Dynamic model of a quad-rotor rotorcraft

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1. Characteristics of the Quad-rotor

The quad-rotor is a useful prototype for learning about aerodynamic phenomena in flying machines that can hover. The quad-rotor shown in Figure 1 has three gyros, which help the remote pilot stabilize the aircraft. Without the gyros the pilot must continually make control input corrections to keep the aircraft hovering at a given position.

Conventional helicopters modify the lift force by varying the collective pitch. These helicopters use a mechanical device known as swashplate to change the rotor blades pitch angle in a cyclic manner so as to obtain the pitch and roll control torques of the vehicle. The swashplate interconnects the servomechanisms and the blades pitch links. In contrast, the quad-rotor does not have a swashplate and has constant pitch blades. In a quad-rotor we can only vary the angular speed of each of the four rotors.

The force f_i produced by motor i is proportional to the square of the angular speed, that is $f_i = k\omega_i^2$. Note that since a motor can only turn in a fixed direction, the produced force f_i is always positive, see Figure 1. The front and the rear motors rotate counter-clockwise, while the other two motors rotate clockwise. With this arrangement, gyroscopic effects and aerodynamic torques tend to cancel in trimmed flight. The main thrust, u,is the sum of the individual thrusts of each motor. The pitch torque is a function of the difference $f_1 - f_3$, the roll torque is a function of $f_2 - f_4$, and the yaw torque is the sum $\tau_{M_1} + \tau_{M_2} + \tau_{M_3} + \tau_{M_4}$, where τ_{M_i} is the reaction torque of motor i due to shaft acceleration and the blades drag. The motor torque is opposed by an aerodynamic drag, τ_{drag}, such that

$$I_{rot}\dot{\omega}_i = \tau_{M_i} - \tau_{drag} \tag{1}$$

where I_{rot} is the moment of inertia of a rotor around its axis.

For definition we have that the aerodynamic drag is given as

$$\tau_{drag} = \frac{1}{2}\rho A v^2, \tag{2}$$

where ρ is the density of the air, A is the frontal area of the moving shape, and v is its velocity relative to the air.

In magnitude, the angular velocity ω is equal to the linear velocity v divided by the radius of rotation r,

$$\omega = \frac{v}{r}. \tag{3}$$

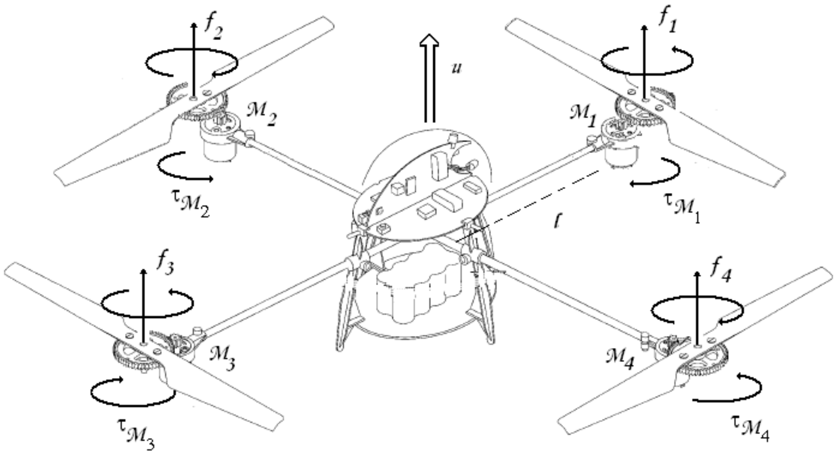

Figura 1: The quad-rotor control inputs. The mini-rotorcraft is controlled by the angular speeds of four electric motors. Each motor produces a thrust and a torque, which combine to generate the main thrust, the yaw torque, the pitch torque, and the roll torque acting on the mini-helicopter.

We can rewrite the aerodynamic drag as

$$\tau_{drag} = k_{drag}\, \omega^2, \tag{4}$$

where $k_{drag} > 0$ is a constant depending on the density of air, the radius, shape of the blade and others factors.

For quasi-stationary manoeuvres, we have $\omega = cte$, then

$$\tau_{M_i} = \tau_{drag}. \tag{5}$$

Forward pitch motion is obtained by increasing the speed of the rear motor M_3 while reducing the speed of the front motor M_1. Likewise, roll motion is obtained using the lateral motors. Yaw motion is obtained by increasing the torque of the front and rear motors (τ_{M_1} and τ_{M_3} respectively) while decreasing the torque of the lateral motors (τ_{M_2} and τ_{M_4} respectively). These motions can be accomplished while keeping the total thrust constant, see Figure 1. In view of its configuration, the quad-rotor has some similarities with the PVTOL aircraft. Indeed, if the roll or pitch and yaw angles are set to zero, the quad-rotor reduces to a PVTOL and can be viewed as two PVTOL's connected such that their axes are orthogonal.

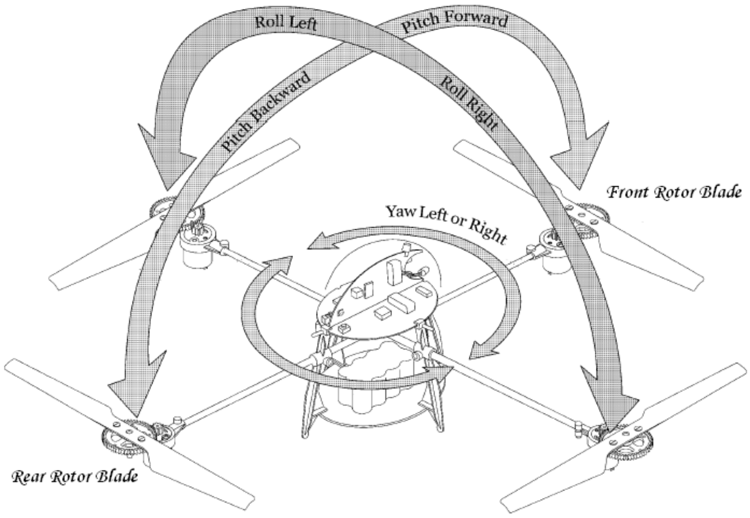

Figura 2: Pitch, roll, and yaw torques of the quad-rotor. The difference between the front rotor blade speed and the rear rotor blade speed produces a pitch torque. The roll torque is produced similarly. The yaw torque is the sum of the torques of each motor.

