Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How perfomant is it ? What about other model such as . . . ?

Using R. R Core Team [2017]

Chapter 1

Upstream concepts

The material covered in the current chapter is meant to be thereafter intensively used, either by being applied in the studied theoretical models or by entering in the construction of the theory-based algorithms.

1.1 Vanilla options

The so-called vanilla options, in opposition to the more complex – not covered in this master thesis – exotic ones, are specifics kind of derivatives coming along with some good-to-know jargons.

First and foremost, as defined in Hull and Basu [2012], an option is a contract between two stakeholders with different interests or at least distinctive motivations, who want to buy or sell a product, which generally is a financial asset called "the underlying." Indeed, someone can look for hedge oneself against risk while the other party wants to make a profit on a speculative move. Likewise other financial contracts, one agrees to buy and the other to sell at a fixed amount of money, namely "the strike price", usually denoted by k with the difference that they would effectively realize the purchase or the sale of the underlying at a future date than the one they enter into the bargain. That date is called "the maturity".

Another key characteristic of options is that they are not symmetric contracts, in the sense that both parties do not have the same rights, depending on their position along with the option's type. Two positions can be taken when entering into such a derivative, either long or short. Going short broadly means purchase the option whereas going long implies writing, or synonymously, selling the derivative. Beyond the position, the contract can be of call or put type.

The following sentences encompass a mix of type (call/put) and position (short/long) combination that gives an overview of all the possible scenarios:

- Someone who is going long into a call gains the right to purchase the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.

- Someone who is going short into a call is forced to sell the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.

- Someone who is going long into a put gains the right to sell the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.

- Someone who is going short into a put is forced to buy the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.

Moreover, vanilla options may be "European" or "American". Latter can be exercised at any time during the whole life of the contract while the European can only be at maturity.

Finally, the payoff provided by all aforementioned European options are summarized throughout equations (1.1) to (1.4), where C_l, C_s, P_l and P_s respectively stand for the long call, short call, long put and short put payoffs. $S(T)$ being the price of the underlying at maturity.

$$C_l = (S(T) - k)^+ \tag{1.1}$$

$$C_s = (k - S(T))^- \tag{1.2}$$

$$P_l = (k - S(T))^+ \tag{1.3}$$

$$P_s = (S(T) - k)^- \tag{1.4}$$

The derivative considered in this master thesis is mainly the European call option, even though the link with European put could somehow be done without being mandatory to exhibit the issue so raised. Therefore such reference to put should only be done if necessary.

1.2 Brownian Motion

All over the present document, the terms "Brownian motion", "Wiener" or "Markov" processes are considered to be equivalent terminology and therefore used as such. However strictly speaking, even though the Brownian motion is in every respect a Wiener process, stricto sensu, the term Markov process is broader. Actually, the more remarkable Markov property is to be a process with independent future increments, putting it in line with the weak form of market efficiency (Hull and Basu [2012]). Whilst, on the other hand, even if the Brownian motion, and thus the Wiener process, share this property, they are defined with the idiosyncrasy to have mean zero and a variance rate of one per unit of time (Hull and Basu [2012]).

Brownian motion is a centerpiece broadly used in subsequent developments. It is notably applied in many models such as in the geometric Brownian motion (chapter 2), in the Black-Scholes-Merton (BSM) equation (chapter 3) and in the Merton mixed jump-diffusion (MJD) and Heston stochastic volatility (HSV) model (**??**).

The Brownian motion is a stochastic process, denoted by $W(t)$ and satisfying $W(0) = 0$, with independent and identically distributed (iid) increments, such as defined by equation (1.5)

$$W(t_{i+1}) - W(t_i) \sim iidN\left(0, t_{i+1} - t_i\right) \tag{1.5}$$

That Markov process is qualified as being time and path dependent in its building blocks. Time dependency implies that like many other functions its value evolves over time. Whereas path dependency means that the Wiener process is also meant to randomly move between two time-steps. Practically, at any time t and from its origin, the Brownian motion may take any value as shown by equation (1.6).

$$W(t) \sim N(0,t) \tag{1.6}$$

As described by Shreve [2004], the construction of such a process could be achieved by following various techniques. Either by computing the differential process (equations (1.7) and (1.8)) for every time-steps and then combining them, by constructing a unique time series in that way; or by resolving the joint moment generating function of the iid random variables' vector $(W(t_i) \dots W(t_m))$, through equation (1.9).

$$dW(t) = \phi(0, t+\epsilon_i) \tag{1.7}$$

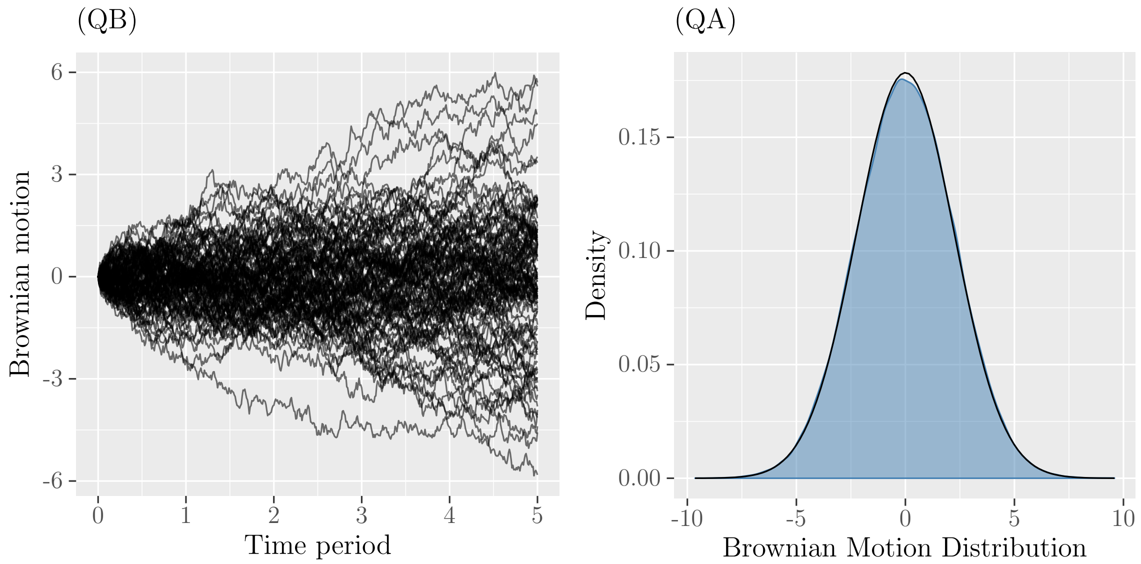
ϕ is a function that generate a random number according to the normal law with parameter $\mu = 0$ and $\sigma^2 = t + \epsilon_i$, with ϵ_i being any arbitrary small delta time step.

$$W(t+\epsilon_i) = W(t) + dW(t) \tag{1.8}$$

$$\varphi(u_1 \dots u_m) = \exp\left(\frac{1}{2}\sum_{i=1}^{m}\left[\sum_{j=i}^{m} u_j\right]^2 (t_i - t_{i-1})\right) \tag{1.9}$$

Figure 1.1 (QA) displays a simulation of one hundred Brownian motions over a time frame of five years. While in turn, figure 1.1 (QB) shows the distribution of the possible outcomes of that experiment, after five years. The black outlined curve is the density of the normal distribution, with mean and variance chose such as exposed by equation (1.5).

Figure 1.1: Multiple Brownian Motion



Note. (QA): Simulation of one hundred Brownian motions using the R package Tedde [2017] with a timeframe of five units (five years) with a time-step of one over one hundred. (QB): Density computed on a simulation of one hundred thousand Brownian motions. The calculation has been made by taking all the final value $W(5)$ and by plotting them as a density function. The Black bell curve is the normal density with mean zero and a standard deviation of square root of five, as expected (**??**).

Lastly, Shreve [2004] demonstrates that a Brownian motion has no tendency to rise or fall as time passes thanks to its martingale property. Therefore, because its initial value is zero, a Wiener process only brings noise by being incorporated into other stochastic series.

1.2.1 Correlated Brownian Motions

Correlated Brownian motions are Wiener processes that related together according to the factor ρ equation (1.10). Following Shreve [2004], these processes happens to be modeled by equations (1.11) and (1.12).

$$dB_1(t)dB_2(t) = \rho(t)dt \tag{1.10}$$

Where $dB_1(t)$ and $dB_2(t)$ are Wiener process

$$B_1(t) = W_1(t) \tag{1.11}$$

$$B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s) \tag{1.12}$$

When ρ is constant over time however, the simple form of equation (1.13) is enough.

$$B_2(t) = \rho W_1(s) + \sqrt{1-\rho^2}W_2(s) \tag{1.13}$$

1.3 Ito's lemma

Based on Taylor approximation theorem, Ito formulae are meant to provide the differential form of such a function $f \circ g$ where the function f can be differentiate with respect to its independent variable whilst g cannot.

$$f(T, W(T)) = f(0, W(0)) + \int_0^T \frac{\partial f(t, W(t))}{\partial t}dt + \int_0^T \frac{\partial f(t, W(t))}{\partial W(t)}dW(t) + \frac{1}{2}\int_0^T \frac{\partial^2 f(t, W(t))}{\partial W(t)^2}dt \tag{1.14}$$

Whilst the equation (1.14) shows the formula for Brownian motion, the **??** represents the form used to differentiate the more complex Ito process (1.15).

$$X(T) = X(0) + \int_0^T \Delta(u)dW(u) + \int_0^T \Theta(u)du \tag{1.15}$$

Where $\Delta(u)$ and $\Theta(u)$ are stochastic process, adapted to a filtration f(t)

$$f(T, X(T)) = \dots \tag{1.16}$$

Ito formulae it thereafter applied to derive such a process like the geometric Brownian motion [REF] or the Black–Scholes–Merton partial differential equation.

1.4 Cox–Ingersoll–Ross

The Cox–Ingersoll–Ross (CIR) stochastic model $R(t)$, defined by the differential equation (1.17)), can be used to simulate interest rates' evolution over time thanks to some of its related properties. Indeed it gives a good fit owing to two of them, namely the mean–reversion and its ability to only be positive (Shreve [2004]).

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t) \tag{1.17}$$

A mean–reverting stochastic process tends to navigate, such a pointy sinusoidal motion, around its mean (see [FIGURE]). The CIR inherits this behaviour from the construction of its differential's drift part, i.e. $(\alpha - \beta R(t))dt$. Indeed when $R(t) = {}^{\alpha}/_{\beta}$, then the drift term $dt = 0$ with the consequence of status quo. In addition, whether $R(t) > {}^{\alpha}/_{\beta}$ or $R(t) < {}^{\alpha}/_{\beta}$, the next value of $R(t+\epsilon) = R(t) + dR(t)$ is pushed back toward ${}^{\alpha}/_{\beta}$. Actually, as showing by equation (1.18), the long–run expected value for the process $R(t)$ is ${}^{\alpha}/_{\beta}$ (Shreve [2004]).

$$\lim_{t\to\infty} \mathbb{E}\, R(t) = \frac{\alpha}{\beta} \tag{1.18}$$

On the other hand, the non-negativity property is explained by the fact that if $R(t) \to 0$ then $dR(t) \simeq \alpha dt > 0$ making $R(t+\epsilon)$ bounced off the x axis, running it away from negative realm (Shreve [2004]).

Ultimately though, the CIR mean–reverting equation (1.17) is used by Heston [1993] in its model in order to drive stochastic interest rates. Heston's model is covered throughout ??.

1.5 Skewness and Kurtosis

1.5.1 Definition

The skewness and kurtosis for a random variable's distribution, are respectivaly the third and fourth moment characterizing either the asymetry and the degree of flatness.

The theoretical moments are defined by equation (1.19) while the empirical ones can be estimated using equation (1.20), with $r = 3$ to compute the skewness or $r = 4$ for the kurtosis.

$$\gamma_r = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^r\right] \tag{1.19}$$

where X is any random variable

$$m_r = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^r \tag{1.20}$$

with $\{x_i\}_{i\in n}$ being a set of outcomes, belonging to the sample set.

1.5.2 Estimation

Sampled skewness and kurtosis formulae exist in different fashion with their precisions depending on the size of the sample along with the skewed of the theoretical distribution to be estimated (Joanes and Gill [1998]).

The chosen method to use to estimate the skewness and kurtosis needs a prudent selection especially for small sized sample. Actually, in order to minimize the mean–squared error and the associated variance, the equations (1.21) and (1.25) are used as an reliable unbiased estimator for sample with normal shaped theoretical distribution.

$$b_{skewness} = \frac{m_3}{S^3} \tag{1.21}$$

$$b_{kurtosis} = \frac{m_4}{S^4} \tag{1.22}$$

where

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^2 \tag{1.23}$$

Conversely, equation (1.24) and **??** are the ones providing a better unbiased estimation than the latters for more skewed distribution, such as for log–normal random variable's sample.

$$G_{skewness} = \frac{k_3}{\sqrt{k_2^3}} \tag{1.24}$$

$$G_{kurtosis} = \frac{k_4}{k_2^2} \tag{1.25}$$

where

$$K_2 = \frac{n}{n-1}m_2 \tag{1.26}$$

$$K_3 = \frac{n}{(n-1)(n-2)}m_3 \tag{1.27}$$

$$K_4 = \frac{n}{\prod_{i=1}^{3} n-i}\left[(n+1)m_4 - 3(n-1)m_2^2\right] \tag{1.28}$$

Consequently the algorithm to apply in order to estimate the skewness and kurtosis from samples should therefore be chosen with respect to the theoretical underpinned distribution.

1.6 Log–return and compounded interest rate

The concept of log–return and continuously compounded interest rates are closely related together as attested by equations (1.29) to (1.31)

$$S(0)e^{Rt} = S(t) \tag{1.29}$$

$$\Longleftrightarrow \ln S(0) + Rt = \ln S(t) \tag{1.30}$$

$$\Longleftrightarrow \ln \frac{S(t)}{S(0)} = Rt \tag{1.31}$$

Where R is the interest rate with continuous compounding, t denotes the time period across which the interest are compounded (in year) and the left–hand side of equation (1.31) stands for the natural logarithm of the stock return occuring during the period t, the so–called log–return.

For what matter in this current document, the focus is set to the sample's log–returns which are used to estimate the volatility term appearing, inter alia, in **??** to simulate geometric Brownian motion.

Following Hull and Basu [2012], if S_i and n respectively denote the stock price at the end of interval i and the total number of observation then the log–return of the ordered sample set $\{S - i\}_{i \in n}$ is give by equation (1.32).

$$u_i = \ln \frac{S_i}{S_{i-1}} \tag{1.32}$$

Whilst the estimation of the standard deviation of all the u_i is defined such as in **??**

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2} \tag{1.33}$$

$$\tag{1.34}$$

Consequently if the volatility of a stochastic process, simulatating a stock price motion, is given by $\sigma\sqrt{t}$, then it could therefore be estimated using market data throughout equation (1.35)

$$\hat{\sigma} = \frac{s}{\sqrt{t}} \tag{1.35}$$

1.7 Arbitrage strategy

The

Chapter 2

The underlying model

2.1 Overview

This chapter highlights one specific model used to model the motion that a stock price time serie would follow. It is the one used in the Black–Scholes–Merton model, Hull and Basu [2012].

