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# Introduction

Talk about what is done to price a vanilla option throughout the BSM method. How does the BSM model is fair under its assumption. What about if we are going beyond ? How performant is it ? What about other model such as ... ?

Using R. R Core Team [2017]

# Chapter 1

## Upstream concepts

The material covered in the current chapter is meant to be thereafter intensively used, either by being applied in the studied theoretical models or by entering in the construction of the theory-based algorithms.

### 1.1 Vanilla options

The so-called vanilla options, in opposition to the more complex - not covered in this master thesis - exotic ones, are specific kind of derivatives coming along with some good-to-know jargons.

First and foremost, as defined in Hull and Basu [2012], an option is a contract between two stakeholders with different interests or at least distinctive motivations, who want to buy or sell a product, which generally is a financial asset called "the underlying." Indeed, someone can look for hedge oneself against risk while the other party wants to make a profit on a speculative move. Likewise other financial contracts, one agrees to buy and the other to sell at a fixed amount of money, namely "the strike price", usually denoted by  $k$  with the difference that they would effectively realize the purchase or the sale of the underlying at a future date than the one they enter into the bargain. That date is called "the maturity".

Another key characteristic of options is that they are not symmetric contracts, in the sense that both parties do not have the same rights, depending on their position along with the option's type. Two positions can be taken when entering into such a derivative, either long or short. Going short broadly means purchase the option whereas going long implies writing, or synonymously, selling the derivative. Beyond the position, the contract can be of call or put type.

The following sentences encompass a mix of type (call/put) and position (short/long) combination that gives an overview of all the possible scenarios:

- Someone who is going long into a call gains the right to purchase the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.
- Someone who is going short into a call is forced to sell the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.
- Someone who is going long into a put gains the right to sell the underlying at a future date for a fixed price. He has to pay some fees to enter into the contract.

- Someone who is going short into a put is forced to buy the underlying at a future date for a fixed price. He receives financial compensation to enter into the contract.

Moreover, vanilla options may be "European" or "American". Latter can be exercised at any time during the whole life of the contract while the European can only be at maturity.

Finally, the payoff provided by all aforementioned European options are summarized throughout equations (1.1) to (1.4), where  $C_l$ ,  $C_s$ ,  $P_l$  and  $P_s$  respectively stand for the long call, short call, long put and short put payoffs.  $S(T)$  being the price of the underlying at maturity.

$$C_l = (S(T) - k)^+ \quad (1.1)$$

$$C_s = (k - S(T))^- \quad (1.2)$$

$$P_l = (k - S(T))^+ \quad (1.3)$$

$$P_s = (S(T) - k)^- \quad (1.4)$$

The derivative considered in this master thesis is mainly the European call option, even though the link with European put could somehow be done without being mandatory to exhibit the issue so raised. Therefore such reference to put should only be done if necessary.

## 1.2 Brownian Motion

All over the present document, the terms "Brownian motion", "Wiener" or "Markov" processes are considered to be equivalent terminology and therefore used as such. However strictly speaking, even though the Brownian motion is in every respect a Wiener process, *stricto sensu*, the term Markov process is broader. Actually, the more remarkable Markov property is to be a process with independent future increments, putting it in line with the weak form of market efficiency (Hull and Basu [2012]). Whilst, on the other hand, even if the Brownian motion, and thus the Wiener process, share this property, they are defined with the idiosyncrasy to have mean zero and a variance rate of one per unit of time (Hull and Basu [2012]).

Brownian motion is a centerpiece broadly used in subsequent developments. It is notably applied in many models such as in the geometric Brownian motion (??), in the Black-Scholes-Merton (BSM) equation (chapter 2) and in the Merton mixed jump-diffusion (MJD) and Heston stochastic volatility (HSV) model (??).

The Brownian motion is a stochastic process, denoted by  $W(t)$  and satisfying  $W(0) = 0$ , with independent and identically distributed (iid) increments, such as defined by equation (1.5)

$$W(t_{i+1}) - W(t_i) \sim iidN(0, t_{i+1} - t_i) \quad (1.5)$$

That Markov process is qualified as being time and path dependent in its building blocks. Time dependency implies that like many other functions its value evolves over time. Whereas path dependency means that the Wiener process is also meant to randomly

move between two time-steps. Practically, at any time  $t$  and from its origin, the Brownian motion may take any value as shown by equation (1.6).

$$W(t) \sim N(0, t) \quad (1.6)$$

As described by Shreve [2004], the construction of such a process could be achieved by following various techniques. Either by computing the differential process (equations (1.7) and (1.8)) for every time-steps and then combining them, by constructing a unique time series in that way; or by resolving the joint moment generating function of the iid random variables' vector  $(W(t_i) \dots W(t_m))$ , through equation (1.9).

$$dW(t) = \phi(0, t + \epsilon_i) \quad (1.7)$$

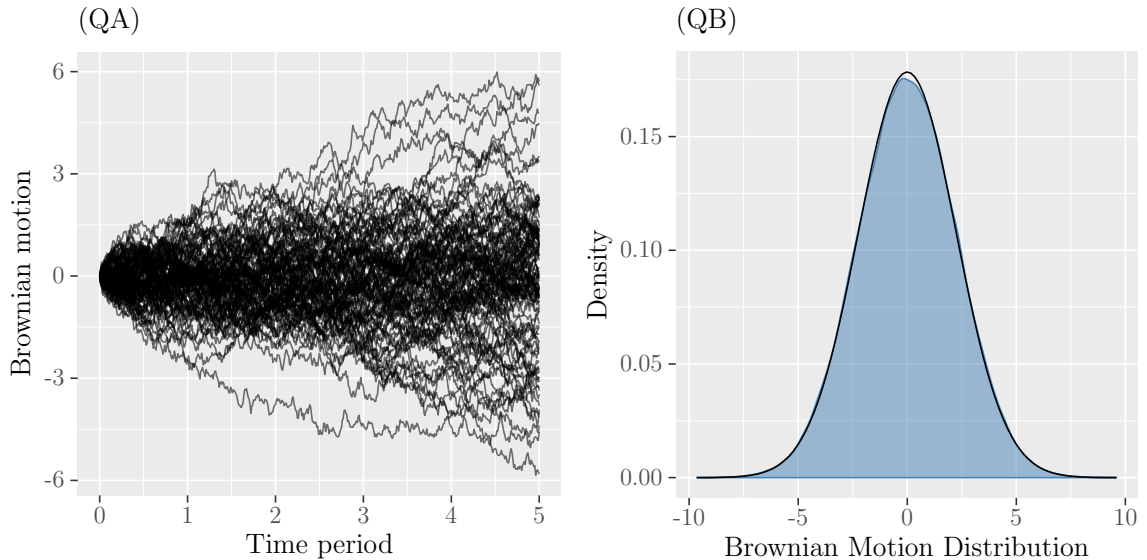
$\phi$  is a function that generate a random number according to the normal law with parameter  $\mu = 0$  and  $\sigma^2 = t + \epsilon_i$ , with  $\epsilon_i$  being any arbitrary small delta time step.

$$W(t + \epsilon_i) = W(t) + dW(t) \quad (1.8)$$

$$\varphi(u_1 \dots u_m) = \exp \left( \frac{1}{2} \sum_{i=1}^m \left[ \sum_{j=i}^m u_j \right]^2 (t_i - t_{i-1}) \right) \quad (1.9)$$

Figure 1.1 (QA) displays a simulation of one hundred Brownian motions over a time frame of five years. While in turn, figure 1.1 (QB) shows the distribution of the possible outcomes of that experiment, after five years. The black outlined curve is the density of the normal distribution, with mean and variance chose such as exposed by equation (1.5).

Figure 1.1: Multiple Brownian Motion



**Notes.** (QA): Simulation of one hundred Brownian motions by using the function *bmotion* of the R package Tedde [2017], with a timeframe of five years long and a discrete time-step of one over one hundred. (QB): The blue filled curve is the density computed on the simulation (QA). The distribution concerns all the final values  $W(T = 5)$ . The Black bell curve represents the normal density with mean zero and a standard deviation of square root of five.

Lastly, Shreve [2004] demonstrates that a Brownian motion has no tendency to rise or fall as time passes thanks to its martingale property. Therefore, because its initial value is zero, a Wiener process only brings noise by being incorporated into other stochastic series.

### 1.2.1 Correlated Brownian motions

Correlated Brownian motions are Wiener processes that related together according to the factor  $\rho$  1.10. Following Shreve [2004], these processes happens to be modeled by equations (1.11) and (1.12).

$$dB_1(t)dB_2(t) = \rho(t)dt \quad (1.10)$$

Where  $dB_1(t)$  and  $dB_2(t)$  are Wiener process

$$B_1(t) = W_1(t) \quad (1.11)$$

$$B_2(t) = \int_0^t \rho(s)dW_1(s) + \int_0^t \sqrt{1 - \rho^2(s)}dW_2(s) \quad (1.12)$$

whenever  $\rho$  is considered as constant over time, however, the simple form given by equation (1.13) is enough to construct such processes.

$$B_2(t) = \rho W_1(s) + \sqrt{1 - \rho^2}W_2(s) \quad (1.13)$$

## 1.3 Ito's lemma

Based on Taylor approximation theorem, Ito formulae are meant to provide the differential form of such a function  $f \circ g$  where the function  $f$  can be differentiated with respect to its independent variable whilst  $g$  cannot.

$$\begin{aligned} f(T, W(T)) = f(0, W(0)) + \int_0^T \frac{\partial f(t, W(t))}{\partial t} dt + \int_0^T \frac{\partial f(t, W(t))}{\partial W(t)} dW(t) \\ + \frac{1}{2} \int_0^T \frac{\partial^2 f(t, W(t))}{\partial W(t)^2} dt \end{aligned} \quad (1.14)$$

While equation (1.14) shows the formula for Brownian motion, the equation (1.16) represents the form used to differentiate the more complex Ito process (1.15).

$$X(T) = X(0) + \int_0^T \Delta(u)dW(u) + \int_0^T \Theta(u)du \quad (1.15)$$

Where  $\Delta(u)$  and  $\Theta(u)$  are stochastic process, adapted to a filtration  $\mathcal{F}(t)$

$$\begin{aligned} f(T, X(T)) = f(0, X(0)) + \int_0^T \frac{\partial f(t, X(t))}{\partial t} dt + \int_0^T \frac{\partial f(t, X(t))}{\partial X(t)} \Delta(t)dW(t) \\ + \int_0^T \frac{\partial f(t, X(t))}{\partial X(t)} \Theta(t)dt + \frac{1}{2} \int_0^T \frac{\partial^2 f(t, X(t))}{\partial W(t)^2} \Delta^2(t)dt \end{aligned} \quad (1.16)$$

Ito formulae are thereafter applied to derive such a process like the geometric Brownian motion or the Black-Scholes-Merton partial differential equation.

## 1.4 Cox-Ingersoll-Ross

The Cox-Ingersoll-Ross (CIR) stochastic model  $R(t)$ , defined by the differential equation (1.17), can be used to simulate interest rates' evolution over time thanks to some of its intrinsic characteristics, namely the mean-reversion and its property to only take positive values (Shreve [2004]).

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\sqrt{R(t)}dW(t) \quad (1.17)$$

According to Shreve [2004], a mean-reverting stochastic process tends to navigate, such a pointy sinusoidal motion, around its mean. The CIR inherits this behaviour from the construction of its differential's drift part, i.e.  $(\alpha - \beta R(t))dt$ . Indeed when  $R(t) = \alpha/\beta$ , then the drift term  $dt = 0$  with the consequence of status quo. In addition, whether  $R(t) > \alpha/\beta$  or  $R(t) < \alpha/\beta$ , the next value of  $R(t + \epsilon) = R(t) + dR(t)$  is pushed back toward  $\alpha/\beta$ . Actually, as showing by equation (1.18), the long-run expected value for the process  $R(t)$  is  $\alpha/\beta$ .

$$\lim_{t \rightarrow \infty} \mathbb{E} R(t) = \frac{\alpha}{\beta} \quad (1.18)$$

On the other hand, depending on Shreve [2004], the non-negativity property is explained by the fact that if  $R(t) \rightarrow 0$  then  $dR(t) \simeq \alpha dt > 0$  making  $R(t + \epsilon)$  bounced off the x axis, running it away from negative realm.

Ultimately, the CIR mean-reverting equation (1.17) is used by Heston [1993] in its pricing model to construct stochastic interest rates. Heston's model is covered in ??.

## 1.5 Skewness and Kurtosis

The skewness and kurtosis of a random variable's distribution, whose respectively are the third and fourth moment, characterize either the asymmetry, for the first, or the degree of flatness, for the second.

The theoretical moments are defined by equation (1.19) while those empirical can be estimated using equation (1.20), with the parameter  $r = 3$  to compute the skewness or  $r = 4$  for the kurtosis.

$$\gamma_r = \mathbb{E} \left[ \left( \frac{X - \mu}{\sigma} \right)^r \right] \quad (1.19)$$

where  $X$  is any random variable

$$m_r = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^r \quad (1.20)$$

with  $\{x_i\}_{i \in n}$  being a set of outcomes belonging to the sample set.



### 1.5.1 Estimation

As demonstrated in Joanes and Gill [1998], sampled skewness and kurtosis formulae can be computed in some different fashions. All of these methods bring their precisions to the calculation of those moments. The precision of the method mostly depends on the size of the sample along with the skewed of the theoretical distribution to be estimated.

The method having to be chosen to estimate the skewness and kurtosis needs a prudent selection especially for a small sample size. Actually, in order to minimize the mean-squared error and the associated variance, the equations (1.21) and (1.22) are used as a reliable unbiased estimator for a sample with normal shaped theoretical distribution.

$$b_{skewness} = \frac{m_3}{S^3} \quad (1.21)$$

$$b_{kurtosis} = \frac{m_4}{S^4} \quad (1.22)$$

where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Conversely, equations (1.23) and (1.24) are the ones providing a better-unbiased estimation for more skewed distributions, such as for log-normal random variable's samples.

$$G_{skewness} = \frac{k_3}{\sqrt{k_2^3}} \quad (1.23)$$

$$G_{kurtosis} = \frac{k_4}{k_2^2} \quad (1.24)$$

where

$$\begin{aligned} K_2 &= \frac{n}{n-1} m_2 \\ K_3 &= \frac{n}{(n-1)(n-2)} m_3 \\ K_4 &= \frac{n}{\prod_{i=1}^3 n-i} \left[ (n+1)m_4 - 3(n-1)m_2^2 \right] \end{aligned}$$

Accordingly, the algorithm to apply in order to estimate the skewness and kurtosis from samples should be chosen with respect to the theoretical underpinned distribution.

## 1.6 Log-return and compounded interest rate

Following Hull and Basu [2012], the concept of log-returns and continuously compounded interest rates are closely related together as attested by equations (1.25) to (1.27).

$$S(0)e^{R_c t} = S(t) \quad (1.25)$$

$$\iff \ln S(0) + R_c t = \ln S(t) \quad (1.26)$$

$$\iff \ln \frac{S(t)}{S(0)} = R_c t \quad (1.27)$$

Where  $R_c$  is the interest rate with continuous compounding,  $t$  denotes the time period across which the interests are compounded ( $t$  is yearly-based) while the left-hand side of equation (1.27) stands for the natural logarithm of the stock return occurring during the period  $t$ , the so-called log-return.

As showed in Hull and Basu [2012], the transformation of an equivalent rate  $R_m$  with a frequency of compounding  $m$  per year into one continuous could be perform thanks to equation (1.28)

$$R_c = m \ln \left( 1 + \frac{R_m}{m} \right) \quad (1.28)$$

For what matters in that master thesis, the focus is set to the sample's log-returns which are used to estimate the volatility term appearing, inter alia, in ?? in order to simulate geometric Brownian motion.

In accordance with Hull and Basu [2012], if  $S_i$  and  $n$  respectively denote the stock price at the end of interval  $i$  and the total number of observations then the log-returns of the ordered sample set  $\{S_i\}_{i \in n}$  is given by equation (1.29).

$$u_i = \ln \frac{S_i}{S_{i-1}} \quad (1.29)$$

Whilst the estimation of the standard deviation of all the  $u_i$  is defined as such in ??

