Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How perfomant is it ? What about other model such as . . . ?

Using R. R Core Team [2017]

Chapter 1

Upstream concepts

1.1 Overview

The material covered in the current chapter is meant to be intensively used further, either in theoretical models or custom–produced algorithm. A few of them have for the only purpose to introduce some concepts subsequently developed in the current master thesis' framework, though.

1.2 Vanilla options

The so–called vanilla options, in opposition to the more complex – not covered in this master thesis – exotic ones, are specifics kind of derivatives coming along with some good–to–know jargons.

First and foremost, an option is a contract between two stakeholders with different interests or at least distinctive motivations, who want to buy or sell a product, broadly a financial asset called the underlying. Indeed, one can look for hedge oneself against risk while the other party wants to make a profit on a speculative move. Likewise other financial contracts, one agrees to buy and the other to sell at a fixed amount of money, namely the strike price denoted by k. The main difference being that they would effectively complete the purchase or the sale of the underlying at a future date than the one they enter into the bargain.

Another key characteristic of options is that they are not symmetric contracts, in the sense that both parties do not have the same rights, depending on their position along with the option's type. Two positions can be taken when entering into such a derivative, long or short. Going short roughly means purchase the option whereas going long implies writing, or synonymously selling, the derivative. Beyond the position, the contract can be of call or put type. The following sentences encompass a unique mix of type–position combination that gives an overview of all the possible scenarios:

- Someone who is going long into a call gains the right to purchase the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.

- Someone who is going short into a call is forced to sell the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.

- Someone who is going long into a put gains the right to sell the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.

- Someone who is going short into a put is forced to buy the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.

Moreover, vanilla options may be European or American. The latter can be exercised at any time during the whole life of the contract while the European can only be at maturity.

Finally, the payoff provided by all aforementioned European options are summarized throughout equations (1.1) to (1.4), where C_l, C_s, P_l and P_s respectively stand for the long call, short call, long put and short put payoffs. $S(T)$ being the price of the underlying at maturity.

$$C_l = (S(T) - k)^+ \tag{1.1}$$
$$C_s = (k - S(T))^- \tag{1.2}$$
$$P_l = (k - S(T))^+ \tag{1.3}$$
$$P_s = (S(T) - k)^- \tag{1.4}$$

The derivative considered in this master thesis document is mainly the European call option, even though the link with European put could somehow be done without being mandatory to exhibit the issue here raised. Therefore such reference to put should only be done when necessary.

1.3 Brownian Motion

All over the present document, Brownian motion, Wiener / Markov processes are considered to be equivalent terminology and therefore used as such. However strictly speaking, even though the Brownian motion is in every respect a Wiener process, stricto sensu, the term Markov process is broader. Indeed, the more remarkable Markov property is to be a process with independent future increments, putting it in line with the weak form of market efficiency (Hull and Basu [2012]). Whilst, on the other hand, even if the Brownian motion, and thus the Wiener process, share this property, they are defined with the idiosyncrasy to have mean zero and a variance rate of one per unit of time (Hull and Basu [2012]).

Brownian motion is a masterpiece broadly used subsequent development. It is notably involved in many algorithms such as the ones for geometric Brownian motion [REF], Black–Scholes–Merton equation [REF], jump–diffusion process [REF] and Heston [REF].

Brownian motion is a continuous function $W(t)$ satisfying $W(0)$ with independent and identically distributed (iid) increments, characterized such as the following equation (1.5)

$$W(t_{i+1}) - W(t_i) \sim iidN\left(0, t_{i+1} - t_i\right) \tag{1.5}$$

This Markov process is qualified as being time and path dependent in its building blocks. Time dependency implies that likewise other functions its value evolves over time. However, the Wiener process is meant to randomly move between two time–steps as well, which directly lie on the path dependency. Practically at any time t, the Brownian motion may take any value as equation (1.6) shows.

$$W(t) \sim N(0, t) \tag{1.6}$$

The construction of such a Markov process could be achieved following various techniques. Either by computing the differential process for every time-steps and then combining it. This procedure gives one possible path of the Brownian motion (see equations (1.7) and (1.8)). Or either by constructing the joint moment generating function of the iid random variables' vector $(W(t_i) \ldots W(t_m))$, following equation (1.9).

$$dW(t) = \phi(0, t + \epsilon_i) \tag{1.7}$$

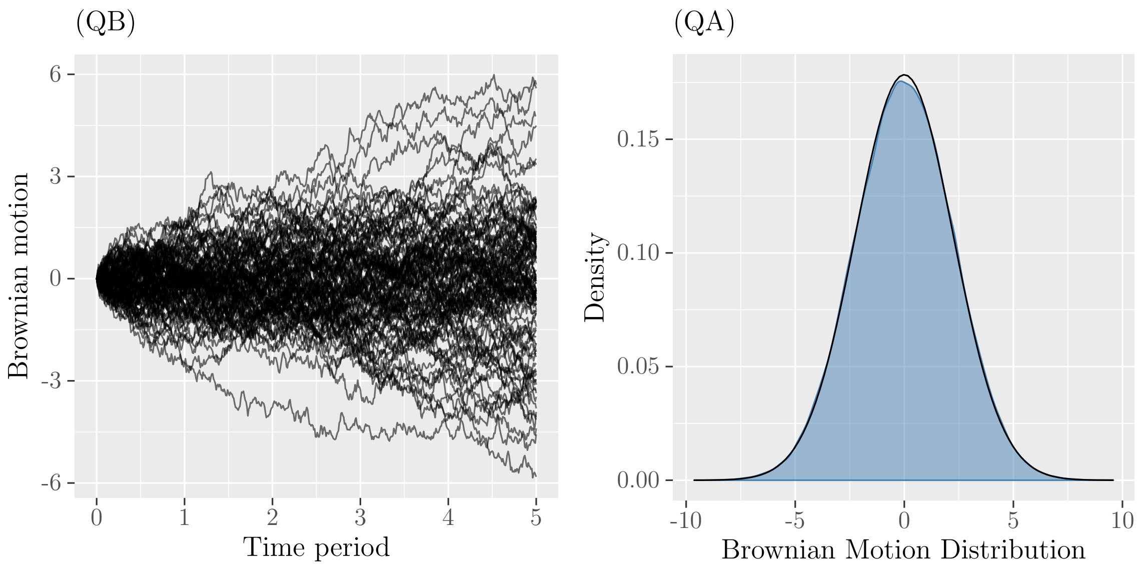
ϕ is a function that generate a random number according to the normal law with parameter $\mu = 0$ and $\sigma^2 = t + \epsilon_i$, with ϵ_i being any arbitrary small delta time step.

$$W(t + \epsilon_i) = W(t) + dW(t) \tag{1.8}$$

$$\varphi(u_1 \ldots u_m) = \exp\left(\frac{1}{2}\sum_{i=1}^{m}\left[\sum_{j=i}^{m} u_j\right]^2 (t_i - t_{i-1})\right) \tag{1.9}$$

Figure 1.1 (QA) displays a simulation of one hundred Brownian motions over a time frame of five years. While in turn figure 1.1 (QB) shows how the experiment's outcomes are distributed after five years, with characteristics such as shown by equation (1.5). The black outlined curve is the density of the normal distribution.

Figure 1.1: Multiple Brownian Motion



Note. (QA): Simulation of one hundred Brownian motions using the R package Tedde [2017] with a timeframe of five units (five years) with a time-step of one over one hundred. (QB): Density computed on a simulation of one hundred thousand Brownian motions. The calculation has been made by taking all the final value $W(5)$ and by plotting them as a density function. The Black bell curve is the normal density with mean zero and a standard deviation of square root of five, as expected (**??**).

