Quantum Notes

June 3, 2025

Section 1

1. $h = 6.63 \times 10^{-34}$Js, $\hbar = \dfrac{h}{2\pi}$
2. Fine structure constant $\alpha = \dfrac{1}{4\pi\epsilon_0}\dfrac{e^2}{\hbar c} \approx \dfrac{1}{137}$
3. Photoelectric effect $E = h\nu = \hbar\omega$, $eV_0 = h\nu - W$
4. de Broglie wavelength $p = h/\lambda, \mathbf{p} = \hbar\mathbf{k}$
5. Angular momentum of electron $L = n\hbar$, n is the principal quantum number
6. $\omega = \dfrac{\hbar k^2}{2m}$
7. $\Phi(\mathbf{r}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = Ae^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - Et)}$ (not really meaningful unless $|\Phi(\mathbf{r}, t)| \to 0$)
8. A localized wavefunction is $\Phi(x,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} g(k)e^{i(kx-\omega t)}\mathrm{d}k$, $g(k,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{+\infty} \Phi(x,t)e^{-i(kx-\omega t)}\mathrm{d}x$, $\displaystyle\int_{\infty}^{\infty} |g(k,t)|^2\mathrm{d}k = \int_{\infty}^{\infty} |\Phi(x,t)|^2\mathrm{d}x$, $g(k)$ is the momentum wavefunction.
9. $\langle x \rangle = \displaystyle\int_{-\infty}^{+\infty} xP(x)dx = \int_{-\infty}^{+\infty} x|\Psi(x)|^2 dx$
10. $\langle x^2 \rangle = \displaystyle\int_{-\infty}^{+\infty} x^2 P(x)dx = \int_{-\infty}^{+\infty} x^2|\Psi(x)|^2 dx$
11. $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$
12. Wavepacket $g(k) = \left(\dfrac{a^2}{\pi}\right)^{1/4} e^{-a^2(k-k_0)^2/2}$,
    - $\Psi(x, 0) = \left(\dfrac{1}{\pi a^2}\right)^{1/4} e^{ik_0x}e^{-x^2/2a}$
    - $\Psi(x, t) = e^{ik_0(x-v_pt)}f(x - v_gt)$
    - $\langle x \rangle = \left(\dfrac{1}{\pi a^2}\right)^{1/2} \displaystyle\int_{-\infty}^{+\infty} xe^{-(x^2/a^2)}dx = 0$
    - $\langle x^2 \rangle = \left(\dfrac{1}{\pi a^2}\right)^{1/2} \displaystyle\int_{-\infty}^{+\infty} x^2e^{-(x^2/a^2)}dx = \dfrac{a^2}{2}$
    - $\langle k \rangle = \left(\dfrac{a^2}{\pi}\right)^{1/2} \displaystyle\int_{-\infty}^{+\infty} ke^{-a^2(k-k_0)^2}dk = k_0$
    - $\left\langle k^2 \right\rangle = \left(\dfrac{a^2}{\pi}\right)^{1/2} \displaystyle\int_{-\infty}^{+\infty} k^2e^{-a^2(k-k_0)^2}dk = \dfrac{1}{2a^2} + k_0^2$
    - $\Delta x = \dfrac{a}{\sqrt{2}}$ and $\Delta p = \dfrac{\hbar}{a\sqrt{2}}$
    - $\Delta x \Delta p = \dfrac{\hbar}{2}$

13. $\Delta x \Delta p_y = 0$ is possible, spatially orthogonal quantities can be measured simultaneously to any precision

14. Time evolution of a gaussian $\omega = \dfrac{\hbar k^2}{2m}$,$g(k) = \left(\dfrac{a^2}{\pi}\right)^{1/4} e^{-a^2(k-k_0)^2/2}$,

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(k) e^{i[kx-\omega(k)t]} dk \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{a^2}{\pi}\right)^{1/4} \int_{-\infty}^{+\infty} e^{-a^2\delta k^2/2} e^{i\left[(k_0+\delta k)x - \left(\omega_0 + \alpha\delta k + \beta(\delta k)^2\right)t\right]} d(\delta k) \\
&\left(\alpha = \frac{\hbar k_0}{m},\ \beta = \frac{\hbar}{2m}\right) \\
|\Psi(x,t)|^2 &= \left(\frac{a^2}{4\pi}\right)^{1/2} \frac{1}{(a^4/4 + h^2t^2/4m^2)^{1/2}} \exp\left\{\frac{-(x-\hbar k_0 t/m)^2}{(a^2 + h^2t^2/m^2a^2)}\right\} \\
\langle x \rangle &= \hbar k_0 t/m = v_g t \\
(\Delta x)^2 &= a^2/2(1 + \hbar^2 t^2/a^4 m^2)
\end{aligned}$$

15. $\Phi(p,t) = \dfrac{1}{\sqrt{\hbar}} g\left(\dfrac{p}{\hbar}, t\right)$,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t) e^{ipx/\hbar} dp,$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Psi(x,t) e^{-ipx/\hbar} dx$$

Gaussian $\Phi(p,0) = \left(\dfrac{a^2}{\pi\hbar^2}\right)^{1/4} \exp^{-a^2(p-p_0)^2/2\hbar^2}$

16. Total energy operator/Hamiltonian $\hat{E} \equiv i\hbar\dfrac{\partial}{\partial t}$

Total momentum operator $\hat{p} \equiv -i\hbar\dfrac{\partial}{\partial x}$, $\hat{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$

Kinetic energy operator $\dfrac{\hat{p}^2}{2m} \equiv -\dfrac{h^2}{2m}\dfrac{\partial^2}{\partial x^2} = -\dfrac{\hbar^2}{2m}\nabla^2$

The eigenvalues of an operator are the possible values that might be returned by an experiment.

