

4750 HW 4

Duncan Wilkie

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Problem 1 (Standard Quantum Limit). Consider a simple Michelson interferometer to measure gravitational waves, with a “DC” readout (that is, not using an RF readout with sidebands, and not at the absolutely dark or absolutely bright fringe).

1. Assuming an operating point with output power $P_{out} \neq 0$, calculate the sensitivity of the photodiode power to the gravitational wave $h = \delta L/L$ (in other words, calculate (dP_{out}/dh)) as a function of the input power P_{in} .
2. (Optional) The average number of photons arriving to a photodetector per unit time is $\bar{n} = P_{out}/(\hbar\omega) = \lambda P_{out}/(hc)$, where λ is the laser wavelength and h is Planck’s constant. However, the actual number of photons will fluctuate due to their quantum nature, with a Gaussian distribution if $\bar{n} \gg 1$: this is “shot noise.” Prove that these fluctuations are read as a noise in the GW readout with a frequency-independent amplitude spectral density

$$h_{shot}(f) = \frac{1}{L} \sqrt{\frac{\hbar c \lambda}{P_{in}}}.$$

3. The average number of photons arriving at each end mirror is $(P_{in}/2)/(\hbar\omega)$, and will push the mirror with radiation pressure. However, that power will also fluctuate, and that will be read as “radiation pressure” noise at the GW readout. Prove that at frequencies well above the suspended pendulum frequency, the noise in the readout will be

$$h_{rp}(f) = \frac{1}{m f^2 L} \sqrt{\frac{\hbar P_{in}}{\pi^2 c \lambda}}.$$

4. (Optional) In a Fabry-Perot Michelson, the shot noise is not frequency-independent because the cavity response is not frequency independent; it is instead

$$h_{shot}(f) = \frac{1}{L} \sqrt{\frac{\hbar c \lambda}{P_{in}}} \sqrt{1 + (f/f_{cp})^2}$$

with f_{cp} being the “cavity pole” of the cavity, $f_{cp} = \pi c/(2\mathcal{F}L)$, and $\mathcal{F} = 450$ is the finesse of the arm cavity. Plot the quadrature sum of the shot noise and radiation pressure noise $h = \sqrt{h_{shot}^2 + h_{rp}^2}$ for 40 kg mirrors, $\lambda = 1064$ nm, $L = 4$ km, for three different input powers: 40 W, 40 kW, 40 MW, for frequencies between 0.1 Hz and 1 kHz (assume the pendulums are very, very long, with resonance frequencies below 0.1 Hz). At what frequencies do each of the sums have their minimum value, and what is the minimum value?

5. Using the expressions in previous parts, prove that the minimum value of the total quantum noise $h = \sqrt{h_{\text{shot}}^2 + h_{\text{rp}}^2}$ for any given input power (this is the “standard quantum limit”) is independent of the input power or the operating point, and is given by

$$h_{\text{SQL}} = \frac{1}{\pi f L} \sqrt{\frac{\hbar}{m}}.$$

Solution. For the first part,

$$\frac{dP_{\text{out}}}{dh} = \frac{dP_{\text{out}}}{d\delta L} \frac{d\delta L}{dh} = \frac{d}{d\delta L} [P_{\text{in}} \sin^2(k\delta L)] \frac{d}{dh} [hL] = kP_{\text{in}} L \sin(2k\delta L).$$

