Hedging performances of the Black-Scholes model in imperfect log-normal world

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Contents

Context: Black–Scholes–Merton model

Black–Scholes–Merton solution

$$c\left(t,x\right)=xN\left(d_{+}\left(\Delta t,x\right)\right)-Ke^{-r\Delta t}N\left(d_{-}\left(\Delta t,x\right)\right)$$

with

$$d_{\pm}\left(\Delta t,x\right)=\frac{1}{\sigma\sqrt{\Delta t}}\left[\log\frac{x}{K}+\left(r\pm\frac{\sigma^{2}}{2}\Delta t\right)\right]$$

Geometric Brownian Motion

$$S\left(t\right)=S\left(0\right)e^{\sigma W(t)+\left(\alpha-\frac{1}{2\sigma^{2}}\right)t}$$

Context: Black–Scholes–Merton assumptions

1. The short-term risk free rate r is known and constant.
2. The stock return involving in the computation of BSM equation is log-normally distributed with constant mean and variance rates.
3. No dividend are provided with the considered share of stock.
4. The option considered within the computation is European.
5. The prices for the bid and ask quotes are identical. It means that there is no bid-ask spread to be considered.
6. Share of stock can be divided into any portions such as needed for the computation.
7. Short selling is allowed with no penalties.

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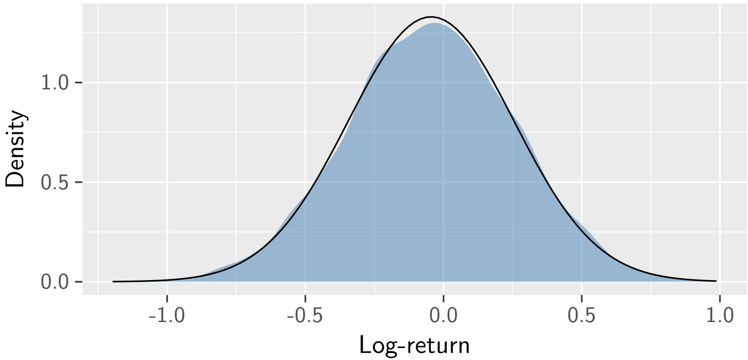
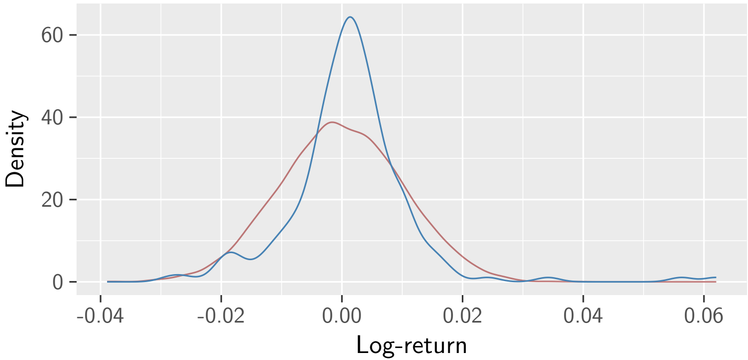
Context: Black–Scholes–Merton log-return density

$$\ln \frac{S(t)}{S(0)} \sim N\left((\alpha - \frac{\sigma^2}{2})t, \sigma^2 t\right)$$

