4141 HW 8

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1

Call the operator G. By the definition of hermicity,

$$\langle Gf|g\rangle = \langle f|Gg\rangle \Leftrightarrow \int_{\mathbb{R}} (Gf(\phi))^* g(\phi)d\phi = \int_{\mathbb{R}} f(\phi)^* Gg(\phi)d\phi \Leftrightarrow \frac{\hbar}{i}\int_{\mathbb{R}} f'(\phi)^* g(\phi)d\phi = \frac{\hbar}{i}\int_{\mathbb{R}} f(\phi)^* g'(\phi)d\phi$$

$$\Leftrightarrow \frac{\hbar}{i}\left(f(\phi)^* g(\phi)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(\phi)^* g'(\phi)d\phi\right) = \frac{\hbar}{i}\left(f(\phi)^* g(\phi)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(\phi)^* g'(\phi)d\phi\right)$$

Therefore, G is Hermitian. Taking the inverse Fourier transform of Gf,

$$\mathcal{F}^{-1}(Gf) = \int_{\mathbb{R}} e^{2\pi i x\phi}\frac{\hbar}{i}f'(\phi)d\phi = \frac{\hbar}{i}f(\phi)e^{2\pi i x\phi}\Big|_{-\infty}^{\infty} - \frac{\hbar}{i}\int_{\mathbb{R}} f(\phi)2\pi i x e^{2\pi i x\phi}d\phi$$

Because f is periodic, the first term is zero, since if one splits the evaluation into a sum of periodic sub-intervals, one obtains zero on each sub-interval. This then becomes

$$= 2\pi\hbar x\int_{\mathbb{R}} f(\phi)e^{2\pi i x\phi}d\phi = 2\pi\hbar x f(x)$$

The expectation value of G is therefore, in position space, the expectation value of the position operator $4\pi^2\hbar^2 x^2$, so I presumably used a heterogeneous Fourier transform definition and the operator is identified with x^2.

2

It's easy to represent H as a matrix if one compares how it acts on $\psi = a|1\rangle + b|2\rangle$ to the action of a matrix on the vector (a, b).

$$H = \epsilon\begin{pmatrix}\langle 1|1\rangle + \langle 2|1\rangle & \langle 1|2\rangle + \langle 2|2\rangle \\ \langle 1|1\rangle - \langle 2|1\rangle & \langle 1|2\rangle - \langle 2|2\rangle\end{pmatrix}$$

By orthonormality, we may write this

$$= \epsilon\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}$$

This corresponds to an eigenvalue equation

$$\det(H - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 - 1 = \lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$$

with multiplicity 2. The eigenvectors may then be immediately computed as $(1-\sqrt{2})|1\rangle + |2\rangle$ and $(1+\sqrt{2})|1\rangle + |2\rangle$, which one may normalize by dividing by the norm if desired.

3

$$\hat{O} = |\psi\rangle\langle\phi|$$

$$\frac{\partial}{\partial x} f(x) = |h\rangle\langle h|g\rangle$$

4

The variance of a probability distribution P is $\Delta P = \langle P\rangle^2 - \langle P^2\rangle$.