Following Shreve [2004], The equation (2.1), called a geometric Brownian motion, features a stochastic process where the only random component is the Brownian motion $W(t)$. Whilst the others parameters, α and σ, stand respectively for the annualized expected return and volatility of the theoretical stock price process, Hull and Basu [2012].

$$S(t) = S(0)\, e^{\sigma W(t) + \left(\alpha - \frac{1}{2\sigma^2}\right)t} \tag{2.1}$$

In order to match with the constraints introduced by the Black and Scholes' pricing formula (**??**), the value the stock price may achieve at the end of a given period, according to constant mean rate of return and volatility, should be random and log–normally distributed (Black and Scholes [1973]).

Furthermore, the path following by any walks of the stock price motion should be a continuous process, as the time frame decrease $\Delta t \to 0$. Therefore in addition to develop the model, all the aforementioned characteristics are shown to be true within this chapter.

2.2 Derivation

In order to get the differential form of equation 2.1, the Itô's formula (section 1.3) is used. By applying the tranformation incurred by Itô, the equations (2.2a) to (2.2b) emerge:

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t) \tag{2.2a}$$

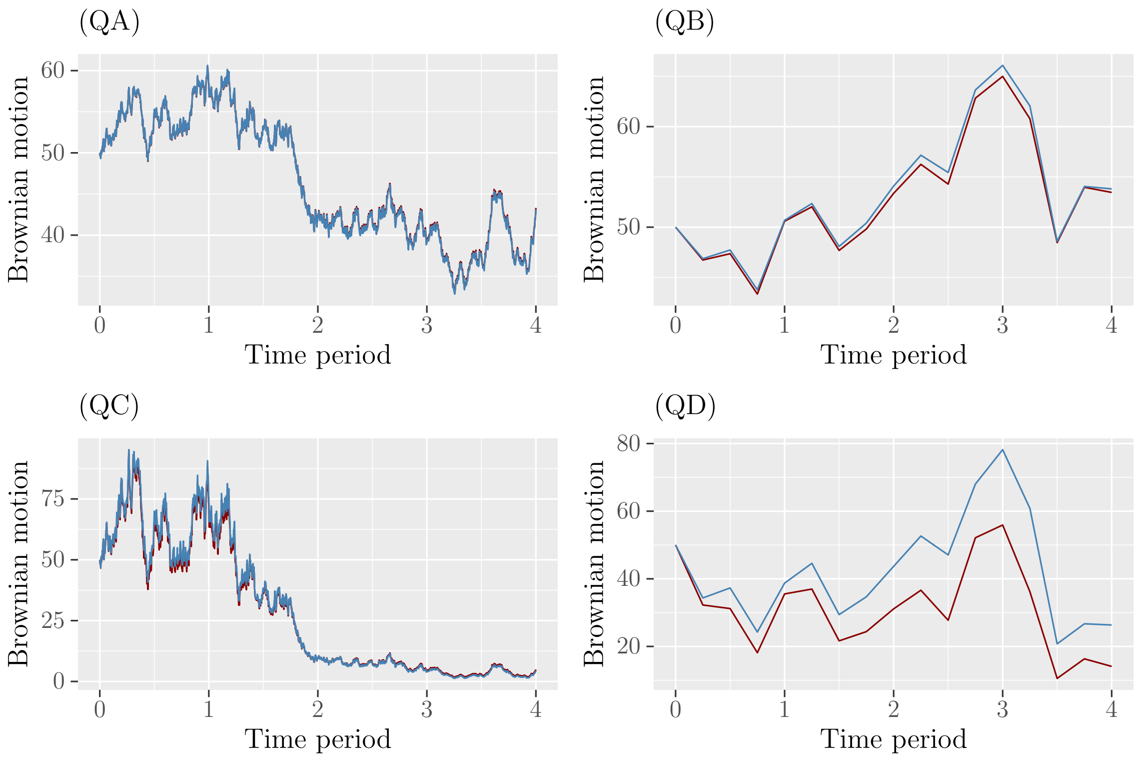
$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) \tag{2.2b}$$

The equation (2.1) shows that any change occuring in the stock price S over a small amount of time is due to the deterministic expected drift rate ($\alpha S(t)$) along with some amounts of added noise included by the random part $(\sigma dW(t))$. The equation (2.1) denotes the stock return stochastic process, Hull and Basu [2012].

$$d \ln S(t) = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW(t) \tag{2.3}$$
$$\ln \frac{S(t)}{S(0)} = (\alpha - \frac{\sigma^2}{2})t + \sigma W(t) \tag{2.4}$$

The **??** exibits the natual logarithm of the stock price return occuring all over the period t. In other words, the expectation of this process, given by $(\alpha - \frac{\sigma^2}{2}) \times t$ happens to be the expected value of the continously compounded rate of return for the aforementioned period of time, Hull and Basu [2012].

Figure 2.1: Accuracy of Itô approximation



The above processes have been constructed over four distinctive groups of two "equivalent" geometric Brownian motions, following equations (2.1) and (2.2a), respectively for the red and blue coloured path. The only parameters that change over the group are the couples (dt, σ) which are set to $QA \equiv (360, 0.2)$, $QB \equiv (4, 0.2)$, $QC \equiv (360, 1)$ and $QD \equiv (4, 1)$. All the processes cover a period of four years and begin with initial value set to 50.

The approximation of the stochastic process **??** achieved using the Itô's lemma depends exclusively on two factors; the volatility parameter σ and the time period occuring between two measures dt. As shown throughout the figure 2.1, the less the volatility or time step, the better the approximation. It is therefore key, during an analysis process to choose an appropriate time step according to a given volatility in order to provide accurate retults.

2.2.1 Distribution of the stock price process

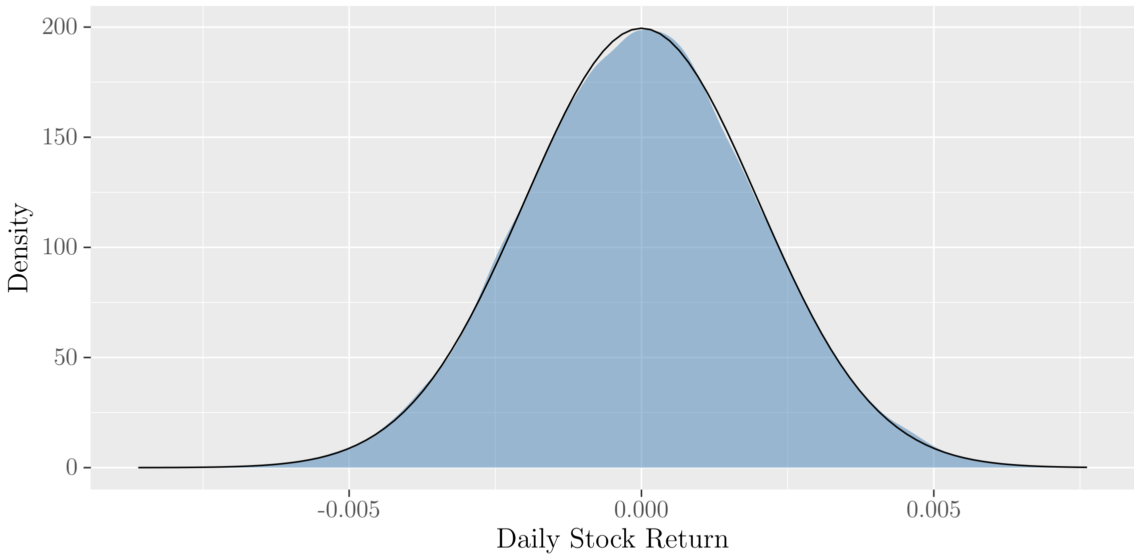
The distributions here developed concern the **??????**. They are described using the first two moments, namely the expectation and variance.

The first distribution (2.5) is related to **??**. What is learnt from it, is that the process followed by **??** has normally distributed returns with expected value and variance relying on the time period dt, Shreve [2004].

$$\frac{dS(t)}{S(t)} \sim N(\alpha dt, \sigma^2 dt) \tag{2.5}$$

According to the (figure 2.2), the normality qualification from equation 2.5 holds.

Figure 2.2: Daily Basis Stock Return Density



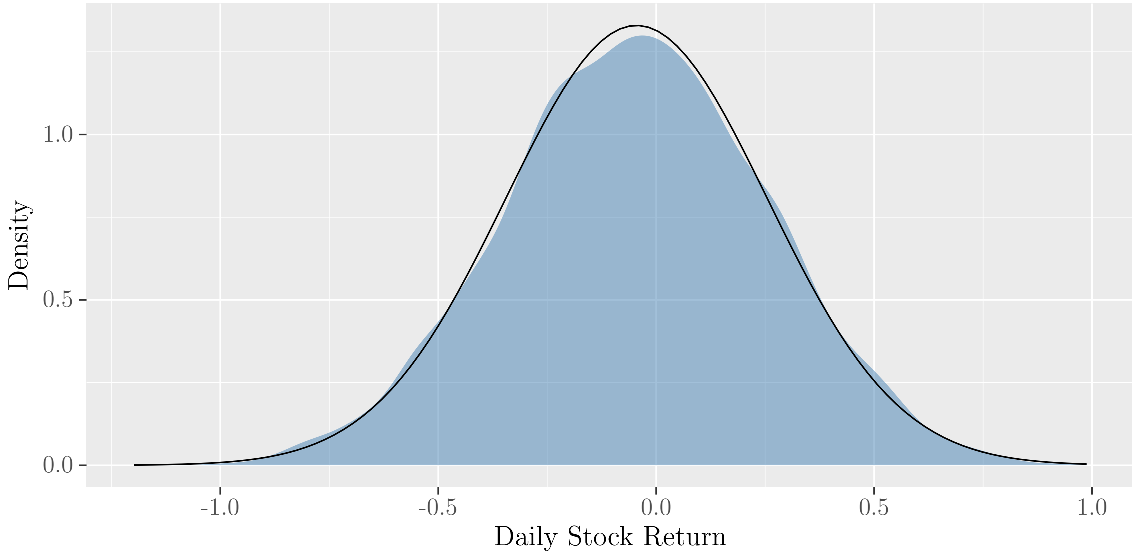
The above density function has been constructed over 10000 samples of a unique process. The samples has been constructed following **??**. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the normal bell curve with mean α and standard deviation of $\sigma \times \sqrt{dt}$. The period between each measure, namely dt has been set to 10000^{-1}.

According to **??**, The distribution of the natural logarithm of the stock price return recorded over a time period of t is characterized by equation (2.6)

$$\ln \frac{S(t)}{S(0)} \sim N((\alpha - \frac{\sigma^2}{2})t, \sigma^2 t) \tag{2.6}$$

From **??**, it could also be shown, by setting $X = \ln S(t)$ and $Y = S(t)$, that the randomly simulated stock price outcomes' distribution, following **??**, match with the log–normal law [REF]. The above point directly echoes the upstream ideal conditions set by Black and Scholes [1973] , especially the one relating the stock price process.

Figure 2.3: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following ??. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the normal bell curve with mean $(\alpha - \frac{\sigma^2}{2}) \times t$ and standard deviation of $\sigma \times \sqrt{dt}$. The time frame is one year ($t = 1$) and period between each measure, namely dt has been set to 500^{-1}.

The following equations shows the relation between the distribution of the log–return and the stock process itself. Let

$$X = \ln S\left(t\right) \sim N\left(\mu \equiv \ln S(0) + \left(\alpha - \frac{\sigma^2}{2}\right)t, \delta^2 \equiv \sigma^2 t\right) \tag{2.7}$$

and

$$Y = S\left(t\right) \tag{2.8}$$

By the existing relation between the log–normal law with respect to the normal law, the following relations emerges.

$$\mathbb{E}\,Y = e^{\ln S(0) + (\alpha - \frac{\sigma^2}{2})t + \frac{\sigma^2 t}{2}} \tag{2.9}$$

$$= S(0)e^{\alpha t} \tag{2.10}$$

and

$$var(Y) = (e^{\sigma^2 t} - 1) * e^{2\mu * \delta^2} \tag{2.11}$$

$$= S(0)^2 e^{2\alpha t}(e^{\sigma^2 t} - 1) \tag{2.12}$$

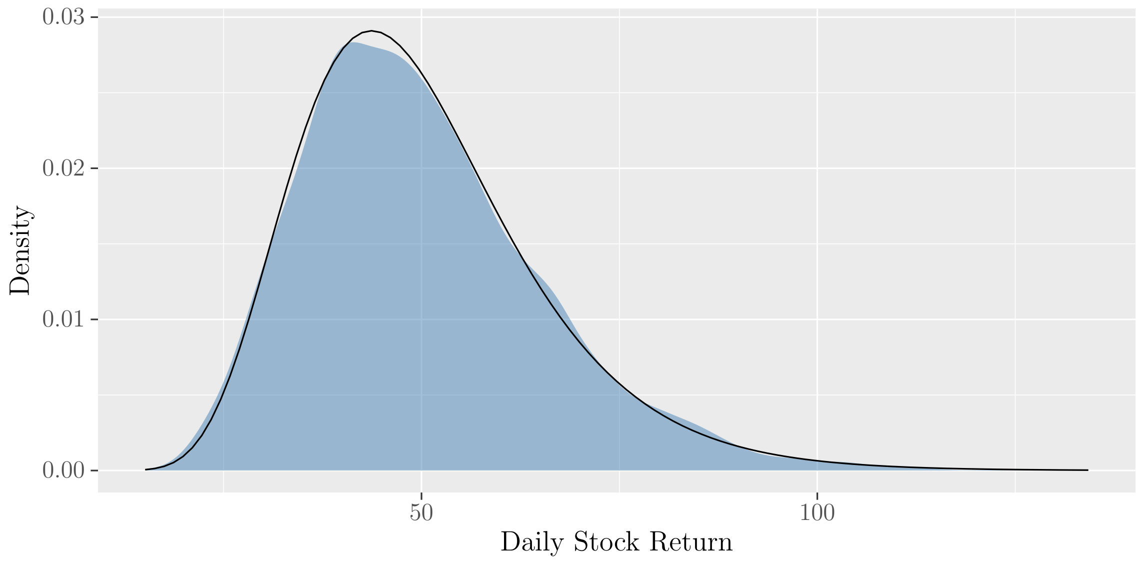
Consequently the stock price random variable $S\left(t\right)$ as relation [REF] for distribution.

$$S\left(t\right) \sim lognormal(\mu, \delta^2) \tag{2.13}$$

$$\ln \frac{S\left(t\right)}{S\left(0\right)} \sim N((\alpha - \frac{\sigma^2}{2})t, \sigma^2 t) \tag{2.14}$$

The ?? and figure 2.4 refers respectively to distributions (2.6) and (2.14).

