Modular Character Theory

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1 Preface

The representation theory of finite groups began with the work of Fröbenius who developed the theory of ordinary characters and was followed by Burnside and Schur who proved ubiquitous results like Burnside's $p^a q^b$ theorem and Schur's Lemma. This theory over fields of characteristic zero has been well studied and leads naturally to question modular representations as we work over a field of prime characteristic. Major developments of the modular theory began with Richard Brauer who used ring and character theoretic arguments to prove many of the results now key to the field with James Green later pioneering modular theoretic results.

This essay concerns itself with modular character theory in which representations may no longer be semisimple and we aim to present cornerstone results on the subject. We begin with a rendition of background material on group algebras and the theorems of Wedderburn and Jordan-Hölder which will lead towards a first glance at Brauer characters as we restrict focus to certain conjugacy classes of the group and generalise ordinary notions. After developing some theory of projective and indecomposable modules we will study the representations of $SL_2(p)$ in characteristic $p > 2$ which will allow us to expose many of these results. The computations will lead naturally to the Green correspondence which explains a correspondence between modules over group algebras of a group and the normalizers of p-subgroups. We will then describe how to connect representations of a group between characteristic 0 and p through p-modular systems and define the notions of the decomposition and Cartan matrices. This will invite some study of the Brauer orthogonality relations and we will provide examples of calculations of these objects in the case of S_5. Later we will provide a brief overview of Young diagrams which give a combinatorial and constructive way to view representations of the symmetric groups and use this theory to calculate the decomposition matrix of S_5 and S_6 in characteristic 5. To finish we will discuss some natural developments and conjectures in the field.

Our approach here will be concise though the works of Webb [24] and Landrock [16] provide largely contained and detailed references on many of the results covered in this essay and more. Landrock [16] in particular develops the theory further while the standard reference text is that of Curtis & Reiner [9]. The treatment of $SL_2(p)$ is largely inspired by Alperin [1] and these referenced texts progress to the theory of blocks and Brauer's main theorems which we will not discuss in this essay though we build up the necessary language to approach these results. The discussion of Young diagrams is greatly inspired by James [14] and will provide a different combinatorial flavour to the mainly algebraic ideas we adopt in much of the essay. We hope this essay provides an enjoyable and illuminating introduction to the study of modular character theory.

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2 Representations and Modules

2.1 Representations

We first begin with the definition of a representation of a finite group.

Definition 2.1. Let G be a finite group and let R be a commutative ring with a 1. A *representation* of G (over R) is a group homomorphism $\rho : G \to GL(V)$ where V is an R-module and $GL(V)$ is the group of invertible R-module homomorphisms $V \to V$. When V is a free R-module $V \cong R^n$, $GL(V) \cong GL_n(R)$ and in this case we say the *degree* of ρ is n.

When V is a free R-module we can write elements of $GL(V)$ in matrix format giving a *matrix representation*. In this essay we will view representations as the action of a group on an R-module.

Definition 2.2. The *group algebra* RG is an R-algebra consisting of linear combinations of elements of G with coefficents in R. Operations are defined as

$$\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g = \sum_{g\in G}(\alpha_g+\beta_g)g \qquad (\sum_{g\in G}\alpha_g g)(\sum_{g\in G}\beta_g g) = \sum_{g\in G}(\sum_{hh'=g}\alpha_h\beta_{h'})g$$

The group algebra is a free R-module with R-basis given by the elements of G.

We can then view representations naturally as RG-modules and this is the viewpoint we will use henceforth.

Lemma 2.3. *There is a 1-1 correspondence of representations of G over R and (left) unital RG-modules.*

Proof: A representation $\rho : G \to GL(V)$ defines an RG-module structure homomorphism on V via $(\sum_{g\in G}\alpha_g g)v = \sum_{g\in G}\alpha_g\rho(g)(v)$. Conversely given an RG-module V we have an R-linear map $\rho(g) : v \to gv$ giving a representation $\rho : G \to GL(V)$.

Generally we can view any ring A as a left A-module via left multiplication of A on itself and we can write this as ${}_AA$ to emphasise A as an A-module. When refering to A as an A-module this is the action we assume. The module ${}_{RG}RG$ will be a key object of our study and we will see it contains the information required to classify finite dimensional representations of G.

Definition 2.4. We say finite dimensional RG-modules V, W are *isomorphic/equivalent* if there is an RG-module isomorphism $f : V \to W$. Since RG has a basis given by elements of G this is the same as an R-linear isomorphism $f : V \to W$ such that $f(gv) = gf(v)$ for all $g \in G$.

We now make explicit what we mean by reducibility and indecomposability:

Definition 2.5. Let V be an R-module. V is *reducible* if there is a submodule $0 \neq W \neq V$ of V and we say V is *irreducible/simple* if it is non zero and not reducible. V is *decomposable* if there exist $V_1, V_2 \neq 0$ such that $V = V_1 \oplus V_2$ else it is *indecomposable*.

2.2 Semisimple modules

An important class of modules are direct sums of simple modules

Definition 2.6. An R-module V is *semisimple* if V is the direct sum of simple R-modules.

Semisimple modules have a number of useful properties

Theorem 2.7. *1. A submodule of a semisimple module is semisimple and a direct summand.*

2. Quotients of semisimple modules are semisimple.

3. A module is semisimple if and only if every submodule is a direct summand.

Proof: For the proof of 3) see Farb & Dennis [10] pg 50.

If V is a simple (left) R-module then V is generated by any non-zero element hence V is the homomorphic image of ${}_RR$ and is the quotient of ${}_RR$ by a (left) maximal ideal of R.

Definition 2.8. The *radical* of R is the ideal $\text{rad}(R)$ consisting of elements which annihilate every simple R-module e.g $\text{rad}(R) = \bigcap_{\text{simple left } R\text{-modules}} \text{ann}_R(V)$ where $\text{ann}_R(V) = \{a \in R|\, aV = 0\}$.

There are important properties satisfied by the radical

Theorem 2.9. *The following are equivalent characterisations of the radical of R, $rad(R)$:*

1. The intersection of all maximal left ideals in R (intersection of all maximal submodules of ${}_RR$)

2. The smallest submodule of ${}_RR$ with the quotient being semisimple

3. The largest nilpotent ideal of R

Proof: See Alperin [1] Chapter 1 Theorem 3. Proposition 4 in the same chapter allows us to describe the radical of an R-module V with similar properties. $\text{rad}(R)V$ is the smallest submodule of V with semisimple quotient and is the intersection of all maximal submodules of V and we will say this is the radical of V, $\text{rad}(V)$.

The radical can also help determine when a module is semisimple

Proposition 2.10. *Suppose R is additionally left artinian. Then $rad(R) = 0 \iff$ every R-module is semisimple $\iff$ every finitely generated R-module is semisimple.*

Proof: See Martin [19] Theorem 3.8.

We have a corresponding module to the radical called the socle.

Definition 2.11. The *socle* of an R-module V, $\text{soc}(V)$ is the sum of all simple modules of U.

Proposition 2.12. *The socle of an R-module V is the largest semisimple submodule of V and the set of $v \in V$ such that $rad(R)v = 0$.*

Proof: See Alperin [1] Chapter I Proposition 5.

We now consider the fundamental result

Theorem 2.13. *(Artin-Wedderburn) Let R be a finite dimensional algebra over a field k (thus $rad(R) = 0$ and every finitely generated R-module is semisimple). Then ${}_RR \cong \bigoplus_{i=1}^{m} S_i^{d_i}$ of simple modules and R is a direct sum of matrix algebras*

$$R = \prod_{i=1}^{m} M_{d_i}(D_i)$$

where D_i are division rings containing k in the centre and are finite dimensional over k.

Proof: For a full proof see the discussion in Webb [24] Chapter 2. There are some important results used in the proof. One result is Schur's Lemma states $\mathrm{End}_R(S) = D^{op}$ for a simple module S_i and division ring D. Also if $V = \oplus_{i=1}^{r} V_i$ as modules then $\mathrm{End}_R(V) \cong M_r(\mathrm{Hom}_R(V_j, V_i))$ such that endomorphisms compose via matrix multiplication.

Remark 2.14. The statement requires the use of division rings however we will see later that when the field is large enough such as when k is algebraically closed we can replace the division rings with k and so in this case for a simple module S, $\mathrm{End}_R(S) \cong k$.

2.3 Composition Series

When our module is not semisimple we can no longer consider only the simple modules to determine the nature of these modules. Indeed we have the result We must instead think of our modules in layers and to do this we will need the Jordan-Hölder theorem.

Definition 2.15. A *composition series* for an R-module M is a series of submodules

$$0 = M_0 \subset M_1 \subset ... \subset M_n = M$$

such that the composition factors M_i/M_{i-1} are simple.

Theorem 2.16. *(Jordan-Hölder) If M is an R-module and M has a composition series then any two composition series are of the same length and up to arrangement the composition factors are pairwise isomorphic. Moreover any series of submodules of M can be refined to a composition series.*

Proof: See Benson [4] Theorem 1.1.4.

The key to understanding modular representations will be to find such composition series and compute the composition factors. An important case is when the composition series is unique in which case we say M is *uniserial*.

Definition 2.17. Recall the radical and socle of an R-module V. We define the radical (Loewy) and socle series of M as

$$V = \mathrm{rad}^0(V) \supseteq \mathrm{rad}^1(V) \supseteq \mathrm{rad}^2(V) \supseteq ...$$
$$0 = \mathrm{soc}^0(V) \subseteq \mathrm{soc}^1(V) \subseteq \mathrm{soc}^2(V) \subseteq ...$$

where $\mathrm{rad}^n(V) = \mathrm{rad}(\mathrm{rad}^{n-1}(V))$ and $\mathrm{soc}^n(V)/\mathrm{soc}^{n-1}(V) = \mathrm{soc}(V/\mathrm{soc}^{n-1}(V))$ are defined recursively.

We call the lengths of these series the radical and socle lengths of V. These lengths coincide and the common length is called the *Loewy length*. When V is uniserial these series are composition series for V.

Proposition 2.18. *Let V be an R-module then the following are equivalent*

1. *V is uniserial (has a unique composition series)*
2. *The successive quotients in the radical series of V are simple*
3. *The successive quotients in the socle series of V are simple*

Proof: See Alperin [1] Chapter 4, Proposition 5. In particular note when V is uniserial $V/\mathrm{rad}(V)$ (the *head* of V) is simple hence appears as the top composition factor in the composition series for V.

3 Brauer Characters I

We make our first foray into modular representations as we discuss representations over fields of positive characteristic. We will need to be more subtle and generalise the notion of a character from the ordinary case . In our treatment here we will work over $\mathbb{C}$ but we will see later that the results are true when working over large enough (splitting) fields. See Webb [24] Chapter 10 for a discussion of analogous results in this general case. For this section we fix a prime $p > 0$ and a field k of characteristic p.

3.1 p'-elements

Definition 3.1. A group element $g \in G$ is a *p-element/p-singular* if its order is a power of p else is a *p'-element/p-regular* when its order is prime to p. When $|G| = p^a m, (p, m) = 1$, $m$is the *p'-part* of $|G|$.

The idea will be to define a character only on these p-regular elements. We note that taking $p = 0$ gives the familiar ordinary definition of characters where all group elements are 'p-regular'. In this case the representations are precisely the ordinary ones

Theorem 3.2. *(Maschke's Theorem) kG is semisimple if char k does not divide G.*

Proof: Benson [4] Corollary 3.6.12.

Proposition 3.3. *Let G be a group. Any $g \in G$ can be written uniquely as $g = xy = yx$ where x is p-singular, y is p-regular*

Proof: If the order of g is $p^a m$ where m is coprime to p. Then by Euclid take $s, t \in \mathbb{Z}$ s.t $sp^a + tm = 1$ aand let $x = g^{tm}, y = g^{sp^a}$. The results are then verified easily.

Remark 3.4. $T^m - 1$ is separable in characteristic p and $T^{mp^a} - 1 = (T^m - 1)^{p^a}$. If we work in a large enough field containing $|G|_{p'}$ roots of unity then for any representation ρ, $\rho(g), \rho(y)$ have the same eigenvalues while the eigenvalues of $\rho(x)$ are all 1 (see Martin [19] Lemma 4.3).

This then motivates us to consider lifting the $|G|_{p'}$ roots of unity in k to $\mathbb{C}$ to consider the eigenvalues of a representation as being in $\mathbb{C}$. The choice of such a lifting is important and will change the following discussion thus from here on we fix a lifting ψ of roots of unity.

3.2 The Brauer Character

Definition 3.5. We define the *Brauer character* $\phi_U : G_{p-reg} \to \mathbb{C}$ as follows. Let $\rho : G \to GL(U)$ be a representation (U a k-module) and g a p-regular element then

$$\phi_U(g) = \sum_{i=1}^{n} \psi(\lambda_i)$$

where λ_i are the eigenvalues of $\rho(g)$.

Note here we assume $\rho(g)$ is diagonalisable so eigenvalues lie in k but choosing g to be p-regular means that $k\langle g\rangle$ is semisimple by Maschke. We have some immediate properties

Lemma 3.6. *U, V, W, S finite dimensional kG-modules. The Brauer Characters satisfy:*

1. $\phi_U(1) = dim_k U$
2. ϕ_U *is a class function on p-regular conjugacy classes*
3. $\phi_U(g^{-1}) = \overline{\phi_U(g)}$
4. $\phi_{U\otimes V} = \phi_U \cdot \phi_V$
5. *If $0 \to U \to V \to W \to 0$ is a short exact sequence of kG-modules then $\phi_V = \phi_U + \phi_W$. In particular if U has composition factors S with multiplicity n_S then $\phi_U = \sum_S n_S\phi_S$*
6. *U, V have the same Brauer character if and only if the eigenvalues for all $g \in G$ under these representations are the same*

Proof: See Webb [24] Proposition 10.1.3 for proofs of (1-5) and Martin [19] Lemma 4.5 for 6). We remark that an algebraic conjugate of a Brauer character is not necessarily a Brauer character. (Landrock [16] pg 64 gives A_8 in characteristic 2 as a counterexample).

