Experimental methods

1. Uncertainty principle in signal processing (purely mathematical) $\Delta f \Delta t = 2\pi$, Δt is time-window/total sample time.

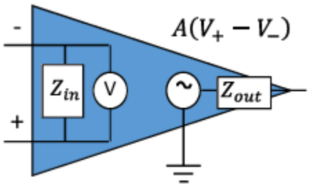
2. $E^2 = p^2c^2 + m_0^2c^4$ because $p = \gamma m_0 v$, $E = \gamma m_0 c^2$

3. Johnson-Nyquist noise: current/voltage fluctuation in resistor due to thermal fluctuations

$$V_{\mathrm{RMS}} = \sqrt{4k_B T R \Delta f}$$

where R is resistance,
Δf is measurement bandwidth.

4. Golden rule

   - Gain G is infinite
   - 

   - Input impedance $Z_{in} = \dfrac{\partial V_{in}}{\partial I_{in}}$ is ∞
   - Output impedance $Z_{out} = \dfrac{\partial V_{out}}{\partial I_{out}}$ is 0
   - The input I_- is 0 (because Z_{in} is infinite)
   - $V_- = V_+$ (only if not saturated, **negative feedback** established)
     (A question on positive feedback & saturated op-amp, results in oscillating output)

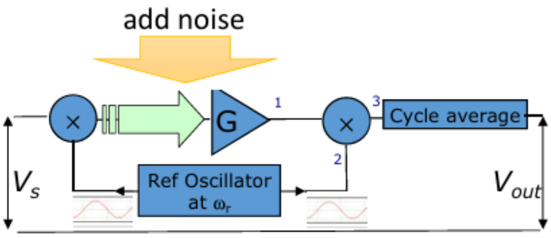
5. Non-ideal op-amp

   - $V_{out} = G(V_+ - V_-)$
   - Gain G not infinite, $10^4 \sim 10^6$
   - Z_{in} not infinite, Z_{out} not 0
   - Gain G is complex and function of frequency
   - Finite slew rate $\dfrac{\partial V_{out}}{\partial t}$
   - An input bias current independent of V_{in}
   - An output bias voltage independent of $(V_+ - V_-)$

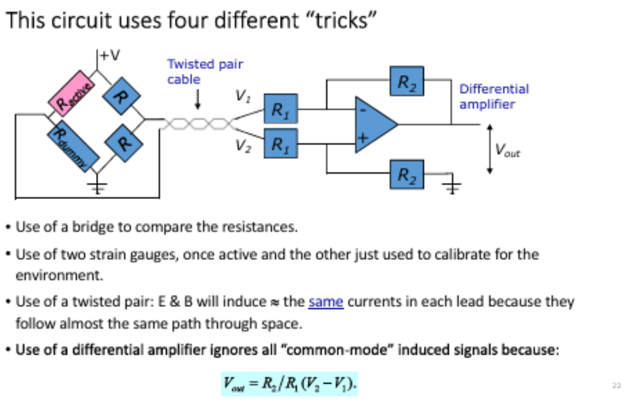
6. - Sd of a set of numbers: $\sqrt{\dfrac{1}{N}\sum_i (x_i - \overline{x})^2}$
   - Estimated Sd of a population, if $\overline{x}$ is also estimated here $\sqrt{\dfrac{1}{N-1}\sum_i (x_i - \overline{x})^2}$ (Bessel's correction)
   - Uncertainty/Sd in mean $= \dfrac{\sum \text{uncertainty of each measurement}}{\sqrt{N}}$
   - Gaussian error propagation $\sigma_f^2 = \left(\dfrac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 \sigma_y^2$

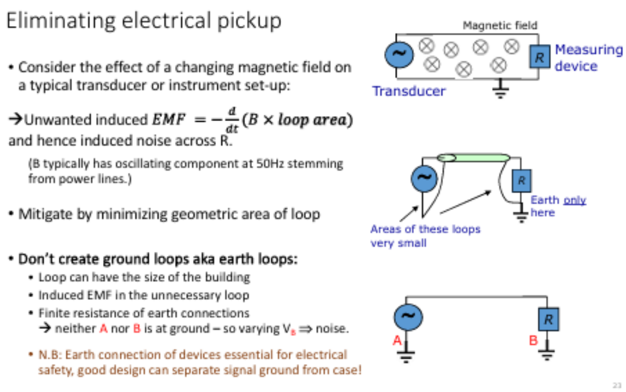
7. Noise

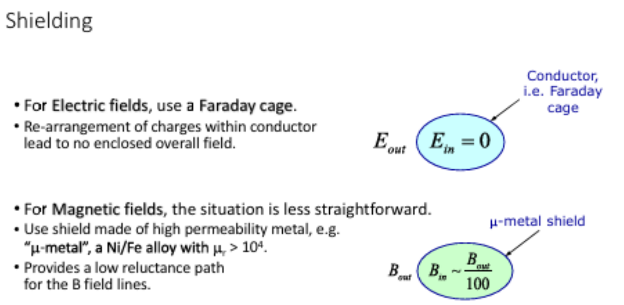
   - White noise (Johnson-Nyquist, shot noise, thermal fluctuations)
   - $1/f$-noise/pink noise, worst at low frequency/DC
   - $1/f^2$-noise/Brownian noise, random walk

