Contents

Introduction

Talk about what is done to price a vanilla option throuhout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How perfomant is it ? What about other model such as . . . ?

Using R. R Core Team [2017]

Chapter 1

Analysis

The objective of the current chapter is to measure the hedging performance of the Black-Scholes model when the stock price evolves in a not log-normal world. The studied models in **??** allowing the course of a time-series to go out of the Black-Scholes frame are the Merton jump-diffusion (MJD) and Heston stochastic volatility(HSV). In order to give realistic results, they have been adjusted in **??**. For both models, two kinds of calibrated parameters have been found out, namely, the risk-neutral and risk-averse ones. Therefore, the "no risk" parameters are used to price options, while those risky serve to generate time-series, thanks to the aforementioned models.

The chosen metric to compare the models' performances is the relative profit and loss (P&L). Latter is explained in **??**.

The analysis is completed in two steps. The first one concerns the BSM model exclusively with all the so related constraints respected. Even though the results of the computations of the options' prices in such an environment do not respect those emerging from reality, the measures of the P&L so computed serve as a benchmark for the other models assessed. In the following part of the analysis, the P&Ls for both models MJD and HSV will be quantified. Those are computed on the delta-hedging strategy. On the one hand by using the delta of Black and Scholes and on the other, by using the appropriate delta of the considered model. Although the Black and Scholes' delta is purposely computed to be applied when the underlying asset is driven by a geometric Brownian motion (GBM), the goal is to estimate how one can go wrong by using it within the real conditions.

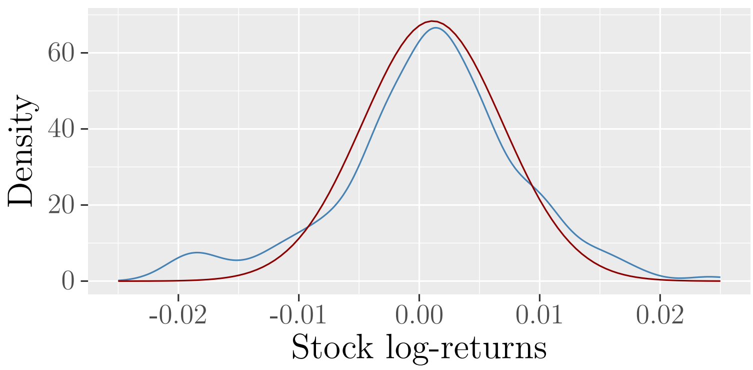
At last, some particular results will be commented.

1.1 The delta-hedging in a log-normal world

In order to use the BSM model for the assessment of the delta-hedging, some of its parameters have to be adjusted, such as it was done for HSV and MJD model.

According to Black and Scholes [1973], only one immutable value for σ exists and the difference between the risk-neutral world, where the options prices are computed, and the risk-averse one, is the replacement of the riskless rate r by the drift rate α. Consequently, to get both α and σ at once, the method here followed is to adjust them according to data providing from the risky world. To do so, similarly as done in **??**, the function *fitdistr* is purposely used. As a reminder, that function needs (i) a probability density function (PDF) to match, (ii) a sample of data and (iii) a list of parameters to be calibrated. Therefore, as defined in Black and Scholes [1973], the normal PDF is the one to be fitted when one deals with the log-returns of the GBM. The so found arguments to use along with the normal PDF are $\{\bar{x} = 0.0013, s = 0.0103\}$, with $\bar{x}$ and s respectively being the estimates of the mean and the standard deviation. However, before using it inside the function *bsm_ts*, to simulate the time-series, they must be turned into drift and variance rate, as explained in **??**. The so adjusted parameters to use within the BSM model are, $\{\alpha = 0.4823, \sigma = 0.1959\}$.