The Brauer character provides identifiability of finite dimensional kG-modules with different composition factors

Theorem 3.7. *(Brauer) Let U, V be finite dimensional kG-modules. Then $\phi_U = \phi_{U'}$ if and only if the multiplicities of simple kG-modules as composition factors of U, V are equal.*

We can then form the Brauer character table of G (mod p) where rows are indexed by the simple kG-modules S and columns indexed by conjugacy classes of p'-elements in G. The entries are $\phi_S(g)$ for p-regular conjugacy class representatives p. Our next focus will be to show this table is square.

3.3 Grothendieck ring

We will construct an algebra isomorphism between a module of rank equal to the number of distinct simple modules and the product of # p-regular conjugacy classes of $\mathbb{C}$-space $\mathbb{C}^{p-reg\ ccls}$. To do so we will need the abelian Grothendieck group $R(G)$ (see Schneider [21] Chapter 1.7 for an explicit construction).

Definition 3.8. The *Grothendieck ring* $R(G)$ has generators $[M]$ where M is an isomorphism class of finite dimensional kG modules. Addition is given by short exact sequences, if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of kG-modules then $[M] = [M'] + [M'']$. Therefore $R(G)$ has a basis $[S_i]$ of isomorphism classes of simple kG-modules S_i. Multiplication is given by $[M] \cdot [N] = [M \otimes_k N]$.

Now we will prove

Theorem 3.9. *The map*

$$\phi : \mathbb{C} \otimes_{\mathbb{Z}} R(G) \to \prod_{p-regccls} \mathbb{C}$$

which extends $[M] \to \phi_M$ is an algebra isomorphism.

which immediately provides the important results

Corollary 3.10. *(Brauer) The number of isomorphism classes of simple kG-modules is the same as the number of p-regular conjugacy classes in G.*

Corollary 3.11. *The irreducible Brauer characters form a basis of class functions on p-regular conjugacy classes.*

Injectivity and surjectivity are proved separately. Injectivity follows from the fact:

Lemma 3.12. *(Injectivity) The irreducible Brauer characters are linearly independent over $\mathbb{C}$*

Proof: See Schneider [21] Lemma 3.2.1.

To prove surjectivity we will introduce the induced module which will give a constructive proof of surjectivity and is an interesting module in its own right.

3.4 Induced Modules

We can regard kG as a $(kG, kH)-$bimodule e.g a left kG-module and right kH-module via left and right multiplication. We can then 'induct' kH-modules to kG.

Definition 3.13. If M is a kH-module the *induced module* M^G is the left kG-module $kG \otimes_{kH} M$.

and the induced module has Brauer character

Lemma 3.14. *The induced module M^G has Brauer character on p'-elements g given by*

$$\phi_{M^G}(g) = \sum_{ccls\ of\ h \in H\ s.t\ h \sim_G g} |C_G(h) : C_H(h)| \phi_M(h)$$

where $h \sim_G g$ means h is conjugate to g in G and thus we deduce

Corollary 3.15. *The map in Theorem 3.9 is surjective.*

Proof : Extend the induction $[M] \to [kG \otimes_{kH} M]$ to a map $\mathbb{C} \otimes_{\mathbb{Z}} R(H) \to \mathbb{C} \otimes_{\mathbb{Z}} R(G)$. Then let $H = \langle g \rangle$ for some p-regular $g \in G$ of order n. Then set $V \in \mathbb{C} \otimes_{\mathbb{Z}} R(H)$ with $V = \frac{1}{n} \sum_{j=1}^{|\langle g \rangle|} e^{-\frac{2\pi i j}{n}} [S_j]$ for simple $k\langle g \rangle$-modules S_j so that under the map $\phi_V(g^t) = \delta_{g^t = g}$. Computation shows that the induced Brauer character ϕ_{V^G} is non-zero only on elements conjugate to g. Thus traversing over all p-regular conjugacy class representatives g gives the result.

We will now exhibit some results we will require later when we show a correspondence between induced and restricted modules.

Lemma 3.16. *Let M_1, M_2 be kH-modules for a subgroup H of G then if M_1 is free/projective M_1^G is also free/projective. Additionally induction commutes with direct sums, $(M_1 \oplus M_2)^G \cong M_1^G \oplus M_2^G$.*

Proof: See Alperin [1] Chapter 8 Lemma 5. There is also a standard result that relates induced and restricted modules.

Theorem 3.17. *(Mackey decomposition) Let H, L be subgroups of G and $L \backslash G / H$ a set of double coset representatives. Then for a kH-module V*

$$(V^G)_L \cong \bigoplus_{s \in L \backslash G / H} (s(V)_{L \cap {}^sH})^L$$

as kH-modules (${}^sH = sHs^{-1}$).

Proof: See Webb [24] Theorem 5.2.1 and Landrock [16] Chapter II Theorem 1.7 for the case with the trace map.

If V is a kG-module then V is naturally a kH-module for a subgroup N of G and we write V_H for the restriction of V to kH. To end this section we state a useful result which will allow us to easily work with normal subgroups.

Theorem 3.18. *(Clifford's Theorem) Let V be a simple kG-module and N a normal subgroup of G. Then V_N is semisimple as a kN-module.*

Proof: Webb [24] Theorem 5.3.1. As an example of its application, by induction on the power of p we deduce that the only simple kG-module of a p-group G is the trivial module.

4 Projective and Indecomposable modules

When kG is not semisimple we cannot decompose kG into simple modules however we can do so into indecomposable modules and we will find a correspondence between the summands and simple modules. First we need a definition on when a module is *projective*.

Proposition 4.1. *Let M be an R-module. The following are equivalent for an R-module P.*

1. *For all surjective homomorphisms $M' \to M$ of R-modules and homomorphism $P \to M$ there exists $P \to M'$ giving a commutative diagram*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists"'] \arrow[d] & \\ M' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

2. *P is a direct summand of a free module*
3. *Every epimorphism $M \to P \to 0$ splits*
4. *The $Hom_R(P, -)$ functor is exact*

If any of these conditions hold we say P is *projective*.

4.1 Indecomposable Modules

The first result of this section will assure us that decompositions into indecomposable modules are unique up to rearrangement.

Theorem 4.2. *(Krull-Schmidt) If A is a finite dimensional k-algebra and M is a finitely generated A-module and*

$$M \cong \bigoplus_{i \in I} M'_i \cong \bigoplus_{j \in J} M''_j$$

are decompositions of M into indecomposable A-modules M'_i, M''_j then there exists a bijection $\phi : I \to J$ such that $M'_i \cong M''_{\phi(i)}$ for all i.

In particular $|I| = |J|$ and decompositions into indecomposable modules are unique up to isomorphism.

Proof: See Landrock [16] (Chapter I Theorem 5.2).

Remark 4.3. Any A-module that is a direct summand of ${}_AA$ is a projective A-module since A is trivially A-free.

Definition 4.4. An element $e \in A$ is an *idempotent* if $e^2 = e$. Idempotents e, f are *orthogonal* if $ef = 0 = fe$ and an idempotent $e \neq 0$ is *primitive* if it cannot be written as $e_1 + e_2$ for orthogonal non zero idempotents e_1, e_2.

Corollary 4.5. *(Fitting) There is a 1-1 correspondence between direct sum decompositions of A into indecomposable projective modules and the primitive orthogonal idempotent decomposition of 1.*

Proof: Write $A = P_1 \oplus ... \oplus P_d$, P_i is a projective indecomposable. The projection map π_i onto P_i is right multiplication by a (necessarily) idempotent element $e_i \in A$ giving $P_i = Ae_i$ and this induces primitive orthogonal idempotent decomposition $1_A = e_1 + ... + e_d$. Conversely if $e \in A$ is an idempotent a decomposition $Ae = I_1 \oplus I_2$ as A-modules induces a decomposition $e = e_1 + e_2, e_i \in A$ where $ee_i = e_i$ then $A_1 \ni e_1e_2 = (e - e_2)e_2 = e_2 - e_2^2$ and similarily we get $e_1 - e_1^2$ so this lies in the intersection of A_1, A_2 thus is zero and we see Ae is indecomposable only if e is primitive.

We then state a fundamental result allowing us to relate indecomposable projective modules to simple modules.

Theorem 4.6. *(Projective Covers) There is 1-1 correspondence of isomorphism classes of indecomposable projective A-modules and isomorphism classes of simple A-modules given by associating the simple module $P/rad(P)$ to the indecomposable projective module P.*

Proof: This is deduced quickly once we have results on idempotent liftings (see later).

Remark 4.7. This theorem has many deep implications. We see $P/\text{rad}(P)$ is simple and if P, Q are both projective indecomposables with the same simple quotient they are isomorphic. Additionally if M is any (not necessarily projective) A-module and $M/\text{rad}(M)$ is simple and isomorphic to $P/\text{rad}(P)$ for some indecomposable projective module P then M is a homomorphic image of P.

Thus given some simple module S we can associate a projective indecomposable module P_S which has radical quotient S. We call P_S a *projective cover* of S.

We can combine this with the following result in the case of some group algebras to see S is the only composition factor of P in this case.

Proposition 4.8. *Let P be a projective indecomposable kG-module for a field k and finite group G. Then*

$$soc(P) \cong P/rad(P)$$

Proof: See Benson [4] Theorem 1.6.3.

We now look to apply these ideas in the case of finding the indecomposable projective modules of the matrix group $SL_2(p)$ over a field of characteristic $p > 2$. We will see that $SL_2(p)$ has some features that will allow us to use the powerful Green correspondence. Our first action will be to find the simple modules of $SL_2(p)$.

5 Towards the indecomposable projective modules of $SL_2(p)$

5.1 Simple kG-modules of $SL_2(p)$

For this section we work in a field of characteristic $p > 2$ and we let $G = SL_2(p)$. The conjugacy classes of G are worked out explicitly in Humphreys [11] and we find p p-regular conjugacy classes so it follows there are p isomorphism classes of simple kG-modules.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of G and define the action of G on the polynomial ring $k[X,Y]$ by $g \cdot f(X,Y) = f(aX+cY, bX+dY)$ for $f(X,Y) \in k[X,Y]$. Now define V_n to be the $n-$dimensional subspace of $k[X,Y]$ of homogeneous polynomials in X,Y of degree $n-1$ with basis $\{X^{n-1}, X^{n-2}Y, ..., Y^{n-1}\}$ and have $V_1 = k$, the trivial kG-module. We then have p distinct kG-modules $V_1, V_2, ..., V_p$ and we will show these are precisely the p simple kG-modules. Let

$$g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and we begin by analysing the structure of the $k\langle g\rangle$-module V_{n+1}.

Lemma 5.1. *V_{n+1} is generated as a $k\langle g\rangle$-module by X^n and has socle kY^n*

Proof: For $0 \leq i \leq n$ let W_{i+1} be the subspace of V_{n+1} of polynomials with terms of X degree at most i with $W_0 = 0$. We claim inductively that any element of $W_{i+1} \backslash W_i$ generates W_{i+1} as a $k\langle g\rangle$-module. The $i = 0$ case is clear so for $i \geq 1$:

Using a binomial expansion we compute $gX^iY^{n-i} = (X+Y)^iY^{n-i} = X^iY^{n-i} + \binom{i}{1}X^{i-1}Y^{n-i+1} + f_i(X,Y)$ where the terms of $f_i(X,Y)$ have X degree at most $i-2$. For $1 \leq i < p$, $\binom{i}{1}$ is non-zero so $gX^iY^{n-i} - X^iY^{n-i} \in W_i \backslash W_{i-1}$ and W_{i+1} are $k\langle g\rangle$-modules. The computation also shows each W_{i+1}/W_i is a trivial $k\langle g\rangle$-module since the coefficent of X^iY^{n-i} is fixed. Thus the $k\langle g\rangle$-module generated by a lift of some element in W_{i+1}/W_i contains all elements in $W_{i+1} \backslash W_i$ and all elements of W_i by induction so the claim follows. Since $W_{n+1} = V_{n+1}$ the first part of the lemma is clear.

Now $\langle g\rangle$ is a Sylow p-subgroup and so the only simple $k\langle g\rangle$ module is trivial and the socle contains elements of V_{n+1} fixed by g. Any such element is contained in $W_{i+1} \backslash W_i$ for some $0 \leq i \leq n$ so generates a module of dimension $i+1$ but trivial modules have dimension 1 hence the only simple module is $W_1 = kY^n$.

Corollary 5.2. *V_{n+1} is simple*

Proof: If W a non zero kG-submodule of V_{n+1} then W is a $k\langle g\rangle$-submodule. kY^n is the unique simple module hence $Y^n \in W$ but a symmetric argument in the previous lemma shows Y^n generates V_{n+1} as a $k\langle h\rangle$-module so $V_{n+1} \subset W$.

We've therefore found all the simple kG-modules (hence the Brauer characters also by lifting eigenvalues of the matrix representations on p-regular conjugacy classes) and so our next goal should be to find the indecomposable projective modules corresponding to these simples. While we could find these (Alperin [1] Chapter 7) directly with the tools we have at hand we will develop theory on the Green Correspondence and exploit this in our computations.