8. Eliminating noise
    - Filter out $1/f$ noise, high pass filter (Switch on and off, if DC)
    - $1/f$ noise, phase sensitive detection with high ω
    - Differential experiment, measure change $\Delta f = f_1 - f_2$
    - Shielding (Electromagnetism, heat)
    - Eliminate source
        - Remote - away from electricity interference/vibration
        - High - above atmosphere
        - Antarctic - Dry/High/Cold
        - Space - All above and gravity free
9. Phase-sensitive detection - eliminate $1/f$ nosie
    - 

    - Modulator at frequency ω_r, left $\cos(\omega_r t + \phi)$, right $\cos(\omega_r t)$
    - Signal V_s, noise $V_n = V_n(\omega)$
    - At 1, $V_1 = V_s G \sin(\omega t + \phi)$
    - At 3, $V_3 = V_s G \sin(\omega t + \phi) \sin(\omega t) = \frac{V_s G}{2}[\cos(\phi) - \cos(2\omega t + \phi)]$
    - $\langle V_3 \rangle = \frac{V_s G}{2} \cos(\phi)$
    - At green arrow and op-amp, noise of frequency ω_n produced
    - $\langle V_n \rangle$ not zero if $\omega = \omega_n$
    - Set ω high, so noise is small
10. Mechanical vibration - air cushion
    Thermal noise - reduce radiation/convection/evaporation - a lid, shiny shielding







Circuits -

11. - Binomial $P(r,p,N) = \frac{N!}{r!(N-r)!}p^r(1-p)^{N-1}$, $\sum_{r=0}^{N} P(r) = 1$, $\langle r\rangle = Np$, $Var(r) = Np(1-p)$
    - Poisson $\frac{\lambda^r}{r!}e^{-\lambda}$, mean=**variance**=$\lambda$
      (shot noise, let $p \to 0$, $N \to \infty$, $\langle r\rangle = \lambda$)
    - Gaussian $\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$, mean μ, variance σ^2
      1,2,3 sd - 68,95,99.7%
      (Johnson noise, thermal fluctuations, $\sim k_B T$)
    - Chi-square $\chi^2(x,n) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}, & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases}$,
      degree of freedom n, mean n, variance $2n$.
12. Likelihood $\prod_i p(y_i|\mathbf{a})$ should be maximized
13. a is parameter(s)
14. Posterior $p(a|\text{data}) = p(\text{data}|a)\frac{p(a)}{p(\text{data})}$, $p(a)$ is prior. ($p(\text{data})$ is a normalizing constant)
15. Choise of prior

- uniform in log space, $p(a) = 1/a$
- uniform $p(a) =$ constant, most common so $p(\text{param}|\text{data}) = p(\text{data}|\text{param})$
- Better, complicated assumption

16. Straight line fitting, constant prior

    - $p(y_i|a) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(y_i - f(x_i|a))^2/2\sigma_i^2}$
    - $L(\mathbf{y}|a) = \prod_i^n p(y_i|a)$
    - to maximize $\ln(L)$, minimize $\chi^2 = \sum_i \left(\dfrac{y_i - f(x_i|a)}{\sigma_i}\right)^2$
    - $f(x_i|a) = mx_i + c$, $a = \mathbf{a} = m, c$
    - $\dfrac{\partial\chi^2}{\partial c} = 0 = \dfrac{1}{N}\dfrac{\partial\chi^2}{\partial c} = \dfrac{1}{N}(-2)\sum_i y_i - (mx_i + c) \implies \overline{y} = m\overline{x} + c$
    - $\dfrac{\partial\chi^2}{\partial m} = 0 = \dfrac{1}{N}\dfrac{\partial\chi^2}{\partial m} = \dfrac{1}{N}(-2)\sum_i x_i(y_i - mx_i - c) \implies \overline{xy} = m\overline{x^2} + c\overline{x}$
    - $m = \dfrac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{xx} - \overline{x}\,\overline{x}} = \dfrac{Cov(x,y)}{Var(x)}$
    - $c = \overline{y} - m\overline{x}$
    - sd of y is $\dfrac{1}{N-2}\sum_i (y_i - (mx_i + c))^2$
    - $\dfrac{\partial\overline{y}}{\partial y_i} = \dfrac{1}{N}, \dfrac{\partial\overline{xy}}{\partial y_i} = \dfrac{x_i}{N}$
    - $\sigma_m = \sigma^2 \sum_i \left(\dfrac{\partial m}{\partial y_i}\right) = \dfrac{\sigma^2}{N(\overline{xx} - \overline{x}\,\overline{x})}$
    - $\sigma_c = \sigma^2 \sum_i \left(\dfrac{\partial c}{\partial y_i}\right) = \dfrac{\sigma^2\overline{xx}}{N(\overline{xx} - \overline{x}\,\overline{x})}$
    - Weighting by $1/\sigma_i^2$ instead of 1, $\overline{y} = \dfrac{\sum_i y_i/\sigma_i^2}{\sum_i 1/\sigma_i^2}$, reduces to $\overline{y} = \dfrac{\sum_i y_i}{\sum_i 1}$ above
    - Assuming:
      - Uniform prior
      - Only errors in y_i
      - y_i Gaussian distribution
    - Remember: Gaussian, min χ^2, $y = mx + c$, $\dfrac{1}{N-2}$, $1/\sigma_i^2$ weighting