The PVTOL is a mathematical model of a flying object that evolves in a vertical plane, see [2]. The aircraft has three degrees of freedom, (x, y, θ) corresponding to its position in the plane and pitch angle. The PVTOL has two independent thrusters that produce a force and a moment and thus is underactuated since it has three degrees of freedom and only two inputs.

1.1. Dynamical Model

In this section we derive a dynamical model of the quad-rotor. This model is obtained by representing the aircraft as a solid body evolving in a three dimensional space and subject to the main thrust and three torques. The main thrust u is shown in Figure 1. The dynamics of the four electric motors are fast thus are neglected.

1.1.1. Euler-Lagrange approach

The generalized coordinates of the rotorcraft are

$$q = (x, y, z, \psi, \theta, \phi) \in \mathbb{R}^6,$$

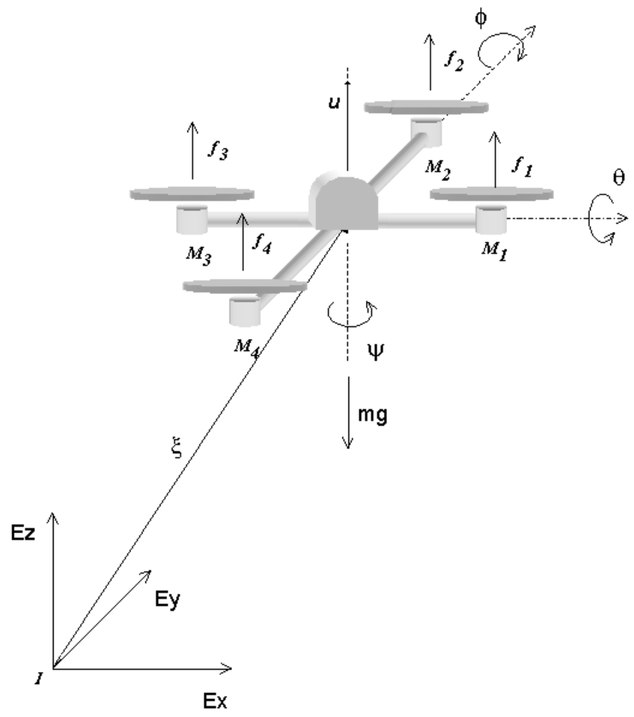

Figura 3: The quad-rotor in an inertial frame. $f_1,\ f_2,\ f_3,\ f_4$ represent the thrust of each motor, ψ, θ, and ϕ represent the Euler angles, and u is the main thrust.

where $\xi = (x, y, z) \in \mathbb{R}^3$ denotes the position vector of the center of mass of the helicopter relative to a fixed inertial frame ($\mathcal{I}$), and $\eta = (\psi, \theta, \phi)\ \in \mathbb{R}^3$ are the Euler angles, ψ is the yaw angle around the z-axis, θ is the pitch angle around the modified y-axis, and ϕ is the roll angle around the modified x-axis (see [4], [5]), which represent the orientation of the rotorcraft.

Define the Lagrangian

$$L(q, \dot{q}) = T_{trans} + T_{rot} - U,$$

where $T_{trans} = \frac{m}{2}\dot{\xi}^T\dot{\xi}$ is the translational kinetic energy, $T_{rot} = \frac{1}{2}\Omega^T\mathbf{I}\ \Omega$ is the rotational kinetic energy, $U = mgz$ is the potential energy of the aircraft, z is the rotorcraft altitude, m denotes the mass of the quad-rotor, Ω is the vector of the angular velocity, $\mathbf{I}$ is the inertia matrix, and g is the acceleration due to gravity. The angular velocity vector ω resolved in the body fixed frame is related to the generalized velocities $\dot{\eta}$ (in the region where the Euler angles are valid) by means of the standard kinematic relationship [6]

$$\Omega = W_\eta \dot{\eta},$$

where

$$W_\eta = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & -\sin\phi & 0 \end{bmatrix}$$

then

$$\Omega = \begin{pmatrix} \dot{\phi} - \dot{\psi} s_\theta \\ \dot{\theta} c_\phi + \dot{\psi} c_\theta s_\phi \\ \dot{\psi} c_\theta c_\phi - \dot{\theta} s_\phi \end{pmatrix}.$$

Define

$$\mathbb{J} = \mathbb{J}(\eta) = W_\eta^T \mathbf{I}\, W_\eta,$$

where

$$\mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}.$$

so that,

$$T_{rot} = \frac{1}{2}\dot{\eta}^T \mathbb{J} \dot{\eta},$$

Thus, the matrix $\mathbb{J} = \mathbb{J}(\eta)$ acts as the inertia matrix for the full rotational kinetic energy of the helicopter expressed directly in terms of the generalized coordinates η.

The model of the full rotorcraft dynamics is obtained from Euler-Lagrange equations with external generalized forces

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = \begin{bmatrix} F_\xi \\ \tau \end{bmatrix},$$

where $F_\xi = R\widehat{F} \in \mathbb{R}^3$ is the translational force applied to the rotorcraft due to main thrust, $\tau \in \mathbb{R}^3$ represents the yaw, pitch, and roll moments, and R denotes the rotational matrix $R(\psi, \theta, \phi) \in SO(3)$ representing the orientation of the aircraft relative to a fixed inertial frame,

$$R = \begin{pmatrix} c_\theta c_\psi & s_\psi s_\theta & -s_\theta \\ c_\psi s_\theta s_\phi - s_\psi c_\phi & s_\psi s_\theta s_\phi + c_\psi c_\phi & c_\theta s_\phi \\ c_\psi s_\theta c_\phi + s_\psi s_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi & c_\theta c_\phi \end{pmatrix}$$

We use c_θ for $\cos\theta$ and s_θ for $\sin\theta$. From Figure 1, it follows that

$$\widehat{F} = \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix},$$

where u is the main thrust directed out of the top of the aircraft expressed as

$$u = \sum_{i=1}^{4} f_i\,,$$

and, for $i = 1, ..., 4$, f_i is the force produced by motor M_i, as shown in Figure 1. Typically, $f_i = k\omega_i^2$, where k_i is a constant and ω_i is the angular speed of the ith motor. The generalized torques are thus

$$\tau = \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix} \triangleq \begin{bmatrix} \sum_{i=1}^{4} \tau_{M_i} \\ (f_2 - f_4)\ell \\ (f_3 - f_1)\ell \end{bmatrix},$$

where ℓ is the distance between the motors and the center of gravity, and τ_{M_i} is the moment produced by motor M_i, $i = 1, ..., 4$, around the center of gravity of the aircraft.