We can plot the total noise with gnuplot:

```
set terminal tikz

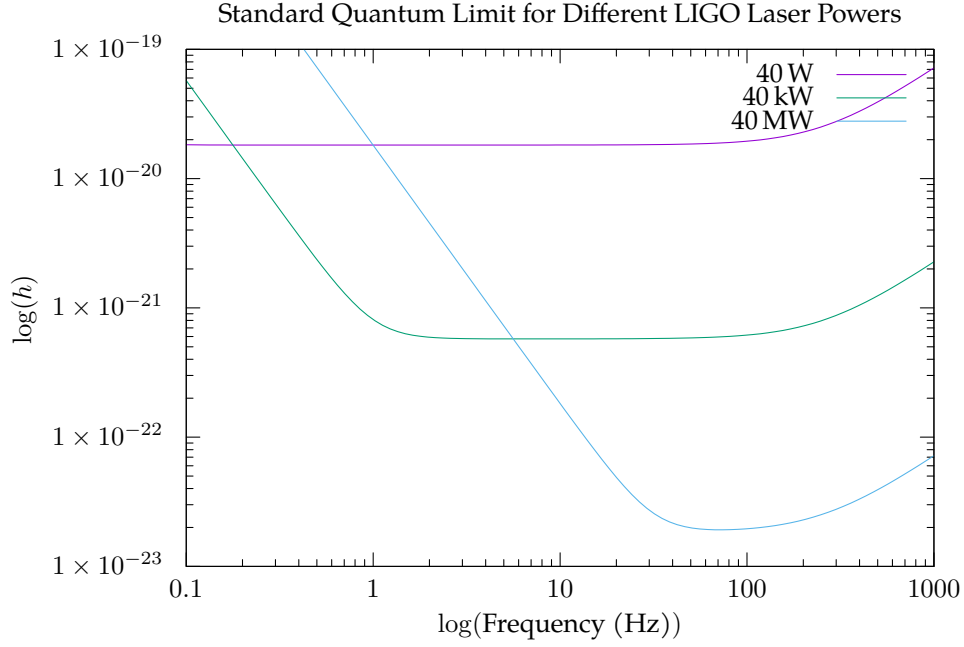
set output "p1.tex"

c = 3e8
h = 6.63e-34
m = 40
l = 1064e-9
L = 4e3
fcp = pi * c / (2 * 450 * L)

shot(f, P) = (1 / L) * sqrt(h * c * l / P) * sqrt(1 + (f / fcp)**2)
radp(f, P) = 1 / (m * f**2 * L) * sqrt(h * P / (pi**2 * c * l))
qnoise(f, P) = sqrt(shot(f, P)**2 + radp(f, P)**2)

set logscale xy
set yrange [1e-23:1e-19]
# set label "\\SI{".gprintf("%.3e", qnoise(1000, 40))."}{" at 1000,qnoise(1000, 40)
# set label "\\SI{".gprintf("%.3e", qnoise(1000, 40e3))."}{" at 1000,qnoise(1000, 40e3)
# set label "\\SI{".gprintf("%.3e", qnoise(1000, 40e6))."}{" at 1000,qnoise(1000, 40e6)
set xlabel "$\\log(\\textrm{Frequency (Hz)})$"
set ylabel "$\\log(h)$" offset -2,0
set title "Standard Quantum Limit for Different LIGO Laser Powers"
plot [f=0.1:1e3] qnoise(f, 40) title '\\SI{40}{W}', \
    qnoise(f, 40e3) title '\\SI{40}{kW}', \
    qnoise(f, 40e6) title '\\SI{40}{MW}'
```

This produces the following:



I wrote function optimization via bisection and central difference in Scheme for an old statistical mechanics assignment; I played around with that to see if there was variation in the flat parts, but the derivative appears to be *zero* there to up to IEEE double-precision.

Writing out the total quantum noise as a function of frequency,

$$h = \sqrt{\frac{hc\lambda}{P_{in}L^2} + \frac{hP_{in}}{m^2f^4L^2\pi^2c\lambda}}$$

Optimizing in P_{in} ,

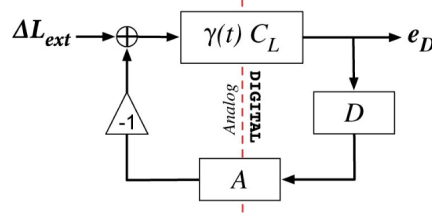
$$\begin{aligned} \frac{dh}{dP_{in}} &= \frac{1}{2} \frac{1}{\sqrt{\frac{hc\lambda}{P_{in}L^2} + \frac{hP_{in}}{m^2f^4L^2\pi^2c\lambda}}} \left(\frac{h}{m^2f^4L^2\pi^2c\lambda} - \frac{hc\lambda}{P_{in}^2L^2} \right) = 0 \\ \Rightarrow \frac{1}{m^2f^4L^2\pi^2c\lambda} &= \frac{c\lambda}{P_{in}^2L^2} \Leftrightarrow P_{in} = mf^2\pi c\lambda \end{aligned}$$