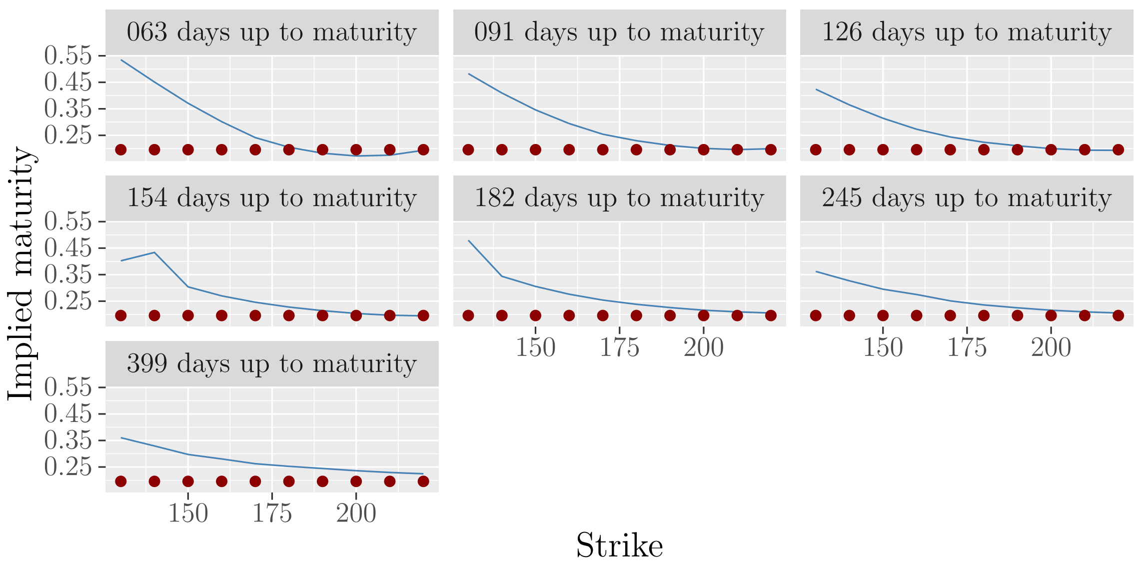
Figure 1.1: Distribution of calibrated BSM time-series



Notes. The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series generated by the function *bsm_ts* taking $\{\tau = 1.0931, \alpha = 0.48229, \sigma = 0.1958\}$ as parameters.

figure 1.1 illustrates the theoretical density curve simulated with the adjusted parameters, while figure 1.2 confronts the blue colored volatility smiles computed from market data with those dotted in red, calculated from data provided by the function *bsm_call* which takes the calibrated σ and the riskless rate r as parameters.

Figure 1.2: BSM volatility smile



Notes. A REFAIRE

At a glance, one can see (i) that the GBM lacks in correctly reproducing the behavior of the market time-series log-returns and (ii) that the BSM equation with only one possible value for the volatility is not enough versatile to fully reproduce the wide range of options prices given by the market.

However, The BSM model is going to serve as a benchmark to compare the hedging performances of the other considered models, namely, MJD and HSV. table 1.1 presents the results of the relative P&Ls got from the delta-hedging processes on European call options with maturities of 3, 6 and 13 months and strikes ranging from 140 to 230. Those outputs are maturities column-wise and grouped by strikes in rows. Furthermore, each row is split into three parts, each of these subsections represents different rebalancing frequency. For instance, the result exhibited in column ”91 dbm [1]” and row ”140 > intraday” give the mean of the relative P&Ls computed on a series of European call options' delta-hedgings with a maturity of 91 days (3 months), a strike of 140 and the rebalancing carried out twice a day.

The dummy time-series of the underlying asset, which will serve the analysis, are depicted in figure 1.3. According to the dimension of table 1.1, any paths of that series helped forty-five times the study since each of them have been involved in the hedge of options with five different strikes having three maturities each, along with three distinctive rebalancing frequencies. The total number of samples is one hundred, and consequently, Table 1.1 summarizes four thousand and five hundred delta-hedging strategies.

Figure 1.3: Sample geometric Brownian motions

Notes. Simulation of one hundred geometric Brownian motions by using the function *bsm_ts* of the R package Tedde [2017], with the following arguments $\{\tau = 1.0931, \alpha = 0.48229, \sigma = 0.1958\}$.

