AARHUS UNIVERSITY

New Directions in Recovering Noisy RSA Keys (A Coding-Theoretic approach)

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Outline

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Motivation

- Side channel Information:
  power consumption, execution time, electromagnetic radiation, sounds, frequencies, temperatures, error messages, faulty outputs, visible light, memory images and cache memory gaps.

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- Side Channel Attacks $\leftrightarrow$ Recovering noisy secret keys.

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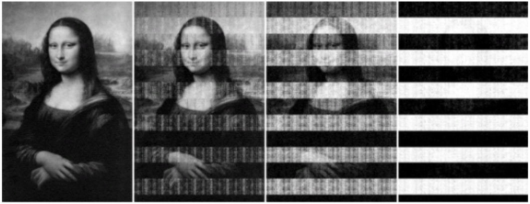

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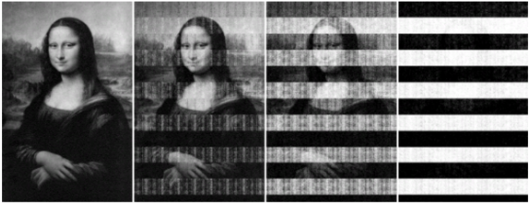

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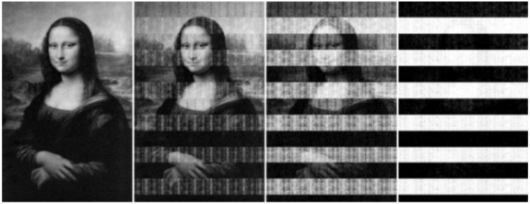

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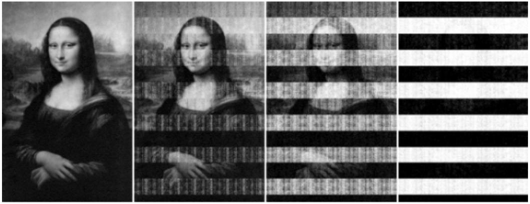

- 0 bits will flip with very low probability ($< 1\%$), but 1 bits will flip with much higher probability which increases with time.
- In a given region the decay is overwhelmingly either $0 \rightarrow 1$ or $1 \rightarrow 0$'

Given a noisy RSA key, is it possible to reconstruct the original key?

Rivest and Shamir (Eurocrypt 1985):
N can be factored given 2/3 of the LSBs of a prime

$1001011011010\overbrace{0001111000101101001101010}$...

Coppersmith (Eurocrypt 1996):
N can be factored given 1/2 of the MSBs of a prime

$\overbrace{10010110110100001111}000101101001101010$...

Boneh et al. (Asiacrypt 1998) :
N can be factored given 1/2 of the LSBs of a prime

$10010110110100001111\overbrace{000101101001101010}$...

Herrmann and May (Asiacrypt 2008):
N can be factored given contiguous blocks of a prime

$100\overbrace{101101}1010000111100010110\overbrace{10011}0101010$...

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$$100\overbrace{1011}011010\overbrace{00}0111\overbrace{100}0\overbrace{101}10\overbrace{1001}10101\overbrace{010}...$$

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- Henecka, May and Meurer (HMM) Algorithm (Crypto 2010): $sk = (p, q, d, d_p, d_q)$ can be found given that all the key bits are subject to errors.

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- The HMM algorithm is designed to work for the symmetric case.
    - In the cold boot scenario, $\alpha := \Pr(0 \rightarrow 1)$ will be extremely very small, while $\beta := \Pr(1 \rightarrow 0)$ may be relatively large, and perhaps even greater than 0.5 in a very degraded case.

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- We propose a Coding-Theoretic Approach using:
  - channel capacity
  - list decoding method
  - random coding techniques
- We derive bounds on the performance of the previous and our new algorithm solving the cold boot problem and more...

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2. State of the Art
3. Our Contributions
4. Experimental Results

Coppersmith method

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- The idea is to build from $f(x)$ a related polynomial $F(x)$ which still has the same solution x_0, with small coefficients.

Factoring with partial knowledge

- Let $N = pq$ and suppose we are given an approximation $\widetilde{p}$ to p.

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- Coppersmith's original version used bivariate polynomials. We present a simpler version following work of Howgrave-Graham, Boneh, Durfee and others.