Lastly, a Brownian motion has no tendency to rise or fall as time passes thanks to its martingale property. Therefore, because its initial value is zero, a Wiener process only brings noise by being incorporated into other stochastic series, such as those proposed further on.

1.3.1 Correlated Brownian Motions

Correlated Brownian motions are Wiener processes that related together according to the factor ρ equation (1.10). Following Shreve [2004], these processes happens to be modeled by equations (1.11) and (1.12).

$$dB_1(t)dB_2(t) = \rho(t)dt \tag{1.10}$$

Where $dB_1(t)$ and $dB_2(t)$ are Wiener process

$$B_1(t) = W_1(t) \tag{1.11}$$

$$B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1-\rho^2(s)}dW_2(s) \tag{1.12}$$

When ρ is constant over time however, the simple form of equation (1.13) is enough.

$$B_2(t) = \rho W_1(s) + \sqrt{1-\rho^2}W_2(s) \tag{1.13}$$

1.4 Ito's lemma

Based on Taylor approximation theorem, Ito formulae are meant to provide the differential form of such a function $f \circ g$ where the function f can be differentiate with respect to its independent variable whilst g cannot.

$$f(T, W(T)) = f(0, W(0)) + \int_0^T \frac{\partial f(t, W(t))}{\partial t}dt + \int_0^T \frac{\partial f(t, W(t))}{\partial W(t)}dW(t) + \frac{1}{2}\int_0^T \frac{\partial^2 f(t, W(t))}{\partial W(t)^2}dt \tag{1.14}$$

Whilst the equation (1.14) shows the formula for Brownian motion, the **??** represents the form used to differentiate the more complex Ito process (1.15).

$$X(T) = X(0) + \int_0^T \Delta(u)dW(u) + \int_0^T \Theta(u)du \tag{1.15}$$

Where $\Delta(u)$ and $\Theta(u)$ are stochastic process, adapted to a filtration f(t)

$$f(T, X(T)) = \dots \tag{1.16}$$

Ito formulae it thereafter applied to derive such a process like the geometric Brownian motion [REF] or the Black–Scholes–Merton partial differential equation.

1.5 Cox–Ingersoll–Ross

The Cox–Ingersoll–Ross (CIR) stochastic model $R(t)$, defined by the differential equation (1.17)), can be used to simulate interest rates' evolution over time thanks to some of its related properties. Indeed it gives a good fit owing to two of them, namely the mean–reversion and its ability to only be positive (Shreve [2004]).

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t) \tag{1.17}$$

A mean–reverting stochastic process tends to navigate, such a pointy sinusoidal motion, around its mean (see [FIGURE]). The CIR inherits this behaviour from the construction of its differential's drift part, i.e. $(\alpha - \beta R(t))dt$. Indeed when $R(t) = {}^{\alpha}/_{\beta}$, then the drift term $dt = 0$ with the consequence of status quo. In addition, whether $R(t) > {}^{\alpha}/_{\beta}$ or $R(t) < {}^{\alpha}/_{\beta}$, the next value of $R(t + \epsilon) = R(t) + dR(t)$ is pushed back toward ${}^{\alpha}/_{\beta}$. Actually, as showing by equation (1.18), the long–run expected value for the process $R(t)$ is ${}^{\alpha}/_{\beta}$ (Shreve [2004]).

$$\lim_{t\to\infty} \mathbb{E}\, R(t) = \frac{\alpha}{\beta} \tag{1.18}$$

On the other hand, the non-negativity property is explained by the fact that if $R(t) \to 0$ then $dR(t) \simeq \alpha dt > 0$ making $R(t + \epsilon)$ bounced off the x axis, running it away from negative realm (Shreve [2004]).

Ultimately though, the CIR mean–reverting equation (1.17) is used by Heston [1993] in its model in order to drive stochastic interest rates. Heston's model is covered throughout ??.

1.6 Skewness and Kurtosis

1.6.1 Definition

The skewness and kurtosis for a random variable's distribution, are respectivaly the third and fourth moment characterizing either the asymetry and the degree of flatness.

The theoretical moments are defined by equation (1.19) while the empirical ones can be estimated using equation (1.20), with $r = 3$ to compute the skewness or $r = 4$ for the kurtosis.

$$\gamma_r = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^r\right] \tag{1.19}$$

where X is any random variable

$$m_r = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^r \tag{1.20}$$

with $\{x_i\}_{i\in n}$ being a set of outcomes, belonging to the sample set.

1.6.2 Estimation

Sampled skewness and kurtosis formulae exist in different fashion with their precisions depending on the size of the sample along with the skewed of the theoretical distribution to be estimated (Joanes and Gill [1998]).

The chosen method to use to estimate the skewness and kurtosis needs a prudent selection especially for small sized sample. Actually, in order to minimize the mean–squared error and the associated variance, the equations (1.21) and (1.25) are used as an reliable unbiased estimator for sample with normal shaped theoretical distribution.

$$b_{skewness} = \frac{m_3}{S^3} \tag{1.21}$$
$$b_{kurtosis} = \frac{m_4}{S^4} \tag{1.22}$$

where

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{\mathrm{x}})^2 \tag{1.23}$$

Conversely, equation (1.24) and **??** are the ones providing a better unbiased estimation than the latters for more skewed distribution, such as for log–normal random variable's sample.

$$G_{skewness} = \frac{k_3}{\sqrt{k_2^3}} \tag{1.24}$$
$$G_{kurtosis} = \frac{k_4}{k_2^2} \tag{1.25}$$

where

$$K_2 = \frac{n}{n-1}m_2 \tag{1.26}$$
$$K_3 = \frac{n}{(n-1)(n-2)}m_3 \tag{1.27}$$
$$K_4 = \frac{n}{\prod_{i=1}^{3} n-i}\left[(n+1)m_4 - 3(n-1)m_2^2\right] \tag{1.28}$$

Consequently the algorithm to apply in order to estimate the skewness and kurtosis from samples should therefore be chosen with respect to the theoretical underpinned distribution.

1.7 Log–return and compounded interest rate

The concept of log–return and continuously compounded interest rates are closely related together as attested by equations (1.29) to (1.31)

$$S(0)e^{Rt} = S(t) \tag{1.29}$$
$$\Longleftrightarrow \ln S(0) + Rt = \ln S(t) \tag{1.30}$$
$$\Longleftrightarrow \ln\frac{S(t)}{S(0)} = Rt \tag{1.31}$$

Where R is the interest rate with continuous compounding, t denotes the time period across which the interest are compounded (in year) and the left–hand side of equation (1.31) stands for the natural logarithm of the stock return occuring during the period t, the so–called log–return.

For what matter in this current document, the focus is set to the sample's log–returns which are used to estimate the volatility term appearing, inter alia, in **??** to simulate geometric Brownian motion.

Following Hull and Basu [2012], if S_i and n respectively denote the stock price at the end of interval i and the total number of observation then the log–return of the ordered sample set $\{S - i\}_{i \in n}$ is give by equation (1.32).

$$u_i = \ln \frac{S_i}{S_{i-1}} \tag{1.32}$$

Whilst the estimation of the standard deviation of all the u_i is defined such as in **??**

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (u_i - \bar{u})^2} \tag{1.33}$$

$$\tag{1.34}$$

Consequently if the volatility of a stochastic process, simulatating a stock price motion, is given by $\sigma\sqrt{t}$, then it could therefore be estimated using market data throughout equation (1.35)

$$\hat{\sigma} = \frac{s}{\sqrt{t}} \tag{1.35}$$

1.8 Arbitrage strategy

The

Chapter 2

The underlying model

2.1 Overview

This chapter highlights one specific model used to model the motion that a stock price time serie would follow. It is the one used in the Black–Scholes–Merton model, Hull and Basu [2012].