Evaluating,

$$h_{SQL} = \sqrt{\frac{hc\lambda}{mf^2\pi c\lambda L^2} + \frac{hmf^2\pi c\lambda}{m^2f^4L^2\pi^2c\lambda}} = \sqrt{\frac{h}{mf^2\pi L^2} + \frac{h}{mf^2L^2\pi}} = \frac{1}{fL} \sqrt{\frac{2h}{m\pi}} = \frac{1}{fL} \sqrt{\frac{4\pi\hbar}{m\pi}} = \frac{2}{fL} \sqrt{\frac{\hbar}{m}}$$

□

Problem 2. Consider the feedback loop used in LIGO to keep the detector operating, with the signal at the antisymmetric port linear in ΔL . For this, the difference in arm length needs to be controlled to have small fluctuations near the operation point: we call this the “DARM loop.” A simple approximation of the loop is shown in the figure below. ΔL_{ext} represents the external disturbances to the arm length difference (due to the seismic noise, thermal noise, and gravitational waves), e_D is the (digital) “error” signal (with units of digital “counts”) we measure at the antisymmetric port, used to generate a control signal c_D that pushes on one or both the end test masses to keep Δl small.



C_L is the sensing function of the dual-recycled Fabry-Perot Michelson interferometer, which is approximated by a simple “cavity pole” transfer function and a constant gain $\gamma = 500$ ct/nm:

$$C(f) = \frac{\tilde{e}_D}{\Delta \tilde{L}} = \frac{500 \text{ ct/nm}}{1 + i f / (350 \text{ Hz})}$$

(in reality, the gain of the sensing function varies due to alignment fluctuations, and we track it by injecting “calibration lines”). The actuation function is the response of the pendulum, which will approximate that of a simple pendulum

$$A(f) = \frac{\Delta \tilde{L}}{\tilde{c}_D} = \frac{1 \text{ pm/ct}}{1 - (f/0.7 \text{ Hz})^2 - i(f/0.7 \text{ Hz})}$$

(We can use a simple pendulum approximation when we push the test mass from the reaction chain at the bottom of the quadruple pendulum, using an electrostatic actuator, and we also need to track its gain with calibration lines. We also need to push on upper masses, which complicates the actual actuation function.)

1. Design a simple digital controller (try first a constant) that results in an open loop gain with unity gain frequency at 100 Hz, and plot the open-loop gain and the closed-loop gain for frequencies between 0.1 Hz and 1 kHz.
2. Prove that we can produce a “calibrated strain” as $\tilde{h} = (\tilde{e}_D / C_L(f) + A(f) \tilde{e}_C) / L$.
3. (Optional) Using the seismic excitation you used in Homework 3 and the digital controller of part (a), plot the amplitude spectral density of the residual displacement ($\Delta L_{ext} - \Delta L_{ctrl}$), and calculate the residual rms as a fraction of the laser wavelength (1064 nm).
4. (Optional) There is sensor noise (shot noise at the photodiode) added to the signal of the optical sensor. Assuming we have an ADC for the photocurrent with gain 10 V/16 384 cts, and 800 W on the beamsplitter, plot the amplitude spectral density of calibrated strain limited by shot noise.

Solution. We seek a $\tilde{D}(f)$ such that

$$|\tilde{G}(f)| = |\tilde{A}(f) \tilde{D}(f) \tilde{C}(f)| = 1$$

at $f = 100 \text{ Hz}$. Taking the ansatz $\tilde{D}(f) = c$,

$$\begin{aligned}
 |c| &= \frac{1}{|\tilde{A}(f)\tilde{C}(f)|} = \frac{1}{\left| \frac{500 \text{ ct/nm}}{1+if/(350 \text{ Hz})} \frac{1 \text{ pm/ct}}{1-(f/0.7 \text{ Hz})^2-i(f/0.7 \text{ Hz})} \right|} \\
 &= \left| \frac{1 - (f/0.7 \text{ Hz})^2 - i(f/0.7 \text{ Hz}) + if/(350 \text{ Hz}) - if(f/0.7 \text{ Hz})^2/(350 \text{ Hz}) + f^2/(245 \text{ Hz})}{0.5 \text{ m}^{-1}} \right| \\
 &= \left| \frac{1 - i\frac{f^3}{172 \text{ Hz}} + f^2(\frac{1}{245 \text{ Hz}} - \frac{1}{0.49 \text{ Hz}^2}) + if(\frac{1}{350 \text{ Hz}} - \frac{1}{0.7 \text{ Hz}})}{0.5 \text{ m}^{-1}} \right| = \left| 2 \text{ m} \left(1 - \frac{i}{172 \text{ Hz}} f^3 - 2.04 f^2 - 1.43 i f \right) \right|
 \end{aligned}$$