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (u_i - \bar{u})^2} \quad (1.30)$$

$$(1.31)$$

Consequently, if the distribution of the log-returns of a stochastic process is normal and is given by equation (1.32), the parameters  $\alpha$  and  $\sigma$  can be assessed by respectively using equations (1.33) and (1.34). This method will be used notably used in chapter 4 to find the calibrated value of some parameters.

$$N \sim \left( \left( \alpha - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right) \quad (1.32)$$

$$\hat{\alpha} = \frac{\bar{x}}{t} + \frac{\hat{\sigma}^2}{2} \quad (1.33)$$

$$\hat{\sigma} = \frac{s}{\sqrt{t}} \quad (1.34)$$

## Chapter 2

# The Black-Scholes-Merton option pricing model

The Black-Scholes-Merton (BSM) model is meant to provide the no-arbitrage price of derivative assets such as European stock option. That model lies in the resolution of a partial differential equation to find the fair price of a derivative at each period of time before maturity. The foundations come from the work of Black and Scholes [1973] and Merton [1973]. While Fischer Black and Myron Scholes had used the capital asset pricing model (CAPM) to resolve the equation, Robert C. Merton had applied a portfolio replication method based on the assumption that the expected return of that portfolio should be equal to the riskfree rate.

To price derivatives by using the model highlighted in this present chapter, Black and Scholes had set some constraints in Black and Scholes [1973]. ?? quotes that assumptions.

One of the most important constraints of BSM is that the price evolution of the underlying asset is modeled by a geometric Brownian motion. Its time continuous form is given by equation (2.1).

$$S(t) = S(0) e^{\sigma W(t) + (\alpha - \frac{1}{2}\sigma^2)t} \quad (2.1)$$

According to Shreve [2004], The GBM features a stochastic process where the only random component is the Brownian motion  $W(t)$ . The others parameters,  $\alpha$  and  $\sigma$ , respectively stand for the drift and volatility rate of the process. That model is fully explored in section 2.2.

Furthermore, in order to match with the constraints underpinned by the model developed by Black and Scholes, the distribution of the underlying asset log-returns has to be normally distributed. Accordingly, the current chapter shows that this prerequisite is met when the price of a security is driven by a GBM.

In that master thesis, the focus would be set to the computation of the pricing of vanilla Call options and the performance of the model when it evolves beyond the boundaries fixed by Black and Scholes. Some discrepancies studied in later chapters will be for instance that the underlying model is not driven by a GBM, the volatility of the stock is not deterministic and varies across time or abrupt and non predictable movements occur during the life of the asset.

Section 2.3 gives the resolution of the BSM equation to price European call option, while section 2.4 introduces some Greek letters, namely,  $(\Delta, \Gamma, \Theta)$ , which are functions that can be used either to price a derivative or to construct a hedging strategy.

## 2.1 Assumptions

Black and Scholes [1973] have provided a framework supported by a bunch of constraints qualified as ideal conditions under which the market would behave in order to make the BSM equation works with accuracy. All of these conditions are here below mentioned.

1. The short-term risk free rate  $r$  is known and constant
2. The stock return involving in the computation of BSM equation is log-normally distributed with constant mean and variance rate
3. No dividend are provided with the considered share of stock
4. The option considered within the computation is european
5. The price for bid and ask quote are identical. It means that there is no bid–ask spread to be considered
6. Share of stock can be divided into any portions such as needed for the computation
7. Short selling is allowed with no penalties

In the study performed purposely for that master thesis, the condition (2), that is the stock return is driven by a GBM, won't be respected. The goal is (i) to analyze how a derivative with another process as an underlying asset could be priced and how these prices reflect those in the market, and (ii) to measure the performance of the hedging if that constraint is not respected. Whilst incidentally, all other requisites will stay as is.

## 2.2 Geometric Brownian motion

The differential form of equation (2.1) is useful to find the properties of its distribution. As demonstrated by Shreve [2004], in order to get it, the Itô's formula (section 1.3) is used with  $S(t)$  as a stochastic process to resolve. By applying that transformation, equations (2.2a) and (2.2b) emerge:

$$dS(t) = \alpha S(t) dt + \sigma S(t) dW(t) \quad (2.2a)$$

$$\frac{dS(t)}{S(t)} = \alpha dt + \sigma dW(t) \quad (2.2b)$$

Equation (2.2a) shows that any change occurring in the stock price  $S(t)$  over a small amount of time is due to its deterministic expected drift rate ( $\alpha S(t)$ ), along with some amounts of added noise incurred by the random part ( $\sigma dW(t)$ ). According to Shreve [2004], such interferences provided by a Brownian motion do not make the process increase or fall over the long-run, i.e., there is no associated drift to them, they only bring volatility. On the other hand, equation (2.2b) denotes the instantaneous return of the asset driven by a GBM.

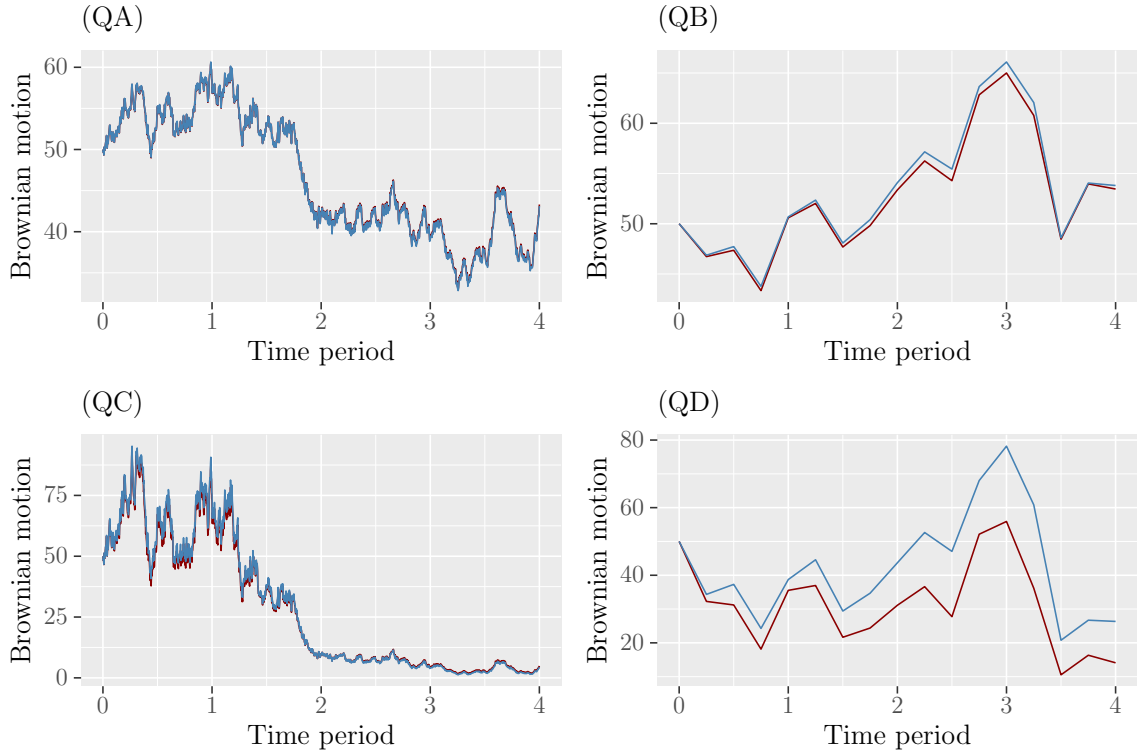
$$d \ln S(t) = \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma dW(t) \quad (2.3)$$

$$\ln \frac{S(t)}{S(0)} = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t) \quad (2.4)$$

Equation (2.4) exhibits the natural logarithm of the stock price return occurring over the period  $t$ . More specifically, as shown in Hull and Basu [2012], the expectation of this process, which is given by  $(\alpha - \frac{\sigma^2}{2})t$ , happens to be the expected value of the continuously compounded rate of return for the aforementioned period of time, since, as previously mentioned, the Brownian motion  $W(t)$  only brings disruptions.

The accuracy of the stochastic process 2.1, when approximated by using the Itô's lemma, depends exclusively on two factors; the volatility parameter  $\sigma$  and the period occurring between two measures  $\tau = t_{i+1} - t_i$ . As shown by figure 2.1, the lower the volatility or shorter the time-step, the better the estimate. It is therefore key, during an analysis process to choose an appropriate time-step according to a given volatility in order to provide accurate results.

Figure 2.1: Accuracy of Itô approximation



**Notes.** The blue line curves are constructed using itô's approximation ?? while equation (2.1) is used to build the red ones. The only parameters that change over the group are the couples  $(dt, \sigma)$  which are set to  $\{dt = 360, \sigma = 0.2\}$  for (QA),  $\{dt = 4, \sigma = 0.2\}$  for (QB),  $\{dt = 360, \sigma = 1\}$  for (QC), and  $\{dt = 4, \sigma = 1\}$  for (QD).

### 2.2.1 Distributions of geometric Brownian motion

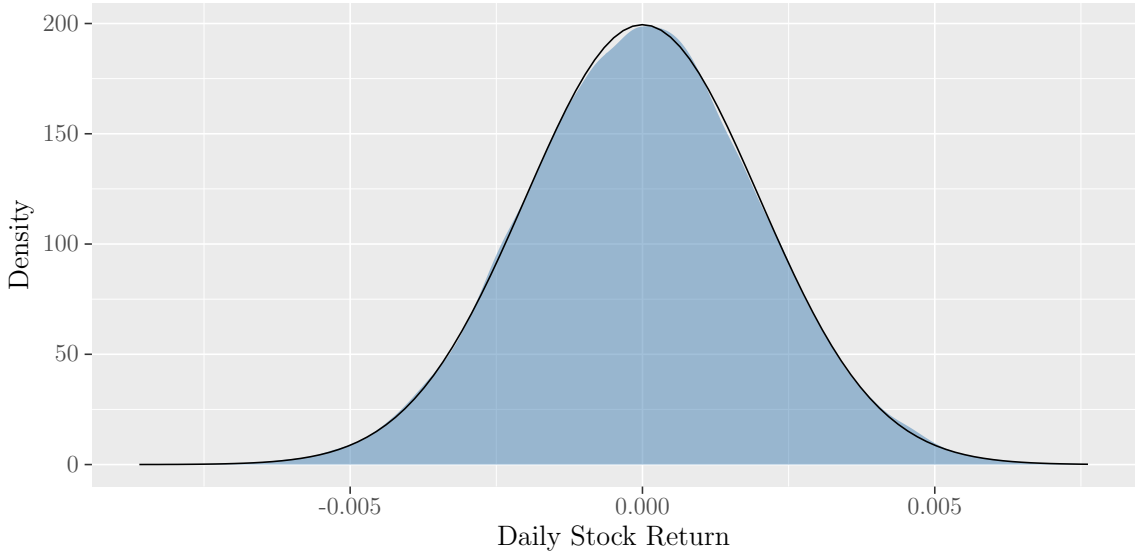
This section delves into the distributions of the process  $S(t)$ . They are described by using the related underlying law together with the relevant moments.

Equation (2.5), provided in Shreve [2004], shows that the process followed by equation (2.1) has normally distributed returns with their expected value and variance proportional to the time period  $dt$ .

$$\frac{dS(t)}{S(t)} \sim N(\alpha dt, \sigma^2 dt) \quad (2.5)$$

Moreover and according to figure 2.2, one can see that the normality qualification of equation (2.5) holds. Indeed, the black bell curve is constructed using the theoretical normal law with  $\alpha dt$  and  $\sigma^2 dt$  respectively as first and second moments, while the blue filled figure is built thanks to empirical results of process 2.2a with the same  $\alpha$  and  $\sigma$  given as parameters.

Figure 2.2: Daily Basis Stock Return Density

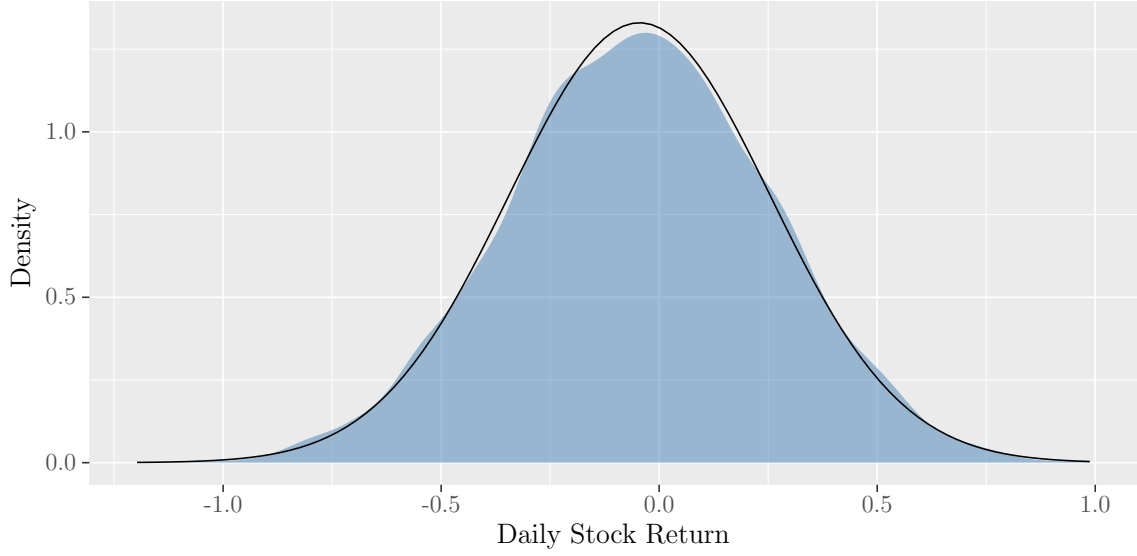


**Notes.** The above blue distribution is constructed over  $10e3$  paths of a unique stochastic process. The samples are built with equation (2.2a). The arguments are adjusted with the following values,  $\alpha = 0$ ,  $\sigma = 30\%$ . The black density belongs to the normal bell curve with mean  $\alpha dt$  and standard deviation of  $\sigma\sqrt{dt}$ . The distance period between each measure, namely  $dt$  has been set to  $10e3^{-1}$ .

According to Shreve [2004], from equation (2.4), the distribution of the natural logarithm of the stock price return, recorded over a period of time of  $\tau$ , turns out to be characterized by equation (2.6).

$$\ln \frac{S(t)}{S(0)} \sim N\left(\left(\alpha - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right) \quad (2.6)$$

Figure 2.3: Daily Basis Stock Return Density



**Notes.** The above blue distribution is constructed over  $10e3$  paths of a unique stochastic process. The samples are built with equation (2.4). The arguments are adjusted with the following values,  $\alpha = 0$ ,  $\sigma = 30\%$ . The black density belongs to the normal bell curve with mean  $(\alpha - \frac{\sigma^2}{2}) \times t$  and standard deviation of  $\sigma \times \sqrt{dt}$ . The distance period between each measure, namely  $dt$  has been set to  $10e3^{-1}$ .

From figure 2.3, the normality of equation (2.5) can be observed. Indeed, the black bell curve is constructed using the theoretical normal law with  $(\alpha - \frac{\sigma^2}{2})t$  and  $\sigma^2 t$  respectively as expectation and variance, while the blue filled figure is built based on empirical results of process 2.4 with the same  $\alpha$  and  $\sigma$  given as parameters.

