Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method.

Chapter 1

The underlying model

1.1 Overview

This chapter highlights one specific model that is used to model the price ticker of a share of stock for any time t in the future.

Equation (1.1) features a stochastic process where the only random component is the Brownian motion $W(t)$. The others parameters of (1.1) are α and σ, which respectively are the mean rate of return and the volatility of a stock price whose path is described by that model.

$$S(t) = S(0)\, e^{\sigma W(t) + \left(\alpha - \frac{1}{2\sigma^2}\right)t} \tag{1.1}$$

The condition underlying by (1.1) is that the mean rate of return and the volatility of the stock price is constant. The path is continuous, there is no jump in the course of the stock price evolution. Moreover it will be shown in subsequent section that the stochastic process (1.1) is log-normally distributed.

1.2 Derivation

The equation (1.1) could be derived using Itô's formula (1.2) in order to get the differential form of that equation. The provided differential formula will be next used to find out the first and second moments of the distribution underpinned by the stock price random process.

$$df(t, W(t)) = \left[\frac{\partial f(t, W(t))}{\partial t} + \frac{1}{2}\frac{\partial^2 f(t, W(t))}{\partial x^2}\right] dt + \frac{\partial f(t, W(t))}{\partial x} dW(t) \tag{1.2}$$

The term $df(t, W(t))$ in (1.2) represents any changes in the value of a function $f(t, W(t))$ occuring over a infinitesimally small move in time dt. The related function is any one involving time along with a Brownian motion. For that matter in the present case, $f(t, W(t)) = S(t)$. By applying the tranformation inccured by Itô, the two following equations (1.3a, 1.3b) emerge:

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t) \tag{1.3a}$$

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) \tag{1.3b}$$

In one hand, the equation (1.3a) shows that any change occuring in the stock price over an small amount of time is due to one fully predictable element – its drift rate $(\alpha S(t))$ – and another one that brings uncertainty.

In the other hand, equation (1.3b) relates the rate of return of the stock price over a small amount of time. Because the uncertainty given is given by the Browinan motion has an expectation of zero, it could be shown that the mean rate of return of a stock price following the model (**??**) is equal to α, as stated through (1.4).

$$\frac{dS(t)}{S(t)} \sim N(\mu dt, \sigma^2 t) \tag{1.4}$$

The figure (1.1) shows that the normality expectation holds, for equation 1.4. The solid line exibits the normal distribution bell curve ($N \sim (0, 40 * 360^{-1})$). While the bins represent the distribution of the increments from a stock processes, with parameters $\alpha = 0$, $\sigma = 40\%$ and $\Delta t = 360^{-1}$.

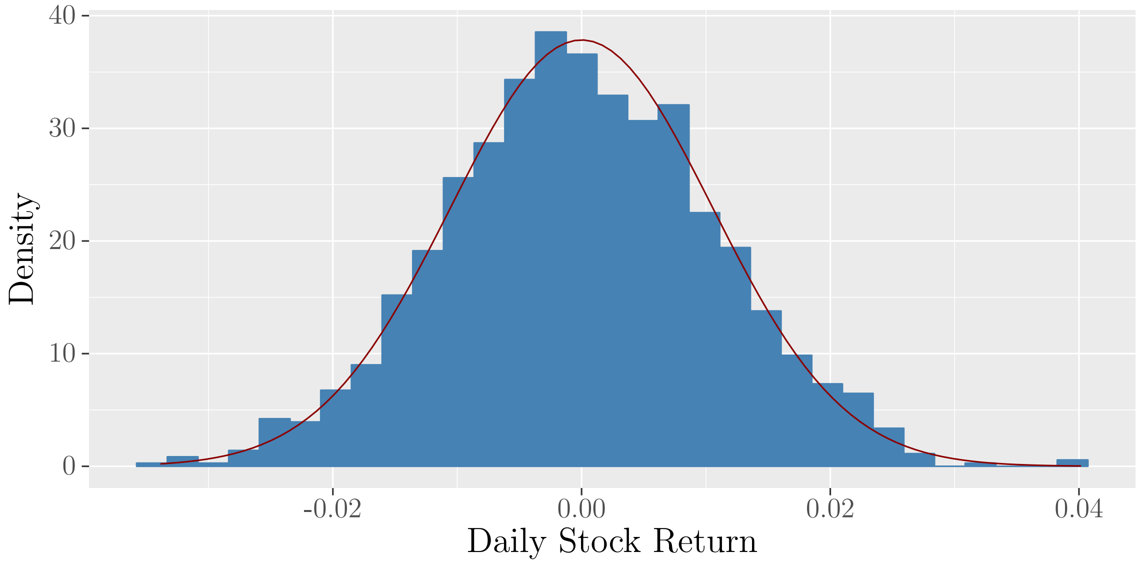

Figure 1.1: Daily Basis Stock Return Density

In other way, the stochastic price motion process (1.1), throughout the Itô approximation, could led to another representation than the one highlighted by equation 1.3a.

$$dS(t) = \alpha S(t)\, \Delta t + \sigma S(t)\, dW(t) \tag{1.5a}$$

$$\frac{\Delta S(t)}{S(t)} = \alpha \Delta t + \sigma \delta W(t) \tag{1.5b}$$

The unique difference between respectively (1.3a) – (1.5a) and (1.3b) – (1.5b) is the frequence at which data are recorded. In the realm of real world, the model described by (1.5a) fairly matches the data pickup requirements in the sense that only discrete measurement can be performed from the bunch of market available data.