5.2 Indecomposable modules for groups with a normal cyclic Sylow p-subgroup

We first take a detour to develop some theory for when a group L has a normal cyclic Sylow p-subgroup H of order p^a with generator x. Recall that uniserial modules are those with unique composition series and they must be indecomposable. Our goal is to show that all indecomposable kL-modules are uniserial and we show this in case where the module is projective also first . We begin by describing some small cases in characteristic p fields.

Example 5.3. *Cyclic p-groups have uniserial group algebras with a unique simple module.* We have a ring isomorphism $kH \cong k[X]/(X^{p^a})$ by extending $x^s \to (X+1)^s$. The submodules of $k[X]/(X^{p^a})$ are precisely $(X^i)/(X^{p^a})$ for $i \leq p^a$ which gives us a unique composition series of length p^a by linearly ordering these submodules by inclusion. We see that composition factors are all isomorphic to $(X^i)/(X^{i+1}) \cong k$ and the only simple module is $k[X]/(X)$. It follows that kH is uniserial with p^a indecomposable modules.

Example 5.4. *Simple $k[L/H]$-modules.* The restriction of any simple kL-module S to H is semisimple by Clifford's Theorem hence by the previous example is the direct sum of simple kH-modules on which H acts trivially. We show the elements of H are the only elements of L which are in the kernel of the L-action on all simple kL-modules. Suppose y was such an element of order prime to p. Then the restriction $kL_{\langle y \rangle}$ is semisimple as a $k\langle y \rangle$-module and the direct sum of $S_{\langle y \rangle}$ modules where S is a simple kL-module. Since y acts trivially on these it acts trivially on kL, a contradiction and the claim is proven. It follows also that the simple $k[L/H]$-modules are precisely the kL-simples.

Remark 5.5. p-groups G are exactly the groups such that kG is indecomposable as a kG-module or equivalently such that the quotient $kG/\text{rad}(kG)$ is one-dimensional. Landrock [16] (Chapter I ,Proposition 9.3).

We've developed some connections between kL, kH and their simple modules and we will now turn to understanding how their projective modules are related which will be the key to showing indecomposable projective kL-modules are uniserial. The first step will be to restrict projective kL-modules to kH.

Proposition 5.6. *Let P be any projective kG-module and H any subgroup of G, then P_H is a projective kH-module.*

Proof: If $g_1, ..., g_{|G:H|}$ are (left) coset representatives for H in G then as a kH-module

$$kG_{kH} \cong k(g_1H) \oplus ... \oplus k(g_{|G:H|}) \cong \bigoplus_{i=1}^{|G:H|} kH$$

since left multiplication by the appropiate g_i defines an isomorphism. By Proposition 4.1, P is a direct summand of a free kG-module which by the previous sentence is a free kH module when restricted to kH and so applying the equivalence again gives the result.

Now combining this with the following lemma and further assuming P is an indecomposable projective kL-module we can complete our analysis and completely decompose P_H as a free kH-module with dim S copies of kH.

Lemma 5.7. *If U is a kG-module then rad(U) =rad(U_H). Moreover if H is cyclic, rad(U) =rad$(1-x)U$.*

Proof: See Alperin [1] (Projective Modules, Lemma 8).

Example 5.8. *Restriction of projective modules to H.* We've seen that P_H is projective and H is a p-group hence P_H is a free kH-module $\bigoplus_{i=1}^m kH$. From Theorem 4.6 we know that $P/\text{rad}(P)$ is isomorphic to some simple S and by the previous lemma the quotient $P_H/\text{rad}(P_H) \cong \bigoplus_{i=1}^m kH/\text{rad}(kH)$ also has dimension dim S. The discussion in Example 5.3 tells us $kH/\text{rad}(kH)$ is the trivial module hence has dimension 1 so m=dim S the radical series of P_H has length p^a with terms $\bigoplus_{i=1}^{\dim S} \text{rad}^j(kH)$. Invoking the previous lemma again we see P has the same radical series.

We now turn to the proof that P is uniserial and the main idea will be to construct the quotients of the radical series of P and show that the successive quotients are simple which implies P is uniserial by Proposition 2.18. First we describe a special case with $S = k$ and projective cover P_k. If $W = \mathrm{rad}(P_k)/\mathrm{rad}^2(P_k)$ and $M = P_k/\mathrm{rad}^2 P_k$, $0 \subset W \subset M$ is the unique composition series for M and M is uniserial. Now if T is any simple kL-module $T \otimes W$ is also simple. Otherwise if W' is the 1 dimensional kL-submodule with the action of $x \in L$ given by the multiplicative inverse of the action of x on W, $W \otimes W' \cong k$ so $(T \otimes W) \otimes W' \cong T$. It follows that $T \otimes W$ can have no proper submodule since T is simple and inductively we see $T \otimes W^{\otimes n}$ is simple. Additionally we claim that $\mathrm{rad}(T \otimes M) \neq 0$ and hence that $T \otimes M$ is not semisimple. By Lemma 5.7 this amounts to finding a non zero $(1-x)(t \otimes m) \in (1-x)(T \otimes M)$. Eariler examples tell us $x \in H$ acts trivially on simple modules so $xt = t$ and for non zero t

$$(1-x)(t \otimes m) = t \otimes m - x(t \otimes m) = t \otimes m - t \otimes xm = t \otimes (1-x)m$$

but $\mathrm{rad}(M) = (1-x)M$ by Lemma 5.7 and is non zero since $\mathrm{rad}(M) = W$ so there exists some m for which $(1-x)m$ is non zero as required.

To summarise the discussion $T \otimes M$ is not semisimple and as $(T \otimes M)/(T \otimes W) \cong T \otimes (M/W) \cong T \otimes k \cong T$ is semisimple $T \otimes W$ contains $\mathrm{rad}(T \otimes M)$. In particular since $T \otimes W$ is simple $T \otimes W = \mathrm{rad}(T \otimes M)$.

Theorem 5.9. *M, W as above and let P be a projective cover of a simple kL-module S. The quotients of successive radicals of P are simple and given by $S, S \otimes W, S \otimes W \otimes W, ..., S \otimes W^{\otimes p^a - 1}$.*

Proof: By Example 5.8 P has a radical series of length p^a and $P/\mathrm{rad}(P) \cong S$. The above discussion shows $\mathrm{rad}(S \otimes M) = S \otimes W$ so $(S \otimes M)/\mathrm{rad}(S \otimes M) \cong S$ and by Theorem 4.6 $S \otimes M$ is the homomorphic image of P. Thus there is a surjective homomorphism from $\mathrm{rad}(P)/\mathrm{rad}^2(P)$ into $S \otimes W$ $=\mathrm{rad}(S \otimes M)/\mathrm{rad}^2(S \otimes M)$ but counting dimensions shows this is an isomorphism.

Now assuming $\mathrm{rad}^n(P)/\mathrm{rad}^{n+1}(P) \cong S \otimes W^{\otimes n}$, $\mathrm{rad}^n(P)$ is the homomorphic image of some projective Q_n with $Q_n/\mathrm{rad}(Q_n) \cong S \otimes W^{\otimes n}$ and so $\mathrm{rad}^{n+1}(P)/\mathrm{rad}^{n+2}(P)$ is the homomorphic image of $\mathrm{rad}(Q_n)/\mathrm{rad}^2(Q_n)$ but by the same argument in the previous paragraph, switching P with Q_n and S with $S \otimes W^n$ we see this quotient is exactly $S \otimes W^{n+1}$. We conclude that $\mathrm{rad}^{n+1}(P)/\mathrm{rad}^{n+2}(P)$ is the homomorphic image of $S \otimes W^{n+1}$ which is simple hence the quotient is either 0 or $S \otimes W^{n+1}$. Since we know the radical series has length p^a this quotient is non zero exactly when $n + 1 < p^a$ and the result follows inductively.

Corollary 5.10. *Projective indecomposable kG-modules are uniserial.*

To finish this section and extend the result to all indecomposable kG-modules we must first consider the dual to projective modules, namely, injective modules which are those that satisfy

Definition 5.11. An A-module I is injective if for any monomorphism $M \rightarrowtail M'$ of A-modules and homomorphism $M \to I$ there exists a homomorphism $M' \to I$ such that the the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{M} \arrow[r] \arrow[d] & \mathrm{M'} \arrow[dl, dashed, "\exists"] \\ & \mathrm{I} & \end{tikzcd}$$

Definition 5.12. An algebra for which all indecomposable projective and injective modules are uniserial is said to be a Nakayama algebra

this is indeed the case when L has a normal cyclic Sylow p-subgroup by the following proposition.

Proposition 5.13. *k a field, L finite and M a kL-module. M is projective if and only if it is injective.*

Proof: See Martin [19] Theorem 8.4.

Therefore our group of interest gives a Nakayama group algebra since projective indecomposables are uniserial and they are precisely the injective indecomposables. Finally we have the last piece of the puzzle:

Proposition 5.14. *The indecomposable modules of a finite dimensional Nakayama algebra over a field are uniserial and the homomorphic image of an indecomposable projective module. In particular the homomorphic images of indecomposable projectives give a complete list of indecomposable modules.*

Proof: See Webb [24] Proposition 11.2.1.

Corollary 5.15. *If L has a normal cyclic Sylow p-subgroup then indecomposable kL-modules are uniserial.*

Remark 5.16. It follows that if M is an indecomposable kL-module then $M/\mathrm{rad}(M) \cong S$ (or 0 if it is simple) since it is the homomorphic image of $P/\mathrm{rad}(P)$ for some indecomposable projective kL-module P. Additionally we see that the composition factors are $S, S \otimes W, S \otimes W \otimes W, \ldots$ by isomorphism theorems.

5.3 Restricting V_n

Recall that $H = \langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \rangle$ is a cyclic Sylow p-subgroup of order p in $G = SL_2(p)$ and that $L = N_G(P) = \{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in G \,|\, ad = 1\}$ is a subgroup of order $p(p-1)$. H is normal in L and so by Corollary 5.15 the indecomposable kL-modules are uniserial. Also the map that takes a matrix in L to its upper left entry in $\mathbb{F}_p^{\times}$ induces an isomorphism $L/H \cong \mathbb{F}_p^{\times}$ which is cyclic and so kL has $p-1$ dimension 1 simple modules. Factoring through H as in Example 5.4 we can label the kL simple modules as S_i for $0 \le i < p-1$ such that for $\alpha = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in L$ the action is multiplication by a^i.

By Theorem 5.9 for any simple kL-module S_i there is a kL-module W which right tensors with S_i to give all composition factors of the projective cover P_i. W is simple so must be some S_j and tensoring with S_j permutes the simple modules of kL. Thus to find W it remains to calculate the composition factors for some V_n restricted to kL, ordered as they appear in the radical series. The computation for $(V_2)_L$ is done in Alperin [1] pg.75, we will do a similar expository computation for $(V_3)_L$ assuming $p > 3$. $(V_3)_L$ is spanned by X^2, XY, Y^2 and $M_1 = k\text{-span}\{XY, Y^2\}$ is a (unique) maximal submodule of dimension 2 while the quotient is isomorphic to S_2 given by the image of the action on X^2. Also, M_1 has a (unique) maximal submodule M_2 spanned by Y^2 isomorphic to S_{-2} and the quotient M_1/M_2 is trivial since the coefficent of XY is fixed by the action of α since $ad = 1$. Thus $(V_3)_L$ is a uniserial (and therefore indecomposable) kL-module with composition factors S_2, S_0, S_{-2}. As $S_i \otimes S_j \cong S_{i+j}$ we get $W \cong S_{-2}$.

To apply the construction of composition factors by tensoring with W we need $(V_i)_L$ to be indecomposable. This will be verified with the following lemma

Lemma 5.17. *Let M be an A-module for an artinian algebra A. If $soc(M) = S$ is simple then M is indecomposable.*

Proof: Since A is artinian any A-module contains a simple module and hence the socle of any submodule U of M is non zero so contains some simple summand of the socle of M thus contains S. Then if $M = U \oplus V$ there is a semisimple submodule of M, $S \oplus S$ giving a contradiction.

Proposition 5.18. *For $1 \leq i < p$ the composition factors of $(V_i)_L$ are $S_{i-1}, S_{i-3}, ..., S_{-(i-1)}$.*

Proof: In Lemma 5.1 we showed that $\mathrm{soc}((V_i)_H) \cong kY^{i-1}$ is a simple 1 dimensional module on which H acts trivially. From Example 5.4 we know that H acts trivially on all simple kL-submodules hence $\mathrm{soc}((V_i)_L)$ is contained in the submodules on which H acts trivially so is the kL-submodule kY^{i-1} isomorphic to $S_{-(i-1)}$. We then have indecomposability and as $\mathrm{soc}((V_i)_L) = kY^{i-1} \cong S_{-(i-1)}$ is the last composition factor the result follows.

6 Green correspondence

Now that we understand the kL-modules $(V_i)_L$ completely we need to develop some theory to bridge the gap between kL and kG modules. The main result we will use to understand this relation is *Green correspondence*. In this section we use the notation that $U|V$ if U is isomorphic to a summand of V as kG-modules.

6.1 Vertices and Sources

We work with an arbitrary subgroup H of G and R a field of complete discrete valuation ring (so Krull-Schmidt holds).

Definition 6.1. An RG-module is *H-free* if it is V^G for some RH-module V and is (relatively) *H-projective* if it is a summand of V^G for some RH-module V.