17. $\hat{E} = \dfrac{\hat{p}^2}{2m} + \hat{V}(x,t) = \hat{E}$

$$\boxed{i\hbar\frac{\partial\Psi(x,t)}{\partial t}} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t)$$

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r},t) + V(\mathbf{r},t)\Psi(\mathbf{r},t)$$

Let $\Psi(x,t) = \Psi(x)T(t) = \Psi(x)e^{-iEt/\hbar}$, time-independent Schrödinger is $-\dfrac{h^2}{2m}\nabla^2\Psi(\boldsymbol{r}) + V(\boldsymbol{r})\Psi(\boldsymbol{r}) = E\Psi(\boldsymbol{r})$

18. If $V(\mathbf{r},t) = V(\mathbf{r})$, $\Psi(\mathbf{r},t) = \psi(\mathbf{r})T(t)$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

19. $i\hbar\dfrac{\partial}{\partial t}\displaystyle\int_{\mathcal{V}} P(\mathbf{r},t)dV = i\hbar\dfrac{\partial}{\partial t}\displaystyle\int_{\mathcal{V}} \Psi^*(\mathbf{r},t)\Psi(\mathbf{r},t)dV$

$$\boxed{\frac{\partial}{\partial t}\int_{\mathcal{V}} P(\mathbf{r},t)dV = -\oint_S \mathbf{J}(\mathbf{r},t)\cdot d\mathbf{s}}$$

$$\frac{\partial P}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0$$

$$\begin{aligned}
\mathbf{J}(\mathbf{r},t) &= \frac{\hbar}{i2m}\left[\Psi^*\big(\mathbf{r},t\big)\nabla\Psi(\mathbf{r},t) - \Psi(\mathbf{r},t)\nabla\Psi^*\big(\mathbf{r},t\big)\right] \\
&= \mathrm{Re}[\Psi^\star(\mathbf{r},t)\frac{\hbar}{im}\boldsymbol{\nabla}\Psi(\mathbf{r},t)] \\
&= \mathrm{Re}[\Psi^\star(\mathbf{r},t)\frac{\hat{\mathbf{p}}}{m}\boldsymbol{\nabla}\Psi(\mathbf{r},t)]
\end{aligned}$$

For $\Psi(x,t) = Ae^{i(kx-\omega t)}$, $\boxed{J = v|A|^2 = \frac{\hbar k}{m}|A|^2}$

20. SHO $V(x) = \frac{1}{2}m\omega^2x^2$, $q = x\sqrt{\frac{m\omega}{\hbar}}$, $\epsilon = \frac{2E}{\hbar\omega}$, $\chi(q) = \psi(q\sqrt{\frac{\hbar}{m\omega}}) = H(q)e^{-q^2/2}$

$$\begin{aligned}
&\frac{\partial^2 H(q)}{\partial q^2} - 2q\frac{\partial H}{\partial q} + (\epsilon - 1)H(q) = 0 \\
&H_n(q) = (-1)^n e^{q^2}\frac{\mathrm{d}^n}{\mathrm{d}q^n}\left(e^{-q^2}\right) \\
&E_n = \left(n + \frac{1}{2}\right)\hbar\omega \\
&\psi_n = A_n H_n\left[x\sqrt{\frac{m\omega}{\hbar}}\right]e^{-\frac{m\omega}{2\hbar}x^2} \\
&A_n = \left[\sqrt{\frac{\hbar}{m\omega}}\sqrt{\pi}2^n n!\right]^{-1/2}
\end{aligned}$$

21. Parity operator $\hat{P}\psi(\mathbf{r}) = \psi(-\mathbf{r})$
22. For bounded systems, $psi(x) = \sum_{n=1}^{\infty} a_n\psi_n(x)$, $|a\rangle = \sum_n a_n |a\rangle_n$
23. Completeness relation (bounded system) $\sum_{n=1}^{\infty}\psi_n(x)\psi_n^*(y) = \delta(x-y)$, $\sum_{n=1}^{N}\mathbf{a}_n\mathbf{a}_n^\dagger = I$, $\sum_n |a\rangle\langle a| = \hat{I}$
24. For unbounded systems, $\psi(x) = \int \phi(k)\chi(x,k)\mathrm{d}k$, $\int_{-\infty}^{\infty}\chi^*(k,x_1)\chi(k,x_2)\mathrm{d}k = \delta(x_2 - x_1)$
25. Commutator $\left[\hat{A},\hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$
26. $A_{nm} \equiv \langle u_n|\hat{A}|u_m\rangle$, $\hat{A} \equiv \sum_{n,m} A_{nm}|u_n\rangle\langle u_m|$ (opeartor as linear combination of outer products)
27. Operators like $\hat{P} = |\psi\rangle\langle\psi|$ is a projection operator
28. The adjoint of an operator is $\hat{A}^\dagger$
    - $\langle u_m|\hat{A}|u_n\rangle = \langle u_n|\hat{A}^\dagger|u_m\rangle^*$
    - Thus $\hat{A}|\psi\rangle = |\phi\rangle \implies \langle\psi|\hat{A}^\dagger = \langle\phi|$
    - $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$
    - Observables are self-adjoint, $\hat{A} = \hat{A}^\dagger$
29. Operators are Hermitian, their expectation values ($\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle$) are real
    - $\hat{x} \equiv x$, position operator
    - $\hat{p} \equiv -i\hbar\frac{\partial}{\partial x}$, momentum operator

- $\dfrac{\hat{p}^2}{2m} \equiv -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}$, kinetic energy operator
- $\hat{E} \equiv i\hbar\dfrac{\partial}{\partial t}$, total energy operator
- $\hat{H} \equiv -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x)$, Hamiltonian

For any observable A there is an associated operator $\hat{A}$ which denotes the act of measurement. The measurements have **0 uncertainty**

30. $[\hat{x}, \hat{p}] = i\hbar$
31. $(\hat{A}\hat{B})^\dagger = \hat{A}\hat{B}$ iff $\hat{A}^\dagger = \hat{A}, \hat{B}^\dagger = \hat{B}, [\hat{A}, \hat{B}] = 0$ (A composite operator is an observable iff its parts are **commute observables**)
32. If $[\hat{A}, \hat{B}] = 0$, they share common eigenvectors; commute means compatible, non-commute means incompatible; compatible operators can be calculated once one of them is measured (e.g. $\hat{p}, \dfrac{\hat{p}^2}{2m}$)
33. The anti-commutator $\{\hat{A}, \hat{B}\}$ of two observables is an observable; the commutator of two observables is anti-Hermitian, so no; however $i[\hat{A}, \hat{B}]$ is an observable
34. $\forall\psi(\langle\psi|0\rangle) = 0$
35. $\boxed{\Delta A \Delta B \geq \frac{1}{2}\left|\langle i\left[\hat{A}, \hat{B}\right]\rangle\right|}$, Deviation $\hat{A}_d = \hat{A} - \langle A\rangle$, $\langle\hat{A}_d^2\rangle = (\Delta A)^2$, $\left[\hat{A}_d, \hat{B}_d\right] = \left[\hat{A}, \hat{B}\right]$, minimize $\langle\phi|\phi\rangle \geq 0$, where $|\phi\rangle = (\hat{A}_d + i\lambda\hat{B}_d)\,|\psi\rangle$
36. Amplitude of normal mode $a(t) = \sqrt{\dfrac{m\omega}{2}}x + i\dfrac{1}{\sqrt{2m\omega}}p$, $\omega a a^* = \dfrac{m\omega^2}{2}x^2 + \dfrac{p^2}{2m} = E$

