Math Cheatsheet

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The Fourier Transform

1. $\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) \mathrm{d}x = \mathcal{F}[f](k)$
   - Real and even ($f^*(x) = f(x) \quad \text{and} \quad f(x) = f(-x)) \implies \tilde{f}^*(k) = \tilde{f}(k)$ ($\tilde{f}$ is **real**)
   - Real and odd $\implies \tilde{f}$ is imaginary ($\tilde{f}^*(k) = -\tilde{f}(k)$)
   - Linearity, $\mathcal{F}[\alpha f(x) + \beta g(x)] = \alpha \mathcal{F}[f(x)] + \beta \mathcal{F}[g(x)]$
   - Rescaling, $\mathcal{F}[f(\alpha x)] = \frac{1}{|\alpha|} \tilde{f}\left(\frac{k}{\alpha}\right)$
   - Translation, $\mathcal{F}[f(x-a)] = e^{-ika} \mathcal{F}[f(x)]$
   - Exponential, $\mathcal{F}[e^{iax} f(x)](k) = \mathcal{F}[f](k-a)$
   - Duality, $\mathcal{F}[\tilde{f}] = f(-k)$
   - For **real** k, $\mathcal{F}[f^*](k) = \mathcal{F}[f](-k)$
   - Symmetry, $f(-x) = \pm f(x) \implies \tilde{f}(-k) = \pm \tilde{f}(k)$
   - Differentiation, $\mathcal{F}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = ik\tilde{f}(k)$ ($\mathcal{I}[k\tilde{f}] = -i\frac{\mathrm{d}f}{\mathrm{d}x}$)
   - $\mathcal{I}\left[\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k}\right] = -ixf(x)$ ($\mathcal{F}[xf] = i\frac{\mathrm{d}\tilde{f}}{\mathrm{d}k}$)
   - Convolution, $\mathcal{F}[f * g] = \sqrt{2\pi}\mathcal{F}[f]\mathcal{F}[g]$
   - $\mathcal{F}[fg] = \frac{1}{\sqrt{2\pi}} \mathcal{F}[f] * \mathcal{F}[g]$
   - Correlation, $\mathcal{F}[f \otimes g](x) = \sqrt{2\pi}\mathcal{F}[f]^*\mathcal{F}[g]$ (Wiener-Khinchin if $g = f$)
   - (TODO: prove the inverse)
   - **Autoconvolution** is $\sqrt{2\pi}\tilde{[f]}^2$, **autocorrelation** is $\sqrt{2\pi}|f|^2$
   - Parseval's theorem $\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \mathrm{d}k$ (inverse of Wiener-Khinchin/delta function)
2. $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \tilde{f}(k) \mathrm{d}k = \mathcal{I}[\tilde{f}](x)$
3. Convolution is $(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x-y) \mathrm{d}y$
4. Correlation is $(f \otimes g) \int_{-\infty}^{\infty} [f(y)]^* g(x+y) \mathrm{d}y$
   - $f(x) \otimes g(x) = f(-x)^* * g(x)$
   - $f(x) \otimes g(x) = g(-x)^* \otimes f(-x)^*$
   - If f is hermitian, $f \otimes g = f * g$
   - If f, g are hermitian, $f \otimes g = g \otimes f$
   - $(f \otimes g) \otimes (f \otimes g) = (f \otimes f) \otimes (g \otimes g)$

5. $\delta(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-t_0)}\mathrm{d}\omega$ (prove $\frac{1}{2\pi}$ by transforming twice)

6. Laplace Transform $\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st}\mathrm{d}t$

f	$\tilde{f}$
$e^{-b\lvert x\rvert},\ b>0$	$\frac{1}{\sqrt{2\pi}}\frac{2b}{k^2+b^2}$
$\frac{1}{x^2+b^2}$	$\sqrt{\frac{\pi}{2b^2}}e^{-b\lvert k\rvert}$
$\frac{1}{\sqrt{2\pi\epsilon^2}\exp\left(-\frac{x^2}{2\epsilon^2}\right)}$	$\frac{1}{\sqrt{2\pi}}exp\left(-\frac{\epsilon^2k^2}{2}\right)$
$\delta(x-a)$	$\frac{1}{\sqrt{2\pi}}e^{-ika}$
$H(x-a)e^{-\epsilon(x-a)}$	$\frac{1}{\sqrt{2\pi}}\frac{e^{-ika}}{\epsilon+ik}$
$H(x-a)$	$\frac{e^{-ika}}{ik\sqrt{2\pi}}$
$H(x+a)H(a-x)$ (tophat)	$\sqrt{\frac{2}{\pi}}\frac{\sin(ak)}{k}$

Vector Calculus

Practice List

1. $(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_jb_k$
2. $\det A = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$
3. $\det(\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k) = \epsilon_{ijk}$
4. $\epsilon_{ijk}\epsilon_{lmn} = \det(\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k)^T\det(\mathbf{e}_l\ \mathbf{e}_m\ \mathbf{e}_n) = \begin{vmatrix}\delta_{il} & \delta_{im} & \delta_{in}\\ \delta_{jl} & \delta_{jm} & \delta_{jn}\\ \delta_{kl} & \delta_{km} & \delta_{kn}\end{vmatrix}$
5. Bear in mind that $\boxed{\delta_{kk} = 3 = \delta_{k1}\delta_{k1}+\delta_{k2}\delta_{k2}+\delta_{k3}\delta_{k3}}$ $(\delta_{11} = 1)$ $(\delta_{lm} = \delta_{l1}\delta_{m1}+\delta_{l2}\delta_{m2}+\delta_{l3}\delta_{m3})$
6. $\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$
7. $\boxed{\epsilon_{ijk}\epsilon_{ijk} = 6}$
8. $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$
9. $\big(\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\big)_i = \big((\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\big)_i$
10. $\nabla f(r) = f'(r)\frac{\mathbf{r}}{r}$, $\boldsymbol{\nabla} r = \frac{\mathbf{r}}{r}$
11. $\nabla f = \mathbf{e}_i\frac{\partial f}{\partial x_j}$, $\nabla\cdot\mathbf{F} = \frac{\partial F_j}{\partial x_j}$, $\nabla\times\mathbf{F} = \epsilon_{ijk}\mathbf{e}_i\frac{\partial F_k}{\partial x_j}$
12. $\|\mathbf{x}\|^2\|\mathbf{y}\|^2 - |\mathbf{x}\cdot\mathbf{y}|^2 \geq 0$
13. $\mathbf{F}\cdot\boldsymbol{\nabla} \neq -\boldsymbol{\nabla}\cdot\mathbf{F}$
14. $$\begin{aligned}
\boldsymbol{\nabla}\cdot(\psi\,\mathbf{F}) &= \psi\,\boldsymbol{\nabla}\cdot\mathbf{F} + (\mathbf{F}\cdot\boldsymbol{\nabla})\psi\,,\\
\boldsymbol{\nabla}\times(\psi\mathbf{F}) &= \psi\,(\boldsymbol{\nabla}\times\mathbf{F}) + (\boldsymbol{\nabla}\psi)\times\mathbf{F}\,,\\
\boldsymbol{\nabla}\cdot(\mathbf{F}\times\mathbf{G}) &= \mathbf{G}\cdot(\boldsymbol{\nabla}\times\mathbf{F}) - \mathbf{F}\cdot(\boldsymbol{\nabla}\times\mathbf{G})\,,\\
\boldsymbol{\nabla}\times(\mathbf{F}\times\mathbf{G}) &= \mathbf{F}\,(\boldsymbol{\nabla}\cdot\mathbf{G}) - \mathbf{G}\,(\boldsymbol{\nabla}\cdot\mathbf{F}) + (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} - (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G}\,,\\
\boldsymbol{\nabla}(\mathbf{F}\cdot\mathbf{G}) &= (\mathbf{F}\cdot\boldsymbol{\nabla})\mathbf{G} + (\mathbf{G}\cdot\boldsymbol{\nabla})\mathbf{F} + \mathbf{F}\times(\boldsymbol{\nabla}\times\mathbf{G}) + \mathbf{G}\times(\boldsymbol{\nabla}\times\mathbf{F})
\end{aligned}$$

15. $\nabla^2\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})$

16. $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) = 0$, $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\mathbf{F}) = 0$

17. (Divergence Theorem)

    - (Vector) $\iiint_V \boldsymbol{\nabla}\cdot\mathbf{u}\mathrm{d}V = \iint_S \mathbf{u}\cdot\mathrm{d}S$
    - (Scalar) $\iiint_V \boldsymbol{\nabla}\phi\mathrm{d}V = \iint_S \phi\mathrm{d}\mathbf{S}$
    - (Generalized Stokes) $\iiint_V \boldsymbol{\nabla}\times\mathbf{A}\mathrm{d}V = \iint_S \hat{\mathbf{n}}\times\mathbf{A}\mathrm{d}S$

18. (Stokes Theorem)

    - $\iint_S \boldsymbol{\nabla}\times\mathbf{u}\cdot\mathrm{d}\mathbf{S} = \oint_C \mathbf{u}\cdot\mathrm{d}\mathbf{r}$
    - (Green's) $\iint_A \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \int_C (u_x\mathrm{d}x + u_y\mathrm{d}y)$

19.

	Cylindrical Polar Coordinates	Spherical Polar Coordinates
q_1	$\rho = (x^2+y^2)^{1/2}$	$r = (x^2+y^2+z^2)^{1/2}$
q_2	$\phi = \tan^{-1}\left(\frac{y}{x}\right)$	$\theta = \tan^{-1}\left(\frac{(x^2+y^2)^{1/2}}{z}\right)$
q_3	z	$\phi = \tan^{-1}(y/x)$

20.