Figure 2.4: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following **??**. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the log–normal curve with mean and standard deviation such as described in [REF]. The time frame is one year ($t = 1$) and period between each measure, namely dt has been set to 500^{-1}.

2.3 Flaws

Even if the **??** is used to model the stock price process, it exists some lacks with the observed real processes. The analysis ([CHA REF]) mainly covers the following discrepancy of the model against empirical results.

The volatility appearing in the aformentioned underlying model (**??**) is constant as time passes. Even if this consideration can be considered as true for short period of time, it is not the case over the long–run Teneng [2011]. Other model substitutes the process defined throughout this chapter with other ones including stochastic volatility processes, making it changing over time. For this purpose, the Heston's model is developed in this current master thesis, which encompasses two stochastic processes, a geometric Brownian motion [???? + REF] for the stock price diffusion model [DIFUSION??] and a mean-reverting CIR model to compute the volatility as a stochastic process. This model is exposed in **??**.

Following [REF and REF], the uderlying log–returns distribution is characterized by the normal law. Consequently, the random variable $S(t)$, at the fixed time t, is normal. However, according to empirical results, the random variable $dS(t)$ do not fit with the normal bell curve (Clark [1973]). Therefore if the price changes are not normally distribution neither is the log–return and thus $S(t)$, at a fixed time t, is not log–normally distributed. Along with the Heston model, another one is considerd; the Merton's jump–diffusion model. Both are able to modify the skewness and kurtosis of the log–return distibution curve. The Merton framework is developed in **??**.

Chapter 3

The Black–Scholes–Merton model

3.1 Overview

In this present section the Black–Scholes–Merton model is explained and developed. Stricly speaking, it would be talked about the BSM (Black–Scholes–Merton) model but not BSM equation. In turn the equation would be refered to as the BS (Black and Scholes) equation. This distinction is made because the model encompasses the derivation of the BS equation, either using the CAPM (Capital Asset Pricing Model) for Black and Scholes [1973] or by setting up a riskless portfolio providing an expected return identiacal to risk-free rate for Merton [1973].

The BSM model is meant to provide the non arbitrage price of derivative assets such as european stock option (Hull and Basu [2012]). In the present master thesis the model derived from the BS formula is focused on the pricing of one of a kind trivial derivative, the vanilla call option.

In order to price call option throughout the model highlighted in this present chapter, some contraints have been set by Black and Scholes [1973]. The section **??** quotes the assumptions made to keep the model usage between boundaries. In latter chapter however it will be shown how the model is relevant by going on the edge and even beyond these constraints.

3.2 Assumptions

Black and Scholes [1973] have provided a framework defining a bunch of constraints qualified as ”ideal conditions” under which the market would behave in order to make the BS equation works with accuracy. All of these conditions are below–mentioned.

1. The short-term risk free rate r is known and constant

2. The stock return involving in the computation of BS equation is lognormally distributed with constant mean and variance rate

3. No dividend are provided with the considered share of stock

4. The option considered within the computation is european

5. The price for bid and ask quote are identical. It means that there is no bid–ask spread to be considerered

6. Share of stock can be divided into any portions such as needed for the computation
7. Short selling is allowed with no penalties

3.3 The Black–Scholes–Merton equation

The pricing method of an option underpinned by the BSM model is closely related to an underlying for which its price $S(t)$ is random and log–normally distributed, such the one developed in chapter 2 throughout equation **??** (Black and Scholes [1973]).

In order to provide a unique fair price to a stock option (e.g. a vanilla european call option) which depends on an underlying such as described above, all the uncertainty associated to the stock price motion has to disappear. To do so, one have to first construct such a portofolio $X(t)$ which encompasses the same source on uncertainty as the option itself, i.e., the geometric brownian motion $S(t)$ and then choosing the adequate position $\Delta(t)$ to take so that all randomness cancels out (Shreve [2004]).

$$dX(t) = \Delta(t)\, dS(t) + r\left(X(t) - \Delta(t)\, S(t)\right) dt \tag{3.1}$$

The goal is to hedge dynamically the position taken in the option. It means that the position has to be frequently rebalanced. Consequently, at any times, the present value of the changes occuring in the portfolio, due to stock price evolution should be equal to the one occuring in the derivative. The only way to achieve this equality is to adapt the delta (Shreve [2004]).

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}c(t,x)\right) \tag{3.2}$$

In that way, one can take a position in the derivatives (short / long) and hedge them by taking $\pm\Delta(t)$ shares of stock. $X(0)$ being the price of the call at time zero

$$X(0) = c(0, S(0)) \tag{3.3}$$

Following the method forementioned, the BSM differential equation is given by equation (3.4), with terminal condition (3.5) and boundary condition (3.6) – (3.7), (Shreve [2004]).

$$rc(t,x) = \frac{\partial c(t,x)}{\partial t} + rx\frac{\partial c(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c(t,x)}{\partial x^2} \tag{3.4}$$

$$c(T,x) = (x-K)^+ \tag{3.5}$$

Whilst the terminal condition focuses on the value taken at maturity, the boundary conditions fix some constraints on the extreme values likely to be taken by the shares of stock at any times during the option life. In that regard, the boundary condition (3.6) shows that an option with an worthless underlying is itself valueless, while whenever the option is deep in the money, simulated with $x = \infty$ (3.7), the value of the derivative is equal to the value of a forward contract involving the same underlying and with the same maturity date.

$$c(t,0) = 0 \tag{3.6}$$

$$\lim_{x\to\infty} \left[c(t,x) - \left(x - e^{-r(T-t)}\right)\right] = 0 \tag{3.7}$$

3.4 Solution for vanilla option pricing method

According to the terminal (3.5) and boundaries conditions (3.6, 3.7), the Black–Scholes–Merton solution for call option happens to be given by equation (3.8), (Shreve [2004]). The right hand side of the equation, $c(t,x)$, denotes the price of a call option depending on the time to maturity and the stock price at that time. In addition to these arguments, two other parameters are required, the strike (k) and the riskless interest rate (r).

$$c(t,x) = xN(d_+(\Delta t, x)) - Ke^{-r\Delta t}N(d_-(\Delta t, x)) \tag{3.8}$$

with

$$d_\pm(\Delta t, x) = \frac{1}{\sigma\sqrt{\Delta t}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\Delta t\right)\right] \tag{3.9}$$

3.5 The greeks

3.5.1 Overview

The Black–Scholes–Merton equation (3.4) can be divided into differents parts. Each one is identified through a greek letter. These letters are the purpose of this section and would be therefore described here.

The Greeks will be next used to show hown the hedge of a call option behave under some variation from the former conditions defined by Black and Scholes. Hence, only the Greeks for call options are considered.

3.5.2 Delta

Delta is the derivative of the call function (3.8) with respect to the stock price, as shown by equation (3.10). It therefore represents the instantaneous rate of change in a call value as the price of its underlying evolves (Hull and Basu [2012]).

$$\Delta(t, S(t)) = \frac{\partial c(t, S(t))}{\partial S(t)} \tag{3.10}$$

Whereas pratically, the derivation of delta for a call is given by equation 3.11, (Shreve [2004]).

$$\Delta_{call}(t, S(t)) = N(d_+(\Delta t, x)) \tag{3.11}$$

At any time t, in order to hedge a short call one should hold $\Delta(t)$ share of stock. Consequently a portfolio comprised of one short position in a call along with Δ shares of stock is said to be delta neutral, because each movement in the stock price is compensated as well by the position in the call as the one in the stock (Hull and Basu [2012]).

The delta neutrality could otherwise be explained using the slope-intercept form of the tangent line below the function $c(t,x)$, keeping t constant. If the stock price is equal to S and the corresponding call price, for a fixed time t and stroke at k, is c, we consequently get (3.12) as equation of the tangent line below $c(t,x)$.

$$y = \frac{\partial c(t, S(t))}{\partial S(t)}(x - S) + c \tag{3.12}$$

According to 3.12, the price of the call, for a stock price S at a fixed time t is given by $y = c$. If – over an infinitesimally small delta time – a positive stock price movement occurs, says that the stock price rises from $S \rightarrow S + \epsilon$. The price of the call is therefore going to change as well, from $y = c$ to $y = \Delta\epsilon + c$. Consequently, in odrer to hedge a short position in the call, Δ shares of stock should be owned. Indeed, by kepping Δ shares, the loss incured by the higher value of the call $y = c + \Delta\epsilon$ will be offset by an inscrease of $\Delta\epsilon$ thanks to the Δ shares held. It makes sense that the hedge of a long call is achieved by setting up a short position in the underlying, according to the same parameter Δ.

The hedge works well for small price movement in the underlying and closely depends on the curvature of the function $c(t,c)$, keeping t constant (Shreve [2004]). It would therefore be interesting to look at the second derivative of the call function with respect to the stock price, in order to get the rate of the rate of change of the call with respect to the underlying price. It is the purpose of the next subsection (3.5.3).

3.5.3 Gamma

Gamma is the second derivative of the option's price function with respect to the underlying price, time keeping constant (3.13).

$$\Gamma(t, S(t)) = \frac{\partial^2 c(t, S(t))}{\partial S(t)^2} \tag{3.13}$$

Whereas pratically, the derivation of gamma for a call is given by equation 3.14, (Shreve [2004]).

$$\Gamma_{call}(t, S(t)) = \frac{1}{\sigma S(t)\sqrt{\Delta t}} N'(d_+(\Delta t, x)) \tag{3.14}$$

It gives the acceleration at which the price of a call moves along with the underlying price, ceteris paribus. It gives information on the curvature of the function to be approximated using the differential form. It is crutial to know how big is the value of gamma in order to adequatly hedge a position in a call. Indeed, if gamma is low, the rebalancing of the hedge does not have to occur frequently but if gamma It gives as information the frequency needed in order to lower the error due to too large price movement. With the delta hedging rule, the more the price moves from its current value the more Consequently, by gathering gamma and delta both together, a given more precise information on hedging against the only stock price movement is achieved.

3.5.4 Theta

Theta is the derivative of the price of an option with respect to the time, stock price unchanges (3.15)

$$\Theta(t) = \frac{\partial c\left(t, S\left(t\right)\right)}{\partial t} \tag{3.15}$$

(Hull), Theta can be used as a proxy for Gamma in a delta neutral portfolio. No need to hedge against time therefore no need to neutralize theta.

3.6 Relation between BSM and the greeks

The Black–Scholes–Merton equation and the greeks are closely related together. In deed the BSM partial derivative formula (3.4) could equally be written using the greeks (3.16)

$$rc\left(t, S\left(t\right)\right) = \Theta + rS\left(t\right)\Delta + \frac{1}{2}\sigma^2 S\left(t\right)^2 \Gamma \tag{3.16}$$

3.7 The delta hedging rules

The purpose of the delta hedging rules is to fully replicate the reverse position taken in an option in order to cover oneself against lost. The technique is achieved by continuously rebalancing its position in order to keep an amount of $\Delta(t)$ share of stock at each time t.

One scenario could be the following. One party write an option at some initial time t_0 at a certain price (given by the Black–Scholes–Merton equation (REF)) $X(0)$, and he / she puts that earning money into a bank account, in which the annual interest rate in action is given by r.

$$X(0) = BSM(S_0, T, \sigma, k) \tag{3.17}$$

Incidentally, following (EQUATION DELTA), he / she computes the delta at time zero and buys exactly that amount of share of stock. Additionaly, he / she divides the whole time frame into n smaller time steps δt in order to subsequently rebalance its position in the stock, according to the evolution of the underlying.

$$\begin{cases} p(0) = \Delta(t_0)S(t_0) \\ p(1) = \left(\Delta(t_1) - \Delta(t_0)\right)S(t_1) \\ p(2) = \left(\Delta(t_2) - \Delta(t_1)\right)S(t_2) \\ \vdots \\ p(i) = \left(\Delta(t_i) - \Delta(t_{i+1})\right)S(t_i) \\ \vdots \\ p(n) = \left(\Delta(t_i) - \Delta(t_{n+1})\right)S(t_n) \end{cases} \tag{3.18}$$

Finally, to see if the hedge works, we value all operation made up to T and sum the whole part

$$
\begin{cases}
P(0) = p(0)e^{r\times(T-t_0)} = p(0)e^{r\times T} \\
P(1) = p(1)e^{r\times(T-t_1)} \\
P(2) = p(2)e^{r\times(T-t_2)} \\
\vdots \\
P(i) = p(i)e^{r\times(T-t_i)} \\
\vdots \\
P(n) = p(n)e^{r\times(T-t_n)} = p(n)
\end{cases}
\tag{3.19}
$$

Now we get the valued portfolio, we have to compare it with something. At the beging we got an amount of money, priced at $X(0)$ which were the Black – Scholes – Merton Price. This amount of money has been put in a money market account with an interest rate of r. At maturity we therefore get the amout of money given by (REFENCE EQUATION).

$$X(T) = X(0)e^{rT} \tag{3.20}$$

At maturity, another operation occurs. If the option is in-the-market, it will be exercised and furthermore $\Delta(T) = 1$, meaning that we have one share of stock to honour the transaction. In exchange we get $\$K$ from the counterparty. It means that in the end we have a positive amount of:

$$X(T) + \Delta(T) * k \tag{3.21}$$

Consequently to check if the hedge works well, we should get the following relation:

$$\sum_{i=0}^{n=T} P(i) \cong X(T) + \Delta(T) * k \tag{3.22}$$

Because the position could vary, we have to adapt the relation above (REF) in order to remove all unit from the comparison. According to hull, a measure of performance of the hedge could be to take the ratio of the standard deviation of the cost to hedge to the option price. In that case, the measure could be comparated among themselve even if the position taken into the underlying difer.

$$\frac{sd(hedge)}{C_0 e^{rT}} \tag{3.23}$$

where hedge is a vector containing n sample generated, and C_0 is the price of the call option at time zero value up to maturity.

Chapter 4

Other Models to be considered

4.1 Overview

4.2 Merton Mixed jump-diffusion Model

In his paper, Merton [1976] provides a model for stock price evolution involving jumps (equation (4.1)).

$$S(t) = S(0)\, e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \tag{4.1}$$

According to Merton [1976], there are two specific sources of uncertainty explained by the model (equation (4.1)). The first one is qualified to be normal, arising repeatedly with low effects and keeping the stock price motion continuous from time to time. These small changes on the price are modeled by a Wiener process, such as it was the case in equation **??**. The cause of these changes is explained by a temporary unbalanced between the supply and demand Merton [1976]. Another type of changes, occuring during the stock lifecycle, are qualified as abnormal. Such "abnormalities" happen less frequently, are unpredictable in their frequency and produce bigger effect on the stock price by giving rise to jumps in the course of stock path and therefore breaking its continuity (Merton [1976]). The jump process is constructed on double basis.