Following Shreve [2004], The equation (2.1), called a geometric Brownian motion, features a stochastic process where the only random component is the Brownian motion $W(t)$. Whilst the others parameters, α and σ, stand respectively for the annualized expected return and volatility of the theoretical stock price process, Hull and Basu [2012].

$$S(t) = S(0)\, e^{\sigma W(t) + \left(\alpha - \frac{1}{2\sigma^2}\right)t} \tag{2.1}$$

In order to match with the constraints introduced by the Black and Scholes' pricing formula (section 3.2), the value the stock price may achieve at the end of a given period, according to constant mean rate of return and volatility, should be random and log–normally distributed (Black and Scholes [1973]).

Furthermore, the path following by any walks of the stock price motion should be a continuous process, as the time frame decrease $\Delta t \to 0$. Therefore in addition to develop the model, all the aforementioned characteristics are shown to be true within this chapter.

2.2 Derivation

In order to get the differential form of equation 2.1, the Itô's formula (section 1.4) is used. By applying the tranformation incurred by Itô, the equations (2.2a) to (2.2b) emerge:

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t) \tag{2.2a}$$

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) \tag{2.2b}$$

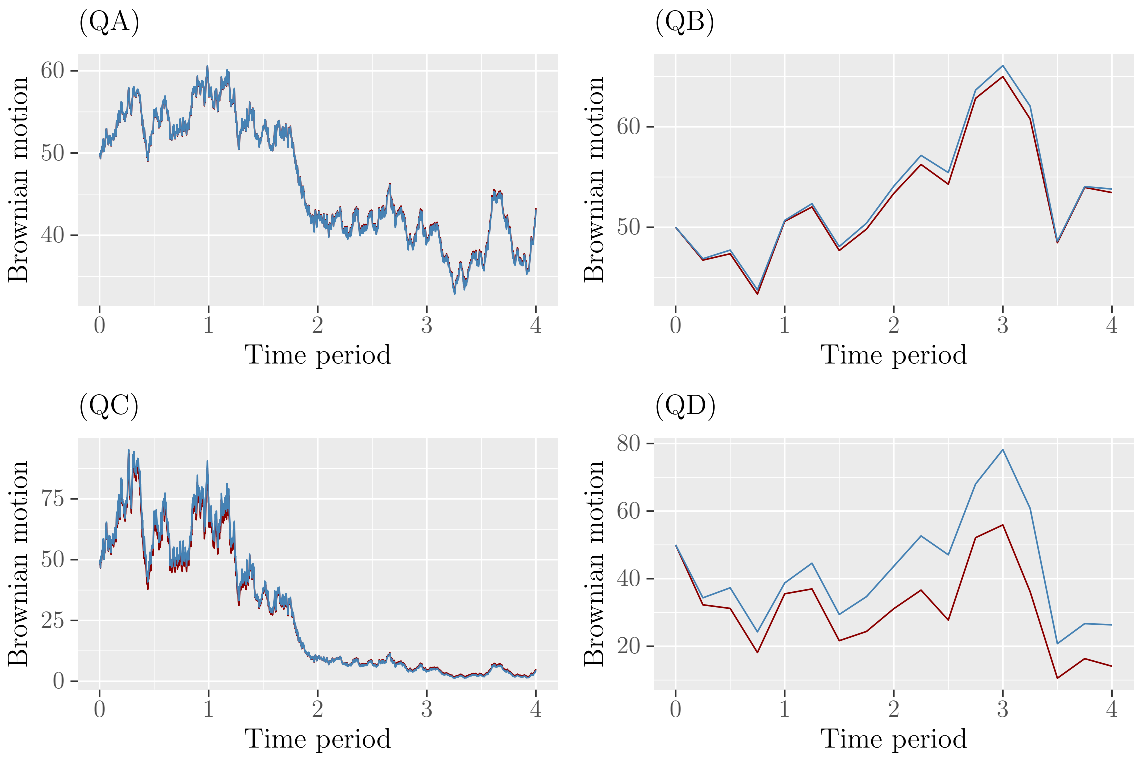
The equation (2.1) shows that any change occuring in the stock price S over a small amount of time is due to the deterministic expected drift rate ($\alpha S(t)$) along with some amounts of added noise included by the random part $(\sigma dW(t))$. The equation (2.1) denotes the stock return stochastic process, Hull and Basu [2012].

$$d \ln S(t) = (\alpha - \frac{\sigma^2}{2})dt + \sigma dW(t) \tag{2.3}$$

$$\ln \frac{S(t)}{S(0)} = (\alpha - \frac{\sigma^2}{2})t + \sigma W(t) \tag{2.4}$$

The **??** exibits the natual logarithm of the stock price return occuring all over the period t. In other words, the expectation of this process, given by $(\alpha - \frac{\sigma^2}{2}) \times t$ happens to be the expected value of the continously compounded rate of return for the aforementioned period of time, Hull and Basu [2012].

Figure 2.1: Accuracy of Itô approximation



The above processes have been constructed over four distinctive groups of two "equivalent" geometric Brownian motions, following equations (2.1) and (2.2a), respectively for the red and blue coloured path. The only parameters that change over the group are the couples (dt, σ) which are set to $QA \equiv (360, 0.2)$, $QB \equiv (4, 0.2)$, $QC \equiv (360, 1)$ and $QD \equiv (4, 1)$. All the processes cover a period of four years and begin with initial value set to 50.

The approximation of the stochastic process **??** achieved using the Itô's lemma depends exclusively on two factors; the volatility parameter σ and the time period occuring between two measures dt. As shown throughout the figure 2.1, the less the volatility or time step, the better the approximation. It is therefore key, during an analysis process to choose an appropriate time step according to a given volatility in order to provide accurate retults.

2.2.1 Distribution of the stock price process

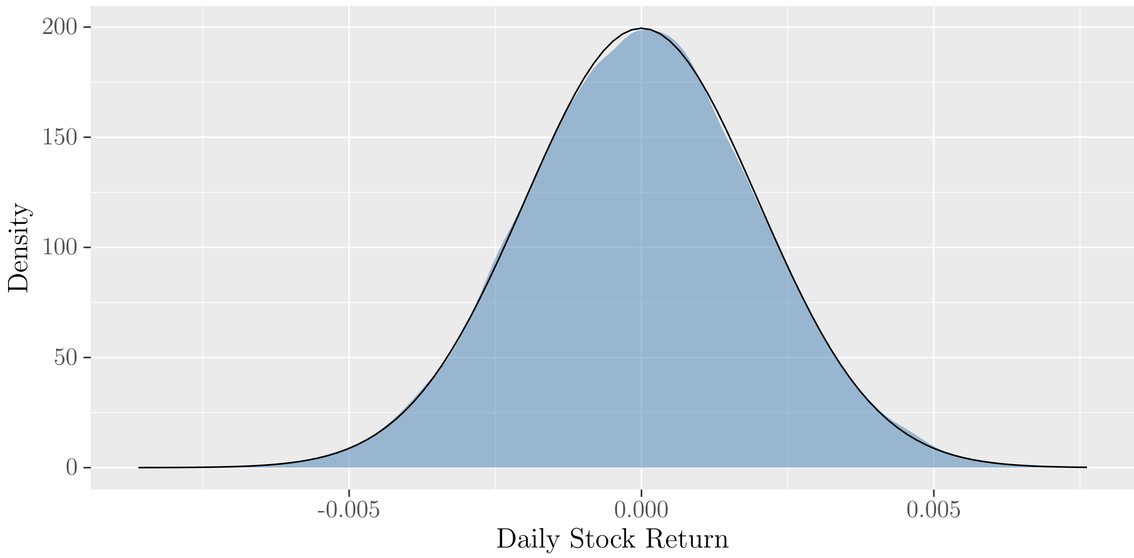
The distributions here developed concern the **??????**. They are described using the first two moments, namely the expectation and variance.