	Cylindrical Polar Coordinates	Spherical Polar Coordinates
x	$\rho\cos\phi$	$r\cos\phi\sin\theta$
y	$\rho\sin\phi$	$r\sin\phi\sin\theta$
z	z	$r\cos\theta$

21.
    - $\mathbf{h}_j = \frac{\partial\mathbf{r}}{\partial q_j} = \frac{\partial x_i}{\partial q_j}\hat{\mathbf{x}}_i$
    - $\mathbf{h}_j = h_j\mathbf{e}_j$, $h_j = \left|\frac{\partial\mathbf{r}}{\partial q_j}\right|$
    - $\mathbf{e}_j = \frac{1}{h_j}\frac{\partial\mathbf{r}}{\partial q_j}$
    - $J \equiv \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} = |\mathbf{h}_1\cdot\mathbf{h}_2\times\mathbf{h}_3|$, $\mathrm{d}V = \mathrm{d}\mathbf{r}_1\times\mathrm{d}\mathbf{r}_2\cdot\mathrm{d}\mathbf{r}_3 = |J|\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3 = h_1h_2h_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$ (chain rule, inverse)
    - (Orthogonality) $\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$, $|\mathrm{d}\mathbf{r}|^2 = \sum_i h_i^2(\mathrm{d}q_i)^2$
    - $\mathrm{d}\mathbf{r} = \sum_i h_j\mathbf{e}_j\mathrm{d}q_j$
    - $\boxed{\boldsymbol{\nabla} = \sum_i \mathbf{e}_i\frac{1}{h_i}\frac{\partial}{\partial q_i}}$
    - (even permutation (i,j,k)) $\mathbf{e}_i = \mathbf{e}_j\times\mathbf{e}_k = h_j\boldsymbol{\nabla}q_j\times h_k\boldsymbol{\nabla}q_k$
    - $\boxed{\boldsymbol{\nabla}\cdot\mathbf{F} = \frac{1}{h_1h_2h_3}\sum_{\text{even perms}}\frac{\partial h_jh_kF_i}{\partial q_i}}$
    - $\nabla^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \frac{1}{h_1h_2h_3}\sum_{\text{even perms}}\frac{\partial}{\partial q_i}\left(\frac{h_jh_k}{h_i}\frac{\partial}{\partial q_i}\right)$

- $\boxed{\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}}$
- $\nabla^2\mathbf{F} = \nabla^2(F_i\mathbf{e}_i)$ $((\nabla^2\mathbf{F})_i \neq \nabla^2 F_i)$

Green's functions

1. Different $\delta_\epsilon(x)$s
   - $\delta_\epsilon(x) = \begin{cases} 0 & x < -\epsilon \\ \frac{1}{2\epsilon} & -\epsilon \leq x \leq \epsilon \\ 0 & \epsilon < x \end{cases}$
   - $\delta_\epsilon(x) = \begin{cases} (x+\epsilon)/\epsilon^2 & -\epsilon < x < 0 \\ (\epsilon - x)/\epsilon^2 & 0 \leq x < \epsilon \\ 0 & \text{otherwise} \end{cases}$
   - $\delta_\epsilon(x) = \frac{\epsilon}{\pi(x^2+\epsilon^2)} = \frac{1}{2\pi}\int_{\infty}^{-\infty} e^{ikx-\epsilon|k|\mathrm{d}k}\mathrm{d}$
   -
2. $\delta(x)$ properties
   - $\int_{-\alpha}^{\beta} \delta(x)\mathrm{d}x = 1,\ \alpha > 0,\ \beta > 0$
   - $\int_{-\infty}^{\infty} \delta(x-\xi)f(x)\mathrm{d}x = f(\xi)$
   - $H'(x) = \delta(x)$
   - $\int_{-\infty}^{\infty} \delta'(x)f(x)\mathrm{d}x = -f'(x)$
   - $\delta(ax) = \frac{\delta(x)}{|a|}$
   - For a nice function $\delta(f(x)) = \sum_i \delta(\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i}(x-x_i)) = \sum_i \frac{\delta(x-x_i)}{\left|\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_i}\right|}$
3. Wronskian $W[y_1, y_2] = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$ is the condition for linearly independent solutions
4. **Initial and boundary conditions** is written as $Ay(a) + By'(a) = E$, $E = 0$ means BC is *homogeneous.*
5. Differentail operator $L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$ (turn $\mathcal{L}$ into this standard form first, coefficient of 2nd order term is 1)
6. Green's function is the solution to $LG(x,\zeta) = \delta(x-\zeta)$
7. $y(x) = \int_a^b G(x,\zeta)f(\zeta)\mathrm{d}\zeta$
8. Green's function properties
   - $G(x,\zeta)$ of L shares the same boundary conditions as $y(x)$, which is the solution to $Ly(x) = f(x)$, by construction
   - $G(x,\zeta)$ is a continuous function in x and ζ
   - $\lim_{\epsilon\to 0}[G(x,\zeta)]_{x=\zeta-\epsilon}^{x=\zeta+\epsilon} = 0$

- $\lim_{\epsilon \to 0} \left[\frac{\partial G}{\partial x} \right]_{x=\zeta-\epsilon}^{x=\zeta+\epsilon} = 1$ (or whatever the coefficient of the 2nd order term is)

(Start from $\boxed{1 = \lim_{\epsilon \to 0} \int_{\zeta-\epsilon}^{\zeta+\epsilon} \delta(\zeta - x) \mathrm{d}x}$, $\mathcal{L}$ is 2nd order, p, q are continuous, and assume $p(x)$ is continuous ★)

9. Writing $G(x, \zeta) = \begin{cases} \alpha_-(\zeta) y_1(x) + \beta_-(\zeta) y_2(x) & \text{for} \quad a \leq x < \zeta \\ \alpha_+(\zeta) y_1(x) + \beta_+(\zeta) y_2(x) & \text{for} \quad \zeta \leq x \leq b, \end{cases}$ it follows immediately that $\begin{pmatrix} y_1(\zeta) & y_2(\zeta) \\ y_1'(\zeta) & y_2'(\zeta) \end{pmatrix} \begin{pmatrix} \alpha_+(\zeta) - \alpha_-(\zeta) \\ \beta_+(\zeta) - \beta_-(\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, if the solutions are independent, $\alpha_+(\zeta) - \alpha_-(\zeta) = -\frac{y_2(\zeta)}{W(\zeta)}$ and $\beta_+(\zeta) - \beta_-(\zeta) = \frac{y_1(\zeta)}{W(\zeta)}$

Partial differential equations

Consider separable solutions
For $x^2 y'' + axy' + y = 0$, try $y = x^r$

Matrices

1. Metric $G_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j$
   - $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^\dagger G \mathbf{w}$
   - $G^\dagger = G$ (Hermitian)
   - $v^\dagger G v \geq 0$ (positive definite)
2. - $\det M = \prod_{i=1}^{n} \lambda_i$ ($\det(AB) = \det A \det B$)
   - $\mathrm{tr}\, M = \sum_{i=1}^{n} \lambda_i$
   - $\mathrm{tr}(M^n) = \mathrm{tr}(\Lambda^n)$ ($\mathrm{tr}(AB) = \mathrm{tr}(BA)$)
3. Unitary matrix $A^\dagger = A^{-1}$
4. Normal matrix $AA^\dagger = A^\dagger A$
5. Hermitian matrices
   - The eigenvalues of an Hermitian matrix are real
   - The eigenvectors of an Hermitian matrix with distinct eigenvalues are orthogonal
   - A Hermitian matrix has n orthogonal linearly independent eigenvectors
   - has n orthonormal eigenvectors
   - Anti-Hermitian and Unitary matrices have imaginary eigenvalues with unit modulus
6. Simplifying quadric surface $x^T A x + b^T x + c = 0$
   - $S = \frac{1}{2}(A + A^T)$, $y^T S y + b^T y + c = 0$ (symmetric thus diagonalizable)
   - Diagonalize, $z^T \Lambda z + b^T z + c = 0$
   - Offset, $x'^T \Lambda x' = k$ (to cancel out second term)
7. Quadric surface names

Coefficients	Quadric Surface
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 > 0,\ k > 0.$	*Ellipsoid*: this includes the case of metric matrices, since S is then positive definite and the λ_i are all positive.
$\lambda_1 = \lambda_2.$	*Surface of revolution* about the z' axis.
$\lambda_1 = \lambda_2 > 0,\ \lambda_3 > 0,\ k > 0.$	*Spheroid*: the surface is a *prolate spheroid* if $\lambda_1 = \lambda_2 > \lambda_3$ and an *oblate spheroid* if $\lambda_1 = \lambda_2 < \lambda_3$.
$\lambda_1 = \lambda_2 = \lambda_3 > 0,\ k > 0.$	*Sphere*.
$\lambda_3 = 0.$	*Cylinder*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 = 0,\ k > 0.$	*Elliptic cylinder*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0,\ k > 0.$	*Hyperboloid of one sheet*.
$\lambda_1 > 0,\ \lambda_2 > 0,\ \lambda_3 < 0,\ k = 0.$	*Elliptical conical surface*.
$\lambda_1 > 0,\ \lambda_2 < 0,\ \lambda_3 < 0,\ k > 0.$	*Hyperboloid of two sheets*.
$\lambda_1 > 0,\ \lambda_2 = \lambda_3 = 0,\ \lambda_1 k \geqslant 0.$	Planes $x' = \pm\sqrt{\frac{k}{\lambda_1}}$.