    - For $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{p}{m}$, $\dfrac{\mathrm{d}p}{\mathrm{d}t} = -m\omega^2 x$
    - $\dfrac{\mathrm{d}a}{\mathrm{d}t} = -i\omega a$
    - $a(t) = a(0)e^{-i\omega t}$,$a(0)\left[\dfrac{m\omega}{2}x(0) + \dfrac{i}{\sqrt{2m\omega}}p(0)\right]$
    - x and p in complex space **scaled** into a circle of radius a
    - $\boxed{\hat{a} = \sqrt{\dfrac{m\omega}{2\hbar}}\hat{x} + i\dfrac{1}{\sqrt{2m\hbar\omega}}\hat{p}}$, $\hat{a}^\dagger = \sqrt{\dfrac{m\omega}{2\hbar}}\hat{x} - i\dfrac{1}{\sqrt{2m\hbar\omega}}\hat{p}$, not Hermitian/observables
        - $\boxed{\left[\hat{a}, \hat{a}^\dagger\right] = 1}$
        - $\left\{\hat{a}, \hat{a}^\dagger\right\} = \dfrac{2}{\hbar\omega}\hat{H}$
        - $\hat{H} = \dfrac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger) = \hbar\omega(\hat{a}^\dagger\hat{a} + \dfrac{1}{2}) = \hbar\omega(\hat{a}\hat{a}^\dagger - \dfrac{1}{2})$
        - $\boxed{\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \dfrac{1}{2}) = \hbar\omega(\hat{N} + \dfrac{1}{2})}$
        - Remember $\boxed{\dfrac{\hbar\omega}{2}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger) = \hat{H} = \dfrac{\hat{p}^2}{2m} + \dfrac{m\omega^2\hat{x}^2}{2}}$ to find out $\hat{a}$ and $\hat{a}^\dagger$
        - Creation/raising operator $\hat{a}^\dagger$: $\boxed{\hat{H}\hat{a}\,|\phi\rangle = (E - \hbar\omega)\hat{a}\,|\phi\rangle}$
        - Annihilation/lowering operator $\hat{a}$: $\boxed{\hat{H}\hat{a}^\dagger\,|\phi\rangle = (E + \hbar\omega)\hat{a}^\dagger\,|\phi\rangle}$
        - The number operator $\boxed{\hat{N} = \hat{a}^\dagger\hat{a}}$ is an observable
        - $\hat{a}\,|\phi_0\rangle = |0\rangle$

37.

$$\boxed{\begin{aligned} \hat{N}\left|\phi_n\right\rangle &= n\left|\phi_n\right\rangle \\ \hat{a}\left|\phi_0\right\rangle &= \left|0\right\rangle \\ \hat{a}\left|\phi_n\right\rangle &= \sqrt{n}\left|\phi_{n-1}\right\rangle \\ \hat{a}^\dagger\left|\phi_n\right\rangle &= \sqrt{n+1}\left|\phi_{n+1}\right\rangle \\ \left\langle\phi_n\right|\hat{a}^\dagger\hat{a}\left|\phi_n\right\rangle &= \left\langle\phi_n\right|\hat{N}\left|\phi_n\right\rangle = n\left\langle\phi_n|\phi_n\right\rangle = n = |c_n|^2\left\langle\phi_{n-1}|\phi_{n-1}\right\rangle = |c_n|^2 \\ \left|\phi_n\right\rangle &= \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\left|\phi_0\right\rangle \Delta x\Delta p = \left(n+\frac{1}{2}\right)\hbar \end{aligned}}$$

38. $\langle A\rangle = \mathrm{Tr}\left[\hat{A}\left|\psi\right\rangle\left\langle\psi\right|\right]$

39. The density operator $\boxed{\hat{O} = \sum_{i=1}^{n} P_i\left|\psi_i\right\rangle\left\langle\psi_i\right|}$, $\langle A\rangle = \sum_{i=1}^{n} P_i\left\langle\psi_i\right|\hat{A}\left|\psi_i\right\rangle = \mathrm{Tr}\left[\sum_{i=1}^{n} P_i\left|\psi_i\right\rangle\left\langle\psi_i\right|\hat{A}\right] = \mathrm{Tr}\left[\hat{O}\hat{A}\right]$, $\boxed{\langle\hat{A}\rangle = \mathrm{Tr}\left[\hat{O}\hat{A}\right]}$, $\boxed{\mathrm{Tr}\left[\hat{O}\right] = 1}$

40. $\hat{O}^\dagger = \hat{O}$

41. $\hat{O}\hat{O} = \sum_{i=0}^{n} P_i^2\left|\psi_i\right\rangle\left\langle\psi_i\right|$. If $\hat{O} = \left|\psi_i\right\rangle\left\langle\psi_i\right|$, $\hat{O}$ is in a pure state.

42. $\hat{X}^n = \sum_i e^{-x_i}\left|\psi_i\right\rangle\left\langle\psi_i\right|$, $F(\hat{X}) = \sum_i F(x_i)\left|\psi_i\right\rangle\left\langle\psi_i\right|$

43. For two particles A and B, the reduced density operator $\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \mathrm{Tr}_B(\left|a_i\right\rangle\left\langle a_j\right| \otimes \left|b_l\right\rangle\left\langle b_m\right|) = \left|a_i\right\rangle\left\langle a_j\right|\mathrm{Tr}(\left|b_l\right\rangle\left\langle b_m\right|) = \left|a_i\right\rangle\left\langle a_j\right| \otimes I_B$ is the partial trace. (Only trace the second qubit)

44. Harmonic oscillator weakly coupled to temperature T: $\hat{\rho} = \frac{1}{Z}e^{-\hat{H}/kT}$, $Z = \mathrm{Tr}\left[e^{-\hat{H}/kT}\right]$, $\mathrm{Tr}(\hat{\rho}) = 1$, $\langle\hat{H}\rangle = \mathrm{Tr}\left[\hat{\rho}\hat{H}\right] = \hbar\omega\left[\frac{1}{2} + \frac{1}{e^{\hbar\omega/kT}-1}\right]$