Firstly, the occurence (i.e. the number of jumps arising throughout a given period of time) is computed thanks to a Poisson–driven process according to a parameter λ. λ denotes the number of jumps per unit of time. Consequently, the probability that a jump occurs during a time range of Δt is equal to λdt (event A, eq. 4.2), whereas the probability that there are no jump during the same range of time is $1 - \lambda dt$, (event B, eq. 4.3) (Matsuda [2004]). While C, eq. 4.4, refers to the event that more than one jump occur during the same small delta time.

$$\mathbb{P}\{A\} \cong \lambda dt \tag{4.2}$$
$$\mathbb{P}\{B\} \cong 1 - \lambda dt \tag{4.3}$$
$$\mathbb{P}\{C\} \cong 0 \tag{4.4}$$

On the other hand, after the occurence comes the size of the jump. Such as the frequency, the importance of the jump can be characterized by a statistic law. Following

?, the log–normal law is used. Matsuda [2004], gives a summary in oder to grips with the concept of jump size equations (4.5) to (4.7).

$$
\begin{aligned}
y_t &\sim lognormal(e^{\mu+\frac{1}{2}\delta^2}, e^{2\mu+\delta^2}(e^{\delta^2}-1)) && (4.5)\\
y_t - 1 &\sim lognormal(\kappa \equiv e^{\mu+\frac{1}{2}\delta^2}-1, e^{2\mu+\delta^2}(e^{\delta^2}-1)) && (4.6)\\
\ln y_t &\sim normal(\mu, \delta^2) && (4.7)
\end{aligned}
$$

with y_t, $y_t - 1$ and $\ln y_t \equiv Y_t$ standing respectively for "absolute price jump size", "relative price jump size" and "log price jump size" (Matsuda [2004]).

The Merton's jump-diffusion process is be able to capture positive / negative skewness (see section 4.2.3) and excess kurtosis (see section 4.2.4) of the log–return density function Merton [1976].

4.2.1 Risk-neutralized process

In order to find the fair price of an option depending on an underlying that follows such a jump-diffusion process, Merton [1976] turns equation (4.1) into one risk-neutral.

$$S\left(t\right) = S\left(0\right) e^{\left(r-\frac{\sigma^2}{2}-\lambda\kappa\right)t+\sigma W(t)+\sum_{i=1}^{N_t} Y_i} \qquad (4.8)$$

Merton [1976] argues in his paper that the jump component of equation (4.1) can be diversified in a well-balanced portfolio and consequently does not need to be risk-neutralized.

However, likewise it was done by Black and Scholes [1973], the drift part of equation (4.1) is risk-neutralized by turning the rate α into its riskfree counterpart r, as shown by equation (4.8).

4.2.2 Graphical representation

Figure 4.1 shows a unique time serie generated using an implementation of equation (4.1). A jump is clearly noticed at day 363. While Table 4.1, which is a subset of the time serie drawn in figure 4.1, illustrates numerically when the jump occurs.

4.2.3 Impact on the skewness log–return

The way to influence the direction of the distibution's shape is achieved by moving the cursor of the expected value of jump impact, in other word, by changing the value of the parameter μ. The figure [REF] shows how the density's shape of the log–return may vary along with this parameter.

4.2.4 Impact on kurtosis log–return

The way to influence the aspect of the distibution's tails is achieved by moving the cursor of the expected value of jump occurence, in other word, by changing the value of the parameter λ. The **??** shows how the distribution's tails of the log–return may vary along with this parameter.

Table 4.1: Merton Mixed jump-diffusion time serie

time periods (days)	stock price
0	50.00
1	49.72
2	49.87
$\vdots$	$\vdots$
323	83.44
324	94.98
$\vdots$	$\vdots$
363	97.41
364	97.07
365	97.00

4.3 Heston stochastic volatility model

In his paper, Heston [1993] tackles with another discrepancy against the real world behaviour introduced by the geometric Brownian motion, namely, its deterministic and immutable volatility σ.

In addition to provide a model where the volatility is stochastic (equation (4.9)), Heston [1993] gives the possibility to make that volatility in correlation with the stock price process (equation (4.10)), according to the parameter ρ defining how the Brownian motions from both processes relates together.

$$dV\left(t\right) = \kappa\left(\theta - V\left(t\right)\right)dt + \sigma\sqrt{V\left(t\right)}dW_V\left(t\right) \tag{4.9}$$

$$dS\left(t\right) = \alpha S\left(t\right)dt + \sqrt{V\left(t\right)}S\left(t\right)dW_S\left(t\right) \tag{4.10}$$

The drift part of the risk stochastic process (4.9 is made up of the long–run mean θ togehter with the mean reversion speed, given by κ, Heston [1993].

$$dW_v\left(t\right)dW_s\left(t\right) = \rho \tag{4.11}$$

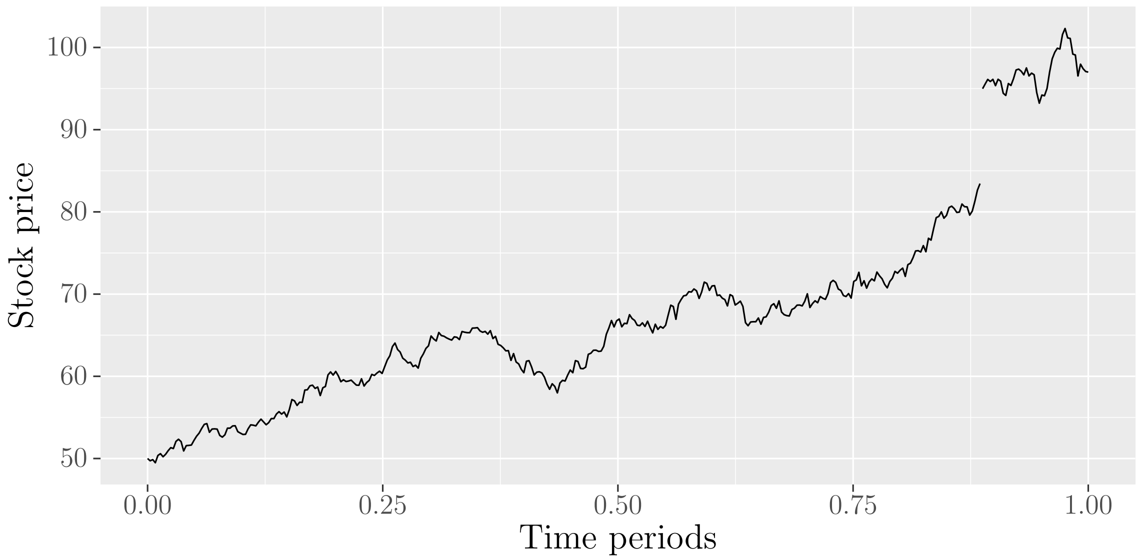
Equation (4.10) represents the evolution of an asset though time given, by its differential form. Such as equation (2.1), developed by Black and Scholes [1973], the parameter α gives the drift rate of return. The difference between both models lies in the way the volatility is perceived. In Heston [1993], the asset volatility is given by the stochastic equation (4.9). More specifically, the volatility thus defined follows a Cox–Ingersoll–Ross process.

Equations (4.9) to (4.10) will be used to build Monte Carlo simulations with the goal of getting samples of dummy stock price motions under real–world assumptions. The framework to use for option pricing purpose is given in section 4.3.3.

4.3.1 Model parameters

Here are described all the parameters appearing in the Heston stochastic volatility model.

Figure 4.1: Merton mixed jump-diffusion time serie



Simulation of one Merton mixed jump-diffusion time serie. Data have been output by the R function *sstoch_jump()* which is an implementation of equation equation (4.1) (see appendix **??**, for more information). The parameters passed to the function are: $S(0) = 50$, $T = 1$ (in year, along with a time step of 365 measures per year), $\sigma = 0.2$, $\alpha = 0.5$, $\lambda = 2$, $\mu = 0.05$, and $\delta = 0.1$.

$S(t)$	Price of the stock at time t
α	Annualized – and deterministic – expected return
$V(t)$	Observed volatility of the stock at time t
κ	Mean-reversion speed
θ	Volatility's long-run mean
σ	Volatility of the volatility

4.3.2 Feller condition

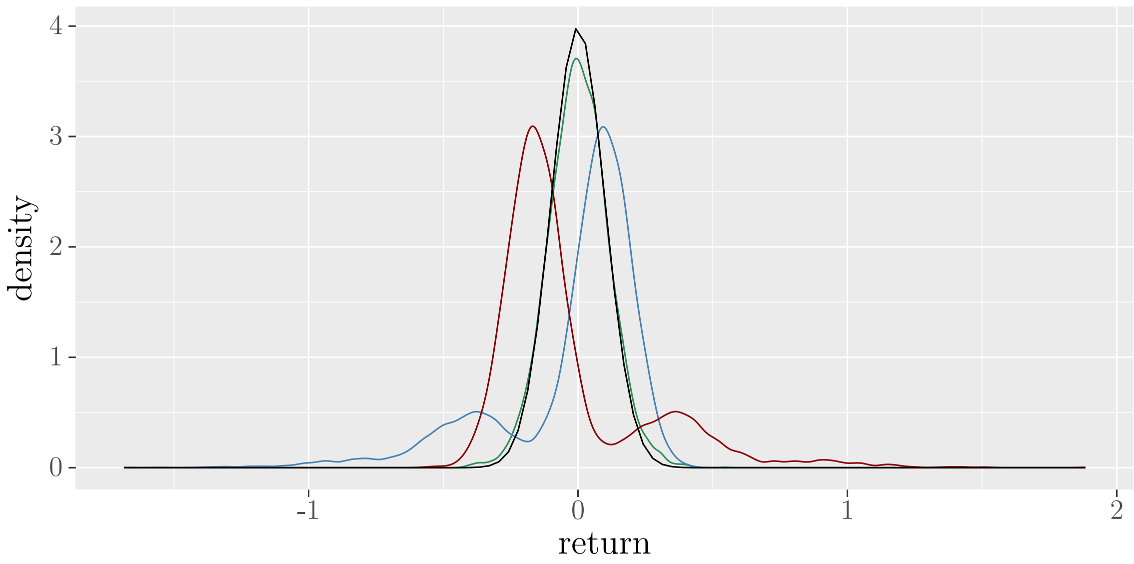
Due to the time discretisation brung by The Monte Carlo simulation, the stochastic process 4.9 may turn out to be negative sometime. If such a negative value is appear at time t, the next value computed for $t + \epsilon$ will raise an error, due to the term $\sqrt{V(t)}$ that obviously does not exist for negative value.

In his paper, Feller [1951] demonstrates that a process such the one described by equation (4.9) does not reach negative value if the following relation 4.12 is respected.

$$
\begin{aligned}
& \lim_{V \to 0} \left(\kappa\theta - V - \frac{1}{2} \frac{\partial (\sigma\sqrt{V})^2}{\partial V} \right) \geq 0 \qquad (4.12) \\
\iff & \lim_{V \to 0} \left(\kappa\theta - V - \frac{1}{2}\sigma^2 \right) \geq 0 \\
\iff & \kappa\theta - \frac{1}{2}\sigma^2 \geq 0 \\
\iff & 2\kappa\theta - \sigma^2 \geq 0 \qquad (4.13)
\end{aligned}
$$

Consequently, if the condition related by equation (4.13) is respected, no negative

Figure 4.2: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (4.1). The only parameter that changes over the group is μ which is set to $(-0.5,\ 0,\ 0.5)$ respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

values would occur by using a Monte Carlo simulation to compute the CIR stochastic volatility.

4.3.3 Risk–neutralized processes

Likewise it has been done by Black and Scholes [1973], Heston [1993] used a risk-neutral framework to price options. To do so, Heston modified the drift parameters of both price and volatility stochastic processes.

The drift part of the price diffusion (equation (4.10)) is risk–neutralized by turning the rate α into its riskless counterpart r, as shown by equation (4.14).

$$dS\left(t\right) = rS\left(t\right)dt + \sqrt{V\left(t\right)}S\left(t\right)dW_S\left(t\right) \tag{4.14}$$

In order to make the volatility process risk–neutralized, Heston added the risk premium parameter, λ, to the drift part of equation (4.9). The risk–neutralized CIR process is given by crefeq:other:hsvvol:riskless.

$$dV\left(t\right) = \kappa^*\left(\theta^* - V\left(t\right)\right)dt + \sigma\sqrt{V\left(t\right)}dW_V\left(t\right) \tag{4.15}$$

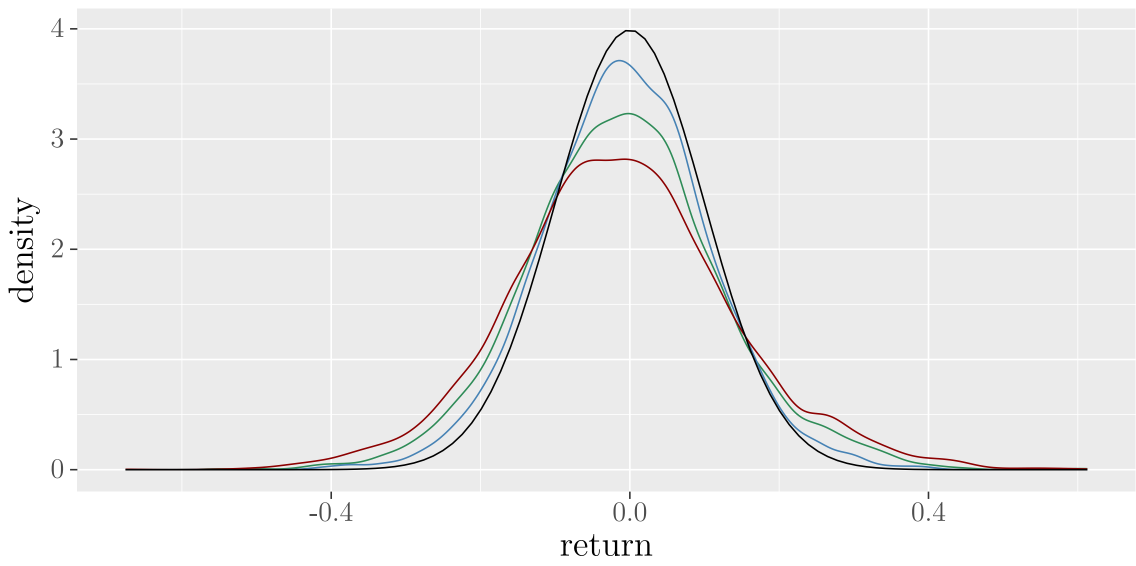
where

$$\kappa^* = \kappa + \lambda \tag{4.16}$$

and

$$\theta^* = \frac{\kappa\theta}{\kappa^*} \tag{4.17}$$

Figure 4.3: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (4.1). The only parameter that changes over the group is λ which is set to (1, 3, 5) respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

Consequently the parameters κ^* and θ^*, which respectively denote the long–run mean and mean–reversion speed, are the ones to estimate while dealing with pricing options purposes.

4.3.4 Graphical representation

Figures 4.4 and 4.5, get a hands-on insight into how the correlation between the underlying Brownian motions of the stock and volatility time series affect both processes. figure 4.4 shows a correlation between the Wiener processes B_1 and B_2 sets to $\rho = -1$, making the two Markov motions perfectly negatively correlated. It directly affects the course of the stocks serie, which is altogether correlated in the same negative direction with respect to the CIR volatility process as well. Likewise, figure 4.5 points out the fully positive correlation occuring between the processes 4.9 and 4.10 whilst the Brownian motions correlation is set to one.