Hull and Basu [2012] shows that the theoretical density of the process  $S(T)$  can be found from equation (2.6) and by applying the properties of the log-normal law. Indeed, equation (2.6) can be transformed such as given by equation (2.7)

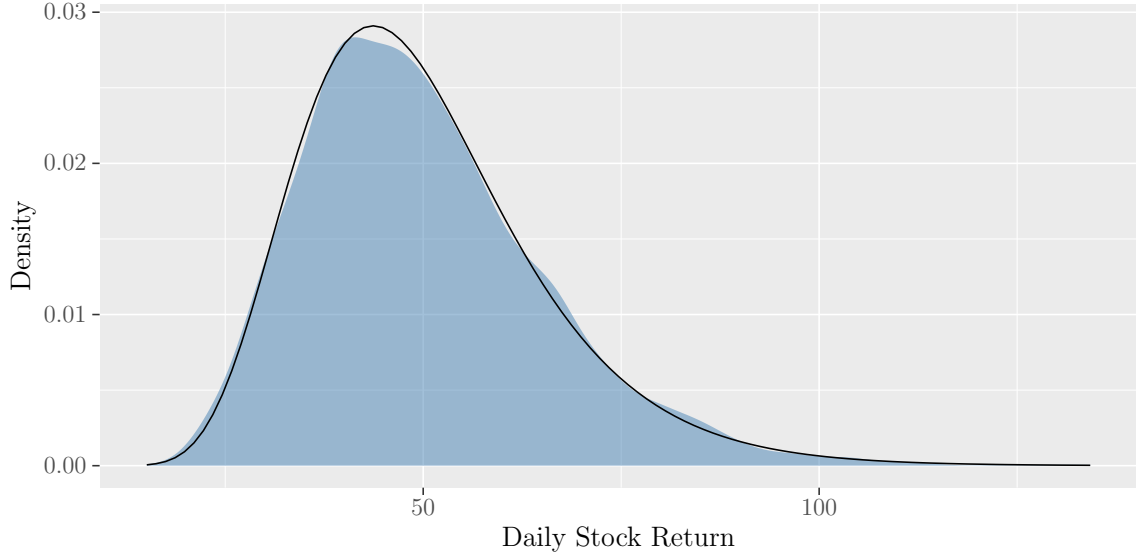
$$\ln S(t) \sim N \left( \ln S(0) + \left( \alpha - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right) \quad (2.7)$$

Any process inside the natural logarithm function characterized by the normal law is de facto determining by the log-normal law. Ultimately by applying the transformation rules of the mean and variance from a normal probability density function to the log-normal parameters, one finds that the process  $S(t)$  is described as follow.

$$S(t) \sim \text{lognormal} \left( S(0)e^{\alpha t}, S(0)^2 e^{2\alpha t} (e^{\sigma^2 t} - 1) \right) \quad (2.8)$$

Figure 2.3 illustrates that the log-normal property of  $S(t)$  is observed. Indeed, the black bell curve is constructed using the theoretical log-normal law with  $S(0)e^{\alpha t}$  and  $S(0)^2 e^{2\alpha t} (e^{\sigma^2 t} - 1)$  respectively as expectation and variance, while the blue filled figure is built upon empirical results of  $S(t)$  with the same  $\alpha$  and  $\sigma$  given as parameters.

Figure 2.4: Daily Basis Stock Return Density



**Notes.** The above blue distribution is constructed over  $10e3$  paths of a unique stochastic process. The samples are built with equation (2.1). The arguments are adjusted with the following values,  $\alpha = 0$ ,  $\sigma = 30\%$ . The black density belongs to the log-normal curve with mean  $S(0)e^{\alpha t}$  and standard deviation of  $S(0)^2 e^{2\alpha t} (e^{\sigma^2 t} - 1)$ . The distance period between each measure, namely  $dt$  has been set to  $10e3^{-1}$ .

## 2.3 The Black-Scholes-Merton equation

Concerning the framework developed in Black and Scholes [1973], the pricing method of an option underpinned by the BSM model is closely related to an underlying for which its price  $S(t)$  is a log-normally distributed stochastic process, such as the GBM be.

As defined in Shreve [2004], to provide a unique fair price to a stock option (e.g., for what matters here, to a vanilla European call) which depends on an underlying driven by a GBM, all the uncertainty associated to the stock price movements has to disappear. To do so, one has first to construct such a portfolio  $X(t)$  which encompasses the same source on uncertainty as for the option itself and then choosing the adequate position  $\Delta(t)$  to take to the underlying asset at each time  $t$ , so that all randomness cancels out.

$$dX(t) = \Delta(t) dS(t) + r(X(t) - \Delta(t) S(t)) dt \quad (2.9)$$

Shreve [2004] shows that the goal is to dynamically hedge the position taken in the option. It means that the position has to be frequently rebalanced. Consequently, at any times, the present value (PV) of the changes occurring in the portfolio, due to the underlying price evolution should be equal to the PV of those incurred by the financial derivative. The only way to achieve this equality is to adapt the delta for each period.

$$d(e^{-rt} X(t)) = d(e^{-rt} c(t, x)) \quad (2.10)$$

That way, one can take a position in the derivative (short/long) and hedge it by taking  $\pm\Delta(t)$  shares of stock.  $X(0)$  being the price of the call at time zero

$$X(0) = c(0, S(0)) \quad (2.11)$$



By following and developing the method above, Shreve [2004] shows that the BSM differential equation is given by equation (2.12), with the terminal (2.13) and boundary conditions (2.14) – (2.15).

$$rc(t, x) = \frac{\partial c(t, x)}{\partial t} + rx \frac{\partial c(t, x)}{\partial x} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 c(t, x)}{\partial x^2} \quad (2.12)$$

$$c(T, x) = (x - K)^+ \quad (2.13)$$

Whilst the terminal condition focuses on the value at maturity, the boundary conditions fix some constraints on the extreme values likely to be taken by the shares of stock at any times during the option life. In that regard, the boundary condition (2.14) shows that any options with a worthless underlying are themselves valueless, while whenever the option is deep-in-the-money, simulated with  $x = \infty$  (2.15), the value of the derivative is equal to the value of a forward contract involving the same underlying and with the same maturity date.

$$c(t, 0) = 0 \quad (2.14)$$

$$\lim_{x \rightarrow \infty} [c(t, x) - (x - e^{-r(T-t)})] = 0 \quad (2.15)$$

As described in Shreve [2004], according to the terminal (2.13) and boundary conditions (2.14, 2.15), the BSM solution for the European calls happens to be given by equation (2.16). The right-hand side of that equation,  $c(t, x)$ , denotes the price of a call option depending on the time before maturity, the volatility of the underlying and its price at that period. In addition to these arguments, two other parameters are required, namely, the strike ( $K$ ) and the riskless interest rate ( $r$ ).

$$c(t, x) = xN(d_+(\Delta t, x)) - Ke^{-r\Delta t}N(d_-(\Delta t, x)) \quad (2.16)$$

with

$$d_{\pm}(\Delta t, x) = \frac{1}{\sigma\sqrt{\Delta t}} \left[ \log \frac{x}{K} + \left( r \pm \frac{\sigma^2}{2} \Delta t \right) \right] \quad (2.17)$$

Consequently, equation (2.16) will be purposely used in this master thesis to compute the price of an option for which the underlying is exclusively driven by a geometric Brownian motion.

## 2.4 The greeks

The Black-Scholes-Merton equation (2.12) can be subdivided into different parts. Each one is identified through a Greek letter ( $\Delta$ ,  $\Theta$ ,  $\Gamma$ ).

The Greeks will afterward be used to show how the hedge of a call option behaves under the conditions defined in Black and Scholes [1973] and beyond. Indeed,  $\Delta$  has a key role in the hedging strategy because it can be used to capture the effect that an instantaneous change in the asset has to the derivative.  $\Gamma$  can remove the error of approximation incurred by a hedge based on  $\Delta$  due to the discretization of the timeframe. Whereas  $\Theta$  is not taken into account for the hedging strategy because it relates to the passing of time and nobody can prevent it.

These letters are described in this section as well as their implication in the present work.

### 2.4.1 Delta

As shown in Shreve [2004], delta is the first derivative of the call function (2.16) with respect to the stock price function, as shown by equation (2.18). It, therefore, represents the instantaneous rate of change of a call's value as the price of its underlying evolves.

$$\Delta(t, S(t)) = \frac{\partial c(t, S(t))}{\partial S(t)} \quad (2.18)$$

Whereas practically, following Shreve [2004], the derivation of delta for a call is given by equation 2.19.

$$\Delta_{call}(t, S(t)) = N(d_+(\Delta t, x)) \quad (2.19)$$

According to Hull and Basu [2012], at each period  $t$ , in order to hedge a short call one should hold  $\Delta(t)$  share of stock. Consequently, a portfolio comprised of one short position in a call along with  $\Delta$  shares of stock is said to be delta-neutral, because each movement in the stock price is compensated between the short position in the call and the long in the stock.

The delta neutrality could otherwise be explained using the slope-intercept form of the tangent line below the function  $c(t, x)$ , keeping  $t$  constant. If the stock price is equal to  $S$  and the corresponding call price, for a fixed time  $t$  and stock at  $k$ , is  $c$ , we consequently get (2.20) as equation of the tangent line below  $c(t, x)$ .

$$y = \frac{\partial c(t, S(t))}{\partial S(t)}(x - S) + c \quad (2.20)$$

According to 2.20, the price of the call, for a stock price  $S$  at a fixed time  $t$  is given by  $y = c$ . If – over an infinitesimally small delta time – a positive stock price movement occurs, e.g., the stock price rises from  $S \rightarrow S + \epsilon$ . The price of the call is, therefore, going to change as well, from  $y = c$  to  $y = \Delta\epsilon + c$ . Consequently, in order to hedge a short position in the call,  $\Delta$  shares of stock should be owned. Indeed, by keeping  $\Delta$  shares, the loss incurred by the higher value of the call  $y = c + \Delta\epsilon$  will be offset by an increase

of  $\Delta\epsilon$  thanks to the number of  $\Delta$  shares held. It makes sense that the hedging of a long call is achieved by setting up a short position in the underlying, according to the same parameter  $\Delta$ .

According to Shreve [2004], the hedge works well for small price movement in the underlying and is closely related to the curvature of the function  $c(t, c)$ , keeping  $t$  constant. It would, therefore, be interesting to look at the second derivative of the call function with respect to the stock price, in order to get the rate of the rate of change (the acceleration) of the call with respect to the underlying price. The next section 2.4.2 devotes to it.

## 2.4.2 Gamma

According to Shreve [2004], Gamma ( $\Gamma$ ) is the second derivative of the option pricing function with respect to the underlying price, time been constant (2.21).

$$\Gamma(t, S(t)) = \frac{\partial^2 c(t, S(t))}{\partial S(t)^2} \quad (2.21)$$

Whereas practically, the derivation of gamma for a call is given by equation 2.22.

$$\Gamma_{call}(t, S(t)) = \frac{1}{\sigma S(t) \sqrt{\Delta t}} N'(d_+(\Delta t, x)) \quad (2.22)$$

It gives the acceleration at which the price of a call moves along with the underlying price, *ceteris paribus*. Therefore, thanks to gamma, the curvature of the function to be approximated using the differential form is known. It can be crucial to at least assess how big is the value of gamma in order to adequately hedge a position in a call. Indeed, if gamma is low, the rebalancing of the hedge does not have to occur as frequently as if it is high because any change in the underlying asset would bring a little move in the derivative price, letting delta be a good approximation of the option evolution with respect to the asset price.

As a portfolio can be delta-neutral, R  man [2017] shows that it can also be gamma-neutral. However, making a portfolio gamma-neutral forces to add another derivative on the same underlying inside it to vanish the gamma, with the consequence to modify the delta of the portfolio. It, therefore, implies to recompute the delta in order to restore the delta-neutrality of the portfolio

## 2.4.3 Theta

Theta is the derivative of the price of an option with respect to the time, stock price being unchanged (2.23).

$$\Theta(t) = \frac{\partial c(t, S(t))}{\partial t} \quad (2.23)$$

According to Hull and Basu [2012], Theta can be used as a proxy for gamma in a delta-neutral portfolio. Indeed, because of its neutrality, delta so disappears of equation (2.24) letting the relation between gamma and theta become clearer, that is, according to the option price and the value of theta, gamma can be assessed.

### 2.4.4 Relation between BSM and the Greeks

The BSM equation and the Greeks closely relate together. Indeed, the BSM partial derivative equation (2.12) could equally be written using the greeks, as shown by equation (2.24).

$$rc(t, S(t)) = \Theta + rS(t)\Delta + \frac{1}{2}\sigma^2 S(t)^2 \Gamma \quad (2.24)$$

## 2.5 The delta hedging rules

The purpose of the delta hedging rules is to fully replicate the reverse position taken in an option to cover oneself against loss. The technique is achieved by continuously rebalancing its position in order to keep an amount of  $\Delta(t)$  share of stock for each period  $t$ .

Consequently to hedge a short position in a European call option, one should construct a portfolio made up of a given amount of share of stock to replicate a long position in the same derivative. The quantity of the underlying asset to keep in that portfolio at any time is provided by the resolution of the delta (2.19) for the appropriate period.

Equation (2.25) shows the relation between a European short call and the delta-neutral portfolio that replicates the reverse position in the derivative for any time  $t \in [0, T]$ , where  $T$  denotes the time to maturity.

$$c(t, S(t)) = \Delta(t)S(t) + e^{r(t-t_0)}c(t_0, S(t_0)) - \left( \sum_{t_i \in ]0, t]} (\Delta(t_i - t_{i-1})S(t_i)) + \Delta(t_0)S(t_0) \right), \forall i \in \mathbb{Z} : i \in \{1, T\} \quad (2.25)$$

where the set  $i$  is the time-step.

## 2.6 Flaws

Even if the equation (2.1) is used to model the stock price process, it exists some lacks with the observed real processes. The analysis ([CHA REF]) mainly covers the following discrepancy of the model against empirical results.

The volatility appearing in the aforementioned underlying model (2.1) is constant as time passes. Even if this consideration can be considered as true for short period of time, it is not the case over the long-run Teneng [2011]. Other model substitutes the process defined throughout this chapter with other ones including stochastic volatility processes, making it changing over time. For this purpose, the Heston's model is developed in this current master thesis, which encompasses two stochastic processes, a geometric Brownian motion [???? + REF] for the stock price diffusion model [DIFUSION??] and a mean-reverting CIR model to compute the volatility as a stochastic process. This model is exposed in ??.

Following [REF and REF], the underlying log-returns distribution is characterized by the normal law. Consequently, the random variable  $S(t)$ , at the fixed time  $t$ , is normal. However, according to empirical results, the random variable  $dS(t)$  do not fit

with the normal bell curve (Clark [1973]). Therefore if the price changes are not normally distributed neither is the log–return and thus  $S(t)$ , at a fixed time  $t$ , is not log–normally distributed. Along with the Heston model, another one is considered; the Merton’s jump–diffusion model. Both are able to modify the skewness and kurtosis of the log–return distribution curve. The Merton framework is developed in ??.

# Chapter 3

## Other Models to be considered

### 3.1 Overview

### 3.2 Merton Mixed jump-diffusion Model

In his paper, Merton [1976] provides a model for stock price evolution involving jumps (equation (3.1)).

$$S(t) = S(0) e^{\left(\alpha - \frac{\sigma^2}{2} - \lambda \kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \quad (3.1)$$

According to Merton [1976], there are two specific sources of uncertainty explained by the model (equation (3.1)). The first one is qualified to be normal, arising repeatedly with low effects and keeping the stock price motion continuous from time to time. These small changes on the price are modeled by a Wiener process, such as it was the case in equation ???. The cause of these changes is explained by a temporary unbalanced between the supply and demand Merton [1976]. Another type of changes, occurring during the stock lifecycle, are qualified as abnormal. Such "abnormalities" happen less frequently, are unpredictable in their frequency and produce bigger effect on the stock price by giving rise to jumps in the course of stock path and therefore breaking its continuity (Merton [1976]). The jump process is constructed on double basis.

