Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How perfomant is it ? What about other model such as . . . ?

Using R. R Core Team [2017]

Chapter 1

Other Models to be considered

1.1 Overview

1.2 Merton Mixed jump-diffusion Model

In his paper, Merton [1976] provides a model for stock price evolution involving jumps (equation (1.1)).

$$S(t) = S(0)\, e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \tag{1.1}$$

According to Merton [1976], there are two specific sources of uncertainty explained by the model (equation (1.1)). The first one is qualified to be normal, arising repeatedly with low effects and keeping the stock price motion continuous from time to time. These small changes on the price are modeled by a Wiener process, such as it was the case in equation **??**. The cause of these changes is explained by a temporary unbalanced between the supply and demand Merton [1976]. Another type of changes, occuring during the stock lifecycle, are qualified as abnormal. Such "abnormalities" happen less frequently, are unpredictable in their frequency and produce bigger effect on the stock price by giving rise to jumps in the course of stock path and therefore breaking its continuity (Merton [1976]). The jump process is constructed on double basis.

Firstly, the occurence (i.e. the number of jumps arising throughout a given period of time) is computed thanks to a Poisson–driven process according to a parameter λ. λ denotes the number of jumps per unit of time. Consequently, the probability that a jump occurs during a time range of Δt is equal to λdt (event A, eq. 1.2), whereas the probability that there are no jump during the same range of time is $1 - \lambda dt$, (event B, eq. 1.3) (Matsuda [2004]). While C, eq. 1.4, refers to the event that more than one jump occur during the same small delta time.

$$\mathbb{P}\{A\} \cong \lambda dt \tag{1.2}$$
$$\mathbb{P}\{B\} \cong 1 - \lambda dt \tag{1.3}$$
$$\mathbb{P}\{C\} \cong 0 \tag{1.4}$$

On the other hand, after the occurence comes the size of the jump. Such as the frequency, the importance of the jump can be characterized by a statistic law. Following

?, the log–normal law is used. Matsuda [2004], gives a summary in oder to grips with the concept of jump size equations (1.5) to (1.7).

$$y_t \sim lognormal(e^{\mu+\frac{1}{2}\delta^2}, e^{2\mu+\delta^2}(e^{\delta^2}-1)) \tag{1.5}$$

$$y_t - 1 \sim lognormal(\kappa \equiv e^{\mu+\frac{1}{2}\delta^2} - 1, e^{2\mu+\delta^2}(e^{\delta^2}-1)) \tag{1.6}$$

$$\ln y_t \sim normal(\mu, \delta^2) \tag{1.7}$$

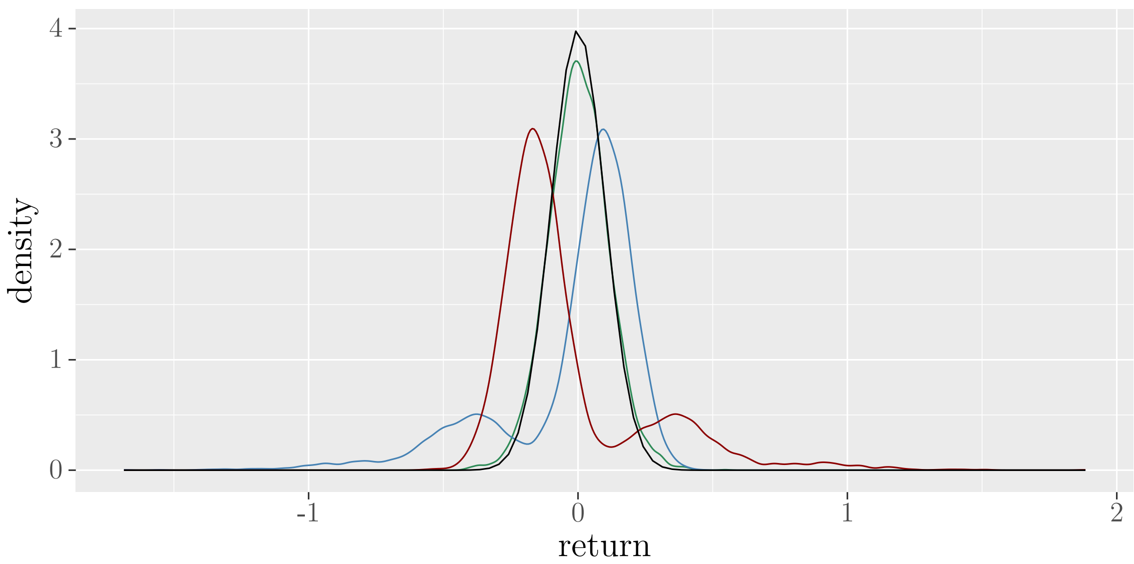
with y_t, $y_t - 1$ and $\ln y_t \equiv Y_t$ standing respectively for "absolute price jump size", "relative price jump size" and "log price jump size" (Matsuda [2004]).