Since the Lagrangian contains no cross terms in the kinetic energy combining $\dot{\xi}$ with $\dot{\eta}$, the Euler-Lagrange equation can be partitioned into dynamics for ξ coordinates and the η coordinates.

The Euler-Lagrange equation for the translation motion is

$$\frac{d}{dt}\left[\frac{\partial L_{\text{trans}}}{\partial \dot{\xi}}\right] - \frac{\partial L_{\text{trans}}}{\partial \xi} = F_\xi,$$

then

$$m\ddot{\xi} + mgE_z = F_\xi,$$

As for the η coordinates we can rewrite

$$\frac{d}{dt}\left[\frac{\partial L_{\text{rot}}}{\partial \dot{\eta}}\right] - \frac{\partial L_{\text{rot}}}{\partial \eta} = \tau,$$

or

$$\frac{d}{dt}\left[\dot{\eta}^T \mathbb{J} \frac{\partial \dot{\eta}}{\partial \dot{\eta}}\right] - \frac{1}{2}\frac{\partial}{\partial \eta}(\dot{\eta}^T \mathbb{J}\dot{\eta}) = \tau,$$

Thus, we obtain

$$\mathbb{J}\ddot{\eta} + \dot{\mathbb{J}}\dot{\eta} - \frac{1}{2}\frac{\partial}{\partial \eta}\left(\dot{\eta}^T \mathbb{J}\dot{\eta}\right) \quad = \quad \tau.$$

Defining the Coriolis /Centripetal vector

$$\bar{V}(\eta, \dot{\eta}) = \dot{\mathbb{J}}\dot{\eta} - \frac{1}{2}\frac{\partial}{\partial \eta}\left(\dot{\eta}^T \mathbb{J}\dot{\eta}\right),$$

we may write

$$\mathbb{J}\ddot{\eta} + \bar{V}(\eta, \dot{\eta}) = \tau,$$

but we can rewrite $\bar{V}(\eta, \dot{\eta})$ as

$$\begin{aligned} \bar{V}(\eta, \dot{\eta}) &= (\dot{\mathbb{J}} - \frac{1}{2}\frac{\partial}{\partial \eta}\left(\dot{\eta}^T \mathbb{J}\right))\dot{\eta}, \\ &= C(\eta, \dot{\eta})\dot{\eta}, \end{aligned}$$

where $C(\eta, \dot{\eta})$ is referred to as the Coriolis terms and contains the gyroscopic and centrifugal terms associated with the η dependence of $\mathbb{J}$.

This yields

$$
\begin{aligned}
m\ddot{\xi} + mgE_z &= F_\xi, \\
\mathbb{J}\ddot{\eta} &= \tau - C(\eta, \dot{\eta})\dot{\eta}.
\end{aligned}
$$

To simplify, we make

$$
\tilde{\tau} = \begin{bmatrix} \tilde{\tau}_\psi \\ \tilde{\tau}_\theta \\ \tilde{\tau}_\phi \end{bmatrix} = \mathbb{J}^{-1}(\tau - C(\eta, \dot{\eta})\dot{\eta}).
$$

Finally we obtain

$$
\begin{aligned}
m\ddot{x} &= -u\sin\theta, && (6) \\
m\ddot{y} &= u\cos\theta\sin\phi, && (7) \\
m\ddot{z} &= u\cos\theta\cos\phi - mg, && (8) \\
\ddot{\psi} &= \tilde{\tau}_\psi, && (9) \\
\ddot{\theta} &= \tilde{\tau}_\theta, && (10) \\
\ddot{\phi} &= \tilde{\tau}_\phi, && (11)
\end{aligned}
$$

where x and y are coordinates in the horizontal plane, z is the vertical position, and $\tilde{\tau}_\psi$, $\tilde{\tau}_\theta$, and $\tilde{\tau}_\phi$ are the yawing moment, pitching moment, and rolling moment, respectively, which are related to the generalized torques τ_ψ, τ_θ, τ_ϕ.

Note that for η coordinates, the nonlinear model is

$$
\mathbb{J}\ddot{\eta} + C(\eta, \dot{\eta})\dot{\eta} = \tau
$$

and this model can be written in the general form as

$$
M(\eta)\ddot{\eta} + C(\eta, \dot{\eta})\dot{\eta} = \tau
$$

where $M(\eta) = \mathbb{J}(\eta) = W_\eta^T \mathbf{I} W_\eta$.

$$
\begin{aligned}
\mathbb{J}(\eta) &= W_\eta^T \mathbf{I} W_\eta \\
&= \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & -\sin\phi & 0 \end{bmatrix}^T \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\theta\cos\phi & -\sin\phi & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -I_{xx}\sin\theta & 0 & I_{xx} \\ I_{yy}\cos\theta\sin\phi & I_{yy}\cos\phi & 0 \\ I_{zz}\cos\theta\cos\phi & -I_{zz}\sin\phi & 0 \end{bmatrix} \\
&= \begin{bmatrix} I_{xx}s^2\theta + I_{yy}c^2\theta s^2\phi + I_{zz}c^2\theta c^2\phi & c\theta c\phi s\phi(I_{yy} - I_{zz}) & -I_{xx}s\theta \\ c\theta c\phi s\phi(I_{yy} - I_{zz}) & I_{yy}c^2\phi + I_{zz}s^2\phi & 0 \\ -I_{xx}s\theta & 0 & I_{xx} \end{bmatrix} \qquad (12)
\end{aligned}
$$