Firstly, the occurrence (i.e. the number of jumps arising throughout a given period of time) is computed thanks to a Poisson-driven process according to a parameter  $\lambda$ .  $\lambda$  denotes the number of jumps per unit of time. Consequently, the probability that a jump occurs during a time range of  $\Delta t$  is equal to  $\lambda \Delta t$  (event  $A$ , eq. 3.2), whereas the probability that there are no jump during the same range of time is  $1 - \lambda \Delta t$ , (event  $B$ , eq. 3.3) (Matsuda [2004]). While  $C$ , eq. 3.4, refers to the event that more than one jump occur during the same small delta time.

$$\mathbb{P}\{A\} \cong \lambda \Delta t \quad (3.2)$$

$$\mathbb{P}\{B\} \cong 1 - \lambda \Delta t \quad (3.3)$$

$$\mathbb{P}\{C\} \cong 0 \quad (3.4)$$

On the other hand, after the occurrence comes the size of the jump. Such as the frequency, the importance of the jump can be characterized by a statistic law. Following

?, the log-normal law is used. Matsuda [2004], gives a summary in order to grips with the concept of jump size equations (3.5) to (3.7).

$$y_t \sim \text{lognormal}(e^{\mu + \frac{1}{2}\delta^2}, e^{2\mu + \delta^2}(e^{\delta^2} - 1)) \quad (3.5)$$

$$y_t - 1 \sim \text{lognormal}(\kappa \equiv e^{\mu + \frac{1}{2}\delta^2} - 1, e^{2\mu + \delta^2}(e^{\delta^2} - 1)) \quad (3.6)$$

$$\ln y_t \sim \text{normal}(\mu, \delta^2) \quad (3.7)$$

with  $y_t$ ,  $y_t - 1$  and  $\ln y_t \equiv Y_t$  standing respectively for "absolute price jump size", "relative price jump size" and "log price jump size" (Matsuda [2004]).

The Merton's jump-diffusion process is able to capture positive / negative skewness (see section 3.2.3) and excess kurtosis (see section 3.2.4) of the log-return density function Merton [1976].

### 3.2.1 Risk-neutralized process

In order to find the fair price of an option depending on an underlying that follows such a jump-diffusion process, Merton [1976] turns equation (3.1) into one risk-neutral.

$$S(t) = S(0) e^{\left(r - \frac{\sigma^2}{2} - \lambda\kappa\right)t + \sigma W(t) + \sum_{i=1}^{N_t} Y_i} \quad (3.8)$$

Merton [1976] argues in his paper that the jump component of equation (3.1) can be diversified in a well-balanced portfolio and consequently does not need to be risk-neutralized.

However, likewise it was done by Black and Scholes [1973], the drift part of equation (3.1) is risk-neutralized by turning the rate  $\alpha$  into its riskfree counterpart  $r$ , as shown by equation (3.8).

### 3.2.2 Graphical representation

Figure 3.1 shows a unique time serie generated using an implementation of equation (3.1). A jump is clearly noticed at day 363. While Table 3.1, which is a subset of the time serie drawn in figure 3.1, illustrates numerically when the jump occurs.

### 3.2.3 Impact on the skewness log-return

The way to influence the direction of the distribution's shape is achieved by moving the cursor of the expected value of jump impact, in other word, by changing the value of the parameter  $\mu$ . The figure [REF] shows how the density's shape of the log-return may vary along with this parameter.

### 3.2.4 Impact on kurtosis log-return

The way to influence the aspect of the distribution's tails is achieved by moving the cursor of the expected value of jump occurrence, in other word, by changing the value of the parameter  $\lambda$ . The ?? shows how the distribution's tails of the log-return may vary along with this parameter.

Table 3.1: Merton Mixed jump-diffusion time serie

time periods (days)	stock price
0	50.00
1	49.72
2	49.87
$\vdots$	$\vdots$
323	83.44
324	94.98
$\vdots$	$\vdots$
363	97.41
364	97.07
365	97.00

### 3.3 Heston stochastic volatility model

In his paper, Heston [1993] tackles with another discrepancy against the real world behaviour introduced by the geometric Brownian motion, namely, its deterministic and immutable volatility  $\sigma$ .

In addition to provide a model where the volatility is stochastic (equation (3.9)), Heston [1993] gives the possibility to make that volatility in correlation with the stock price process (equation (3.10)), according to the parameter  $\rho$  defining how the Brownian motions from both processes relates together.

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_V(t) \quad (3.9)$$

$$dS(t) = \alpha S(t)dt + \sqrt{V(t)}S(t)dW_S(t) \quad (3.10)$$

The drift part of the risk stochastic process (3.9 is made up of the long-run mean  $\theta$  together with the mean reversion speed, given by  $\kappa$ , Heston [1993].

$$dW_v(t)dW_s(t) = \rho \quad (3.11)$$

Equation (3.10) represents the evolution of an asset though time given, by its differential form. Such as equation (2.1), developed by Black and Scholes [1973], the parameter  $\alpha$  gives the drift rate of return. The difference between both models lies in the way the volatility is perceived. In Heston [1993], the asset volatility is given by the stochastic equation (3.9). More specifically, the volatility thus defined follows a Cox–Ingersoll–Ross process.

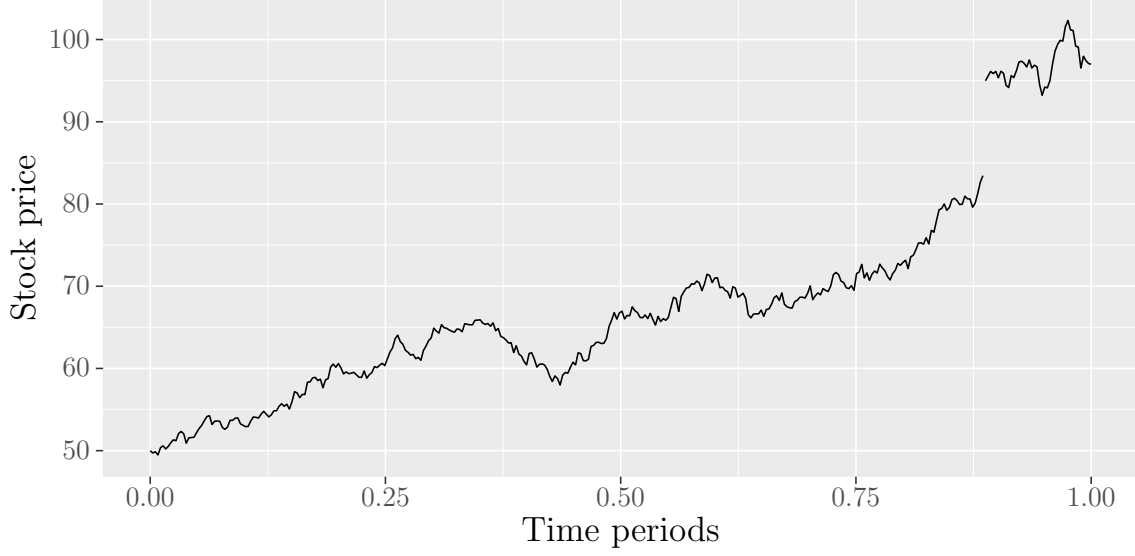
Equations (3.9) to (3.10) will be used to build Monte Carlo simulations with the goal of getting samples of dummy stock price motions under real-world assumptions. The framework to use for option pricing purpose is given in section 3.3.3.

#### 3.3.1 Model parameters

Here are described all the parameters appearing in the Heston stochastic volatility model.



Figure 3.1: Merton mixed jump-diffusion time serie



Simulation of one Merton mixed jump-diffusion time serie. Data have been output by the R function *sstoch\_jump()* which is an implementation of equation (3.1) (see appendix ??, for more information). The parameters passed to the function are:  $S(0) = 50$ ,  $T = 1$  (in year, along with a time step of 365 measures per year),  $\sigma = 0.2$ ,  $\alpha = 0.5$ ,  $\lambda = 2$ ,  $\mu = 0.05$ , and  $\delta = 0.1$ .

$S(t)$	Price of the stock at time $t$
$\alpha$	Annualized – and deterministic – expected return
$V(t)$	Observed volatility of the stock at time $t$
$\kappa$	Mean-reversion speed
$\theta$	Volatility's long-run mean
$\sigma$	Volatility of the volatility

### 3.3.2 Feller condition

Due to the time discretisation brought by The Monte Carlo simulation, the stochastic process 3.9 may turn out to be negative sometime. If such a negative value appears at time  $t$ , the next value computed for  $t + \epsilon$  will raise an error, due to the term  $\sqrt{V(t)}$  that obviously does not exist for negative value.

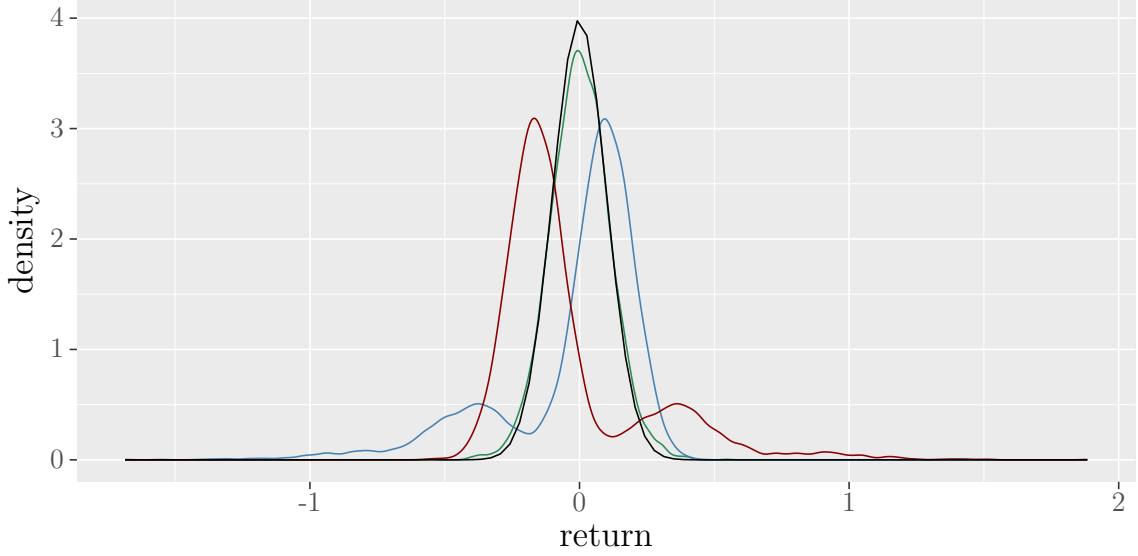
In his paper, Feller [1951] demonstrates that a process such the one described by equation (3.9) does not reach negative value if the following relation 3.12 is respected.

$$\lim_{V \rightarrow 0} \left( \kappa\theta - V - \frac{1}{2} \frac{\partial(\sigma\sqrt{V})^2}{\partial V} \right) \geq 0 \quad (3.12)$$

$$\begin{aligned} &\iff \lim_{V \rightarrow 0} \left( \kappa\theta - V - \frac{1}{2} \sigma^2 \right) \geq 0 \\ &\iff \kappa\theta - \frac{1}{2} \sigma^2 \geq 0 \\ &\iff 2\kappa\theta - \sigma^2 \geq 0 \end{aligned} \quad (3.13)$$

Consequently, if the condition related by equation (3.13) is respected, no negative

Figure 3.2: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples each. All samples have been constructed following equation (3.1). The only parameter that changes over the group is  $\mu$  which is set to  $(-0.5, 0, 0.5)$  respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of  $\sqrt{dt} \times \sigma$ .

values would occur by using a Monte Carlo simulation to compute the CIR stochastic volatility.

### 3.3.3 Risk-neutralized processes

Likewise it has been done by Black and Scholes [1973], Heston [1993] used a risk-neutral framework to price options. To do so, Heston modified the drift parameters of both price and volatility stochastic processes.

The drift part of the price diffusion (equation (3.10)) is risk-neutralized by turning the rate  $\alpha$  into its riskless counterpart  $r$ , as shown by equation (3.14).

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW_S(t) \quad (3.14)$$

In order to make the volatility process risk-neutralized, Heston added the risk premium parameter,  $\lambda$ , to the drift part of equation (3.9). The risk-neutralized CIR process is given by crefeq:other:hsvvol:riskless.

$$dV(t) = \kappa^*(\theta^* - V(t))dt + \sigma\sqrt{V(t)}dW_V(t) \quad (3.15)$$

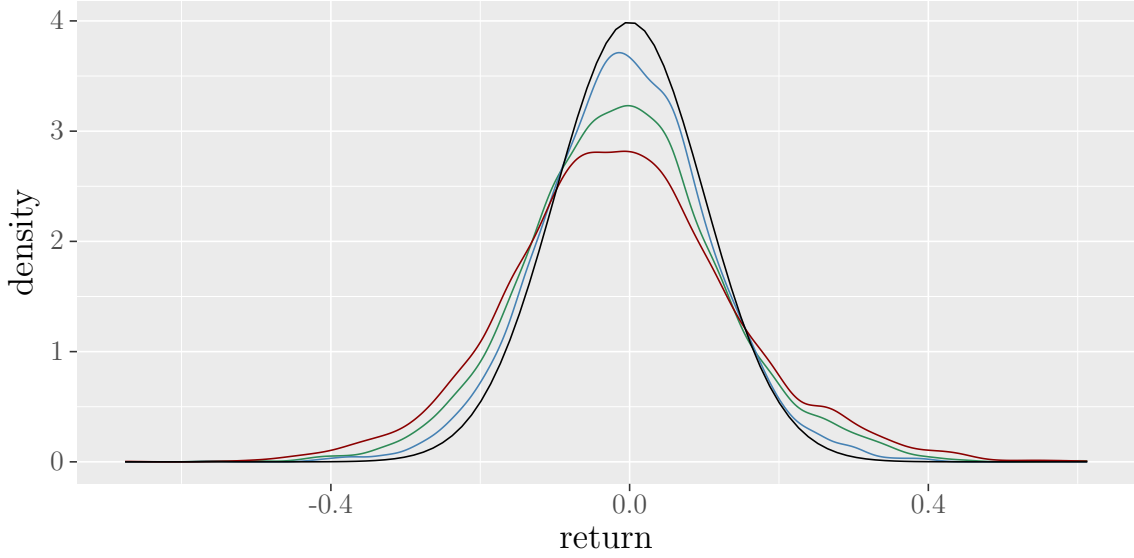
where

$$\kappa^* = \kappa + \lambda \quad (3.16)$$

and

$$\theta^* = \frac{\kappa\theta}{\kappa^*} \quad (3.17)$$

Figure 3.3: Merton: Daily Basis Stock Return Density



The above density function has been constructed over three distinctive groups of 5000 samples each. All samples have been constructed following equation (3.1). The only parameter that changes over the group is  $\lambda$  which is set to (1, 3, 5) respectively for the blue, green and red density function. The black density belongs to the normal curve with mean 0 and standard deviation of  $\sqrt{dt} \times \sigma$ .

Consequently the parameters  $\kappa^*$  and  $\theta^*$ , which respectively denote the long-run mean and mean-reversion speed, are the ones to estimate while dealing with pricing options purposes.

### 3.3.4 Graphical representation

Figures 3.4 and 3.5, get a hands-on insight into how the correlation between the underlying Brownian motions of the stock and volatility time series affect both processes. figure 3.4 shows a correlation between the Wiener processes  $B_1$  and  $B_2$  sets to  $\rho = -1$ , making the two Markov motions perfectly negatively correlated. It directly affects the course of the stocks serie, which is altogether correlated in the same negative direction with respect to the CIR volatility process as well. Likewise, figure 3.5 points out the fully positive correlation occurring between the processes 3.9 and 3.10 whilst the Brownian motions correlation is set to one.