The Merton's jump-diffusion process is be able to capture positive / negative skewness (see section 1.2.1) and excess kurtosis (see section 1.2.2) of the log–return density function Merton [1976].

1.2.1 Impact on the skewness log–return

The way to influence the direction of the distibution's shape is achieved by moving the cursor of the expected value of jump impact, in other word, by changing the value of the parameter μ. The figure [REF] shows how the density's shape of the log–return may vary along with this parameter.

Figure 1.1: Merton: Daily Basis Stock Return Density

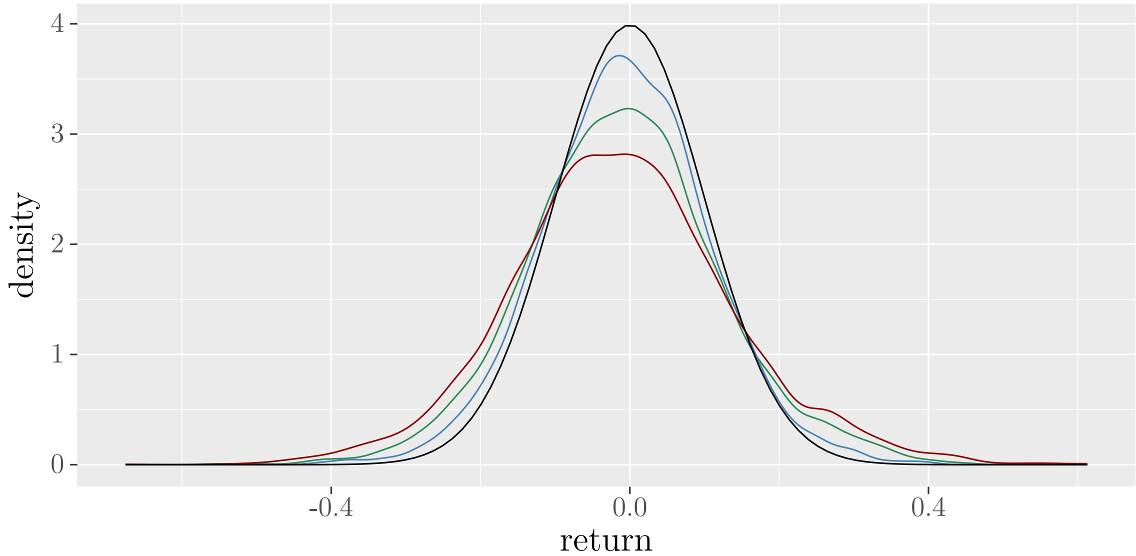


The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (1.1). The only parameter that changes over the group is μ which is set to $(-0.5,\ 0,\ 0.5)$ respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

1.2.2 Impact on kurtosis log–return

The way to influence the aspect of the distibution's tails is achieved by moving the cursor of the expected value of jump occurence, in other word, by changing the value of the parameter λ. The **??** shows how the distribution's tails of the log–return may vary along with this parameter.

Figure 1.2: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (1.1). The only parameter that changes over the group is λ which is set to (1, 3, 5) respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

1.3 Heston stochastic volatility model

In his paper, Heston [1993] tackles with another discrepancy against the real world behaviour introduced by the geometric Brownian motion, namely, its deterministic and immutable volatility σ.

