4123 HW 4

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1a

The Lagrangian for this system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

The modified Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} + \lambda \frac{\partial C}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = m\ddot{x}$$

and

$$\frac{\partial L}{\partial y} + \lambda \frac{\partial C}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow mg + 2\lambda y = m\ddot{y}$$

Adding these together and solving for λ, we obtain

$$mg + 2\lambda(x + y) = m(\ddot{x} + \ddot{y}) \Leftrightarrow \lambda = \frac{m}{2}\frac{\ddot{x} + \ddot{y} - g}{x + y}$$

1b

Differentiating the constraint twice,

$$C = 0 \Rightarrow x\dot{x} + y\dot{y} = 0 \Rightarrow x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0$$

Solving for $\ddot{x}$ and $\ddot{y}$,

$$\ddot{x} = -\frac{y\ddot{y} + \dot{x}^2 + \dot{y}^2}{x}$$

$$\ddot{y} = -\frac{x\ddot{x} + \dot{x}^2 + \dot{y}^2}{y}$$

Adding,

$$\ddot{x} + \ddot{y} = -\left(\frac{y^2\ddot{y} + y\dot{x}^2 + y\dot{y}^2 + x^2\ddot{x} + x\dot{x}^2 + x\dot{y}^2}{xy}\right)$$

$$= \frac{-1}{xy}\left(x^2\ddot{x} + y^2\ddot{y} + (x + y)(\dot{x}^2 + \dot{y}^2)\right)$$

Plugging this in to the equation for λ,

$$\lambda = -\frac{m}{2}\left(\frac{x^2\ddot{x} + y^2\ddot{y}}{x^2 y + y^2 x} + \frac{\dot{x}^2 + \dot{y}^2}{xy} + \frac{g}{x+y}\right)$$

Multiplying by $1 = \frac{l^2}{l^2}$ and noting that $l^2 = x^2 + y^2 = (x+y)^2 - 2xy$,

$$=$$

2a

If $U = 0$, the effective potential is

$$U_{\text{eff}}(r) = U_{\text{cf}} = \frac{l^2}{2\mu r^2} \approx \frac{l^2}{2m_2 r^2}$$

The approximation is from $m_1 \gg m_2$. The resulting Lagrangian in the CM frame (which is approximately the m_2 frame for the same reason) is

$$L = \frac{1}{2}m_2 \dot{r}^2 - \frac{l^2}{2m_2 r^2}$$

The resulting Euler-Lagrange equation of motion is

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Leftrightarrow \frac{l^2}{m_2 r^3} = m_2 \ddot{r} \Leftrightarrow \frac{l^2}{m_2^2} = r^3 \ddot{r}$$

This is a second-order autonomous equation with no first-derivative dependence, and so can be solved as

$$t(r) + c_1 = \pm\int \frac{dr}{\sqrt{2\int \frac{l^2}{m_2^2 r^3}dr + c_2}} = \pm\sqrt{\frac{m_2^2}{l^2}}\int \frac{dr}{\sqrt{c_2 - \frac{1}{r^2}}} = \pm\sqrt{\frac{m_2^2}{l^2}}\frac{r\sqrt{c_2 - \frac{1}{r^2}}}{c_2}$$

$$\Rightarrow \sqrt{r^2 c_2 - 1} = \pm\left(c_2 t\sqrt{\frac{l^2}{m_2^2}} + c_1 c_2\right)$$

$$\Leftrightarrow r = \sqrt{1 + \left(t\sqrt{\frac{l^2}{m_2^2}} + c\right)^2}$$