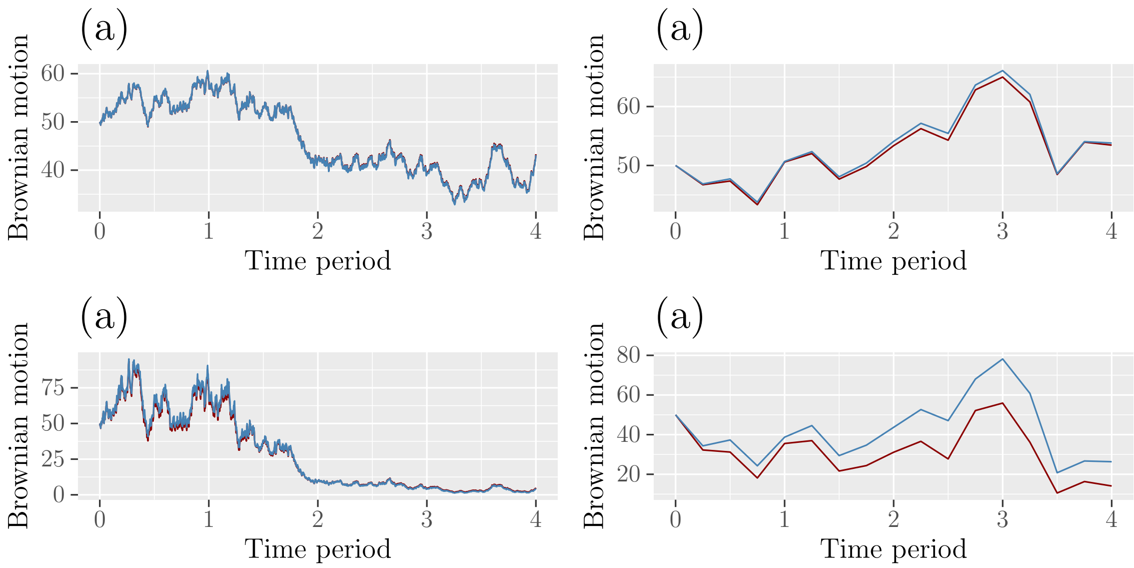

Figure 1.2: Accuracy of Itô approximation

The figure (1.2) is purposed to show how precise is the estimation made by using the approximation (1.3a) against (1.1) as $\Delta t \to 0$. Four graphs are bundled, each one represents a stock price motion. The blue line is made by the stricky application of (1.1) while the dark red one is gotten thanks to the Itô's approximation (1.3a). They vary among themselves by, in turn, either a different value for the variance parameter or a different recording frequency or both. In one hand, horizontally the stock volatility change whilst the time duration between two measures remains identical. While in the other hand, vertically, whether the volatilities are equal, the time steps are differents. Indeed the measurement is simulated with a daily basis for the two charts above whereas fixed quarterly for the bottom ones. It is obvious, following these representations that the less the volatility and time step, the better the approximation.

1.2.1 Distribution of the stock price process

According to Itô's approximation and by setting $f(t, W(t))$ to be $\ln S(t)$ it is showed that the natural logarith of $S(t)$ is normaly distributed with mean $\left(\alpha - \frac{\sigma^2}{2}\right)\Delta t$ and variance $\sigma^2 \Delta t$. Because every process with normally distributed logarithm are log–normally distributed, and following the relationship between the two laws, $S(t)$ is consequently log–normally distributed with mean () and var ().

MEAN VAR

+ FIGURE

1.2.2 Flaws

1.3 Volatility issues

1.4 Jump issues

Chapter 2

The Black–Scholes–Merton model

2.1 OverView

The Black–Schole–Merton equation is meant to provide the non arbitrage price of any kind of derivative assets. In the present document the model derived by the Black–Scholes–Merton formula is focus on the pricing of one of a kind trivial derivative, the call vanilla option.

In order to price this derivative throughout the model highlighted in this present chapter, some contraints are set. The section 2.2 exibits the assumptions made to keep the model between boundaries. In latter chapter however it will be shown how the model is relevant by going on the edge on that constraints and even beyond.

2.2 Assumptions

As previously stated, the Black–Scholes–Merton is a framework to define the non arbitrage price for call option. As a model it is somehow far away from reality in a bunch of way. The list below itemize all the contraint attached to the model.