In addition to provide a model where the volatility is stochastic (equation (1.8)), Heston [1993] gives the possibility to make that volatility in correlation with the stock price process (equation (1.9)), according to the parameter ρ defining how the Brownian motions from both processes relates together.

$$dV\left(t\right) = \kappa\left(\theta - V\left(t\right)\right)dt + \sigma\sqrt{V\left(t\right)}dW_V\left(t\right) \tag{1.8}$$

$$dS\left(t\right) = \alpha S\left(t\right)dt + \sqrt{V\left(t\right)}S\left(t\right)dW_S\left(t\right) \tag{1.9}$$

The drift part of the risk stochastic process (1.8 is made up of the long–run mean θ togehter with the mean reversion speed, given by κ, Heston [1993].

$$dW_v\left(t\right)dW_s\left(t\right) = \rho \tag{1.10}$$

Equation (1.9) represents the evolution of an asset though time given, by its differential form. Such as **??**, developed by Black and Scholes [1973], the parameter α gives the drift rate of return. The difference between both models lies in the way the volatility is perceived. In Heston [1993], the asset volatility is given by the stochastic equation (1.8). More specifically, the volatility thus defined follows a Cox–Ingersoll–Ross process.

Equations (1.8) to (1.9) will be used to build Monte Carlo simulations with the goal of getting samples of dummy stock price motions under real–world assumptions. The framework to use for option pricing purpose is given in section 1.3.3.

1.3.1 Model parameters

Here are described all the parameters appearing in the Heston stochastic volatility model.

$S(t)$	Price of the stock at time t
α	Annualized – and deterministic – expected return
$V(t)$	Observed volatility of the stock at time t
κ	Mean-reversion speed
θ	Volatility's long-run mean
σ	Volatility of the volatility

1.3.2 Feller condition

Due to the time discretisation brung by The Monte Carlo simulation, the stochastic process 1.8 may turn out to be negative sometime. If such a negative value is appear at time t, the next value computed for $t+\epsilon$ will raise an error, due to the term $\sqrt{V(t)}$ that obviously does not exist for negative value.

In his paper, Feller [1951] demonstrates that a process such the one described by equation (1.8) does not reach negative value if the following relation 1.11 is respected.

$$
\begin{aligned}
&\lim_{V\to 0}\left(\kappa\theta - V - \frac{1}{2}\frac{\partial(\sigma\sqrt{V})^2}{\partial V}\right) \geq 0 && (1.11)\\
\iff &\lim_{V\to 0}\left(\kappa\theta - V - \frac{1}{2}\sigma^2\right) \geq 0\\
\iff &\kappa\theta - \frac{1}{2}\sigma^2 \geq 0\\
\iff &2\kappa\theta - \sigma^2 \geq 0 && (1.12)
\end{aligned}
$$

Consequently, if the condition related by equation (1.12) is respected, no negative values would occur by using a Monte Carlo simulation to compute the CIR stochastic volatility.

1.3.3 Risk–neutralized processes

Likewise it has been done by Black and Scholes [1973], Heston [1993] used a risk-neutral framework to price options. To do so, Heston modified the drift parameters of both price and volatility stochastic processes.

The drift part of the price diffusion (equation (1.9)) is risk–neutralized by turning the rate α into its riskless counterpart r, as shown by equation (1.13).

$$dS\left(t\right) = rS\left(t\right)dt + \sqrt{V\left(t\right)}S\left(t\right)dW_S\left(t\right) \tag{1.13}$$

In order to make the volatility process risk–neutralized, Heston added the risk premium parameter, λ, to the drift part of equation (1.8). The risk–neutralized CIR process is given by crefeq:other:hsvvol:riskless.

$$dV\left(t\right) = \kappa^*\left(\theta^* - V\left(t\right)\right)dt + \sigma\sqrt{V\left(t\right)}dW_V\left(t\right) \tag{1.14}$$

where

$$\kappa^* = \kappa + \lambda \tag{1.15}$$

and

$$\theta^* = \frac{\kappa\theta}{\kappa^*} \tag{1.16}$$

Consequently the parameters κ^* and θ^*, which respectively denote the long–run mean and mean–reversion speed, are the ones to estimate while dealing with pricing options purposes.