Now fix an indecomposable RG-module U . Let $A(U) = \{Q \leq G | U \text{ is } Q\text{-projective}\}$ and suppose the set was non-empty. We could then choose a subgroup Q 'minimal' in $A(U)$, that is, U is Q-projective but not Q'-projective for any subgroup $Q' \leq Q$. Additionally we could find an indecomposable RQ-module S such that $U|S^G$. We say Q is a *vertex* of U and S is a *source* of U and can view Q as a measure of how close to projective U is (in particular projectivity is equivalent to being 1-projective). It turns out we can always do this and these objects are unique up to conjugacy.

Theorem 6.2. *Let U be an indecomposable RG-module. Then*

1. *There is a vertex of U, Q in G unique up to conjugacy. Moreover U is H-projective where H is a subgroup of G if and only if H contains a conjugate of Q.*

2. *There is a source S, unique up to conjugacy in $N_G(Q)$*

3. *If the p' part of $|G|$ is invertible in R vertices are p-subgroups*

Proof: See Benson [4] Proposition 3.10.2.

We see that assuming the p'-part of $|G|$ is invertible, if H contains a Sylow p-subgroup (every p-group is contained in some Sylow p-group) any RG-module is H-projective. It is further true that the vertex of the trivial RG-module is a Sylow p-subgroup (Webb [24] Proposition 11.6.2 3).

From now on we assume that the p'-part of $|G|$ is invertible.

6.2 Green correspondence

Theorem 6.3. *(Green correspondence) Let R be a field of characteristic p or a complete discrete valuation ring with a characteristic p residue field. Let H be a subgroup of G containing the normaliser $N_G(Q)$ for some subgroup Q. Suppose U, V are indecomposable RG, RL modules respectively with vertex Q.*

1. *In any indecomposable RH-module decomposition of U_H there is a unique indecomposable summand $f(U)$ with vertex Q. Moreover $U_H \cong f(U) \oplus X$ where each summand of X has a vertex in $\mathcal{Y} = \{Y \leq G | Y \leq {}^xQ \cap H$ for some $x \in G \backslash H\}$*
2. *In any indecomposable kG-module decomposition of V^G there is a unique indecomposable summand $g(V)$ with vertex Q. Moreover $V^G \cong g(V) \oplus Y$ where each summand of Y has a vertex in $\mathcal{X} = \{X \leq G | X \leq {}^xQ \cap Q$ for some $x \in G \backslash H\}$*
3. *As written above $gf(U) \cong U$ and $fg(V) \cong V$.*

Proof (Sketch) (Benson [4] Theorem 3.12.2): 1) Let S be a source of U then $S^G \cong V \oplus V'$ such that V is indecomposable and $U|V^G$. Then by Mackey decomposition and Krull-Schmidt $(V^G)_H \cong V \oplus V^*$ where V^* is a sum of modules projective relative to subgroups in $\mathcal{Y}$ (*) so V is the only summand with Q as a vertex (if ${}^xQ \cap H = {}^aQ$ for some $a \in H$ then $a^{-1}x \in N_G(Q)$ and $x \in aH = H$ so this cannot happen by our choice of x). Also U_H contains some summand with vertex Q since induction commutes with direct sums and $U|(U_H)^G$. Thus have $f(U) = V$.
2) There is a summand U of V^G such that V is a summand of U_H. Then $V^G \cong U \oplus U'$. Let W be an indecomposable summand of U' and suppose it does not have a vertex in $\mathcal{X}$. V has vertex Q so by definition W is relatively Q-projective so by Theorem 6.2 has vertex $Q' \leq Q$. Any source of W is a summand of $W_{Q'}$ and so a summand of $D_{Q'}$ where $D|W_H$. But then D is not projective relative to any subgroup in $\mathcal{Y}$ meaning that neither is U'_H which contradicts the claim (*) and it follows that summands of U' have vertices in $\mathcal{X}$. The claim is proved taking $g(V) = U$.
3) use 1), 2).

6.3 Trivial Intersections

For the remainder of this section we will consider a special case of the Green correspondence, namely where G has a Sylow p-subgroup P such that for all $g \in G, P \cap {}^gP = P$ or 1 . We first refine the Green correspondence in this case, working also with the group algebras kG, kL over a field k of characteristic p.

Corollary 6.4. *Let U, V be kG, kL-modules respectively where $L = N_G(P)$ with non-trivial vertex (non-projective). Then $U_L \cong V \oplus Y$, $V^G \cong U \oplus X$ where Y, X are projective kL, kG-modules respectively.*

Proof: If is further known that in the case of Green Correspondence that $g(V) = U$ by a theorem of Burry-Carlson-Puig (Benson [4] Theorem 3.12.3). For $x \notin L = N_G(P), {}^xP \cap P = 1$ and $\mathcal{X} = \{\{1\}\}$ so X is projective. Also for $x \notin L, {}^xP \cap L = 1$ is a p-subgroup of L which then must be conjugate by some $a \in L$ to some subgroup of P however $ax \notin L$ so this is a contradiction as then $asQ(as)^{-1} \cap Q \neq 1$. It follows that Y has trivial vertex and is projective.

We now turn our attention to finding short exact sequences which will allow us to find the indecomposable projective modules of V_n, the simple $k(SL_2(p))$-modules. The main driver is the important corollary

Corollary 6.5. *Let U_1, U_2 be non projective indecomposable kG-modules with corresponding kL-modules V_1, V_2. Then $0 \to U_1 \to U \to U_2 \to 0$ is a non-split short exact sequence iff there is a non split s.e.s $0 \to V_1 \to V \to V_2 \to 0$.*

Proof: See Alperin [1] Chapter 10 Corollary 2. In particular the short exact sequences are related by induction and restriction.

Theorem 6.6. *There are non splitting short exact sequences of kG-modules*

$$0 \to V_{p-1-i} \to V \to V_i \quad 1 \leq i < p-1$$

$$0 \to V_{p+1-i} \to V \to V_i \to 0 \quad 1 < i \leq p-1$$

Proof: We will prove the first result. For the second see Alperin [1] Section III Corollary 5. Recall that if U is an indecomposable kL-module and $U/\mathrm{rad}(U) \cong S_j$ the composition factors are $S_j, S_{j-2}, \ldots$. Hence let U be the indecomposable kL-module of dimension $p-1$ with $U/\mathrm{rad}(U) \cong S_{i-1}$. It follows that $U/\mathrm{rad}^i(U)$ has dimension i and also has radical quotient S_{i-1} by isomorphism theorems. Also $U/\mathrm{rad}^i(U) \cong (V_i)_L$. Then the radical quotient of $\mathrm{rad}^i(U)$ is the composition factor that follows the $\mathrm{soc}(U/\mathrm{rad}^i(U))$ which is $S_{-(i-1)}$ and this must be $S_{-i-1} = S_{(p-1-i)-1}$. dim $\mathrm{rad}^i(U) = p-1-i$ so $\mathrm{rad}^i(U)$ is precisely $(V_{p-1-i})_L$. We have therefore found the short exact sequence

$$0 \to (V_{p-1-i})_L \to U \to (V_i)_L \to 0$$

and U is indecomposable so this sequence must not split. The result then follows by the previous corollary.

6.4 Indecomposable projective kG-modules for V_n

We apply the previous theorem letting P_i be a projective cover of S_i.Setting $i = 1$ in the first sequence gives the existence of a module U_1 such that $U_1/V_{p-2} \cong V_1$. Since V_{p-2} is simple it is the radical of U_1 and so U_1 is the homomorphic image of P_1, the indecomposable projective module corresponding to S_1. It follows that $U_1 \cong P_1/\ker\phi_1$ for some surjective homomorphism $\phi_1 : P_1 \to U_1$ and $\mathrm{rad}(P_1/\ker\phi_1) =$ (ker ϕ_1+ $\mathrm{rad}(P_1))/\ker\phi_1$ but $\ker\phi_1$ is contained in some maximal module of P_1 while $\mathrm{rad}(P_1)$ is the unique maximal hence $\ker\phi_1 \subset \mathrm{rad}(P_1)$. It follows that $\mathrm{rad}(U_1) \cong \mathrm{rad}(P_1/\ker\phi_1) \cong \mathrm{rad}(P_1)/\ker\phi_1 \cong V_1$. Similarily setting $i = p-1$ in the second sequence gives a $U_{p-1}/\mathrm{rad}(U_{p-1}) \cong V_{p-1}$ and again the same argument gives $\mathrm{rad}(P_{p-1})/\ker\phi_{p-1} \cong V_{p-1}$ for some homomorphism ϕ_2.

Recalling the socle of P_i is also S_i in the case of group algebras, dim $P_1 \geq$ dim V_1 + dim V_{p-2} + dim $V_1 = 1 + p - 2 + 1 = p$ and similarily dim $P_{p-1} \geq p - 1 + 2 + p - 1 = 2p$.

Now using both sequences in the case $1 < i < p-1$ and employing the same arguments we find submodules U_-, U_+ such that $\mathrm{rad}(P_i)/U_- \cong V_{p-1-i}$ and $\mathrm{rad}(P_i)/U_+ \cong V_{p+1-i}$ and since U_+, U_- are different maximal ideals $V_{p-1-i} \cong \mathrm{rad}(P_i)/U_- \cong (U_+ + U_-)/U_- \cong U_+/(U_+ \cap U_-)$ by isomorphism theorems therefore dim $P_i \geq i + (p+1-i) + (p-1-i) + i = 2p$. We now have lower bounds on the dimensions of each projective indecomposable P_i and to relate these to the regular representation we need the following facts

Proposition 6.7. *Assume k is algebraically closed. In a decomposition of kG as a free kG module into a direct sum of indecomposable modules, each isomorphism type of indecomposable module occurs the same number of times as the dimension of the corresponding simple module.*

Lemma 6.8. *Assume k is algebraically closed. If G has a Sylow p-subgroup of order p^a then every projective kG-module has dimension divisible by p^a.*

And we can finally find all the composition factors of the P_i

Theorem 6.9. *The composition factors of the projective indecomposables P_i are as follows:*

- $P_1 : V_1, V_{p-2}, V_1$
- $P_i : V_i, V_{p-1-i}, V_{p+1-i}, V_i \quad 1 < i < p-1$
- $P_{p-1} : V_{p-1}, V_2, V_{p-1}$
- $P_p : V_p$

Proof: Combining the previous results we have

$$\begin{aligned}
\dim\, kG &= p(p-1)(p+1) \\
&= \sum_{i=1}^{p} \dim V_i \cdot \dim P_i \qquad \textit{Proposition } 6.7 \\
&\geq p + \sum_{i=2}^{p-2} i \cdot 2p + p - 1 \cdot 2p + p \cdot \dim\, P_i \\
&\geq (p^3 - p^2 - p) + p \cdot p \qquad \textit{Lemma } 6.8 \\
&= p(p-1)(p+1)
\end{aligned}$$

Forcing equality throughout and giving the result.

6.5 The Cartan Matrix

The Cartan Matrix will allow us to display these composition factors of indecomposable projective modules in a matrix. Assuming A is a finite dimensional algebra, if S, T are simple A-modules and P_T is the projective cover of T then define the *Cartan invariant* c_{ST} to be

$$c_{ST} = \text{the composition factor of } S \text{ in } P_T$$

Thus indexing rows and columns by isomorphism classes of simple A-modules the Cartan invariants give a matrix C where the S, T entry is c_{ST} called the *Cartan matrix*.

Example 6.10. Using the composition factors from Theorem 6.9 we can form the Cartan Matrix for $SL_2(P)$. Let $B_1 = \{V_1, V_{p-2}, V_3, V_{p-4}, ...\}$ an ordered set of the odd dimensional simples (excluding p), $B_2 = \{V_2, V_{p-1}, V_4, ...\}$ of even dimensional, $B_3 = \{V_p\}$. Then the ordered union gives a basis in which the matrix is a block diagonal matrix

Figure 1: Cartan Matrix of $SL_2(p)$

$$\begin{pmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_1 = M_2 = \begin{pmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & . & . & . & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 3 \end{pmatrix}$$

Remark 6.11. Here the Cartan matrix is symmetric and block diagonal and indeed this is always true. It is also known that the determinant of C is a p-power (Brauer (1941)), see Alperin, Collins and Sibley [2] for a proof independent of the use of characters.

7 Decomposition and Cartan Matrices

We now turn our attention to understanding how to relate KG and kG modules where K has charateristic 0 and k has prime characteristic. It will be key to understand an intermediary PID with a unique maximal ideal so that we can 'reduce' our representations to positive characteristic.

7.1 Splitting Fields

Let $\mathcal{O}$ be a *discrete valuation ring (DVR)*, that is, a principal ideal domain with a unique non zero maximal ideal (π). It is common to assume that $\mathcal{O}$ is complete since π-adic completion does not change the residue field $\mathcal{O}/\pi\mathcal{O}$. It is known that any non zero element of $x \in \mathcal{O}$ can be written (uniquely) as $x = u\pi^a$ for some unit $u \in \mathcal{O}$ and a a non negative integer. We may define a valuation $v_\pi : \mathcal{O} \to \mathbb{Z}_{\geq 0}$ on $\mathcal{O}$ by $v_\pi(x) = a$. It is also a fact that the field of fractions K of $\mathcal{O}$ has elements of the form $u\pi^a$ for $a \in \mathbb{Z}$ so the valuation can be extended to K. Combining with the residue field defines a system.

Definition 7.1. A *p-modular system* $(K, \mathcal{O}, k)$ consists of a DVR $\mathcal{O}$ with maximal ideal (π), its field of fractions K of characteristic 0 and residue field $k = \mathcal{O}/\pi\mathcal{O}$ of characteristic p.