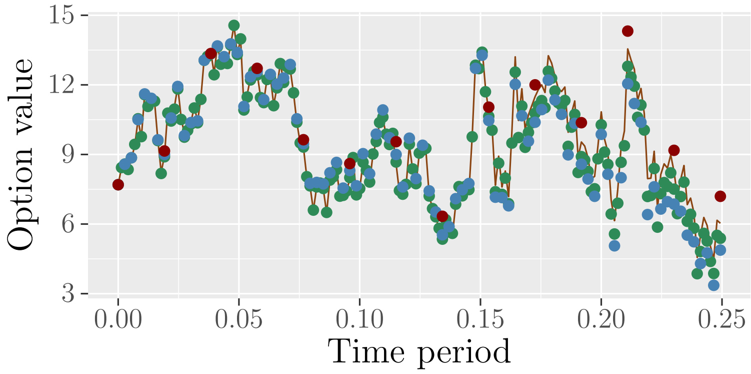
From table 1.1 one can observe that the balancing frequency has a positive impact on the quality of the hedge. Indeed, the more frequent balancing, the better hedging. Figure 1.4 shows an extract of the delta-hedging processes with different balancing frequencies. The continuous brown line represents the course of the option price and the green, blue and red dots respectively exhibits the associated delta-neutral portfolios' values with a rebalancing frequency of twice a day, once a day and once a week.

In the light of the above figure 1.4, and as confirmed by the table 1.1, a portfolio more regularly balanced will give a better result in the BSM delta-hedging strategy, within a log-normal world.

The vast majority of these results shows that with a less rebalancing frequency, the average value of P&Ls is negative, meaning that the delta-neutral portfolios underperform with respect to the option itself. The reason is due to the option's gamma. As a reminder, gamma gives the acceleration of any changes in the call function with respect to the stock price. With a positive gamma, and it is always positive for a vanilla stock option within the BSM model, the function $c(S(t), t)$, with t constant, is concave up. As Shreve [2004] shows, the delta-neutral portfolio is tangent below the curve of the call function. Therefore, due to the convexity of the latter, an instantaneous change of the asset price, either by increasing or decreasing, always make the related delta-neutral portfolio suffer and hence deflates. That kind of portfolio is called short gamma.

[1] days before maturity

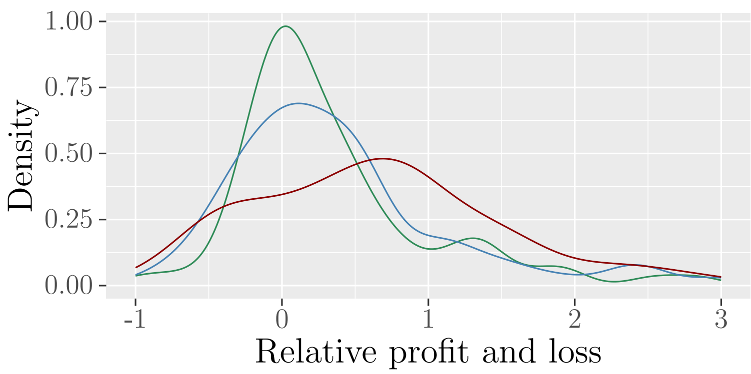
Figure 1.4: Delta-neutral portfolio with different frequency of balancing



Notes.

On other note, the average worst result comes from the coverage of deep-out-of-the-money options, weekly rebalanced. However, by opting for a more frequent rebalancing strategy, the mean of the relative P&Ls reduces. Indeed, it goes from 31.7% to 2% by choosing a rhythm of twice a day instead of once a week. Figure 1.5 shows the distribution of the above-mentioned P&Ls with a portfolio balancing applied either daily (blue curve), twice (green curve) a day or once a week (red curve). As expected, the more frequently rebalanced portfolios show less variance and more value near zero for the associated P&Ls.

Figure 1.5: Sample geometric Brownian motions



Notes.

The more disruptive observation is given by the averaging P&Ls of the hedging of deep-out-of-the-money, weekly rebalanced options with 91 days before maturity, namely, 0.317. Unlike the other P&Ls, this one is positive, meaning that the delta-neutral portfolios that replicate a position in the long European call won on average, which contradicts the previous statement stating the opposite. The underpinned reason is explained by a high value of options' theta, i.e., the first derivative of the call function with respect to time, all other things staying unchanged. Indeed, theta gives as information the effect of time on the derivative. As theta is always negative, then when time passes, all other things being equal, the derivative value decreases. Therefore, the more the value of theta is high in absolute value, the faster the option is going to lose value with respect to time. Consequently, the adverse effect of theta on the option has a positive relative impact on