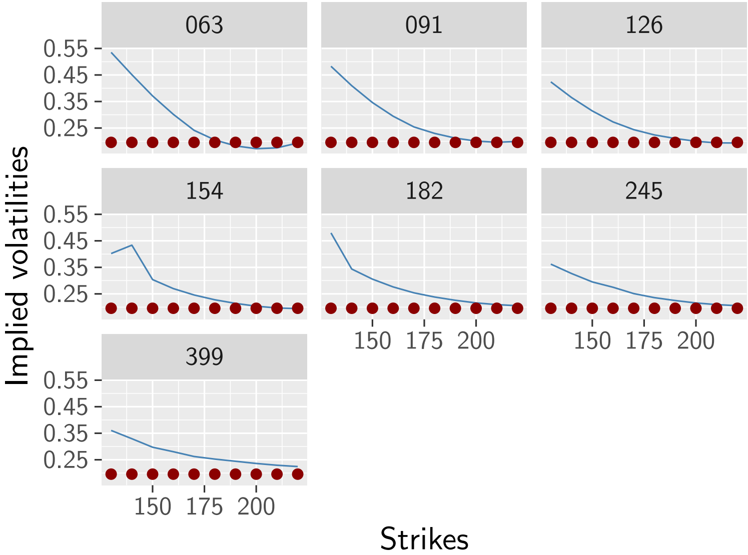


Context: Black–Scholes–Merton log-return density

The Apple case



Context: Black–Scholes–Merton volatility surface



Context: Summary

- ~~Normality of log-returns~~
- Options' price implied volatility
- Geometric Brownian Motion should be replaced

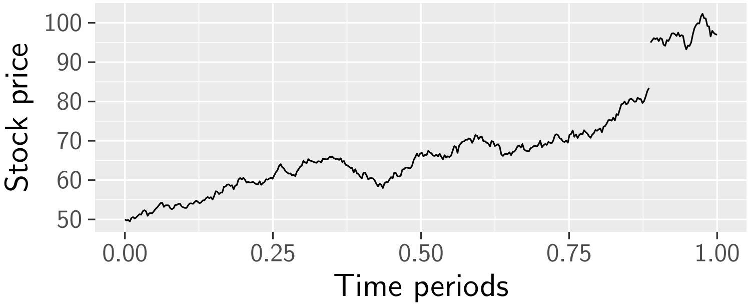
Other models

The Merton's Jump-Diffusion Model (MJD)

MJD stochastic process

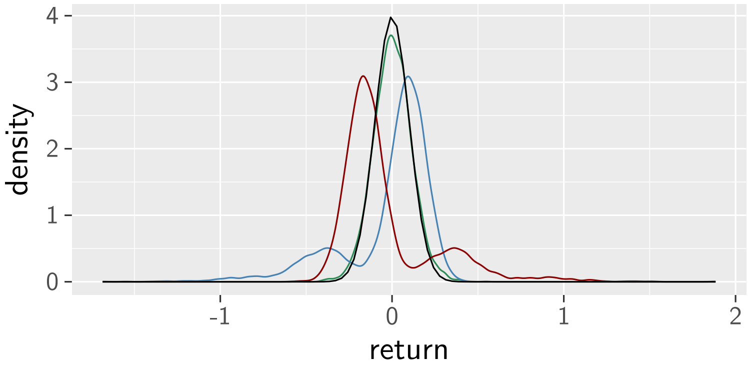
$$S\left(t\right) = S\left(0\right)e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i}$$

Graphical representation



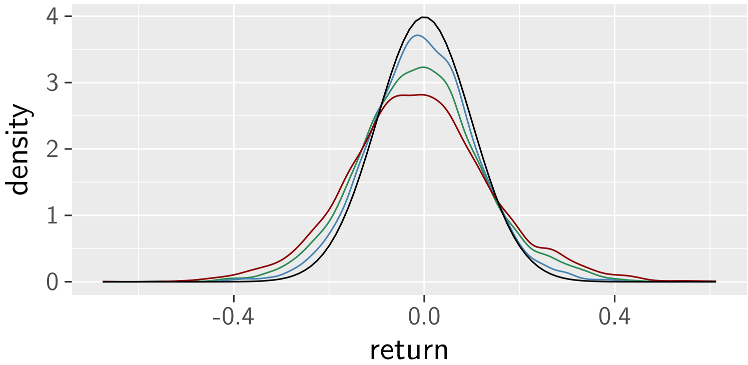
MJD: log-return density

Impact on the skewness (μ)



MJD: log-return density

Impact on the kurtosis (λ)