To obtain de Coriolis matrix, we use

$$
\frac{d}{dt}\left[\frac{\partial L_{\text{rot}}}{\partial \dot{\eta}}\right] - \frac{\partial L_{\text{rot}}}{\partial \eta} = \tau,
$$

or

$$\frac{d}{dt}\left[\Omega^T \mathbf{I} \frac{\partial \Omega}{\partial \dot{\eta}}\right] - \Omega^T \mathbf{I} \frac{\partial \Omega}{\partial \eta} = \tau,$$

then

$$\frac{\partial \Omega}{\partial \dot{\eta}} = \begin{bmatrix} -s_\theta & 0 & 1 \\ c_\theta s_\phi & c_\phi & 0 \\ c_\theta c_\phi & -s_\phi & 0 \end{bmatrix}$$

thus

$$\begin{aligned}
\Omega^T \mathbf{I} \frac{\partial \Omega}{\partial \dot{\eta}} &= \begin{pmatrix} \dot{\phi} - \dot{\psi} s_\theta \\ \dot{\theta} c_\phi + \dot{\psi} c_\theta s_\phi \\ \dot{\psi} c_\theta c_\phi - \dot{\theta} s_\phi \end{pmatrix}^T \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} -s_\theta & 0 & 1 \\ c_\theta s_\phi & c_\phi & 0 \\ c_\theta c_\phi & -s_\phi & 0 \end{bmatrix} \\
&= \begin{bmatrix} \dot{\phi} - \dot{\psi} s_\theta & \dot{\theta} c_\phi + \dot{\psi} c_\theta s_\phi & \dot{\psi} c_\theta c_\phi - \dot{\theta} s_\phi \end{bmatrix} \begin{bmatrix} -I_{xx} s_\theta & 0 & I_{xx} \\ I_{yy} c_\theta s_\phi & I_{yy} c_\phi & 0 \\ I_{zz} c_\theta c_\phi & -I_{zz} s_\phi & 0 \end{bmatrix} \\
&= \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= -I_{xx}(\dot{\phi} s_\theta - \dot{\psi} s_\theta^2) + I_{yy}(\dot{\theta} c_\theta s_\phi c_\phi + \dot{\psi} c_\theta^2 s_\phi^2) + I_{zz}(\dot{\psi} c_\theta^2 c_\phi^2 - \dot{\theta} c_\theta s_\phi c_\phi) \\
b_2 &= I_{yy}(\dot{\theta} c_\phi^2 + \dot{\psi} c_\theta s_\phi c_\phi) - I_{zz}(\dot{\psi} c_\theta s_\phi c_\phi - \dot{\theta} s_\phi^2) \\
b_3 &= I_{xx}(\dot{\phi} - \dot{\psi} s_\theta)
\end{aligned}$$

Differentiating $\Omega^T \mathbf{I} \frac{\partial \Omega}{\partial \dot{\eta}}$, we obtain

$$\begin{aligned}
\dot{b}_1 &= -I_{xx}(\ddot{\phi} s_\theta + \dot{\phi}\dot{\theta} c_\theta - \ddot{\psi} s_\theta^2 - 2\dot{\psi}\dot{\theta} s_\theta c_\theta) + I_{yy}(\ddot{\theta} c_\theta s_\phi c_\phi - \dot{\theta}^2 s_\theta s_\phi c_\phi - \dot{\theta}\dot{\phi} c_\theta s_\phi^2 + \dot{\theta}\dot{\phi} c_\theta c_\phi^2 + \ddot{\psi} c_\theta^2 s_\phi^2 \\
&\quad -2\dot{\psi}\dot{\theta} s_\theta c_\theta s_\phi^2 + 2\dot{\psi}\dot{\phi} c_\theta^2 s_\phi c_\phi) + I_{zz}(\ddot{\psi} c_\theta^2 c_\phi^2 - 2\dot{\psi}\dot{\theta} s_\theta c_\theta c_\phi^2 - 2\dot{\psi}\dot{\phi} c_\theta^2 s_\phi c_\phi - \ddot{\theta} c_\theta s\phi c_\phi + \dot{\theta}^2 s_\theta s_\phi c_\phi \\
&\quad +\dot{\theta}\dot{\phi} c_\theta c_\phi^2 - \dot{\theta}\dot{\phi} c_\theta c_\phi^2) \\
\dot{b}_2 &= I_{yy}(\ddot{\theta} c_\phi^2 - 2\dot{\theta}\dot{\phi} s_\phi c_\phi + \ddot{\psi} c_\theta s_\phi c_\phi - \dot{\psi}\dot{\theta} s_\theta s_\phi c_\phi + \dot{\psi}\dot{\phi} c_\theta c_\phi^2 - \dot{\psi}\dot{\phi} c_\theta s_\phi^2) - I_{zz}(\ddot{\psi} c_\theta s_\phi c_\phi - \dot{\psi}\dot{\theta} s_\theta s_\phi c_\phi \\
&\quad -\dot{\psi}\dot{\phi} c_\theta s_\phi^2 + \dot{\psi}\dot{\phi} c_\theta c_\phi^2 - \ddot{\theta} s_\phi^2 - 2\dot{\theta}\dot{\phi} s_\phi c_\phi) \\
\dot{b}_3 &= I_{xx}(\ddot{\phi} - \ddot{\psi} s_\theta - \dot{\psi}\dot{\theta} c_\theta)
\end{aligned}$$

In another hand, we have

$$\frac{\partial \Omega}{\partial \eta} = \begin{bmatrix} 0 & -\dot{\psi} c_\theta & 0 \\ 0 & -\dot{\psi} s_\theta s_\phi & -\dot{\theta} s_\phi + \dot{\psi} c_\theta c_\phi \\ 0 & -\dot{\psi} s_\theta c_\phi & -\dot{\psi} c_\theta s_\phi - \dot{\theta} c_\phi \end{bmatrix}$$

then

$$\begin{aligned}
\Omega^T \mathbf{I} \frac{\partial \Omega}{\partial \eta} &= \begin{bmatrix} \dot{\phi} - \dot{\psi} s_\theta \\ \dot{\theta} c_\phi + \dot{\psi} c_\theta s_\phi \\ \dot{\psi} c_\theta c_\phi - \dot{\theta} s_\phi \end{bmatrix}^T \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 0 & -\dot{\psi} c_\theta & 0 \\ 0 & -\dot{\psi} s_\theta s_\phi & -\dot{\theta} s_\phi + \dot{\psi} c_\theta c_\phi \\ 0 & -\dot{\psi} s_\theta c_\phi & -\dot{\psi} c_\theta s_\phi - \dot{\theta} c_\phi \end{bmatrix} \\
&= \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
h_1 &= 0 \\
h_2 &= -I_{xx}(\dot\psi\dot\phi c_\theta - \dot\psi^2 s_\theta c_\theta) - I_{yy}(\dot\psi\dot\theta s_\theta s_\phi c_\phi + \dot\psi^2 s_\theta c_\theta s_\phi^2) - I_{zz}(\dot\psi^2 s_\theta c_\theta c_\phi^2 - \dot\psi\dot\theta s_\theta s_\phi c_\phi) \\
h_3 &= I_{yy}(-\dot\theta^2 s_\phi c_\phi - \dot\psi\dot\theta c_\theta s_\phi^2 + \dot\psi\dot\theta c_\theta c_\phi^2 + \dot\psi^2 c_\theta^2 s_\phi c_\phi) \\
&\quad + I_{zz}(-\dot\psi^2 c_\theta^2 s_\phi c_\phi + \dot\psi\dot\theta c_\theta s_\phi^2 - \dot\psi\dot\theta c_\theta c_\phi^2 + \dot\theta^2 s_\phi c_\phi)
\end{aligned}$$

