Cheatsheet

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1. $$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{D}&=\rho\\ \boldsymbol{\nabla}\times\mathbf{E}&=-\dot{\mathbf{B}}\\ \boldsymbol{\nabla}\cdot\mathbf{B}&=0\\ \boldsymbol{\nabla}\times\mathbf{H}&=\mathbf{J}+\dot{\mathbf{D}}\end{aligned}$$

2. $\mathbf{F}=q\mathbf{E}+q\mathbf{v}\times\mathbf{B}$
3. - $\mathbf{p}=q\mathbf{a}$
   - $\mathbf{m}=I\int_S \mathrm{d}\mathbf{S}$
   - $\mathbf{G}=\mathbf{p}\times\mathbf{E}$, $\mathbf{G}=\mathbf{m}\times\mathbf{B}$
   - $U=-\mathbf{p}\cdot\mathbf{E}$, $U=-\mathbf{m}\cdot\mathbf{B}$
   - $\mathbf{F}=-\boldsymbol{\nabla}U(\mathbf{r})$, $\mathbf{F}=(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$, $\mathbf{F}=(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}$
4. $U_E=\frac{1}{2}\epsilon\epsilon_0|\mathbf{E}(\mathbf{r})|^2=\frac{1}{2}\mathbf{D}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})$
   $U_B=\frac{1}{2}\mathbf{B}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r})$
5. $\mathbf{D}=\epsilon_0\mathbf{E}+\mathbf{P}=\epsilon_0(1+\chi)\mathbf{E}$, $\mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M})=\mu_0(1+\chi_m)\mathbf{H}$
6. $\boxed{\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{F})=\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{F})-\nabla^2\mathbf{F}}$
   - $\mathrm{d}\mathbf{B}=\frac{\mu_0 I}{4\pi R^2}\mathrm{d}\mathbf{l}\times\hat{\mathbf{R}}$
   - $\nabla^2\mathbf{E}=\mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$
     $\nabla^2\mathbf{H}=\mu_0\epsilon_0\frac{\partial^2\mathbf{H}}{\partial t^2}$ (No charge, no current)
     $\frac{E_x}{B_y}=\frac{\omega}{k}=v=\frac{1}{\sqrt{\epsilon\epsilon_0\mu\mu_0}}$
     $\frac{E_x}{H_y}=Z=\sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}$
     $\mathbf{N}=\mathbf{E}\times\mathbf{H}$
     $\mathbf{R}=\frac{\mathbf{N}}{c}$
7. $\mathbf{m}=V\mathbf{M}=I\int \mathrm{d}S$
8. Scalar potential $\mathbf{B}=\mu_0\mathbf{H}=-\mu_0\boldsymbol{\nabla}\phi_m$, vector potential $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}, \boldsymbol{\nabla}\cdot\mathbf{A}=0, -\nabla^2\mathbf{A}=\mu_0\mathbf{J}$
9. $Z=\frac{V}{I}=\sqrt{\frac{L}{C}}$
10. $r=\frac{V_r}{V_i}=\frac{Z_t-Z}{Z_t+Z}$, $t=\frac{V_t}{V_i}=\frac{2Z_t}{Z_t+Z}$

1. $\mathbf{J} = \sum \mathbf{r} \times \mathbf{p} = \sum \mathbf{r} \times m(\boldsymbol{\omega} \times \mathbf{r}) = \sum m(r^2\boldsymbol{\omega} - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})) = \sum m(\boldsymbol{r}^T\boldsymbol{r}\mathbf{1} - \boldsymbol{r}\boldsymbol{r}^T)\boldsymbol{\omega} = \boldsymbol{I}\boldsymbol{\omega} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \boldsymbol{\omega}$

2. For body frame S, $\boldsymbol{G} = \left[\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right]_S + \boldsymbol{\omega} \times \boldsymbol{J}$, Euler's equations are

$$\begin{aligned} G_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ G_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ G_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}$$