Table 1.1: Hedging with BSM: Relative P&L

		91 dbm[a]	182 dbm	399 dbm
	intraday	0	0	0
140	daily	0	0	0
	weekly	0	0	0
	intraday	0	0	0
160	daily	0	0	0
	weekly	-0.001	-0.001	-0.003
	intraday	0	0.001	-0.001
186	daily	-0.005	-0.002	-0.004
	weekly	-0.01	-0.019	-0.021
	intraday	0.022	0.008	-0.001
200	daily	-0.002	-0.005	-0.006
	weekly	-0.007	-0.052	-0.037
	intraday	0.02	0.042	-0.007
230	daily	0.022	-0.063	-0.022
	weekly	0.317	-0.285	-0.136

[a]dbm: days before maturity

the hedging-portfolio and can overwhelm the negative effect of gamma. Table 1.2 lists all the P&Ls' results of those hedging scenarios.

Table 1.2: Worst relative P&L for BSM

	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
[0]	1.01	8.52	-0.52	0.67	-15.42	0.55	3.62	0.35	2.45	2.15
[10]	2.55	0.85	1.61	0.06	-1.68	-4.93	3.16	0.61	0.27	0.97
[20]	-9.32	0.38	3.34	-0.43	-9.27	0.60	-0.37	-0.27	0.27	-0.57
[30]	1.54	0.83	17.43	1.04	-37.44	1.09	1.48	1.73	0.92	0.77
[40]	1.17	0.29	-2.33	0.45	-0.51	8.98	-3.76	-0.55	0.83	0.69
[50]	-8.85	0.18	-0.96	0.92	4.38	0.29	2.95	4.55	5.29	23.50
[60]	-0.11	4.71	3.83	-0.26	3.70	-2.95	1.28	7.00	0.80	0.15
[70]	-0.21	11.00	0.54	0.23	0.71	-38.71	15.04	7.75	-0.49	0.85
[80]	-15.18	1.62	5.40	-9.23	-0.43	1.39	0.65	1.31	-7.34	1.53
[90]	-0.02	3.19	-1.68	13.05	-0.12	-6.70	-0.34	8.68	0.69	2.18

In order to understand what happens to those P&Ls let us analyze the hedging process behind the one with the highest positive value. Latter is highlighted in table 1.2. Henceforth, the computed values of theta for the options causing that P&L result are listed in table 1.3.

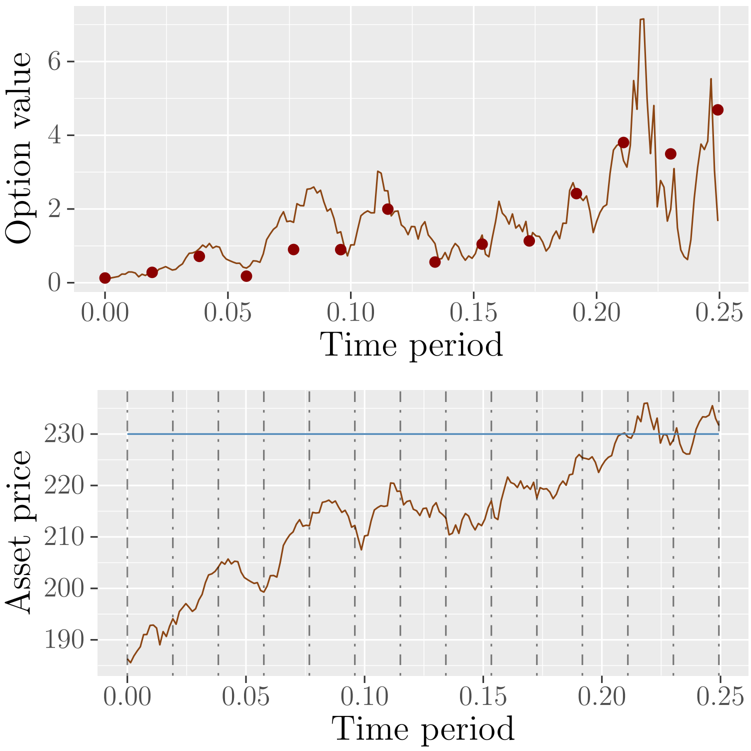
The lowest values of theta appear at the end of the life of the derivative. The reason, as shown through figure 1.6, is due to the fact that at the time near maturity, the values of the option's theta are computed when the option is just in-the-money, letting by consequence the time with a big influence on its value.

By resuming the solution of theta given in Shreve [2004], for the European call option, one gets the following equations.

