UCL
Université catholique de Louvain

Louvain School of Management (LSM)

LOUVAIN
School of Management

# Hedging performances of the Black-Scholes model in imperfect log-normal world

Mémoire réalisé par
**Anthony Tedde**

Promoteur(s)
**Frédéric Vrins**

Lecteur
**Isabelle Platten**

Année académique 2017-2018
**Master en Sciences de gestion**

# Abstract

The study developed in the current master thesis concerns the Black-Scholes-Merton (BSM) method initially intended to give a fair price to options for which a geometric Brownian motion (GBM) drives the underlying asset. Indeed this model works remarkably well provided that the underpinned assumptions are respected with the most restrictive one stating that the distribution of the log-returns of the underlying asset has to be normal. However, empirical results show that this constraint of log-normality does not hold for real-life trials and consequently break the more key hypothesis of the aforementioned model.

The primary purpose of the analysis developed in that work is to assess how wrong goes the BSM equation while the underlying asset is not a GBM. To that end, others models will be investigated, namely, the Merton jump diffusion (MJD) and Heston stochastic volatility (HSV) models. Each of it bears a new characteristic. The MJD brings jumps to the paths of the underlying processes whilst HSV comes with a volatility parameter that evolves with times unlike the deterministic one inherent to BSM. These specificities affect the log-returns distributions that are no more normal.

Accordingly, once calibrated, those models will be used to simulate a world where the assumption of log-normality of the BSM model does not hold anymore. The study case explored is to measure how the BSM equation will react when its central assumption is broken. To do this, the performance of the delta-neutral portfolios of BSM, MJD and HSV processes will be benchmarked.

As a subanalysis, another goal of that master thesis is to examine if, with a unique set of parameters, the HSV and MJD models can reproduce the volatility smiles constructed from the options prices provided by the market. A conclusion would be that those models are versatile enough to cover a broader range of market prices than the BSM model itself.

# Acknowledgements

First and foremost, I want to address all my gratitude to Anne, and our two wonderful children, Elsa and Valentin, for their patience and huge support. They gave me all the strength needed to go beyond my self-motivation.

Many thanks to all of my family and precious friends that make the life more pleasant. I especially think of Nonna Emma, Mario, Josiane, Maud, Eddy, Christelle, Sébastien, and Pierre.

Thank you mum and dad.

Last but not least, I want to sincerely thank Pr. Frédéric Vrins for trusting me. It was a real pleasure to meet you.

# Contents


**Abstract** **i**

**Acknowledgements** **ii**

**1 Other Models to be considered** **1**

1.1 Merton Mixed jump-diffusion Model . . . . . 1

1.1.1 Risk-neutralized process . . . . . 3

1.1.2 Graphical representation . . . . . 3

1.1.3 Impact on the skewness log–return . . . . . 3

1.1.4 Impact on kurtosis log–return . . . . . 4

1.2 Heston stochastic volatility model . . . . . 4

1.2.1 Model parameters . . . . . 5

1.2.2 Feller condition . . . . . 6

1.2.3 Risk–neutralized processes . . . . . 7

1.2.4 Graphical representation . . . . . 8

1.2.5 Impact on log–return density's skewness . . . . . 8

1.2.6 Impact on kurtosis density return . . . . . 9

1.3 Option pricing method . . . . . 10

1.3.1 Probabilistic approach . . . . . 10

1.3.2 Characteristic function for Merton Mixed jump–diffusion model . . 12

1.3.3 Characteristic function for Heston stochastic volatility model . . . . 13

**Bibliography** **14**

**Appendices** **15**

# List of Figures

1.1 Merton mixed jump-diffusion time-series . . . . . . . . . . . . . . . . . . . . 4
1.2 Merton returns density: Skewness . . . . . . . . . . . . . . . . . . . . . . . . 5
1.3 Merton returns density: Kurtosis . . . . . . . . . . . . . . . . . . . . . . . . 6
1.4 Heson process with negatively correllated Brownian motions . . . . . . . . . 9
1.5 Heson process with positively correllated Brownian motions . . . . . . . . . 10
1.6 Log-returns skewness with Heston . . . . . . . . . . . . . . . . . . . . . . . . 11
1.7 Log-returns kurtosis with Heston . . . . . . . . . . . . . . . . . . . . . . . . 12

# List of Tables

1.1 Merton Mixed jump-diffusion time-series . . . . . . . . . . . . . . . . . . . . . 3

# Chapter 1

# Other Models to be considered

In this chapter are explored models that will substitute the geometric Brownian motion for the stock prices evolution along with the related option pricing method.

The first considered model is the Merton jump-diffusion (MJD) that brings jump to the course of a stochastic process while the second is the Heston stochastic volatility model that allows the volatility parameter to evolve through time.

At the end of that chapter, a method developed by Heston [1993] illustrates how to compute the prices of options based on such underlying by using their characteristic function.

## 1.1 Merton Mixed jump-diffusion Model

In his paper, Merton [1976] provides a model for stock price evolution involving jumps (equation (1.1)).

$$S\left(t\right) = S\left(0\right) e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \tag{1.1}$$

According to Merton [1976], there are two specific sources of uncertainty explained by the model (equation (1.1)).