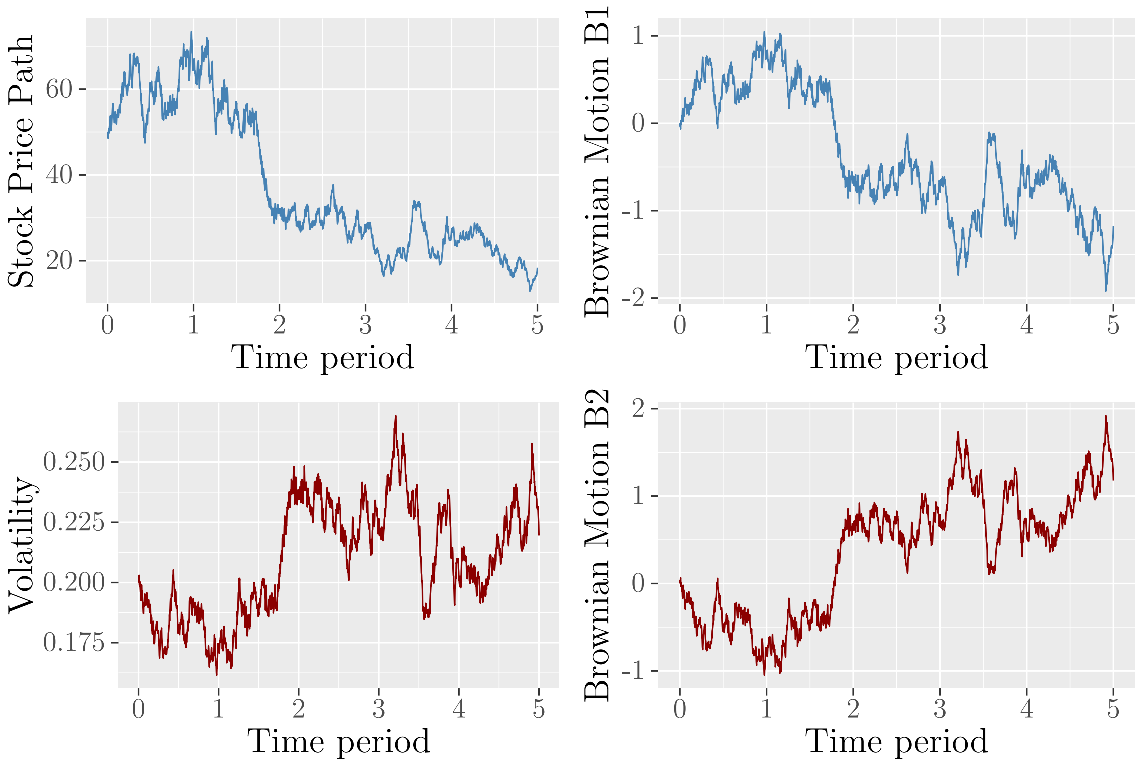
The usage of the aforementioned Heston model lies in the fact that the correlation between the CIR and asset processes' Brownian motions would notably explain the spot return skewness whereas the kurtosis of the distribution may be affected by the volatility parameter σ of the volatility stochastic process (equation (4.9)), (Heston [1993]) It may consequently be consistent with what happens in the equity market, namely a sharp decrease in equity price implies an increase in stock volatility (Crisóstomo [2015]).

4.3.5 Impact on log–return density's skewness

Through the Heston stochastic volatility model, the skewness of the distribution of continuously compounded spot return may be affected by the parameter ρ.

When a positive correlation exists between both Brownian motions, an increase in

Figure 4.4: Heson Framework Using Negative Correllated Brownian Motions



Simulation of Heston time series. Data have been output by the R function *heston()* which is an implementation of equation equations (4.9) to (4.11) (see appendix **??**, for more information). The parameters passed to the function are: $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 measures per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$ and $\rho = -1$.

the volatility implies a rise in the asset price whereas a decrease in the volatility tend to lower the asset price. In other words, when the uncertainty is high and consequently the changes in the asset price are numerous, these latter tend to be positive. That is why the distribution of the spot return is offset to the left with a right fat tail when ρ is positive.

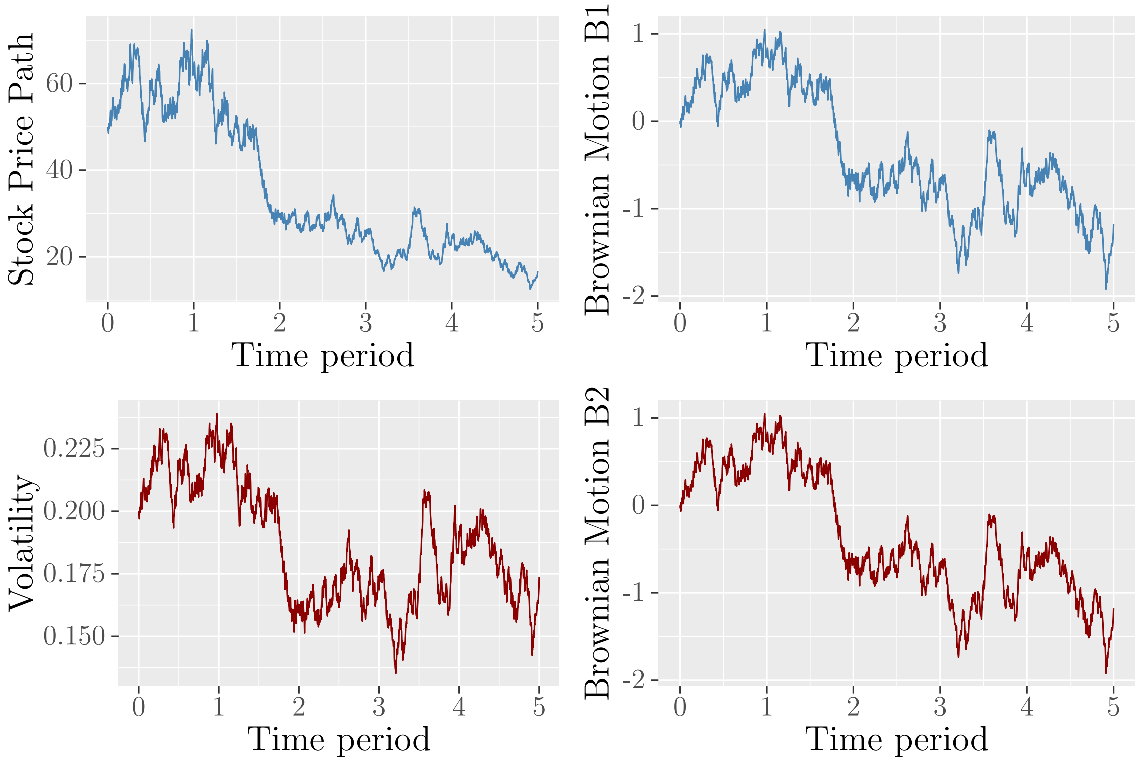
The opposite relation is noticed with negative correlation, namely, lower prices relate to higher volatility generating a left fat tail in the log–return distribution (Heston [1993]). These statements are observed in figure 4.6.

4.3.6 Impact on kurtosis density return

Following Heston [1993], the kurtosis of the distribution of the spot return may be affected by the parameter σ, which represent the volatility of the volatility.

First and foremost, following equation (4.9) if $\sigma = 0$, the volatility V of the Heston model turns out to be deterministic and equation (4.10) becomes a geometric Brownian motion with normal distribution for the time series' log-returns. Otherwise, Heston [1993] showed that by rising σ, the kurtosis of the spot returns increases. Consequently, within the Heston stochastic volatility model, the bigger σ, the fatter the tail, ceteris paribus. These statements are observed in figure 4.7.

Figure 4.5: Heson Framework Using Positive Correllated Brownian Motion



Simulation of Heston time series. Data have been output by the R function *heston()* which is an implementation of equation equations (4.9) to (4.11) (see appendix **??**, for more information). The parameters passed to the function are: $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 measures per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$ and $\rho = 1$.

4.4 Option pricing method

As shown by section 4.2 and section 4.3, the frameworks developed by Merton [1976] and Heston [1993] drastically change the distribution of any underlying assets following such processes. Therefore, the pricing method to be used must also be adapted in order to take that update into account.

In his paper, Heston [1993] developed a techinque to price options using the characteristic function of the underlying asset. Furthermore, according to Crisóstomo [2015] that method could be used to price every options provided that the underlying's characteristic is known.

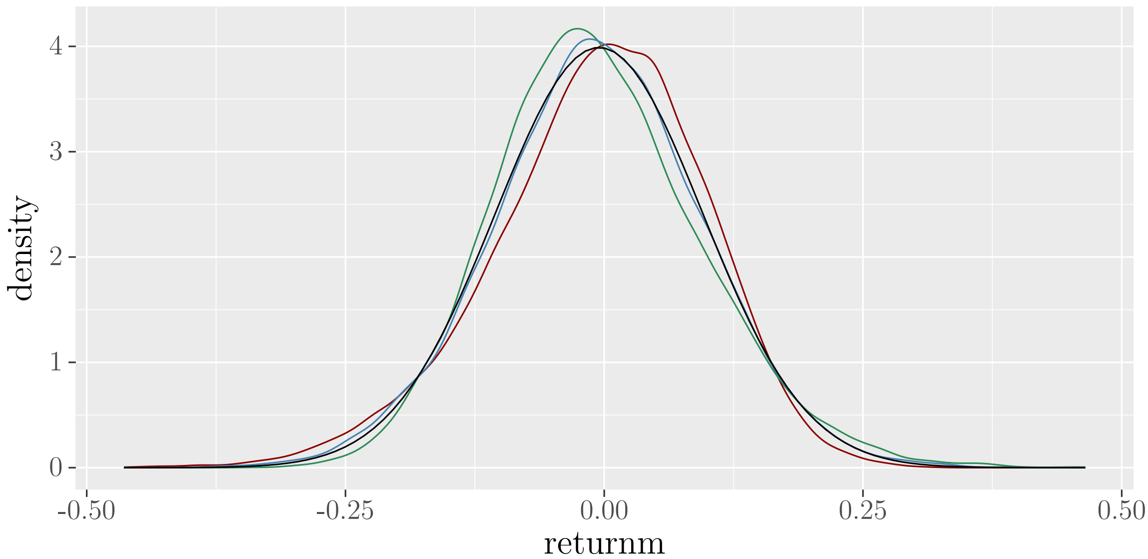
4.4.1 Probabilistic approach

Heston [1993] proposed the solution given by equation (4.18) to price a european call option.

$$C(t) = S(t)P_1 - e^{-r(T-t)}KP_2 \tag{4.18}$$

Through this method, the european call price at time t, namely $C(t)$, is computed thanks to equation (4.18), where $S(t)$ and $e^{r(T-t)}$ respectively stand for the stock price and the present value of the strike at that time t.

Figure 4.6: Merton: Log–return skewness on Heston



The above densities function (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (4.10) for the stock data and on equation (4.9) for the related volatility (see function *heston()* on appendix **??** for more information). The only parameter that changes over the groups is ρ which is set to -0.5, 1, 0.5. The log-return densities of these groups are respectively represented by the red, green and blue outlined density functions. The black density represents to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log-price return cover a period of one year with a time step of 500.

$$P_1(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi - i)}{i\phi\psi(x, V, t; -i)}\right) d\phi \tag{4.19}$$

$$P_2(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi)}{i\phi}\right) d\phi \tag{4.20}$$

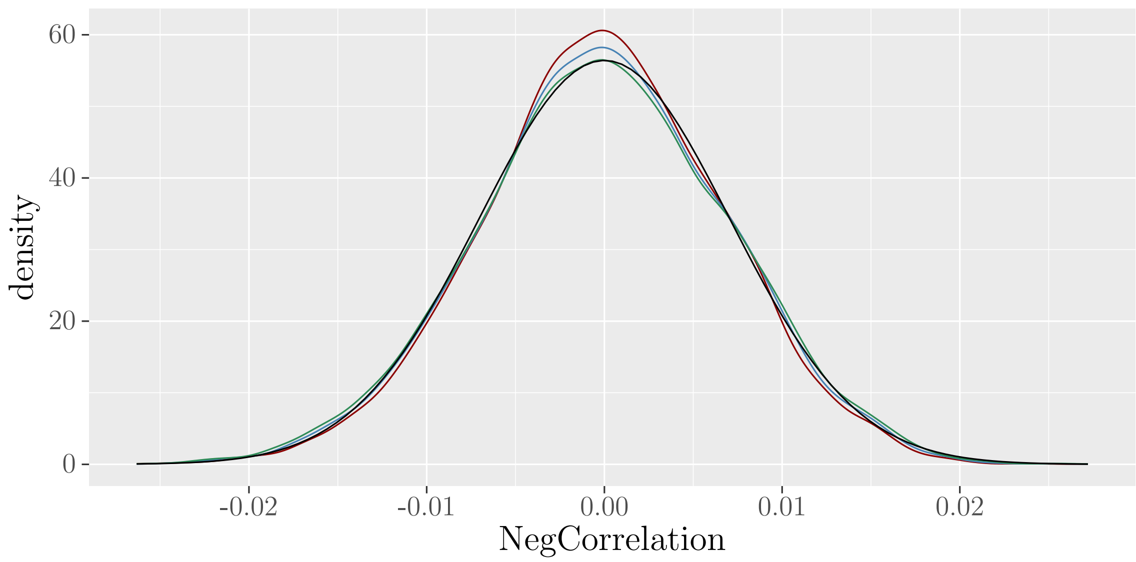
Following the development in Crisóstomo [2015], **??** both are probability quantities that involve the underlying characteristic function, namely $\psi(x, V, t; \phi)$. Once these quantities are computed, they are substituted in equation (4.18) in order to get the call price at time t.

The characteristic functions for the Merton jump–diffusion (section 4.2) and Heston stochastic volatility (section 4.3) models are developed sections 4.4.2 and 4.4.3

4.4.2 Characteristic function for Merton Mixed jump–diffusion model

Matsuda [2004] demonstrates that the characteristic function of the Merton mixed jump-diffusion process is given by equation (4.21). The baseline equation to construct the Merton's process characteristic function is the risk-neutralized version, namely equation (4.8).