The Euler-Lagrange equations for the torques is

$$\tau = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} \dot b_1 - h_1 \\ \dot b_2 - h_2 \\ \dot b_3 - h_3 \end{bmatrix}$$

where

$$\begin{aligned}
\tau_1 &= -I_{xx}(\ddot\phi s_\theta + \dot\phi\dot\theta c_\theta - \ddot\psi s_\theta^2 - 2\dot\psi\dot\theta s_\theta c_\theta) + I_{yy}(\ddot\theta c_\theta s_\phi c_\phi - \dot\theta^2 s_\theta s_\phi c_\phi - \dot\theta\dot\phi c_\theta s_\phi^2 + \dot\theta\dot\phi c_\theta c_\phi^2 + \ddot\psi c_\theta^2 s_\phi^2 \\
&\quad -2\dot\psi\dot\theta s_\theta c_\theta s_\phi^2 + 2\dot\psi\dot\phi c_\theta^2 s_\phi c_\phi) + I_{zz}(\ddot\psi c_\theta^2 c_\phi^2 - 2\dot\psi\dot\theta s_\theta c_\theta c_\phi^2 - 2\dot\psi\dot\phi c_\theta^2 s_\phi c_\phi - \ddot\theta c_\theta s\phi c_\phi + \dot\theta^2 s_\theta s_\phi c_\phi \\
&\quad +\dot\theta\dot\phi c_\theta s_\phi^2 - \dot\theta\dot\phi c_\theta c_\phi^2) \\
\tau_2 &= I_{xx}(\dot\psi\dot\phi c_\theta - \dot\psi^2 s_\theta c_\theta) + I_{yy}(\ddot\theta c_\phi^2 - 2\dot\theta\dot\phi s_\phi c_\phi + \ddot\psi c_\theta s_\phi c_\phi + \dot\psi\dot\phi c_\theta c_\phi^2 - \dot\psi\dot\phi c_\theta s_\phi^2 + \dot\psi^2 s_\theta c_\theta s_\phi^2) \\
&\quad -I_{zz}(\ddot\psi c_\theta s_\phi c_\phi - \dot\psi^2 s_\theta c_\theta c_\phi^2 - \dot\psi\dot\phi c_\theta s_\phi^2 + \dot\psi\dot\phi c_\theta c_\phi^2 - \ddot\theta s_\phi^2 - 2\dot\theta\dot\phi s_\phi c_\phi) \\
\tau_3 &= I_{xx}(\ddot\phi - \ddot\psi s_\theta - \dot\psi\dot\theta c_\theta) - I_{yy}(-\dot\theta^2 s_\phi c_\phi - \dot\psi\dot\theta c_\theta s_\phi^2 + \dot\psi\dot\theta c_\theta c_\phi^2 + \dot\psi^2 c_\theta^2 s_\phi c_\phi) \\
&\quad -I_{zz}(-\dot\psi^2 c_\theta^2 s_\phi c_\phi + \dot\psi\dot\theta c_\theta s_\phi^2 - \dot\psi\dot\theta c_\theta c_\phi^2 + \dot\theta^2 s_\phi c_\phi)
\end{aligned}$$

or

$$\begin{aligned}
\tau_1 &= \ddot\psi(I_{xx}s_\theta^2 + I_{yy}c_\theta^2 s_\phi^2 + I_{zz}c_\theta^2 c_\phi^2) + \ddot\theta(I_{yy}c_\theta s_\phi c_\phi - I_{zz}c_\theta s_\phi c_\phi) - \ddot\phi I_{xx}s_\theta \\
&\quad +\dot\psi(I_{xx}\dot\theta s_\theta c_\theta + I_{yy}(-\dot\theta s_\theta c_\theta s_\phi^2 + \dot\phi c_\theta^2 s_\phi c_\phi) - I_{zz}(\dot\theta s_\theta c_\theta c_\phi^2 + \dot\phi c_\theta^2 s_\phi c_\phi)) \\
&\quad +\dot\theta(I_{xx}\dot\psi s_\theta c_\theta - I_{yy}(\dot\theta s_\theta s_\phi c_\phi + \dot\phi c_\theta s_\phi^2 - \dot\phi c_\theta c_\phi^2 + \dot\psi s_\theta c_\theta s_\phi^2) \\
&\quad +I_{zz}(\dot\phi c_\theta s_\phi^2 - \dot\phi c_\theta c_\phi^2 - \dot\psi s_\theta c_\theta c_\phi^2 + \dot\theta s_\theta s_\phi c_\phi)) \\
&\quad -\dot\phi(I_{xx}\dot\theta c_\theta - I_{yy}\dot\psi c_\theta^2 s_\phi c_\phi + I_{zz}\dot\psi c_\theta^2 s_\phi c_\phi) \\
\tau_2 &= \ddot\psi(I_{yy}c_\theta s_\phi c_\phi - I_{zz}c_\theta s_\phi c_\phi) + \ddot\theta(I_{yy}c_\phi^2 + I_{zz}s_\phi^2) \\
&\quad +\dot\psi(-I_{xx}\dot\psi s_\theta c_\theta + I_{yy}\dot\psi s_\theta c_\theta s_\phi^2 + I_{zz}\dot\psi s_\theta c_\theta c_\phi^2) \\
&\quad -\dot\theta(I_{yy}\dot\phi s_\phi c_\phi - I_{zz}\dot\phi s_\phi c_\phi) \\
&\quad +\dot\phi(I_{xx}\dot\psi c_\theta + I_{yy}(-\dot\theta s_\phi c_\phi + \dot\psi c_\theta c_\phi^2 - \dot\psi c_\theta s_\phi^2) + I_{zz}(\dot\psi c_\theta s_\phi^2 - \dot\psi c_\theta c_\phi^2 + \dot\theta s_\phi c_\phi)) \\
\tau_3 &= -\ddot\psi I_{xx}s_\theta + \ddot\phi I_{xx} \\
&\quad -\dot\psi(I_{yy}\dot\psi c_\theta^2 s_\phi c_\phi - I_{zz}\dot\psi c_\theta^2 s_\phi c_\phi) \\
&\quad +\dot\theta(-I_{xx}\dot\psi c_\theta + I_{yy}(\dot\theta s_\phi c_\phi + \dot\psi c_\theta s_\phi^2 - \dot\psi c_\theta c_\phi^2) - I_{zz}(\dot\psi c_\theta s_\phi^2 - \dot\psi c_\theta c_\phi^2 + \dot\theta s_\phi c_\phi))
\end{aligned}$$