The first one is qualified as normal, repeatedly arising with low effects and keeping the stock price motion continuous from time to time. These small changes on the price are modeled by a Wiener process, such as it was the case in equation **??**. The cause of these fluctuations is explained by a temporary unbalanced between the supply and demand Merton [1976].

Another type of changes, occurring during the stock lifecycle, is qualified as abnormal by Merton [1976]. Such "abnormalities" happen less frequently, are unpredictable in their frequency and produce bigger effects on the stock price by giving rise to jumps during the course of the stock path, and therefore breaking its continuity. The jump process is constructed on a double basis.

Firstly, the occurrence (i.e. the number of jumps arising throughout a given period of time) is computed thanks to a Poisson-driven process according to a parameter $\lambda$. $\lambda$ denotes the number of jumps per unit of time. Consequently, the probability that a jump occurs during a time range of $\Delta t$ is equal to $\lambda dt$ (event $A$, eq. 1.2), whereas the probability that there are no jump during the same range of time is $1 - \lambda dt$, (event $B$, eq. 1.3) (Matsuda [2004]). While event C, from equation (1.4), refers to the event that more than one jump occurs during the same small delta time.

$$\mathbb{P}\{A\} \cong \lambda dt \tag{1.2}$$

$$\mathbb{P}\{B\} \cong 1 - \lambda dt \tag{1.3}$$

$$\mathbb{P}\{C\} \cong 0 \tag{1.4}$$

On the other hand, after the occurrence, the size of the jump matters. Such as the frequency, a statistic law characterizes the importance of the jump. Following Merton [1976], the log-normal law is used. Matsuda [2004] offers the equations (1.5) to (1.7) to summarize the law and the parameters that describe the jump intensity.

$$y_t \sim lognormal(e^{\mu + \frac{1}{2}\delta^2}, e^{2\mu+\delta^2}(e^{\delta^2} - 1)) \tag{1.5}$$

$$y_t - 1 \sim lognormal(\kappa \equiv e^{\mu + \frac{1}{2}\delta^2} - 1, e^{2\mu+\delta^2}(e^{\delta^2} - 1)) \tag{1.6}$$

$$\ln y_t \sim normal(\mu, \delta^2) \tag{1.7}$$

with $y_t$, $y_t - 1$ and $\ln y_t \equiv Y_t$ standing respectively for "absolute price jump size", "relative price jump size" and "log price jump size" (Matsuda [2004]).

The Merton's jump-diffusion process is be able to capture positive / negative skewness (see section 1.1.3) and excess kurtosis (see section 1.1.4) of the log-return density function, in accordance with Merton [1976].

### 1.1.1 Risk-neutralized process

In order to find the fair price of an option depending on an underlying that follows such a jump-diffusion process, Merton [1976] turns equation (1.1) into one risk-neutral.

$$S(t) = S(0)\, e^{\left(r - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \tag{1.8}$$

Merton [1976] argues in his paper that the jump component of equation (1.1) can be diversified in a well-balanced portfolio and consequently does not need to be risk-neutralized.

However, likewise it was done by Black and Scholes [1973], the drift part of equation (1.1) is risk-neutralized by turning the rate $\alpha$ into its riskfree counterpart $r$, as shown by equation (1.8).

### 1.1.2 Graphical representation

Figure 1.1 shows a unique time-series generated using an implementation of equation (1.1). A jump is clearly marked at day 363. While table 1.1, which is a subset of the time-series drawn in figure 1.1, illustrates numerically when the jump occurs.

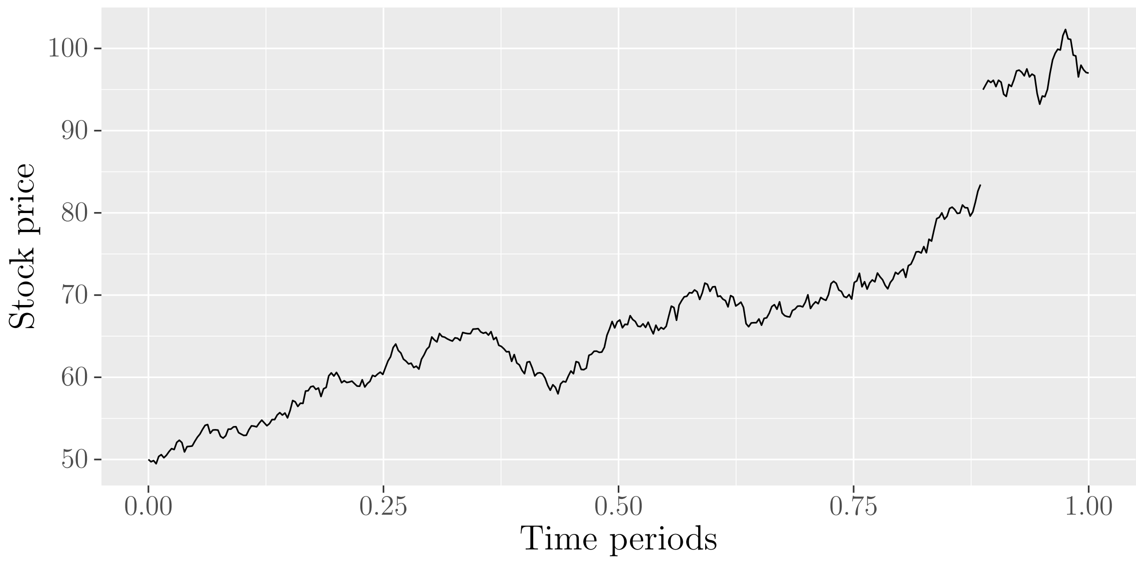
Table 1.1: Merton Mixed jump-diffusion time-series