Example 7.2. An important example that helps us to relate representations in characteristic 0 with those in characteristic p are the local fields K_p, the field of fractions of the p-adic completion $\mathcal{O}_p = \varinjlim_n \mathcal{O}/p^n$ where $\mathcal{O}$ is the ring of integers of a number field K and p is some prime lying above a rational prime in $\mathbb{Z}$. Then $(K_p, \mathcal{O}_p, \mathcal{O}_p/p\mathcal{O}_p \cong \mathcal{O}/p)$ is a p-modular system.

We will first impose a condition on the fields K, k henceforth, namely that the p-modular system is *splitting for* G. This will simplify our calculations and allow us to fix the decomposition and Cartan matrices.

Definition 7.3. If F is a field and A is a finite dimensional F-algebra then an extension E of F is a *splitting field* for A if and only if for all simple $E \otimes_F A$-modules $\mathrm{End}_{E \otimes_F A} \cong E$. We say a field E is a splitting field for a finite group G if E is a splitting field for the group algebra EG.

Theorem 7.4. *A a finite dimensional algebra over a field k. Then A has a splitting field of finite degree over k and for a finite group G splitting fields are finite degree extensions of $\mathbb{Q}$ or $\mathbb{F}_p$.*

Proof: Webb [24] Theorem 9.2.6.

We deduce from the Theorem that simple kG-modules can be written over a finite field $\mathbb{F}_{p^n}$ when k has characteristic p and a finer result from Brauer (Webb [24] Theorem 9.2.7) tells us a field is a splitting field for G if it contains a primitive m-th root of unity, where m is the exponent of G.

Definition 7.5. A splitting p-modular system for G (where G is finite) is a p-modular system $(K, \mathcal{O}, k)$ where both K, k are splitting fields for G.

Definition 7.6. Given a p-modular system $(K, \mathcal{O}, k)$, A a finitely generated algebra over $\mathcal{O}$ and M a (finitely generated) $K \otimes_{\mathcal{O}} A$ module, by an *$\mathcal{O}$-form* we mean a $\mathcal{O}$-free A-module $\widehat{M}$ such that $M \cong K \otimes_{\mathcal{O}} \widehat{M}$.

These $\mathcal{O}$-forms always exist and can be constructed. If V is a $K \otimes_{\mathcal{O}} A$-module then we have a K-basis for V, $v_1, ..., v_d$ say. Let W be the A-module generated by the basis which is necessarily free over $\mathcal{O}$ since $\mathcal{O}$ is a PID. Let $w_1, ..., w_r$ be a $\mathcal{O}$-basis of W. Any K-linear relation becomes a $\mathcal{O}$-linear relation by clearing denominators hence $w_1, ..., w_r$ also form a K-basis for V thus W is an $\mathcal{O}$-form for V.

Furthermore the choice of $\mathcal{O}$-form (induced by the choice of basis for V) will not affect our results relating to group algebras as described later in Theorem 7.8. It follows that we can 'reduce' any KG-module by writing it in $\mathcal{O}$ then taking the result modulo π.

7.2 Decomposition Matrix

Given a p-modular system $(K, \mathcal{O}, k)$ and A a finite dimensional algebra over $\mathcal{O}$ we use the conventions $A^K = K \otimes_{\mathcal{O}} A$ and where (π) is the maximal ideal of $\mathcal{O}$ we let $\bar{A} = A/\pi A$. We assume further A^K is semisimple.

Definition 7.7. A^K semisimple. Let $V_1, ..., V_n$ be a full set of isomorphism classes of simple A^K-modules and $S_1, ..., S_m$ the simple $\bar{A}$-modules then we define the *decomposition numbers* $\{d_{ij}\}$ as the multiplicity of S_j as a composition factor of the reduction mod (π) of an $\mathcal{O}$-form of V_i. We say the matrix D with entries d_{ij} is the *decomposition matrix*.

We see that the decomposition number depends on the choice of $\mathcal{O}$-form however in the case of group algebras and a splitting p-modular system this choice has no effect, namely from the following result

Theorem 7.8. *Given a splitting p-modular system $(K, \mathcal{O}, k)$ if W, W' are $\mathcal{O}$-forms for a KG-module V then the composition factors of*

$$k \otimes_{\mathcal{O}} W = W/\pi W \text{ and } k \otimes_{\mathcal{O}} W' = W'/\pi W'$$

are the same.

Proof: Easily proved using Brauer Characters (Martin [19] Theorem 7.2). A general proof for the case where W, W' are A-modules and equality in composition factors of $\bar{A}$-modules $W/\pi W, W'/\pi W$ is given in Serre (1967) [22] III.2.2.

Before we can relate the decomposition numbers and cartan invariants we need to see how to relate indecomposable projective $\bar{A}$-modules to those over A. To do so we will need to lift idempotents.

7.3 Idempotent Lifting

Recall that primitive idempotent decompositions of 1 correspond to the projective indecomposable modules hence to lift projective covers from $\bar{A}$ if suffices to lift the corresponding idempotent. We can also lift in other contexts, for example from $A/\text{rad}(A)$ (where by Artin-Wedderburn we can decompose into elementary matrices) to A.

Proposition 7.9. Idempotent Lifting 1 *Let N be a nilpotent ideal in the ring A. If $1 = e_1 + ... + e_d$ is primitive idempotent decomposition in A/N then there exists a primitive idempotent decomposition in A, $1 = f_1 + ... + f_d$ with $f_i + N = e$ mod N. If e_i is conjugate to e_j then f_i is conjugate to f_j. Moreover if f is an idempotent of A then $f + N$ is primitive if and only if f is primitive.*

Proof: See Benson [4] Theorem 1.7.3, Corollary 1.7.4 and Martin [19] Corollary 9.7.

We will need to work further to complete our ability to lift idempotents. e.g for the case in of π-adic completions as in Example 7.2.

Theorem 7.10. Idempotent lifting 2 *Let A be a finitely generated algebra over a complete DVR $\mathcal{O}$ (so is π-adically complete). If $1_{\bar{A}} = e_1 + ... + e_d$ is a primitive orthogonal decomposition then there is $1_A = f_1 + ... + f_d$ a corresponding primitive orthogonal decomposition in A where f_i are lifts of e_i and e_i, e_j are conjugate if and only if f_i, f_j are.*

First we show how to lift an idempotent $g_1 \in \bar{A}$ to A. Given The kernel of the natural map $A/\pi^n \to A/\pi^{n-1}A$ is $\pi^{n-1}A/\pi^n A$ which is nilpotent. Thus we can apply the first Idempotent Lifting result for nilpotent ideals to get an idempotent $g_i \in A/\pi^n$ which lifts $g_{i-1} \in A/\pi^{n-1}A$. Since A is π-adically complete this gives an idempotent $f \in A$. Conjugacy follows with primitive properties by the same proofs in the previous proposition.

Now let $1_{\bar{A}} = e_1 + ... + e_d$ into primitive orthogonal idempotents. First set $h_1 = 1$ and define h_i inductively for $i > 2$ as the lift of $e_i + ... + e_d$ to an idempotent in $h_{i-1}Ah_{i-1}$. Thus $h_ih_{i+1} = h_ih_ih_{i+1}h_i = h_ih_{i+1}h_i = h_{i+1} = h_{i+1}h_i$. Then define the lift of e_i to be $h_i - h_{i+1}$. We have orthogonality since for $i < j, f_j = h_{i+1}f_jh_{i+1}$ by the previous argument so $f_if_j = (h_i - h_{i+1})h_{i+1}f_jh_{i+1} = (h_ih_{i+1} - h_{i+1})f_jh_{i+1} = 0$. Finally $f_1 + ... + f_d = (h_1 - h_2) + (h_2 - h_3) + ... + (h_d - 0) = h_1 = 1$.

We can therefore lift a primitive idempotent decomposition from $k \cong \mathcal{O}/\pi\mathcal{O}$ to $\mathcal{O}$ and it follows projective indecomposable modules in the decomposition of $\bar{A}$ lift to a decomposition of A and in particular the projective cover $\bar{A}$-module P_i lifts to a unique projective cover in A (up to isomorphism) $\widehat{P}_i$.

7.4 $C = D^TD$

Now we turn to the major result of this section. We know that the decomposition matrix gives $\mathbb{Z}$-linear relations between modular simples and reductions of ordinary simples as well as ordinary simples of lifts of modular projective indecomposables while the Cartan matrix relates modular simples of modular projective indecomposable so it seems plausible that they may be related. This is made explicit in the following theorem.

Theorem 7.11. *Let $(K, \mathcal{O}, k)$ be a p-modular system, A finite dimensional algebra over $\mathcal{O}$ and assume that A^K is semisimple. If K, k are splitting fields for $A^K, \bar{A}$ respectively then the Cartan Matrix C of $\bar{A}$ and decomposition Matrix D are related by*

$$C = D^TD$$

Proof: Let $P_1, ..., P_d$ be projective $\bar{A}$-modules corresponding to simple $\bar{A}$-modules S_i which lift to some $\widehat{P}_i$ over A. Then

$$\begin{aligned}
[P_i : S_j] &= \dim_k(\mathrm{Hom}_{\bar{A}}(P_j, P_i)) && \text{(A)}\\
&= \mathrm{rank}_{\mathcal{O}}(\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)) && \text{(B)}\\
&= \dim_K(K \otimes_{\mathcal{O}} \mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)) && \text{(C)}\\
&= \dim_K(\mathrm{Hom}_{A^K}(K \otimes_{\mathcal{O}} \widehat{P}_j, K \otimes_{\mathcal{O}} \widehat{P}_i)) && \text{(D)}\\
&= \sum_{s=1}^{d} d_{si}d_{sj} && \text{(E)}
\end{aligned}$$

and the claim is proven.

Let's discuss the reasoning behind the steps:
(A): We have the more general result

Lemma 7.12. *Let k be a splitting field for a finite dimensional algebra $\bar{A}$ then for finite generated $\bar{A}$-mod M*

$$dim_k Hom_{\bar{A}}(P_i, M) = [M : S_i]$$

Proof: We induct on the composition length of M. Indeed if M is simple then a non zero map from P_i to M must be onto, thus a quotient of P_i is isomorphic to the simple M but P_i has a unique maximal ideal thus the kernel of such a map is exactly the radical of P_i. It follows that every such map corresponds to a map $P_i/\mathrm{rad}(P_i) \cong S_i \to M$ so the base case follows from Schur's Lemma.

For M not simple a top composition factor induces a short exact sequence

$$0 \to M' \to M \to S \to 0$$

for some simple composition factor S and since P_i is projective, the $\mathrm{Hom}_{\bar{A}}(P_i, -)$ functor is exact giving the short exact sequence

$$0 \to \mathrm{Hom}_{\bar{A}}(P_i, M') \to \mathrm{Hom}_{\bar{A}}(P_i, M) \to \mathrm{Hom}_{\bar{A}}(P_i, S) \to 0$$

$\dim_k$ is additive and $[M : S_i] = [M' : S_i] + [S : S_i]$ by Jordan-Holder thus the claim follows by induction.

Remark 7.13. The proof still holds when k is replaced by the division ring corresponding to P_i in the statement of Schur's Lemma. (Benson [4] Lemma 1.7.6)

(B) First note every map in $\mathrm{Hom}_{\bar{A}}(P_j, P_i))$ is liftable to A. For some primitive idempotents e_i, e_j, $P_i = \bar{A}e_i, \bar{A}e_j$ and so $\mathrm{Hom}_{\bar{A}}(P_j, P_i) = \mathrm{Hom}_{\bar{A}}(Ae_j, Ae_i) \cong e_j\bar{A}e_i$ where the last isomorphism is given by $\alpha \to \alpha(e_j)$ (Landrock [16] Lemma 5.4). By idempotent lifting this lifts to an A-module f_jAf_i for idempotent lifts f_j, f_i and employing the reverse of the isomorphism we get $\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$ of lifted projective covers so every map is liftable.

We deduce $\mathrm{Hom}_{\bar{A}}(P_j, P_i))$ corresponds to the maps of $\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$ which factor through $\pi\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$ but $\mathcal{O}$ is a PID so $\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$ is a free $\mathcal{O}$-module and

$$\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)/\pi\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i) \cong \mathcal{O}^{\oplus n}/\pi\mathcal{O}^{\oplus n} \cong (\mathcal{O}/\pi\mathcal{O})^{\oplus n}$$

as $\mathcal{O}$-modules where n is the $\mathcal{O}$-rank of $\mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$.

(C) Note if $\widehat{M}$ is a $\mathcal{O}$-form a free $\mathcal{O}$-basis $v_1, ..., v_d$ for $\widehat{M}$ induces a K-basis for $K \otimes_{\mathcal{O}} \widehat{M}$ through $1 \otimes_{\mathcal{O}} v_1, ..., 1 \otimes_{\mathcal{O}} v_d$. It is clearly generating and any linear relation $\sum_{i=1}^{d} \alpha_i(1 \otimes_{\mathcal{O}} v_i)$ corresponds to $\alpha \otimes_{\mathcal{O}} \sum_{i=1}^{d} \beta_i v_i$ for some $\alpha, \alpha_i \in K, \beta_i \in \mathcal{O}$ since K is a field.

(D) Discussion of the basis in (C) induces an embedding $K \otimes_{\mathcal{O}} \mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$ in $\mathrm{Hom}_{A^K}(K \otimes_{\mathcal{O}} \widehat{P}_i, K \otimes_{\mathcal{O}} \widehat{P}_j)$. Conversely $K = \mathcal{O}[\frac{1}{\pi}]$ (and in particular every element in $K\backslash\mathcal{O}$ is of the form $u\pi^{-a}$ for $u \in \mathcal{O}^\times, a \in \mathbb{N}$ so the image of some $\alpha \in \mathrm{Hom}_{A^K}(K \otimes_{\mathcal{O}} \widehat{P}_i, K \otimes_{\mathcal{O}} \widehat{P}_j)$ is spanned by $\{u_1\pi^{-n_1} \otimes_{\mathcal{O}} m_1, ..., u_d\pi^{-n_d} \otimes_{\mathcal{O}} m_d\}$ so for some large enough n, $\pi^n\alpha$ is spanned by $\{1 \otimes_{\mathcal{O}} m'_1, ..., 1 \otimes_{\mathcal{O}} m'_d\}$ hence we can identify a unique $\pi^n\alpha \in \mathrm{Hom}_A(\widehat{P}_j, \widehat{P}_i)$.