8. All eigenvalues of a nilpotent matrix are 0
9. The Rayleight-Ritz variational principle. The first variation of $\lambda(x) = \dfrac{x^T S x}{x^T x}$ is 0 for all possible δx when $Sx = \lambda(x)x$

1 Elementary analysis

1. Limit, series, partial sum, absolute convergence $\implies$ (conditional) convergence
2. Tests for convergence
   - Comparison test (between two series)
   - D'Alembert's ratio test
   - Cauchy's test $\lim\limits_{r\to\infty} u_r^{1/r} < 1$
3. Taylor $f(x_0 + h) = \displaystyle\sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x_0)$
4. $\dfrac{\mathrm{d}f}{\mathrm{d}z} \equiv f'(z) = \lim\limits_{\delta z\to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$ same by any route in the complex plane
5. Cauchy-Riemann equations $f(z) = u(x,y) + iv(x,y)$, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$
   - u, v are harmonic functions, $\nabla^2 u = \nabla^2 v = 0$
   - u, v are conjugate harmonic functions, $\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = 0$
6. C^1 complex functions are analytic
7. $f(z)\ (z - z_0)^N$, zero of order N at z_0, $f(z)\ (z - z_0)^{-N}$, pole of order N
8. Laurent series $f(z) = \displaystyle\sum_{-\infty}^{\infty} a_n (z - z_0)^n$, on annulus $\alpha < |z - z_0| < \beta$, infinite $n < 0$ means essential singularity
9. Radius of convergence: power series $f(z) = \displaystyle\sum_{r=0}^{\infty} a_r z^r$ converges for $z = z_1$, then it converges absolutely for $|z - z_0| < |z_1 - z_0|$
   - $\lim\limits_{r\to\infty} \left|\dfrac{a_{r+1}}{a_r}\right| = \dfrac{1}{R}$
   - $\lim\limits_{r\to\infty} |a_r|^{1/r} = \dfrac{1}{R}$

Series solution of ODE

1. Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

2. $W' + p(x)W = 0$, $W(x) = C\exp\left(-\int^x p(\zeta)\mathrm{d}\zeta\right)$,

   $y_2(x) = y_1(x)\int^x \frac{W(\eta)}{y_1(\eta)^2}\mathrm{d}\eta = y_1(x)\int^x \frac{C}{y_1(\eta)^2}\exp\left(-\int^\eta p(\zeta)\mathrm{d}\zeta\right)$

3. $y'' + p(z)y' + q(z) = 0$

4. Ordinary point $p(z),\ q(z)$ analytic at $z = z_0$, regular singular point $(z-z_0)p(z), (z-z_0)^2q(z)$ analytic at $z = z_0$

5. $z = z_0$ is ordinary point, two independent solutions like $y = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad |z-z_0| < R$

6. $z = z_0$ is regular singular point, $y_1 = z^{\sigma_1}\sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad a_0 \neq 0, \sigma \in \mathbb{C}$

7. Indicial equation $\sigma(\sigma-1) + p_0\sigma + q_0 = 0$, $p_0 = \lim_{z\to z_0}((z-z_0)p(z))$, $q_0 = \lim_{z\to z_0}((z-z_0)^2q(z))$

8. If $\sigma_1 - \sigma_2 \in \mathbb{R}$, $\mathrm{Re}(\sigma_1) \geq \mathrm{Re}(\sigma_2)$ another solution is $y_2 = ky_1\log z + z^{\sigma_2}\sum_0^\infty b_n z^n$

9. Variation of parameters

Sturm-Liouville

1. Norm of $u(x)$ is $||u||^2 = \langle u|u\rangle = \int_\alpha^\beta |u(x)|^2\mathrm{d}x$ is real and ≥ 0.

2. If $\langle u|\mathcal{L}v\rangle = \langle \mathcal{L}u|v\rangle = \langle v|\mathcal{L}u\rangle^*$ if boundary terms are 0, called **self-adjoint**.

3. Sturm-Liouville operator defined on $\alpha \leq x \leq \beta$ is $\mathcal{L} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(\rho(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + \sigma(x)$, σ, ρ are real, $\forall \alpha < x < \beta (\rho > 0)$
   - $\langle u|\mathcal{L}v\rangle = \langle v|\mathcal{L}u\rangle^* + [\rho(vu^{*\prime} - u^*v')]_\alpha^\beta$ means formally self-adjoint (differ by a constant)
   - If $[\rho(vu^{*\prime} - u^*v')]_\alpha^\beta = 0$, $\mathcal{L}$ is self-adjoint
   - Might not work if $\mathcal{L}$ not defined on $x \in [\alpha, \beta]$ e.g.$[-1, 1]$ for Legendre's equation $(1-x^2)y'' - 2xy' + l(l+1)y = 0$, solutions x and $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ not orthogonal

4. Inner product with a weight function $\langle u|v\rangle_w = \int_\alpha^\beta w(x)u^*(x)v(x)\mathrm{d}x$

5. $\mathcal{L} = w\tilde{\mathcal{L}}$ for a second order operator $\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[a(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] - b(x)\frac{\mathrm{d}}{\mathrm{d}x} - c(x)$
   - $w(x)$ is real and positive
   - $w\tilde{\mathcal{L}} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(aw\frac{\mathrm{d}}{\mathrm{d}x}\right) + (aw' - bw)\frac{\mathrm{d}}{\mathrm{d}x} - wc$
   - Let $aw' = bw$, $w(x) = Ce^{\int^x \frac{b(\zeta)}{a(\zeta)}\mathrm{d}\zeta}$
   - Note that $w(x), a(x), b(x), c(x)$ are real (by definition)
   - $\langle u|\tilde{\mathcal{L}}v\rangle_w = \langle \tilde{\mathcal{L}}v|u\rangle_w + [wa(vu^{*\prime} - u^*v')]_\alpha^\beta$
   - i.e. $\boxed{(\lambda_u - \lambda_v)\int_a^b u^*vw\mathrm{d}x = \left[\rho(x)(u^{*\prime}v' - u^*v')\right]_a^b}$

- $\tilde{\mathcal{L}}y = \lambda y \implies \mathcal{L}y = \lambda w y$

6. If $\{y_n\}$ is a complete set of **orthonormal** eigenfunctions,

   the completeness relation is $\boxed{\sum_{n=1}^{\infty} y_n(x) y_n^*(x') = \frac{1}{w(x')}\delta(x - x')}$

   - $f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$
   - $\langle y_n | y_m \rangle_w = \delta_{nm}$
   - $a_n = \langle y_n | f \rangle_w$
   - $f(x) = \sum_{n=1}^{\infty} \langle y_n | f \rangle_w y_n(x) = \int_{\alpha}^{\beta} f(x') \left[w(x') \sum_{n=1}^{\infty} y_n(x) y_n^*(x') \right] dx'$
   - Recall that $\mathcal{L}G(x, x') = \delta(x - x')$, $y(x) = \int_{\alpha}^{\beta} G(x, x') f(x') \mathrm{d}x'$
   - $\boxed{G(x, x') = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n^*(x')}$ such that $\mathcal{L}G(x, x') = w(x) \sum_{n=1}^{\infty} y_n(x) y_n^*(x') = \frac{w(x)}{w(x')}\delta(x - x') = \delta(x - x')$ ($\mathcal{L} = \mathcal{L}_x$ acts on $y(x)$ only instead of $y^*(x')$)
   - $G(x, x') = G^*(x, x')$, if $\mathcal{L}$ has zero eigenvalue $G(x, x')$ will not exist

7. Bessel's inequality $\|f\|_w^2 \geq \sum_{n=1}^{N} |a_n|^2$

   - $f(x) \approx \sum_{n=1}^{N} a_n y_n$, let $a_n = u + iv$
   - Error

$$
\begin{aligned}
E &= \left| f(x) - \sum_{n=1}^{N} a_n y_n \right|_w^2 \\
&= \|f\|_w^2 - \sum_{n=1}^{N} [a_n^* \langle y_n | f \rangle_w + a_n \langle f | y_n \rangle_w] + \sum_{n=1}^{N} \sum_{n=1}^{N} a_n^* a_m \langle y_n | y_m \rangle_w \\
&= \|f\|_w^2 - \sum_{n=1}^{N} [(u - iv)(\langle y_n | f \rangle_w)^* + (u + iv)\langle f | y_n \rangle_w] + \sum_{n=1}^{N} |a_n|^2 \\
&= \|f\|_w^2 + \sum_{n=1}^{N} (u^2 + v^2) - \sum_{n=1}^{N} [2u \operatorname{Re}\{\langle y_n | f \rangle_w\} + 2v \operatorname{Im}\{\langle y_n | f \rangle_w\}]
\end{aligned}
$$

   - If $\frac{\partial E}{\partial u} = 0 = \frac{\partial E}{\partial v}$, $a_n = \langle y_n | f \rangle_w$, $E = \|f\|_w^2 - \sum_{n=1}^{N} |a_n|^2 \geq 0$, becomes equality at $N \to \infty$

Calculus of variation

1. The functional $G[y] : \{y_k\} \to \mathbb{R} = \int_{\alpha}^{\beta} f(y, y'; x) \mathrm{d}x$
2. $\delta G = \int_{\alpha}^{\beta} \delta y(x) \frac{\delta G}{\delta y(x)} \mathrm{d}x + \ldots = \left[\delta y \frac{\partial f}{\partial y'} \right]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \delta y \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) \right] \mathrm{d}x + \ldots$
3. Euler Lagrange equation $\frac{\delta G}{\delta y(x)} = 0 = \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right)$, $\frac{\partial f}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right)$

4. **First integral** if integrand $f(y, y', x) = f(y, y')$ does not depend on x,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \\ &= \frac{\partial f}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) &= \frac{\partial f}{\partial x} = 0\end{aligned}$$

$$\boxed{y'\frac{\partial f}{\partial y'} - f = c}$$

5. Sturm-Liouville
   - $F[y] = \langle y|\mathcal{L}y\rangle = \int_\alpha^\beta y^*\mathcal{L}y\mathrm{d}x = \int_\alpha^\beta \left[\rho|y'|^2 + \sigma|y|^2\right]\mathrm{d}x$
   - $G[y] = \langle y|y\rangle_w = \int_\alpha^\beta w|y|^2\mathrm{d}x$
   - $\frac{\delta F}{\delta y} = 2\mathcal{L}y$, $\frac{\delta G}{\delta y} = 2wy$
   - $\Lambda[y] = \frac{\langle y|\mathcal{L}y\rangle}{\langle y|y\rangle_w} = \frac{F[y]}{G[y]}$
   - $\frac{\delta\Lambda}{\delta y} = \frac{1}{G}\left[\frac{\delta F}{\delta y} - \Lambda\frac{\delta G}{\delta y}\right] = \frac{2}{G}\left[\mathcal{L}y - \Lambda wy\right]$
   - At extremum, $\mathcal{L}y = \lambda wy = \Lambda[y]wy$, $\Lambda[y]$ is extremized by eigenfunctions of $\tilde{\mathcal{L}} = w^{-1}\mathcal{L}$ and the eigenvalues λ are its extremal values

6.