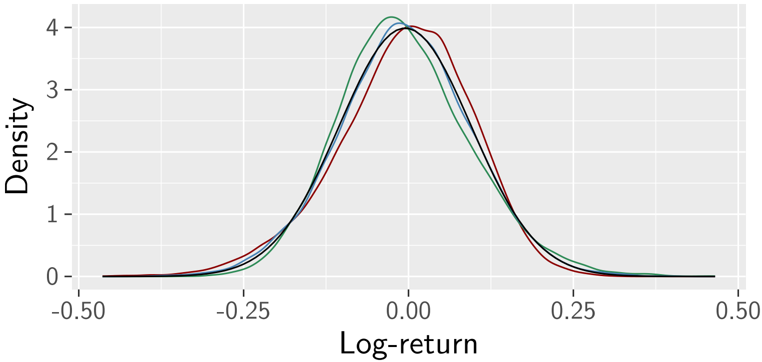
The Heston stochastic volatility model (HSV)

HSV stochastic process

$$
\begin{aligned}
dV(t) &= \kappa(\theta - V(t))\,dt + \sigma\sqrt{V(t)}dW_V(t) \\
dS(t) &= \alpha S(t)\,dt + \sqrt{V(t)}S(t)\,dW_S(t) \\
\rho &= dW_v(t)\,dW_s(t)
\end{aligned}
$$

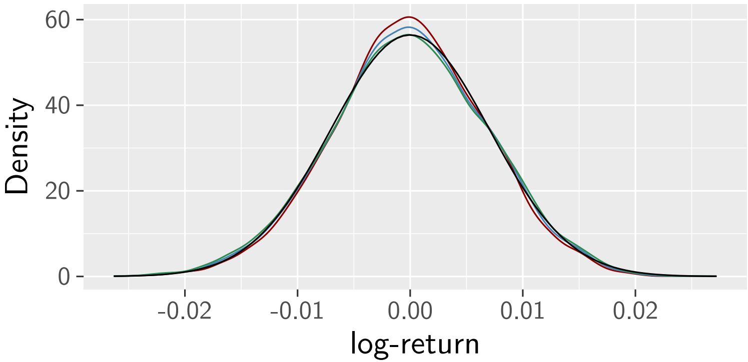
MJD: log-return density

Impact on the skewness (ρ)



MJD: log-return density

Impact on the kurtosis (σ)



Heston probabilistic approach

$$c(t) = S(t)P_1 - e^{-r(T-t)}KP_2$$

With

$$P_1(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi - i)}{i\phi\psi(x, V, t; -i)}\right) d\phi$$

$$P_2(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi)}{i\phi}\right) d\phi$$

Methodology

Calibration

Template data

- Apple stocks (01/01/2017 - 31/12/2017)
- Apple stocks call options (18/05/2018)

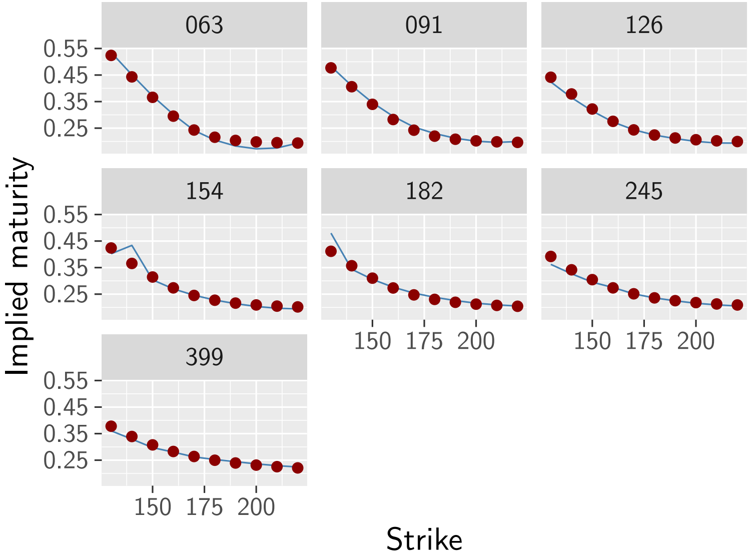
Calibration of the option valuation models

- Concerns the models used to price the options
- least-square non-linear analysis