(E) A^K is semisimple hence $K \otimes_{\mathcal{O}} \widehat{P}_i, K \otimes_{\mathcal{O}} \widehat{P}_j$ are semisimple. Letting the simple A^K-modules be V_i with corresponding $\mathcal{O}$-forms W_i, by the preceding arguments and the remark in Theorem 7.8 on the equality of multiplicities of simple modules in different reduced $\mathcal{O}$-forms

$$d_{ij} = [W_i/\pi W_i : S_j] = \dim_k \mathrm{Hom}_{\bar{A}}(P_j, W_i/\pi W_i) = \mathrm{rank}_{\mathcal{O}} \mathrm{Hom}_A(\widehat{P}_j, W_i) = \dim_K \mathrm{Hom}_{A^K}(K \otimes_{\mathcal{O}} \widehat{P}_j, V_i)$$

which by standard results (Schur) in ordinary representation theory is the multiplicity of the simple V_i in $K \otimes_{\mathcal{O}} \widehat{P}_j$. The claim follows from standard results.

Remark 7.14. When we do not have splitting assumptions the proof gives
$c_{ij} = \sum_s d_{si}d_{sj} \dim_k \mathrm{End}_{\bar{A}}(S_j)/\dim_k \mathrm{End}_{A^K}(V_i)$ where the extra factor comes in our discussion in (E) of the multiplicity of V_i.

Remark 7.15. When a group G is direct product $H \times K$, all irreducible matrix representations of G are Kronecker products of irreducible representations of H, K so the Decomposition matrix (resp. Cartan) of G is the product of the Decomposition matrices (resp. Cartan) for H, K respectively. Brauer & Nesbitt [5] VI.

We can now compute the decomposition matrix for $SL_2(p)$ from the Cartan Matrix we found earlier in Example 6.10 (working over splitting fields). We remark that it is not usually possible to find the decomposition matrix solely the theorem and the Cartan matrix and this is a special case.

Example 7.16. *The decomposition matrix of $SL_2(p)$.* The relation $C = D^T D$ completely determines the decomposition matrix since it must have entries in $\mathbb{Z}_{\geq 0}$. For the non-trivial Cartan block the corresponding decomposition block is

$$\begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & ... & ... & \\ & & & 1 & 1 \\ & & & & 1 \\ & & & & 1 \end{pmatrix}$$

Remark 7.17. A result (Brauer (1971) [6], Proposition 6G) states that that when the *'defect group'* of a *'block'* is abelian and the inertial index of the block is 1, all decompostion numbers are 1.

8 Brauer Characters II

For this section we will work over the group algebras given by a p-modular system $(K, \mathcal{O}, k)$. First let us relate these to the decomposition numbers and cartan invariants we have worked with previously. For a group G we let G_{reg} be the set of p-regular elements in G. For this section we will assume the following notation. Assume K, k are splitting, let $\chi_1, ..., \chi_n : G \to K$ be the ordinary irreducible characters and $\phi_1, ..., \phi_m : G_{reg} \to K$ the irreducible Brauer characters corresponding to irreducible kG-modules $S_1, ..., S_m$. Let the characters of $K \otimes_{\mathcal{O}} \widehat{P}_j$ where $\widehat{P}_j$ is a lift to $\mathcal{O}G$ of a projective kG cover P_j be φ_j

Proposition 8.1. *We have the following relations (where defined):*

$$i)\ \chi_i = \sum_{j=1}^{m} d_{ij}\phi_j \qquad ii)\ \varphi_i = \sum_{j=1}^{m} c_{ij}\phi_j \qquad iii)\ \varphi_j = \sum_{i=1}^{n} d_{ij}\chi_i$$

Proof: i), ii) are clear. The third result follows from discussion in (E) above on the decomposition number. It follows that given the decomposition matrix and either of the ordinary and modular character tables we can construct the other. Similarily given the Cartan matrix and the Brauer character table we can read off the characters of the indecomposable projectives.

8.1 Brauer Orthogonality Relations

Orthogonality relations are invaluable in constructing the ordinary character tables. We will exhibit similar results in the modular case with Brauer characters.

As for ordinary characters we will define a bilinear form on the vector space of p-regular class functions, Cl_K^{p-reg} by

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{|G|} \sum_{g \in G_{reg}} \phi_1(g^{-1})\phi_2(g)$$

In particular the arguments from the proof $C = D^T D$, results on ordinary characters and the semisimplicity of $k\langle g \rangle$ for a p-regular g show $\dim_k \mathrm{Hom}_{kG}(P_i, U) = \langle \varphi_i, \phi_U \rangle$. (Webb [24] Proposition 10.2.1).

Theorem 8.2. Brauer row orthogonality *The sets, of Brauer characters of irreducible kG-modules and of Brauer characters of indecomposable projective kG-modules are bases for Cl_K^{p-reg} are dual, that is,*

$$\langle \varphi_i, \phi_j \rangle = \delta_{ij}$$

Proof: We've already seen that the irreducible Brauer characters form a basis. We show $\varphi_1, ..., \varphi_m$ span Cl_K^{p-reg}. Fix p-regular conjugacy class representatives $g_1, ..., g_m$ and define a basis of class functions for Cl_K^{p-reg} given by $\lambda_i = \sum_{j=1}^n \chi_j(g_i^{-1})\chi_j$ for $i = 1, ..., m$ which span by ordinary column orthogonality. Now we know for any $i, k\langle g_i \rangle$ is semisimple since g_i has order coprime to p and so the induction to G of any simple $k\langle g_i \rangle$ module is projective hence the character is that of a direct summand of $\mathcal{O}G$ and thus in the K-span of $\varphi_1, ..., \varphi_m$. This is true for any i so it follows from *Brauer's theorem on induced characters* (Serre [23] Section 10.5, Theorem 19) that λ_i is in K-span of $\varphi_1, ..., \varphi_m$ also. Duality follows from the fact $\dim_k \mathrm{Hom}_{kG}(P_i, S_j) = \delta_{ij}$.

Theorem 8.3. Brauer column orthogonality

$$\sum_{j=1}^{m} \langle \varphi_j(g_a)\phi_j(g_b^{-1}) \rangle = \delta_{ab}|C_G(g_a)|$$

Proof: Apply Proposition 8.1 (ii) then $C = D^T D$ and use ordinary character theory.

Proposition 8.4. *Let ϕ be a Brauer character and $g_1, ..., g_d$ p-regular conjugacy class representatives, then*

- $\sum_{i=1}^d \phi(g_i)$ *is a non-negative integer*
- $\prod_{i=1}^d \phi(g_i)$ *is an integer*

Proof: [15] Chapter 4, Proposition 1.16.

We now have relations on the Brauer characters however we need to know the Brauer character of the indecomposable projective modules to use the relations so they are generally not as useful in the ordinary case (where the indecomposable projective modules are precisely the simple modules).

8.2 Blocks of defect zero

With a splitting p-modular system $(K, \mathcal{O}, k)$ for a group G, a ring summand of kG is said to be a *block of defect zero* if it has a projective simple module. This defining property has many useful consequences and for us it will be that the block has a unique simple module and it is the reduction of a unique ordinary irreducible module. The projective simple will then let us use the orthogonality relations described above. We have in fact already touched blocks of defect zero in the form of the Steinberg representation V_p in our discussion of the simples and indecomposable projectives of $SL_2(p)$. The main result tells us exactly when simple KG-modules reduce to simple kG-modules.

Theorem 8.5. *$(K, \mathcal{O}, k)$ a splitting p-modular system and G a finite group of order $p^a m, (m,p) \neq 1$. Let M be a KG-module of dimension n with $\mathcal{O}$-form U. Then M is a simple KG-module and $p^a | n$ if and only if the kG-module $U/\pi U$ is projective and simple.*

Proof: See Webb [24] Theorem 9.6.1.

Remark 8.6. We can further say there is a correspondence of blocks of $\mathcal{O}G$ with a single irreducible character and irreducible characters of G with dimension divisible by p^a. Furthermore such irreducible characters vanish on p-singular elements. (Landrock [16] Chapter I Corollary 16.2)

8.3 The Brauer characters of S_5 in characteristic 2

S_5 has order $2^3 \cdot 3 \cdot 5$ and by the previous proposition there are no projective simple Brauer characters. The conjugacy classes in symmetric groups are given by cycle types and S_5 has the ordinary character table

S_5	e	(12)	(123)	(1234)	(12345)	(12)(34)	(12)(345)
χ_1	1	1	1	1	1	1	1
χ_2	1	-1	1	-1	1	1	-1
χ_3	4	2	1	0	-1	0	-1
χ_4	4	-2	1	0	-1	0	1
χ_5	5	1	-1	-1	0	1	1
χ_6	5	-1	-1	1	0	1	-1
χ_7	6	0	0	0	1	-2	0

There are 3 2-regular conjugacy classes given by elements of odd order. Restricting the ordinary character table to these elements gives

S_5	e	(123)	(12345)
χ_1^β	1	1	1
χ_3^β	4	1	-1
χ_5^β	5	-1	0
χ_7^β	6	0	1

We first show χ_3 remains irreducible on reduction. It corresponds to the natural permutation action of S_5 on $\{(x_1, x_2, ..., x_5)^T : x_1 + ... + x_5 = 0\}$. There is a basis

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

and actions given by

$$(123) : v_1 \to v_2,\ v_2 \to -v_1 - v_2,\ v_3 \to v_1 + v_2 + v_3,\ v_4 \to v_4$$

$$(12345) : v_1 \to v_2,\ v_2 \to v_3,\ v_3 \to v_4,\ v_4 \to -(v_1 + v_2 + v_3 + v_4)$$

with irreducible characters

$$\mathrm{Tr}\begin{pmatrix}0 & -1 & 1 & 0\\ 1 & -1 & 1 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = 1, \mathrm{Tr}\begin{pmatrix}0 & 0 & 0 & -1\\ 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\end{pmatrix} = -1$$

We claim it is irreducible on reduction mod 2. It suffices to show we can generate $v_1, ..., v_4$ given any of the following elements $v_a, v_a + v_b, v_a + v_b + v_c, v_1 + ... + v_4$ for distinct integers $a, b, c \in \{1, 2, 3, 4\}$. Under a suitable choice of 5-tuple we may assume $a = 1, b = 2, c = 3$. Then v_1 is clear (apply (12345) consecutively). $(123)^{-1} \cdot (v_1 + v_2) = v_1$, $(123)^{-1} \cdot (v_1 + v_2 + v_3) = v_3$ and $(12345)^{-1} \cdot (v_1 + ... + v_4) = v_4$ so we have irreducibility of χ_3^β.

It remains to find the last irreducible character. $\chi_7^\beta = \chi_5^\beta + \chi_1^\beta$ so we need only focus on χ_5^β. We claim χ_5^β is not irreducible and will use the column and row orthogonality relations to verify this. Rewrite $\chi_1^\beta = \phi_1, \chi_3^\beta = \phi_2$ and let $\chi_3^\beta = \lambda$. Let the corresponding projective cover characters be $\varphi_1, \varphi_2, \varphi_3$ with φ_3 corresponding to the unfound irreducible. Since the order of the Sylow 2-subgroups is 8 they have corresponding dimensions $8a, 8b, 8c$ for positive integers a, b, c.

First by Brauer column orthogonality if n is the degree of the last irreducible

$$120 = |C_G(e)| = 8a \cdot 1 + 8b \cdot 4 + 8c \cdot n \implies a + 4b + nc = 15$$

Suppose now that λ is irreducible so that $n = 5$, then we have the equation $a + 4b + 5c = 15$. Since there are no projective simple modules, the multiplicity of any irreducible Brauer character is at least 2 in its projective indecomposable. Thus $8c \geq 5 \cdot 2$ so $c \geq 2$ and since $a, b > 0$ we must have $c = 2$ therefore the projective indecomposable corresponding to λ has dimension 16. By Proposition 8.1 and the discussion the character φ_3 is then one of the following

$$\varphi_3 = 3\lambda + \phi_1 \text{ or } 2\lambda + 6\phi_1 \text{ or } 2\lambda + \phi_2 + 2\phi_1$$

We can compute the inner products

$$\langle\lambda, \lambda\rangle = \frac{1}{120}(5 \cdot 5 + 20 \cdot -1) = \frac{1}{24}, \quad \langle\phi_1, \lambda\rangle = -\frac{1}{8}, \quad \langle\phi_2, \lambda\rangle = 0$$

Then simple computation shows that for any of the choices above $\langle\varphi_3, \lambda\rangle \neq 1$ which contradicts Brauer row orthogonality so λ is not a simple character.

It is clear that the last irreducible is a factor of λ and since it must not be a 1-degree character, the last irreducible character is of the form $\lambda - k\phi_1$ for $k = 1, 2, 3$. Since $\lambda(e) + \lambda((123)) + \lambda((12345)) = 5 + -1 + 0 = 4$ and$\phi(e) + \phi((123)) + \phi((12345)) = 3$ by Proposition 8.4 on sums of rows of the Brauer character table the last irreducible is $\lambda - \phi_1$. This completes the Brauer character table and thus the decomposition and Cartan matrices from which we can read off the characters of the indecomposable projectives.