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- The idea is to form a lattice corresponding to polynomials which have a root modulo p and to use Coppersmith method.

Example

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- These all have the same small solution x_0 modulo p.
- We build the lattice corresponding to these polynomials.

Example

The lattice has basis matrix:

$$\begin{pmatrix} N & 0 & 0 & 0 \\ \widetilde{p} & X & 0 & 0 \\ 0 & \widetilde{p}X & X^2 & 0 \\ 0 & 0 & \widetilde{p}X^2 & X^3 \end{pmatrix}$$

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- The polynomial has the root $x = 7$ over Z.
- We can check that $p = \widetilde{p} + 7 = 2837$ is a factor of N.

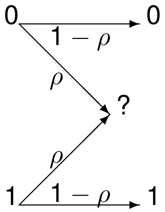
Coppersmith method

Theorem

Let $N = pq$ with $p \approx q$ and suppose we are given the high order half of the bits of p then one can factor N in polynomial time.

Heninger & Shacham

- Can be considered as an erasure channel.



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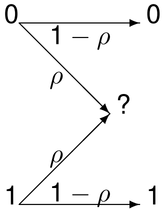

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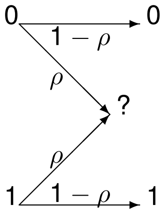

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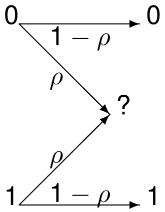

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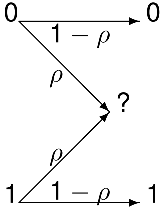

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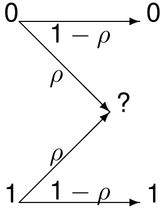

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    - Recovery by growing a search tree in a bit-by-bit fashion, starting with the least significant bits.
    - Prune the search tree removing the partial solutions which do not match with the known key bits.
- The algorithm always succeeds. However, the algorithm will blow up if only few bits are known.

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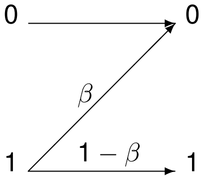

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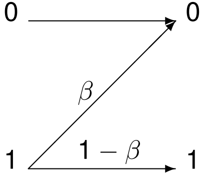

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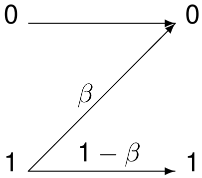

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- Fails if there is a $0 \to 1$ flip.

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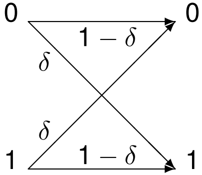

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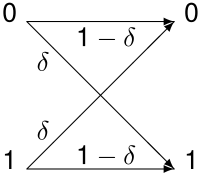

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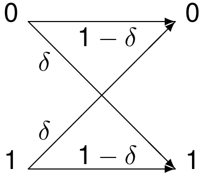

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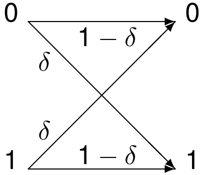

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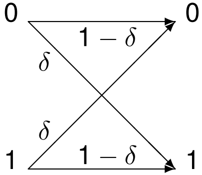

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- Consider *t* bit-slices at a time of possible solutions to the algebraic relations.
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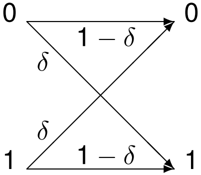

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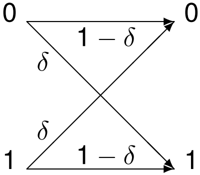

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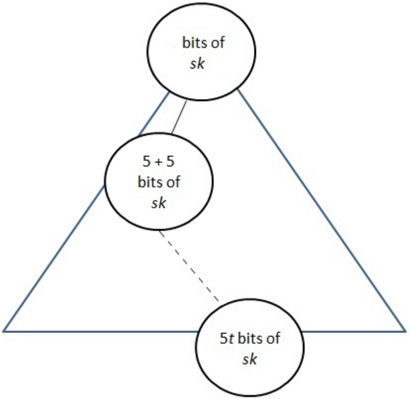
  2. Compute the Hamming distance between the candidate solutions and the noisy key, keeping all candidates for which this metric is less than some carefully chosen threshold C
- The algorithm will fail if the correct solution is rejected. In addition, if C is set too loosely then there is a large number of candidate solutions.