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}t} &= \frac{\partial L}{\partial t} + \sum_{i=1} N\dot{q}_i\frac{\partial L}{\partial q_i} + \ddot{q}_i\frac{\partial L}{\partial \dot{q}_i} \\ &= \frac{\partial L}{\partial t} + \sum_{i=1} N\dot{q}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + \ddot{q}_i\frac{\partial L}{\partial \dot{q}_i} \\ &= \frac{\partial L}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1} N\dot{q}_i\frac{\partial L}{\partial \dot{q}_i} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[L + \sum_{i=1}^N \dot{q}_i\frac{\partial L}{\partial \dot{q}_i}\right] &= \frac{\partial L}{\partial t} = \frac{\mathrm{d}H}{\mathrm{d}t}\end{aligned}$$

7. Time invariance of L is energy (H) conservation, q invariance of L is $\frac{\partial L}{\partial \dot{q}}$ conservation (generalized momentum)
8. $\phi(x, y, \lambda) = f(x, y) - \lambda p(x, y)$, $\Phi_\lambda[y] = G[y] - \lambda P[y]$

Laplace Poisson

1. Find constants with **boundary conditions**, implicit conditions for **physical** solutions, **continuity of derivative** and **continuity of solution**
2. Poisson's equation $\nabla^2\Phi = \rho(\mathbf{x})$, $\rho = 0$ is Laplace's equation
   - Diffusion $\kappa\nabla^2 u = \frac{\partial u}{\partial t} - S(\mathbf{x})$, in steady state $\frac{\partial u}{\partial t} = 0$. $\nabla^2 u = -\frac{S(\mathbf{x})}{\kappa}$, flux $\mathbf{F} = -\kappa\boldsymbol{\nabla}u$
   - $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_q(\mathbf{x})/\epsilon_0$, $\boldsymbol{\nabla}\times\mathbf{E} = 0 \implies \mathbf{E} = -\boldsymbol{\nabla}\Phi$, $\nabla^2\Phi = -\rho_q(\mathbf{x})\epsilon_0$
   - $\nabla^2\Phi = 4\pi G\rho_m(\mathbf{x})$

- 3D Schrödinger
- Ideal fluid (irrotational, incompressible) $\nabla \times u = 0 \implies u = \nabla\Phi$, continuity (incompressible) $\rho\nabla \cdot u = -\frac{\rho}{t} = S(\mathbf{x})$,$\nabla^2\Phi = 0$

3. Cylindrical $\Psi = \Psi(r, \phi)$, $x = r\cos\phi, y = r\sin\phi$, $\nabla^2\Psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} = 0$

$$\boxed{\Psi = A_0 + B_0\phi + C_0 \ln r + D_0\phi \ln r + \sum_{n=1}^{\infty}(A_n r^n + C_n r^{-n})\cos n\phi + \sum_{n=1}^{\infty}(B_n r^n + D_n r^{-n})\sin n\phi}$$

4. Axisymmetric Spherical $\Psi = \Psi(r, \theta)$, $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, $\nabla^2\Psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) = 0$

$$\boxed{\Psi(r, \theta) = \sum_{l=0}^{\infty}(A_l r^l + B_l r^{-l-1})P_l(\cos\theta)}$$

5.

$P_1(x)$	1
$P_2(x)$	x
$P_3(x)$	$\frac{1}{2}(3x^2 - 1)$
$P_4(x)$	$\frac{1}{2}(5x^3 - 3x)$

6. The solution to Poisson's equation is unique with Dirichlet ($\Phi(\mathbf{r}) = f(\mathbf{r})$) or Neumann ($\frac{\partial\Phi}{\partial n} = \mathbf{n} \cdot \nabla\Phi = f(\mathbf{r})$) boundary conditions on a surface S.
   Prove using difference of two solutions and $\nabla \cdot (\Psi\nabla\Psi) = \nabla\Psi \cdot \nabla\Psi + \Psi\nabla \cdot (\nabla\Psi)$

7. Green's function

$$\begin{cases} \nabla^2 G(\mathbf{r}, \mathbf{r}')_{\mathbf{r}} = \delta^{(3)}(\mathbf{r} - \mathbf{r}'), & \mathbf{r} \textit{ in } V \\ (\text{Dirichlet})\ G = 0, & \mathbf{r} \textit{ on } S \\ (\text{Neumann})\ \frac{\partial G}{\partial n} = \frac{1}{A}, & \mathbf{r} \textit{ on } S \end{cases}$$

where $A = \oint_S \mathrm{d}S$.

   If V is all of space, G is the **fundamental solution**.

8. The fundamental solution in 3D: $\nabla^2 G = \delta^{(3)}(\mathbf{r} - \mathbf{r}')$, $G \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty$
   - Spherically symmetric, $G = G(\mathbf{r})$
   - $\nabla^2 G = (r^2G')'/r^2 = 0$, $G = \frac{C}{r} + A$, $A = 0$
   - S is a sphere of radius ϵ; $\left.\frac{\partial G}{\partial r}\right|_{r=\epsilon} = -\frac{C}{\epsilon^2}$
   - $\int_{r<\epsilon} \nabla^2 G \mathrm{d}V = \oint_{r=\epsilon} \nabla G \cdot \mathbf{n}\mathrm{d}S = -\frac{C}{\epsilon^2}\oint_{r=\epsilon} \mathrm{d}S = -4\pi C$
   - $\nabla^2 G = \delta^{(3)}(\mathbf{r} - \mathbf{r}') = -4\pi C\delta^{(3)}(\mathbf{r} - \mathbf{r}')$, $C = -\frac{1}{4\pi}$
   - $\boxed{G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}}$

9. The fundamental solution in 2D: $\nabla^2 G = \delta^{(2)}(\mathbf{r} - \mathbf{r}')$, $|\nabla G| \to 0 \quad \text{as} \quad |\mathbf{r}| \to \infty$ (or G vanishes in a finite radius)

- Circularly symmetric, $G = G(\mathbf{r})$
- $(rG')'/r = 0 \implies G = C\ln r + A$, $r \neq 0$
- S is a circle with radius ϵ; $\left.\dfrac{\partial G}{\partial r}\right|_{r=\epsilon} = \dfrac{C}{\epsilon}$
- $\displaystyle\int_{r<\epsilon} \nabla^2 G \mathrm{d}A = \oint_{r\epsilon} \boldsymbol{\nabla} G \cdot \mathbf{n} \mathrm{d}l = \frac{C}{\epsilon}\oint_{r=\epsilon} \mathrm{d}l = 2\pi C$
- $\nabla^2 G = \delta^{(2)}(\mathbf{r}-\mathbf{r}') = 2\pi C\delta^{(2)}(\mathbf{r}-\mathbf{r}')$, $C = \dfrac{1}{2\pi}$
- $\boxed{G(\mathbf{r},\mathbf{r}') = \dfrac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'| + C}$

10. Method of images: **Dirichlet** boundary condition — charge of **opposite** sign, to cancel out at the boundary; **Neumann** — charge of **same** sign, to make $\dfrac{\partial G}{\partial n} = 0$ at the boundary
11. For sphere and circle, it is equivalent to a inverse point with a particular strength
12. $\mathbf{F} = \Phi\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Phi$, $\boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Psi) = \boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Psi + \Phi\nabla^2\Psi$ leads to Green's theorem

$$\boxed{\int_V (\Phi\nabla^2\Psi - \Psi\nabla^2\Phi)\mathrm{d}V = \oint_S (\Phi\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Phi)\cdot\mathrm{d}S = \oint_S \left(\Phi\frac{\partial\Psi}{\partial n} - \Psi\frac{\partial\Phi}{\partial n}\right)}$$

(replace V and S to A and l in 2D)

- (Dirichlet) For $\begin{cases}\nabla^2\Phi & = \rho(\mathbf{r}), \quad \text{in } V\\ \Phi(\mathbf{r}) & = f(\mathbf{r}), \quad \text{on } S\end{cases}$
- Let $\color{red}{\Psi = G}$,

$$\begin{aligned}\int_V (\Phi\nabla^2 G - G\nabla^2\Phi)\mathrm{d}V &= \oint_S (\Phi\boldsymbol{\nabla}G - G\boldsymbol{\nabla}\Phi)\cdot\mathbf{n}\mathrm{d}S\\ \int_V \Phi\delta^{(3)}(\mathbf{r}-\mathbf{r}')\mathrm{d}V &= \int_V G\rho\mathrm{d}V + \oint_S f\frac{\partial G}{\partial n}\mathrm{d}S\\ \color{red}{\Phi(\mathbf{r}')} &= \int_V \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}V + \oint_S f(\mathbf{r})\frac{\partial G}{\partial n}\mathrm{d}S\\ \color{red}{\Phi(\mathbf{r}')} &= \int_{\Re^3} \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}V = \int_{\Re^3}\frac{\rho_q(\mathbf{r})}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}\mathrm{d}V\end{aligned}$$

The last line is for all space (sphere of radius ∞)
Turn it into Laplace by setting $\forall\mathbf{r}\in V(\rho(\mathbf{r}) = 0)$

- (Neumann) For $\begin{cases}\nabla^2\Phi & = \rho(\mathbf{r}), \quad \text{in } V\\ \dfrac{\partial\Phi}{\partial n} & = f(\mathbf{r}), \quad \text{on } S\end{cases}$
- Let $G = \Psi$. $\left.\dfrac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n}\right|_{\mathbf{r}\in S} = \dfrac{1}{A}$,

$$\begin{aligned}\Phi(\mathbf{r}') &= \int_V \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}V + \frac{1}{A}\oint_S \Phi(\mathbf{r})\mathrm{d}S - \oint_S f(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}S\\ \Phi(\mathbf{r}') &= \int_V \rho(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}V - \oint f(\mathbf{r})G(\mathbf{r},\mathbf{r}')\mathrm{d}S\end{aligned}$$

where the second line follows from finite surface integral of $\Phi(\mathbf{r})$ and infinite A of V being all space.