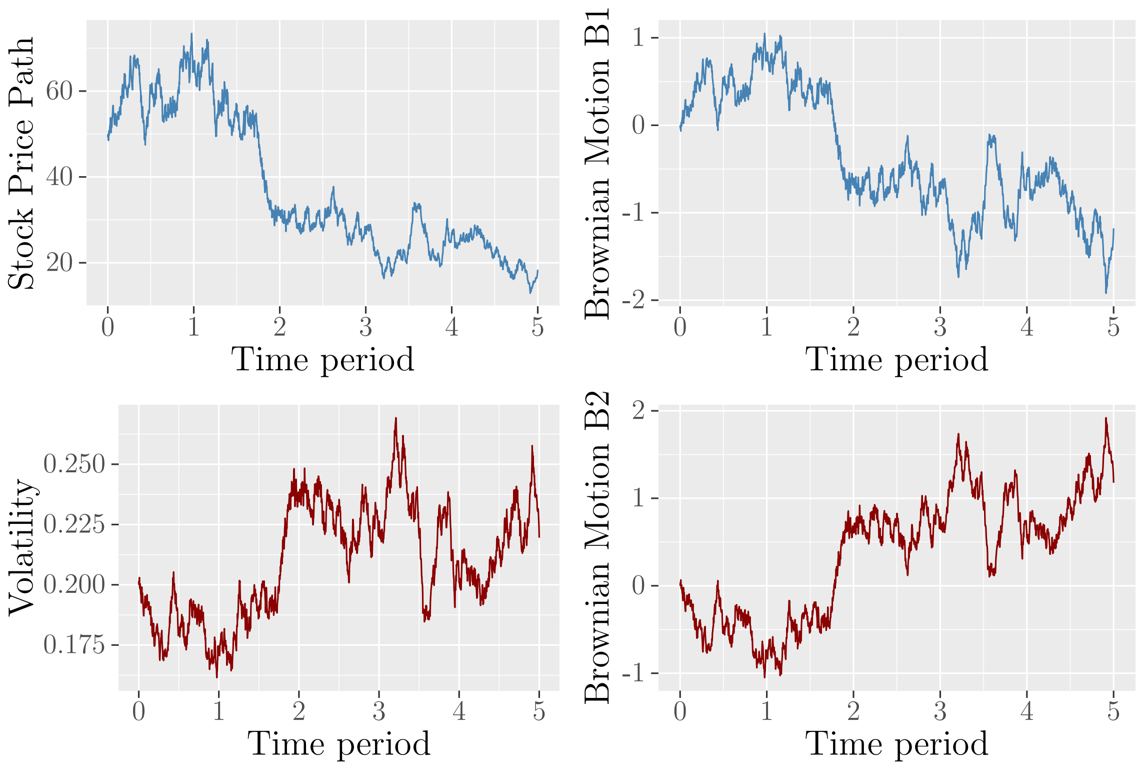
1.3.4 Graphical representation

Figures 1.3 and 1.4, get a hands-on insight into how the correlation between the underlying Brownian motions of the stock and volatility time series affect both processes.

figure 1.3 shows a correlation between the Wiener processes B_1 and B_2 sets to $\rho = -1$, making the two Markov motions perfectly negatively correlated. It directly affects the course of the stocks serie, which is altogether correlated in the same negative direction with respect to the CIR volatility process as well.

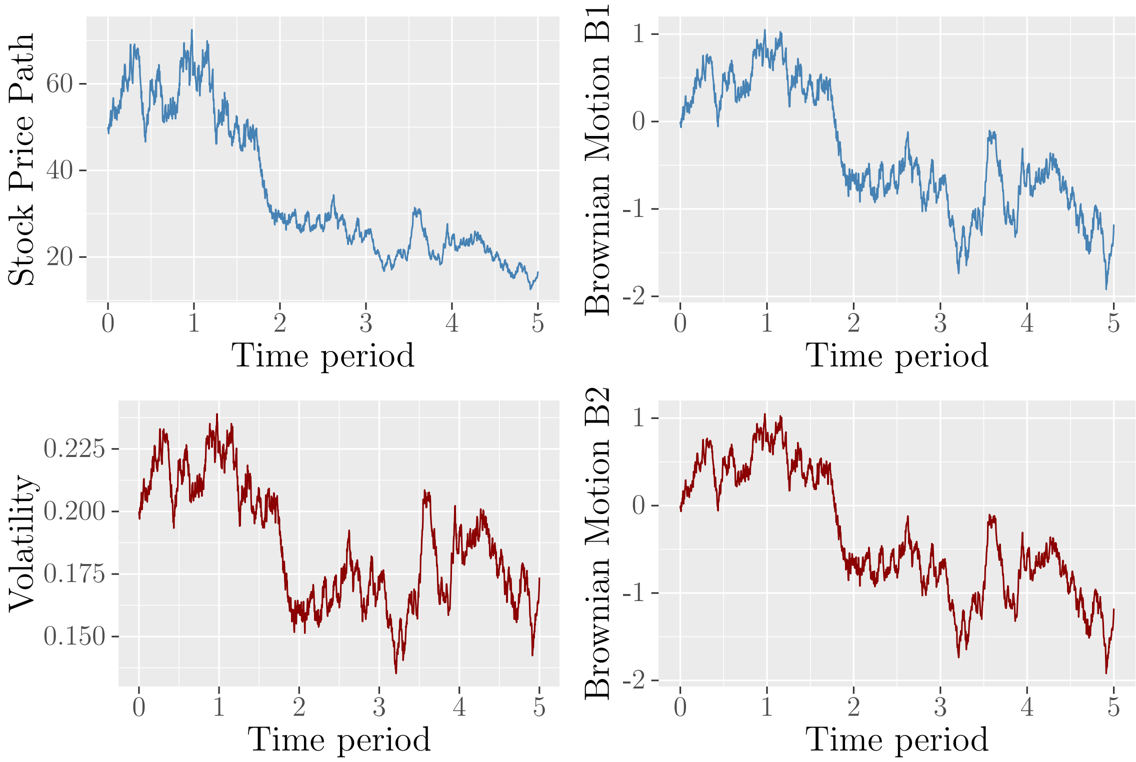
Likewise figure 1.4 points out the fully positive correlation occuring between the processes equation (1.8) and equation (1.9) whilst the Brownian motions correlation is set to one.