17. χ^2 test, for binned data (defined differently as above), $\chi^2 = \sum_i \dfrac{(O_i - E_i)^2}{E_i}$ =degree of freedom=No.data-No.parameter

18. Other tests

    - Non-parametric statistics - look at your data
    - Runs test: non-random pattern despite χ^2 fits well?
    - Sign test: x and y have same distribution?
    - Mann-Whitney test: 2 samples from same distribution?
    - Kolmogorov-Smirnov test: 2 distributions different?

19.

Oscillations

1. $\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F}{m}$, $\omega_0 = \sqrt{\frac{k}{m}}, \gamma = \frac{b}{m}, Q = \frac{sqrtmk}{b}$

2. $L\ddot{q} + R\dot{q} + \frac{1}{C}q = V(t)$, $\omega_0 = \frac{1}{\sqrt{LC}}\gamma = R/L, Q = \frac{1}{R}\sqrt{\frac{L}{C}}$

3. $Q = \frac{\omega_0}{\gamma}$

4. Response function
   - $x = \mathcal{R}(x_0 e^{i\omega t})$, $x_0 = Ae^{i\phi_x}$, $F = \mathcal{R}(F_0 e^{i\omega t})$, $F_0 = Be^{i\phi_F}$
   - $R(\omega) = \frac{x_0}{F_0} = \frac{1}{m[(\omega_0^2 - \omega^2) + i\gamma\omega]}$
   - $|R| = \frac{1}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$
   - $\arg(R) = \arctan\left[\frac{-\gamma\omega}{(\omega_0^2 - \omega^2)}\right]$
   - Resonance $\omega_a = \omega_0\sqrt{1 - \frac{\gamma^2}{2\omega_0^2}} = \omega_0\sqrt{1 - \frac{1}{2Q^2}}$
   - $v_0 = i\omega x$, $\left|\frac{v_0}{F_0}\right| = \frac{1}{m\sqrt{\frac{(\omega_0^2-\omega^2)^2}{\omega^2} + \gamma^2}}$
   - $\arg\left(\frac{v_0}{F_0}\right) = \arctan\left[\frac{\omega_0^2 - \omega^2}{\gamma\omega}\right]$, $\omega_v = \omega$
   - $a_0 = \frac{-F_0}{m[(\omega_0^2 - \omega^2)/\omega^2 + \frac{i\gamma}{\omega}]}$
   - $\omega_{\text{acc}} - \omega_0\left(1 - \frac{1}{2Q^2}\right)^{-1/2}$, $\omega_a\omega_{\text{acc}} = \omega_0^2$
   - $\langle P\rangle = \frac{1}{2}\Re(F_0 v_0^*) = \frac{1}{2}\Re v_0 v_0^* m[(\omega_0^2 - \omega^2)/i\omega + \gamma] = \frac{1}{2}m\gamma|v_0|^2 = \frac{1}{2}b|v_0|^2 = \langle P_{\text{dissipated}}\rangle$
   - Mechanical Impedance $Z = \frac{F_0}{v_0} = m\left[\frac{\omega_0^2 - \omega^2}{i\omega} + \gamma\right]$, $\langle P\rangle = \frac{1}{2}|v_0|^2\Re(Z)$

5. $\boxed{\Re(A)\Re(B) = \frac{1}{2}(A + A^*)\frac{1}{2}(B + B^*) = \frac{1}{2}\Re(AB + AB^*)}$

6. $P = Fv = \Re(F_0 e^{i\omega t})\Re(v_0 e^{i\omega t}) = \frac{1}{2}\Re(F_0 v_0 e^{2i\omega t} + F_0 v_0^*)$, $\langle P\rangle = \frac{1}{2}\Re(|F_0||v_0|e^{i(\phi_F - \phi_v)}) = \frac{1}{2}|F_0||v_0|\cos(\phi_F - \phi_v)$

7. $\langle P\rangle = \frac{1}{2}\Re(V_0 I_0^*) = \frac{1}{2}|I_0|^2 R$

8. $\omega_P = \omega_v = \omega_0$

9. At $\omega = \omega_\pm$, $|v_{\max}|/|v_0| = 1/\sqrt{2}$, $\Delta\omega = \omega_+ - \omega_- = \gamma$, $Q = \frac{\omega_0}{\Delta\omega}$

10. Two coherent cosine driving forces, $x = A_1\cos(\omega t + \alpha_1 + \phi) + A_2\cos(\omega t + \alpha_2 + \phi)$, $A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)$