Structure of a subtree



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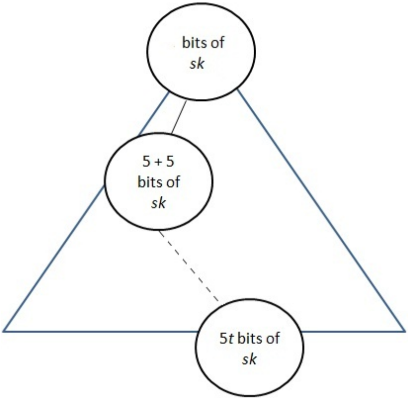

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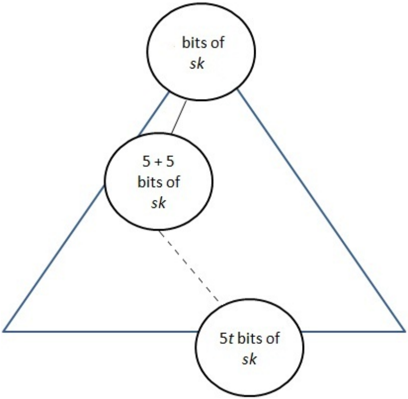

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- Each leaf contains $5t$ fresh bits of $sk = (p, q, d, d_p, d_q)$.
- Iterate for $n/(2t)$ rounds. All leaves in the last subtrees contain all $n/2$ bits of p, q, d, d_p and d_q.

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- We show how to recast the problem of noisy RSA key recovery as a problem in coding theory.

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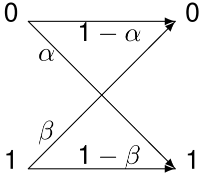
Our Contributions

- We consider the general non-symmetric channel



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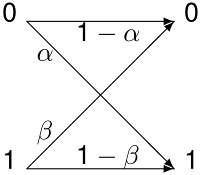
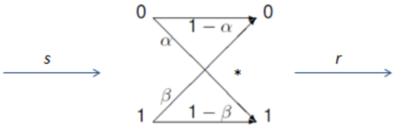


- This will allow us to model the real cold-boot scenario as well as those of Heninger & Shacham and Henecka, May & Meurer.

Channel model



- Code $\mathcal{C}$:
  The set of 2^t candidates, with one codeword s being selected and transmitted over a noisy channel, resulting in a received word r).

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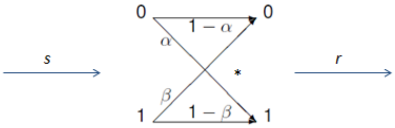

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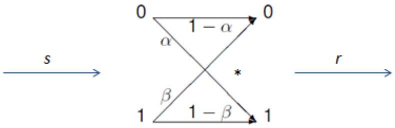
- This code has rate $R \geq 1/m$, ($m = 2, 3, 5$) and lenght mt.

Channel model



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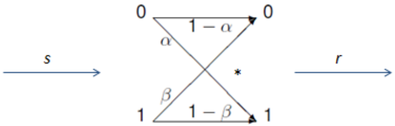

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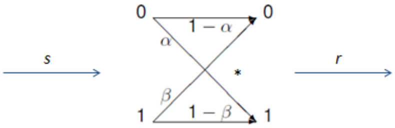

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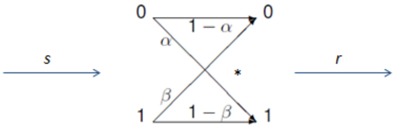

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Initial Steps (à la HS)

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- $d_p = d \mod p - 1$ and $q_p^{-1} = q^{-1} \mod p$
- Make RSA congruences explicit:

$$\begin{aligned} N &= pq \\ ed &= k\phi(N) + 1 \\ ed_p &= k_p(p-1) + 1 \\ ed_q &= k_q(q-1) + 1 \end{aligned}$$

for some constants k, k_p and k_q.