Figure 1.3: Heson Framework Using Negative Correllated Brownian Motions



The above stochastic processes have been constructed following 1.8–**??**. The Brownian motion correlation is set to $\rho = -1$. The other parameters are set with the following values; $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$.

Figure 1.4: Heson Framework Using Positive Correllated Brownian Motion



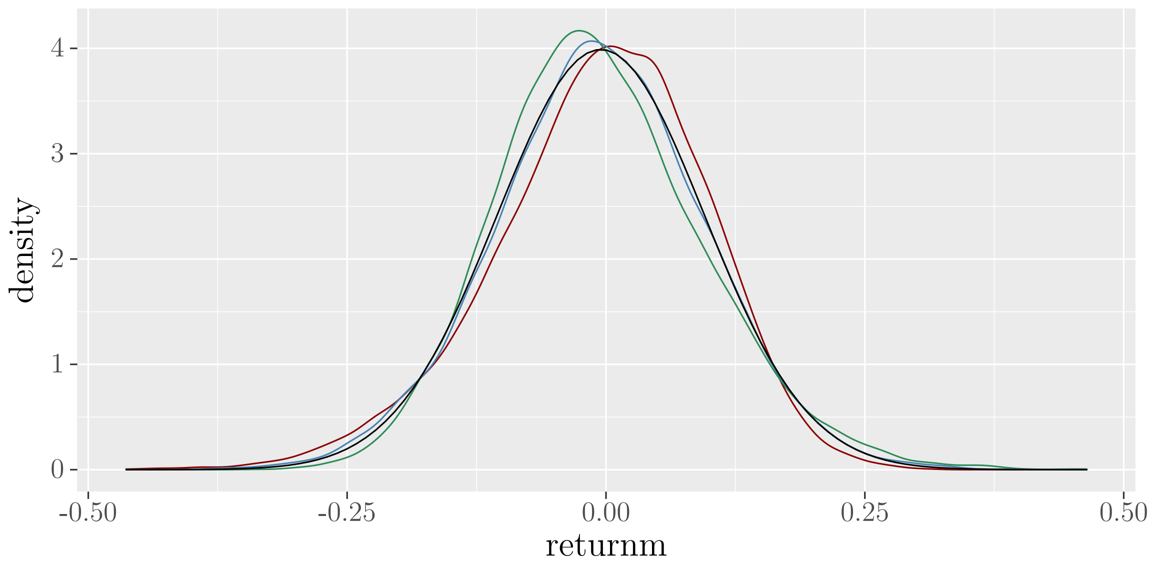
The above stochastic processes have been constructed following 1.8–??. The Brownian motion correlation is set to $\rho = 1$. The other parameters are set with the following values; $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$.

The usage of the forementioned Heston model lies in the fact that the correlation between the CIR and asset processes' Brownian motions would notably explain the spot return skewness whereas the kurtosis of the distribution may be affected by the volatility parameter σ of the volatility stochastic process (equation (1.8)), (Heston [1993]) It may consequently be consistent with what happens in equity market, namely sharp decrease in equity price implies increase in stock volatility (Crisóstomo [2015]).