11. $\frac{\partial^2\Psi}{\partial t^2} = v^2\frac{\partial^2\Psi}{\partial x^2}$, phase speed $v = \sqrt{T/\rho}$

12. $\Psi(x,t) = \Re(Ae^{i(\omega t - kx)})$, k is the wavenumber

13. $\boxed{\frac{\partial^2\Psi}{\partial t^2} = v^2\nabla^2\Psi}$

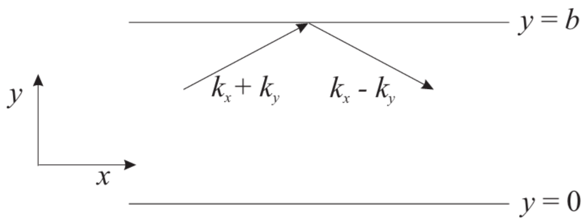
14. $\Psi(\mathbf{r},t) = \Re(Ae^{i\omega - i\mathbf{k}\cdot\mathbf{t}})$, $\mathbf{k}$ is the wavevector

15. Spherical waves $v^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) = \frac{\partial^2 \Psi}{\partial r^2}$, $\Psi(r,t) = \frac{f(r \pm vt)}{r} = \Re(\frac{Ae^{i\omega t - ikr}}{r})$
16. Cylindrical waves $v^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Psi}{\partial r}\right) = \frac{\partial^2 \Psi}{\partial r^2}$, for $r \gg \lambda$, $\Psi(r,t) \approx \Re(\frac{f(r \pm vt)}{\sqrt{r}})$
17. Wave impedance (can be **negative**)
    - $\boxed{Z = \frac{\text{transverse driving force}}{\text{transverse velocity}} = \frac{-T\frac{\partial \Psi}{\partial x}}{\frac{\partial \Psi}{\partial t}} = \frac{T}{v} = \sqrt{T\rho} = \rho v}$
    - Power=transverse force×transverse velocity, $\langle P \rangle = \frac{1}{2}\Re(Fu^*) = \frac{1}{2}\Re(Z)|u|^2$
    - $u = i\omega A_0 e^{i(\omega t - kx)} \implies \boxed{\langle P \rangle = \frac{1}{2}\Re(Z)\omega^2 A_0^2}$
    - $\frac{\mathrm{dKE}}{\mathrm{d}t} = \frac{1}{2}\rho\left(\frac{\partial \Psi}{\partial t}\right)^2$, $\frac{\mathrm{dPE}}{\mathrm{d}x} = \frac{1}{2}T\left(\frac{\partial \Psi}{\partial x}\right)^2$, $v^2 = \frac{\omega^2}{k^2} = \frac{T}{\rho} \implies \mathrm{KE} = \mathrm{PE}$
    - $\langle P \rangle = \frac{\mathrm{d}\langle E \rangle}{\mathrm{d}x} v = \frac{\mathrm{d}\langle \mathrm{KE} \rangle + \langle \mathrm{PE} \rangle}{\mathrm{d}x} = \frac{1}{2}\rho v \omega^2 A_0^2 = \frac{1}{2} Z\omega^2 A_0^2$ (Note $\Re(e^{ix})^2 = \Re(e^{2ix} + 1) \neq \Re(e^{2ix})$)
18. Transverse rope wave at a boundary, same frequency, different wavelength
    - $A_1 e^{i(\omega t - k_1 x)}, B_1 e^{i(\omega t + k_1 x)}, A_2 e^{i(\omega t - k_2 x)}$, incident, reflected, transmitted
    - Continuous displacement Ψ, $A_1 + B_1 = A_2$
    - Continuous transverse force $\boxed{-T\frac{\partial \Psi}{\partial x} \propto Z\Psi}$, $Z_1(A_1 - B_1) = Z_2 A_2$
    - $r = \frac{B_1}{A_1} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$, $\tau = \frac{A_2}{A_1} = \frac{2Z_1}{Z_1 + Z_2}$
    - $R = \frac{\langle P_{B_1} \rangle}{\langle P_{A_1} \rangle} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$, $T = \frac{\langle P_{A_2} \rangle}{\langle P_{A_1} \rangle} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2}$, $R + T = 1$
    - Antiphase $Z_2 = \infty$, in-phase $Z_2 = 0$, matched $Z_1 = Z_2$
19. Longitudinal wave in gas, for displacement a,
    - Adiabatic process $pV^\gamma = pV^{C_p/C_V} = \text{constant}$
    - $\boxed{-\frac{\partial \Psi_p}{\partial x} = \rho\frac{\partial^2 a}{\partial t^2}}$
    - $\boxed{\Psi_p = \mathrm{d}p = -\gamma p \frac{\mathrm{d}\Delta V}{\mathrm{d}V} = -\gamma p \frac{\partial a}{\partial x}}$
    - $\frac{\partial \Psi_p}{\partial x} = -\gamma p \frac{\partial^2 a}{\partial x^2} - \gamma \frac{\partial p}{\partial x}\frac{\partial a}{\partial x} \approx -\gamma p \frac{\partial^2 a}{\partial x^2}$
    - $\boxed{\frac{\partial^2 a}{\partial x^2} = \frac{\rho}{\gamma p}\frac{\partial^2 a}{\partial t^2} = v^2 \frac{\partial^2 a}{\partial t^2}}$, $v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma nRTV}{Vm}} = \sqrt{\frac{\gamma RT}{M}}$,
    where M is the molar mass.
    - $\frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}k_B T \implies v_{\mathrm{rms}} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3RT}{M}} \geq v$ (phase speed of sound wave is slightly smaller than rms speed of gas)
    - $a = a_0 e^{i\omega t - ikx}$, $\Psi_p = -\gamma p \frac{\partial a}{\partial x} = i\gamma pka$
    - Acoustic impedance $\boxed{\mathcal{L} = \sqrt{\gamma p \rho} = v\rho = \frac{\gamma p}{v}}$, impedance $Z = \frac{\text{force}}{\text{velocity}} = \frac{\Delta S \Psi_p}{\dot{a}} = \Delta S \mathcal{L}$
    - Pressure amplitude $A = i\gamma pka_0$ is the amplitude of pressure $\Psi_p = Ae^{i\omega t - ikx} = i\gamma pka_0 e^{i\omega t - ikx}$
    - Intensity $I = \frac{1}{2}\Re(\Psi_p \dot{a}^*) = \frac{1}{2}\gamma pk\omega a_0^2 = \frac{|A|^2}{2\mathcal{L}} = \frac{A_{\mathrm{rms}}^2}{\mathcal{L}} = \frac{1}{2}\mathcal{L}\omega^2|a_0|^2$ (mean P per unit area)