| time periods (days) | stock price |
|---|---|
| 0 | 50.00 |
| 1 | 49.72 |
| 2 | 49.87 |
| ⋮ | ⋮ |
| 323 | 83.44 |
| 324 | 94.98 |
| ⋮ | ⋮ |
| 363 | 97.41 |
| 364 | 97.07 |
| 365 | 97.00 |

### 1.1.3 Impact on the skewness log–return

The way to influence the direction of the distribution's shape is achieved by moving the cursor of the expected value of jump impact, in other words, by changing the value of the

Figure 1.1: Merton mixed jump-diffusion time-series




**Notes.** Simulation of one Merton jump-diffusion time-series. Data have been output by the R function *mjd_ts* which is an implementation of equation equation (1.1) (see appendix **??**, for more information). The parameters passed to the function are: $S(0) = 50$, $T = 1$ (in year, along with a time step of 365 measures per year), $\sigma = 0.2$, $\alpha = 0.5$, $\lambda = 2$, $\mu = 0.05$, and $\delta = 0.1$.

parameter $\mu$. figure 1.2 shows how the density's shape of the log-return may vary together with this parameter.

### 1.1.4 Impact on kurtosis log–return

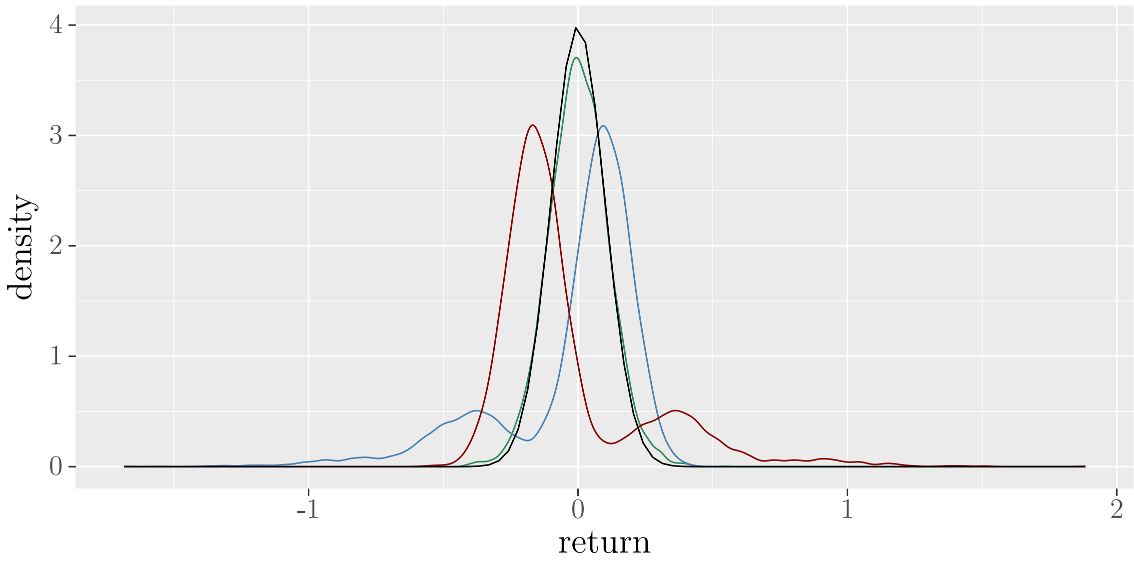
The way to influence the aspect of the distribution's tails is achieved by moving the cursor of the expected value of jump occurrence, in other words, by changing the value of the parameter $\lambda$. The **??** shows how the distribution's tails of the log-return may vary together with this parameter.

## 1.2 Heston stochastic volatility model

In his paper, Heston [1993] tackles with another discrepancy against the real world behavior introduced by the geometric Brownian motion, namely, its deterministic and immutable volatility $\sigma$.

Besides, to provide a model where the volatility is stochastic (equation (1.9)), Heston [1993] gives the possibility to make that volatility in correlation with the stock price process (equation (1.10)), according to the parameter $\rho$ defining how the Brownian motions from both processes relate together.