45. $\left|\psi(t)\right\rangle = \sum_n c_n(t)\left|\phi_n\right\rangle$, $i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle = \hat{H}\left|\psi\right\rangle$, $\sum_n\left[i\hbar\frac{\mathrm{d}c_n(t)}{\mathrm{d}t} - c_n(t)E_n\right]\left|\phi_n\right\rangle = 0$, $c_n(t) = c_n(0)e^{-iE_nt/\hbar}$, $\boxed{\left|\psi\right\rangle = \sum_n c_n(0)e^{-iE_nt/\hbar}\left|\phi_n\right\rangle} = e^{-\hat{H}t/\hbar}\sum_n c_n(0\left|\phi\right\rangle_n)$, $\omega_n = E_n/\hbar$

46. Time shift operator $\boxed{\hat{U}(t,t_0)} = e^{-\hat{H}(t-t_0)/\hbar} = \boxed{\sum_n e^{-i\omega_n(t-t_0)}\left|\psi_n\right\rangle\left\langle\psi_n\right|}$, $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = \hat{I}$, $\left[\hat{U},\hat{H}\right] = 0$

47. $\left|\psi(t)\right\rangle = e^{-i\hat{H}t/\hbar}\left|\psi(0)\right\rangle$, $\left|\psi(t)\right\rangle = \hat{U}(t,t_0)\left|\psi(t_0)\right\rangle$

48. Ehrenfest's theorem $\boxed{\frac{\mathrm{d}\langle\hat{A}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\langle\left[\hat{H},\hat{A}\right]\rangle + \langle\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}\rangle}$

49. $\Delta E\Delta A \geq \frac{1}{2}\left|\langle i\left[\hat{H},\hat{A}\right]\rangle\right| = \frac{\hbar}{2}\left|\frac{\mathrm{d}\langle\hat{A}\rangle}{\mathrm{d}t}\right|$, define $\Delta t = \frac{\Delta A}{\left|\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}\right|}$, $\Delta E\Delta t \geq \frac{\hbar}{2}$

50. For a conserved observable ($\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = 0$), its expectation value is constant if $\left[\hat{A},\hat{H}\right] = 0$

51. $\langle\hat{A}\rangle = \sum_{n,m} c_m^* c_n e^{i(E_m-E_n)t/\hbar}A_{mn}$, where $A_{mn} = \left\langle\phi_m\right|\hat{A}\left|\phi_n\right\rangle$

52. $\left[\hat{A}, \hat{B}^l\right] = \left[\hat{A}, \hat{B}^{l-1}\hat{B}\right] = \left[\hat{A}, \hat{B}^{l-1}\hat{B}\right] = \left[\hat{A}, \hat{B}^{l-1}\right]\hat{B} + \hat{B}^{l-1}\left[\hat{A}, \hat{B}\right]$ (distribute while keeping order)
    - Remember, use cross terms $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B}^l - \hat{B}^l\hat{A} = \hat{A}\hat{B}^l - \hat{B}\hat{A}\hat{B}^{l-1} + \hat{B}\hat{A}\hat{B}^{l-1} - \hat{B}^l\hat{A} = \left[\hat{A}, \hat{B}\right]\hat{B}^{l-1} + \hat{B}\left[\hat{A}, \hat{B}^{l-1}\right]$
    - $\left[\hat{x}, \hat{p}^l\right] = i\hbar l\hat{p}^{l-1} = i\hbar\frac{\partial}{\partial\hat{p}}\hat{p}^l$, $\left[\hat{p}, \hat{x}^l\right] = -i\hbar l x^{l-1} = -i\hbar\frac{\partial}{\partial\hat{x}}\hat{x}^l$ (induction)
    - $[\hat{x}, F(\hat{p})] = i\hbar\frac{\partial F(\hat{p})}{\partial\hat{p}}, [\hat{p}, G(\hat{x})] = -i\hbar\frac{\partial G(\hat{x})}{\partial\hat{x}}$
    - $\boxed{[\hat{x}, F(\hat{p}, \hat{x})] = i\hbar\frac{\partial F(\hat{p}, \hat{x})}{\partial\hat{p}}}$ (Taylor series)
    - $\boxed{[\hat{p}, F(\hat{p}, \hat{x})] = -i\hbar\frac{\partial F(\hat{p}, \hat{x})}{\partial\hat{x}}}$
    - $\left[\hat{A}, F(\hat{A})\right] = \hat{0}$, $[\hat{x}, V(\hat{x})] = \hat{0}$, $\hat{V} = V(\hat{x})$ is an observable. But $[\hat{p}, V(\hat{x})] = -i\hbar\frac{\mathrm{d}V(\hat{x})}{\mathrm{d}\hat{x}} \neq \hat{0}$

53. - $\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\langle\left[\hat{H}, \hat{x}\right]\rangle = \frac{\langle\hat{p}\rangle}{m}$
    - $\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = -\langle\frac{\mathrm{d}V(\hat{x})}{\mathrm{d}\hat{x}}\rangle = \langle F(\hat{x})\rangle$
    - $\langle F(\hat{x})\rangle \neq F(\langle\hat{x}\rangle)$

54. Heisenberg picture $\hat{A}^H(t) = e^{i\hat{H}t/\hbar}\hat{A}e^{-i\hat{H}t/\hbar} = \hat{U}^\dagger(t, t_0)\hat{A}\hat{U}(t, t_0)$, $\langle A(t)\rangle = \langle\psi(0)|\,\hat{A}^H\,|\psi(0)\rangle$ (observable/variable changes instead of state vector/wavefunction)

55. Heisenberg equation $\boxed{i\hbar\frac{\mathrm{d}\hat{A}^H}{\mathrm{d}t} = \left[\hat{A}^H, \hat{H}\right]}$, for time-independent observable $\hat{A}$

56. von-Neumann equation $\boxed{i\hbar\frac{\mathrm{d}\hat{O}}{\mathrm{d}t} = \left[\hat{H}, \hat{O}\right]}$

57. $\Psi(\mathbf{r}, t) = \Psi(x, y, z, t) = \Psi(r, \theta, \phi, t)$, $P(\mathbf{r}) = |\Psi(x, y, z, t)|^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = |\Psi(r, \theta, \phi, t)|^2 r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$, plane wave $\Psi(\mathbf{r}, t) \propto e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t} = e^{i\mathbf{p}\cdot\mathbf{r}/\hbar - i\omega t}$

58. $[\hat{r}_j, \hat{p}_k] = i\hbar\delta_{jk}$

59. Angular momentum operator $\boxed{\hat{\mathbf{L}}^\dagger = \hat{\mathbf{L}} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \begin{pmatrix} \hat{y}\hat{p}_z - \hat{z}\hat{p}_y \\ \hat{z}\hat{p}_x - \hat{x}\hat{p}_z \\ \hat{x}\hat{p}_y - \hat{y}\hat{p}_x \end{pmatrix}}$

60.