- dBA $= 10\log_{10}\left(\frac{I}{I_{ref}}\right) = 20\log_{10}\left(\frac{o_{\text{rms}}}{p_{\text{ref}}}\right)$ (p_{ref} is a rms value)

20. **Longitudinal** sound waves
    - $\Psi_p = -K\frac{\partial a}{\partial x}$, K is elastic modulus for any medium
      - In adiabatic gas, $K = B = \gamma p$ (B is the bulk modulus), correct assumption
      - In isothermal gas, $K = B = p$, bad assumption for sound in air
      - In solid, $K = E$, Young's modulus
    - $v = \sqrt{\frac{K}{\rho}}$, gas $v = \sqrt{\frac{\gamma p}{\rho}}$, solid $v = \sqrt{\frac{E}{\rho}}$, non-dispersive
    - $\mathcal{L} = \frac{K}{v} = \sqrt{K\rho}$

21. Damped waves on a string, damping force on dx is $-\beta\dot{\psi}\mathrm{d}x$
    - $\boxed{\frac{\partial^2\Psi}{\partial t^2} + \frac{\beta}{\rho}\frac{\partial\Psi}{\partial t} = v^2\frac{\partial^2\Psi}{\partial x^2}}$, $\Gamma = \frac{\beta}{\rho}$
    - trial solution $\psi = \psi_0 e^{i(\omega t - kx)}$, ω is real, k is complex in general ($k = k_r - ik_i$)
    - $\frac{\partial^2\Psi}{\partial t^2} + \Gamma\frac{\partial\Psi}{\partial t} = v^2\frac{\partial^2\Psi}{\partial x^2}$, $\omega^2 - i\Gamma\omega = v^2k^2$
    - $k_r^2 - k_i^2 = \frac{\omega^2}{v^2}$, $2k_rk_i = \frac{\Gamma\omega}{v^2}$
    - Light damping $\Gamma \ll \omega$, $k_r \approx \omega/v$, $k_i \approx \Gamma/(2v)$
    - Heavy damping $k_r \approx k_i \approx \left(\frac{\Gamma\omega}{2v^2}\right)^{1/2}$
    - Find both by solving quadratic and approximating square root at the last step
    - The **characteristic impedance** $Z = \frac{F}{v} = \frac{-T\Psi'}{\dot{\Psi}} = \frac{Tk}{\omega} = \frac{T}{\omega}(k_r - ik_i)$
    - Light damping $Z(\omega) = \frac{T}{v}\left(1 - \frac{i\Gamma}{2\omega}\right) = Z_0\left(1 - \frac{i\Gamma}{2\omega}\right)$, where Z_0 is Z with zero damping
    - Heavy damping $Z(\omega) = Z_0(1-i)\sqrt{\frac{\Gamma}{2\omega}}$
    - Reflection coefficient $r = \frac{Z_1 - Z_2}{Z_1 + Z_2}$
    - From undamped (Z_0) to damped (Z)
      - $r(\omega) \approx \frac{i\Gamma}{4\omega}$
      - $r(\omega) \approx -1$
    -