The non linear model can be written as

$$M(\eta)\ddot\eta + C(\eta,\dot\eta)\dot\eta = \tau$$

where

$$M(\eta) = \begin{bmatrix} I_{xx}s_\theta^2 + I_{yy}c_\theta^2 s_\phi^2 + I_{zz}c_\theta^2 c_\phi^2 & c_\theta c_\phi s_\phi(I_{yy} - I_{zz}) & -I_{xx}s_\theta \\ c_\theta c_\phi s_\phi(I_{yy} - I_{zz}) & I_{yy}c_\phi^2 + I_{zz}s_\phi^2 & 0 \\ -I_{xx}s_\theta & 0 & I_{xx} \end{bmatrix} \quad (13)$$

and

$$C(\eta, \dot{\eta}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

where

$$\begin{aligned}
c_{11} &= I_{xx}\dot{\theta}s_\theta c_\theta + I_{yy}(-\dot{\theta}s_\theta c_\theta s_\phi^2 + \dot{\phi}c_\theta^2 s_\phi c_\phi) - I_{zz}(\dot{\theta}s_\theta c_\theta c_\phi^2 + \dot{\phi}c_\theta^2 s_\phi c_\phi) \\
c_{12} &= I_{xx}\dot{\psi}s_\theta c_\theta - I_{yy}(\dot{\theta}s_\theta s_\phi c_\phi + \dot{\phi}c_\theta s_\phi^2 - \dot{\phi}c_\theta c_\phi^2 + \dot{\psi}s_\theta c_\theta s_\phi^2) \\
&\quad + I_{zz}(\dot{\phi}c_\theta s_\phi^2 - \dot{\phi}c_\theta c_\phi^2 - \dot{\psi}s_\theta c_\theta c_\phi^2 + \dot{\theta}s_\theta s_\phi c_\phi) \\
c_{13} &= -I_{xx}\dot{\theta}c_\theta + I_{yy}\dot{\psi}c_\theta^2 s_\phi c_\phi - I_{zz}\dot{\psi}c_\theta^2 s_\phi c_\phi \\
c_{21} &= -I_{xx}\dot{\psi}s_\theta c_\theta + I_{yy}\dot{\psi}s_\theta c_\theta s_\phi^2 + I_{zz}\dot{\psi}s_\theta c_\theta c_\phi^2 \\
c_{22} &= -I_{yy}\dot{\phi}s_\phi c_\phi + I_{zz}\dot{\phi}s_\phi c_\phi \\
c_{23} &= I_{xx}\dot{\psi}c_\theta + I_{yy}(-\dot{\theta}s_\phi c_\phi + \dot{\psi}c_\theta c_\phi^2 - \dot{\psi}c_\theta s_\phi^2) + I_{zz}(\dot{\psi}c_\theta s_\phi^2 - \dot{\psi}c_\theta c_\phi^2 + \dot{\theta}s_\phi c_\phi) \\
c_{31} &= -I_{yy}\dot{\psi}c_\theta^2 s_\phi c_\phi + I_{zz}\dot{\psi}c_\theta^2 s_\phi c_\phi \\
c_{32} &= -I_{xx}\dot{\psi}c_\theta + I_{yy}(\dot{\theta}s_\phi c_\phi + \dot{\psi}c_\theta s_\phi^2 - \dot{\psi}c_\theta c_\phi^2) - I_{zz}(\dot{\psi}c_\theta s_\phi^2 - \dot{\psi}c_\theta c_\phi^2 + \dot{\theta}s_\phi c_\phi) \\
c_{33} &= 0
\end{aligned}$$

Note that $M(\eta)$ obtained in (12) and (13) are the same and is a symmetric matrix, such that

$$\begin{aligned}
\det(M(\eta)) &= (I_{xx}s_\theta^2 + I_{yy}c_\theta^2 s_\phi^2 + I_{zz}c_\theta^2 c_\phi^2)(I_{xx}I_{yy}c_\phi^2 + I_{xx}I_{zz}s_\phi^2) \\
&\quad - c_\theta^2 c_\phi^2 s_\phi^2 (I_{yy} - I_{zz})^2 I_{xx} - I_{xx}^2 s_\theta^2 (I_{yy}c_\phi^2 + I_{zz}s_\phi^2) \\
&= I_{xx}^2 I_{yy} s_\theta^2 c_\phi^2 + I_{yy}^2 I_{xx} c_\theta^2 s_\phi^2 c_\phi^2 + I_{zz}I_{xx}I_{yy}c_\phi^4 c_\theta^2 + I_{xx}^2 I_{zz} s_\theta^2 s_\phi^2 \\
&\quad + I_{zz}^2 I_{xx} c_\theta^2 s_\phi^2 c_\phi^2 + I_{zz}I_{xx}I_{yy}c_\theta^2 s_\phi^4 - c_\theta^2 c_\phi^2 s_\phi^2 (I_{yy} - I_{zz})^2 I_{xx} \\
&\quad - I_{xx}^2 s_\theta^2 (I_{yy}c_\phi^2 + I_{zz}s_\phi^2) \\
&= I_{zz}I_{xx}I_{yy}c_\theta^2(s_\phi^4 + c_\phi^4) + (I_{yy}^2 + I_{zz}^2 - (I_{yy} - I_{zz})^2) I_{xx} c_\theta^2 s_\phi^2 c_\phi^2 \\
&= I_{zz}I_{xx}I_{yy}c_\theta^2(s_\phi^4 + c_\phi^4) + 2I_{yy}I_{zz}I_{xx}c_\theta^2 s_\phi^2 c_\phi^2 \\
&= I_{zz}I_{xx}I_{yy}c_\theta^2(s_\phi^2 + c_\phi^2)^2 \\
&= I_{zz}I_{xx}I_{yy}c_\theta^2
\end{aligned}$$