Evaluating this at $f = 100 \text{ Hz}$, $|c| = 42500$. The open-loop gain is the expression $|G(f)|$ above, and the closed-loop gain is $H(f) = \frac{1}{1+G(f)}$. Plotting with gnuplot,

```

set terminal tikz
set output 'p2.tex'

c0 = 500e9 # ct / m
fc = 350   # Hz
fa = 0.7   # Hz
a0 = 1e-12 # m / ct
D = 42500
i = {0, 1}

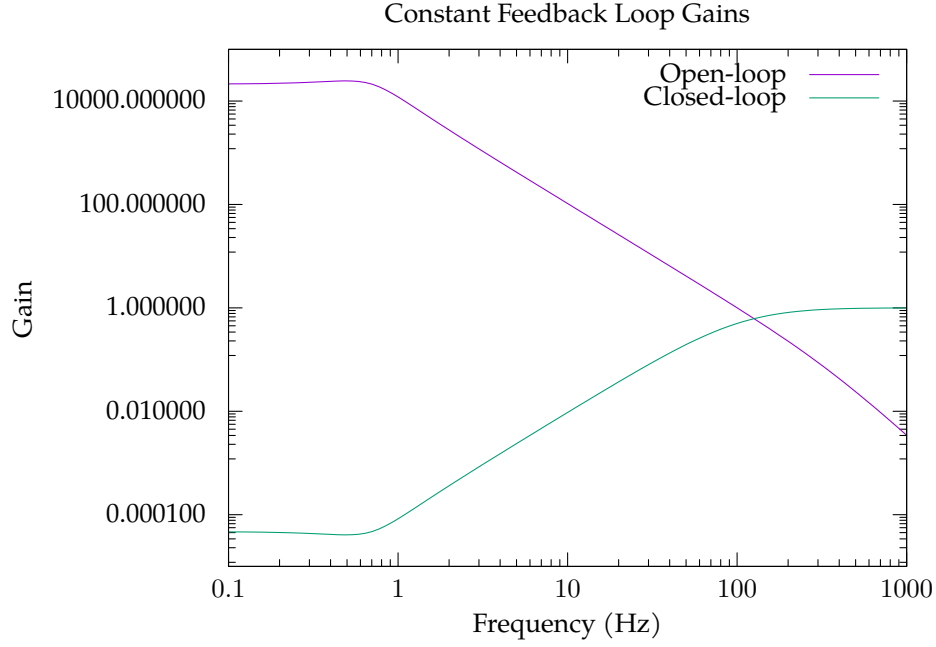
C(f) = c0 / (1 + i * (f / fc))
A(f) = a0 / (1 - (f / fa)**2 - i * (f / fa))

G(f) = abs(A(f) * C(f) * D)
H(f) = abs(1 / (1 + G(f)))

set title 'Constant Feedback Loop Gains'
set xlabel 'Frequency (Hz)'
set ylabel 'Gain'
set logscale xy
plot [f=0.1:1e3] G(f) title 'Open-loop', H(f) title 'Closed-loop'

```

results in



The loop gives

$$e_D = C(f)(\Delta L_{ext} - DA(f)) \Leftrightarrow \Delta L_{ext} = hL = \frac{e_D}{C(f)} + DA(f) \Rightarrow h = \left(\frac{e_D}{C(f)} + DA(f) \right) / L.$$

□

Problem 3. Let's use a simple Michelson detector at the dark port using phase modulation of the input laser and an RF readout.

1. Consider the input laser modulated, and expanded using Bessel functions of the first kind

$$E = E_0 e^{i(\omega t + \Gamma \sin(\Omega t))} = E_0 e^{i\omega t} \sum_{n=-\infty}^{\infty} J_n(\Gamma) e^{in\Omega t}$$

Given an average arm length L , a time-independent macroscopic (Shnupp) arm length difference $\Delta L \gg \lambda$ and a very small fluctuation in arm length difference $\delta L \ll \lambda$ from a dark operating point where $kL = n\pi$, calculate the output power to first order in δL .