Figure 4.7: Merton: Log–return skewness on Heston



The above densities function (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (4.10) for the stock data and on equation (4.9) for the related volatility (see function *heston()* on appendix **??** for more information). The only parameter that changes over the groups is σ which is set to 0, 0.2, 0.4. The log-return densities of these groups are respectively represented by the green, blue and red outlined density curves. The black density represents to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log-price return cover a period of one year with a time step of 500.

$$\psi^{merton}(\phi) = e^{\lambda(T-t)\left(e^{i\mu\phi-\frac{\delta^2\phi^2}{2}}-1\right)+i\phi\left(\ln S(t)+\left(r-\frac{\sigma^2}{2}-\lambda\kappa\right)(T-t)\right)-\sigma^2\frac{\phi^2}{2}(T-t)} \tag{4.21}$$

where

$$\kappa = e^{\mu+\frac{\delta^2}{2}} - 1 \tag{4.22}$$

According to the method given by Heston [1993], the characteristic function 4.21 will be used inside equations (4.19) and (4.20) in order to compute the quantities P_1 and P_2 that could be thereafter replaced inside equation (4.18) to find the european call price $C(t)$ corresponding to a stock price process $S(t)$ driven by the Merton mixed jump–diffusion model.

4.4.3 Characteristic function for Heston stochastic volatility model

Following the development proposed by Gatheral and Taleb [2006], Crisóstomo [2015] provided the Heston characteristic function (equation (4.23)) based on the process $\ln S(t)$.

$$\psi^{heston}\left(\ln(S(t)), V(t), t; \phi\right) = e^{C(T-t,\phi)\theta+D(T-t,\phi)V(t)+i\phi\ln\left(S(t)e^{r(T-t)}\right)} \tag{4.23}$$

where

$$C(\tau,\phi) = \kappa\left(r_-\tau - \frac{2}{\sigma^2}\ln\left(\frac{1-ge^{-h\tau}}{1-g}\right)\right)$$
$$D(\tau,\phi) = r_-\frac{1-e^{-h\tau}}{1-ge^{-h\tau}}$$

and

$$r_\pm = \frac{\beta \pm h}{\sigma^2}; h = \sqrt{\beta^2 - 4\alpha\gamma}$$
$$g = \frac{r_-}{r_+}$$
$$\alpha = -\frac{\phi^2}{2} - \frac{i\phi}{2}; \beta = \kappa - \rho\sigma i\phi; \gamma = \frac{\sigma^2}{2}$$

Equation (4.23) can be directly used inside equations (4.19) and (4.20) in order to compute the quantities P_1 and P_2 that could be thereafter replaced inside equation (4.18) to find the european call price $C(t)$ corresponding to a stock price process $S(t)$ driven by the Heston stochastic volatility model.

Chapter 5

Methodology

5.1 Overview

The objective of this master thesis is to measure the performances of the Black-Scholes-Merton pricing method when the assumption of normality for the log-returns distribution is not met.

In order to measure such performances, I will (i) compute the options price by using in turn the models BSM, Merton mixed jump-diffusion and Heston stochastic volatility models. Thereafter, I will (ii) compare the implied volatility of those computed prices split by maturities and by models. Finally, I will (iii) construct delta-neutral portfolios to measure the hedging performance of the aforementioned models.

Instead of exclusively using market data, whether to gather option prices or to build the delta-neutral portfolio, I will rather construct some theoretical based algorithm. Depending on the framework to explore, the option prices will be computed either by using the BSM equation, if the underlying process relates to a geometric Brownian motion or by using the method developed by Heston [1993], it the underlying process relates to the model MJD or HSV. To assess the delta-neutral portfolio, I will need time series to measure its evolution across time. those series will be simulated based on the theories developed by Merton [1976] and Heston [1993] in order to respectively obtain paths with jumps and others with stochastic volatility. Consequently, I have built some functions in the R language to perform those tasks. table 5.1 is a summary of a few of them used in that chapter and in the analysis. More details on them are given in appendix **??**.

Table 5.1: R functions dealing with options and time series

Function name	Arguments	Purpose
bsm_call	$\{S(0), T, k, r, \sigma\}$	Compute the BSM price of an option
mjd_call	$\{S(0), T, k, r, \lambda, \mu, \delta\sigma\}$	Compute the Merton price of an option
hsv_call	$\{S(0), T, k, r, V(0), \theta, \kappa, \sigma, \rho\}$	Compute the heston price of an option
bsm_ts	$\{S(0), T, \sigma, \alpha, dt\}$	Simulate BSM time series
mjd_ts	$\{S(0), T, \sigma, \alpha, \lambda, \mu, \delta dt\}$	Simulate MJD time series
hsv_ts	$\{S(0), T, V(0), \alpha, \rho, \kappa, \theta, \sigma, dt\}$	Simulate HSV time series

Even though I won't construct my hedge directly on market data, I need those data to calibrate the parameters to pass into the functions listed in table 5.1. Therefore, the functions with the objective to provide european call price will be calibrated with the Apple european call market data while, to stay consistent, the functions that simulate time series will be benchmarked using the Apple stock data available on the market. This process of calibration is fully explained at section 5.2.

5.2 Calibration

Whenever one deals with functions aimed to reproduce some real-life experiments, the calibration process is crucial because it gives to the functions the capacity to act within appropriate boundaries.

The process of calibration which will be applied in the current section concerns two distinctive groups of parameters. Those intended to the functions that compute the European call price and those used by the functions that simulate the hypothetical stock market movements. Consequently, the methods to calibrate both kinds of arguments differ, mainly because the options prices calculation must be performed under risk-neutral environment and the delta hedging is measured on time-series evolving under a risk-averse world.

To do so, I am going to use the available option price data to calibrate the arguments aimed to the functions that compute the price of the latter. Whilst I am going to use the stock's historical data to estimate the right values for the time-series output-related functions. The option market prices and stock data were downloaded using the package *quantmod*, developed by Ryan and Ulrich [2018] which uses Yahoo! finance as provider. The datasets so downloaded are available in appendix **??**.

Section 5.2.1 explains how I am going to operate for the options' parameters, whereas section 5.2.2 shows the procedure I am going to follow for the assets simulations' arguments.

5.2.1 Option prices based calibration

In accordance to Heston [1993] and Crisóstomo [2015], provided that the characteristic functions of the MJD and HSV models are known, the European call option price of such underlying processes can be computed using equation (4.18). Although known, these characteristics functions (equations (4.21) and (4.23)) need some parameters to work and these parameters need appropriates values to best fit with what is observed in reality. That is why both functions need to be benchmarked with referential values before being used.

To do so, I am going to apply the same method followed by Crisóstomo [2015], namely, iteratively minimizing the difference between the options market prices and those generated by the functions that compute the option prices based on the characteristic functions. As the market data comes with a large number of maturities and stikes, tables 5.2 and 5.3 list those considered during the analysis.

Table 5.2: Taking into account maturities during calibration stage

63	91	126	154	182	245	399

The optimization method used is the least-square non-linear analysis. To work, such a method need (i) a function that returns dummy data, (ii) a dataset of data to serve as a template, (iii) a cost function to minimize and (iv) the parameters to optimize.

Table 5.3: Taking into account strikes during calibration stage

130	140	150	160	170	180	190	200	210	220

The functions that return the artificial data are those exhibited in table 5.1, that is to say, *mjd_call* and *hsv_call* for the computation of call price using MJD and HSV, respectively. The dataset template is the one in **??** (see appendix **??**). While the cost function is given by equation (5.1), the parameters to assess depend on the underlying model (either MJD or HSV).

$$\begin{aligned} &\left(C^{mkt}_{K,T} - C^{h|m}_{K,T}(arguments)\right)^2 \\ &\forall K \in \{130, 140, 150, 160, 170, 180, 190, 200, 210, 220\}, \\ &T \in \{63, 91, 126, 154, 182, 245, 399\} \end{aligned} \tag{5.1}$$

Where the subscripts K and T respectively stand for strike price and maturity date, while the superscripts mkt denotes the "market price" and $h|m$ refers to either Heston or Merton process based prices.

Consequently, the equation (5.1) is that to be minimized using the least-square non-linear analysis, for all strikes and maturities. The output of this process will eventually be the calibrated arguments that maximize the performance of the model with respect to what is observed in the reality.

One difficulty when dealing with such an algorithm is that the least-square non-linear approach may return several best-fit sets of arguments due to the existence of multiple local minima in the cost function. Therefore, in order to choose among all the outputs provided by the optimization method, I will select those that ensure that the pricing function gives the most of its returns within the bid-ask spread of the option data, as shown by equation (5.2).

$$C^{mkt}_{bid} \leq C^{h|m}_{K,T}(arguments) \leq C^{mkt}_{ask} \tag{5.2}$$

Even though the cost function to be optimized is the same for both HSV and MJD pricing option procedures, the arguments to calibrate are different. sections 5.2.3 and 5.2.4 respectively illustrate the MJD and HSV calibration process and results.

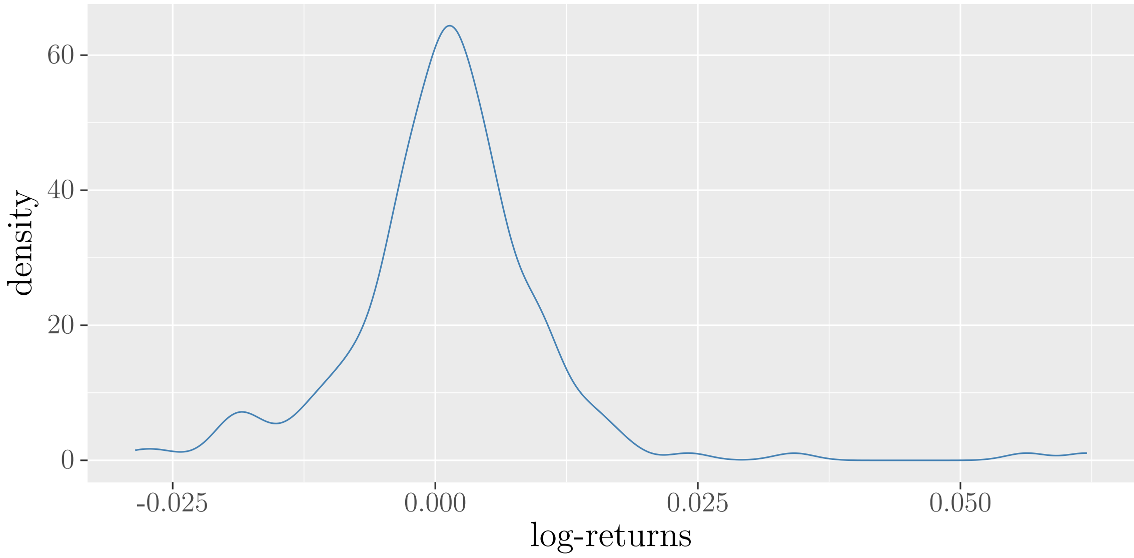
5.2.2 Asset prices based calibration

In order to calibrate the parameters to pass to the MJD and HSV model, to generate all the dummy times series that will serve for the analysis of the delta hedging, I am going to use the historical market data on the Apple stock as a template.

To perform such an upstream analysis, I will use an approximation method to estimate the arguments with which the distribution of the log-returns generated by both MJD and HSV models better fit the density curve of the historical log-return. The list of arguments to estimate will be smaller than the one needed to option calibration because I will start with the set of the already assessed parameters from options data and only (re)calibrate those risk-neutralized.

To get the historical data, I will once more use the package *quantmod*, developed by Ryan and Ulrich [2018] which uses Yahoo! finance as a provider. The dataset so downloaded is available in appendix **??** and concerns the daily stock price from 18th May 2017 to 18th May 2018. figure 5.1 shows the density curve generated by the log-return of the historical data.

Figure 5.1: Historical Apple stock Log-returns distribution



The above densities function is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018.

The procedures to calibrate the arguments for MJD and HSV processes are respectively explained at sections 5.2.3 and 5.2.4.

5.2.3 Merton's model calibration

Calibration of parameters for option pricing

In order to use the function *mjd_call* based on equation (4.18) to compute the price of European call options on an underlying with increments driven by the MJD model, the parameters $\{\lambda, \mu, \delta, \sigma\}$ must be calibrated with respect to the available option market data. Consequently, the so estimated parameters will be in line with the measures that make equation (4.1) risk-neutral.

To do so, the aforementioned least-square non-linear analysis will be used together with data on Apple call option as a template. In that respect, the theoretical models to calibrate are given by equations (4.18) to (4.20) along with equation (4.21), implemented by the R function *mjd_call*.

Moreover, to stay in a range of acceptable values, the parameters of the MJD model should lie between some defined boundaries.

Indeed, σ and δ are going to range between 0 and 1. Otherwise, no limit is fixed for both μ and λ, letting the frequency of jumps and their average magnitudes fully free. Those boundaries are chosen to minimize the computational iterations and make the least-square non-linear analysis more effective while letting it with enough amplitude in its possibilities.

According to those constraints and due to the presence of multiple local minima, the *lsqnonlin* function returns the **??** as the whole sets of parameters that make the function *mjd_call* better fit with reality.

Table 5.4: Best estimates for MJD call option model

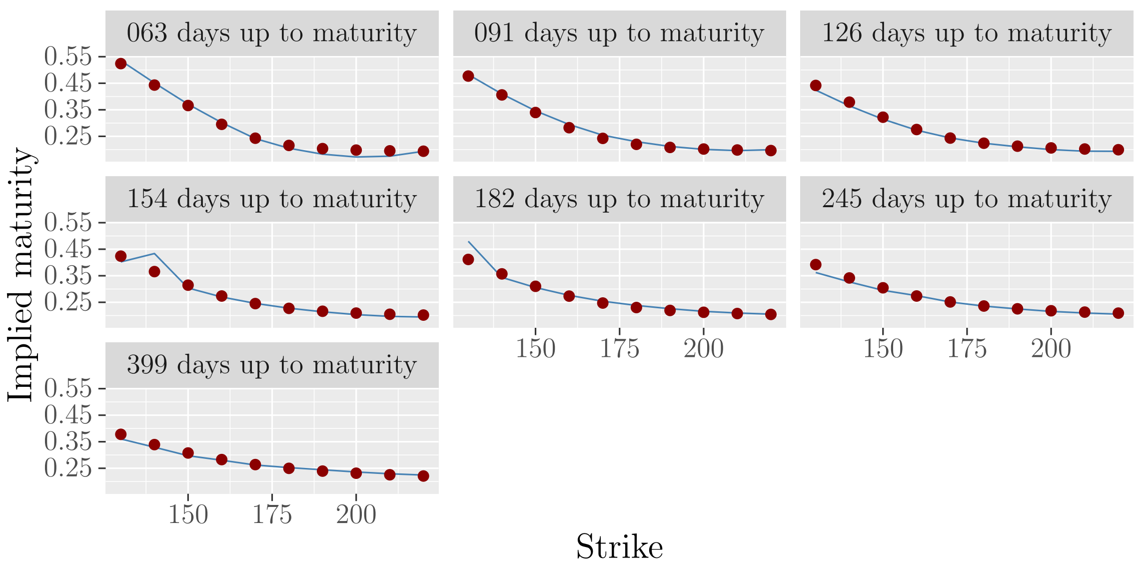
lambda	mu	delta	sigma
0.07915	-0.37583	0.08771	0.18651
0.11606	-0.30724	0.09993	0.18445
0.14411	-0.24174	0.12023	0.18676
0.06839	-0.37680	0.18072	0.18870
0.10312	-0.29738	0.19545	0.18583
0.05698	-0.38007	0.28030	0.19043
0.08216	-0.28573	0.27560	0.18822

Otherwise, The set 5.3 is the one making respond the model the best with what is observed in reality. Indeed when passing that set to the R function *call_merton*, for all strikes of table 5.3 and maturities of table 5.2, more than 68% of the so generated prices are within the bid-ask spread of the Apple option historical data.

$$\left\{\begin{array}{rcl} \lambda & = & 0.1031218, \\ \mu & = & -0.2973769, \\ \delta & = & 0.1954519, \\ \sigma & = & 0.1858289 \end{array}\right\} \tag{5.3}$$

?? confronts the blue colored volatility smiles computed from market data, with those dotted in red, calculated from the function *mjd_call*, which takes the items of the set 5.3 as parameters.