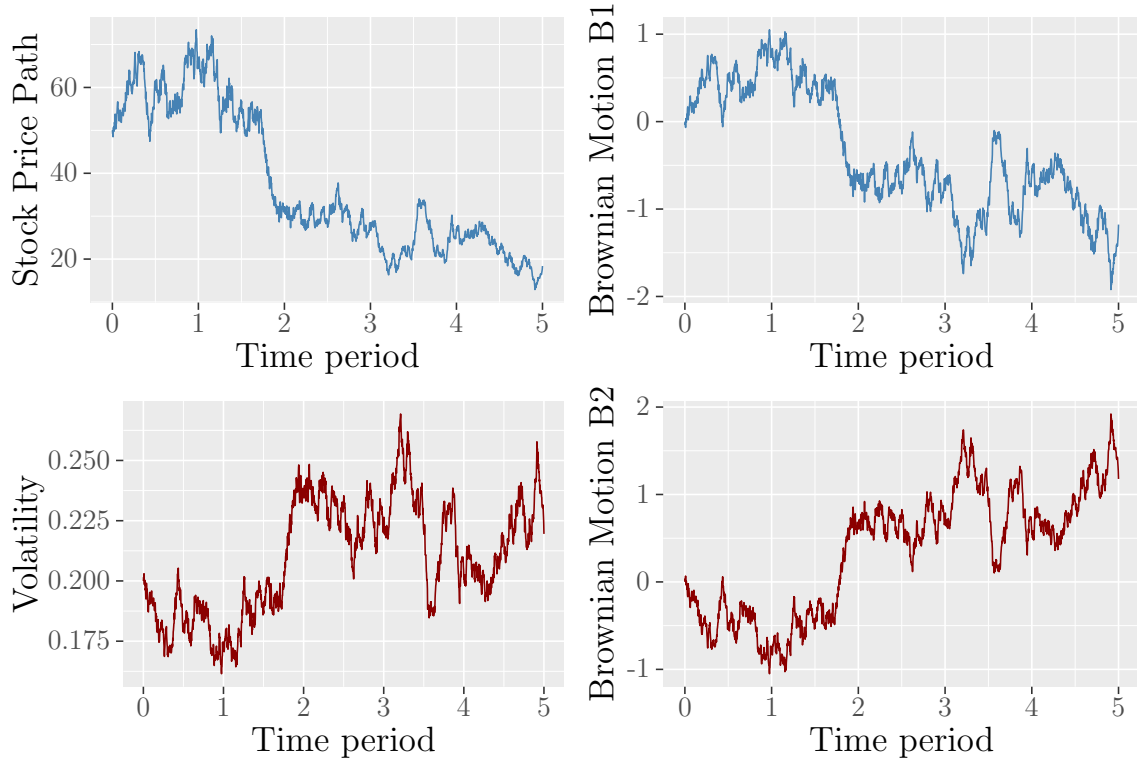
The usage of the aforementioned Heston model lies in the fact that the correlation between the CIR and asset processes' Brownian motions would notably explain the spot return skewness whereas the kurtosis of the distribution may be affected by the volatility parameter  $\sigma$  of the volatility stochastic process (equation (3.9)), (Heston [1993]) It may consequently be consistent with what happens in the equity market, namely a sharp decrease in equity price implies an increase in stock volatility (Crisóstomo [2015]).

### 3.3.5 Impact on log-return density's skewness

Through the Heston stochastic volatility model, the skewness of the distribution of continuously compounded spot return may be affected by the parameter  $\rho$ .

When a positive correlation exists between both Brownian motions, an increase in

Figure 3.4: Heson Framework Using Negative Correlated Brownian Motions



Simulation of Heston time series. Data have been output by the R function *heston()* which is an implementation of equations (3.9) to (3.11) (see appendix ??, for more information). The parameters passed to the function are:  $S(0) = 50$ ,  $V(0) = 0.2$ ,  $T = 5$  (years, along with a time step of 365 measures per year),  $\alpha = 0$ ,  $\kappa = 0.5$ ,  $\Theta = 0.2$ ,  $\sigma = 0.1$  and  $\rho = -1$ .

the volatility implies a rise in the asset price whereas a decrease in the volatility tend to lower the asset price. In other words, when the uncertainty is high and consequently the changes in the asset price are numerous, these latter tend to be positive. That is why the distribution of the spot return is offset to the left with a right fat tail when  $\rho$  is positive.

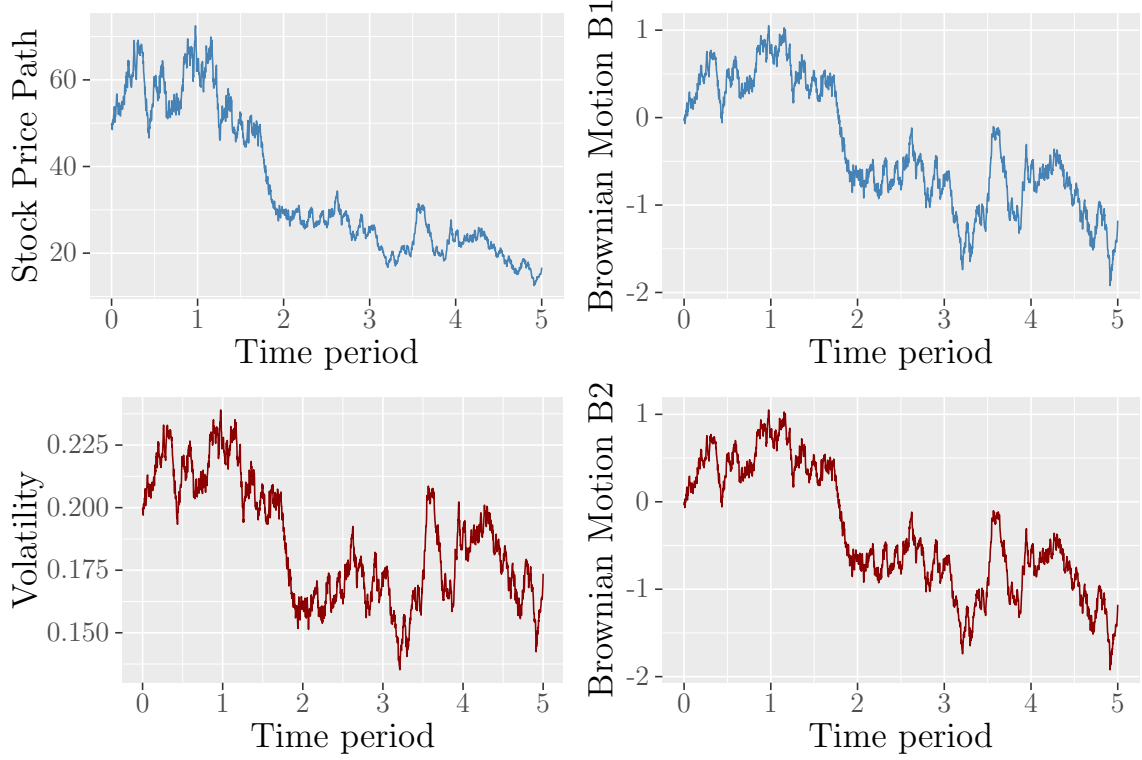
The opposite relation is noticed with negative correlation, namely, lower prices relate to higher volatility generating a left fat tail in the log-return distribution (Heston [1993]). These statements are observed in figure 3.6.

### 3.3.6 Impact on kurtosis density return

Following Heston [1993], the kurtosis of the distribution of the spot return may be affected by the parameter  $\sigma$ , which represent the volatility of the volatility.

First and foremost, following equation (3.9) if  $\sigma = 0$ , the volatility  $V$  of the Heston model turns out to be deterministic and equation (3.10) becomes a geometric Brownian motion with normal distribution for the time series' log-returns. Otherwise, Heston [1993] showed that by rising  $\sigma$ , the kurtosis of the spot returns increases. Consequently, within the Heston stochastic volatility model, the bigger  $\sigma$ , the fatter the tail, ceteris paribus. These statements are observed in figure 3.7.

Figure 3.5: Heston Framework Using Positive Correlated Brownian Motion



Simulation of Heston time series. Data have been output by the R function *heston()* which is an implementation of equations (3.9) to (3.11) (see appendix ??, for more information). The parameters passed to the function are:  $S(0) = 50$ ,  $V(0) = 0.2$ ,  $T = 5$  (years, along with a time step of 365 measures per year),  $\alpha = 0$ ,  $\kappa = 0.5$ ,  $\Theta = 0.2$ ,  $\sigma = 0.1$  and  $\rho = 1$ .

### 3.4 Option pricing method

As shown by section 3.2 and section 3.3, the frameworks developed by Merton [1976] and Heston [1993] drastically change the distribution of any underlying assets following such processes. Therefore, the pricing method to be used must also be adapted in order to take that update into account.

In his paper, Heston [1993] developed a technique to price options using the characteristic function of the underlying asset. Furthermore, according to Crisóstomo [2015] that method could be used to price every options provided that the underlying's characteristic is known.

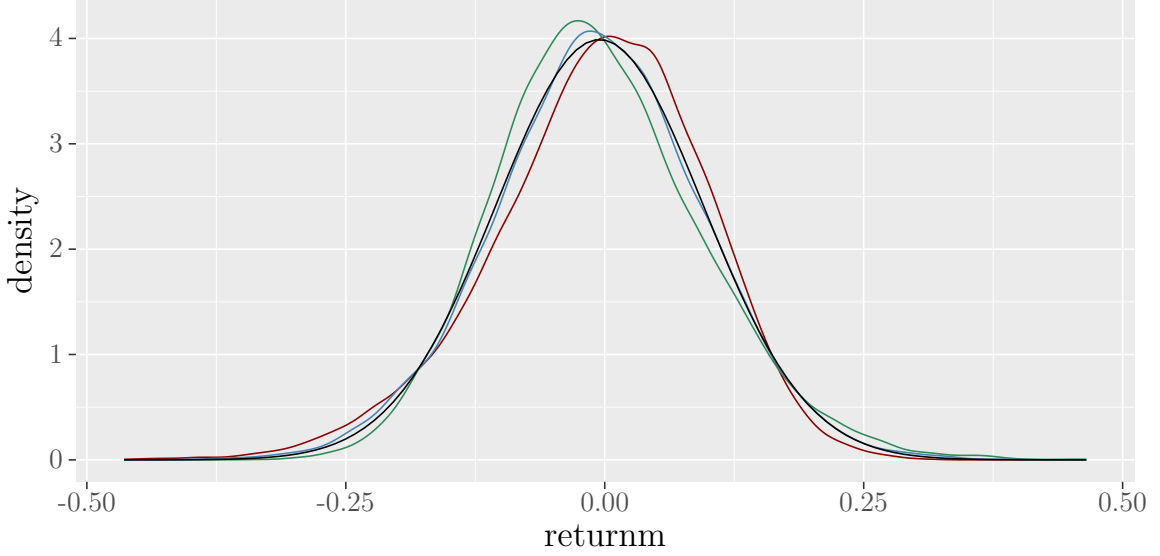
#### 3.4.1 Probabilistic approach

Heston [1993] proposed the solution given by equation (3.18) to price a european call option.

$$C(t) = S(t)P_1 - e^{-r(T-t)}KP_2 \quad (3.18)$$

Through this method, the european call price at time  $t$ , namely  $C(t)$ , is computed thanks to equation (3.18), where  $S(t)$  and  $e^{-r(T-t)}$  respectively stand for the stock price and the present value of the strike at that time  $t$ .

Figure 3.6: Merton: Log-return skewness on Heston



The above densities function (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (3.10) for the stock data and on equation (3.9) for the related volatility (see function *heston()* on appendix ?? for more information). The only parameter that changes over the groups is  $\rho$  which is set to  $-0.5$ ,  $1$ ,  $0.5$ . The log-return densities of these groups are respectively represented by the red, green and blue outlined density functions. The black density represents to the normal bell curve with mean  $-\frac{\theta}{2}$  and standard deviation of  $\sqrt{\theta}$ . The log-price return cover a period of one year with a time step of 500.

$$P_1(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{e^{-i\phi \ln K} \psi(x, V, t; \phi - i)}{i\phi \psi(x, V, t; -i)} \right) d\phi \quad (3.19)$$

$$P_2(x, V, t; \ln K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{e^{-i\phi \ln K} \psi(x, V, t; \phi)}{i\phi} \right) d\phi \quad (3.20)$$

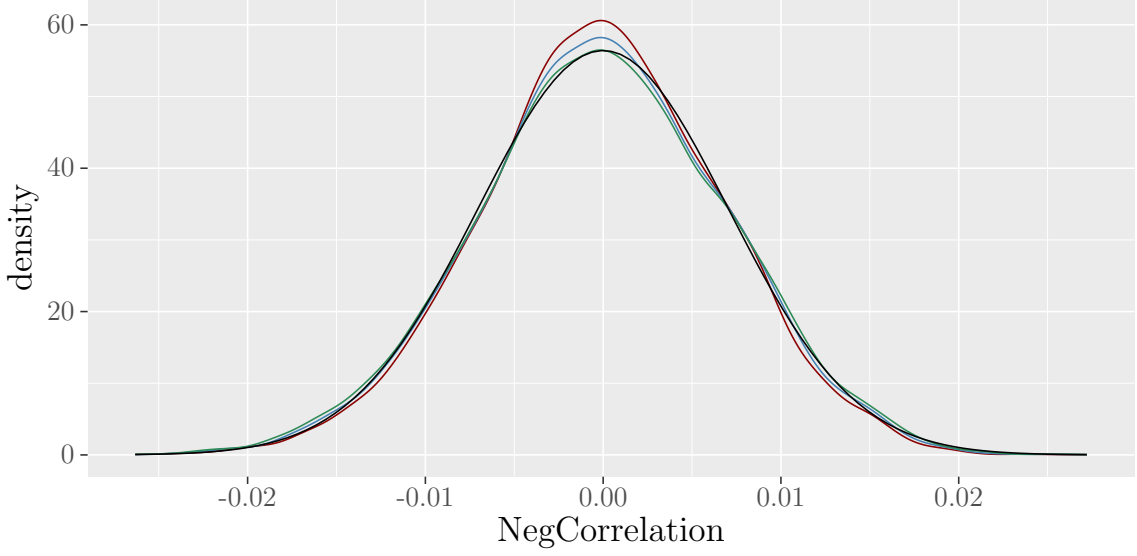
Following the development in Crisóstomo [2015], ?? both are probability quantities that involve the underlying characteristic function, namely  $\psi(x, V, t; \phi)$ . Once these quantities are computed, they are substituted in equation (3.18) in order to get the call price at time  $t$ .

The characteristic functions for the Merton jump–diffusion (section 3.2) and Heston stochastic volatility (section 3.3) models are developed sections 3.4.2 and 3.4.3

### 3.4.2 Characteristic function for Merton Mixed jump–diffusion model

Matsuda [2004] demonstrates that the characteristic function of the Merton mixed jump–diffusion process is given by equation (3.21). The baseline equation to construct the Merton’s process characteristic function is the risk-neutralized version, namely equation (3.8).

Figure 3.7: Merton: Log-return skewness on Heston



The above densities function (red, blue and green) are constructed over three distinctive groups of 10000 samples each. The black curve density is theoretical. All samples are generated by an algorithm based on equation (3.10) for the stock data and on equation (3.9) for the related volatility (see function *heston()* on appendix ?? for more information). The only parameter that changes over the groups is  $\sigma$  which is set to 0, 0.2, 0.4. The log-return densities of these groups are respectively represented by the green, blue and red outlined density curves. The black density represents to the normal bell curve with mean  $-\frac{\theta}{2}$  and standard deviation of  $\sqrt{\theta}$ . The log-price return cover a period of one year with a time step of 500.

$$\psi^{merton}(\phi) = e^{\lambda(T-t)\left(e^{i\mu\phi - \frac{\delta^2\phi^2}{2}} - 1\right) + i\phi\left(\ln S(t) + \left(r - \frac{\sigma^2}{2} - \lambda\kappa\right)(T-t)\right) - \sigma^2\frac{\phi^2}{2}(T-t)} \quad (3.21)$$

where

$$\kappa = e^{\mu + \frac{\delta^2}{2}} - 1 \quad (3.22)$$

According to the method given by Heston [1993], the characteristic function 3.21 will be used inside equations (3.19) and (3.20) in order to compute the quantities  $P_1$  and  $P_2$  that could be thereafter replaced inside equation (3.18) to find the european call price  $C(t)$  corresponding to a stock price process  $S(t)$  driven by the Merton mixed jump–diffusion model.