Figure 2: Brauer character table, decomposition and Cartan matrices for S_5 in characteristic 2

S_5	e	(123)	(12345)
χ_1^β	1	1	1
χ_3^β	4	1	-1
χ_5^β	4	-2	-1

$$D_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 8 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 3 \end{pmatrix}$$

8.4 The Brauer characters of S_5 in characteristic 3

There are five 5-regular conjugacy classes hence 5 Brauer irreducibles. The 1 dimensional characters remain are distinct modulo 3 and since $|S_5/A_5| = 2$ these are all the 1 dimensional Brauer irreducibles. Moreover the Sylow 3-subgroups have order 3 which divides 6 so χ_7 reduces to a Brauer irreducible. Restricting the ordinary character table to these conjugacy classes gives the table:

S_5	e	(12)	(1234)	(12345)	(12)(34)
ϕ_1	1	1	1	1	1
ϕ_2	1	-1	-1	1	1
ϕ_3	4	2	0	-1	0
ϕ_4	4	-2	0	-1	0
ϕ_5	5	1	-1	0	1
ϕ_6	5	-1	1	0	1
ϕ_7	6	0	0	1	-2

We first show ϕ_3, ϕ_4 reduce to Brauer irreducibles. Suppose they have a 1 dimensional composition factor. Then there is a degree 3 Brauer character which evaluates to -2 on (12345) but a simple calculation shows the sum of any 3 5-th roots of unity in $\mathbb{C}$ cannot be -2, a contradiction. Else suppose they are the sum of two degree 2 Brauer characters, these must surely be distinct because $\phi_3((12345)) = -1$ while Brauer characters are algebraic integers and $-1/2$ is not. It follows $\phi_3((12345)) = \lambda((12345)) + \mu((12345)) = -1$ for some Brauer characters $\lambda \neq \mu$. However we also have Brauer characters $\phi_2 \cdot \lambda, \phi_2 \cdot \mu$ and $|\{\lambda((12345)), \mu((12345)), -\mu((12345)), -\lambda((12345))\}| > 2$ so this cannot be the case as there are only 5 irreducible Brauer characters. We conclude ϕ_3, ϕ_4 are irreducible complete the Brauer character table.

Figure 3: Brauer Character Table, decomposition and Cartan matrices for S_5 in characteristic 3

S_5	e	(12)	(1234)	(12345)	(12)(34)
ϕ_1	1	1	1	1	1
ϕ_2	1	-1	-1	1	1
ϕ_3	4	2	0	-1	0
ϕ_4	4	-2	0	-1	0
ϕ_7	6	0	0	1	-2

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad C_3 = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

9 Young diagrams

We could have found the modular character data of S_5 in characteristic 5 using the aforementioned methods and results however we will demonstrate the powerful combinatoric methods in the theory of Young diagrams that will allow us to construct all ordinary and modular irreducible representations as well as give strong partial results on the modular decomposition matrices. This section is largely influenced by the book by G.D.James [14] which we direct the reader to for a comprehensive introduction to the topic and many more results and examples. Our approach will be to expose the general theory on Young diagrams and guide the reader to James [14] for the proofs. We make the convention that symmetric group elements will be read from left to right such that $(12)(23) = (132)$ and not (123).

9.1 Definitions

Recall elements of the symmetric group can be written uniquely as a product of disjoint cycles and that cycle types determine the conjugacy classes of S_n. We can define further

Definition 9.1. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...$ be non-negative integers, then $\lambda = (\lambda_1, \lambda_2, ...)$ is a *partition* of n if $\sum_{i=1}^{n} \lambda_i = n$

We will generally ignore zeros and abbreviate repeated elements of the partition as powers e.g a partition $(5, 3, 2, 2, 1, 0, 0, ...)$ is written as $(5, 3, 2^2, 1)$.

Definition 9.2. For a partition λ of n, we define a *diagram* $[\lambda] = \{(i, j)|i \geq 1, 1 \leq j \leq \lambda_i\}$ of nodes (i, j) while rows and columns are given by nodes with the same i, j value respectively. The *conjugate diagram* $[\lambda']$ of $[\lambda]$ is given by interchanging the rows and columns.

and S_n acts naturally on these node wise.

It will be helpful for us to define orderings on these diagrams. We have a partial order called the *dominance order* and a total order called the *dictionary order* defined as:

Definition 9.3. If λ, μ are partitions of n, λ *dominates* μ ($\lambda \trianglerighteq \mu$) if for all $j, \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i$ and the dictionary order $>$ is given by $\lambda > \mu$ if the least j for which $\lambda_j \neq \mu_j$ satisfies $\lambda_j > \mu_j$.

Definition 9.4. A λ-*tableau* t is obtained from $[\lambda]$ by replacing each node with a unique integer in $1, 2, ..., n$ (in one of $n!$ ways). For a tableau t, its *row stabilizer* R_t and *column stablizer* C_t are the subgroups of S_n fixing the rows and columns of t setwise, respectively.

We have an equivalence class on λ-tableaux $t_1 \sim t_2$ if and only if $t_1\pi = t_2$ for some $\pi \in R_{t_1}$ e.g tableaux are equivalent if the elements of their rows are the same setwise. An equivalence class is called a *tabloid* $\{t\}$.

Definition 9.5. (Hooks) Let $[\lambda]$ be a diagram.

- The (i, j)-*hook* of $[\lambda]$ contains the $\lambda_i - j$ nodes to the right of (i, j) and all $\lambda'_j - i$ below (including (i, j)).
- The *length* h_{ij} of the (i, j)-hook is $\lambda_i + \lambda'_j + 1 - i - j$. Replacing the nodes in $[\lambda]$ with their corresponding hook length gives the hook graph.
- A *skew-hook* is a connected part of the edge of $[\lambda]$ which gives a proper diagram when removed.

There is a natural 1-1 correspondence of hooks and skew-hooks by associating the (i, j)-hook with the skewhook starting at the rim of the i-th row and ending at the j-th column and for a skew-hook we say its *leg-length* is the number of nodes in the leg (below the node) of the corresponding hook . A *skew r-hook* is a skew-hook containing r nodes.

9.2 Specht Modules

We now turn our attention to the construction of kS_n-modules. A partition λ of n induces a *Young subgroup* S_λ of S_n

$$S_\lambda = S_{1,2,...,\lambda_1} \times S_{\lambda_1+1,...,\lambda_1+\lambda_2} \times ...$$

and a corresponding kS_n-module M^λ given by an ordering of unordered tuples. e.g $M^{(n-2,2)}$ consists of unordered pairs $ij, i \neq j$. We can see that M^λ has a k-basis given by the λ-tabloids and is generated as a kS_n module by any tabloid. It is clear that the dimension of M^λ is $\frac{n!}{\lambda_1!\lambda_2!...}$ and that S_λ is the kernel of the action. We can also define a bilinear form on M^λ extending $\langle\{t_1\},\{t_2\}\rangle = \delta_{12}$.

Definition 9.6. For a tableau t we define a signed column sum $\kappa_t = \sum_{\sigma\in C_t}(\operatorname{sgn}\sigma)\sigma$, a signed sum of elements of the column stabilizer. We therefore have a polytabloid $e_t = \{t\}k_t$ and the kS_n span of polytabloids over all λ-tableau t in M^λ is the *Specht Module* S^λ.

Now we let a *standard tableau* be a tableau t such that numbers increase of rows and columns. Then the polytabloid formed from the equivalence class containing t is the polytabloid e_t (Note polytabloids depend only t not $\{t\}$). It is a fact (James [14] Theorem 8.4) that e_t over standard tableau t forms a basis for S^λ and further over $\mathbb{Q}$ such a basis is even a $\mathbb{Z}$-basis for $S^\lambda_{\mathbb{Q}}$. We can then form a *Gram matrix* induced by this basis and the bilinear form above (the i,jth entry is $\langle e_i, e_j\rangle$ for some ordering $1,...,d$ of the standard polytabloids).

Definition 9.7. Define g^λ to be the greatest common divisor of entries in the Gram matrix, or equivalently

$$g^\lambda = gcd\{\langle e_t, e_{t^*}\rangle | e_t, e_{t^*} \text{ are polytabloids in } S^\lambda_{\mathbb{Q}}\}$$

and it is further known that $p|g^\lambda$ if and only if λ is p-singular (James [14] Corollary 10.5).

These Specht Modules will be vital to our study as they are precisely the ordinary irreducibles.

Theorem 9.8. *(Ordinary Irreducibles of S_n) Let k be a field of characteristic 0. The Specht (kS_n-)modules are self-dual and irreducible. In particular they are all ordinary irreducible representiations.*

Proof: James [14] Theorem 4.12.

The dimension of S^λ can be nicely characterised as

Theorem 9.9. *(Frame, Robinson and Thrall)*

$$dim\ S^\lambda = n!\frac{\prod_{i<k}(h_{il}-h_{kl})}{\prod_i h_{il}!} = \frac{n!}{\prod \text{hook lengths in } [\lambda]}$$

Example 9.10. The ordinary character table of S_5, rows indexed by S^λ and columns by conjugacy classes.

S_5	(5)	(4,1)	(3,2)	(3,1)	$(2^2,1)$	$(2,1^3)$	(1^5)
(5)	1	1	1	1	1	1	1
(4,1)	-1	0	-1	1	0	2	4
(3,2)	0	-1	1	-1	1	1	5
$(3,1^2)$	1	0	0	0	-2	0	6
$(2^2,1)$	0	1	-1	-1	1	-1	5
$(2,1^3)$	-1	0	1	1	0	-2	4
(1^5)	1	-1	-1	1	1	-1	1

9.3 Irreducible Modules in Char p

Recall that a conjugacy class of S_n is p-regular if the order of elements in the class is coprime to p. A partition λ is *p-singular* if for some i $\lambda_{i+1} = ... = \lambda_{i+p} > 0$ and *p-regular* otherwise. It is true that the number of p-regular conjugacy classes is equal to the number of p-regular partitions (James [14] Lemma 10.2).

We've already seen that $S^\lambda \cap S^{\lambda\perp}$ is the maximal submodule of S^λ and it is further true that in a field of positive characteristic $S^\lambda \subset S^{\lambda\perp}$ if and only if p divides g^λ which we saw can only be if λ is p-singular. This motivates the following definition

Definition 9.11. If λ is p-regular define $D^\lambda = S^\lambda/(S^\lambda \cap S^{\lambda\perp})$.

These will be key to our construction of the decomposition matrices, namely since

Theorem 9.12. *Let k be a field of characteristic $p > 0$. The irreducible kS_n-modules are D^λ, varying λ over all p-regular partitions of n. Also they are self dual and every field is splitting for S_n.*

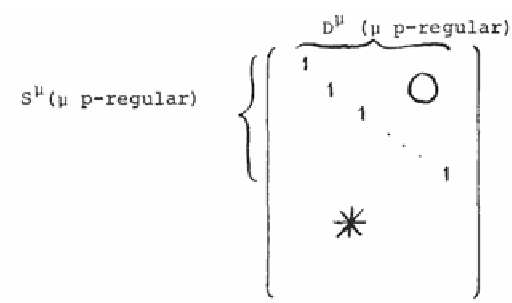
Proof: James [14] Theorem 11.5.

It is a general result that dim $M/(M \cap M^\perp)$ is the rank of the Gram matrix written in a basis for M and so we can compute the dimension of D^λ in prime characteristic by computing the rank of the Gram matrix in the standard basis of S^λ reduced mod p.

9.4 Useful results on decomposition matrices

Module theory (2nd Isomorphism Theorem) and previous results show that all composition factors of S^λ have the form D^μ for $\mu \rhd \lambda$. Combining with our result in Theorem 9.12 we begin this subsection with the result

Proposition 9.13. *For a prime p, ordering p-regular partitions before p-singular partitions in dictionary order the decomposition matrix for S_n has the form*



This shows there is a diagonal line of 1's with 0's above which we can immediately write down in the decomposition matrix We also have a computationally useful result.

Theorem 9.14. *(Murnaghan-Nakayama Rule) If v is a partition of $n-r$ then the generalised character corresponding to*

$$\sum\{(-1)^i[\lambda] | [\lambda]\backslash[v] \text{ is a skew } r\text{-hook of leg-length } i\}$$

vanishes on classes not containing an r-cycle.

Proof: See James [14] Theorem 21.7.

We say a partition is a hook representation/partition if the Young diagram looks like a hook e.g the partition is of the form $(a, 1^b)$.

Theorem 9.15. *Over a field of odd characteristic p*

- *If $p \nmid n$ then all hook representations of S_n remain irreducible mod p and they are pairwise non isomorphic*
- *If $p|n$ then part of the p-modular decomposition matrix looks like*

$$\begin{array}{l|cccccc}
(n) & 1 & & & & & \\
(n-1,1) & 1 & 1 & & & \text{\Large 0} & \\
(n-1,1^2) & & 1 & 1 & & & \\
\vdots & & & & \ddots & & \\
(2,1^{n-2}) & \text{\Large 0} & & & & 1 & 1 \\
(1^n) & & & & & & 1
\end{array}$$

Proposition 9.16. *(James) The multiplicity of $D^{(n-j,j)}$ as a composition factor of $S^{(n-m,m)}$ is 1 if we have p-adic decompositions*

$$n - 2j + 1 = a_0 + a_1 p + ... + a_r p^r$$
$$m - j = b_0 + b_1 p + ... + b_s p^s$$

with $s < r$ and for each i,$b_i = 0$ or $b_i = a_i$. Otherwise the multiplicity is 0.