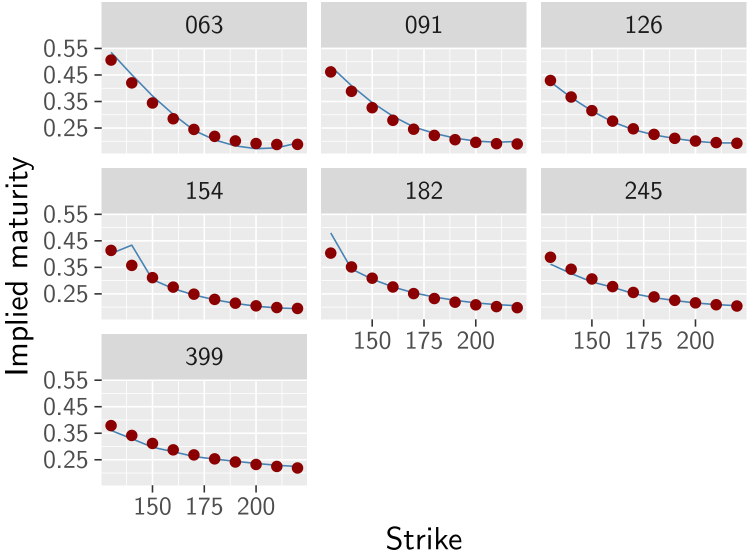
Calibration of the time-series generation models

- Concerns the time-series used to simulate stock prices evolutions
- Fit optimization based method

MJD: Implied volatility



HSV: Implied volatility



Delta hedging

Construction of the delta-neutral portfolio at T=0

$$p(t_0) = \Delta^{m \oplus h}(t_0) S(t_0)$$

Portfolio balancing

$$p(t_i) = \left(\Delta^{m \oplus h}(t_i) - \Delta^{m \oplus h}(t_{i-1}) \right) S(t_i),$$

$\forall i \in \mathbb{Z} : i \in [1, T]$

Delta hedging

Measurement of the performances

$$P\&L = e^{-rT}\frac{\pi(S(T),T)}{c(S(0),0)}$$

Where

$$\begin{aligned}\pi(S(t),t) &= \Delta(t)S(t) + e^{rt}c(S(t_0),t_0) \\ &- \sum_{i\in\mathbb{Z}:i\in[1,t]}\left(e^{r(t-t_i)}p(t_i)\right) - c(S(t),t)\end{aligned}$$

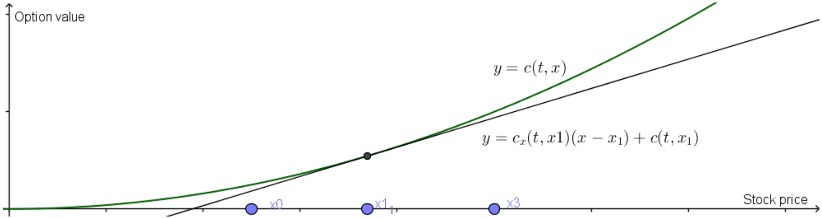
Analysis and results

Analysis and results: BSM

- Balancing frequency
- Negative P&L
    - Gamma

		91 dbm	182 dbm	399 dbm
	intraday	0	0	0
140	daily	0	0	0
	weekly	0	0	0
	intraday	0	0	0
160	daily	0	0	0
	weekly	-0.001	-0.001	-0.003
	intraday	0	0.001	-0.001
186	daily	-0.005	-0.002	-0.004
	weekly	-0.01	-0.019	-0.021
	intraday	0.022	0.008	-0.001
200	daily	-0.002	-0.005	-0.006
	weekly	-0.007	-0.052	-0.037
	intraday	0.02	0.042	-0.007
230	daily	0.022	-0.063	-0.022
	weekly	0.317	-0.285	-0.136

Analysis and results: BSM



- Short Gamma
- Long Theta

Analysis and results: BSM

- Balancing frequency
- Negative effect on P&L
  - Short Gamma
- Positive effect on P&L
  - Long Theta