Figure 1.2: Merton returns density: Skewness




**Notes.** The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (1.1). The only parameter that changes over the group is $\mu$ which is set to $(-0.5, 0, 0.5)$ respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

$$dV(t) = \kappa(\theta - V(t))\,dt + \sigma\sqrt{V(t)}dW_V(t) \tag{1.9}$$

$$dS(t) = \alpha S(t)\,dt + \sqrt{V(t)}S(t)\,dW_S(t) \tag{1.10}$$

The drift part of the risk stochastic process (1.9 is made up of the long–run mean $\theta$ togehter with the mean reversion speed, given by $\kappa$, Heston [1993].

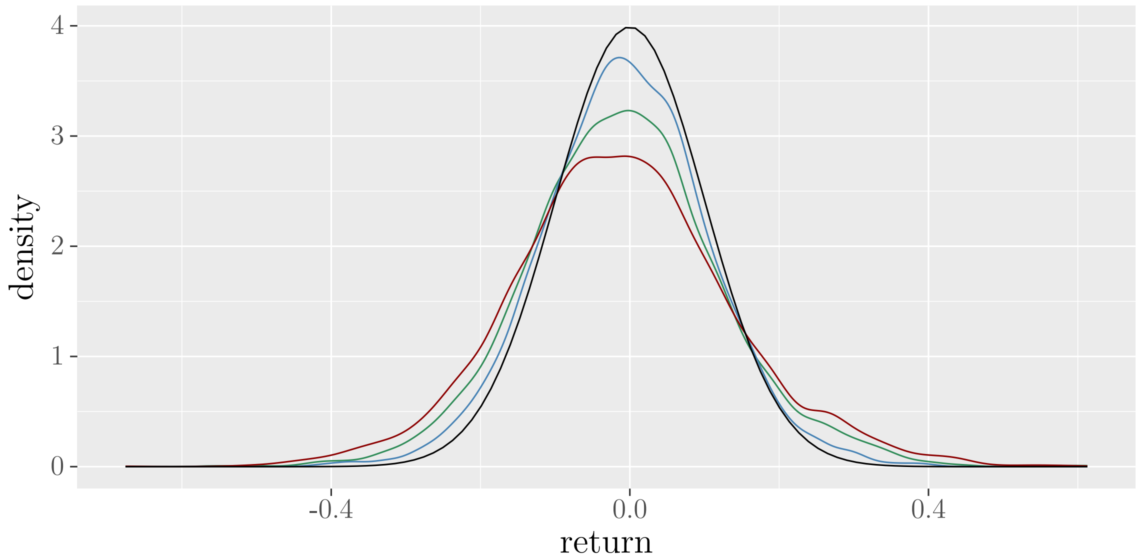
$$dW_v(t)\,dW_s(t) = \rho \tag{1.11}$$

Equation (1.10) represents the evolution of an asset though time, given by its differential form. Such as **??**, developed by Black and Scholes [1973], the parameter $\alpha$ gives the drift rate. The difference between both models lies in the way the volatility is perceived. In Heston [1993], the asset volatility is given by the stochastic equation (1.9). More specifically, the volatility so defined follows a Cox-Ingersoll-Ross process.

### 1.2.1 Model parameters

Here are described all the parameters appearing in the Heston stochastic volatility model.

Figure 1.3: Merton returns density: Kurtosis




**Notes.** The above density function has been constructed over three distinctive groups of 5000 samples eachs. All samples have been constructed following equation (1.1). The only parameter that changes over the group is $\lambda$ which is set to (1, 3, 5) respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of $\sqrt{dt} \times \sigma$.

| | |
|---|---|
| $S(t)$ | Price of the stock at time t. |
| $\alpha$ | Annualized – and deterministic – expected return. |
| $V(t)$ | Observed volatility of the stock at time t. |
| $\kappa$ | Mean-reversion speed. |
| $\theta$ | Volatility's long-run mean. |
| $\sigma$ | Volatility of the volatility. |

### 1.2.2 Feller condition

Due to the time discretization brought by a simulation, the stochastic process 1.9 may turn out to be sometimes negative. If such a value appears at time $t$, the next value computed for $t + \epsilon$ will raise an error, due to the term $\sqrt{V(t)}$ that does not exist for a negative value.

In his paper, Feller [1951] demonstrates that a process such the one described by equation (1.9) does not reach negative values if the following relation 1.12 is respected.

$$\lim_{V\to 0}\left(\kappa\theta - V - \frac{1}{2}\frac{\partial(\sigma\sqrt{V})^2}{\partial V}\right) \geq 0 \tag{1.12}$$

$$\Longleftrightarrow \lim_{V\to 0}\left(\kappa\theta - V - \frac{1}{2}\sigma^2\right) \geq 0$$

$$\Longleftrightarrow \kappa\theta - \frac{1}{2}\sigma^2 \geq 0$$

$$\Longleftrightarrow 2\kappa\theta - \sigma^2 \geq 0 \tag{1.13}$$

Consequently, if the condition related by equation (1.13) is respected, no negative value would occur by using any time-discretized simulation to compute the CIR stochastic volatility.

### 1.2.3 Risk–neutralized processes

Likewise it has been done by Black and Scholes [1973], Heston [1993] used a risk-neutral framework to price options. To do so, Heston modified the drift parameters of both price and volatility stochastic processes.