1.3.5 Impact on log–return density's skewness

The skewness of the distribution of continuously compounded spot return may be affected by the parameter ρ. When a positive correlation exists between the two Brownian motion, an increase in the asset price implies more variance than the one recorded for lower stock price. Consequently a fat log–return distribution's right–hand tail is noticed. The opposite relation is noticed with negative correlation, namely, lower prices bring higher volatility generating a fatter left–hand tail in the log–return distribution (Heston [1993]). These statements are observed in [REF].

Figure 1.5: Merton: Log–return skewness on Heston



The above density function has been constructed over three distinctive groups of 10000 samples eachs. All samples have been constructed following equation (1.9), with stochastic volatility defined by equation (1.8). The only parameter that changes over the group is ρ which is set to $(-0.5, 1, 0.5)$ respectively for the red, green and blue outlined density functions. The black density belongs to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log–price return cover a period of one year with a time step of 500.

1.3.6 Impact on kurtosis density return

1.4 Option pricing method

As shown by section 1.2 and section 1.3, the frameworks developed by Merton [1976] and Heston [1993] drastically change the distribution of any underlying assets following such processes. Therefore, the pricing method to be used must also be adapted in order to take that update into account.

In his paper, Heston [1993] developed a techinque to price options using the characteristic function of the underlying asset. Furthermore, according to Crisóstomo [2015] that method could be used to price every options provided that the underlying's characteristic is known.

1.4.1 Probabilistic approach

Heston [1993] proposed the solution given by equation (1.17) to price a european call option.

$$C(t) = S(t)P_1 - e^{-r(T-t)}KP_2 \tag{1.17}$$

Through this method, the european call price at time t, namely $C(t)$, is computed thanks to equation (1.17), where $S(t)$ and $e^{r(T-t)}$ respectively stand for the stock price and the present value of the strike at that time t.

$$P_1(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln K}\psi(x, V, t; \phi - i)}{i\phi\psi(x, V, t; -i)}\right)d\phi \tag{1.18}$$

$$P_2(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi\ln K}\psi(x, V, t; \phi)}{i\phi}\right)d\phi \tag{1.19}$$

Following the development in Crisóstomo [2015], **??** both are probability quantities that involve the underlying characteristic function, namely $\psi(x, V, t; \phi)$. Once these quantities are computed, they are substituted in equation (1.17) in order to get the call price at time t.

The characteristic functions for the Merton jump–diffusion (section 1.2) and Heston stochastic volatility (section 1.3) models are developed sections 1.4.2 and 1.4.3

1.4.2 Characteristic function for Merton Mixed jump–diffusion model

1.4.3 Characteristic function for Heston stochastic volatility model

Following the development proposed by Gatheral and Taleb [2006], Crisóstomo [2015] provided the Heston characteristic function (equation (1.20)) based on the process $\ln S(t)$.

$$\psi^{heston}(\ln(S(t)), V(t), t; \phi) = e^{C(T-t,\phi)\theta + D(T-t,\phi)V(t) + i\phi\ln\left(S(t)e^{r(T-t)}\right)} \tag{1.20}$$

where

$$C(\tau, \phi) = \kappa\left(r_-\tau - \frac{2}{\sigma^2}\ln\left(\frac{1 - ge^{-h\tau}}{1 - g}\right)\right)$$
$$D(\tau, \phi) = r_-\frac{1 - e^{-h\tau}}{1 - ge^{-h\tau}}$$

and

$$r_\pm = \frac{\beta \pm h}{\sigma^2}; h = \sqrt{\beta^2 - 4\alpha\gamma}$$
$$g = \frac{r_-}{r_+}$$
$$\alpha = -\frac{\phi^2}{2} - \frac{i\phi}{2}; \beta = \kappa - \rho\sigma i\phi; \gamma = \frac{\sigma^2}{2}$$

Equation (1.20) can be directly used inside equations (1.18) and (1.19) in order to compute the quantities P_1 and P_2 that could be thereafter replaced inside equation (1.17) to find the european call price $C(t)$ corresponding to a stock price process $S(t)$ driven by the Heston stochastic volatility model.