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  (trick from [Boneh, Durfee, Frankel 98])
- If e is prime we can find k_p, k_q:

$$k_p^2 - (k(N-1)+1)k_p - k \equiv 0 \mod e$$

Initial Steps (à la HS)

Define $\tau(x) := \max\{i \in \mathbb{N} : 2^i \mid x\}$ such as $2^{\tau(k_p)+1} | k_p(p-1)$, $2^{\tau(k_q)+1} | k_q(q-1)$ and $2^{\tau(k)+2} | k\phi(N)$. Then:

$$\begin{aligned} d_p &\equiv e^{-1} \mod 2^{\tau(k_p)+1} \\ d_q &\equiv e^{-1} \mod 2^{\tau(k_q)+1} \\ d &\equiv e^{-1} \mod 2^{\tau(k)+2}. \end{aligned}$$

This allows us to correct the least significant bits of d, d_p and d_q. Furthermore we can calculate *slice*(0), where we define

$$\mathit{slice}(i) := (p[i], q[i], d[i + \tau(k)], d_p[i + \tau(k_p)], d_q[i + \tau(k_q)]).$$

with $x[i]$ denoting the i-th bit of the string x.

Initial Steps (à la HS)

- Obtaining a solution (p', q', d', d'_p, d'_q) from *slice*(0) to *slice*$(i-1)$ then the bits in *slice*(i) (p, q, d, d_p, d_q) are related as follows:

$$
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- Multivariate Hensel's Lemma gives values for the c_i.
- Previous bits give us constraints on future bits.
- We perform t Hensel lifts to generate 2^t candidate partial solutions.
  Filter these according to some criterion.
  Repeat on remaining candidates.

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- This can be calculated as

$$\arg \max_{1 \leq i \leq M2^t} \left((1-\alpha)^{n^i_{00}} \alpha^{n^i_{01}} (1-\beta)^{n^i_{11}} \beta^{n^i_{10}} \right)$$

Our Algorithm

Algorithm 1: Pseudo-code of our Algorithm

Data: (N, e), $\widetilde{sk} = (\tilde{p}, \tilde{q}, \tilde{d}, \tilde{d}_p, \tilde{d}_q)$ α, β.

Initialization Phase:
Find (k, k_p, k_q) given (N, e);
Find *slice*(0) given (e, k, k_p, k_q);
Create a *list* and add *slice*(0);
Lifting phase:
for stage $= 1$ ***to*** $n/(2t)$ **do**
    for i $= 1$ ***to*** L **do**
        Replace each partial solution i from the *list* with a set of 2^t candidate solutions s_i obtained by Hensel lifting;
        Pruning Phase:
        Calculate the log-likelihood $\log \Pr(r|s_i)$ for each entry s_i on list;
        Add the L entries in list having the highest log-likelihoods and delete the remainder;

Finalization Phase: Find one candidate that satisfies all the RSA equations;
Output : *sk*

Our algorithm does not quite implement ML decoding at each stage.
Our algorithm has *deterministic* polynomial running time $O(L2^t n/2t)$ and *deterministic* memory consumption $O(L2^t)$ or $(O(t + L))$.
The running time in all our experiments was $O(2^t)$ per stage rather than $O(L2^t)$ because of multi-threading.

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- For fixed L and m, for our code, this holds provided $1/m$ is strictly less than the capacity as $t \to \infty$.
- Apply to the maximum likelihood list decoding rule.
- Our strong randomness assumption is not true for our code due to the Hensel lifting.

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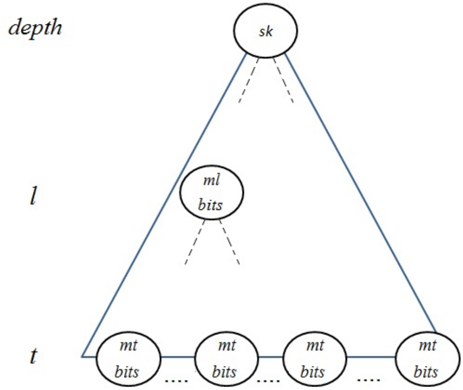

The Binary Symmetric Channel

- When sk is of the form (p, q, d, d_p, d_q), the code rate is at least $1/5$ (we have 2^t codewords and length $5t$).

sk	R	δ	HMM
(p, q, d, d_p, d_q)	$1/5$	0.243	0.237
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Important

When $\delta \geq 0.243$ it can be shown that no algorithm can list decode using a polynomially-sized list.