The drift part of the price diffusion (equation (1.10)) is risk-neutralized by turning the rate $\alpha$ into its riskless counterpart $r$, as shown by equation (1.14).

$$dS\left(t\right) = rS\left(t\right)dt + \sqrt{V\left(t\right)}S\left(t\right)dW_S\left(t\right) \tag{1.14}$$

In order to make the volatility process risk-neutralized, Heston added the risk premium parameter, $\lambda$, to the drift part of equation (1.9). Crefeq:other:hsvvol:riskless gives the so risk-neutralized CIR process.

$$dV\left(t\right) = \kappa^*\left(\theta^* - V\left(t\right)\right)dt + \sigma\sqrt{V\left(t\right)}dW_V\left(t\right) \tag{1.15}$$

where

$$\kappa^* = \kappa + \lambda \tag{1.16}$$

and

$$\theta^* = \frac{\kappa\theta}{\kappa^*} \tag{1.17}$$

Consequently, the parameters $\kappa^*$ and $\theta^*$, which respectively denote the long-run mean and mean-reversion speed, are the ones to estimate while dealing with HSV pricing options purposes.

### 1.2.4 Graphical representation

Figures 1.4 and 1.5, give a hands-on insight in how the correlation between the underlying Brownian motions of the stock and volatility time series affect both processes. figure 1.4 shows a correlation between the Wiener processes $B_1$ and $B_2$ sets to $\rho = -1$, making the two Markov motions perfectly negatively correlated. It directly affects the course of the stocks series, which is altogether correlated in the same negative direction with respect to the CIR volatility process as well. Likewise, figure 1.5 points out the fully positive correlation occuring between the processes 1.9 and 1.10 whilst the Brownian motions correlation is set to one.

As shown by Heston [1993], the usage of the aforementioned Heston model lies in the fact that the correlation between the CIR and asset processes' Brownian motions would notably explain the spot return skewness whereas the kurtosis of the distribution may be affected by the volatility parameter $\sigma$ of the stochastic volatility process (equation (1.9)). It may consequently be consistent with what happens in the equity market, namely a sharp decrease in equity price implies an increase in stock volatility (Crisóstomo [2015]).

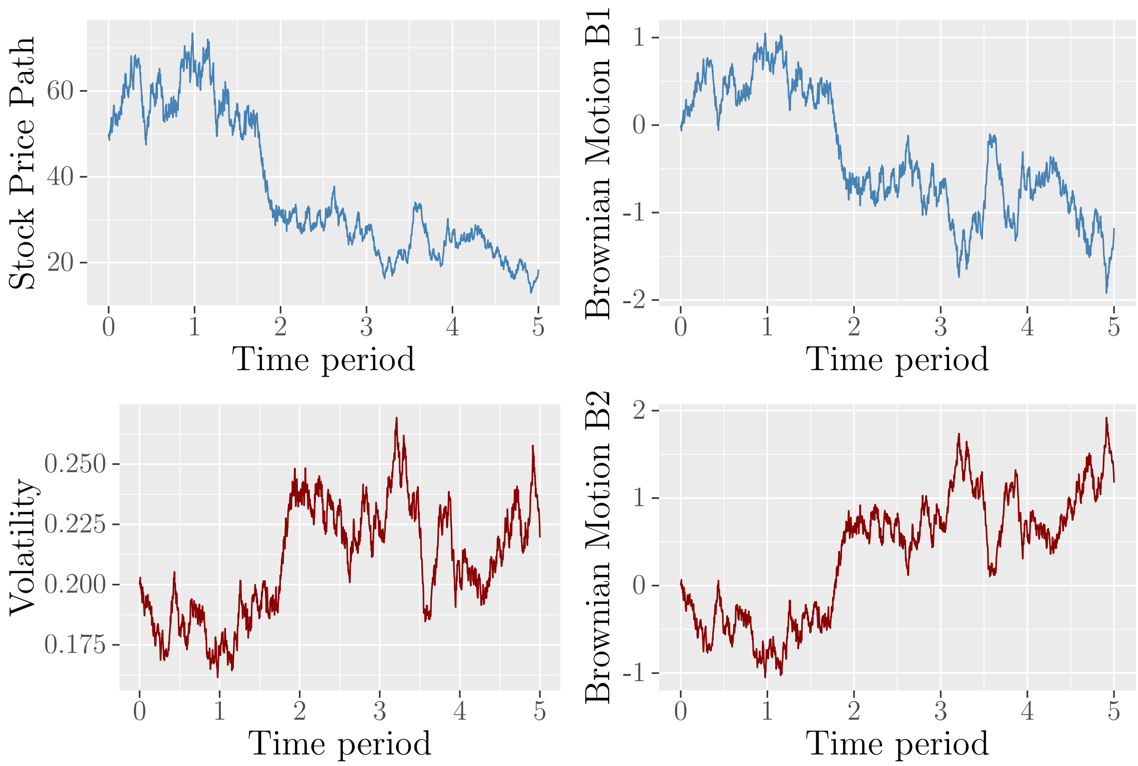
### 1.2.5 Impact on log–return density's skewness

Through the Heston stochastic volatility model, the skewness of the distribution of continuously compounded spot return may be affected by the parameter $\rho$.