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$
$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$
$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

only one component can be measured accurately, usually chosen to be $\hat{L}_z$

61. $\hat{L}^2 = \hat{\mathbf{L}}\cdot\hat{\mathbf{L}} = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, $\left[\hat{L}_x^2, \hat{L}_z\right] = -i\hbar(\hat{L}_x\hat{L}_y + \hat{L}_y\hat{L}_x) = -\left[\hat{L}_y^2, \hat{L}_z\right]$

$$\left[\hat{L}^2, \hat{L}_x\right] = 0$$
$$\left[\hat{L}^2, \hat{L}_y\right] = 0$$
$$\left[\hat{L}^2, \hat{L}_z\right] = 0$$

$\left[\hat{r}^2, \hat{L}_x\right] = \left[\hat{p}^2, \hat{L}_x\right] = 0$

62. Ladder operators (again not observables)

$$\boxed{\begin{aligned}\hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \hat{L}_-^\dagger \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \hat{L}_+^\dagger,\end{aligned}}$$

$\boxed{\left[\hat{L}_z, \hat{L}_\pm\right] = \hbar \hat{L}_\pm}$, $\hat{L}_z\hat{L}_\pm = \hat{L}_\pm\hat{L}_z \pm \hbar\hat{L}_\pm$, For eigenvectors of $\hat{L}_z$, $\hat{L}_z\left|\phi_\alpha\right\rangle = \alpha\hbar\left|\phi_\alpha\right\rangle$, $\hat{L}_z\hat{L}_\pm\left|\phi_\alpha\right\rangle$ $= (\hat{L}_\pm\hat{L}_z \pm \hbar\hat{L}_\pm)\left|\phi_\alpha\right\rangle = (m_l \pm 1)\hbar\hat{L}_\pm\left|\phi_\alpha\right\rangle$

63. $\hat{L}_+\left|\phi_\alpha\right\rangle$ and $\hat{L}_-\left|\phi_\alpha\right\rangle$ are the eigenstate of $\hat{L}_z$ with eigenvalue $(\alpha+1)\hbar$ and $(\alpha-1)\hbar$

64. $\left[\hat{L}^2, \hat{L}_\pm\right] = \left[\hat{L}^2, \hat{L}_x \pm i\hat{L}_y\right] = 0$, $\hat{L}^2\hat{L}_\pm\left|\phi_\alpha\right\rangle = \hat{L}_\pm\hat{L}^2\left|\phi_\alpha\right\rangle = \Lambda\hbar^2\hat{L}_\pm\left|\phi_\alpha\right\rangle$

65. $m_l = 0, \pm 1, \ldots, \pm l,$

66. $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$

67.

$$\begin{aligned}\hat{L}_-\left|l, m_l\right\rangle &= C_{l,m_l}\left|l, m_l - 1\right\rangle, \quad C_{l,m_l} = \hbar\sqrt{l(l+1) - m_l(m_l-1)} \\ \hat{L}_+\left|l, m_l\right\rangle &= D_{l,m_l}\left|l, m_l + 1\right\rangle, \quad D_{l,m_l} = \hbar\sqrt{l(l+1) - m_l(m_l+1)}\end{aligned}$$

68. $L_z = m_e vr$, $T = 2\pi r/v$, magnetic dipole moment $\mu_z = IA = \frac{-e}{T}\pi r^2 = \boxed{\gamma L_z} = -\frac{e\hbar}{2m_e}m_l =$ $\boxed{-\mu_B m_l}$, $\boxed{\gamma = -\frac{e}{2m_e}}$ is the gyromagnetic ratio, $\boxed{\mu_B = \frac{e\hbar}{2m_e}}$ is Bohr's magneton

69.

$$\hat{p}_r = -i\hbar\frac{\partial}{\partial r} \tag{1}$$

$$\hat{p}_\theta = -i\hbar\frac{1}{r}\frac{\partial}{\partial\theta} \tag{2}$$

$$\hat{p}_\phi = -i\hbar\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{3}$$

$$\hat{L}_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \tag{4}$$

$$\hat{L}_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{5}$$

$$\boxed{\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}} \tag{6}$$

$$\boxed{0 = [\hat{L}_x, \hat{r}] = [\hat{L}_y, \hat{r}] = [\hat{L}_z, \hat{r}]} \tag{7}$$

$$\hat{L}_\pm = \hbar e^{\pm i\phi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) \tag{8}$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{9}$$

$$\boxed{\frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2}} \tag{10}$$

$$\hat{H} = -\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{L}^2}{2mr^2} + V(r) \qquad \text{Central potential} \tag{11}$$

$$\boxed{0 = [\hat{L}_x, \hat{H}(r)] = [\hat{L}_y, \hat{H}(r)] = [\hat{L}_z, \hat{H}(r)]} \qquad \text{Central potential} \tag{12}$$

70. $\hat{L}_z Y_{l,m_l}(\theta,\phi) = -i\hbar\frac{\partial}{\partial\phi}Y_{l,m_l} = m_l\hbar Y_{l,m_l}$, $\boxed{Y_{l,m_l}(\theta,\phi) = F_{l,m_l}(\theta)e^{im_l\phi}}$

71. $\hat{L}^2 Y_{l,m_l}(\theta,\phi) = l(l+1)\hbar^2 Y_{l,m_l}$

72. $\hat{H}\psi_{n,l,m_l}(r,\theta,\phi) = E\psi_{n,l,m_l}$, $\psi_{n,l,m_l} = R_{n,l}(r)Y_{l,m_l}(\theta,\phi)$

$n \geq 1$	principal quantum number
$l = 0, 1, 2, \cdots, n-1$	orbital angular momentum quantum number
$m_l = -l, \cdots, 0, \cdots, l$	magnetic angular momentum quantum number
$s = \frac{1}{2}, 1, 0, 2$	spin
$m_s = -s, \cdots, s$	spin magnetic quantum number
$j = \|l-s\|, \|l-s\|+1, \ldots, l+s$	total angular momentum quantum number
$m_j = -j, \ldots, j$	total magnetic angular momentum quantum number