Figure 5.2: Implied volatility of Apple option prices computed with MJD



The blue curves represent the implied volatility computed from the option market data on Apple while the ....

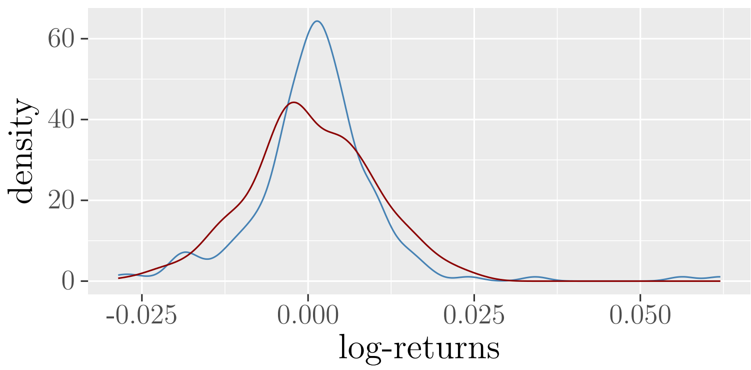
Consequently, the set 5.3 is the one I will use together with the *mjd_call* function whenever I have to compute option price using the MJD model on Apple data for the purpose of that master thesis.

Calibration of parameters for time series

The set 5.3 is calibrated under the risk-neutral world and cannot, therefore, be used as is to simulate the dummy time-series that will be used in the analysis aimed to measure the delta hedging performance. Furthermore, in addition to those parameters, the drift rate α has to be estimated as well. It was not present in the set 5.3 because it is not taken into account in the computation of the options prices.

Figure 5.3 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *mjd_ts()* fed with the risk-neutral parameters5.3, for the red one.

Figure 5.3: Historical and MJD related Apple stock Log-returns distribution



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series genereated by the function *hsv_ts* taking the risk-neutral parameters 5.3 as arguments.

The goal of that section is to find the risk-averse parameters that make both the distributions of the log-returns generated from market data or from *mjd_ts* fit together.

Practically, to do so, I am using an approximation algorithm developed by Venables and Ripley [2002] which is directly available in the R language through the function *fitdistr* from the R package *MASS*. The goal of that algorithm is the find the parameters of a density function that make it reproduce the distribution of a sample of data. To do the job, that algorithm needs (i) a density function to be fitted along with (ii) a sample of random data as a template. As defined in Matsuda [2004], the density of the log-returns generated by the MJD model is given by **??**.

$$P_t(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\phi x + F_t(\phi)} d\phi \tag{5.4}$$

where

$$F_t(\phi) = \frac{\kappa\theta}{\sigma^2}\gamma t - \frac{2\kappa\theta}{\sigma^2}\ln\left(\cosh\frac{\Omega t}{2} + \frac{\Omega^2 - \gamma^2 + 2\kappa\gamma}{2\kappa\Omega}\sinh\frac{\Omega t}{2}\right)$$

and

$$\Omega = \sqrt{\gamma^2 + \sigma^2(\phi^2 - i\phi)}, \gamma = \kappa + i\rho\phi\sigma$$

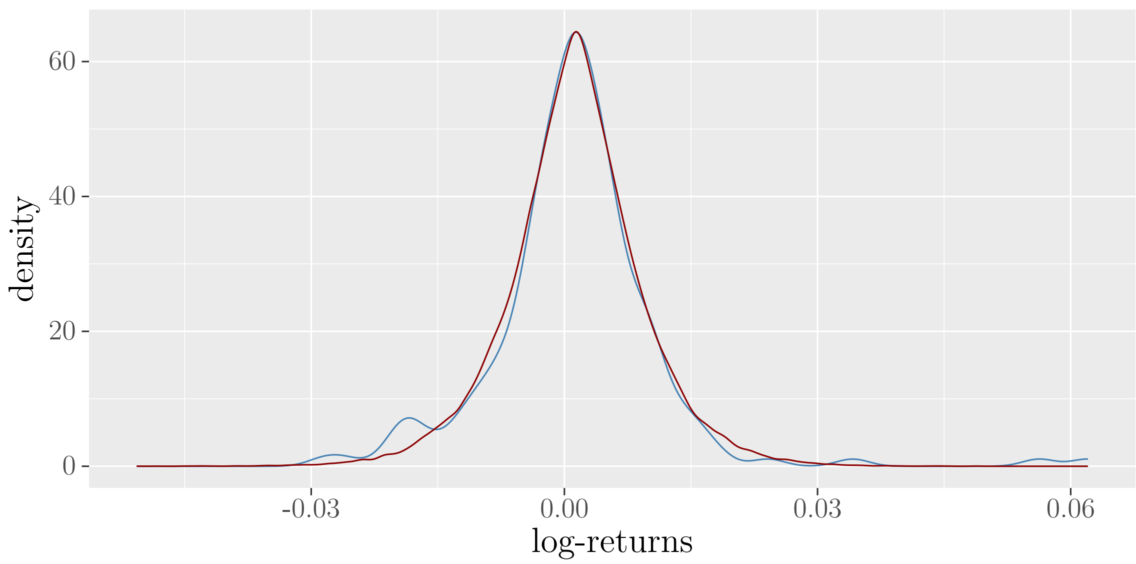
The parameters to be guessed are....

By applying the optimization function together with **??** as a density function, the Apple stock data [REF] as a template and the variable λ as a cursor, the optimization algorithm outputs $\lambda = 5.4883278229$ as the best fit argument for the risk premium. Therefore, the set of the risk-averse parameters is fully described here below in 5.8.

$$\left\{ \begin{array}{lcl} V(0) & = & 0.03798218, \\ \theta & = & 0.02054013, \\ \sigma & = & 0.50378803, \\ \rho & = & -0.39877827, \\ \kappa & = & 9.489383, \\ \alpha & = & 0.4822917 \end{array} \right\} \tag{5.5}$$

Figure 5.7 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *hsv_ts()* fed with the risk-averse parameters5.8, for the red one.

Figure 5.4: Historical and HSV related Apple stock Log-returns distribution in the risk-averse world



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series genereated by the function *hsv_ts* taking the risk-averse parameters 5.8 as arguments.

Consequently, the set 5.8 is the one I will use together with the *hsv_ts()* function whenever I have to compute time series using the HSV model within the real world constraints for the purpose of that master thesis.

5.2.4 Heston's model calibration

Calibration of parameters for option pricing

In order to use the function *hsv_call* based on equation (4.18) to compute the price of European call options on an underlying with increments driven by the HSV model, the parameters $(V(0), \kappa, \theta, \sigma, \rho)$ must be calibrated with respect to the available option market data. Consequently, the so estimated parameters will be in line with the risk-neutral measure, or more specifically κ and θ, which are the only ones adapted to make equation (4.15) risk-neutral.

To do so and similarly as it was done for the MJD option model, the least-square non-linear analysis will be used together with data on Apple call option as a template. In that respect, the theoretical models to calibrate are given by equations (4.18) to (4.20) along with equation (4.23), implemented by the R function *hsv_call*.

Moreover, to stay in a range of acceptable values, the parameters of the HSV model should lie between some defined boundaries.

De facto, the mean-reversion speed (κ), the long-run variance (θ) and the volatility of the volatility (σ) need to take values that together respect the Feller's condition (see section 4.3.2). Additionally, σ should range between 0 and 1 and, according to **?** κ should be positive to avoid mean aversion. Although the correlation coefficient ρ may take any value from $[-1, 1]$, typically the correlation between the stock price increments and their intrinsic volatility is negative. Consequently, I set the borders of ρ as being $]0, -1[$.

According to those constraints and due to the presence of multiple local minima, the *lsqnonlin* function returns the table 5.5 as the whole sets of parameters that make the function *hsv_call* better fit with reality.

Table 5.5: Best estimates for HSV call option model

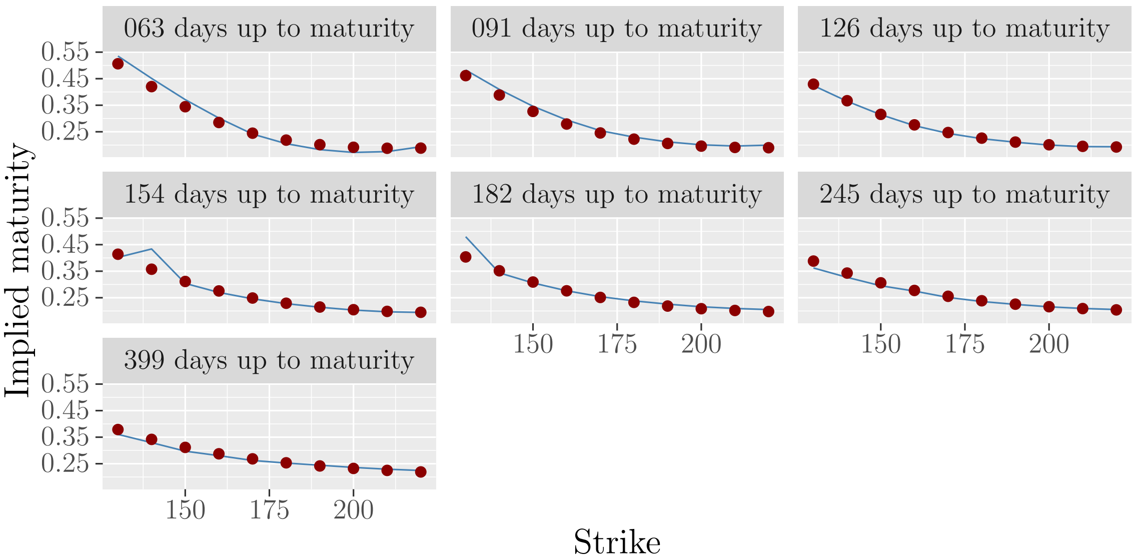
v0	theta	sigma	rho	kappa
0.03835	0.05144	0.30882	-0.40228	1.99821
0.03851	0.05142	0.29539	-0.49338	1.99931
0.03850	0.05241	0.30601	-0.59979	1.99843
0.03774	0.04832	0.30687	-0.40213	3.00036
0.03827	0.04954	0.40690	-0.40189	3.00011
0.03883	0.04798	0.30747	-0.50211	3.00015
0.03910	0.04948	0.40736	-0.50193	3.00005
0.03695	0.04703	0.30697	-0.40288	4.00092
0.04449	0.04443	0.40682	-0.40289	4.00079
0.03798	0.04872	0.50379	-0.39878	4.00106
0.04091	0.04651	0.40705	-0.50227	4.00097
0.03740	0.04754	0.30583	-0.60126	4.00096
0.03890	0.04792	0.40565	-0.60086	4.00091
0.03661	0.04604	0.30659	-0.40295	5.00152
0.04034	0.04547	0.40591	-0.40298	5.00153
0.04111	0.04558	0.50652	-0.40298	5.00144
0.04034	0.04720	0.60647	-0.40250	5.00138
0.03921	0.04709	0.50753	-0.50170	5.00152
0.03789	0.04656	0.30488	-0.80072	5.00164

Otherwise, The set 5.6 is the one making respond the model the best with what is observed in reality. Indeed when passing that set to the R function *call_heston*, for all strikes of table 5.3 and maturities of table 5.2, more than 70% of the so generated prices are within the bid-ask spread of the Apple option historical data.

$$\left\{ \begin{array}{lcl} V(0) & = & 0.03798218, \\ \theta & = & 0.04871543, \\ \sigma & = & 0.50378803, \\ \rho & = & -0.39877827, \\ \kappa & = & 4.00105546 \end{array} \right\} \tag{5.6}$$

figure 5.5 confronts the blue colored volatility smiles computed from market data, with those dotted in red, calculated from the function *hsv_call*, which takes the items of the set 5.6 as parameters.

Figure 5.5: Implied volatility of Apple option prices computed with HSV



The blue curves represent the implied volatility computed from the option market data on Apple while the ....

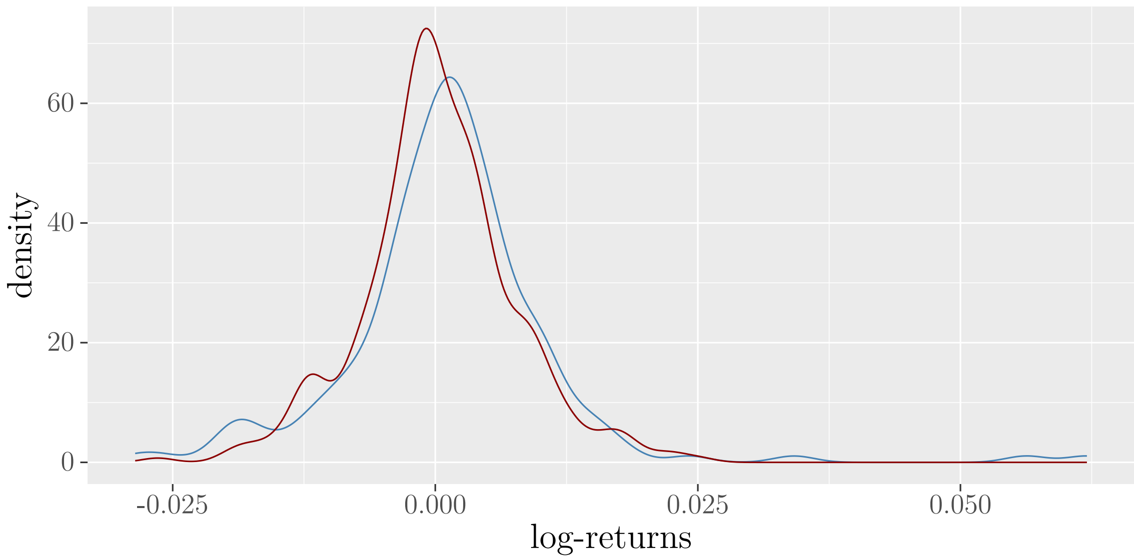
Consequently, the set 5.6 is the one I will use together with the *hsv_call* function whenever I have to compute option price using the HSV model on Apple data for the purpose of that master thesis.

Calibration of parameters for time series

The set 5.6 is calibrated under the risk-neutral world and cannot, therefore, be used as is to simulate the dummy time-series that will be used in the analysis aimed to measure the delta hedging performance. Furthermore, in addition to those parameters, the drift rate α has to be estimated as well. It was not present in the set 5.6 because it is not taken into account in the computation of the options prices.

