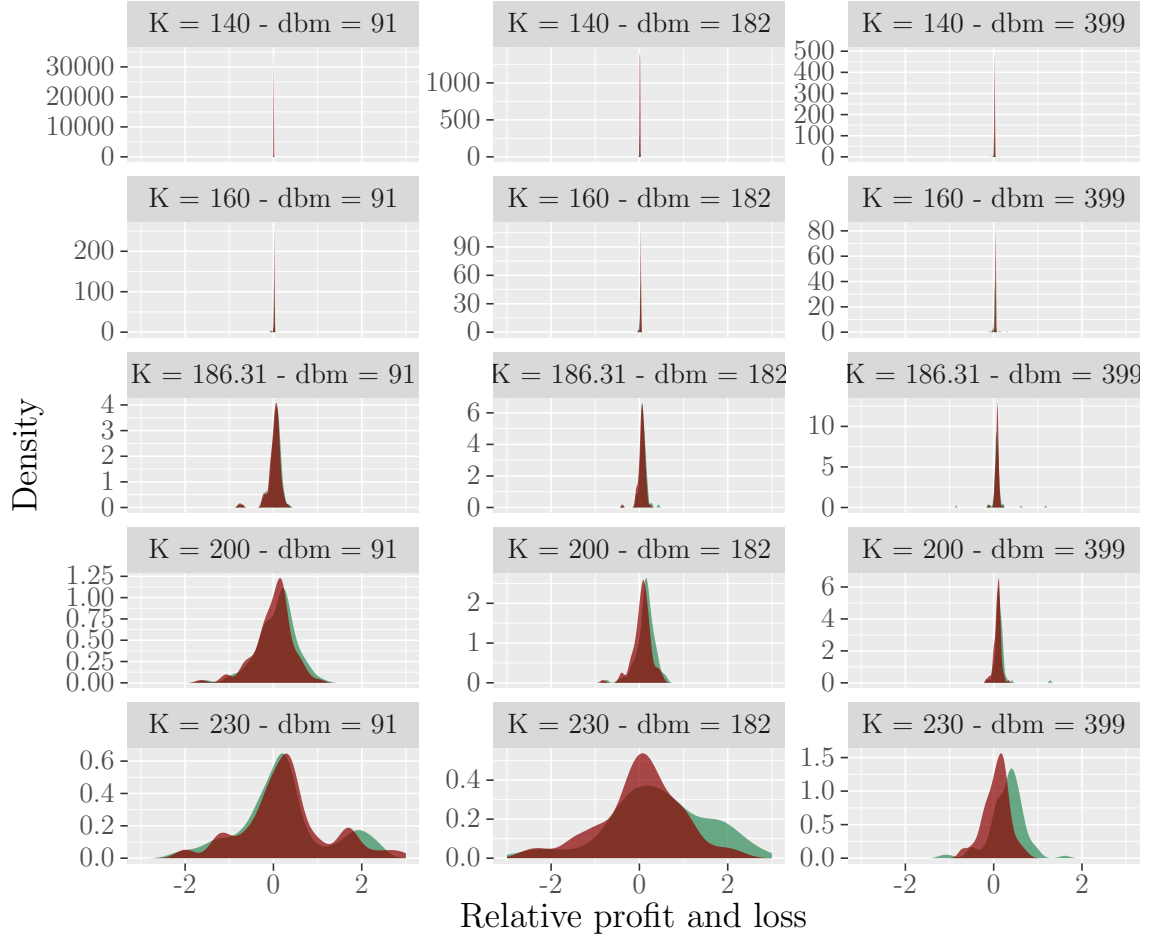
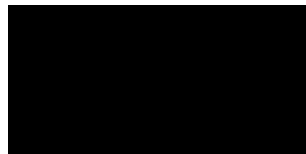**Notes.**

Table 1.5: Hedging with HSV: Relative P&L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}
	intraday	0	0.002	0.011	0.011	0.009	0.038
140	daily	-0.001	0.002	0.01	0.011	0.009	0.038
	weekly	0.001	0.002	0	0.011	0.008	0.038
	intraday	0.009	0.028	0.023	0.073	0.042	0.143
160	daily	0.008	0.028	0.025	0.072	0.036	0.143
	weekly	0.008	0.028	0.019	0.073	0.036	0.143
	intraday	0.158	0.252	0.159	0.392	0.153	0.524
186	daily	0.15	0.245	0.195	0.391	0.156	0.522
	weekly	0.117	0.241	0.158	0.378	0.139	0.519
	intraday	0.459	-0.298	0.43	0.146	0.279	0.546
200	daily	0.433	-0.361	0.42	0.126	0.255	0.544
	weekly	0.268	-0.659	0.369	0.005	0.246	0.498
	intraday	2.136	-0.527	1.884	-2.452	1.01	-0.235
230	daily	1.948	-1.197	1.893	-2.655	0.989	-0.224
	weekly	1.407	-2.152	1.547	-2.402	0.917	-0.353

Figure 1.9: Sample geometric Brownian motions

(K = 140, dbm = 182)

(K = 160, dbm = 182)

(K = 186, dbm = 182)

(K = 200, dbm = 182)

(K = 230, dbm = 182)

Notes.

Figure 1.10: Sample geometric Brownian motions

Notes.

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