Therefore, $M(\eta)$ is positive definite for all $\theta \neq n\pi/2$, $\forall\ n = 1, 3, 5....$

Differentiating M, we have

$$\begin{aligned}
\dot{M}_{11} &= 2I_{xx}\dot{\theta}s_\theta c_\theta - 2I_{yy}\dot{\theta}s_\theta c_\theta s_\phi^2 + 2I_{yy}\dot{\phi}c_\theta^2 s_\phi c_\phi - 2I_{zz}\dot{\theta}s_\theta c_\theta c_\phi^2 - 2I_{zz}\dot{\phi}c_\theta^2 s_\phi c_\phi \\
\dot{M}_{12} &= \dot{M}_{21} = -I_{yy}\dot{\theta}s_\theta s_\phi c_\phi + I_{yy}\dot{\phi}c_\theta c_\phi^2 - I_{yy}\dot{\phi}c_\theta s_\phi^2 + I_{zz}\dot{\theta}s_\theta s_\phi c_\phi - I_{zz}\dot{\phi}c_\theta c_\phi^2 + I_{zz}\dot{\phi}c_\theta s_\phi^2 \\
\dot{M}_{13} &= \dot{M}_{31} = -I_{xx}\dot{\theta}c_\theta \\
\dot{M}_{22} &= -2I_{yy}\dot{\phi}s_\phi c_\phi + 2I_{zz}\dot{\phi}s_\phi c_\phi \\
\dot{M}_{23} &= \dot{M}_{32} = 0 \\
\dot{M}_{33} &= 0
\end{aligned}$$

The matrix

$$P = \dot{M} - 2C$$

is given by

$$
\begin{aligned}
P_{11} &= 0 \\
P_{12} &= -I_{yy}\dot{\phi}c_\theta c_\phi^2 + I_{yy}\dot{\phi}c_\theta s_\phi^2 - I_{zz}\dot{\theta}s_\theta s_\phi c_\phi + I_{zz}\dot{\phi}c_\theta c_\phi^2 - I_{zz}\dot{\phi}c_\theta s_\phi^2 \\
&\quad -2I_{xx}\dot{\psi}s_\theta c_\theta + 2I_{yy}\dot{\psi}s_\theta c_\theta s_\phi^2 + 2I_{zz}\dot{\psi}s_\theta c_\theta c_\phi^2 \\
P_{13} &= I_{xx}\dot{\theta}c_\theta - 2I_{yy}\dot{\psi}c_\theta^2 s_\phi c_\phi + 2I_{zz}\dot{\psi}c_\theta^2 s_\phi c_\phi \\
P_{21} &= -P_{12} \\
P_{22} &= 0 \\
P_{23} &= -2I_{xx}\dot{\psi}c_\theta - 2I_{yy}(-\dot{\theta}s_\phi c_\phi + \dot{\psi}c_\theta c_\phi^2 - \dot{\psi}c_\theta s_\phi^2) - 2I_{zz}(\dot{\psi}c_\theta s_\phi^2 - \dot{\psi}c_\theta c_\phi^2 + \dot{\theta}s_\phi c_\phi) \\
P_{31} &= -P_{13} \\
P_{32} &= -P_{23} \\
P_{33} &= 0
\end{aligned}
$$

which is skew-symmetric, this is

$$
P = \begin{bmatrix} 0 & P_{12} & P_{13} \\ -P_{12} & 0 & P_{23} \\ -P_{13} & -P_{32} & 0 \end{bmatrix}
$$

1.2. Newton-Euler approach

The general motion of a rigid body in space is a combination of translational and rotational motions.

Consider a rigid body moving in inertial space. The body is undergoing both rotations and translations.

Let us consider earth fixed frame $\mathcal{I}$ and body fixed frame $\mathcal{A}$, as seen in Figure 3. The center of mass and the body fixed frame origin are assumed to coincide. Using Euler angles parametrization, the airframe orientation in space is given by a rotation $\mathcal{R}$ from $\mathcal{A}$ to $\mathcal{I}$, where $\mathcal{R} \in \mathcal{SO}3$ is the rotation matrix. The dynamics of a rigid body under external forces applied to the center of mass and expressed in earth fixed frame are in Newton-Euler formalism:

$$\begin{aligned} \dot{\xi} &= v \\ m\dot{v} &= f \\ \dot{\mathcal{R}} &= \mathcal{R}\hat{\Omega} \\ \mathbf{I}\dot{\Omega} &= -\Omega \times \mathbf{I}\Omega + \tau \end{aligned} \tag{14}$$

where $\xi = (x, y, z)^T$ denote the position of the centre of mass of the airframe in the frame $\mathcal{I}$ relative to a fixed origin, $v \in \mathcal{I}$ denote the linear velocity expressed in the inertial frame, $\Omega \in \mathcal{A}$ denote the angular velocity of the airframe expressed in the body fixed frame. m denote the mass of the rigid object and $\mathbf{I} \in \mathbb{R}^{3\times 3}$ denote the constant inertia matrix around the centre of mass (expressed in the body fixed frame $\mathcal{A}$). $\hat{\omega}$ denotes the skew-symmetric matrix of the vector ω. $f \in \mathcal{I}$ represents the vector of the principal non-conservative forces applied to the object; including thrusts T_f and drag terms associated with the rotors. $\tau \in \mathcal{A}$ is derived from differential thrust associated with pairs of rotors along with aerodynamic effects and gyroscopic effects.

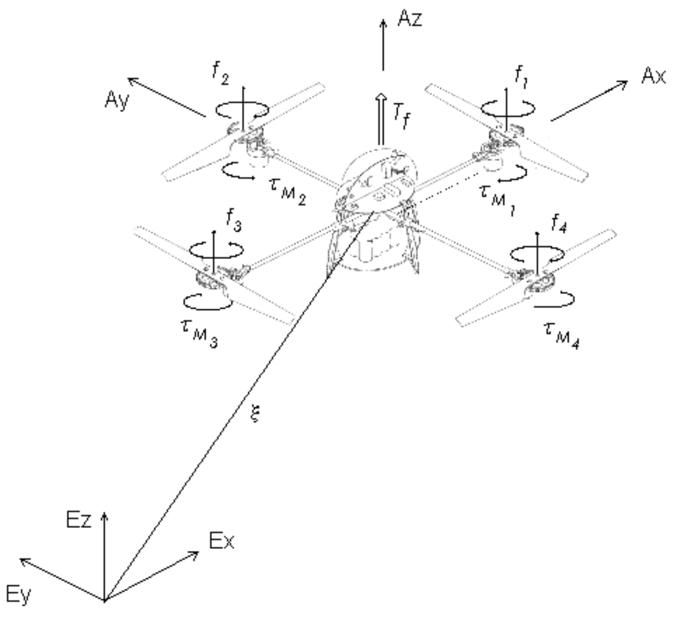

Figura 4: The quad-rotor in an inertial frame. f_i represent the thrust of motor i and T_f is the main thrust.