Proof: See James [14] Theorem 24.15.

9.5 S_5 in characteristic 5

We will now use the results and theory we have developed to find the decomposition matrix for S_5 in characteristic 5. The only p-singular partition is (1^5) so writing partitions in dictionary order we immediately find part of the decomposition matrix from Proposition 9.13:

D_5	(5)	(4,1)	(3,2)	(3,1)	$(2^2,1)$	$(2,1^3)$
(5)	1	0	0	0	0	0
(4,1)		1	0	0	0	0
(3,2)			1	0	0	0
$(3,1^2)$				1	0	0
$(2^2,1)$					1	0
$(2,1^3)$						1

Then we work out the dimensions of the Specht modules and the modular irreducibles using the hook length formula and the Gram matrices and add these to our table along with the S^{1^5} module

	D_5	1 (5)	3 (4,1)	5 (3,2)	3 (3,1^2)	5 (2^2, 1)	1 (2, 1^3)
1	(5)	1	0	0	0	0	0
4	(4,1)		1	0	0	0	0
5	(3,2)			1	0	0	0
6	(3,1^2)				1	0	0
5	(2^2, 1)					1	0
4	(2, 1^3)						1
1	(1^5)						

Dimensional considerations then show that adding the matrix subdiagonal 1's in the matrix for hook representations in Theorem 9.15 ($p = 5|n = 5$) completes the decomposition matrix

	D_5	1 (5)	3 (4,1)	5 (3,2)	3 (3,1^2)	5 (2^2, 1)	1 (2, 1^3)
1	(5)	1	0	0	0	0	0
4	(4,1)	1	1	0	0	0	0
5	(3,2)	0	0	1	0	0	0
6	(3,1^2)	0	1	0	1	0	0
5	(2^2, 1)	0	0	0	0	1	0
4	(2, 1^3)	0	0	0	1	0	1
1	(1^5)	0	0	0	0	0	1

and we can therefore compute the Cartan Matrix from which the characters of the indecomposable projective modules follow easily

$$C_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

Finally we can find the Brauer Character table for S_5 in characteristic 5 as

S_5	e	(12)	(12)(34)	(123)	(123)(45)	(1234)
(5)	1	1	1	1	1	1
(4,1)	3	1	-1	0	-2	-1
(3,2)	5	1	1	-1	1	-1
(3,1^2)	3	-1	-1	0	-2	-1
(2^2,1)	5	-1	1	-1	-1	1
(2, 1^3)	1	-1	1	1	-1	-1

The character of $D^{(4,1)}$ can be computed easily as $\chi^{(4,1)} = \phi^{(5)} + \phi^{(4,1)}$ on 5-regular elements and so the modular character is determined by subtracting $\phi^{(5)}$ from $\chi^{(4,1)}$. Similar considerations determine the other Brauer characters.

9.6 Decomposition matrix of S_6 in characteristic 5

The case of S_6 in characteristic 3 is worked out in James [14] Example 24.3 while the case of characteristic 2 is constructed in the Appendix of James [14]. We will construct the decomposition matrix in characteristic 5 where the only 5-singular partition is (1^5). In the opposite case to S_5 in characteristic 5 we can use part i) of Theorem 9.15 to understand that the hook representations remain simple.

	D_5	(6)	(5,1)	(4,2)	$(4,1^2)$	(3^2)	(3,2,1)	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$
1	(6)	1	0	0	0	0	0	0	0	0	0
5	(5,1)	0	1	0	0	0	0	0	0	0	0
9	(4,2)			1	0	0	0	0	0	0	0
10	$(4,1^2)$	0	0	0	1	0	0	0	0	0	0
5	(3^2)					1	0	0	0	0	0
16	$(3,2,1)$						1	0	0	0	0
10	$(3,1^3)$	0	0	0	0	0	0	1	0	0	0
5	(2^3)								1	0	0
9	$(2^2,1^2)$									1	0
5	$(2,1^4)$	0	0	0	0	0	0	0	0	0	1
1	(1^6)										

Now we use Proposition 9.16 to determine the factors of $S^{(4,2)}$. $D^{(5,1)}$ is not a composition factor since in the same notation $n-2j+1 = 6-2+1 = 5 = 0+5$ while $m-j = 1$ (so $a_0 \neq b_0$ and $b_0 \neq 0$) but $D^{(6)}$ is since in this case $n-2j+1 = 6-0+1 = 7 = 2+1\cdot 5$ and $m-j = 2-0 = 2$. This completes the composition factors of $S^{(4,2)}$. Similar calculations shows $S^{(3^2)}$ remains irreducible.

We then use the Murnaghan-Nakayama rule with partition $[v] = [1]$ and $r = 5$ to find

$$\chi^{(6)} + \chi^{(1^6)} + \chi^{(3,2,1)} = \chi^{(4,2)} + \chi^{(2^2,1^2)} \tag{A}$$

on 5-regular classes. We know $\chi^{(6)} = \phi^{(6)}, \chi^{(1^6)} = \phi^{(1^6)}$ are irreducible while we know the decomposition of $S^{(4,2)}$ so we deduce $\chi^{(4,2)}$ shares a factor of $\phi^{(4,2)}$ with $\chi^{(3,2,1)}$ and $\chi^{(2^2,1^2)}$ has a factor of $\chi^{(1^6)}$. We summarise this with the current decomposition matrix:

	D_5	(6)	(5,1)	(4,2)	$(4,1^2)$	(3^2)	(3,2,1)	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$
1	(6)	1	0	0	0	0	0	0	0	0	0
5	(5,1)	0	1	0	0	0	0	0	0	0	0
9	(4,2)	1	0	1	0	0	0	0	0	0	0
10	$(4,1^2)$	0	0	0	1	0	0	0	0	0	0
5	(3^2)	0	0	0	0	1	0	0	0	0	0
16	$(3,2,1)$			1			1	0	0	0	0
10	$(3,1^3)$	0	0	0	0	0	0	1	0	0	0
5	(2^3)								1	0	0
9	$(2^2,1^2)$									1	0
5	$(2,1^4)$	0	0	0	0	0	0	0	0	0	1
1	(1^6)										

We finally employ dimension calculations of D^λ and add these to our matrix

D_5	1	5	8	10	5	8	10	5	1	5
	(6)	(5,1)	(4,2)	$(4,1^2)$	(3^2)	(3,2,1)	$(3,1^3)$	(2^3)	$(2^2,1^2)$	$(2,1^4)$

to immediately deduce the rows for $S^{(3,2,1)}, S^{(2^3)}$ are completed. From Equation (A) we find $\chi^{(2^2,1^2)} = \phi^{(2^2,1^2)} + \phi^{(3,2,1)}$ and since $\chi^{(1^6)}$ is a factor of $S^{2^2,1^2}$ we complete the decomposition matrix.

	D_5	1 (6)	5 (5,1)	8 (4,2)	10 $(4,1^2)$	5 (3^2)	8 (3,2,1)	10 $(3,1^3)$	5 (2^3)	1 $(2^2,1^2)$	5 $(2,1^4)$
1	(6)	1	0	0	0	0	0	0	0	0	0
5	(5,1)	0	1	0	0	0	0	0	0	0	0
9	(4,2)	1	0	1	0	0	0	0	0	0	0
10	$(4,1^2)$	0	0	0	1	0	0	0	0	0	0
5	(3^2)	0	0	0	0	1	0	0	0	0	0
16	$(3,2,1)$	0	0	1	0	0	1	0	0	0	0
10	$(3,1^3)$	0	0	0	0	0	0	1	0	0	0
5	(2^3)	0	0	0	0	0	0	0	1	0	0
9	$(2^2,1^2)$	0	0	0	0	0	1	0	0	1	0
5	$(2,1^4)$	0	0	0	0	0	0	0	0	0	1
1	(1^6)	0	0	0	0	0	0	0	0	1	0

10 Conclusion

We provided a brief introduction to modular character theory, largely focused on kG-modules and understood these through indecomposable projective modules and composition series. This lead us to Brauer characters and the notion of p-regular elements along with decomposition and Cartan matrices which gave us a malleable way to represent the information on these kG-modules with p-modular splitting systems providing a bridge between different characteristics. We gave explicit computations for $SL_2(p)$ and the symmetric groups S_5, S_6 using the Green correspondence, Brauer orthogonality relations, Young diagrams and general module theory. In fact the groups $SL_2(p^n)$ are well understood also [3] and larger groups or those with smaller top composition factor may be of interest to the reader next.

There has a vast amount of further development of this theory namely through *central idempotents* and the language of *blocks* and *defect groups* which measure the complexity of the algebra of the blocks. We may view the blocks themselves as equivalences classes of simple modules of an algebra A such that simple A-modules S, T have $S \sim T$ if there is a sequence of simple modules $S = S_1, S_2, ..., S_n = T$ where S_i, S_{i+1} are both composition factors of the same indecomposable projective module. In particular if e is a block and $0 \to U \to V \to W \to 0$ is a short exact sequence of A-modules then V belongs to e if and only if U, W do also and we see a connection to Theorem 6.6. Many texts describe this theory such as Landrock [16], Alperin [1] and others and some key baseline results include Brauer's main theorems which provides a correspondence between blocks of kG of defect group D and blocks of $kN_G(D)$ with defect group D. The Brauer correspondence is even explicit, described by the celebrated Brauer morphism and is compatible with the Green correspondence we illustrated.

Much work has been been inspired by the many problems and outstanding conjectures in the field such as Brauer's $k(B)$ conjecture and Alperin's Weight Conjecture. Problems like the Alperin-McKay conjecture and Brauer's Height Zero Conjecture have recently been proved with the former by Britta Späth & Marc Cabanes [8] through a reduction to the inductive Alperin-McKay condition on finite simple groups (Isaacs, Malle, Navarro [13]) while the remaining open 'only if' implication of Brauer's Height Zero Conjecture was solved by Malle, Navarro, Fry, Tiep [18]. These conjectures are examples of *local-global* principles [17] which describe the relationship of structural properties of a block of a finite group in characteristic p and *local subgroups*, subgroups $N_G(P)$ for non-trivial p-subgroups P. Many further results give information on the ordinary character table, especially pertaining to the nature of Sylow p-subgroups and a concise treatment is given by Sambale [20]. Another important tool that we have not discussed here is the use of homological algebra (covered well by Benson [4]), in particular the Ext functor provides answers to more general questions on extensions in the form of short exact sequences as in Theorem 6.6. For example the short exact sequences we proved are rephrased in Section 12.2 of Humphreys [12] in the language of homological algebra and these tools form the statement of Broué's Abelian Defect Group Conjecture [7] which has many deep consequences in modular representation theory.

References

[1] J.L. Alperin, *Local Representation Theory*, Cambridge, 1986.

[2] J.L. Alperin, M.J. Collins and D.A. Sibley, Projective Modules, Filtrations and Cartan Invariants, *Bulletin of the London Mathematical Society* **16** (1984), 416-420.

[3] H.H. Andersen, J. Jorgensen, P. Landrock, The Projective Indecomposable Modules of $SL(2, p^n)$, *Proceedings of the London Mathematical Society* **s3-46** (1983), 38-52.

[4] D.J. Benson, *Representations and cohomology*, Volume 1, Cambridge University Press, 1991.

[5] R. Brauer & C. Nesbitt, On the Modular Characters of Groups, *Annals of Mathematics* **42** (1941), no. 2, 556-590.

[6] R. Brauer, Some applications of the theory of blocks of characters of finite groups IV, *J. Algebra* **17** (1971), 489-521.

[7] M. Broué, (1990) Isométries parfaites, types de blocs, catégories dérivées, *Astérisque* **181-182** (1990), 61-92.

[8] M. Cabanes and B. Späth, The McKay conjecture on character degrees, *Annals of Mathematics* (2025), to appear.

[9] C.W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, John Wiley and Sons, 1962.

[10] B. Farb and R.K. Dennis, *Noncommutative Algebra*, GTM Volume 144, Springer, 1993.

[11] J.E. Humphreys, Representations of SL(2,p), *AMM* **82** (1975), no.1, 21-39.

[12] J.E. Humphreys, *Modular Representations of Finite Groups of Lie Type*, Cambridge University Press, 2005.

[13] I.M. Isaacs, G. Malle, G. Navarro, A reduction theorem for the McKay conjecture, *Invent.math* **170** (2007), 33-101.

[14] G.D. James, *The representation theory of the symmetric groups*, Springer LNM 682, 1978.

[15] G. Karpilovksy and L. Nachbin, *North-Holland Mathematics Studies*, Volume 182, 1995.

[16] P. Landrock, *Finite Group Algebras and Their Modules*, London Mathematical Society Lecture Note Series 84, Cambridge University Press, 1983.

[17] G. Malle, *Local-global conjectures in the representation theory of finite groups*, arXiv, https://arxiv.org/abs/1512.01145, 2015.

[18] G. Malle, G. Navarro, A.A.S. Fry, P.H. Tiep, Brauer's Height Zero Conjecture, *Annals of Mathematics* **200** (2024), 557-608.

[19] S. Martin, *Modular Representations of Finite Groups*, Lectures, Cambridge Part III Mathematics, 2025.

[20] B. Sambale, Character tables and defect groups, *J. Algebra* **562** (2020), 323-340.

[21] P. Schneider, *Modular representation theory of finite groups*, Springer, 2013.

[22] J.P. Serre, *Representation lineares des groupes finis*, Hermann, Paris, 1967.

[23] J.P. Serre *Linear Representations of Finite Groups*, Springer, 1977.

[24] P.J. Webb, *A course in finite group representation theory*, Cambridge University Press, 2016.