		91 dbm	182 dbm	399 dbm
	intraday	0	0	0
140	daily	0	0	0
	weekly	0	0	0
	intraday	0	0	0
160	daily	0	0	0
	weekly	-0.001	-0.001	-0.003
	intraday	0	0.001	-0.001
186	daily	-0.005	-0.002	-0.004
	weekly	-0.01	-0.019	-0.021
	intraday	0.022	0.008	-0.001
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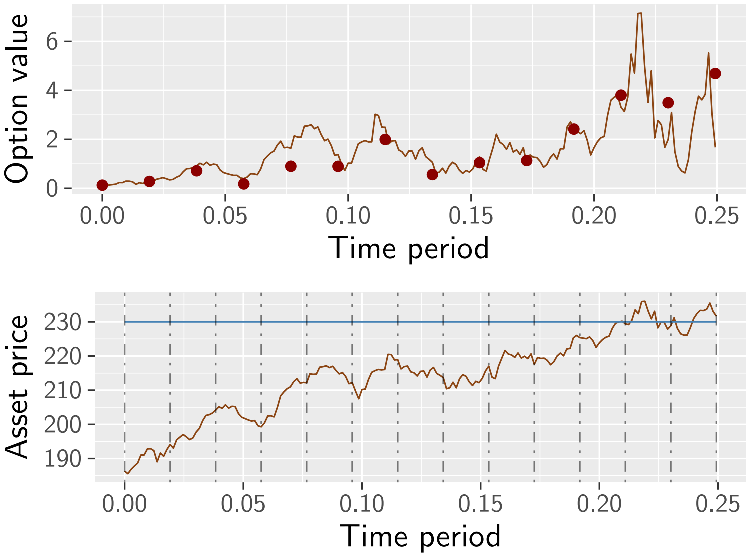


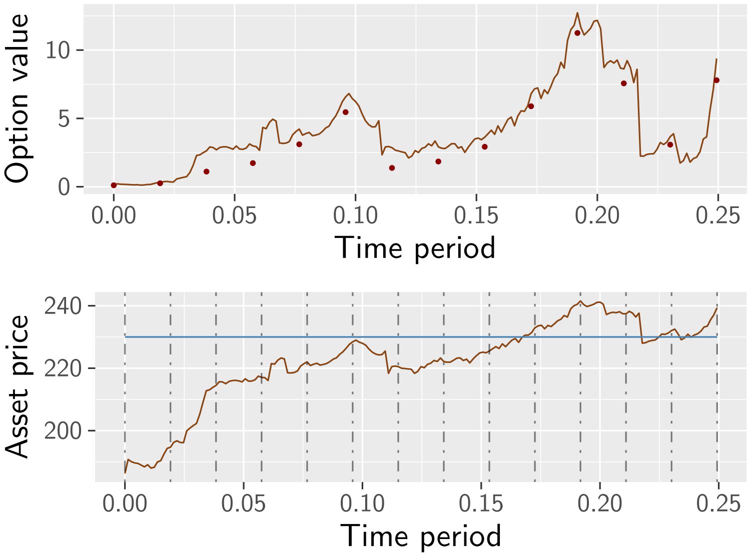
Analysis and results: Merton

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}	Δ_{mrt}	Δ_{bsm}
	intraday	0.004	0.006	0.011	0.012	0.01	0.021
140	daily	0.002	0.006	0.008	0.012	0.016	0.021
	weekly	0.004	0.006	0.006	0.011	0.007	0.021
	intraday	0.011	0.018	0.021	0.029	0.025	0.042
160	daily	0.016	0.018	0.022	0.029	0.019	0.042
	weekly	0.013	0.016	0.018	0.026	0.018	0.04
	intraday	0.036	0.021	0.078	0.055	0.079	0.074
186	daily	0.039	0.022	0.072	0.055	0.068	0.074
	weekly	0.014	-0.008	0.055	0.037	0.057	0.061
	intraday	0.072	-0.002	0.139	0.061	0.13	0.086
200	daily	0.06	-0.013	0.131	0.057	0.115	0.085
	weekly	-0.02	-0.1	0.083	0.005	0.085	0.053
	intraday	0.955	0.331	0.444	-0.061	0.301	0.063
230	daily	1.098	0.466	0.409	-0.091	0.261	0.054
	weekly	-0.741	-1.335	0.085	-0.438	0.174	-0.088