### 3.4.3 Characteristic function for Heston stochastic volatility model

Following the development proposed by Gatheral and Taleb [2006], Crisóstomo [2015] provided the Heston characteristic function (equation (3.23)) based on the process  $\ln S(t)$ .

$$\psi^{heston}(\ln(S(t)), V(t), t; \phi) = e^{C(T-t, \phi)\theta + D(T-t, \phi)V(t) + i\phi \ln(S(t)e^{r(T-t)})} \quad (3.23)$$

where

$$C(\tau, \phi) = \kappa \left( r_- \tau - \frac{2}{\sigma^2} \ln \left( \frac{1 - g e^{-h\tau}}{1 - g} \right) \right)$$

$$D(\tau, \phi) = r_- \frac{1 - e^{-h\tau}}{1 - g e^{-h\tau}}$$

and

$$r_{\pm} = \frac{\beta \pm h}{\sigma^2}; h = \sqrt{\beta^2 - 4\alpha\gamma}$$

$$g = \frac{r_-}{r_+}$$

$$\alpha = -\frac{\phi^2}{2} - \frac{i\phi}{2}; \beta = \kappa - \rho\sigma i\phi; \gamma = \frac{\sigma^2}{2}$$

Equation (3.23) can be directly used inside equations (3.19) and (3.20) in order to compute the quantities  $P_1$  and  $P_2$  that could be thereafter replaced inside equation (3.18) to find the european call price  $C(t)$  corresponding to a stock price process  $S(t)$  driven by the Heston stochastic volatility model.



# Chapter 4

## Methodology

The objective of this master thesis is to measure the performances of the Black-Scholes-Merton (BSM) pricing method when the assumption of normality for the log-returns distribution is not met. The analysis will be focused on the Apple stock data and its related calls options.

The methodology followed to measure such performances consists of three main parts. First, The computation of the options price by using, in turn, the BSM, Merton mixed jump-diffusion (MJD) and Heston stochastic volatility (HSV) models. After that, the implied volatility of those computed prices, split by models and maturities, are going to be compared. Finally, the construction of delta-neutral portfolios in order to measure the hedging performances of the models mentioned above.

Instead of exclusively using market data, whether to collect the options prices or the stock time-series, theoretically based algorithms have been purposely created. Depending on the framework to explore, the options prices will be either computed by using the BSM equation (equation (2.16)), if the underlying process relates to a geometric Brownian motion or by using the method developed by Heston [1993] (equation (3.18)), if the underlying process relates to the model MJD or HSV. In order to assess the delta-neutral portfolio, time-series will be needed to measure its evolution across time. Those series will be simulated based on the theories developed by Merton [1976] and Heston [1993] to respectively obtain paths with jumps and others with stochastic volatility as a correlated process. Consequently, functions have been created in the R language to perform those tasks. Table 4.1 is a summary of a few used in that chapter and in the analysis. More details on them are given in appendix ??.

Even though the hedging won't directly be constructed on market data, they are needed to calibrate the parameters to pass into the functions above listed in table 4.1. Therefore, the functions with the objective to calculate the European calls prices will be calibrated with the Apple European calls market data and, to be consistent, those that simulate time-series will be adjusted using the Apple stock data available on the market. These processes of calibration are fully explained in section 4.1.

### 4.1 Calibration

Whenever one deals with functions aimed to reproduce some real-life experiments, the calibration process is crucial because it gives to the functions the capacity to act within appropriate boundaries.

The process of calibration that will be applied in the current section concerns two

Table 4.1: R functions dealing with options and time series

Function name	Arguments	Purpose
bsm_call	$\{S(0), T, k, r, \sigma\}$	Compute the BSM price of an option
mjd_call	$\{S(0), T, k, r, \lambda, \mu, \delta\sigma\}$	Compute the Merton price of an option
hsv_call	$\{S(0), T, k, r, V(0), \theta, \kappa, \sigma, \rho\}$	Compute the heston price of an option
bsm_delta	$\{S(0), T, k, r, \sigma\}$	Compute the BSM delta of an option
mjd_delta	$\{S(0), T, k, r, \lambda, \mu, \delta\sigma\}$	Compute the Merton delta of an option
hsv_delta	$\{S(0), T, k, r, V(0), \theta, \kappa, \sigma, \rho\}$	Compute the heston delta of an option
bsm_ts	$\{S(0), T, \sigma, \alpha, dt\}$	Simulate BSM time series
mjd_ts	$\{S(0), T, \sigma, \alpha, \lambda, \mu, \delta dt\}$	Simulate MJD time series
hsv_ts	$\{S(0), T, V(0), \alpha, \rho, \kappa, \theta, \sigma, dt\}$	Simulate HSV time series

distinctive groups of parameters. Those intended to the functions that compute the European call price and those used by the functions that simulate the possible stock market movements. Consequently, the methods to adjust both kinds of arguments differ, mainly because the options prices calculation must be performed under risk-neutral environment and the delta hedging is measured on time-series evolving in a risk-averse world.

Therefore, the market available options prices data will be used to calibrate the functions that compute options prices, while those on the Apple stock's prices evolution will serve to estimate the right values to be passed as parameters for the time-series output-related functions. The option market prices and stock data were downloaded using the package *quantmod*, developed by Ryan and Ulrich [2018] which uses Yahoo finance as a provider. The datasets so downloaded are available in appendix ??.

Section 4.1.1 explains how to operate for the options' parameters, whereas section 4.1.2 shows the followed procedure to adjust the assets simulations' arguments.

#### 4.1.1 Option prices based calibration

In accordance with Heston [1993] and Crisóstomo [2015], provided that the characteristic functions of the MJD and HSV models are known, the European calls prices of such underlying processes can be computed using equation (3.18). Although known, these characteristic functions (equations (3.21) and (3.23)) need that their parameters are fed with appropriate values to best fit with what is observed in reality. That is why both functions will be adjusted with referential values before being used.

To do so, the method followed by Crisóstomo [2015] will be applied. It consists of finding the arguments that minimize the square of the difference between the options prices provided by the market and those generated by a pricing function. Such a procedure can be resource consuming because it is iterative. Consequently to avoid endless computations and since the market comes with a large number of data, the maturities, and strikes listed in tables 4.2 and 4.3 will be those considered during the analysis.

As for Crisóstomo [2015], the optimization method used is the least-square non-linear analysis. To work, such a method needs (i) a function that returns dummy data – i.e., the one to be adjusted –, (ii) a dataset that serves as a template, (iii) a cost function

Table 4.2: Maturities took into account during the calibration stage

63 91 126 154 182 245 399

Table 4.3: Strikes took into account during the calibration stage

130 140 150 160 170 180 190 200 210 220

to minimize, and (iv) a set of parameters to optimize. To perform that analysis, an algorithm developed by Borchers [2018] is directly available in the R language through the function *lsqnonlin* from the R package *pracma*.

The functions that output the artificial data are those exhibited in table 4.1, that is to say, and for what matters here, *mjd\_call* and *hsv\_call* for the computation of European calls prices using MJD and HSV, respectively. The template of options data is available in ?? (see appendix ??). While the cost function is given by equation (4.1), the parameters to assess depend on the underlying model (either MJD or HSV).

$$cst = \sum_{k \in K} \sum_{t \in T} (C_{k,t}^{mkt} - C_{k,t}^{m \oplus h}(\text{arguments}))^2, \quad (4.1)$$

$$\text{with } K = \{130, 140, 150, 160, 170, 180, 190, 200, 210, 220\},$$

$$T = \{63, 91, 126, 154, 182, 245, 399\}$$

On the one hand, The subscripts  $K$  and  $T$  respectively stand for the sets of strike prices and maturity dates, while on the other hand, the superscript  $mkt$  denotes the "market price" whilst the one  $m \oplus h$  refers to either MJD (m) or HSV (h) process based prices.

Consequently, equation (4.1) is that to be minimized by the least-square non-linear analysis, for all strikes and maturities. The outputs of this process will eventually be the calibrated arguments that maximize the performance of the model with respect to what is observed in reality.

One difficulty when dealing with such an algorithm is that the least-square non-linear analysis approach may return several best-fit sets of arguments due to the existence of multiple local minima in the cost function. Therefore, in order to select only one set among all the results provided by the optimization method, will be chosen that with which, at strikes and maturity dates equal, the pricing function gives the more of its outputs within the bid-ask spread of the corresponding options whose price is giving by the market, as shown by equation (4.2).

$$C_{bid}^{mkt} \leq C_{K,T}^{m \oplus h}(\text{arguments}) \leq C_{ask}^{mkt} \quad (4.2)$$

Even though the cost function to be optimized is the same for both HSV and MJD options pricing procedures, the parameters to adjust are different. sections 4.1.3 and 4.1.4 respectively illustrate the MJD and HSV related calibration processes along with the associated results.

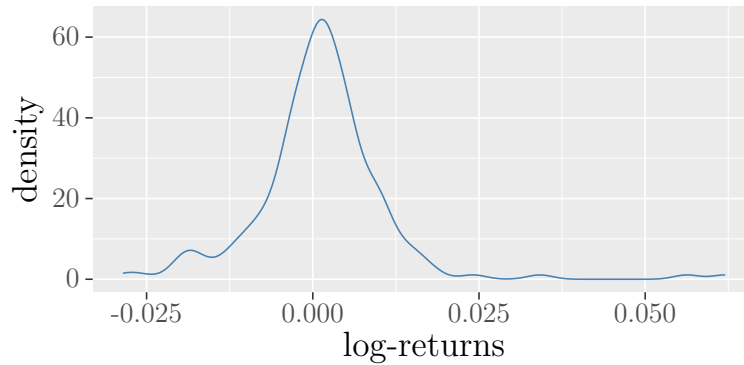
### 4.1.2 Asset prices based calibration

To calibrate the parameters to pass to the MJD and HSV models aimed to generate all the dummy times-series that will serve for the analysis of the delta hedging, historical market data on the Apple stock will be used as a template.

To perform such an upstream analysis, an approximation method to estimate the arguments with which the distribution of the log-returns generated by both MJD and HSV models better fit the density curve of the historical log-returns will be applied. The list of arguments to adjust is smaller than the one needed to option calibration because the starting point is the set of the already calibrated parameters gotten from options data and will only be readjusted, those they are risk-neutralized.

To get the historical data, I will once more use the package *quantmod*, developed by Ryan and Ulrich [2018] which uses Yahoo finance as a provider. The dataset so downloaded is available in appendix ?? and concerns the daily stock price on Apple from 18th May 2017 to 18th May 2018. figure 4.1 shows the density curve generated by the log-returns of historical data.

Figure 4.1: Historical Apple stock Log-returns distribution



**Notes.** The above density function is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. It represents the empirical distribution of its daily log-returns observed during that period.

The functions that output the dummy time-series are those exhibited in table 4.1, namely, *mjd\_ts* and *hsv\_ts* for the generation of MJD and HSV processes, respectively. The procedures to calibrate the arguments for those functions are explained in sections 4.1.3 and 4.1.4.

### 4.1.3 Merton's model calibration

#### Calibration of parameters for option pricing

In order to use the function *mjd\_call* based on equation (3.18) to compute the price of European call options on an underlying with increments driven by the MJD model, the parameters  $\{\lambda, \mu, \delta, \sigma\}$  must be calibrated with respect to the available market data.

To do so, the aforementioned least-square non-linear analysis will be used together with data on Apple call option as a template. In that respect, the theoretical models to calibrate are given by equations (3.18) to (3.20) along with equation (3.21), implemented by the R function *mjd\_call*.

Moreover, to stay in a range of acceptable values, the parameters of the MJD model should lie between some defined boundaries.

Indeed,  $\sigma$  and  $\delta$ , as standard deviation parameters, must range between 0 and 1. Otherwise, no limit is fixed for both  $\mu$  and  $\lambda$ , letting the frequency of jumps and their average magnitudes fully free.

According to those constraints and due to the presence of multiple local minima, the *lsqnonlin* function returns the ?? as the whole sets of parameters that make the function *mjd\_call* better fit with reality.

Table 4.4: Best estimates for MJD call option model

lambda	mu	delta	sigma
0.07915	-0.37583	0.08771	0.18651
0.11606	-0.30724	0.09993	0.18445
0.14411	-0.24174	0.12023	0.18676
0.06839	-0.37680	0.18072	0.18870
0.10312	-0.29738	0.19545	0.18583
0.05698	-0.38007	0.28030	0.19043
0.08216	-0.28573	0.27560	0.18822

However, the set of arguments 4.3 is the one making respond the model the best with what is observed in reality. Indeed, when passing that set to the R function *call\_merton*, for all strikes and maturities from tables 4.2 and 4.3, more than 68% of the so generated prices are within the bid-ask spread of the Apple option historical data.

$$\left\{ \begin{array}{l} \lambda = 0.1031218, \\ \mu = -0.2973769, \\ \delta = 0.1954519, \\ \sigma = 0.1858289 \end{array} \right\} \quad (4.3)$$

Figure 4.2 confronts the blue colored volatility smiles computed from market data, with those dotted in red, calculated from data provided by the function *mjd\_call*, which takes the items of the set 4.3 as parameters.

Consequently, for the purpose of that master thesis, the set 4.3 is the one that will be used together with the function *mjd\_call* whenever performing the computation of option prices for the MJD model.

### Calibration of parameters for time series

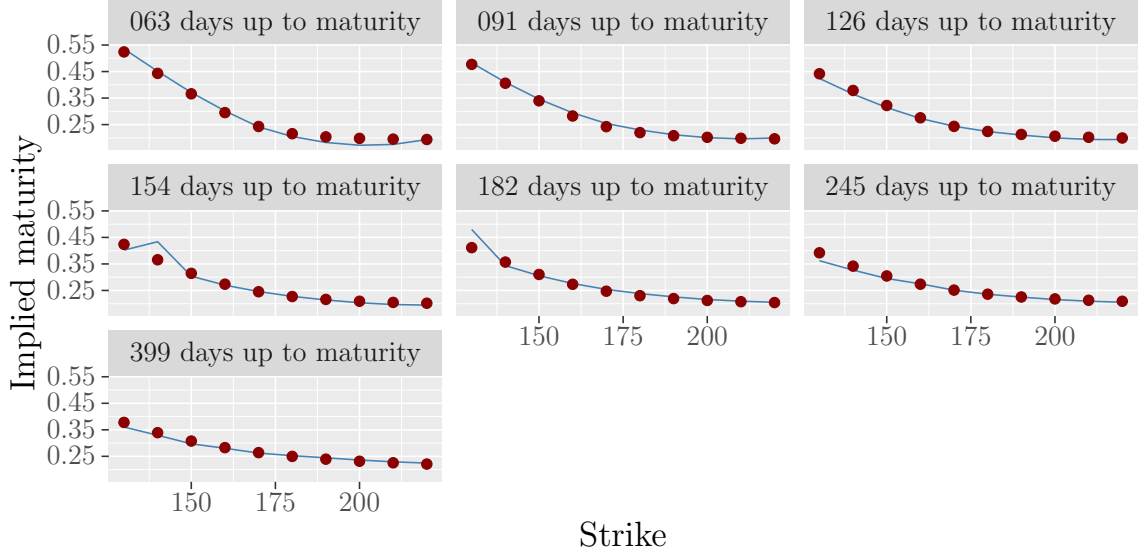
The set 4.3 is calibrated under the risk-neutral world and cannot, therefore, be used as is to simulate the dummy time-series that will serve the analysis aimed to measure the delta hedging performance. Furthermore, in addition to those parameters, the drift rate  $\alpha$  has to be estimated as well. It was not present in the set 4.3 because it was not taken into account in the computation of the options prices.

Figure 4.3 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *mjd\_ts()* fed with the risk-neutral parameters 4.3, for the red one.

The goal of that section is to find the risk-averse parameters that make both the distributions of the log-returns generated from market data or from *mjd\_ts* fit together.