When a positive correlation exists between both Brownian motions, an increase in the volatility implies a rise in the asset price whereas a decrease in the volatility tends to lower the asset price. In other words, when the uncertainty is high, and consequently the changes in the asset price are numerous, these latter tend to be positive. That is why the distribution of the spot return is offset to the left with a right fat tail when $\rho$ is positive.

The opposite relation is noticed with negative correlation, namely, lower prices relate to

Figure 1.4: Heson process with negatively correllated Brownian motions




**Notes.** Simulation of Heston time-series. Data have been output by the R function *hsv_ts* which is an implementation of equation equations (1.9) to (1.11) (see appendix **??**, for more information). The parameters passed to the function are: $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 measures per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$ and $\rho = -1$.

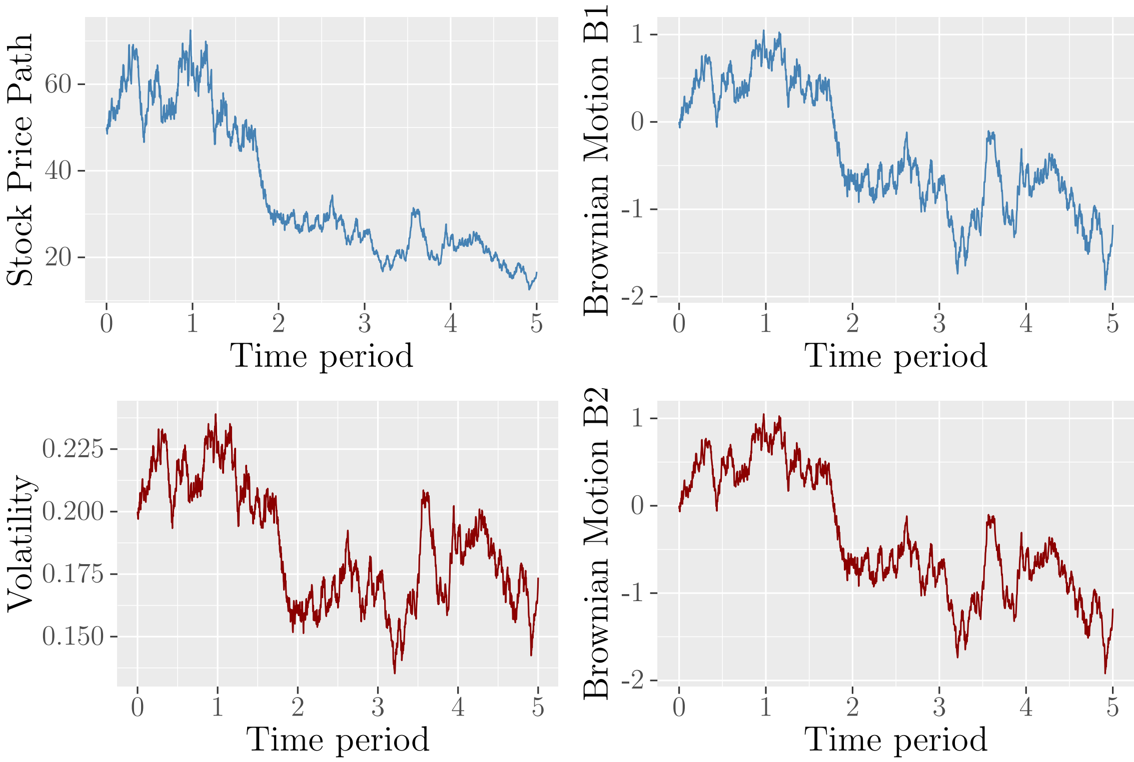
higher volatility generating a left fat tail in the log–return distribution (Heston [1993]). These statements are observed in figure 1.6.

### 1.2.6 Impact on kurtosis density return

Following Heston [1993], the kurtosis of the distribution of the spot return may be affected by the parameter $\sigma$, which represent the volatility of the volatility.

First and foremost, following equation (1.9) if $\sigma = 0$, the volatility V of the Heston model turns out to be deterministic and equation (1.10) becomes a geometric Brownian motion with normal distribution for the time-series' log-returns. Otherwise, Heston [1993] showed that by raising $\sigma$, the kurtosis of the spot returns increases. Consequently, within the Heston stochastic volatility model, the bigger $\sigma$, the fatter the tail, ceteris paribus. These statements are observed in figure 1.7.

Figure 1.5: Heson process with positively correllated Brownian motions




**Notes.** Simulation of Heston time-series. Data have been output by the R function *hsv_ts* which is an implementation of equation equations (1.9) to (1.11) (see appendix **??**, for more information).. The parameters passed to the function are: $S(0) = 50$, $V(0) = 0.2$, $T = 5$ (years, along with a time step of 365 measures per year), $\alpha = 0$, $\kappa = 0.5$, $\Theta = 0.2$, $\sigma = 0.1$ and $\rho = 1$.