Table: Hedging with MJD: Relative P&L

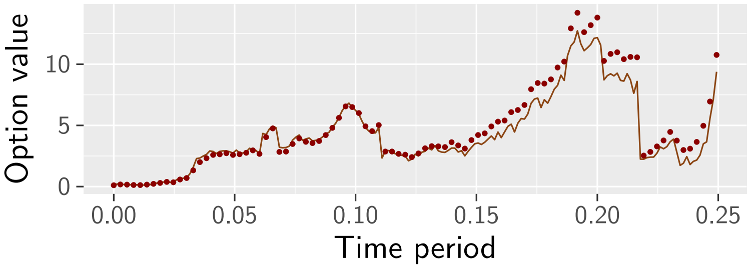
Analysis and results: Merton

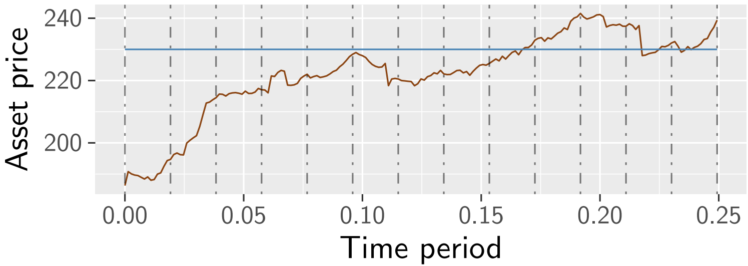
Weekly rebalanced portfolio(K = 230, dbm = 91)



Analysis and results: Merton

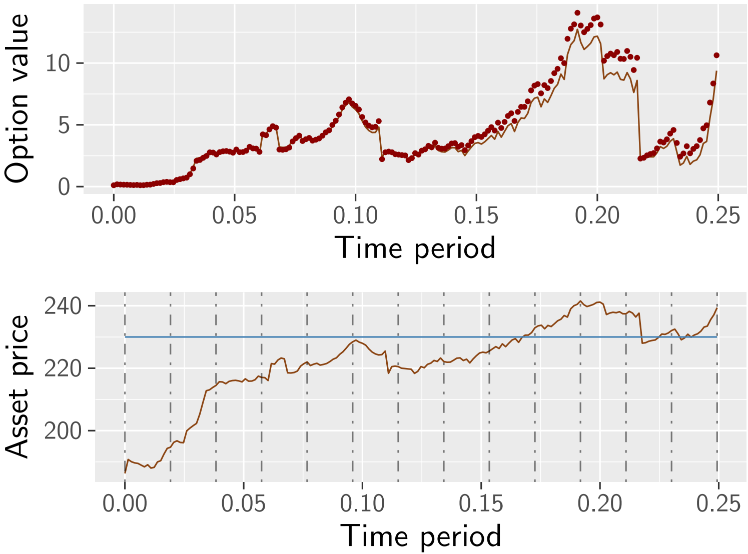
Daily rebalanced portfolio(K = 230, dbm = 91)