73. Symmetry $Y_{l,m_l}(\theta,\phi) = Y_{l,m_l}(\theta,\phi+2\pi)$, $e^{i2\pi m_l} = 1$, satisfied because m_l is an integer

74.

$$\hat{L}_+ |l,l\rangle = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)Y_{l,l}(\theta,\phi) = 0$$

$$\frac{\partial Y_{l,l}}{\partial\theta} = -i\cot\theta\frac{\partial Y_{l,l}}{\partial\phi} = l\cot\theta Y_{l,l}$$

$$Y_{l,l}(\theta,\phi) = F_{l,l}(\theta)e^{il\phi}$$

$$\sin\theta\frac{\partial F_{l,l}}{\partial\theta} = l\cos\theta F_{l,l}$$

$$F_{l,l}(\theta) = C(\sin\theta)^l$$

$$\boxed{Y_{l,l}(\theta,\phi) = C(\sin\theta)^l e^{il\phi}}$$

75. $F_{l,m_l-1} = \frac{-\hbar}{C_{l,m_l}}\left(\frac{\partial F_{l,m_l}}{\partial\theta} + m_l\cot\theta F_{l,m_l}\right)$

76. $\int_0^\infty\int_0^\pi\int_0^{2\pi}|\psi(r,\theta,\phi)|^2 r^2\sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$
$\int|Y_{l,m_l}(\theta,\phi)|^2\mathrm{d}\Omega = \int_0^\pi\int_0^{2\pi}|Y_{l,m_l}|^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi = 1$
$P(r)\mathrm{d}r = |R_{n,l}(r)|^2 r^2\mathrm{d}r$

77. Spherical harmonics

- $Y_{l,-m_l}(\theta,\phi) = (-1)^{m_l}Y_{l,m_l}(\theta,\phi) = Y_{l,m_l}(\theta,\pi-\phi)$
- $Y_{l,m_l}(\pi-\theta,\pi+\phi) = (-1)^l Y_{l,m_l}(\theta,\phi)$

78. Diatomic molecules

- $E = \frac{L^2}{2I} = \frac{l(l+1)\hbar^2}{2I}$ with $2l+1$ degeneracy
- $P_n = \frac{e^{-\beta E_n}}{\sum_{n=1}^\infty e^{-\beta E_n}}$, $\beta = \frac{1}{k_B T}$
- $\langle E\rangle = \frac{\sum_l \frac{l(l+1)\hbar^2}{2I}(2l+1)e^{-\frac{l(l+1)\hbar^2}{2Ik_BT}}}{\sum_l(2l+1)e^{-\frac{l(l+1)\hbar^2}{2Ik_BT}}}$
- The characteristic temperature $\theta_{rot} = \frac{\hbar^2}{2Ik_B}$
- $C_{rot} = N\frac{\partial\langle E\rangle}{\partial T}$

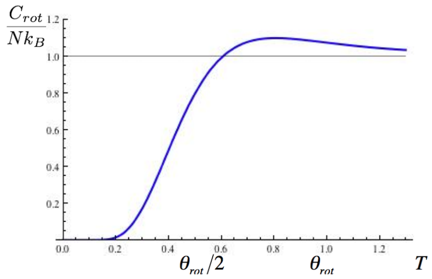

Figure 9.5: Rotational specific heat normalised to R, where $N = n_{\text{mol}} N_A$ is the number of molecules, and n_{mol} is the number of moles.

-
- $\lim_{T\to 0} C_{rot} = 0$
- $\lim_{T\to\infty} C_{rot} = R = N_A k_B$

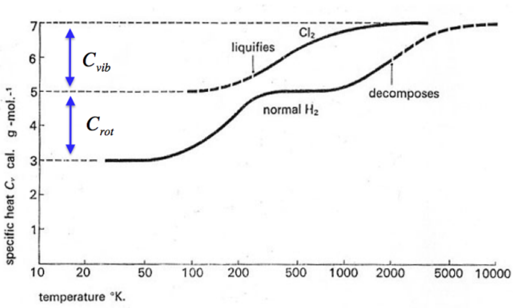

Figure 9.6: Total heat capacities of hydrogen and chlorine.

-
- Heavier the atom, greater the I, smaller the θ_{rot}
- unlike C_{vib}, C_{rot} has **lower** characteristic temperature, thus generally present in diatomic gas (θ_{rot} < freezing point) for elements heavier than D_2 (Deuterium)

79. $\hat{L}^2\psi = l(l+1)\hbar^2\psi$, combining with (10)
    - $-\frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_{n,l}}{\partial r}\right) + \frac{l(l+1)\hbar^2}{2mr^2}R_{n,l} + V(r)R_{n,l} = ER_{n,l}$
    - substitution $U_{n,l}(r) = rR_{n,l}(r)$, $-\frac{\hbar^2}{2m}\frac{\partial^2 U_{n,l}}{\partial r^2} + \left[\frac{l(l+1)\hbar^2}{2mr^2} + V(r)\right]U_{n,l} = EU_{n,l}$
    - 1D Schrödinger, with $V(r)$ replaced by $\frac{\langle\hat{L}^2\rangle}{2mr^2} + V(r)$
    - Hydrogen-like atom (one electron)
        - $V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$
        - $\boxed{E_n = -\frac{\hbar^2}{2m}\frac{Z^2}{a_0^2}\frac{1}{n^2}} = -\frac{13.6Z^2}{n^2}\text{eV}$,
          where $a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = \frac{\hbar}{\alpha mc} = 0.53\times 10^{-10}$ is the Bohr radius.
        - $A = \frac{2m}{\hbar^2}\frac{Ze^2}{4\pi\epsilon_0}$, $\kappa^2 = -\frac{2mE_n}{\hbar^2}$, $\kappa = \frac{Z}{na_0}$
        - $R_{n,l} = \frac{U_{n,l}}{r} \propto \sum_{q=0}^{n-(l+1)} c_q r^{q+l} e^{-\kappa r}$
        - $c_{q+1} = c_q\frac{2\kappa(q+l+1-A/2\kappa)}{(q+1)[q+2(l+1)]}$
        - $$\boxed{\begin{aligned} n &\geq 1 \\ l &= 0,1,2,\cdots,n-1 \\ m_l &= -l,-l+1,\cdots,-1,0,1,\cdots,l-1,l \end{aligned}}$$