Cartesian tensors

1. A vector $\mathbf{v}$ is a set of numbers v_i defined wrt. a set of orthonormal basis vectors $\mathbf{e}_i$ by $\boxed{v_i' = L_{ij}v_j}$, where $L_{ij} = \mathbf{e}_i'\cdot\mathbf{e}_j$ ($\boxed{L_{ij} \text{ is } \textit{orthogonal}, L^TL = I}$)

2. $\boldsymbol{\nabla} = \mathbf{e}_i\partial_i$, $\partial_i \equiv \dfrac{\partial}{\partial x_i}$, $\partial x'_i = L_{ij}\partial x_j$ (by chain rule) ($\boldsymbol{\nabla}$ is therefore a vector, but only in Cartesian coordinates, where covectors same as vectors)

3. a Cartesian axial vector (pseudo-vector) a is a set of coefficients a_i defined wrt. a set of orthonormal basis vectors $\mathbf{e}_i$, s.t. a'_i wrt. another orthonormal basis $\mathbf{e}'_i$ are given by $a'_i = \det(L)L_{ij}a_j$ (same as vector under proper transformation $\det(L) = 1$. Not reversed after improper rotations $\det(L) = -1$)

4. A Cartesian tensor T of order n has n indices $T_{i_n \ldots i_n}$, defined wrt. a set of orthonormal basis vectors $\mathbf{e}_i$ that transforms like $\mathbf{e}'_i = L_{ij}\mathbf{e}_j$. $\boxed{T'_{i_1 \ldots i_n} = L_{i_1 j_1} \ldots L_{i_n j_n} T_{i_1 \ldots i_n}}$

5. Likewise, a Cartesian pseudo-tensor E of order n is $\boxed{E'_{i_1 \ldots i_n} = \det(L) L_{i_1 j_1} \ldots L_{i_n j_n} E_{i_1 \ldots i_n}}$

6. δ_{ij} is a tensor, ϵ_{ijk} is a pseudo-tensor

7. Linear combination of order n tensors are order n tensors (closed under addition)

8. Tensor $n \otimes$ tensor $m \to$ tensor $n + m$; Tensor $n \otimes$ pseudo-tensor $m \to$ pseudo-tensor $n + m$

9. Contraction reduces order n to $n - 2$

10. **Symmetric** - same after swapping two indices; **Antisymmetric** - $\times(-1)$ after swapping. Symmetry is invariant of coordinate system. For symmetric T_{ijk}, antisymmetric E_{pqr}, $T_{ijk}E_{ijr} = 0$

11. Any 2nd order tensor expressed as symmetric and antisymmetric tensors $T_{ij} = S_{ij} + A_{ij}$, $S_{ij} = (T_{ij} + T_{ji})/2$, $A_{ij} = (T_{ij} - T_{ji})/2$

12. Pseudo-vectors are equivalent to 2nd order antisymmetric tensors $\omega_k = \dfrac{1}{2}\epsilon_{ijk}A_{jk}$,

$$A_{ij} = \epsilon_{ijk}\omega_k = \begin{vmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{vmatrix}$$

13. Symmetric tensors are sums of a constant and a traceless symmetric tensor $\tilde{S} = S - \dfrac{\mathrm{Tr}(S)}{3}\mathbb{I}$, $\mathrm{Tr}\left(\tilde{S}\right) = 0$

14. Isotropic tensors have same components in all frames $\boxed{T'_{ijk\ldots} = T_{ijk\ldots}}$. The most general isotropic tensors:
    - 0th order, scalar (all)
    - 1th order, only $\mathbf{0}$ (for both vector & pseudovector)
    - 2nd order, $\lambda\delta_{ij}$, λ is a scalar
    - 3rd order, $\lambda\epsilon_{ijk}$
    - 4th order, $\lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}$

15. isotropic $\neq$ homogeneous

Contour integration

1. Complex derivative $\dfrac{\mathrm{d}f}{\mathrm{d}z} = \lim\limits_{\delta z \to 0} \dfrac{f(z + \delta z) - f(z)}{\delta z}$ is the same for all δz, with $f(z) = u(x, y) + iv(x, y)$, the *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

(Or simply $\dfrac{\partial f}{\partial z^*} = 0 = \dfrac{\partial(u + iv)}{\partial(x - iy)}$), and u, v are harmonic, $\nabla^2 u = \nabla^2 v = 0$, and conjugate harmonic $\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = 0$

2. Cauchy's theorem $\boxed{\oint_C f(z)\mathrm{d}z = 0}$ on a simply connected region R without singularity/$f(z)$ being analytic (C^1); path independent from a to b, $\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z$, if no singularity between C_1, C_2

3. Laurent series $\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n}$, on **annulus** $\alpha < |z-z_0| < \beta$

4. The **residue** of a pole is a_{-1} (coefficient in Laurent series) For a simple pole, $\mathrm{res}_{z=z_0} f(z) = \lim_{z\to z_0}[(z-z_0)f(z)]$; for an N-pole, $\boxed{\mathrm{res}_{z=z_0} = \lim_{z\to z_0}\left[\frac{1}{(N-1)!}\frac{\mathrm{d}^{(N-1)}}{\mathrm{d}z^{N-1}}[(z-z_0)^N f(z)]\right]}$; or just use Laurent series (for essential singularity)

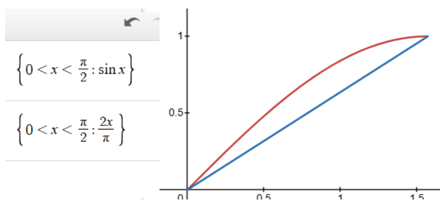
5. Residue theorem $\boxed{\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{res} f(z_k)}$, C is **anticlockwise** and encloses all z_k (hint: Laurent series' negative powers around singularity z_k, shrink C to a circle around it/or $\oint_C f(z)\mathrm{d}z$ subtract (or add clockwise) integrals of these circles $= 0$)

6. For *analytic* $f(z)$ on R, Cauchy's formula $\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\mathrm{d}z}$, $f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\mathrm{d}z$, where simple closed anticlockwise C. So *analytic complex function is infinitely differentiable.*

7. Branch cut for multivalued functions
   - $\ln(z) = \ln r + i(2n\pi + \theta)$
   - $z^{1/a} = re^{i(2n\pi+\theta)/a}$, $a > 1$

8. Jordan's lemma $\lim_{R\to\infty}\int_{C_R} g(z)e^{i\lambda z}\mathrm{d}z = 0$ if
   - $g(z) \to 0$
   - C_R is upper semicircle, $\lambda > 0$, or
   - C_R is lower semicircle, $\lambda < 0$

9. Very important inequality in complex analysis $0 < \theta < \frac{\pi}{2}$, $\sin x > \frac{2x}{\pi}$

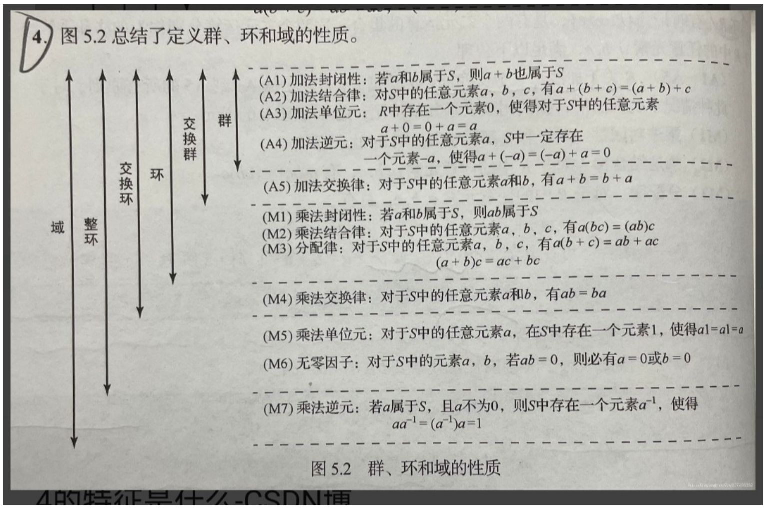


10. Gaussian integration lemma $\forall a \in \mathbb{C}$, $\int_{-\infty}^{\infty} e^{-(u+a)^2}\mathrm{d}u = \sqrt{\pi}$

Small Oscillations

1. $\mathcal{L} = T - V = \frac{1}{2}T_{ij}q_iq_j - \frac{1}{2}V_{ij}q_iq_j$
2. $T_{ij}\ddot{q}_j + V_{ij}q_j = 0$
3. $(-\omega^2\mathbf{T} + \mathbf{V})\mathbf{Q} = \mathbf{0}$, $\mathbf{q} = \mathbf{Q}\sin(\omega(t-t_0))$ or $Q_i = q_i(t-t_0)$ if $\omega = 0$
4. $\mathbf{Q}$ is **generalized engenvector**. $\alpha^{(n)}(t) = Q_i^{(n)}T_{ij}q_j(t) = A^{(n)}\sin\omega_n(t-t_0^{(n)})$ is a **normal coordinate**. (A variable substitution to get a simple $\sin\left(t - t_0^{(n)}\right)$)
5. **Orthonormality** $(\mathbf{Q}^{(m)})^T\mathbf{T}\mathbf{Q}^{(n)} = \delta_{mn}$