Practically, to do so, an approximation algorithm developed by Venables and Ripley [2002] which is directly available in the R language through the function *fitdistr* from the R package *MASS* will be purposely used. The goal of that algorithm is the find the parameters of a density function that make it reproduce the distribution of a sample of

Figure 4.2: Implied volatility of Apple option prices computed with MJD



**Notes.** The implied volatilities that represent the above volatility smiles have been computed by using an iterative method on the BSM equation to solve  $\sigma$ . The parameters given to the function *bsm\_call* were the stock price  $S$ , the strike price  $K$ , the riskless rate  $r$  and the time to maturity  $T$ , letting  $\sigma$  unknown. The method used was to find the root of  $C(S(0), K, T, r; \sigma) - C_{K,T}$ , where  $C_{K,T}$  is either the market price or the price giving by the function *mjd\_call*, and  $C(S(0), K, T, r; \sigma)$  is the function *bsm\_call* with  $\sigma$  as a cursor.

data. To do the job, that algorithm needs (i) a density function along with (ii) a sample of random data as a template. As defined in Matsuda [2004], the density of the log-returns of time-series generated by the MJD model is given by equation (4.4).

$$P_t(x) = \sum_{i=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^i}{i!} N \left[ x; \left( \alpha - \frac{\sigma^2}{2} - \lambda \kappa \right) t + i\mu, \sigma^2 t + i\delta^2 \right] \quad (4.4)$$

where

$$\kappa = e^{\mu + \frac{1}{2}\delta^2} - 1$$

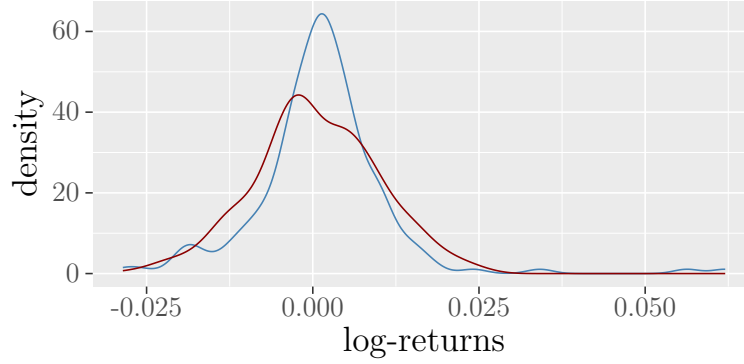
The term  $e^{-\lambda t} (\lambda t)^i / i!$  gives the probability that  $\lambda$  jumps occur during the time period  $t$ , while  $N$  denotes the normal probability density function.

The parameters to be assessed are the drift and variance rates of the stock prices process, namely,  $\alpha$  and  $\sigma$ , the frequency of the jumps occurrence  $\lambda$  and their intensity parameters, that is,  $\mu$  and  $\delta$ . All of those parameters are fully described in section 3.2.

By giving to the optimization function the equation (4.4) as a density function, the Apple stock data ?? as a template and the set of variables  $\{\alpha, \sigma, \lambda, \mu, \delta\}$   $\lambda$  as a cursor, that algorithm outputs the set 4.5 as the best fit arguments for the real world parameters.

$$\left\{ \begin{array}{l} \alpha = 0.4816753642, \\ \sigma = 0.1020368856, \\ \lambda = 99.5434345733, \\ \mu = -0.0006791266, \\ \delta = 0.0161092106, \end{array} \right\} \quad (4.5)$$

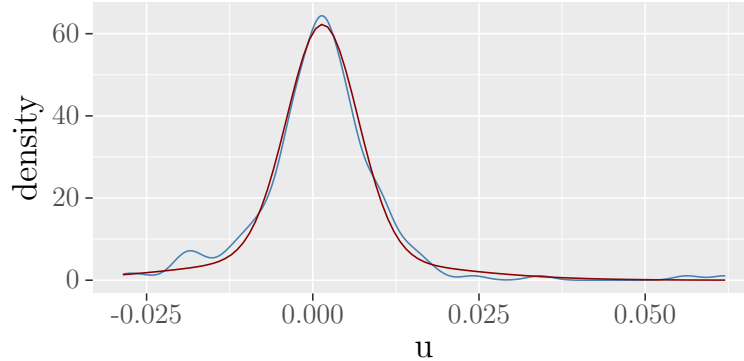
Figure 4.3: Historical and MJD related Apple stock Log-returns distribution



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series generated by the function *hsv\_ts* taking the risk-neutral parameters 4.3 as arguments.

Figure 4.4 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time series generated by the function *mjd\_ts()* fed with the risk-averse parameters 4.5, for the red one.

Figure 4.4: Historical and MJD related Apple stock Log-returns distribution in the risk-averse world



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series generated by the function *mjd\_ts* taking the risk-averse parameters 4.5 as arguments.

Consequently, for the purpose of that master thesis, the set 4.5 is the one that will be used together with the function *mjd\_ts()* whenever performing the computation of time series using the MJD model within the real world constraints.

#### 4.1.4 Heston's model calibration

##### Calibration of parameters for option pricing

In order to use the function *hsv\_call* based on equation (3.18) to compute the price of European call options on an underlying with increments driven by the HSV model, the parameters  $\{V(0), \kappa, \theta, \sigma, \rho\}$  must be calibrated with respect to the available market data.

Therefore, the so estimated parameters will be in line with the risk-neutral measure, or more specifically  $\kappa$  and  $\theta$ , which are the only ones needed to be adapted to make equation (3.15) risk-neutral.

To do so and similarly as it was done for the MJD option model, the least-square non-linear analysis will be used together with data on Apple call option as a template. In that respect, the theoretical models to calibrate are given by equations (3.18) to (3.20) along with equation (3.23), implemented by the R function *hsv\_call*.

Moreover, to stay in a range of acceptable values, the parameters of the HSV model should lie between some defined boundaries.

De facto, the mean-reversion speed ( $\kappa$ ), the long-run variance ( $\theta$ ) and the volatility of the volatility ( $\sigma$ ) need to take values that together respect the Feller's condition (see section 3.3.2). Additionally,  $\sigma$  must range between 0 and 1 and, according to ?  $\kappa$  should be positive to avoid mean aversion. Although the correlation coefficient  $\rho$  may take any value from  $[-1, 1]$ , typically the correlation between the stock price increments and their intrinsic volatility is negative. Consequently, the boundaries of  $\rho$  are set as  $]0, -1[$ .

According to those constraints and due to the presence of multiple local minima, the *lsqnonlin* function returns the table 4.5 as the whole sets of parameters that make the function *hsv\_call* better fit with reality.

Table 4.5: Best estimates for HSV call option model

v0	theta	sigma	rho	kappa
0.03851	0.05142	0.29539	-0.49338	1.99931
0.03774	0.04832	0.30687	-0.40213	3.00036
0.03910	0.04948	0.40736	-0.50193	3.00005
0.03798	0.04872	0.50379	-0.39878	4.00106
0.03740	0.04754	0.30583	-0.60126	4.00096
0.03661	0.04604	0.30659	-0.40295	5.00152
0.03921	0.04709	0.50753	-0.50170	5.00152

However, the set of arguments 4.6 is the one making respond the model the best with what is observed in reality. Indeed when passing that set to the R function *call\_heston*, for all strikes and maturities from tables 4.2 and 4.3 more than 70% of the so generated prices are within the bid-ask spread of the historical data.

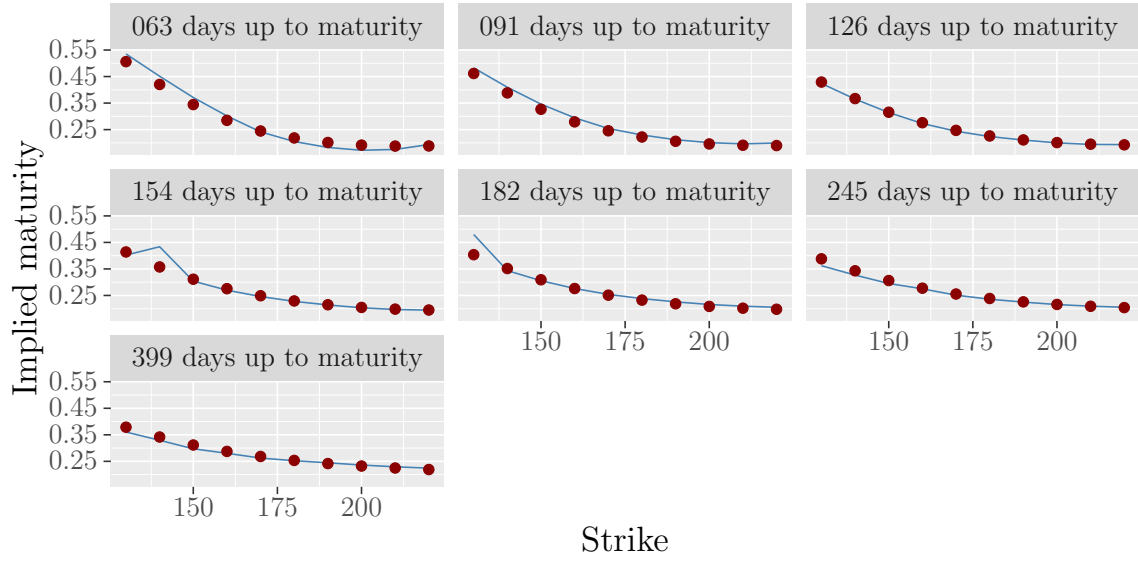
$$\left\{ \begin{array}{lcl} V(0) & = & 0.03798218, \\ \theta & = & 0.04871543, \\ \sigma & = & 0.50378803, \\ \rho & = & -0.39877827, \\ \kappa & = & 4.00105546 \end{array} \right\} \quad (4.6)$$

Figure 4.5 confronts the blue colored volatility smiles computed from market data, with those dotted in red, calculated from data provided by the function *hsv\_call* which takes the items of the set 4.6 as parameters.

Consequently, for the purpose of that master thesis, the set 4.6 is the one that will be used together with the function *hsv\_call* function whenever performing the computation of option prices for the HSV model.



Figure 4.5: Implied volatility of Apple option prices computed with HSV



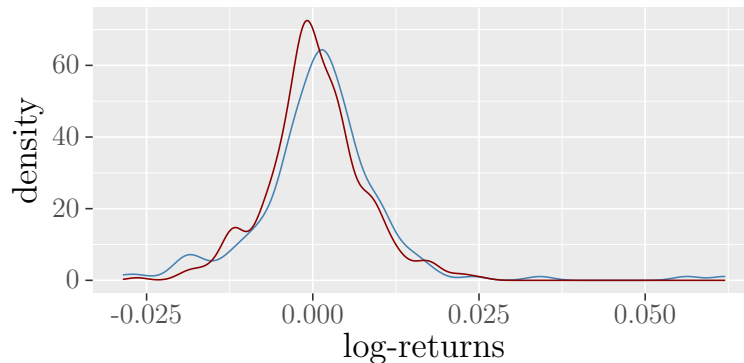
The blue curves represent the implied volatility computed from the option market data on Apple while the .....

### Calibration of parameters for time series

The set 4.6 is calibrated under the risk-neutral world and cannot, therefore, be used as is to simulate the dummy time-series that will serve the analysis aimed to measure the delta hedging performance. Furthermore, in addition to those parameters, the drift rate  $\alpha$  has to be estimated as well. It was not present in the set 4.6 because it was not taken into account in the computation of the options prices.

Figure 4.6 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical Apple stock data for the blue curve or from dummy time-series generated by the function *hsv\_ts()* fed with the risk-neutral parameters 4.6, for the red one.

Figure 4.6: Historical and HSV related Apple stock Log-returns distribution



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series generated by the function *hsv\_ts* taking the risk-neutral parameters 4.6 as arguments.

The goal of that section is to find the risk-averse parameters that make both the

distributions of the log-returns generated from market data or from *hsv\_ts* fit together. To do so, and according to the differences between equations (3.14) and (3.15) and equations (3.9) and (3.10), the parameters to modify are the drift rate  $r \rightarrow \alpha$ , the mean reversion speed  $\kappa^* \rightarrow \kappa$  and the long-run volatility  $\theta^* \rightarrow \theta$ . Accordingly, the correlation parameter  $\rho$  and the volatility of the volatility  $\sigma$  stay unchanged from the risk-neutral to risk-averse world.

The drift rate  $\alpha$  has been estimated by following the method exhibited at section 1.6. From the market empirical log-returns' first and second moments, the value 0.4822917 has been found for  $\alpha$ .

According to equations (3.16) and (3.17), in order to calculate  $\kappa$  and  $\theta$ , the risk premium  $\lambda$  has to be assessed. Practically, such as it was the case for the MJD model, the function *fitdistr* from the R package *MASS* will be purposely used. As defined in Dragulescu and Yakovenko [2002], the density of the log-returns generated by the HSV model is given by equation (4.7).

$$P_t(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\phi x + F_t(\phi)} d\phi \quad (4.7)$$

where

$$F_t(\phi) = \frac{\kappa\theta}{\sigma^2} \gamma t - \frac{2\kappa\theta}{\sigma^2} \ln \left( \cosh \frac{\Omega t}{2} + \frac{\Omega^2 - \gamma^2 + 2\kappa\gamma}{2\kappa\Omega} \sinh \frac{\Omega t}{2} \right)$$

and

$$\Omega = \sqrt{\gamma^2 + \sigma^2(\phi^2 - i\phi)}, \gamma = \kappa + i\rho\phi\sigma$$

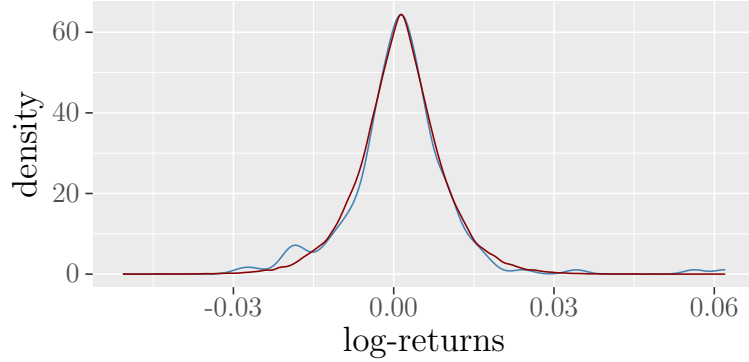
By giving to the optimization function the equation (4.7) as a density function, the Apple stock data ?? as a template and the variable  $\lambda$  as a cursor, that algorithm outputs  $\lambda = 5.4883278229$ , as the best fit argument, for the risk premium. The set of the risk-averse parameters is fully described here below in 4.8.

$$\left\{ \begin{array}{lcl} V(0) & = & 0.03798218, \\ \theta & = & 0.02054013, \\ \sigma & = & 0.50378803, \\ \rho & = & -0.39877827, \\ \kappa & = & 9.489383, \\ \alpha & = & 0.4822917 \end{array} \right\} \quad (4.8)$$

Figure 4.7 shows the empirical density curves illustrating the distributions of the log-returns computed either from historical data for the blue curve or from dummy time-series generated by the function *hsv\_ts()* fed with the risk-averse parameters 4.8, for the red one.

Consequently, for the purpose of that master thesis, the set 4.8 is the one that will be used together with the function *hsv\_ts()* whenever performing the computation of time series using the HSV model within the real world constraints.

Figure 4.7: Historical and HSV related Apple stock Log-returns distribution in the risk-averse world



The above blue density curve is constructed over the historical data of the Apple share of stock price evolution from 18th May 2017 to 18th May 2018. while the red curve is constructed from time-series generated by the function *hsv\_ts* taking the risk-averse parameters 4.8 as arguments.

## 4.2 Delta hedging

In order to measure the hedging performance of the MJD and HSV models, I will (i) compute the prices of some options and then virtually sell them at that price, (ii) construct the delta-neutral portfolio at time zero and thereafter continuously rebalance it, in accordance with the underlying asset price evolutions and (iii) measure the overall cost of the hedge as a performance indicator.

### 4.2.1 Determination of the option price

The computation of the option price at time zero will depend on the underlying model to be hedged. Indeed, the function *mjd\_call* will be used to price options for which the underlying asset price evolution is driven by the MDJ model, whereas *mjd\_call* will be used for HSV processes.

As explain as an introduction, I will virtually sell the options I price. Therefore, as the writer of the options, I will raise \$  $X(0) = C_K^{m\oplus h}(0)$ . That cash will directly be put into a money market account to make it grow at a constant riskless rate all along the duration of the hedging process, namely, during the whole life of the option.

The T-bills will serve as riskless rate. According to the available data on <https://www.treasury.gov/>, quotes of these are given by table 4.6.

Table 4.6: Treasury bill quotes on 18 May 2018

T-bills			
4 weeks	13 weeks	26 weeks	52 weeks
1.66%	1.90%	2.09%	2.30%

The rates of table 4.6 are annual-base, consequently to get their countinuously compounded counterpart, EQUATION X can be used to give those given by table 4.7.