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For list decoding it can be shown that, on average, an exponential list of candidates will need to be considered when the code rate exceeds capacity.

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The Z-channel

The capacity is:

$$C_Z(\beta) = \log_2(1 + (1-\beta)\beta^{\frac{\beta}{1-\beta}}).$$

sk	*R*	β
(p, q, d, d_p, d_q)	$1/5$	0.666
(p, q, d)	$1/3$	0.486
(p, q)	$1/2$	0.304

The True Cold Boot Setting

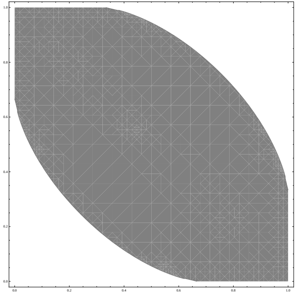

Figure: *x*-axis is α, *y*-axis is β.

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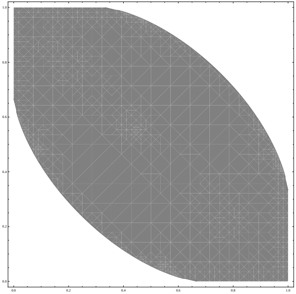

Figure: *x*-axis is α, *y*-axis is β.

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- When $sk = (p, q)$ the capacity bound on β is 0.298.

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- The error rate is $\lambda/(\kappa+\lambda)$.
- According to our capacity analysis $\lambda/(\kappa+\lambda) \leq 0.5$ since this is a special case of the erasure channel.

Outline

1. Motivation
2. State of the Art
3. Our Contributions
4. Experimental Results

The Erasure Channel

ρ	0.1	0.2	0.3	0.4	0.5	0.6
Success rate	1	1	1	1	1	1
Keys examined	512	512	516	527	553	627
liftings	511	511	513	520	536	593
Time per trial (s)	0.00235	0.0023	0.00232	0.00234	0.00236	0.00259

ρ	0.7	0.77	0.78	0.79	0.8
Success rate	1	1	0.98	0.77	0.4
Keys examined	971	167762	263835	923938	2875484
liftings	910	167634	263959	912849	2829735
Time per trial (s)	0.00409	0.783	1.18	4.18	13.1

Table: Experimental results for the erasure channel. Capacity bound on ρ is 0.8.

The Erasure Channel

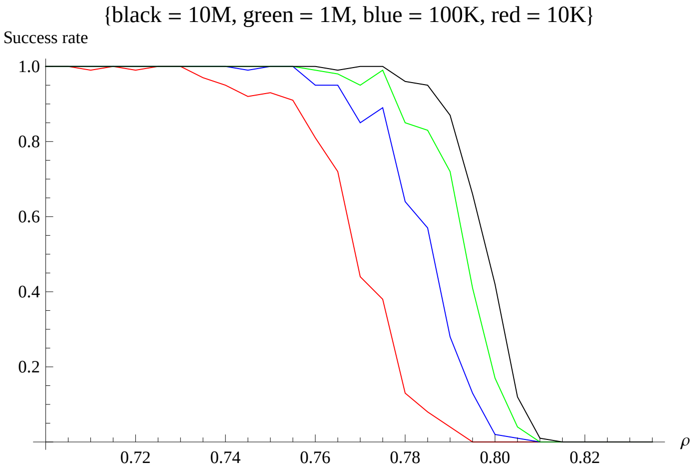
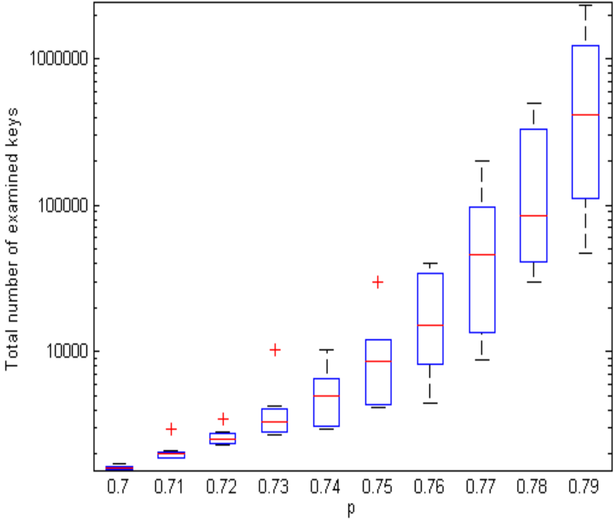


Figure: Graph showing achievable error rates based on different panic sizes.