Group theory

1. A group $(G, *)$ is a set G and a binary operation $*$ satisfying
   - Identity axiom, $\forall g \in G \exists e,\ e * g = g * e = g$
   - Inverses axiom, $\forall g \in G \exists h \in G,\ h * g = g * h = e$
   - Associativity axiom, $\forall g, h, k \in G,\ (g * h) * k = g * (h * k)$
   - Closedness/Closure, $g_i * g_j \in G$
2. Examples
   - nonzero complex numbers under multiplication $(\mathbb{C}, \times)$
   - 2×2 invertible matrices/general linear group of degree 2 over real numbers under multiplication $(GL_2(\mathbb{R}), *)$
   - 2×2 real matrices under addition $M_2(\mathbb{R}, +)$
   - Integers under addition $(\mathbb{Z}, +)$
   - **Symmetric groups/general permutation groups** S_n/Σ_N, $|S_n| = n!$, defined by $(S, *)$, where S is the set of all permutations of $\{1, 2, \ldots, n\}$, $*$ is a bijection from $\{1, 2, \ldots, n\}$ to itself.
   - $U(n)$, unitary $n \times n$ matrices
   - $GL(n, \mathbb{C})$, $n \times n$ invertible matrices
   - S_n symmetric group/permutation group, $|S_n| = n!$
   - D_n dihedral group, $|D_n| = 2n$, n rotations by $\dfrac{360°}{n}$, n reflections
   - Small abelian and non-abelian groups (link)
3. **Order** of a group G - number of elements in it
4. $g^q = e$, q is the **order** of group element g (if does not exist, infinite order)
5. A **cyclic** group satisfies $G = \{g^n : n \in \mathbb{Z}\}$
6. g_1, g_2 are **generators** if $G = \{\prod_n g_n : n \in \mathbb{Z}, g_n \in \{g_1, g_2\}\}$
7. A group is **abelian** if every two elements commute. Cyclic groups are abelian.
8. 

4. 图 5.2 总结了定义群、环和域的性质。

(A1) 加法封闭性：若a和b属于S，则$a+b$也属于S
(A2) 加法结合律：对S中的任意元素a，b，c，有$a+(b+c)=(a+b)+c$
(A3) 加法单位元：R中存在一个元素0，使得对于S中的任意元素 $a+0=0+a=a$
(A4) 加法逆元：对于S中的任意元素a，S中一定存在一个元素$-a$，使得$a+(-a)=(-a)+a=0$
(A5) 加法交换律：对于S中的任意元素a和b，有$a+b=b+a$
(M1) 乘法封闭性：若a和b属于S，则ab属于S
(M2) 乘法结合律：对于S中的任意元素a、b、c，有$a(bc)=(ab)c$
(M3) 分配律：对于S中的任意元素a，b，c，有$a(b+c)=ab+ac$ $(a+b)c=ac+bc$
(M4) 乘法交换律：对于S中的任意元素a和b，有$ab=ba$
(M5) 乘法单位元：对于S中的任意元素a，在S中存在一个元素1，使得$a1=a1=a$
(M6) 无零因子：对于S中的元素a，b，若$ab=0$，则必有$a=0$或$b=0$
(M7) 乘法逆元：若a属于S，且a不为0，则S中存在一个元素a^{-1}，使得 $aa^{-1}=(a^{-1})a=1$

图 5.2 群、环和域的性质

9. D_n, n-fold dihedral groups, order $2n$. reflections about diagonal
10. Group table: For $g_1 g_2$, apply column g_2 then row g_1. Each row/column is a **complete rearrangement/derangements** of another.
11. A subgroup is a subset that's also a group
12. Let $(G, *)$, $(H, \times)$ be groups

- A **group homomorphism** $f : G \to H$ is a function such that $\forall x, y \in G,\ f(x * y) = f(x) \times f(y)$ (preserve group operation)
- A **group isomorphism** is a bijective group homomorphism
- (Example of homomorphism but not isomorphism: a subgroup, chosen carefully)

13. If $(H, \times)$ is an isomorphism of $(G, *)$, H are $n \times n$ invertible matrices, $\times$ is matrix multiplication, group H is a **faithfaul representation** of G

14. Two elements are **conjugate**, $g_1 \sim g_2$ iff $\exists g,\ g_2 = g g_1 g^{-1}$

15. **Conjugacy classes** of a group are disjoint classes of elements, where the elements in each of them are mutually conjugate. The conjugacy class of g is $\text{Cl}(g) = \{hgh^{-1} | h \in G\}$

16. Conjugacy classes for D_n (nth order dihedral groups) are
    - $\{e\}$
    - $\{R^2, \ldots, R^n\}$
    - $\{m_1, \ldots, m_n\}$

17. A **normal subgroup** H is a subgroup that consists entirely of conjugacy classes of G. G is a subgroup. A **proper normal subgroup** is such group with $H \neq G$. $I = e \in G$.

18. A group G, a subgroup $H = \{I, h_1, h_2 \ldots\}$ of G, $g \in G$, a **left coset** of H in G is $gH = \{g, gh_1, gh_2, \ldots\}$, a **right coset** of H in G is $Hg = \{g, h_1 g, h_2 g, \ldots\}$.

19. Properties
    - $(g_1 g_2)^{-1} = g_2^{-1} g_1^{-1}$
    - $g_1 \sim g_2 \implies g_2 \sim g_1$
    - $g_1 \sim g_2, g_2 \sim g_3 \implies g_1 \sim g_3$
    - The identity e of any group is a conjugacy class by itself
    - Each element of an abelian group is in a class by itself
    - g and g^{-1} may or may not be in the same conjugacy class
    - The left and right coset of any subgroup H of G are identical if G is abelian
    - The left and right coset of any *normal* subgroup H are always identical ($gH = Hg$)
      $gh \in gH$, $ghg^{-1} = h_1 \in H$ because H is normal, $gh = h_1 g \in Hg$. vice versa.
    - Order is the same in cosets $|gH| = |Hg| = |H|$
      elements of cosets are still distinct
    - Two cosets are either disjoint or identical
      For Hg_1, Hg_2, if $h_1 g_1 = h_2 g_2$, $Hg_1 = Hh_1^{-1} h_2 g_2 = Hg_2$ because $h_1^{-1} h_2 \in H$
    - Two cosets Hg_1 and Hg_2 are identical iff $g_1 g_2^{-1} \in H$
    - Every element of G is in some coset
    - The subgroup H and its left cosets partition G
      Because $I \in H$, $\forall g \in G$, $g \in Hg$
    - (**Lagrange's theorem**) $\boxed{\text{If } H \text{ is a subgroup of } G,\ |G| = n|H|,\ n \in \mathbb{Z}^+}$

      where $n = |G : H|$ is the index of H in G,
      the number of distinct left/right cosets of H in G.
    - The order of every $g \in G$ divides $|G|$
      Each element generates a cyclic subgroup of the same order.
      If $|G|$ is prime, $G = C_n$ the cyclic group.

20. All order 4 groups are isomorphic to the cyclic group C_4 or the Vierergruppe/Klein four-group

21. The **kernel** K of group G is the set of all k such that $\Phi(k) = I_H$ (kinda a "nullspace" for groups)

22. For a homomorphic map $\Phi : G \mapsto H$ between two groups $(G, *)$, $(H, \times)$,

- $\Phi(g_1 * g_2) = \Phi(g_1) \times \Phi(g_2)$
- (Identity maps to identity) $\Phi(I_G) = I_H$
- (Inverses maps to inverses) $\Phi(g^{-1}) = [\Phi(g)]^{-1}$
- Example: $(\mathbb{R}, +)$ and $(U(1), \times)$, where $U(x) = \{c \in \mathbb{C} \mid |c| = 1\}$, $\Phi(x) = e^{ix}$, $\Phi(x + 2\pi) = \Phi(x)$ so only surjective, not bijective. Kernel $K = \{2n\pi \mid n \in \mathbb{Z}\}$

23. $\boxed{\text{The kernel is a \textbf{normal} subgroup}}$ of G because
    - It is closed, $\forall k_1, k_2 \in K,\ k_1 * k_2 \in K$
    - $I_G \in K$ (identity maps to identity)
    - inverse exists, $\forall k \in K,\ k^{-1} \in K$
    - $\forall k \in K,\ \Phi(gkg^{-1}) = \Phi(g)\Phi(k)\Phi(g^{-1}) = \Phi(g)I_H[\Phi(g)]^{-1} = I_H \implies gkg^{-1} \in K$
24. The **product** of two cosets is the set of all products of two elements from each set $C_1 \times C_2 = \{c_1c_2 \mid c_1 \in C_1, c_2 \in C_2\}$
25. The **direct product** of two groups is similar but forms ordered pairs. $G \times H = \{(g, h) \mid g \in G, h \in H\}$
    And $(g_1, h_1) \cdot (g_2, h_2) = (g_1 * g_2, h_2 \times h_2)$
    Example: $D_4 = C_2 \times C_2$ (D_4 is Klein four group, C_2 is 2nd order cyclic group)
26. For a normal subgroup K of G, product of cosets $|(g_1K)(g_2K)| = |K| \neq |K|^2$ (not all distinct)
    For a $g \in (g_1K)(g_2K)$, $g = g_1k_1g_2k_2 = g_1(g_2g_2^{-1})k_1g_2k_2 = g_1g_2k_1'k_2 = g_1g_2k_3 \in g_1g_2K$
27. For a cosets of non-normal groups, $|(g_1K)(g_2K)| \neq |K|$
28. A **quotient group**, G/K, for a normal subgroup K is a group of its cosets in G.
    Example: for D_3, $H = \{e, r, r^2\}$, $D_3/H = \{H, sH\}$, because $H \cdot H = H, H \cdot sH = sH$, $sH \cdot sH = H$, it's isomorphic to C_2
29. (**Factorization theorem**)(Proof)
    $\boxed{\text{If } K \text{ is the kernel of a homomorphism } \Phi : G \mapsto H, \text{ then } G/K \text{ is isomorphic to } H.}$
30. **Cayley's theorem**: every order-N finite group is isomorphic to a subgroup of S_n (S_n reminds me of power sets)
31. S_{n-1} is a subgroup of S_n (fix one item)
32. An n-cycle in S_n is a permutation that acts only on the positions $p_r, r = 1, 2, \ldots, n < N$ written as $(p_1, p_2, \ldots, p_n)$ which stands for $\begin{pmatrix} p_1 & p_2 & \cdots & p_n \\ p_2 & p_3 & \cdots & p_1 \end{pmatrix}$
33. Any permutation can be decomposed uniquely into disjoint cycles.
    $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 6 & 1 & 4 & 3 \end{pmatrix} = (1\ 2\ 5\ 4)\,(3\ 6) = (3\ 6)\,(1\ 2\ 5\ 4)$$
34. A two cycle is a transposition/swap. $S_2 = C_2$
35. A permutation is odd/even if it is a product of odd/even permutations. An n-cycle can be decomposed into $(n-1)$ 2-cycles (decomposition *not unique, not disjoint* in general). n cycles's parity depends on $n - 1$.
36. The **cycle shape** of an element S_n is the set of numbers $\{n_2, n_3, \ldots\}$ specifying the number of 2-cycles, 3-cycles,... in the **unique** decomposition into **disjoint** cycles.
37. The disjoint cycles are the conjugacy classes of S_n.
38. G is the direct product of H and J ($G = J \times H = H \times J$) if
    - H and J are **normal subgroups** of G
    - H and J are **disjoint**, apart from the identity
    - G is **generated only by** elements of H and J
39. If $G = H \times J$, G/H is isomorphic to J
    $G/H = \{jH \mid j \in J\}$, $\Phi : j \mapsto jH$ is a homomorphism because H is normal ($j_1j_2H = (j_1H)(j_2H)$)
    $j_1 \neq j_2 \implies j_1H \neq j_2H$, Φ is 1-1 so also isomorphic.