– The energy degeneracy of state n is $\boxed{2\sum_{l=0}^{n-1}(2l+1) = 2n^2}$, factor of 2 due to spin

80. Two body problem $\hat{V} = \hat{V}(|\mathbf{r}_b - \mathbf{r}_a|) = \hat{V}(|\mathbf{r}|)$, then

- $\hat{H} = \frac{\hat{\mathbf{p}}_a^2}{2m_a} + \frac{\hat{\mathbf{p}}_b^2}{2m_b} + \hat{V}(\mathbf{r}_a, \mathbf{r}_b)$
- Centre of mass position $\hat{\mathbf{R}} = \frac{m_a\hat{\mathbf{r}}_a + m_b\hat{\mathbf{r}}_b}{m_a + m_b}$
- Relative position $\hat{\mathbf{r}} = \hat{\mathbf{r}}_b - \hat{\mathbf{r}}_a$
- $$\begin{aligned}\hat{\mathbf{r}}_a &= \frac{m_b}{m_a + m_b}\hat{\mathbf{r}} + \hat{\mathbf{R}} \\ \hat{\mathbf{r}}_b &= -\frac{m_a}{m_a + m_b}\hat{\mathbf{r}} + \hat{\mathbf{R}}\end{aligned}$$
- CoM momentum $\hat{\mathbf{P}} = i\hbar\frac{\partial}{\partial \mathbf{R}} = i\hbar\left(\frac{\partial \hat{r}_a}{\partial \hat{R}}\frac{\partial}{\partial \hat{r}_a} + \frac{\partial \hat{r}_b}{\partial \hat{R}}\frac{\partial}{\partial \hat{r}_b}\right) = \hat{\mathbf{p}}_a + \hat{\mathbf{p}}_b$
- relative momentum $\hat{\mathbf{p}} = i\hbar\frac{\partial}{\partial \hat{\mathbf{r}}} = \frac{m_a m_b}{m_a + m_b}\left(\frac{\hat{\mathbf{p}}_a}{m_a} - \frac{\hat{\mathbf{p}}_a}{m_b}\right)$
- $[\hat{\mathbf{R}}, \hat{\mathbf{p}}] = [\hat{\mathbf{r}}, \hat{\mathbf{P}}] = 0$, $[\hat{\mathbf{R}}, \hat{\mathbf{P}}] = [\hat{\mathbf{r}}, \hat{\mathbf{p}}] = i\hbar$
- $\frac{\mathrm{d}\langle\hat{\mathbf{P}}\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\langle\left[\hat{H}, \hat{\mathbf{P}}\right]\rangle = 0$
- $\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + \frac{\hat{\mathbf{p}}}{2\mu} + \hat{V}(r)$, $\frac{1}{\mu} = \frac{1}{m_a} + \frac{1}{m_b}$
- CoM $\hat{H}_{CoM} = -\frac{\hbar^2}{2m}\nabla_R^2$
- Relative $\hat{H}_r = -\frac{\hbar^2}{2m}\nabla_r^2 + V(r)$
- $\boxed{\psi(\mathbf{R}, \mathbf{r}) = U(\mathbf{R})u(\mathbf{r})}$, $\frac{1}{U(\mathbf{R})}\hat{H}_{CoM}U(\mathbf{R}) + \frac{1}{u(\mathbf{r})}\hat{H}_r u(\mathbf{r}) = E\psi(\mathbf{R}, \mathbf{r})$
- $\hat{H}_{CoM}U(\mathbf{R}) = E_{CoM}U(\mathbf{R})$, $\hat{H}_r u(r) = E_r u(r)$
- $\psi(\mathbf{r}_a, \mathbf{r}_b) = \boxed{\psi(\mathbf{R}, \mathbf{r}) = e^{i\mathbf{P}\cdot\mathbf{R}/\hbar}\psi_{n,l,m_l}(\mathbf{r})}$

81. Spin $\hat{\mathbf{S}} = \mathbf{i}\hat{S}_x + \mathbf{j}\hat{S}_y + \mathbf{k}\hat{S}_z$, analogous to $\hat{\mathbf{L}}$

82.

$$\begin{aligned}\left[\hat{S}_x, \hat{S}_y\right] &= i\hbar\hat{S}_z \\ \left[\hat{S}_y, \hat{S}_z\right] &= i\hbar\hat{S}_x \\ \left[\hat{S}_z, \hat{S}_x\right] &= i\hbar\hat{S}_y\end{aligned}$$

83. Because of the commutation relationships, $\left[\hat{S}^2, \hat{S}_z\right] = 0$, $\hat{S}_z|\psi\rangle = m_s\hbar|\psi\rangle$, $\hat{S}^2|\psi\rangle = s(s+1)\hbar^2$

84. $(2s+1)$ possible values, $m_s = -s, -s+1\cdots, s-1, s$, s is integer or half-integer, $2s+1$ is odd or even

85. Electron: $2s+1 = 2$, $s = 1/2$, $\langle\hat{S}_z\rangle = \pm\hbar/2$, $\langle\hat{S}\rangle = \sqrt{s(s+1)}\hbar = \frac{\sqrt{3}}{2}\hbar$, $\hat{S}_z|\uparrow\rangle = +\frac{\hbar}{2}|\uparrow\rangle$ is spin up, $\hat{S}_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$ is spin down

86. $\boxed{\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y}$,

- $$\boxed{\begin{aligned}\langle\hat{S}_z\rangle = \pm\hbar/2\hat{S}_+\left|s,m_s\right\rangle &= \hbar\sqrt{s(s+1)-m_s(m_s+1)}\left|s,m_s+1\right\rangle \\ \hat{S}_-\left|s,m_s\right\rangle &= \hbar\sqrt{s(s+1)-m_s(m_s-1)}\left|s,m_s-1\right\rangle\end{aligned}}$$
- Which results in
$$\boxed{\begin{aligned}\hat{S}_+\left|\uparrow\right\rangle &= \left|0\right\rangle \\ \hat{S}_+\left|\downarrow\right\rangle &= \hbar\left|\uparrow\right\rangle \\ \hat{S}_-\left|\uparrow\right\rangle &= \hbar\left|\downarrow\right\rangle \\ \hat{S}_-\left|\downarrow\right\rangle &= \left|0\right\rangle\end{aligned}}$$
- $\boxed{\hat{S}_x = \frac{\hat{S}_+ + \hat{S}_-}{2}, \hat{S}_y = \frac{\hat{S}_+ - \hat{S}_-}{2i},\ \hat{S}_x\left|\uparrow\right\rangle = \frac{\hbar}{2}\left|\downarrow\right\rangle, \hat{S}_x\left|\downarrow\right\rangle = \frac{\hbar}{2}\left|\uparrow\right\rangle}$,
$$\boxed{\begin{aligned}\left|\chi_\pm\right\rangle &= \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle) \\ \hat{S}_x\left|\chi_\pm\right\rangle &= \frac{\hbar}{2}\left|\chi_\pm\right\rangle \\ \left|\phi_\pm\right\rangle &= \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle \pm i\left|\downarrow\right\rangle) \\ \hat{S}_y\left|\phi_\pm\right\rangle &= \frac{\hbar}{2}\left|\phi_\pm\right\rangle\end{aligned}}$$