- The stock price process involving in the computation of Black–Scholes–Merton is lognormally distributed with mean and variance as described in 1.2.1
- Short selling is not forbidden as hands-on market opperations
- The price for bid and ask quote are identical. It means that there is no bid–ask spread to be considerered
- The underlying does not provide dividend at any time
- No arbitrage opportunity are implied
- Are constant, the risk-free intererst rate (r), the stock volatility (σ) and, even though no considered in the Black–Scholes–Merton solution for the purpose of vanilla option pricing, also is the stock rate of return (α)

[TODO: Send reader to METHODOLOGY chapter where is explain how the analysis will go away from these constraints]

2.3 The Black–Sholes–Merton equation

The Black–Scholes–Merton model provides a backward parabolic differential equation (2.1) with terminal condition () and boundary contirion () , () with the purpose of finding out the fair price of an option provided that assumptions (2.2) are respected.

Because the dummy variable x and t are dummies, the equation (2.1) involved no uncertainty. As it is shown through chapter (GREEKS REF), the uncertainty underlying to the model is removed thank to the delta hedging rule.

$$rc\left(t,x\right) = \frac{\partial c\left(t,x\right)}{\partial t} + rx\frac{\partial c\left(t,x\right)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c\left(t,x\right)}{\partial x^2} \tag{2.1}$$

The terminal condition (2.2) is intended to support that the value of a call option at maturity equals its payoff.

$$c\left(T,x\right) = \left(x-K\right)^+ \tag{2.2}$$

Whilst the terminal condition focuses on the value taken at maturity, the bondaries condition fix some constraints on the extreme values likely to be taken by the shares of stock at any time during the optio like. In that regard, the boundary condition (2.3) shows that an option with an worthless underlying is itself valueless while whenever the option is deep in the money, simulated with $x = \infty$ (2.4), the value of the derivative is equal to the value of a forward contract involving the same underlying and with the same maturity date.

$$c\left(t,0\right) = 0 \tag{2.3}$$

$$\lim_{x\to\infty}\left[c\left(t,x\right) - \left(x - e^{-r(T-t)}\right)\right] = 0 \tag{2.4}$$

2.4 Solution for vanilla option pricing method

According to the terminal (2.2) and boundaries conditions (2.3, 2.4), the Black–Scholes–Merton solution for call option happens to be given by equation (**??**). This equation takes two arguments, being the time and the stock price at that time. In addition to these, two other parameters are required, the strike (k) and the riskless interest rate (r). The solution provided by the equation is the fair price of a vanilla call, at each time t, given a precise maturity date and depending on a stock price. The stock price is simulated by (1.1) or with its approximation (**??**).

$$c\left(t,x\right) = xN\left(d_+\left(\Delta t,x\right)\right) - Ke^{-r\Delta t}N\left(d_-\left(\Delta t,x\right)\right) \tag{2.5}$$

with,

$$d_\pm\left(\Delta t,x\right) = \frac{1}{\sigma\sqrt{\Delta t}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\Delta t\right)\right] \tag{2.6}$$

2.5 The greeks

2.5.1 Overview

The Black–Scholes–Merton equation (2.1) can be divided into differents parts. Each one is identified through a greek letter. These letters are the purpose of this section and would be therefore described here.

2.5.2 Delta

Delta represents the derivative of the call function $(c\,(t, S\,(t)))$ with respect to the stock price, as shown with equation (2.7). Delta represents therefore the instantaneous rise or fall of a call price while the price of its undlerlying changes, over an infinitesimally small gap of time, dt.

$$\Delta(t) = \frac{\partial c\,(t, S\,(t))}{\partial S\,(t)} \tag{2.7}$$

At any time t, a hedged portfolio for a short position in a call option would be to hold $\Delta(t)$ share of stock, in order to protect the whole portfolio from an adverse stock movement.

Taking the slope-intercept form of the tangent line below the function $c\,(t, x)$, keeping t constant. Consider that the stock price is equal to S and the corresponding call price, at a fixed time t and stroke at k, is c. we therefore get (2.8) as equation of the tangent line below $c\,(t, x)$.

$$y = \frac{\partial c\,(t, S\,(t))}{\partial S\,(t)}(x - S) + c \tag{2.8}$$

According to 2.8, the price of the call, for a stock price S at a fixed time is given by $y = c$. Con If – over an infinitesimally small delta time – a positive movement on the stock price is now considered, says that the stock price rises from $S \to S + \epsilon$. the price of the call also is going to change from $y = c$ to $y = \Delta\epsilon + c$. Consequently, in odrer to hedge a short position, Δ share of stocks should be detain. Indeed, by getting Δ shares, the loss incured by the higher value of the call $y = c + \Delta\epsilon$ will be offset by an inscrease of $\Delta\epsilon$ thanks to the Δ shares held. It makes sense that a hege on a long position is achieve by setting up a short position in the underlying, according to the same parameter Δ.

The hedge works well for small price movement in the underlying and is closely related to the curvature of the function $c\,(t, c)$, keeping t constant. It would therefore be interesting to look at the second derivative of the call function with respect to the stock price. It is the purpose of the next subsection.

2.5.3 Gamma

Chapter 3

The Greeks

3.1 The purpose of the Greeks

3.2 Delta

3.3 Theta

3.4 Gamma

3.5 Vega

3.6 Rho

3.7 An hedging strategy involving the Greeks and Black–Scholes–Merton

TEST