Representations

1. A n-dimensional **representation** of a group G is a homomorphism from G to a subgroup of $GL(n, \mathbb{C})$. It's **faithful** if it's also isomorphic, otherwise **unfaithful**.
2. **Regular representation** G are matrices $D(g)$ generated by
$G = \{g_1 = e, g_2, \dots, g_N\}$, $g^{-1}G = \{g_{p(1)}, g_{p(2)}, \dots, g_{p(N)}\}$, $D_{ij} = \delta_{p(i),j} = \delta_{i,p^{-1}(j)}$
(Row order of $D(g)$ is element order of $g^{-1}G$, $D(e) = \mathbf{I}$)
3. Two sets of matices are **equivalent** if they are similar by an invertible matrix S (similarity transformation)
4. If complex conjugate matrix $D^*(g) \sim D(g)$ for all g, can you make $D(g)$ entirely real? It's unknown, if you can, it's real. Otherwise, D is pseudo-real.
5. **Direct sum** of representations $D^{(1)}(g) \oplus D^{(2)}(g) = \begin{pmatrix} D^{(1)}(g) & 0 \\ 0 & D^{(2)}(g) \end{pmatrix}$ is a block-diagonal matrix.
6. **Direct product** of representations $\otimes$, equations are cumbersome, better just see its picture.
7. The **character** of a representation is $\{\mathrm{Tr}(D(g)) \mid g \in G\}$
   - Due to property of trace, equivalent representation have identical character
   - g in the same *conjugacy class* have the same character
   - Character of regular representation $\{|G|, 0, 0, \dots, 0\}$
8. Invariant subspace of a set of linear operations $\{A_i\}$ is
$W \subseteq V$ such that $x \in W, \mathbf{A}_i x \in W$, $\forall A_i \in \{A_i\}$.

   Remark. higher dimensional "eigenvector"-like stuff
9. A representation $\{D(g)\}$ is irreducible representation (**irrep**) iff $\{0\}$ and $V = \mathbb{C}^n$ are the only invariant subspaces, where dimension of $D(g)$ is n.
10. A 2D non-abelian representation is irreducible (diagonal matrices are abelian)
11. The group-invariant inner product $[\mathbf{x}, \mathbf{y}] = \sum_{g \in G} (\mathbf{D}(g)\mathbf{x}, \mathbf{D}(g)\mathbf{y})$, where $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\dagger \mathbf{y}$
    - $(\mathbf{D}(h)\mathbf{x}, \mathbf{D}(h)\mathbf{y}) = [\mathbf{x}, \mathbf{y}]$, $h \in G$
    - Length is $[\mathbf{x}, \mathbf{x}]^{1/2}$
    - Group representations $D(g)$ **unitary** using this inner product (because length preserved), $GL(n, \mathbb{C})$ similar to $U(n)$
    - Unitary matrices have orthogonal eigenvectors, thus "mutually orthogonal invariant subspaces"
    - The representations of finite groups can always be taken to be unitary with respect to this inner product, if we use **orthonormal basis**.
12. All 1D (unfaithful) represetations are irreducible (no subspace that's not $\{0\}$ or itself). Any two different 1D representations are inequivalent.
13. Character table of **irreps** (irreducible representations) of D_4

	I	R^2	R	R^3	m_1	m_2	m_3	m_4
$\chi_{d^{(1)}}$	1	1	1	1	1	1	1	1
$\chi_{d^{(2)}}$	1	1	1	1	-1	-1	-1	-1
$\chi_{d^{(3)}}$	1	1	-1	-1	1	1	-1	-1
$\chi_{d^{(4)}}$	1	1	-1	-1	-1	-1	1	1
$\chi_{d^{(5)}}$	2	-2	0	0	0	0	0	0

- χ_i^J denotes i-th column, J-th row in the character table above
- First four unfaithful 1D, last one faithful 2D
- Vertical lines separates conjugacy classes
- Column in different conjugacy classes are orthogonal
- Rows are orthogonal (gridlock orthogonality)
- rows sum to 0, except for the 1st one ★
- (**Theorem 1**) The number of inequivalent irreps ρ = the number of **conjugacy classes** c ★
  (Horizontal, vertical, both 5 cells, counting each conjugacy class as one)
- (**Theorem 2**)

  Sum of **squared** dimensions of inequivalent irreps in a conjugacy class $= |G| = \sum_{J=1}^{\rho} d_J^2$ ★

  (only last/fifth representation 2D, others 1D. $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8$)
  A special case of column orthogonality.
- (**Theorem 3**) The dimension d_J of each irrep **divides** $|G|$ ★
- (Corollary of 2) Every irrep has dimension $\leq (|G| - 1)^{1/2}$,
  groups with $|G| < 5$ cannot have have 2d irrep, they're abelian.
  Only $|C_5| = 5$ because 5 is prime,
  so smallet non-abelian group is D_3, $|D_3| = 6$.

14. (**Schur's 1st lemma**) For two irreps with matrices $D_1(g) : U \to U$, $D_2(g) : V \to V$, define the *intertwining operator* $T : U \to V$ to be

$$\forall g \in G, \quad TD_1(g) = D_2(g)T,$$

then

- $T = \mathbf{0}$. The irreps can be equivalent or inequivalent.
- T is invertible, and the irreps must be equivalent. $U = V$ have same dimensions★

15. (**Schur's 2nd lemma**) A matrix that commutes with all $D(g)$ of an irrep is $T = \lambda \mathbf{I}$, $\lambda \in \mathbb{C}$★

16. (**The grand orthogonality theorem**) Let $D^J(g) : G \to V_J$ be matrices of an irrep, $V_J = \mathrm{GL}_d(\mathbb{C})$, J loops through all **inequivalent** irreps, $d = \dim V_J$

$$\frac{1}{|G|} \sum_{g \in G} (D_{ij}^J(g^{-1})) D_{kl}^K(g) = \frac{1}{d} \delta_{jk} \delta_{il} \delta^{JK}$$

where $\delta^{JJ} = 1$ is not summed over ★.

17.

$$\boxed{\sum_{g \in G} D_{ij}^\dagger(g) D_{kl}(g) = \frac{|G|}{d} \delta_{il} \delta_{jk}}$$

18. g in same conjugacy class have the same trace/character
so character $\chi(g) = \operatorname{Tr} D(g)$ is a *class function*.

19. (**Row orthogonality**) Take trace on boths sides $(i = j, k = l)$

$$\sum_{g \in G} \left(\chi^J(g)\right)^* \chi^K(g) = \sum_{i=1}^{c} |C_i| (\chi_i^J)^* \chi_i^K = |G| \delta^{JK} = \begin{cases} |G|, & \text{if} \quad i = j \\ 0, & \text{if} \quad i \neq j \end{cases}$$

where $|C_i|$ is size of C_i,
χ_i is $\chi(g)$ for g in C_i,
C_i is the i-th conjugacy class.

Remark. χ are c-dimensional vectors, for ρ inequivalent irreps, we have ρ distinct χ. $\rho \leq c$ for linear independence

20. (**Column orthogonality**) ★

$$\sum_{J=1}^{\rho}(\chi_i^J)^*\chi_j^J = \begin{cases}\dfrac{|G|}{|C_i|}, & \text{if} \quad i=j \\ 0, & \text{if} \quad i\neq j\end{cases}$$

Remark. χ are ρ dimensional vectors, we have c distinct such vectors, $c \leq \rho$.

21. ★ $\rho = c$ because of column and row orthogonality, $c \leq \rho$ and $\rho \leq c$.

22. $\chi_1^J = d_J$; for each irrep J, $\chi_1 = \chi(e) = Tr(I^{(d_J)}) = d_J$ is the dimension of the representation.

23. ★ From column orthogonality, the first column (identity e), $\sum_{J=1}^{\rho}|\chi_1^J|^2 = \sum_{J=1}^{\rho} d_J^2 = \dfrac{|G|}{|C_1|}$, $|C_1|$ is the size of first conjugacy class, $\{e\}$, $|C_1| = 1$.