The first distribution (2.5) is related to **??**. What is learnt from it, is that the process followed by **??** has normally distributed returns with expected value and variance relying on the time period dt, Shreve [2004].

$$\frac{dS\,(t)}{S\,(t)} \sim N(\alpha dt, \sigma^2 dt) \tag{2.5}$$

According to the (figure 2.2), the normality qualification from equation 2.5 holds.

Figure 2.2: Daily Basis Stock Return Density



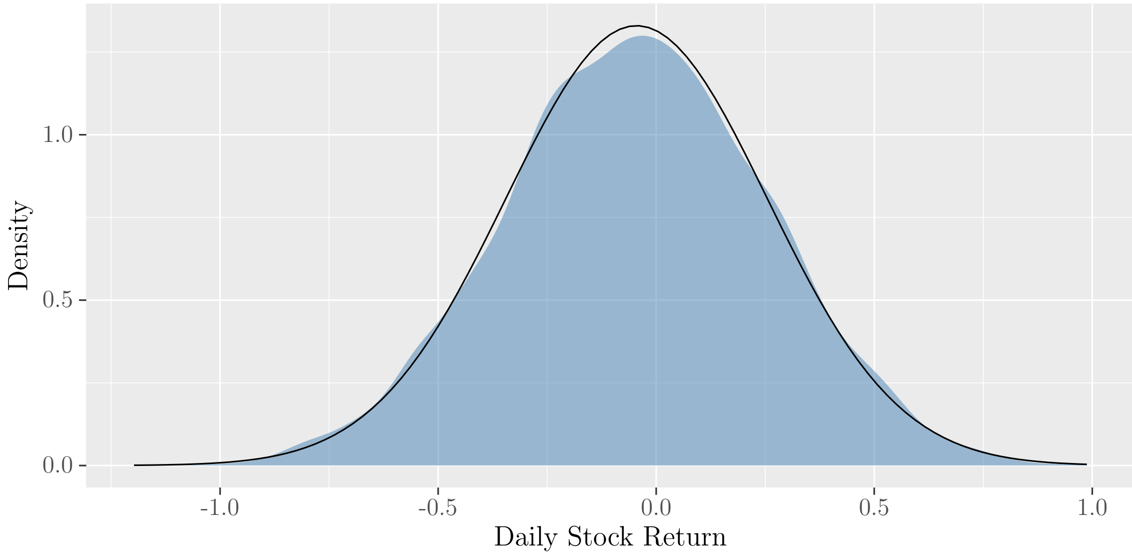
The above density function has been constructed over 10000 samples of a unique process. The samples has been constructed following **??**. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the normal bell curve with mean α and standard deviation of $\sigma \times \sqrt{dt}$. The period between each measure, namely dt has been set to 10000^{-1}.

According to **??**, The distribution of the natural logarithm of the stock price return recorded over a time period of t is characterized by equation (2.6)

$$\ln \frac{S\,(t)}{S\,(0)} \sim N((\alpha - \frac{\sigma^2}{2})t, \sigma^2 t) \tag{2.6}$$

From **??**, it could also be shown, by setting $X = \ln S\,(t)$ and $Y = S\,(t)$, that the randomly simulated stock price outcomes' distribution, following **??**, match with the log–normal law [REF]. The above point directly echoes the upstream ideal conditions set by Black and Scholes [1973] , especially the one relating the stock price process.

Figure 2.3: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following **??**. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the normal bell curve with mean $(\alpha - \frac{\sigma^2}{2}) \times t$ and standard deviation of $\sigma \times \sqrt{dt}$. The time frame is one year ($t = 1$) and period between each measure, namely dt has been set to 500^{-1}.

The following equations shows the relation between the distribution of the log–return and the stock process itself. Let

$$X = \ln S(t) \sim N\left(\mu \equiv \ln S(0) + \left(\alpha - \frac{\sigma^2}{2}\right) t, \delta^2 \equiv \sigma^2 t\right) \tag{2.7}$$

and

$$Y = S(t) \tag{2.8}$$

By the existing relation between the log–normal law with respect to the normal law, the following relations emerges.

$$\begin{aligned} \mathbb{E}\, Y &= e^{\ln S(0) + (\alpha - \frac{\sigma^2}{2})t + \frac{\sigma^2 t}{2}} && (2.9) \\ &= S(0) e^{\alpha t} && (2.10) \end{aligned}$$

and

$$\begin{aligned} var(Y) &= (e^{\sigma^2 t} - 1) * e^{2\mu * \delta^2} && (2.11) \\ &= S(0)^2 e^{2\alpha t}(e^{\sigma^2 t} - 1) && (2.12) \end{aligned}$$

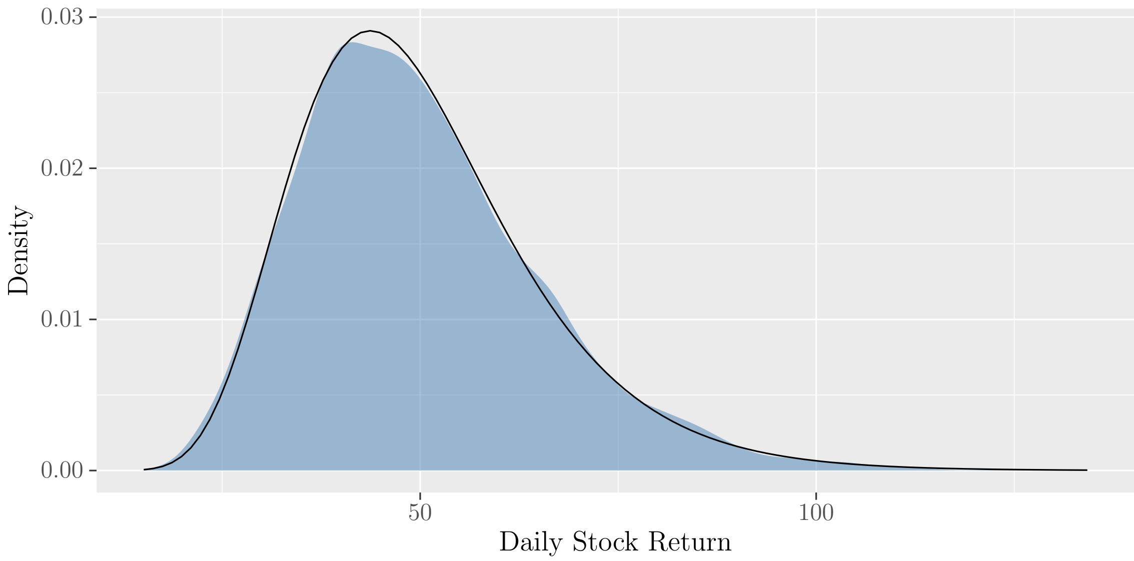
Consequently the stock price random variable $S(t)$ as relation [REF] for distribution.

$$S(t) \sim lognormal(\mu, \delta^2) \tag{2.13}$$

$$\ln \frac{S(t)}{S(0)} \sim N((\alpha - \frac{\sigma^2}{2})t, \sigma^2 t) \tag{2.14}$$

The **??** and figure 2.4 refers respectively to distributions (2.6) and (2.14).