The Erasure Channel



Partial knowledge of p and q

κ	λ	Total unknown bits	Max stack size	Keys Examined	Time (s)
2	2	510	135	31740	0.0571
4	4	508	137	44369	0.0782
6	6	508	47	15948	0.0285
8	8	504	138	172403	0.3
10	10	504	30	7942	0.014
12	12	496	48	16887	0.0292
14	14	492	61	59174	0.105
16	16	496	140	9200234	12.1
18	18	502	79	404272	0.711
20	20	484	81	1004018	1.78
22	22	496	39	25207	0.0441
24	24	496	47	134339	0.237
26	26	478	100	11521189	152

Table: Experimental results for the block-wise erasure channel with $\kappa = \lambda$.

Partial knowledge of p and q

κ	λ	Total unknown bits	Max stack size	Keys Examined	Time (s)
16	2	112	1	512	0.000907
16	4	192	1	512	0.000905
16	6	272	1	512	0.000902
16	8	336	1	512	0.000914
16	10	384	21	832	0.00141
16	12	432	39	2075	0.00345
16	14	464	108	225767	0.381
16	16	496	140	9200234	12.1
16	17	512	16	869974	1.67

Table: Experimental results for the block-wise erasure channel with $\kappa = 16$ and increasing λ.

Cold Boot Scenario

β	0.1	0.2	0.3	0.4	0.5	0.55	0.6	0.61
t	6	6	8	12	16	18	18	18
L	4	4	8	8	16	32	64	64
S.Pr	1	1	0.97	0.97	0.66	0.31	0.09	0.04

Table: Success probabilities for the true cold-boot case with $\alpha = 0.001$. Capacity bound on β is 0.658.

Cold Boot Scenario

β	0.1	0.15	0.20	0.25	0.30	0.35	0.40	0.43
t	6	10	14	16	18	18	18	18
L	4	16	16	16	16	16	32	64
$S.Pr$	0.99	0.99	0.98	0.96	0.63	0.55	0.12	0.04

Table: Success probabilities for the true cold-boot case with $\alpha = 0.001$ and $sk = (p, q, d)$. Capacity bound on β is 0.479.

β	0.05	0.1	0.15	0.20	0.26
t	10	12	16	18	18
L	8	8	16	32	64
$S.Pr$	0.95	0.83	0.68	0.29	0.06

Table: Success probabilities for the true cold-boot case with $\alpha = 0.001$ and $sk = (p, q)$. Capacity bound on β is 0.298.

Heninger & Shacham Setting

ρ	0.2	0.3	0.4	0.46	0.5	0.55	0.6	0.62	0.63
t	6	8	12	16	18	18	18	18	18
L	4	8	8	8	16	16	16	64	64
$S.Pr$	1	1	0.98	0.87	0.81	0.43	0.13	0.07	0.03

Table: Success probabilities for the idealized cold boot case ($\alpha = 0$). Capacity bound on β is 0.666.

Henecka, May & Meurer Setting

δ	0.08	0.12	0.16	0.18	0.19	0.2	0.21	0.22
t	6	10	16	18	18	18	18	18
L	4	8	32	32	32	32	32	64
$S.Pr$	1	0.93	0.84	0.60	0.38	0.20	0.08	0.04

Table: Success probabilities for the symmetric case ($(\alpha, \beta) = (\delta, \delta)$). Capacity bound on δ is 0.243.

Henecka, May & Meurer Setting

Table: Success probabilities for the symmetric case, $\alpha = \beta$.

α	0.06	0.08	0.12	0.16	0.19	0.2	0.21	0.22
HMM	0.48	0.5	0.5	0.35	0.24	0.21	-	-
ML	1	1	0.93	0.84	0.38	0.20	0.08	0.04

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- We use the converse to Shannon's theorem, derive bounds on list decoding to establish limits on the noise levels.
- For practical RSA key sizes our algorithm outperforms the previous approaches.
- Ours is the first algorithm to solve the motivating cold-boot problem.