Appendices

Appendix A

R functions catalogue

For this master thesis, I created two R packages to help me in the analysis and in the related experiments. These packages are available, under open-source software license, through my GitHub account. Names are "Random walk" (https://github.com/AnthonyTedde/RandomW and "Stock price simulator" (https://github.com/AnthonyTedde/StockPriceSimulator).

The R package "Random walk" contains functions that simulate time-discretized Brownian motions. It is widely used inside "Stock price simulator", mainly to add noise in the simulation of stock price time series.

Unlike "Random walk", the package "Stock price simulator" is more multipurpose. The algorithms I developed inside range from the simulation of stock price path to the computation of option fair price and include hedging strategies, Greeks computation, Itô's approximation, characteristic functions and so on.

Section A.1 describes the functions inside the "Stock price simulator" package that simulate time series by using a time discretization approximation.

A.1 Time series simulation

A.1.1 Geometric Brownian motion

Based on ??, this function provides one possible path of a time-discretized version of the geometric Brownian motion. It outputs a R data.frame comprised of two columns, namely "time_periods" and "stock_price_path", which respectively denotes the period expressed in year and the corresponding value of the stock at that time.

function

```
> sstock(initial_stock_price, time_to_maturity,
        seed, scale, sigma, alpha)
```

Arguments

initial_stock_price	Price of the stock at time zero.
time_to_maturity	Duration of the simulation, expressed in year.
seed	Parameter that fixes initial value of the pseudo random number generation in order to get reproducible experiment.
scale	Number of time steps. For instance scale = 365 would mean daily measurement.
sigma	Annualized volatility rate.
alpha	Annualized drift rate.

examples

```
> s <- sstock(initial_stock_price = 100,
        time_to_maturity = 1,
        seed = 1,
        scale = 52,
        sigma = .2,
        alpha = .15)

> print(s)
```

Table A.1: A sample of time-discretized geometric Brownian motion

time_periods (Year)	stock_price_path
0.00	100.00
0.02	98.52
0.04	99.27
0.06	97.24
0.08	101.90
⋮	⋮
0.88	122.33
0.90	123.88
0.92	126.87
0.94	126.79
0.96	130.25
0.98	132.03
1.00	130.13

A.1.2 Heston stochastic volatility

Based on equations (1.8) to (1.10), this function provides one possible path of a time-discretized version of the Heston stochastic volatility model. It outputs a R data.frame comprised of five columns. There are "time_periods" ,"stock_price_path" and "CIR", which respectively denotes the period expressed in year the corresponding values of the stock and of the volatility at that time. In addition, the two more variables "B1" and "B2" stand for the Brownian motions that bring the noise in both stochastic processes $S(t)$ and $V(t)$.

function

```
> heston(initial_stock_price, initial_volatility, time_to_maturity,
        seed, scale, alpha, rho, kappa, theta, sigma)
```

Arguments

initial_stock_price	Price of the stock at time zero.
initial_volatility	Volatility of the stock at time zero.
time_to_maturity	Duration of the simulation, expressed in year.
seed	Parameter that fixes initial value of the pseudo random number generation in order to get reproducible experiment.
scale	Number of time steps. For instance scale = 365 would mean daily measurement.
alpha	Annualized drift rate.
rho	Correlation between the Stock price and volatility processes.
kappa	Mean-reversion speed.
theta	Volatility's long-run mean.
sigma	Volatility of the volatility.

examples

```
> s <- heston(initial_stock_price = 100,
              initial_volatility = .2,
              time_to_maturity = 1,
              seed = 1,
              scale = 52,
              alpha = .15,
              rho = -.8,
              kappa = 2,
              theta = .2,
              sigma = .1)

> print(s)
```

Table A.2: A sample of time-discretized Heston stochastic volatility process

time_periods	stock_price_path	B1	B2	CIR
0.00	100.00	0.00	0.00	0.20
0.02	96.40	-0.09	0.10	0.20
0.04	97.79	-0.06	-0.02	0.20
0.06	93.02	-0.18	0.20	0.21
0.08	102.68	0.04	0.18	0.21
⋮	⋮	⋮	⋮	⋮
0.92	142.21	0.59	0.03	0.20
0.94	141.64	0.57	-0.01	0.20
0.96	149.70	0.70	-0.10	0.19
0.98	153.75	0.75	-0.22	0.19
1.00	148.56	0.67	-0.14	0.19

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