Figure 2.4: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following **??**. The arguments are set with the following value, $\alpha = 0$, $\sigma = 30\%$. The black density belongs to the log–normal curve with mean and standard deviation such as described in [REF]. The time frame is one year ($t = 1$) and period between each measure, namely dt has been set to 500^{-1}.

2.3 Flaws

Even if the **??** is used to model the stock price process, it exists some lacks with the observed real processes. The analysis ([CHA REF]) mainly covers the following discrepancy of the model against empirical results.

The volatility appearing in the aformentioned underlying model (**??**) is constant as time passes. Even if this consideration can be considered as true for short period of time, it is not the case over the long–run Teneng [2011]. Other model substitutes the process defined throughout this chapter with other ones including stochastic volatility processes, making it changing over time. For this purpose, the Heston's model is developed in this current master thesis, which encompasses two stochastic processes, a geometric Brownian motion [???? + REF] for the stock price diffusion model [DIFUSION??] and a mean-reverting CIR model to compute the volatility as a stochastic process. This model is exposed in section 4.3.

Following [REF and REF], the uderlying log–returns distribution is characterized by the normal law. Consequently, the random variable $S(t)$, at the fixed time t, is normal. However, according to empirical results, the random variable $dS(t)$ do not fit with the normal bell curve (Clark [1973]). Therefore if the price changes are not normally distribution neither is the log–return and thus $S(t)$, at a fixed time t, is not log–normally distributed. Along with the Heston model, another one is considerd; the Merton's jump–diffusion model. Both are able to modify the skewness and kurtosis of the log–return distibution curve. The Merton framework is developed in section 4.2.

Chapter 3

The Black–Scholes–Merton model

3.1 OverView

In this present section the Black–Scholes–Merton model is explained and developed. Stricly speaking, it would be talked about the BSM (Black–Scholes–Merton) model but not BSM equation. In turn the equation would be refered to as the BS (Black and Scholes) equation. This distinction is made because the model encompasses the derivation of the BS equation, either using the CAPM (Capital Asset Pricing Model) for Black and Scholes [1973] or by setting up a riskless portfolio providing an expected return identiacal to risk-free rate for Merton [1973].

The BSM model is meant to provide the non arbitrage price of derivative assets such as european stock option (Hull and Basu [2012]). In the present master thesis the model derived from the BS formula is focused on the pricing of one of a kind trivial derivative, the vanilla call option.

In order to price call option throughout the model highlighted in this present chapter, some contraints have been set by Black and Scholes [1973]. The section **??** quotes the assumptions made to keep the model usage between boundaries. In latter chapter however it will be shown how the model is relevant by going on the edge and even beyond these constraints.

3.2 Assumptions

Black and Scholes [1973] have provided a framework defining a bunch of constraints qualified as ”ideal conditions” under which the market would behave in order to make the BS equation works with accuracy. All of these conditions are below–mentioned.

1. The short-term risk free rate r is known and constant
2. The stock return involving in the computation of BS equation is lognormally distributed with constant mean and variance rate
3. No dividend are provided with the considered share of stock
4. The option considered within the computation is european
5. The price for bid and ask quote are identical. It means that there is no bid–ask spread to be considerered

6. Share of stock can be divided into any portions such as needed for the computation

7. Short selling is allowed with no penalties

3.3 The Black–Scholes equation

The pricing method of an option underpinned by the BSM model is closely related to an underlying for which its price $S(t)$ is random and log–normally distributed, such the one developed in chapter 2 throughout equation **??** (Black and Scholes [1973]).

In order to provide a unique fair price to a stock option (e.g. a vanilla european call option) which depends on an underlying such as described above, all the uncertainty associated to the stock price motion has to disappear. To do so, one have to first construct such a portofolio $X(t)$ which encompasses the same source on uncertainty as the option itself, i.e., the geometric brownian motion $S(t)$ and then choosing the adequate position $\Delta(t)$ to take so that all randomness cancels out (Shreve [2004]).

$$dX(t) = \Delta(t)\, dS(t) + r\left(X(t) - \Delta(t)\, S(t)\right) dt \tag{3.1}$$

The goal is to hedge dynamically the position taken in the option. It means that the position has to be frequently rebalanced. Consequently, at any times, the present value of the changes occuring in the portfolio, due to stock price evolution should be equal to the one occuring in the derivative. The only way to achieve this equality is to adapt the delta (Shreve [2004]).

$$d\left(e^{-rt}X(t)\right) = d\left(e^{-rt}c(t,x)\right) \tag{3.2}$$

In that way, one can take a position in the derivatives (short / long) and hedge them by taking $\pm\Delta(t)$ shares of stock. $X(0)$ being the price of the call at time zero

$$X(0) = c(0, S(0)) \tag{3.3}$$

Following the method forementioned, the BSM differential equation is given by equation (3.4), with terminal condition (3.5) and boundary condition (3.6) – (3.7), (Shreve [2004]).

$$rc(t,x) = \frac{\partial c(t,x)}{\partial t} + rx\frac{\partial c(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c(t,x)}{\partial x^2} \tag{3.4}$$

$$c(T,x) = (x-K)^+ \tag{3.5}$$

Whilst the terminal condition focuses on the value taken at maturity, the boundary conditions fix some constraints on the extreme values likely to be taken by the shares of stock at any times during the option life. In that regard, the boundary condition (3.6) shows that an option with an worthless underlying is itself valueless, while whenever the option is deep in the money, simulated with $x = \infty$ (3.7), the value of the derivative is equal to the value of a forward contract involving the same underlying and with the same maturity date.

$$c(t,0) = 0 \tag{3.6}$$

$$\lim_{x\to\infty} \left[c(t,x) - \left(x - e^{-r(T-t)}\right)\right] = 0 \tag{3.7}$$

3.4 Solution for vanilla option pricing method

According to the terminal (3.5) and boundaries conditions (3.6, 3.7), the Black–Scholes–Merton solution for call option happens to be given by equation (3.8), (Shreve [2004]). The right hand side of the equation, $c(t,x)$, denotes the price of a call option depending on the time to maturity and the stock price at that time. In addition to these arguments, two other parameters are required, the strike (k) and the riskless interest rate (r).

$$c(t,x) = xN\left(d_+(\Delta t, x)\right) - Ke^{-r\Delta t}N\left(d_-(\Delta t, x)\right) \tag{3.8}$$

with

$$d_\pm(\Delta t, x) = \frac{1}{\sigma\sqrt{\Delta t}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\Delta t\right)\right] \tag{3.9}$$

3.5 The greeks

3.5.1 Overview

The Black–Scholes–Merton equation (3.4) can be divided into differents parts. Each one is identified through a greek letter. These letters are the purpose of this section and would be therefore described here.

The Greeks will be next used to show hown the hedge of a call option behave under some variation from the former conditions defined by Black and Scholes. Hence, only the Greeks for call options are considered.

3.5.2 Delta

Delta is the derivative of the call function (3.8) with respect to the stock price, as shown by equation (3.10). It therefore represents the instantaneous rate of change in a call value as the price of its underlying evolves (Hull and Basu [2012]).

$$\Delta(t, S(t)) = \frac{\partial c(t, S(t))}{\partial S(t)} \tag{3.10}$$

Whereas pratically, the derivation of delta for a call is given by equation 3.11, (Shreve [2004]).

$$\Delta_{call}(t, S(t)) = N\left(d_+(\Delta t, x)\right) \tag{3.11}$$

At any time t, in order to hedge a short call one should hold $\Delta(t)$ share of stock. Consequently a portfolio comprised of one short position in a call along with Δ shares of stock is said to be delta neutral, because each movement in the stock price is compensated as well by the position in the call as the one in the stock (Hull and Basu [2012]).