(Because $\boldsymbol{J} = I\boldsymbol{\omega} = \sum_{i=1}^{3} I_i\omega_i\hat{e}_i$, $\boldsymbol{G} = \sum_{i=1}^{3} I_i\dot{\omega}_i\hat{e}_i + I_i\omega_i\frac{\mathrm{d}\hat{e}_i}{\mathrm{d}t}$ and $\boxed{\frac{\mathrm{d}\hat{e}_i}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{e}_i}$)

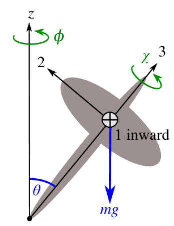
3. $T = \frac{1}{2}\sum m(\boldsymbol{\omega} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) = \frac{1}{2}\sum m\boldsymbol{\omega} \cdot (\boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})) = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{J} = \frac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega}$

4. $\Omega_b \equiv \frac{I_1 - I_3}{I_1}\omega_3$

5. $\Omega_s = \frac{\dot{\omega}_1}{|\boldsymbol{\omega}|\sin\theta_s} = \frac{J}{I_1}$

6. $\Omega_b \sin\theta_b = \Omega_s \sin\theta_s$ (Poinsot)

7. Symmetric top with Euler angles (θ, ϕ, χ)
   - $\boldsymbol{\omega} = \dot{\phi}\hat{e}_z + \dot{\theta}\hat{e}_1 + \dot{\chi}\hat{e}_3$
   - In body frame S, $\boldsymbol{\omega} = (\dot{\theta}, \dot{\phi}\sin\theta, \dot{\chi} + \dot{\phi}\cos\theta)$
     $\boldsymbol{J} = (I_1\dot{\theta}, I_1\dot{\phi}\sin\theta, I_3(\dot{\chi} + \dot{\phi}\cos\theta))$
   - Keep $\omega_3 = \dot{\chi} + \dot{\phi}\cos\theta$, $J_z = J_3\cos\theta + J_2\sin\theta$ constant
   - We get $\dot{\phi} = \Omega_s$, $\dot{\chi} = \Omega_b$



8. For simplicity, only consider principal axis to express e and τ as vectors

9. Strain is $e = \delta l/l$, stress is $\tau = -P = -F/A$, for isotropic material, $E\mathbf{e} = \begin{pmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{pmatrix}\boldsymbol{\tau}$, σ is Poisson ratio, $e_1 = e_2 = e_3 = \frac{\tau(1-2\sigma)}{E}$, $\frac{\delta V}{V} \approx e_1 + e_2 + e_3 = \frac{3\tau(1-2\sigma)}{E}$, $\boxed{B = \frac{E}{3(1-2\sigma)}}$

10. $\boldsymbol{\tau} = \begin{pmatrix} 1 & -\sigma & -\sigma \\ -\sigma & 1 & -\sigma \\ -\sigma & -\sigma & 1 \end{pmatrix}^{-1} E\mathbf{e} = \frac{E}{(\sigma+1)(1-2\sigma)}\begin{pmatrix} 1-\sigma & \sigma & \sigma \\ \sigma & 1-\sigma & \sigma \\ \sigma & \sigma & 1-\sigma \end{pmatrix}\mathbf{e} = \lambda(e_1+e_2+e_3) + 2G\boldsymbol{e} = \lambda\,\mathrm{Tr}(\boldsymbol{e})\boldsymbol{I} + 2G\boldsymbol{e}$, Lamé's constant $\boxed{\lambda \equiv \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, G = \frac{E}{2(1+\sigma)}}$, $\lambda = B - \frac{2}{3}G$

11. $W = -\frac{\mathrm{d}F}{\mathrm{d}x}, F = -\frac{\mathrm{d}B}{\mathrm{d}x}, B = \frac{EI}{R} = EIy'' = \int y \cdot \tau \mathrm{d}A$, $\tau = Ee = E\frac{y}{R}$, $I = \int y^2 \mathrm{d}A$

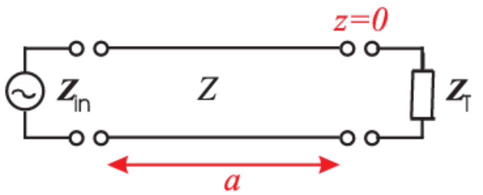
12. $\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\mathbf{v}$