Table 1.3: Worst relative P&L for BSM

Time	0.00	0.02	0.04	0.06	0.08	0.10	0.12	...
Theta	-1.83	-3.80	-8.50	-5.26	-13.88	-13.84	-20.39	...
Time	...	0.13	0.15	0.17	0.19	0.21	0.23	0.25
Theta	...	-14.89	-19.05	-19.84	-35.54	-47.81	-65.66	Inf

Figure 1.6: European call option with higher theta as time goes to maturity



Notes.

$$\Theta = -rKe^{-r(T-t)}N\Big(d_-(T-t, S(t))\Big) - \frac{\sigma S(t)}{2\sqrt{T-t}}N'\Big(d_+(T-t, S(t))\Big) \tag{1.1}$$

with

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right] \tag{1.2}$$

With N and N^- respectively standing for the normal cumulative distribution function (CDF) and for the normal probability density function (PDF).

Intuitively, according to equation (1.1) As $\tau\downarrow \implies \sigma S(t)/(2\sqrt{T-t})\uparrow$ and $N'(d_+(T-t, S(t)))$ gets its higher value when $d_+(T-t, S(t)) \to 0$. Also, by equation (1.2), for any given τ sufficiently small, the function d_+ is the nearest to zero as $x \to S(t)$. Figure 1.6 shows that it is exactly what happens for the observed option.

1.2 Merton jump-diffusion performance measuring

Table 1.4 is arranged the same way than table 1.1 but except that the columns are subdivided to give the information on the relative P&L obtained by either hedging with the delta MJD or BSM. For instance, the result exhibited in the column "91 dbm $>$ $delta_{bsm}$" and row "140 $>$ intraday" gives the mean relative P&L computed on a series of delta-hedging on European call options with a maturity of 91 days (3 months), a strike of 140, a rebalancing done twice a day and with $\Delta(t)$ computed using the BSM equation, while the output in the column "91 dbm $>$ $delta_{mjd}$" and row "140 $>$ intraday", give the same information but with $\Delta(t)$ computed by using the Merton equation.

The paths of the samples time-series of the underlying asset are all displayed in figure 1.12. Any of them was used ninety times in the analysis because all have been involved in the hedge of options with five different strikes having three maturities each, along with three distinctive rebalancing frequencies and depending on two deltas as well. The total number of samples was one hundred, and consequently, Table 1.4 summarizes nine thousand delta-hedging strategies.

Figure 1.7: Sample geometric Brownian motions

Notes.

From table 1.1, the effect of the rebalancing frequency is not as clear as it was with the hedging of the BSM model in the log-normal world. At first sight, however, one can observe that with the increase of the balancing rhythm comes a rise in the value of the delta-neutral portfolio, on average, which even tends to systematically outperform the European call. Nonetheless, the information given by table 1.1 is only the mean of all the computed P&Ls and therefore lacks in exhaustiveness.

In order to get more comprehension of that result, let us focus on the distribution of the P&Ls. Latters are displayed in figure 1.8 where the green, blue and red-filled densities respectively denote the P&Ls' distributions when the delta-neutral portfolios are balanced twice a day, daily or weekly.

Figure 1.8: Sample geometric Brownian motions

Notes.

In the light of figure 1.8, one can observe that as the rhythm of balancing increased, the variance of the averaged relative profits and losses decrease with the most of the distribution so concentrated around the mean.

Likewise, the time to maturity and the strike price both affect the distributions of the P&Ls.

Table 1.4: Hedging with MJD: Relative P&L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}
	intraday	0.004	0.006	0.011	0.012	0.01	0.021
140	daily	0.002	0.006	0.008	0.012	0.016	0.021
	weekly	0.004	0.006	0.006	0.011	0.007	0.021
	intraday	0.011	0.018	0.021	0.029	0.025	0.042
160	daily	0.016	0.018	0.022	0.029	0.019	0.042
	weekly	0.013	0.016	0.018	0.026	0.018	0.04
	intraday	0.036	0.021	0.078	0.055	0.079	0.074
186	daily	0.039	0.022	0.072	0.055	0.068	0.074
	weekly	0.014	-0.008	0.055	0.037	0.057	0.061
	intraday	0.072	-0.002	0.139	0.061	0.13	0.086
200	daily	0.06	-0.013	0.131	0.057	0.115	0.085
	weekly	-0.02	-0.1	0.083	0.005	0.085	0.053
	intraday	0.955	0.331	0.444	-0.061	0.301	0.063
230	daily	1.098	0.466	0.409	-0.091	0.261	0.054
	weekly	-0.741	-1.335	0.085	-0.438	0.174	-0.088