The delta neutrality could otherwise be explained using the slope-intercept form of the tangent line below the function $c(t,x)$, keeping t constant. If the stock price is equal to S and the corresponding call price, for a fixed time t and stroke at k, is c, we consequently get (3.12) as equation of the tangent line below $c(t,x)$.

$$y = \frac{\partial c(t, S(t))}{\partial S(t)}(x - S) + c \tag{3.12}$$

According to 3.12, the price of the call, for a stock price S at a fixed time t is given by $y = c$. If – over an infinitesimally small delta time – a positive stock price movement occurs, says that the stock price rises from $S \rightarrow S + \epsilon$. The price of the call is therefore going to change as well, from $y = c$ to $y = \Delta\epsilon + c$. Consequently, in odrer to hedge a short position in the call, Δ shares of stock should be owned. Indeed, by kepping Δ shares, the loss incured by the higher value of the call $y = c + \Delta\epsilon$ will be offset by an inscrease of $\Delta\epsilon$ thanks to the Δ shares held. It makes sense that the hedge of a long call is achieved by setting up a short position in the underlying, according to the same parameter Δ.

The hedge works well for small price movement in the underlying and closely depends on the curvature of the function $c(t,c)$, keeping t constant (Shreve [2004]). It would therefore be interesting to look at the second derivative of the call function with respect to the stock price, in order to get the rate of the rate of change of the call with respect to the underlying price. It is the purpose of the next subsection (3.5.3).

3.5.3 Gamma

Gamma is the second derivative of the option's price function with respect to the underlying price, time keeping constant (3.13).

$$\Gamma(t, S(t)) = \frac{\partial^2 c(t, S(t))}{\partial S(t)^2} \tag{3.13}$$

Whereas pratically, the derivation of gamma for a call is given by equation 3.14, (Shreve [2004]).

$$\Gamma_{call}(t, S(t)) = \frac{1}{\sigma S(t)\sqrt{\Delta t}} N'(d_+(\Delta t, x)) \tag{3.14}$$

It gives the acceleration at which the price of a call moves along with the underlying price, ceteris paribus. It gives information on the curvature of the function to be approximated using the differential form. It is crutial to know how big is the value of gamma in order to adequatly hedge a position in a call. Indeed, if gamma is low, the rebalancing of the hedge does not have to occur frequently but if gamma It gives as information the frequency needed in order to lower the error due to too large price movement. With the delta hedging rule, the more the price moves from its current value the more Consequently, by gathering gamma and delta both together, a given more precise information on hedging against the only stock price movement is achieved.

3.5.4 Theta

Theta is the derivative of the price of an option with respect to the time, stock price unchanges (3.15)

$$\Theta(t) = \frac{\partial c\left(t, S\left(t\right)\right)}{\partial t} \tag{3.15}$$

(Hull), Theta can be used as a proxy for Gamma in a delta neutral portfolio. No need to hedge against time therefore no need to neutralize theta.

3.6 Relation between BSM and the greeks

The Black–Scholes–Merton equation and the greeks are closely related together. In deed the BSM partial derivative formula (3.4) could equally be written using the greeks (3.16)

$$rc\left(t, S\left(t\right)\right) = \Theta + rS\left(t\right)\Delta + \frac{1}{2}\sigma^2 S\left(t\right)^2 \Gamma \tag{3.16}$$

Chapter 4

Other Models to be considered

4.1 Overview

4.2 Merton Mixed jump-diffusion Model

In his paper, Merton [1976] provides a model for stock price evolution involving jumps (equation (4.1)).

$$S(t) = S(0)\, e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \tag{4.1}$$

According to Merton [1976], there are two specific sources of uncertainty explained by the model (equation (4.1)). The first one is qualified to be normal, arising repeatedly with low effects and keeping the stock price motion continuous from time to time. These small changes on the price are modeled by a Wiener process, such as it was the case in equation **??**. The cause of these changes is explained by a temporary unbalanced between the supply and demand Merton [1976]. Another type of changes, occuring during the stock lifecycle, are qualified as abnormal. Such "abnormalities" happen less frequently, are unpredictable in their frequency and produce bigger effect on the stock price by giving rise to jumps in the course of stock path and therefore breaking its continuity (Merton [1976]). The jump process is constructed on double basis.

Firstly, the occurence (i.e. the number of jumps arising throughout a given period of time) is computed thanks to a Poisson–driven process according to a parameter λ. λ denotes the number of jumps per unit of time. Consequently, the probability that a jump occurs during a time range of Δt is equal to λdt (event A, eq. 4.2), whereas the probability that there are no jump during the same range of time is $1 - \lambda dt$, (event B, eq. 4.3) (Matsuda [2004]). While C, eq. 4.4, refers to the event that more than one jump occur during the same small delta time.

$$\mathbb{P}\{A\} \cong \lambda dt \tag{4.2}$$
$$\mathbb{P}\{B\} \cong 1 - \lambda dt \tag{4.3}$$
$$\mathbb{P}\{C\} \cong 0 \tag{4.4}$$

On the other hand, after the occurence comes the size of the jump. Such as the frequency, the importance of the jump can be characterized by a statistic law. Following

?, the log–normal law is used. Matsuda [2004], gives a summary in oder to grips with the concept of jump size equations (4.5) to (4.7).

$$y_t \sim lognormal(e^{\mu+\frac{1}{2}\delta^2}, e^{2\mu+\delta^2}(e^{\delta^2}-1)) \tag{4.5}$$
$$y_t - 1 \sim lognormal(\kappa \equiv e^{\mu+\frac{1}{2}\delta^2}-1, e^{2\mu+\delta^2}(e^{\delta^2}-1)) \tag{4.6}$$
$$\ln y_t \sim normal(\mu, \delta^2) \tag{4.7}$$

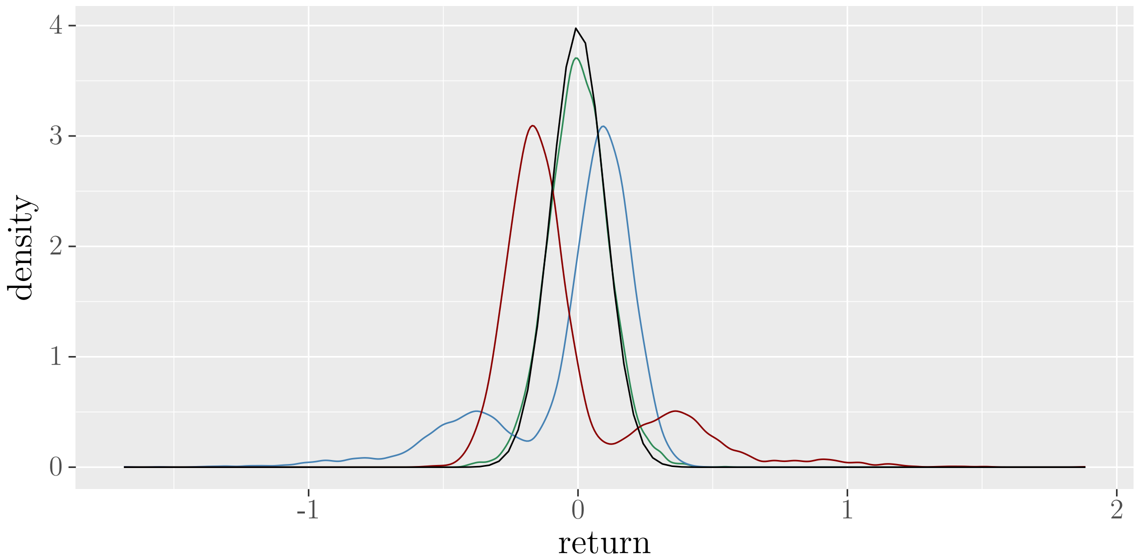
with y_t, $y_t - 1$ and $\ln y_t \equiv Y_t$ standing respectively for "absolute price jump size", "relative price jump size" and "log price jump size" (Matsuda [2004]).