Thereafter by applying the bootstrap method along with linear interpolation, one can find the maturity-wise related riskless rates of the Apple call option used for the current analysis. Those rates are given by table 4.8.

Table 4.7: Treasury bill quotes on 18 May 2018

T-bills			
4 weeks	13 weeks	26 weeks	52 weeks
1.659%	1.896%	2.068%	2.274%

Table 4.8: Maturities explored during the hedging performance measurement

Maturity(in days)	63	91	126	154	182	245	399
Riskless rate	1.817%	1.896%	1.962%	2.015%	2.068%	2.139%	2.311%

### 4.2.2 Construction of the delta-neutral portfolio

At time zero, the built of the delta-neutral portfolio is achieved by the purchase of  $\Delta(0)$  share of stock multiplying by the number of assets involving in the derivative product. Simply put, if one wants to construct such a portfolio to cover against the possible losses incurred by the selling of a call option on one hundred shares, the number of underlying to buy at that time would be of  $\Delta(t) \times 100$ . For the sake of clarity, in the analysis I will perform, it is implicit that any European call option involves the purchase of only one underlying at maturity, though. Consequently, at time zero, the cost of such a portfolio is given by equation (4.9).

$$p(t_0) = \Delta^{m \oplus h}(t_0)S(t_0) \quad (4.9)$$

Where the superscripts m (Merton) and h (Heston) respectively stand for the computation of MJD and HSV delta.

The objective of that portfolio is to accurately replicate the reverse position taken in a derivative, i.e., to reproduce a position in a long European call in the case that matters for that analysis. To be efficient in that role, it must be frequently rebalanced.

Accordingly, in order to do so, I will observe the underlying's time-series at every period of time  $\delta t = 1days$  and compute the updated value of  $\Delta(t)$  at each of those observational timesteps. Therefore, the requisite expenses aimed to keep the portfolio delta-neutral will evolve over time, as shown here below by equation (4.10).

$$p(t_i) = \Delta^{m \oplus h}(t_i - t_{i-1})S(t_i), \forall i \in \mathbb{Z} : i \in \{1, T\} \quad (4.10)$$

The aforementioned time-series that will be daily scrutinized are obviously hypothetical. They will be generated by the functions *mjd\_ts* and *hsv\_ts* for which the risk-averse parameters will be passed in.

### 4.2.3 Computation of the delta

By definition, the delta is the first derivative of the option's pricing function with respect to the stock price. It consequently, represents the instantaneous rate of change in the option value as the price of its underlying evolves, such so illustrated by equation (2.18).

Although Black and Scholes [1973] gives to delta the value  $N(d_+(\Delta t, x))$ , with  $d_+$  explained by equation (2.17), latter can only be used within the framework developed by Black and Scholes [1973], that is, if the underlying is driven by a geometric Brownian

motion, the volatility is deterministic, or if no jump occurs. Accordingly, the solution to compute the delta provided by Black and Scholes [1973] cannot be used for such the MJD and HSV models.

By following the definition of delta and because the pricing function of a European call option, for which the underlying asset price evolution is driven by either the MDJ or HSV models, is given by equation (3.18), then, the equation (4.11) gives the solution to compute the delta for such models.

$$\Delta^{m\oplus h}(t, S(t)) = P_1^{m\oplus h} + S(t) \frac{\partial P_1^{m\oplus h}}{\partial S(t)} - e^{-r(T-t)} K \frac{\partial P_2^{m\oplus h}}{\partial S(t)} \quad (4.11)$$

where

$$\begin{aligned} \frac{\partial P_1^{m\oplus h}}{\partial S(t)} = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{i\phi \exp(-i\phi \ln K)}{-\phi^2 \psi^{m\oplus h}(-i)^2} \right. \\ \left. \times \left( \frac{\partial \psi^{m\oplus h}(\phi - i)}{\partial S(t)} \psi^{m\oplus h}(-i) - \psi^{m\oplus h}(\phi - i) \frac{\partial \psi^{m\oplus h}(-i)}{\partial S(t)} \right) \right] d\phi \end{aligned} \quad (4.12)$$

and

$$\frac{\partial P_2^{m\oplus h}}{\partial S(t)} = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left( \frac{\exp(-i\phi \ln K)}{i\phi} \frac{\partial \psi^{m\oplus h}(\phi)}{\partial S(t)} \right) d\phi \quad (4.13)$$

Hence, the equation (4.11) is the one to solve for both MJD and HSV models to find them the delta at time  $t$ . Equations (4.12) and (4.13) can be solved by finding the first derivatives of the characteristic functions of MJD and HSV processes with respect to the stock price. By the way, it turns out that both of these have the same form. Equation (4.14) gives the solution of these derivatives, the term  $\psi^{m\oplus h}(x)$  have to be replaced by the appropriate characteristic function.

$$\frac{\partial \psi^{m\oplus h}(x)}{\partial S(t)} = \psi^{m\oplus h}(x) \frac{ix}{S(t)} \quad (4.14)$$

By replacing all the derivative operations inside equations (4.12) and (4.13) by equation (4.14), one finds the solved version given by equations (4.15) and (4.16).

$$\frac{\partial P_1^{m\oplus h}}{\partial S(t)} = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{\exp(-i\phi \ln K) \psi^{m\oplus h}(\phi - i)}{S(t) \psi^{m\oplus h}(-i)} \right] d\phi \quad (4.15)$$

$$\frac{\partial P_2^{m\oplus h}}{\partial S(t)} = \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left[ \frac{\exp(-i\phi \ln K) \psi^{m\oplus h}(\phi)}{S(t)} \right] d\phi \quad (4.16)$$

Consequently, in the extent that the term  $\psi^{m\oplus h}(\phi)$  is replaced by the appropriate characteristic function, substituting equations (4.15) and (4.16) into 4.11 gives 4.17 that allows the computation of the delta of an option for which the underlying asset is driven by either a MJD or HSV process.

$$\Delta^{m\oplus h}(t, S(t)) = P_1^{m\oplus h} + \frac{S(t)}{\pi} \int_0^\infty \text{Re} \left[ \frac{\exp(-i\phi \ln K) \psi^{m\oplus h}(\phi - i)}{S(t) \psi^{m\oplus h}(-i)} \right] - \frac{e^{-r(T-t)} K}{\pi} \int_0^\infty \text{Re} \left[ \frac{\exp(-i\phi \ln K) \psi^{m\oplus h}(\phi)}{S(t)} \right] \quad (4.17)$$

In that respect, the R functions *mjd\_delta* and *hsv\_delta*, respectively compute the MJD and HSV related deltas at time  $t$  by taking the risk-neutral parameters.

#### 4.2.4 Measuring the hedging performance

Finally, to analyze how much the hedging works, one method is to measure its overall cost. To do so, the really first thing is to value at maturity all the buying and selling operations carried out in order to keep delta-neutral the previously constructed portfolio and add them together.

$$p_{tot} = \sum_{i=0}^T p(t_i) e^{T-t_i}, \forall i \in \mathbb{Z} : i \in \{1, T\} \quad (4.18)$$

On the other hand, at maturity, the amount of money  $X(0)$  received when entering into the contract at time zero will have continuously grown up at the constant riskless rate  $r$ , as described by equation (4.19).

$$X(T) = X(0) * e^{rT} \quad (4.19)$$

Incidentally, if the option is in-the-market at maturity date, it first implies that it will be exercised, so increasing the wealth of the short position in the derivative by  $K$ , and then that  $\Delta(T) = 1$ , meaning that the writer of the option has exactly one share of stock in his / her portfolio to honour the transaction. Otherwise, the option is out-of-the-market, inducing that no transaction occurs and that  $\Delta(T) = 0$ . Consequently, at maturity, someone with a short position in the derivative, i.e., the one tempting to be replicated in the current analysis, will receive a positive cash inflow of  $X(T) + \Delta(T)K$ .

Hence, if the hedge works well, the relation exhibited by 4.20 should be respected.

$$\sum_{i=0}^{n=T} P(i) \cong X(T) + \Delta(T) * k \quad (4.20)$$

Because the position could vary, we have to adapt the relation above (REF) in order to remove all unit from the comparison. According to hull, a measure of performance of the hedge could be to take the ratio of the standard deviation of the cost to hedge to the option price. In that case, the measure could be comparated among themselves even if the position taken into the underlying differ.

$$\frac{sd(hedge)}{C_0 e^{rT}} \quad (4.21)$$

where hedge is a vector containing  $n$  sample generated, and  $C_0$  is the price of the call option at time zero value up to maturity.

# Chapter 5

## Analysis

The objective of the current chapter is to measure the hedging performance of the Black-Scholes model when the stock price evolves in a non-log-normal world. The studied models allowing the course of a time-series to go out of the Black-Scholes frame are the Merton jump-diffusion (MJD) and Heston stochastic volatility(HSV) models. In order to give realistic results, they have been adjusted in ???. For both models, two kinds of parameters have been found, namely, the risk-neutral and risk-averse ones. Therefore, the "no risk" parameters are used to price option, while the risky serve to generate time-series thanks to those models.

The chosen metric to compare the models is the "relative profit and loss" (P&L). Latter is explained in ???.

The analysis is completed in two steps. The first concerns exclusively the BSM model with all the related constraints respected. Even though the computation of the options' prices in such an environment does not respect those emerging from reality, the measure of the P&L so computed serve as a benchmark for the other models. The next and last part of the analysis measures the P&L for both models MJD and HSV. Those P&L are computed using the delta-hedging strategy. On one hand by using the delta of Black and Scholes and on the other, by using the appropriate delta of the considered model. Although the Black and Scholes delta is purposely computed to be applied when the underlying asset is driven by a geometric Brownian motion (GBM), the goal is to assess how one can go wrong by using it within the real conditions.

At last, some particular results are commented.

### 5.1 Measuring performance in a log-normal world

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Table 5.1: Hedging with BSM: Relative P&amp;L

		91 dbm <sup>a</sup>	182 dbm	399 dbm
	intraday	0	0	0
140	daily	0	0	0
	weekly	0	0	-0.001
	intraday	0	0	0
160	daily	0	-0.001	-0.001
	weekly	-0.003	-0.005	-0.005
	intraday	-0.005	0.001	-0.002
186	daily	-0.01	-0.007	-0.005
	weekly	-0.066	-0.034	-0.019
	intraday	-0.015	-0.003	-0.005
200	daily	-0.047	-0.019	-0.011
	weekly	-0.229	-0.101	-0.039
	intraday	0.296	0.009	-0.03
230	daily	-0.045	-0.059	-0.039
	weekly	-1.305	-0.322	-0.149

<sup>a</sup>dbm: days before maturity

Table 5.2: Hedging with MJD: Relative P&amp;L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		$\Delta_{mrt}$	$\Delta_{bsm}$	$\Delta_{mrt}$	$\Delta_{bsm}$	$\Delta_{mrt}$	$\Delta_{bsm}$
	intraday	0.004	0.006	0.011	0.012	0.01	0.021
140	daily	0.002	0.006	0.008	0.012	0.016	0.021
	weekly	0.004	0.006	0.006	0.011	0.007	0.021
	intraday	0.011	0.018	0.021	0.029	0.025	0.042
160	daily	0.016	0.018	0.022	0.029	0.019	0.042
	weekly	0.013	0.016	0.018	0.026	0.018	0.04
	intraday	0.036	0.021	0.078	0.055	0.079	0.074
186	daily	0.039	0.022	0.072	0.055	0.068	0.074
	weekly	0.014	-0.008	0.055	0.037	0.057	0.061
	intraday	0.072	-0.002	0.139	0.061	0.13	0.086
200	daily	0.06	-0.013	0.131	0.057	0.115	0.085
	weekly	-0.02	-0.1	0.083	0.005	0.085	0.053
	intraday	0.955	0.331	0.444	-0.061	0.301	0.063
230	daily	1.098	0.466	0.409	-0.091	0.261	0.054
	weekly	-0.741	-1.335	0.085	-0.438	0.174	-0.088



Table 5.3: Hedging with HSV: Relative P&L

Strikes	frequency	91 dbm		182 dbm		399 dbm	
		$\Delta_{hsv}$	$\Delta_{bsm}$	$\Delta_{hsv}$	$\Delta_{bsm}$	$\Delta_{hsv}$	$\Delta_{bsm}$
140	intraday	0	0.002	0.011	0.011	0.009	0.038
	daily	-0.001	0.002	0.01	0.011	0.009	0.038
	weekly	0.001	0.002	0	0.011	0.008	0.038
160	intraday	0.009	0.028	0.023	0.073	0.042	0.143
	daily	0.008	0.028	0.025	0.072	0.036	0.143
	weekly	0.008	0.028	0.019	0.073	0.036	0.143
186	intraday	0.158	0.252	0.159	0.392	0.153	0.524
	daily	0.15	0.245	0.195	0.391	0.156	0.522
	weekly	0.117	0.241	0.158	0.378	0.139	0.519
200	intraday	0.459	-0.298	0.43	0.146	0.279	0.546
	daily	0.433	-0.361	0.42	0.126	0.255	0.544
	weekly	0.268	-0.659	0.369	0.005	0.246	0.498
230	intraday	2.136	-0.527	1.884	-2.452	1.01	-0.235
	daily	1.948	-1.197	1.893	-2.655	0.989	-0.224
	weekly	1.407	-2.152	1.547	-2.402	0.917	-0.353

# Bibliography

- Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.
- Hans W. Borchers. *pracma: Practical Numerical Math Functions*, 2018. URL <https://CRAN.R-project.org/package=pracma>. R package version 2.1.4.
- Peter K Clark. A subordinated stochastic process model with finite variance for speculative prices. *Econometrica: journal of the Econometric Society*, pages 135–155, 1973.
- Ricardo Crisóstomo. An analysis of the heston stochastic volatility model: Implementation and calibration using matlab. *arXiv preprint arXiv:1502.02963*, 2015.
- Adrian A Dragulescu and Victor M Yakovenko. Probability distribution of returns in the heston model with stochastic volatility. *Quantitative Finance*, 2(6):443–453, 2002.
- William Feller. Two singular diffusion problems. *Annals of Mathematics*, 54(1):173–182, 1951. ISSN 0003486X. URL <http://www.jstor.org/stable/1969318>.
- J. Gatheral and N.N. Taleb. *The Volatility Surface: A Practitioner’s Guide*. Wiley Finance. Wiley, 2006. ISBN 9780470068250. URL <https://books.google.be/books?id=9Y8rWE6mLOEC>.
- Steven L Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343, 1993.
- John C Hull and Sankarshan Basu. *Options, futures, and other derivatives*. Pearson Education India, 2012.
- D. N. Joanes and C. A. Gill. Comparing measures of sample skewness and kurtosis. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 47(1):183–189, 1998. ISSN 00390526, 14679884. URL <http://www.jstor.org/stable/2988433>.
- Kazuhisa Matsuda. Introduction to merton jump diffusion model. *Department of Economics. The Graduate Center, The City University of New York*, 2004.
- Robert C Merton. Theory of rational option pricing. *The Bell Journal of economics and management science*, pages 141–183, 1973.
- Robert C Merton. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics*, 3(1-2):125–144, 1976.

- R Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2017. URL <https://www.R-project.org/>.
- J.R.M. Röman. *Analytical Finance: Volume II: The Mathematics of Interest Rate Derivatives, Markets, Risk and Valuation*. Springer International Publishing, 2017. ISBN 9783319525846. URL <https://books.google.be/books?id=SCpBDwAAQBAJ>.
- Jeffrey A. Ryan and Joshua M. Ulrich. *quantmod: Quantitative Financial Modelling Framework*, 2018. URL <https://CRAN.R-project.org/package=quantmod>. R package version 0.4-13.
- Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media, 2004.
- Anthony Tedde. *RandomWalk: Generate Random Walks*, 2017. R package version 0.0.0.9000.
- Dean Teneng. Limitations of the black-scholes model. *Collection of Papers 2011–Volume I*, page 143, 2011.
- W. N. Venables and B. D. Ripley. *Modern Applied Statistics with S*. Springer, New York, fourth edition, 2002. URL <http://www.stats.ox.ac.uk/pub/MASS4>. ISBN 0-387-95457-0.