On the one hand, concerning the effect of the maturity on the shape of the distributions, the observations are split into two categories. The first concentrates the computed measures when $0 < K/S(t) < 1$ at time zero, that is, all the [deep]-in-the-money options, while the second gathers all the others, namely, [deep]-out-of|at-the-money. To observe how such results affect the distribution, figure 1.8 is zoomed-in for the interesting P&Ls' density curves of the first considered category, so giving figure 1.9.

Figure 1.9: Sample geometric Brownian motions

Notes.

From figure 1.9, it can be observed that as the time to maturity increase for options with a strike below the underlying asset price at original date, the volatility of the hedging portfolios performance also increase. Whereas, according to figure 1.8, the inverse behavior is observed for the volatility of the performance of options with the higher strike price, i.e., the further the maturity, le lower the uncertainty about the associated P&Ls.

On the other hand, as shown by figures 1.8 and 1.9 and table 1.4, the performances of poorly rebalanced portfolios are better for options the more in-the-money. Such a behavior is explain by a weaker gamma for the deep-in-the-money options. Indeed, as time passes, the prices of the underlying assets tend to get more valuable in the vast majority, as illustrated by figure 1.12. Therefore, the associated options with smaller strikes continue to gain value but at a lower rate because the incidences of the stock price changes become less impacting. The inverse reasoning is to be applied to deep-out-of-the-money options. Effectively, the initial price of such options is near zero, so are the associated deltas. However, with high growth time-series as described in table 1.4, the collateral effect is that they rapidly affect the option price, just as the values of the delta are influenced. Consequently, according to and for the measured options, gamma is lower for in-the-money and higher for out-of-the-money options.

figure 1.10 shows the distribution of the P&Ls when the hedging is built with BSM or MJD deltas. The red densities concern the data related to BSM deltas while those in green represent the distributions of the performance metrics for the MJD delta-hedging.

Figure 1.10: Sample geometric Brownian motions

Notes.

According to the results of table 1.4 and figure 1.10, hedging in the real life with delta BSM seems not so bad with respect to the gotten measures. Let us dive more into deeply the delta-hedging by observing what happens behind the hood.

Figure 1.11 illustrates how the hedgings work depending on the strike prices and the deltas computed. The small circles represent the replicated portfolios at each rebalancing time, whereas the underpinned black lines are the course of the options' values as time passes. The blue and red circles respectively stand for the MJD and BSM delta-hedging portfolios' values.

Figure 1.11: Sample geometric Brownian motions

Notes.

1.3 Heston stochastic volatility performance measuring

In Table 1.5, the inforamtion are completely organized the same way that in table 1.5.

Figure 1.12 illustrates all the dummy time-series that will be considered for the analysis of delta-hedging using HSV as a stochastic process for the underlying asset.

Figure 1.12: Sample geometric Brownian motions

Notes.

Table 1.5: Hedging with HSV: Relative P&L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}
	intraday	0	0.002	0.011	0.011	0.009	0.038
140	daily	-0.001	0.002	0.01	0.011	0.009	0.038
	weekly	0.001	0.002	0	0.011	0.008	0.038
	intraday	0.009	0.028	0.023	0.073	0.042	0.143
160	daily	0.008	0.028	0.025	0.072	0.036	0.143
	weekly	0.008	0.028	0.019	0.073	0.036	0.143
	intraday	0.158	0.252	0.159	0.392	0.153	0.524
186	daily	0.15	0.245	0.195	0.391	0.156	0.522
	weekly	0.117	0.241	0.158	0.378	0.139	0.519
	intraday	0.459	-0.298	0.43	0.146	0.279	0.546
200	daily	0.433	-0.361	0.42	0.126	0.255	0.544
	weekly	0.268	-0.659	0.369	0.005	0.246	0.498
	intraday	2.136	-0.527	1.884	-2.452	1.01	-0.235
230	daily	1.948	-1.197	1.893	-2.655	0.989	-0.224
	weekly	1.407	-2.152	1.547	-2.402	0.917	-0.353

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