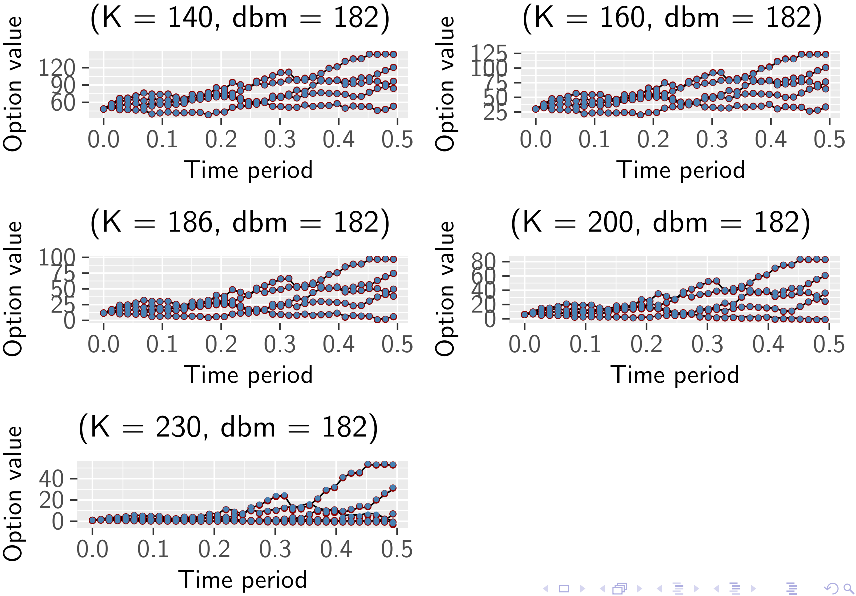


Analysis and results: Merton

Intradaily rebalanced portfolio($K = 230$, dbm $= 91$)



Analysis and results: Merton

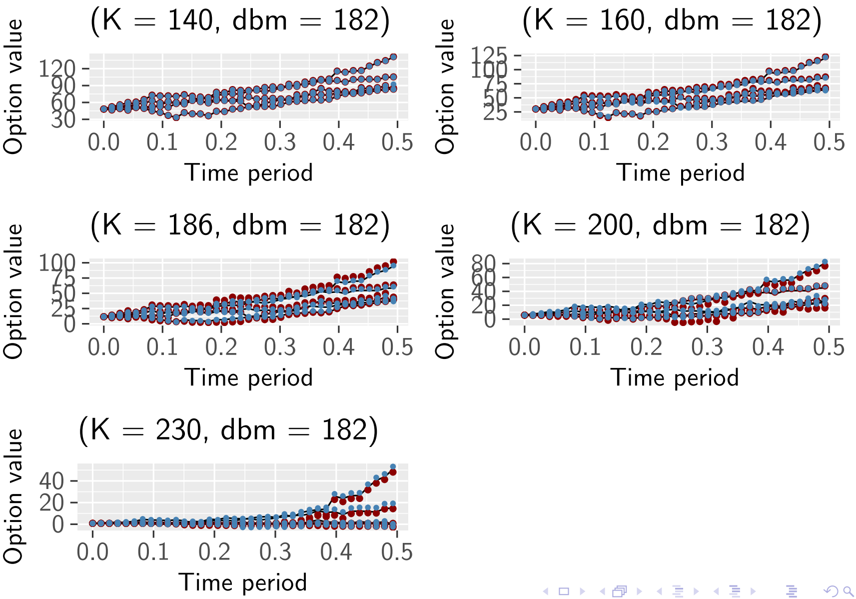


Analysis and results: Heston

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}	Δ_{hsv}	Δ_{bsm}
	intraday	0	0.002	0.011	0.011	0.009	0.038
140	daily	-0.001	0.002	0.01	0.011	0.009	0.038
	weekly	0.001	0.002	0	0.011	0.008	0.038
	intraday	0.009	0.028	0.023	0.073	0.042	0.143
160	daily	0.008	0.028	0.025	0.072	0.036	0.143
	weekly	0.008	0.028	0.019	0.073	0.036	0.143
	intraday	0.158	0.252	0.159	0.392	0.153	0.524
186	daily	0.15	0.245	0.195	0.391	0.156	0.522
	weekly	0.117	0.241	0.158	0.378	0.139	0.519
	intraday	0.459	-0.298	0.43	0.146	0.279	0.546
200	daily	0.433	-0.361	0.42	0.126	0.255	0.544
	weekly	0.268	-0.659	0.369	0.005	0.246	0.498
	intraday	2.136	-0.527	1.884	-2.452	1.01	-0.235
230	daily	1.948	-1.197	1.893	-2.655	0.989	-0.224
	weekly	1.407	-2.152	1.547	-2.402	0.917	-0.353

Table: Hedging with HSV: Relative P&L

Analysis and results: Merton



Conclusion

- Possibility to reproduce volatility smiles
- Better performances for in/at-the-money
- Effects of Gamma / Theta not negligible for out-of-the-money
- Usage of deltas Heston / Merton if possible

How to go further

- Consider a wider range of assets
- Accurately measure the effect of Gamma / Theta