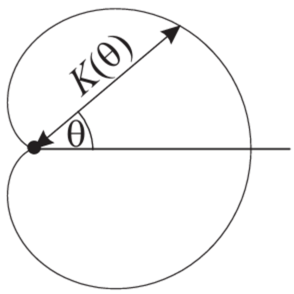
22. Waveguide on a membrane
    - 

    - $\Psi_A = Ae^{i(\omega t - k_xx - k_yy)}$
    - $\Psi_B = -Ae^{i(\omega t - k_xx + k_yy)}$
    - $\Psi = \Psi_A + \Psi_B = -2iA\sin(k_yy)e^{i(\omega t - k_xx)}$
    - Boundary condition: $\Psi = 0$ at $y = 0, b$

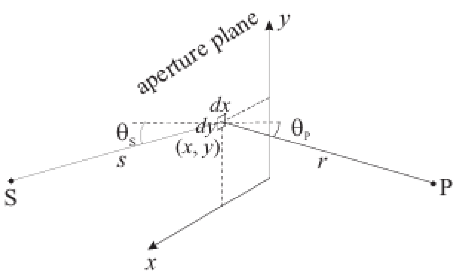
- $k_y = \frac{n\pi}{b}$
- $k^2 = k_x^2 + k_y^2$
- Dispersion relation: $\boxed{\omega^2 = v^2\left(k_x^2 + \frac{n^2\pi^2}{b^2}\right)}$
- Phase velocity $\boxed{v_p = \frac{\omega}{k_x} = \frac{\omega}{\sqrt{\left(\frac{\omega^2}{v^2} - \frac{n^2\pi^2}{b^2}\right)}}}$
- Group velocity $\boxed{v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{v^2}{\omega}\sqrt{\left(\frac{\omega^2}{v^2} - \frac{n^2\pi^2}{b^2}\right)}}$
- If $k_x^2 < 0$, k_x is imaginary, the wave is **evanescent**

23. Wave equation $\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}$
24. $c_0 = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, $n = \frac{c_0}{c} = \sqrt{\mu_r\epsilon_r}$
25. $Z = \frac{|\mathbf{E}|}{|\mathbf{H}|} = \frac{|\mathbf{E}|}{|\mathbf{B}/\mu|} = c\mu = \sqrt{\frac{\mu}{\epsilon}} = Z_0\sqrt{\frac{\mu_r}{\epsilon_r}}$
26. $\boxed{Z = \frac{Z_0}{n}}$, $\boxed{Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}} \approx 376.730\Omega$ (Z_0 is impedance of free space)
27. Reflection of EM wave $r = \frac{n_1 - n_2}{n_1 + n_2}$
28. Optics
    - Quantum Electrodynamics: full theory of EM interaction with matter, complicated, only used for simple systems
    - Maxwell's Equations: Large number of photons, hard to compute except for special boundary conditions
    - Physical Optics: aka scalar wave theory, ignore polarization, simplify b.c. Use Huygens' construction of secondary waves. (What we use in this course)
    - Ray Optics: Ignore wave properties, "corpuscular theory".
29. Huygens' Principle: each point on a wavefront acts as a source of secondary wavelets which propagate, overlap, interfere, and thus carry the wavefront forward
30. Huygens-Fresnel Principle
    - Obliquity/inclination factor $K(\theta) = \frac{1 + \cos\theta}{2}$

      David Miller: dipole; Forrest Anderson: dirac delta functions
    - The secondary wavelets have an amplitude $-i/\lambda$ relative to the primary wave
    - No physical foundation, but experimentally verified

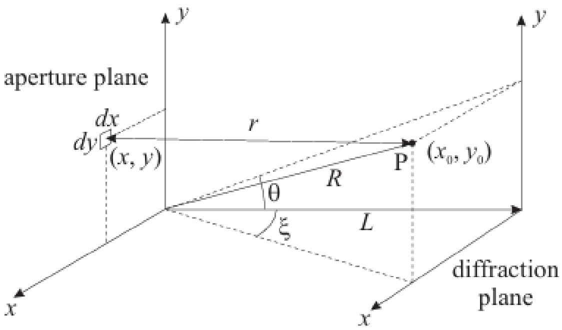


31. The diffraction integral

    - 

- $\Psi(\mathbf{r}, t) = \mathrm{Re}(\psi(\mathbf{r}e^{-i\omega t}))$
- $\psi_{\text{incident}} = \dfrac{a_S e^{iks}}{s}$
- Obliquity factor $\boxed{K(\theta) = \dfrac{\cos\theta_S + \cos\theta_P}{2}}$ (Proof)
- Fresnel-Kirchhoff diffraction Integral $\psi_P = \displaystyle\iint_\Sigma -\frac{i}{\lambda} h(x,y) K(\theta) \frac{a_S e^{ik(s+r)}}{sr} \mathrm{d}x\mathrm{d}y$
- Key take-away: plane wave on aperture produces $\boxed{\psi_P \propto \dfrac{e^{ikr}}{r}}$