13. $\rho\frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla}P + \rho\mathbf{g} + \eta\nabla^2\mathbf{v}$ ($\eta = 0$, Euler's equation)

14. $P + \frac{1}{2}\rho v^2 + \rho\phi = C$

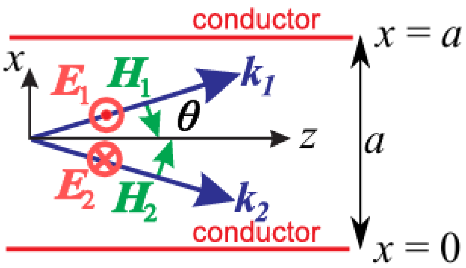
15. $\Phi = v_0 \cos\theta \left(r + \frac{a^3}{2r^2} \right)$
16. Magnus effect $\mathbf{F} = \rho\mathbf{v} \times \kappa$, $\kappa = 2\pi r v_\theta$ is circulation around cylinder
17. Coriolis force $2m\omega v \sin\theta$

Examples to memorize

1. Plasma
   - Electron $m_e \frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
   - $B_y = E_x/c$, $c \ll |\mathbf{v}|$, $\mathbf{B}$ ignored
   - $\mathbf{r} = \frac{e}{m_e\omega^2}\mathbf{E_0}e^{i(kz-\omega t)}$
   - Electron and lattice - dipole $\mathbf{p} = -e\mathbf{r} = -\frac{e^2}{m_e\omega^2}\mathbf{E_0}e^{i(kz-\omega t)}$
   - Dipole moment per unit volume $\mathbf{P} = -\frac{Ne^2}{m_e\omega^2}\mathbf{E} = \epsilon_0\chi\mathbf{E}$
   - $\epsilon = 1 + \chi = 1 - \frac{Ne^2}{m_e\epsilon_0\omega^2} = \boxed{1 - \frac{{\omega_p}^2}{\omega^2}}$, $\omega_p = \sqrt{\frac{N}{m_e\epsilon_0}}e$
   - Below ω_p, $\epsilon < 0$, n is imaginary, reflect
2. Conductor
   - Currents form $\mathbf{J} = \sigma\mathbf{E}$, $\sigma \sim 10^7 \gg 1$
   - $\nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}} = \sigma\mathbf{E} + \epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = (\sigma - i\omega\epsilon\epsilon_0)\mathbf{E} = -i\omega\epsilon'\epsilon_0\mathbf{E}$
   - Effective dielectric constant $\epsilon' = \epsilon - \frac{\sigma}{i\omega\epsilon_0} \approx \boxed{i\frac{\sigma}{\omega\epsilon_0}}$
   - $n = \sqrt{\epsilon'\mu} = \pm\frac{1+i}{\sqrt{2}}\sqrt{\frac{\sigma\mu}{\omega\epsilon_0}}$
   - $E = E_0e^{i(\omega t - kz)}$, $c/n = \frac{\omega}{k}$
   - $k = \frac{n\omega}{c} = \frac{1+i}{\sqrt{2}}\sqrt{\sigma\mu_0\mu\omega} = \frac{1+i}{\delta}$, skin depth $\boxed{\delta = \sqrt{\frac{2}{\sigma\omega\mu_0\mu}}}$
   - $E = E_0e^{-z/\delta + i(z/\delta - \omega t)}$, exponential decay wrt. skin depth δ
3. The skin effect
   - $\mathbf{E}$ along wire (x direction), $\mathbf{J} = \sigma\mathbf{E}$. z is radial direction
   - $J_x(z) = J_0e^{-z/\delta + i(z/\delta - \omega t)}$
   - Approximate I at small δ, $I = \int_0^\infty J_x(z)(2\pi a)\mathrm{d}z = \pi a J_0\delta(1+i)e^{-i\omega t}$
   - $\langle I^2 \rangle = (\pi a J_0 \delta)^2$
   - $\mathrm{d}P = \frac{J^2\mathrm{d}A}{\sigma}$, $P = \frac{J_0^2\pi a\delta}{2\sigma}$
   - $R = \frac{P}{\langle I^2\rangle} = \frac{P}{\langle I^2\rangle} = \frac{1}{2\pi a\delta\sigma} = \frac{1}{\sigma A'}$
   - Skin effect: at high ω, small δ, $\boxed{\text{effective area}=2\pi a\delta}$ (annulus of thickness of skin depth)
4. Input impedance is impedance measured at the input (position matters)
   - 