The Merton's jump-diffusion process is be able to capture positive / negative skewness (see section 4.2.1) and excess kurtosis (see section 4.2.2) of the log–return density function Merton [1976].

4.2.1 Impact on the skewness log–return

The way to influence the direction of the distibution's shape is achieved by moving the cursor of the expected value of jump impact, in other word, by changing the value of the parameter μ. The figure [REF] shows how the density's shape of the log–return may vary along with this parameter.

Figure 4.1: Merton: Daily Basis Stock Return Density

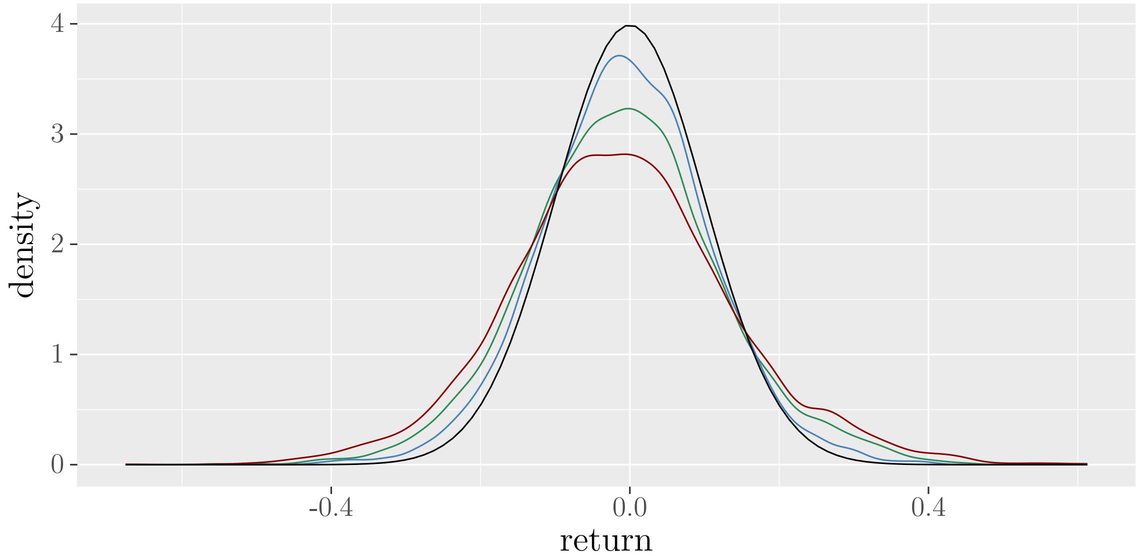


The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (4.1). The only parameter that changes over the group is μ which is set to $(-0.5,\ 0,\ 0.5)$ respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

4.2.2 Impact on kurtosis log–return

The way to influence the aspect of the distibution's tails is achieved by moving the cursor of the expected value of jump occurence, in other word, by changing the value of the parameter λ. The **??** shows how the distribution's tails of the log–return may vary along with this parameter.

Figure 4.2: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (4.1). The only parameter that changes over the group is λ which is set to (1, 3, 5) respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

4.3 Heston stochastic volatility model

In his paper, Heston [1993] tackles with another discrepancy against the real world behaviour introduced by the geometric Brownian motion, namely, its deterministic and immutable volatility σ.

In addition to provide a model where the volatility is stochastic (equation (4.8)), Heston [1993] gives the possibility to make this volatility in correlation with the stock price process (equation (4.9)), according to the parameter ρ defining how the Brownian motions from both processes relates together.

$$dV(t) = \kappa(\theta - V(t))\,dt + \sigma\sqrt{V(t)}dW_V(t) \tag{4.8}$$

$$dS(t) = \alpha S(t)\,dt + \sqrt{V(t)}S(t)\,dW_S(t) \tag{4.9}$$

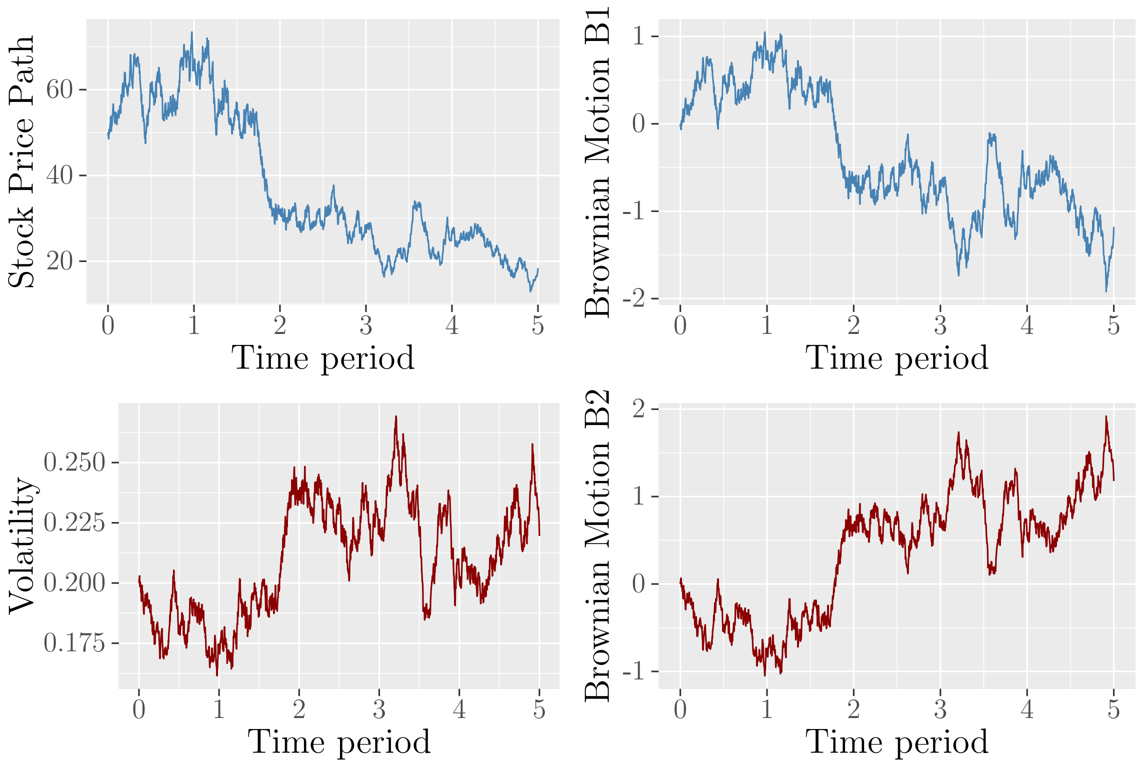
The drift part of the risk stochastic process (4.8 is made up of the long–run mean θ togehter with the mean reversion speed, given by κ, Heston [1993].

$$dW_v(t)\,dW_s(t) = \rho \tag{4.10}$$

The framework [REF + REF], simulating the behavior of an asset, includes a mean reverting Cox–Ingersoll–Ross(CIR) stochastic process (equation (4.8)) as volatility component.