32. Fraunhofer diffraction

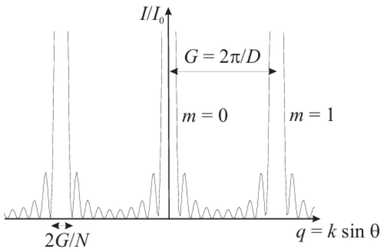
- 

- $r^2 = L^2 + (x_0 - x)^2 + (y_0 - y)^2 = R^2\left(1 - 2\dfrac{x_0 x + y_0 y}{R^2} + \dfrac{x^2+y^2}{R^2}\right)$
- $R^2 = L^2 + x_0^2 + y_0^2$
- $r \approx R - \dfrac{x_0 x + y_0 y}{R} + \dfrac{x^2 + y^2}{2R}$
- Condition for Fraunhofer $\boxed{\dfrac{k(x^2+y^2)}{2R} \ll \pi}$ (or $R \gg \dfrac{\rho^2}{\lambda}$)
- Ignore obliquity factor, $\psi_P \propto \displaystyle\iint_\Sigma h(x,y) \exp\left[-ik\left(\frac{x_0}{R}x + \frac{y_0}{R}y\right)\right] \mathrm{d}x\mathrm{d}y$
- 1D aperture, $\psi_P \propto \displaystyle\int \boxed{h(y)e^{-iqy}\mathrm{d}y}$, $q = ky_0/R = k\sin\theta$

33. Fraunhofer, angular resolution of circular aperture $\theta = \dfrac{1.22\lambda}{d}$.

34. Diffraction grating of N slits' resolution at n-th maxima $\dfrac{\delta\lambda}{\lambda} = \dfrac{1}{nN}$

- slit separation D, aperture screen distance R, $p = k\sin\theta = \dfrac{2\pi}{\lambda}\dfrac{x_0}{R}$
- Width of n-th peak is first peak at $x_0 = 0$ to position of first zero
- $h(y) = \displaystyle\sum_{m=0}^{N-1} \delta(x - mD)$
- $\psi_P \propto \displaystyle\int h(y)e^{-ipx}\mathrm{d}x = \sum_{m=0}^{N-1} e^{-ipmD} = \frac{1 - e^{-ipND}}{1 - e^{-ipD}} = \frac{\sin\left(\frac{pND}{2}\right)}{\sin\left(\frac{pD}{2}\right)} e^{-ip(N-1)D/2}$
- $I = |\psi_P|^2 \propto \left(\dfrac{\sin\left(\frac{pND}{2}\right)}{\sin\left(\frac{pD}{2}\right)}\right)^2$
- Let $I = I_0$ at $N = 1$, $I = I_0\left(\dfrac{\sin\left(\frac{pND}{2}\right)}{\sin\left(\frac{pD}{2}\right)}\right)^2$

- Peaks at $\frac{pD}{2} = n\pi$, if not a peak and $\frac{pND}{2} = n\pi$, a zero, there are $N-1$ zeros between peaks. Peak height $N^2 I_0$ by l'Hopital's rule.
- 1/2-Peak-width (first zero) is $pND/2 = kNDx_0/2R = \pi$, $x_0 = \frac{\lambda R}{DN}$
- In the extreme case, n-th peak of $\lambda + \delta\lambda$ falls on minimum of n-th peak of λ
- n-th maximum, $D\sin\theta = D\frac{x_0}{R} = n\lambda$, $x_0 = \frac{n\lambda}{D}$
- $\frac{n(\lambda+\delta\lambda)R}{D} - \frac{n\lambda R}{D} = \frac{\lambda R}{DN}$, $\boxed{\frac{\delta\lambda}{\lambda} = \frac{1}{nN}}$
- Higher order peaks separates lights better (higher resolution), but might not be obtained.
- More slits N on grating increases resolution, but reduces x_0 so more difficult to measure. (Rayleigh Criterion)
- 

35.

$f(t)$	$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \mathrm{d}t$
$\delta(t - t_0)$	$\frac{e^{-i\omega t_0}}{\sqrt{2\pi}}$
$f(t) = \mathrm{int}(\lvert t\rvert \leq 1/2)$ (top-hat)	$\frac{\mathrm{sinc}(\omega/2)}{\sqrt{2\pi}}$
$e^{-t^2/2}$ (Gaussian)	$e^{-\omega^2/2}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$ (comb)	$\frac{\sqrt{2\pi}}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} n)$

36. Babinet's principle: The diffracted intensities of an aperture and its complement are the same except at the origin

$$\psi_a \propto \iint_A e^{-i(px+qy)} \mathrm{d}x\mathrm{d}y$$

$$\psi_b \propto \iint_{\text{all space}} e^{-i(px+qy)} \mathrm{d}x\mathrm{d}y - \psi_a \propto \boxed{\delta(p,q) - \psi_a}$$

37. On-axis Fresnel diffraction ($R \sim \rho^2/\lambda$, $p = q = 0$)

$$\psi_P \propto \iint_\Sigma h(x,y) \exp\left(ik\frac{x^2+y^2}{2R}\right) \mathrm{d}x\mathrm{d}y$$