- $V_i = V_1 e^{-i(kz-\omega t)}$, $V_r = rV_1 e^{-i(-kz-\omega t)}$, $\dfrac{V_i}{I_i} = Z$, $\dfrac{V_r}{I_r} = -Z$
- $Z_{\text{in}} = \left.\dfrac{V_i + V_r}{I_i + I_r}\right|_{r=a} = \dfrac{e^{ika} + re^{-ika}}{e^{ika} - re^{-ika}} Z$, $\dfrac{Z_{\text{in}}}{Z} = \dfrac{Z_t \cos(ka) + iZ \sin(ka)}{Z \cos(ka) + iZ_t \sin(ka)}$
- Short-circuit, $Z_t = 0$, $\dfrac{Z_{\text{in}}}{Z} = i \tan(ka)$
- Open-circuit, $Z_t \to \infty$, $\dfrac{Z_{\text{in}}}{Z} = -i \cot(ka)$
- Quater-wavelength, $a = \lambda/4, ka = \pi/2$, $\dfrac{Z_{\text{in}}}{Z} = \dfrac{Z}{Z_t}$

5. (Non-TEM) Parallel plate waveguide



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- $k^2 = k_x^2 + k_z^2$, $k_z = k_g$, $k_x = \dfrac{m\pi}{a}$ (standing wave)

6. Rectangular waveguide

- General TE_{mn}, transverse electric, n and m in x,y direction
- $(k_x, k_y, k_z) = \left(\dfrac{m\pi}{a}, \dfrac{n\pi}{b}, k_g\right)$
-

$$
\begin{aligned}
E_x &= A_0 k_y \cos(k_x x) \sin(k_y y) \cos(k_z z - \omega t) \\
E_x &= -A_0 k_y \sin(k_x x) \cos(k_y y) \cos(k_z z - \omega t) \\
E_z &= 0
\end{aligned}
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- $\dfrac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$
- $k_z = k_g$, if imaginary, evanescent

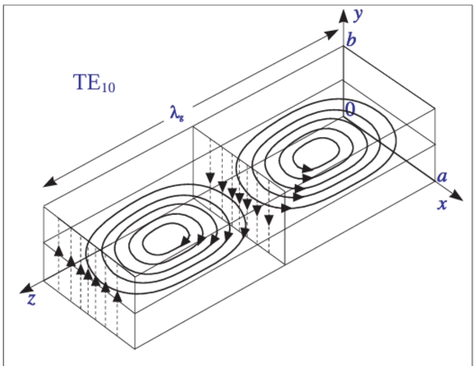

Figure 8.17: The TE₁₀ mode in a rectangular waveguide: ***E*** (dotted line) and ***H*** (solid line) fields.

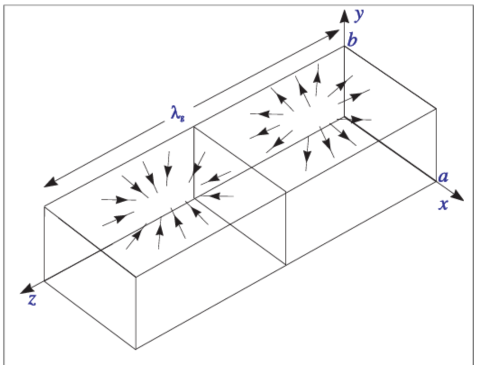

Figure 8.18: The currents associated with the TE₁₀ mode of a rectangular waveguide.

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