4132 HW 7

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1a

Taking the Lorenz gauge,

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \Rightarrow \rho = 0$$

$$\Box^2 \vec{A} - -\mu_0 \vec{J} \Rightarrow \nabla^2 \left(\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}\right) - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\right)\hat{r} = -\mu_0\vec{J}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{-1}{2\pi\epsilon_0}\frac{qt}{r^3}\hat{r}\right) - 0 = -\mu_0\vec{J}$$

$$\Rightarrow -\frac{qt}{2\pi\epsilon_0 r^2}\frac{-1}{r^2}\hat{r} = -\mu_0\vec{J}$$

$$\Rightarrow \vec{J} = -\frac{qt}{2\pi\epsilon_0\mu_0 r^4}\hat{r}$$

1b

Under the given gauge,

$$V' = V - \frac{\partial\lambda}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

and

$$\vec{A}' = \vec{A} + \nabla\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} = \frac{1}{2\pi\epsilon_0}\frac{qt}{r^2}\hat{r}$$

The first is simply the scalar potential due to a stationary point charge of magnitude q, and the second is radially outward and therefore curl-free, resulting in a zero magnetic field, also consistent with a stationary point charge.

2

Applying the forced wave equation ansatz given in the text,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}} \frac{J(\vec{r}', t)}{\mathscr{r}} dV$$

$$= \frac{\mu_0}{4\pi}\left(\int_{-b}^{-a}\frac{k\left(t-\frac{c}{\mathscr{r}}\right)}{\mathscr{r}}\hat{x}dx+\int_{a}^{b}\frac{k\left(t-\frac{c}{\mathscr{r}}\right)}{\mathscr{r}}\hat{x}dx+a\int_0^{\pi}\frac{k(t-\frac{c}{\mathscr{r}})}{\mathscr{r}}(-\hat{\theta})d\theta+b\int_0^{2\pi}\frac{k(t-\frac{c}{\mathscr{r}})}{\mathscr{r}}(\hat{\theta})d\theta\right)$$

The x and θ dependence of $\mathscr{r}$ may be found from the relation $\mathscr{r} = r - r' = (x)$

3

The scalar potential is

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{|\mathscr{r}|c - \mathscr{r}\cdot v}$$

The position of the charge at time t is, in cylindrical coordinates, $r' = a\hat{r} + \omega t\hat{\theta}$. We may then write

$$\mathscr{r} = r - r' = (r-a)\hat{r} + (\theta - \omega t)\hat{\theta} + z\hat{z}$$

The velocity of the charge in cylindrical coordinates is

$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} = a\omega\hat{\theta}$$

Plugging this in to the scalar potential,

$$V(r,\theta,z,t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{a\omega(\omega t-\theta) + c\sqrt{(r-a)^2+(\theta-\omega t)^2+z^2}}$$

On the z-axis, $r = \theta = 0$, so

$$V(z,t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{at\omega^2 - c\sqrt{a^2+\omega^2t^2+z^2}}$$

The vector potential is

$$\vec{A}(\vec{r},t) = \frac{v}{c^2}V(\vec{r},t) = \frac{a\omega\hat{\theta}}{c^2}\frac{1}{4\pi\epsilon_0}\frac{qc}{a\omega(\omega t-\theta)+c\sqrt{(r-a)^2+(\theta-\omega t)^2+z^2}}$$

On the z-axis, this becomes

$$\vec{A}(z,t) = \frac{1}{4\pi\epsilon_0 c}\frac{a\omega q}{at\omega^2 - c\sqrt{a^2+\omega^2t^2+z^2}}\hat{\theta}$$

4

The Liénard-Wiechert potentials are, writing everything in one dimension and applying $\mathscr{r} = r - r'$ with x' as the position of the charge,

$$V(x,t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(x-x')c-(x-x')\dot{x}'} = \frac{1}{4\pi\epsilon_0}\frac{qc}{(c-\dot{x}')(x-x')}$$

and

$$\vec{A}(x,t) = \frac{v}{c^2}V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{\dot{x}'q}{c(c-\dot{x}')(x-x')}$$

Taking the gradient of the first yields

$$\vec{E}(x,t) = -\frac{qc}{4\pi\epsilon_0(c-\dot{x})}\frac{1}{(x-x')^2}\hat{x} = \frac{1}{4\pi\epsilon_0}\frac{-qc}{c-v}\frac{1}{\mathscr{r}^2}$$

Taking the