87. $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$
88. $$\begin{aligned}\left[\hat{J}_x, \hat{J}_y\right] &= i\hbar\hat{J}_z \\ \left[\hat{J}_y, \hat{J}_z\right] &= i\hbar\hat{J}_x \\ \left[\hat{J}_z, \hat{J}_x\right] &= i\hbar\hat{J}_y\end{aligned}$$
89. Thus, $\left[\hat{J}^2, \hat{J}_z\right] = 0$, $\hat{J}_z\left|\psi\right\rangle = m_j\hbar\left|\psi\right\rangle$, $\hat{J}^2\left|\psi\right\rangle = j(j+1)\hbar^2\left|\psi\right\rangle$, $m_j = -j, \dots, j$
90. $\left|j, m_j\right\rangle = \sum_{l,m_l,s,m_s} C_{l,m_l,s,m_s}\left|l, m_l\right\rangle\left|s, m_s\right\rangle$
91. Total ladder operators $\hat{J}_\pm = \hat{L}_\pm + \hat{S}_\pm$, $\hat{J}_\pm\left|j, m_j\right\rangle = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)}\left|j, m_j \pm 1\right\rangle$
92.

Spin	1/2	0	1
Electron	•		
Proton	•		
Neutron	•		
Quark	•		
Photon			•
Phonon		•	
Higgs boson		•	

93. Particle exchange operator
    - $\hat{P}_{ij}\left|\xi_i, \xi_j\right\rangle = \left|\xi_j, \xi_i\right\rangle$
    - $\langle P\rangle = 1$
    - $\hat{P}_{ij}^\dagger\hat{P}_{ij} = \hat{I} = \hat{P}_{ij}^2$
    - $\frac{\mathrm{d}\langle P\rangle}{\mathrm{d}t} = \frac{i}{\hbar}\langle[\hat{H}, \hat{P}]\rangle = 0$
    - $[\hat{A}, \hat{P}_{ij}] = 0$, $\hat{P}_{ij}$ compatible with all observables

94. The *symmetrisation postulate*: states of N identical particles are either exchange symmetric or exchanged antisymmetric

95. The *spin statistics theorem*: integer spin particles are bosons $p_B(\epsilon) = \dfrac{1}{\exp[(\epsilon-\mu)/k_BT]-1}$, half-dd-integer spin particles are fermions $p_F(\epsilon) = \dfrac{1}{\exp[(\epsilon-\mu)/k_BT]+1}$

96. Two particle basis $|\alpha\rangle|\alpha\rangle$, $|\alpha\rangle|\beta\rangle$, $|\beta\rangle|\alpha\rangle$, $|\beta\rangle|\beta\rangle$ can be expressed as $|\alpha\rangle|\alpha\rangle$, $|s\rangle = \dfrac{1}{\sqrt{2}}[|\alpha\rangle|\beta\rangle + |\beta\rangle|\alpha\rangle]$, $|a\rangle = \dfrac{1}{\sqrt{2}}[|\alpha\rangle|\beta\rangle - |\beta\rangle|\alpha\rangle]$, $|\beta\rangle|\beta\rangle$. $|a\rangle$ is antisymmetric ($\hat{P}_{\alpha\beta}|a\rangle = -|a\rangle$), the others are symmetric ($\hat{P}_{\alpha\beta}|s\rangle = |s\rangle$)

97. N bosons placed in n states, each having m_n bosons
    - $|1,2,1\rangle$, $|1,1,2\rangle$ and $|2,1,1\rangle$ are the same state, to use previous notations, we add them up and normalize
    - The state is described by $\dfrac{2!1!}{3!}[|1,1,2\rangle + |1,2,1\rangle + |2,1,1\rangle]$
    - generally this symmetric/boson state is
      $|a\ldots r\ldots z\rangle^s = \sqrt{\dfrac{\prod_n(m_n!)}{N!}}\sum_P |\xi^a\rangle\cdots|\xi^r\rangle\cdots|\xi^z\rangle$

      where P loops through all permutations.

98. N fermions
    - for fermions, each occupy a unique state ($m_n = 1$ only)
    - $|1,1,2\rangle$ does not exist; cannot have two fermions being 1
    - $|123\rangle^a = |1,2,3\rangle + |2,3,1\rangle + |3,1,2\rangle - |1,3,2\rangle - |3,2,1\rangle - |2,1,3\rangle$
    - $|a\ldots r\ldots z\rangle^a = \dfrac{1}{\sqrt{N!}}\sum_P(-1)^{f(P)}|\xi^a\rangle\cdots|\xi^r\rangle\cdots|\xi^z\rangle = \dfrac{1}{\sqrt{N!}}\begin{vmatrix} |\xi^1\rangle & |\xi^1\rangle & \cdots & |\xi^1\rangle \\ |\xi^2\rangle & |\xi^2\rangle & \cdots & |\xi^2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ |\xi^N\rangle & |\xi^N\rangle & \cdots & |\xi^N\rangle \end{vmatrix}$

      where $f(P)$ is the number of swaps needed to get permutation P from original order, the determinant is the *Slater determinant.*

99. Davisson & Germer, electrons have de Broglie wavelength (they diffract off a lattice with Bragg reflection, $d\sin\theta = n\lambda$)

100. Bohr's atom, discrete angular momentum makes electrons stay in stable orbit

101. Planck's blackbody, discrete photon energy/cavity mode produces correct equation, solved the ultaviolet catastrophe in Rayleight-Jeans law

102. Photoelectric effect, work function, intensity useless frequency important

103. Stern-Gerlach (2 splits, not odd number of 2l+1 splits, due to spin)