1.2.1. Translational force and gravitational force

The only forces acting in the corps are given by the translational force T_f and the gravitational force g. The thrust T_f applied to the vehicle is (see Figure 4)

$$T_f = \sum_{i=1}^{4} f_i \tag{15}$$

where the lift f_i generated by a rotor in free air can be modelled as $f_i = k\omega_i^2$ in the z–direction, where $k > 0$ is a constant and ω_i is the angular speed of the ith motor. Rewriting the above, we have

$$T_f = k\left(\sum_{i=1}^{4} \omega_i^2\right). \tag{16}$$

Then

$$F = \begin{bmatrix} 0 \\ 0 \\ T_f \end{bmatrix},$$

The gravitational force applied to the vehicle is

$$f_g = -mgE_z, \tag{17}$$

This yields

$$f = R_{E_z} T_f + f_g. \tag{18}$$

1.2.2. Torques

Due to the rigid rotor constraint the dynamics of each rotor disk around its axis of rotation can be treated as a decoupled system in the generalized variable ω_i denoting angular velocity of a rotor around its axis. The torque exerted by each electrical motor is denoted τ_{M_i}. The motor torque is opposed by an aerodynamic drag $\tau_{drag} = k_\tau\ \omega_i^2$.

Using Newton's second law we have

$$I_M \dot{\omega}_i = -\tau_{drag} + \tau_{M_i},$$

where I_M is the angular moment of the i-th motor and $k_\tau > 0$ is a constant for quasi-stationary manoeuvres in free.

In steady state, that is, when $\dot{\omega}_i = 0$, the yaw torque is

$$\tau_{M_i} = k_\tau \omega_i^2. \tag{19}$$

The torque applied on the vehicle's body along an axis is the difference between the torque generated by each propeller on the other axis. Pitch motion is obtained by increasing the speed of the rear motor M_3 while reducing the speed of the front motor M_1. Likewise, roll motion is obtained using the lateral motors. Yaw motion is obtained by increasing the torque of the front and rear motors, τ_{M_1} and τ_{M_3} respectively, while decreasing the torque of the lateral motors, τ_{M_2} and τ_{M_4}. These motions can be accomplished while keeping the total thrust constant.

The generalized torques are thus

$$\tau_{\mathcal{A}} = \begin{bmatrix} \sum_{i=1}^{4} \tau_{M_i} \\ (f_2 - f_4)\ell \\ (f_3 - f_1)\ell \end{bmatrix} = \begin{bmatrix} \tau_\psi \\ \tau_\theta \\ \tau_\phi \end{bmatrix},$$

where ℓ is the distance between the motors and the center of gravity.

Rewriting the above,

$$\tau_\psi = k_\tau(\omega_1^2 + \omega_3^2 - \omega_2^2 - \omega_4^2) \tag{20}$$

$$\tau_\theta = \ell k(\omega_2^2 - \omega_4^2) \tag{21}$$

$$\tau_\phi = \ell k(\omega_3^2 - \omega_1^2) \tag{22}$$

where τ_ψ, τ_θ, τ_ϕ are the generalized torques (yawing moment, pitching moment, and rolling moment).

Each rotor may be thought of as a rigid disk rotating around the axis E_z in the body-fixed-frame with angular velocity ω_i. The axis of rotation of the rotor is itself moving with the angular velocity of the airframe. This leads to the following gyroscopic torques applied to the airframe

$$\begin{aligned} \tau_{G_{\mathcal{A}}} &= -\sum_{i=1}^{4} I_M(\omega \times E_z)\omega_i \\ &= -(\omega \times E_z)\sum_{i=1}^{4} I_M\omega_i \end{aligned} \tag{23}$$

This yields

$$\tau = \tau_{\mathcal{A}} + \tau_{G_{\mathcal{A}}}. \tag{24}$$

Rewriting (14), we obtain

$$\begin{aligned} \dot{\xi} &= v \\ m\dot{v} &= R_{E_z}T_f - mgE_z \\ \dot{\mathcal{R}} &= \mathcal{R}\hat{\Omega} \\ \mathbf{I}\dot{\Omega} &= -\Omega \times \mathbf{I}\Omega + \tau_{\mathcal{A}} + \tau_{G_{\mathcal{A}}} \end{aligned} \tag{25}$$

1.3. Newton's equation to Lagrange's equations

We use the classical *yaw*, *pitch* and *roll* Euler angles (ψ, θ, ϕ), commonly used in aeronautics applications [4, 5]. The relationship between the Euler angles used and the rotation matrix is

$$R = \begin{pmatrix} c_\theta c_\psi & s_\psi s_\theta & -s_\theta \\ c_\psi s_\theta s_\phi - s_\psi c_\phi & s_\psi s_\theta s_\phi + c_\psi c_\phi & c_\theta s_\phi \\ c_\psi s_\theta c_\phi + s_\psi s_\phi & s_\psi s_\theta c_\phi - c_\psi s_\phi & c_\theta c_\phi \end{pmatrix},$$

we use c_θ for $\cos\theta$ and s_θ for $\sin\theta$.

We can partitioner (25) into the dynamics for the ξ coordinates and the η dynamics. Rewriting for the ξ dynamics, we obtain

$$\ddot{\xi} = \frac{1}{m}(R_{E_z}T_f - gE_z)$$

where

$$R_{E_z} = \begin{pmatrix} -s_\theta \\ c_\theta s_\phi \\ c_\theta c_\phi \end{pmatrix}$$

Seeing Figures 3 and 4 we obtain $u = T_f$, this yields

$$\ddot{x} = -\frac{1}{m}u\sin\theta, \tag{26}$$

$$\ddot{y} = \frac{1}{m}u\cos\theta\sin\phi, \tag{27}$$

$$\ddot{z} = \frac{1}{m}(u\cos\theta\cos\phi - g), \tag{28}$$

Note that ecs. (26)-(28) are the same that ecs. (6)-(8).

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