## 1.3 Option pricing method

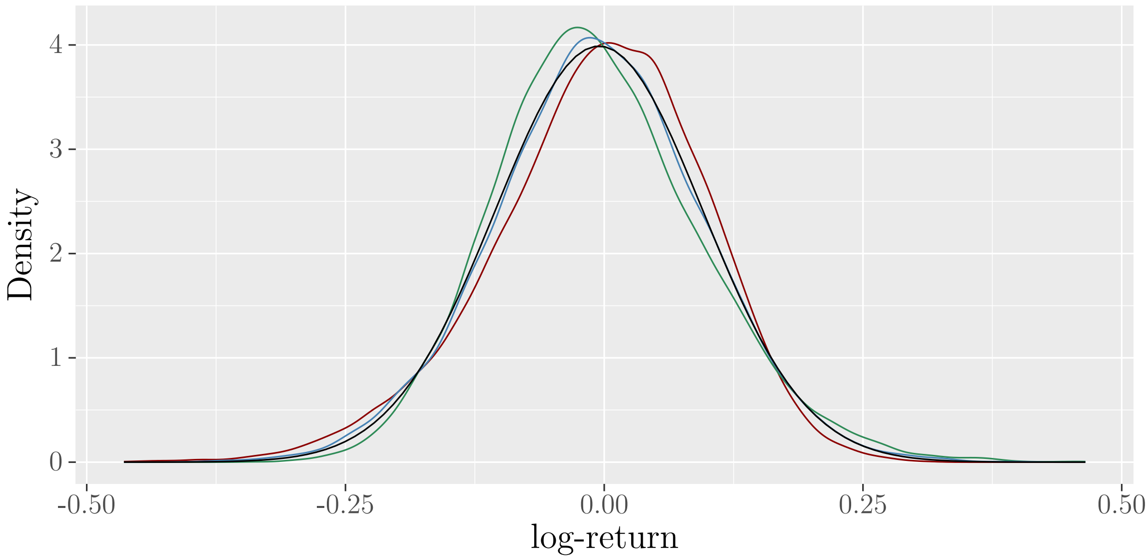
As shown by section 1.1 and section 1.2, the frameworks developed by Merton [1976] and Heston [1993] drastically change the distribution of any underlying assets following such processes. Therefore, the pricing method to be used must also be adapted in order to take that update into account.

In his paper, Heston [1993] developed a technique to price options using the characteristic function of the underlying asset. Furthermore, according to Crisóstomo [2015] that method could be used to price any option provided that the underlying's characteristic function is known.

### 1.3.1 Probabilistic approach

Heston [1993] proposed the solution given by equation (1.18) to price a european call option.

Figure 1.6: Log-returns skewness with Heston




**Notes.** The above density functions (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (1.10) for the stock data and on equation (1.9) for the related volatility (see function *hsv_ts()* on appendix **??** for more information). The only parameter that changes over the groups is $\rho$ which is set to $-0.5$, 1, 0.5. The log-return densities of these groups are respectively represented by the red, green and blue outlined density functions. The black density represents to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log-price return cover one year with a time step of 500.

$$c(t) = S(t)P_1 - e^{-r(T-t)}KP_2 \tag{1.18}$$

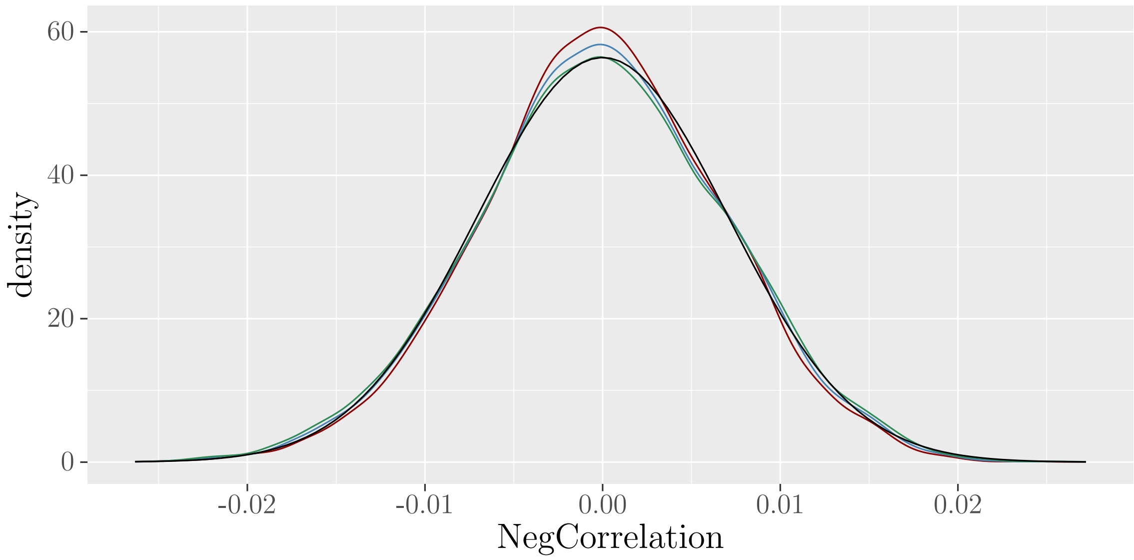
Through this method, the european call price at time $t$, namely $c(t)$, is computed thanks to equation (1.18), where $S(t)$ and $e^{r(T-t)}$ respectively stand for the stock price and the present value of the strike at that time $t$.