From figures 4.3 and 4.4, it is gotten an hands-on insight on how correlated Brownian motions work on the way the Heston's stock price and volatility framework interact with each other. In figure 4.3 the correlation between the Wiener processes is set to $\rho = -1$, making the two Markov motion perfectly negatively correlated. It directly affects the course of the stocks serie, which is altogether correlated in the same negative direction with respect to the CIR volatility process as well. Likewise figure 4.4 points out the positive correlation occuring between the processes 4.8 and 4.9 whilst the Brownian motion correlation is set to one.

Figure 4.3: Heson Framework Using Negative Correllated Brownian Motions



The above stochastic processes have been constructed following 4.8–??. The Brownian motion correlation is set to $\rho = -1$. The other parameters are set with the following values; $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$.

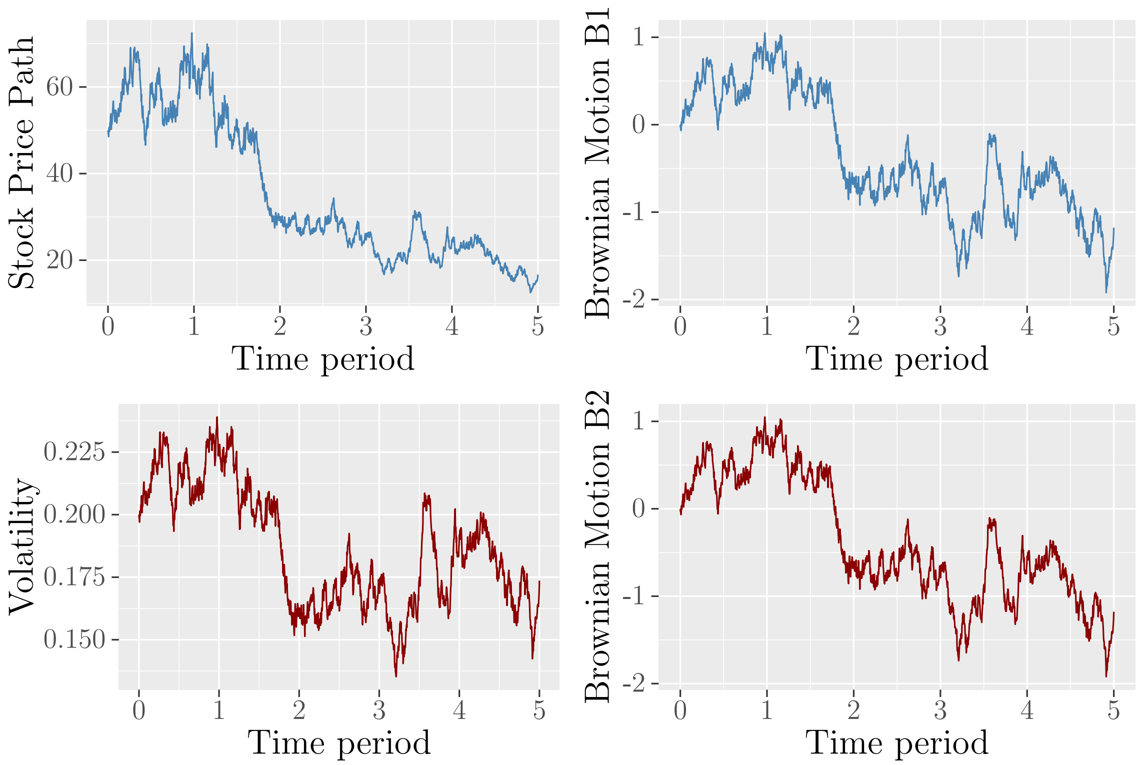
The usage of the forementioned Heston model lies in the fact that the correlation between the CIR and asset processes' Brownian motions would notably explain the spot return skewness whereas the kurtosis of the distribution may be affected by the volatility parameter σ of the volatility stochastic process (equation (4.8)), (Heston [1993]) It may consequently be consistent with what happens in equity market, namely sharp decrease in equity price implies increase in stock volatility (Crisóstomo [2015]).

4.3.1 Impact on log–return density's skewness

The skewness of the distribution of continuously compounded spot return may be affected by the parameter ρ. When a positive correlation exists between the two Brownian motion, an increase in the asset price implies more variance than the one recorded for lower stock price. Consequently a fat log–return distribution's right–hand tail is noticed. The opposite relation is noticed with negative correlation, namely, lower prices bring higher volatility generating a fatter left–hand tail in the log–return distribution (Heston [1993]). These statements are observed in [REF].

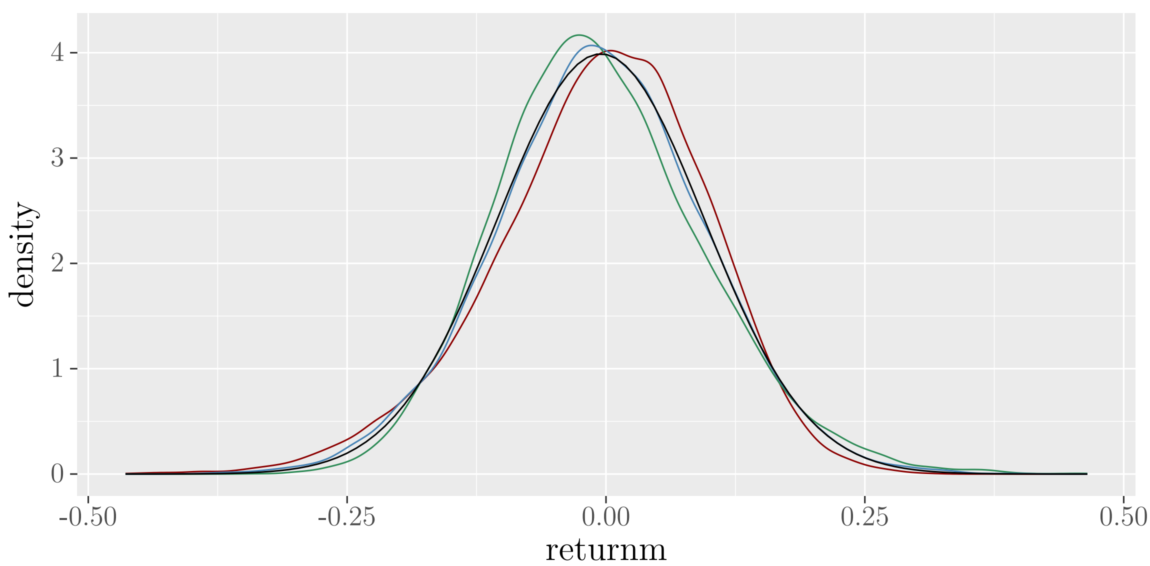
4.3.2 Impact on kurtosis density return

Figure 4.4: Heson Framework Using Positive Correllated Brownian Motion



The above stochastic processes have been constructed following 4.8–**??**. The Brownian motion correlation is set to $\rho = 1$. The other parameters are set with the following values; $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$.

Figure 4.5: Merton: Log–return skewness on Heston



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following equation (4.9), with stochastic volatility defined by equation (4.8). The only parameter that changes over the group is ρ which is set to $(-0.5, 1, 0.5)$ respectively for the red, green and blue outlined density functions. The black density belongs to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log–price return cover a period of one year with a time step of 500.

Chapter 5

Methodology

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