38. Rectangular aperture

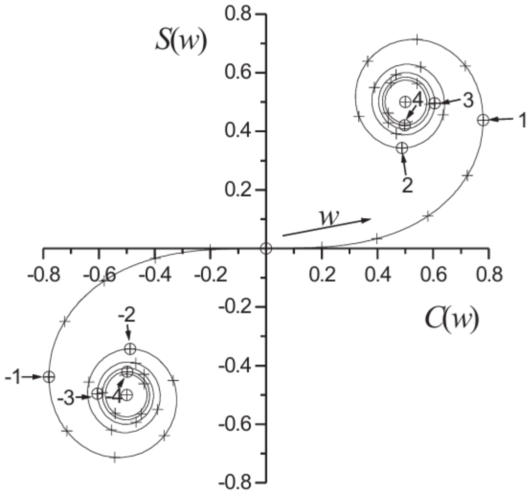
$$\psi_P \propto \int_{x_1}^{x_2} \exp\left(\frac{ikx^2}{2R}\right) \mathrm{d}x \int_{y_1}^{y_2} \exp\left(\frac{iky^2}{2R}\right) \mathrm{d}y$$

$$u = x\sqrt{\frac{2}{\lambda R}}, \quad v = y\sqrt{\frac{2}{\lambda R}}$$

$$\psi_P \propto \int_{u_1}^{u_2} \exp\left(\frac{i\pi u^2}{2}\right) \mathrm{d}u \int_{v_1}^{v_2} \exp\left(\frac{i\pi v^2}{2}\right) \mathrm{d}v$$

$$C(w) = \int_0^w \cos\left(\frac{\pi u^2}{2}\right) \mathrm{d}u$$

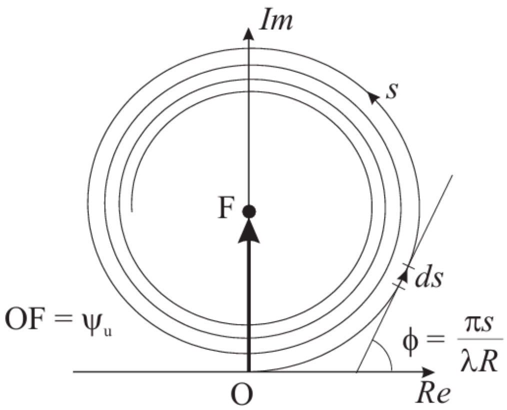
$$C(w) = \int_0^w \sin\left(\frac{\pi u^2}{2}\right) \mathrm{d}u$$



$\mathrm{d}l^2 = \mathrm{d}w$ on the Cornu spiral
$w \to \infty$, $\psi_P = 0.5(1+i)$
$I = |\psi_P|^2 \propto$ distance between 2 points on spiral

39. Circular aperture

$$\psi_P \propto \int_{s=0}^{s=r_a^2} \frac{K(s)}{(a^2+s)^{1/2}(b^2+s)^{1/2}} \exp\left(\frac{i\pi s}{\lambda R}\right) \pi \mathrm{d}s, \quad \boxed{s = \rho^2}$$



Fresnel half-period zones, $\boxed{\sqrt{(n-1)\lambda R} \le \rho \le \sqrt{n\lambda R}}$, predicts poisson's spot, also a *highly chromatic* lens with focal length $f = \frac{\rho_1^2}{\lambda} = \frac{\rho^n}{n\lambda}$
Many focal point corresponding to different number of half-period zones

40. Interference with broadband light

41. Thin film interference

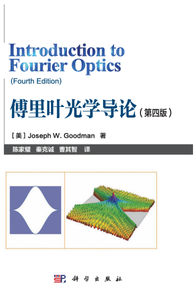
42. Fabry-Perot etalon

43. Left/Right polarization: right/left hand, thumb along propagation direction (IEEE Engineer), or the opposite/towards source (Experimentalist)
(Sketch polarization in 3D)

Proofs

1. ★

    - Proof a bit long, read through this great book next time (chap11 on holography too!)



2. ★

- The plane wave has amplitude ψ_s at the aperture
- Perpendicular distance from P to aperture is L, is large
- $\psi_P = \iint_\Sigma K(\theta_S, \theta_P) \frac{A\psi_s e^{ikr}}{r} \rho \mathrm{d}\rho \mathrm{d}\phi$
- $L^2 + \rho^2 = r^2$, $\rho \mathrm{d}\rho = r\mathrm{d}r$, r goes from L to $R(\phi)$
- $R(\phi)$ varies rapidly as ϕ changes because L is large
- $\psi_P = \int_0^{2\pi} \int_L^{R(\phi)} \frac{A\psi_s e^{ikr}}{r} r\mathrm{d}r\mathrm{d}\phi = \frac{A\psi_s}{ik} \int_0^{2\pi} e^{ikR(\phi)} \mathrm{d}\phi - \frac{2\pi A}{ik} \psi_s e^{ikL}$
- $e^{ikR(\phi)}$ oscillates quickly, first integral vanishes
- $\psi_P = \psi_s$, $\frac{-2\pi A}{ik} = 1$, $\boxed{A = -i/\lambda}$