Figure 5.6 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *hsv_ts()* fed with the risk-neutral parameters5.6, for the red one.

Figure 5.6: Historical and HSV related Apple stock Log-returns distribution



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series genereated by the function *hsv_ts* taking the risk-neutral parameters 5.6 as arguments.

The goal of that section is to find the risk-averse parameters that make both the distributions of the log-returns generated from market data or from *hsv_ts* fit together. To do so, and according to the differences between equations (4.14) and (4.15) from equations (4.9) and (4.10), the parameters to modify are the drift rate $r \to \alpha$, the mean reversion speed $\kappa^* \to \kappa$ and the long-run volatility $\theta^* \to \theta$. Accordingly, the correlation parameter ρ and the volatility of the volatility σ stay unchanged from the risk-neutral to risk-averse world.

I followed the method exhibited at section 1.6 to get a rough estimation of the drift rate α. From the log-returns' first and second moments, I find the value 0.4822917 for the estimate of α.

According to equations (4.16) and (4.17), in order to calculate κ and θ, the risk premium λ has to be guessed. Practically, to do so, I am using an approximation algorithm developed by Venables and Ripley [2002] which is directly available in the R language through the function *fitdistr* from the R package *MASS*. The goal of that algorithm is the find the parameters of a density function that make it reproduce the distribution of a sample of data. To do the job, that algorithm needs (i) a density function to be fitted along with (ii) a sample of random data as a template. As defined in Dragulescu and Yakovenko [2002], the density of the log-returns generated by the HSV model is given by equation (5.7).

$$P_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\phi x + F_t(\phi)} d\phi \tag{5.7}$$

where

$$F_t(\phi) = \frac{\kappa\theta}{\sigma^2}\gamma t - \frac{2\kappa\theta}{\sigma^2} \ln\left(\cosh\frac{\Omega t}{2} + \frac{\Omega^2 - \gamma^2 + 2\kappa\gamma}{2\kappa\Omega}\sinh\frac{\Omega t}{2}\right)$$

and

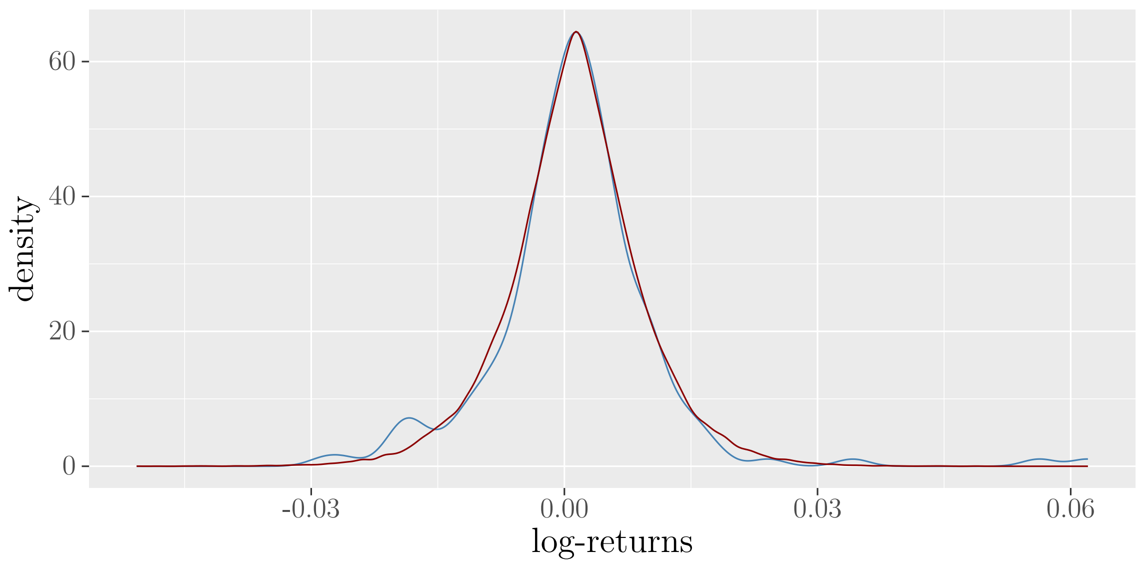
$$\Omega = \sqrt{\gamma^2 + \sigma^2(\phi^2 - i\phi)}, \gamma = \kappa + i\rho\phi\sigma$$

By applying the optimization function together with equation (5.7) as a density function, the Apple stock data [REF] as a template and the variable λ as a cursor, the optimization algorithm outputs $\lambda = 5.4883278229$ as the best fit argument for the risk premium. Therefore, the set of the risk-averse parameters is fully described here below in 5.8.

$$\left\{ \begin{array}{lcl} V(0) & = & 0.03798218, \\ \theta & = & 0.02054013, \\ \sigma & = & 0.50378803, \\ \rho & = & -0.39877827, \\ \kappa & = & 9.489383, \\ \alpha & = & 0.4822917 \end{array} \right\} \tag{5.8}$$

Figure 5.7 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *hsv_ts()* fed with the risk-averse parameters5.8, for the red one.

Figure 5.7: Historical and HSV related Apple stock Log-returns distribution in the risk-averse world



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series genereated by the function *hsv_ts* taking the risk-averse parameters 5.8 as arguments.

Consequently, the set 5.8 is the one I will use together with the *hsv_ts()* function whenever I have to compute time series using the HSV model within the real world constraints for the purpose of that master thesis.

5.3 Delta hedging

In order to measure the hedging performance of the MJD and HSV models, I will (i) compute the prices of some options and then virtually sell them at that price, (ii) construct the delta-neutral portfolio at time zero and thereafter continuously rebalance it, in accordance with the underlying asset price evolutions and (iii) measure the overall cost of the hedge as a performance indicator.

5.3.1 Determination of the option price

The computation of the option price at time zero will depend on the underlying model to be hedged. Indeed, the function *mjd_call* will be used to price options for which the underlying asset price evolution is driven by the MDJ model, whereas *mjd_call* will be used for HSV processes.

As explain as an introduction, I will virtually sell the options I price. Therefore, as the writer of the options, I will raise \$ $X(0) = C_K^{m/h}(0)$. That cash will directly be put into a money market account to make it grow at a constant riskless rate all along the duration of the hedging process, namely, during the whole life of the option.

The T-bills will serve as riskless rate. According to the available data on `https://www.treasury.gov/`, quotes of these are given by table 5.6.

Table 5.6: Treasury bill quotes on 18 May 2018

T-bills			
4 weeks	13 weeks	26 weeks	52 weeks
1.66%	1.90%	2.09%	2.30%

The rates of table 5.6 are annual-base, consequently to get their countinuously compounded counterpart, EQUATION X can be used to give those given by table 5.7.

Table 5.7: Treasury bill quotes on 18 May 2018

T-bills			
4 weeks	13 weeks	26 weeks	52 weeks
1.659%	1.896%	2.068%	2.274%

Thereafter by applying the bootstrap method along with linear interpolation, one can find the maturity-wise related riskless rates of the Apple call option used for the current analysis. Those rates are given by table 5.8.

Table 5.8: Maturities explored during the hedging performance measurement

Maturity(in days)	63	91	126	154	182	245	399
Riskless rate	1.817%	1.896%	1.962%	2.015%	2.068%	2.139%	2.311%

5.3.2 Construction of the delta-neutral portfolio

At time zero, the built of the delta-neutral portfolio is achieved by the purchase of $\Delta(0)$ share of stock multiplying by the number of assets involving in the derivative product.

Simply put, if one wants to construct such a portfolio to cover against the possible losses incurred by the selling of a call option on one hundred shares, the number of underlying to buy at that time would be of $\Delta(t) \times 100$. For the sake of clarity, in the analysis I will perform, it is implicit that any European call option involves the purchase of only one underlying at maturity, though. Consequently, at time zero, the cost of such a portfolio is given by equation (5.9).

$$p(t_0) = \Delta^{m/h}(t_0)S(t_0) \tag{5.9}$$

Where the superscripts m (Merton) and h (Heston) respectively stand for the computation of MJD and HSV delta.

The objective of that portfolio is to accurately replicate the reverse position taken in a derivative, i.e., to reproduce a position in a long European call in the case that matters for that analysis. To be efficient in that role, it must be frequently rebalanced.

Accordingly, in order to do so, I will observe the underlying's time-series at every period of time $\delta t = 1 days$ and compute the updated value of $\Delta(t)$ at each of those observational timesteps. Therefore, the requisite expenses aimed to keep the portfolio delta-neutral will evolve over time, as shown here below by equation (5.10).

$$p(t_i) = \Delta^{m/h}(t_i - t_{i-1})S(t_i), \forall i \in \mathbb{Z} : i \in \{1, T\} \tag{5.10}$$

The aforementioned time-series that will be daily scrutinized are obviously hypothetical. They will be generated by the functions *mjd_ts* and *hsv_ts* for which the risk-averse parameters will be passed in.

5.3.3 Computation of the delta

By definition, the delta is the first derivative of the option's pricing function with respect to the stock price. It consequently, represents the instantaneous rate of change in the option value as the price of its underlying evolves, such so illustrated by equation (3.10).

Although Black and Scholes [1973] gives to delta the value $N\left(d_+\left(\Delta t, x\right)\right)$, with d_+ explained by equation (3.9), latter can only be used within the framework developed by Black and Scholes [1973], that is, if the underlying is driven by a geometric Brownian motion, the volatility is deterministic, or if no jump occurs. Accordingly, the solution to compute the delta provided by Black and Scholes [1973] cannot be used for such the MJD and HSV models.

By following the definition of delta and because the pricing function of a European call option, for which the underlying asset price evolution is driven by either the MDJ or HSV models, is given by equation (4.18), then, the equation (5.11) gives the solution to compute the delta for such models.

$$\Delta^{m/h}\left(t, S(t)\right) = P_1^{m/h} + S(t)\frac{\partial P_1^{m/h}}{\partial S(t)} - e^{-r(T-t)}K\frac{\partial P_2^{m/h}}{\partial S(t)} \tag{5.11}$$

where

$$\frac{\partial P_1^{m/h}}{\partial S(t)} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{i\phi\exp\left(-i\phi\ln K\right)}{-\phi^2\psi^{m/h}(-i)^2}\right.$$
$$\left.\times\left(\frac{\partial\psi^{m/h}(\phi-i)}{\partial S(t)}\psi^{m/h}(-i)-\psi^{m/h}(\phi-i)\frac{\partial\psi^{m/h}(-i)}{\partial S(t)}\right)\right]d\phi \quad (5.12)$$

and

$$\frac{\partial P_2^{m/h}}{\partial S(t)} = \frac{1}{\pi}\int_0^\infty Re\left(\frac{\exp\left(-i\phi\ln K\right)}{i\phi}\frac{\partial\psi^{m/h}(\phi)}{\partial S(t)}\right)d\phi \quad (5.13)$$

Hence, the equation (5.11) is the one to solve for both MJD and HSV models to find them the delta at time t. Equations (5.12) and (5.13) can be solved by finding the first derivatives of the characteristic functions of MJD and HSV processes with respect to the stock price. By the way, it turns out that both of these have the same form. Equation (5.14) gives the solution of these derivatives, the term $\psi^{m/h}(x)$ have to be replaced by the appropriate characteristic function.

$$\frac{\partial\psi^{m/h}(x)}{\partial S(t)} = \psi^{m/h}(x)\frac{ix}{S(t)} \quad (5.14)$$

By replacing all the derivative operations inside equations (5.12) and (5.13) by equation (5.14), one finds the solved version given by equations (5.15) and (5.16).

$$\frac{\partial P_1^{m/h}}{\partial S(t)} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{\exp\left(-i\phi\ln K\right)\psi^{m/h}(\phi-i)}{S(t)\psi^{m/h}(-i)}\right]d\phi \quad (5.15)$$
$$\frac{\partial P_2^{m/h}}{\partial S(t)} = \frac{1}{\pi}\int_0^\infty Re\left[\frac{\exp\left(-i\phi\ln K\right)\psi^{m/h}(\phi)}{S(t)}\right]d\phi \quad (5.16)$$

Consequently, in the extent that the term $\psi^{m/h}(\phi)$ is replaced by the appropriate characteristic function, substituting equations (5.16) and (5.16) into 5.11 gives 5.17 that allows the computation of the delta of an option for which the underlying asset is driven by either a MJD or HSV process.

$$\Delta^{m/h}\left(t,S(t)\right) = P_1^{m/h} + \frac{S(t)}{\pi}\int_0^\infty Re\left[\frac{\exp\left(-i\phi\ln K\right)\psi^{m/h}(\phi-i)}{S(t)\psi^{m/h}(-i)}\right]$$
$$-\frac{e^{-r(T-t)}K}{\pi}\int_0^\infty Re\left[\frac{\exp\left(-i\phi\ln K\right)\psi^{m/h}(\phi)}{S(t)}\right] \quad (5.17)$$

In that respect, the R functions *mjd_delta* and *hsv_delta*, respectively compute the MJD and HSV related deltas at time t by taking the risk-neutral parameters.

5.3.4 Measuring the hedging performation

Finally, to analyze how much the hedging works, one method is to measure its overall cost. To do so, the really first thing is to value at maturity all the buying and selling operations carried out in order to keep delta-neutral the previously constructed portfolio and add them together.

$$p_{tot} = \sum_{i=0}^{T} p(t_i)e^{T-t_i}, \forall i \in \mathbb{Z} : i \in \{1, T\} \tag{5.18}$$

On the other hand, at maturity, the amount of money $X(0)$ received when entering into the contract at time zero will have continuously grown up at the constant riskless rate r, as described by equation (5.19).

$$X(T) = X(0) * e^{rT} \tag{5.19}$$

Incidentally, if the option is in-the-market at maturity date, it first implies that it will be exercised, so increasing the wealth of the short position in the derivative by K, and then that $\Delta(T) = 1$, meaning that the writer of the option has exactly one share of stock in his / her portfolio to honour the transaction. Otherwise, the option is out-of-the-market, inducing that no transaction occurs and that $\Delta(T) = 0$. Consequently, at maturity, someone with a short position in the derivative, i.e., the one tempting to be replicated in the current analysis, will receive a positive cash inflow of $X(T) + \Delta(T)K$.

Hence, if the hedge works well, the relation exhibited by 5.20 should be respected.

$$\sum_{i=0}^{n=T} P(i) \cong X(T) + \Delta(T) * k \tag{5.20}$$

Because the position could vary, we have to adapt the relation above (REF) in order to remove all unit from the comparison. According to hull, a measure of performance of the hedge could be to take the ratio of the standard deviation of the cost to hedge to the option price. In that case, the measure could be comparated among themselve even if the position taken into the underlying difer.

$$\frac{sd(hedge)}{C_0 e^{rT}} \tag{5.21}$$

where hedge is a vector containing n sample generated, and C_0 is the price of the call option at time zero value up to maturity.

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