$$P_1(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi - i)}{i\phi\psi(x, V, t; -i)}\right) d\phi \tag{1.19}$$

$$P_2(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-i\phi \ln K}\psi(x, V, t; \phi)}{i\phi}\right) d\phi \tag{1.20}$$

Following the development in Crisóstomo [2015], both equations (1.19) and (1.20) are probability quantities that involve the underlying characteristic function, namely $\psi(x, V, t; \phi)$.

Figure 1.7: Log-returns kurtosis with Heston




**Notes.** The above density functions (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (1.10) for the stock data and on equation (1.9) for the related volatility (see function *hsv_ts()* on appendix **??** for more information). The only parameter that changes over the groups is $\sigma$ which is set to 0, 0.2, 0.4. The log-return densities of these groups are respectively represented by the green, blue and red outlined density curves. The black density represents to the normal bell curve with mean $-\frac{\theta}{2}$ and standard deviation of $\sqrt{\theta}$. The log-price return cover one year with a time step of 500.

Once these quantities are computed, they are substituted in equation (1.18) in order to get the call price at time $t$.

The characteristic functions for the Merton jump-diffusion (section 1.1) and Heston stochastic volatility (section 1.2) models are developed sections 1.3.2 and 1.3.3

### 1.3.2 Characteristic function for Merton Mixed jump–diffusion model

Matsuda [2004] demonstrates that the characteristic function of the Merton mixed jump-diffusion process is given by equation (1.21). The baseline equation to construct the Merton's process characteristic function is the risk-neutralized version, namely equation (1.8).

$$\psi^{merton}(\phi) = e^{\lambda(T-t)\left(e^{i\mu\phi-\frac{\delta^2\phi^2}{2}}-1\right)+i\phi\left(\ln S(t)+\left(r-\frac{\sigma^2}{2}-\lambda\kappa\right)(T-t)\right)-\sigma^2\frac{\phi^2}{2}(T-t)} \tag{1.21}$$

where

$$\kappa = e^{\mu+\frac{\delta^2}{2}} - 1 \tag{1.22}$$

According to the method given by Heston [1993], the characteristic function 1.21 will be used inside equations (1.19) and (1.20) in order to compute the quantities $P_1$ and $P_2$ that could be thereafter replaced inside equation (1.18) to find the european call price $c(t)$ corresponding to a stock price process $S(t)$ driven by the Merton mixed jump-diffusion model.

### 1.3.3 Characteristic function for Heston stochastic volatility model

Following the development proposed by Gatheral and Taleb [2006], Crisóstomo [2015] provided the Heston characteristic function (equation (1.23)) based on the process $\ln S(t)$.

$$\psi^{heston}\left(\ln(S(t)), V(t), t; \phi\right) = e^{C(T-t,\phi)\theta + D(T-t,\phi)V(t) + i\phi\ln\left(S(t)e^{r(T-t)}\right)} \tag{1.23}$$

where

$$C(\tau,\phi) = \kappa\left(r_-\tau - \frac{2}{\sigma^2}\ln\left(\frac{1-ge^{-h\tau}}{1-g}\right)\right)$$
$$D(\tau,\phi) = r_-\frac{1-e^{-h\tau}}{1-ge^{-h\tau}}$$

and

$$r_\pm = \frac{\beta \pm h}{\sigma^2}; h = \sqrt{\beta^2 - 4\alpha\gamma}$$
$$g = \frac{r_-}{r_+}$$
$$\alpha = -\frac{\phi^2}{2} - \frac{i\phi}{2}; \beta = \kappa - \rho\sigma i\phi; \gamma = \frac{\sigma^2}{2}$$

Equation (1.23) can be directly used inside equations (1.19) and (1.20) in order to compute the quantities $P_1$ and $P_2$ that could be thereafter replaced inside equation (1.18) to find the european call price $c(t)$ corresponding to a stock price process $S(t)$ driven by the Heston stochastic volatility model.

# Bibliography


Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.

Ricardo Crisóstomo. An analysis of the heston stochastic volatility model: Implementation and calibration using matlab. *arXiv preprint arXiv:1502.02963*, 2015.

William Feller. Two singular diffusion problems. *Annals of Mathematics*, 54(1):173–182, 1951. ISSN 0003486X. URL `http://www.jstor.org/stable/1969318`.

J. Gatheral and N.N. Taleb. *The Volatility Surface: A Practitioner's Guide*. Wiley Finance. Wiley, 2006. ISBN 9780470068250. URL `https://books.google.be/books?id=9Y8rWE6mLOEC`.

Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343, 1993.

Kazuhisa Matsuda. Introduction to merton jump diffusion model. *Department of Economics. The Graduate Center, The City University of New York*, 2004.

Robert C Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2):125–144, 1976.

# Appendices