24. $\chi(D_1(g) \oplus D_2(g)) = \chi(D_1(g)) + \chi(D_1(g))$,
$\chi(D_1(g) \otimes D_2(g)) = \chi(D_1(g))\chi(D_1(g))$

25. (**Decomposition** of a reducible representation)

$$\mathbf{S}\mathbf{D}(g)\mathbf{S}^{-1} = \bigotimes_{J=1}^{\rho}\left(\mathbf{I}^{(n_J)} \otimes \mathbf{D}^J(g)\right)$$

where $\mathbf{I}^{(n)}$ is $n \times n$ identity matrix,
J loops through ρ inequivalent irreps,
n_J is number of times J-th irrep $\mathbf{D}^J(g)$ occured.

26. Take trace on both sides, $\chi(g) = \sum_{J=1}^{\rho} n_J\chi^J(g)$

27. $\boxed{n_J = \langle\chi,\chi^J\rangle = \dfrac{1}{|G|}\sum_{i}^{c}|C_i|\chi(g)(\chi^J(g))^* = \dfrac{1}{|G|}\sum_{g\in G}\chi(g)^*\chi^J(g)}$ is always an integer (multiplicity of χ^J in χ)

28. ★ Let χ^1 be the trivial irrep (all 1s, so $\chi^1(g) = 1$), $\langle\chi^J,\chi^1\rangle = \dfrac{1}{|G|}\sum_{g\in G}\chi^J(g) = \delta^{J1} \implies \sum_{g\in G}\chi^J(g) = |G|\delta^{J1}$

Other tricks

1. Remove poles/infinities by introducing a small value, and set it to 0 after all the calculations
Examples

   - (Lecture notes) Evaluate $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\mathrm{d}k$ by considering $\dfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-\epsilon|k|}\mathrm{d}k$
   - (Lecture notes) Prove $\mathcal{F}[H(x-a)] = \dfrac{e^{-ika}}{\sqrt{2\pi}ik}$ with a similar technique. (Also relate $\mathcal{F}[f]$ with $\mathcal{F}[f']$)
   - (Problem sheet 1)

     A time-independent magnetic field $\boldsymbol{B}(\boldsymbol{r})$ is given by

     $$\boldsymbol{B} = \frac{\mu_0 I}{2\pi}\frac{\boldsymbol{e}_z \times \boldsymbol{r}}{x^2+y^2},$$

     where μ_0 is the magnetic permeability and I is a constant. Using Cartesian coordinates, calculate the electric current density $\boldsymbol{J}$ given by the steady Maxwell equation $\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0\boldsymbol{J}$. Also evaluate $\oint_C \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r}$, where C is a circle of radius a in the plane $z=0$ and centred on $x=y=0$. Discuss whether Stokes's theorem applies in this situation.

- (Wikipedia) Sturm-Liouville operator $L = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right] + q(x)$
- (Wikipedia) Boundary condition operator $\mathbf{D}u = \begin{pmatrix} A_1\frac{\mathrm{d}}{\mathrm{d}x} + B_1\Big|_{x=0} \\ A_2\frac{\mathrm{d}}{\mathrm{d}x} + B_2\Big|_{x=l} \end{pmatrix}$
- (Wikipedia) d'Alembert operator $\Box = \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\nu\partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

Examples

1. $\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}\mathrm{d}x = \pi$, answer, a theorem
2. (hint: Feymann's trick) $\int_{-\infty}^{\infty}\frac{x^2e^x}{(e^x+1)^2}\mathrm{d}x = \pi^2/3$, answer
3. Scale solution Φ of $\nabla^2\Phi = 0$ to fit similar boundary conditions: $\Phi\left(x, \frac{y}{a}\right)$ is wrong. $\Phi\left(\frac{x}{a}, \frac{y}{a}\right)$ is correct (Lent question sheet 4, Q1)

Proofs

1. ★

$$
\begin{aligned}
1 = \lim_{x\to 0}\int_{\zeta-\epsilon}^{\zeta+\epsilon}\delta(x-\zeta)\mathrm{d}x &= \lim_{x\to 0}\int_{\zeta-\epsilon}^{\zeta+\epsilon}\mathcal{L}G(x,\zeta)\mathrm{d}x \\
&= \lim_{x\to 0}\int_{\zeta-\epsilon}^{\zeta+\epsilon}\left(\frac{\partial^2 G}{\partial x^2} + p(x)\frac{\partial G}{\partial x} + qG\right)\mathrm{d}x \\
&= \lim_{x\to 0}\int_{\zeta-\epsilon}^{\zeta+\epsilon}\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x} + pG\right) + \left(-\frac{\partial p}{\partial x} + q\right)G\mathrm{d}x \\
&= \lim_{x\to 0}\int_{\zeta-\epsilon}^{\zeta+\epsilon}\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial x} + pG\right)\mathrm{d}x \\
&= \lim_{x\to 0}\left[\frac{\partial G}{\partial x} + pG\right]_{\zeta-\epsilon}^{\zeta+\epsilon}
\end{aligned}
$$

because p, q are continuous.

Thus G is continuous, satisfies the same boundary condition as y and $\lim_{x\to 0}\left[\frac{\partial G}{\partial x}\right]_{\zeta-\epsilon}^{\zeta+\epsilon} = 1$.

2. Schur's first lemma ★
   - Show Kernel$(T) \subseteq U$ and Image$(T) \subseteq V$ are invariant subspaces of $\{D_1(g) \mid g \in G\}$ and $\{D_2(g) \mid g \in G\}$ by using definitions on elements in them
   - If $T \neq \mathbf{0}$, Kernel$(T) = \{0\}$, Image$(T) = V$, so T is one-to-one and onto, invertible
   - $D_1(g) = T^{-1}D_2(g)T$, the two irreps are equivalent
3. Schur's second lemma ★
   - $D(g)T = TD(g)$
   - whether T is singular or not, $T - \lambda I$ is singular, where λ is one of its eigenvalues
   - $D(g)T - \lambda D(g) = TD(g) - \lambda D(g)$, $D(g)(T-\lambda I) = (T-\lambda I)D(g)$
   - use first lemma, $T - \lambda I$ is invertible or 0
   - $T - \lambda I = 0$, $T = \lambda I$
   - Better explanantion here

4. Grand orthogonality theorem ★

   - Let $d = \dim V_J$, M be any $d \times d$ matrix, $T = \sum_{g\in G} D^J(g^{-1})MD^K(g)$
   - $gG = G$, element on whole group causes a derangement, $D^J(g)T = T = TD^K(g)$
   - If $J = K$, use Schur's second lemma, $T = \lambda(M)\mathbf{I}\delta^{JK}$
   - $\sum_{g\in G} D^J_{ij}(g^{-1})M_{jk}D^K_{kl}(g) = \lambda(M)\delta_{il}\delta^{JK}$, λ depends on M
   - $\sum_{g\in G} D^J_{ij}(g^{-1})D^K_{kl}(g) = \lambda_{jk}\delta_{il}\delta^{JK}$, where $\lambda_{jk} = \lambda(M^{jk})$, M^{jk} has 1 at M_{jk}, otherwise 0
   - $$\begin{aligned}\lambda_{jk}\delta_{ii}\delta^{JJ} &= \sum_{g\in G} D^J_{ij}(g^{-1})D^J_{ki}(g) = \sum_{g\in G} D^J_{ij}(g^{-1})D^J_{jk}(g)D^J_{jk}(g^{-1})D^J_{ki}(g)\\ &= \sum_{g\in G} D^J_{ik}(e)D^J_{ki}(e) = \sum_{g\in G} D_{jk}(e) = \sum_{g\in G} \delta_{jk} = |G|\delta_{jk}\end{aligned}$$
     $\delta_{ii} = d$, $\delta^{JJ} = 1$ because J is not summed over
   - $\lambda_{jk} = \frac{1}{d}|G|\delta_{jk}$
   - $D(g^{-1}) = (D(g))^{-1} = (D(g))^\dagger$ because D is unitary wrt. group invariant inner product
   - $\frac{1}{|G|}\sum_{g\in G}(D^J_{ij}(g^{-1}))^\dagger D^K_{kl}(g) = \frac{1}{d}\delta_{jk}\delta_{il}\delta^{JK}$

5. Column orthogonality of character table ★

   - Not obvious but here we are
   - $\langle \chi^J_i, \chi^K_i\rangle = \frac{1}{|G|}\sum_{i=1}^{c}|C_i|(\chi^J_i)\chi^K_i = \delta^{JK}$
   - Define character table matrix $\mathbf{X}$ using $\mathbf{X}_{iJ} = \chi^J_i$ (subscript horizontal/G, superscript vertical/C_i)
   - $\mathbf{I} = \mathbf{X}\frac{1}{|G|}\begin{pmatrix}|C_1| & \cdots & 0\\ 0 & \ddots & 0\\ 0 & \cdots & |C_c|\end{pmatrix}\mathbf{X}^T = \mathbf{X}\mathbf{D}\mathbf{X}^T$
   - $\mathbf{X}$ is $\rho \times c$, $\mathbf{D}$ is $c \times c$, so it fits. $\rho \neq c$ in general, yet.
   - $\mathbf{XDX}^T = I, \mathbf{X}^T = \mathbf{D}^{-1}\mathbf{X}^{-1}, \mathbf{X}^T\mathbf{X} = \mathbf{D}^{-1} = |G|\begin{pmatrix}\frac{1}{|C_1|} & \cdots & 0\\ 0 & \ddots & 0\\ 0 & \cdots & \frac{1}{|C_c|}\end{pmatrix}$
   - Products of columns of $\mathbf{X}$ goes to diagonal
   - take $(\cdot)_{ij}$ on both sides, $\sum_{J=1}^{\rho}\chi^J_i\chi^J_j = \begin{cases}\frac{|G|}{|C_i|} \text{ or } \frac{|G|}{|C_j|}, & \text{if } i = j\\ 0, & \text{if } i \neq j\end{cases}$

6. ★ Proof here

   - Because I just want to sleep, see link above
   - I would rather just start over everything reading abstract algebra in the future