2. Assume we use a low-pass filter to eliminate terms oscillating with 2Ω frequencies or higher, but keep terms oscillating with Ω . Write down the output power as $P_{out} = P_{DC} + A \sin(\Omega t) + B \cos(\omega T)$ (again, to first order in δL).
3. (Optional) What's the shot noise produced by the power on the photodiode? Consider the DC term and the oscillating terms.

4. Now let's demodulate the signal. Assuming we convert the power in the photodiode into a voltage $V_{out} = \eta Z P_{out}$ (η is the quantum PD efficiency in A/W, Z is impedance used in the readout in Ohms), calculate the demodulated signals $V_Q = V_{out} \sin(\Omega t + \phi)$ and $V_I = V_{out} \cos(\Omega t + \phi)$, where ϕ is a phase we will tune in the last part.
5. Write down the expressions for the demodulated signals V_Q, V_I after using another low-pass filter to eliminate terms oscillating at $n\Omega$.
6. Find the phase ϕ that makes one of the demodulated signals zero, and the other proportional to δL : we can now read a GW signal with a simple Michelson operating at the (carrier) dark port!

Solution. Asymptotically, $J_n(\Gamma) \rightarrow \frac{1}{(n+1)!} \left(\frac{\Gamma}{2}\right)^n$ as $\Gamma \rightarrow 0$, so for larger $|n|$ the contribution of the term is smaller. Accordingly, we take only the carrier and the first sidebands.

$$E = E_0 e^{i\omega t} (J_0(\Gamma) + J_1(\Gamma) e^{i\Omega t} - J_1(\Gamma) e^{-i\Omega t}) = E_0 e^{i\omega t} (J_0(\Gamma) + 2J_1(\Gamma) \cos(\Omega t)).$$

We have that

$$E_{out}^0 = J_0 E_0 e^{i\omega t} e^{-2ikL} \sin(k\Delta l),$$

$$E_{out}^\pm = \pm i J_1 E_0 e^{i\omega t} e^{\pm i\Omega t} e^{-2ikL} e^{\mp 2i\Omega L/c} \sin([\omega \pm \Omega]\Delta L/c)$$

Near the dark fringe, to first order in δL , $\sin(k\Delta l) \approx k\delta L$ and $\sin([\omega \pm \Omega]\Delta L/c) \approx (\omega \pm \Omega)\delta L/c$. The total output power is

$$P_{out} = |E_{out}|^2 = |E_{out}^0|^2 + |E_{out}^+|^2 + |E_{out}^-|^2 + \text{Re}(E_{out}^0 (E_{out}^+ + E_{out}^-)^*) + \text{Re}(E_{out}^+ E_{out}^-)$$

The magnitude-squared of E_{out}^0 is just the DC operating power; using the above expressions,

$$P_{out} = P_{DC} + J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \cos(\Omega t + \phi) + J_1^2 \sin^2(\Omega\Delta L/c) \cos(2\Omega t + \phi')$$

If a low-pass filter is introduced, it prunes out the last term of the output power:

$$P'_{out} = P_{DC} + J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \cos(\Omega t + \phi)$$

Using the sum-to-product identity for cosine,

$$= P_{DC} + J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) (\cos(\Omega t) \cos(\phi) + \sin(\Omega t) \sin \phi)$$

$$= P_{DC} + \left[J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \cos(\phi) \right] \cos(\Omega t) + \left[J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \sin \phi \right] \sin(\Omega t)$$

Accordingly, in the photodiode readout we have

$$V_{out} = \eta Z \left(P_{DC} + \left[J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \right] [\cos(\phi) \cos(\Omega t) + \sin \phi \sin(\Omega t)] \right)$$

The corresponding demodulated signals are merely this times $\sin(\Omega t + \phi) = \sin(\Omega t) \cos \phi + \cos(\Omega t) \sin \phi$ and $\cos(\Omega t + \phi) = \cos(\Omega t) \cos \phi - \sin(\Omega t) \sin \phi$. In each case, there is some simplification via the Pythagorean and sine double-angle identities to

$$V_Q = \eta Z \left(P_{DC} \sin(\Omega t + \phi) + \left[J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega\Delta L/c) \right] \left[\frac{1}{2} \sin(2\Omega t) + \frac{1}{2} \sin(2\phi) \right] \right)$$

and

$$V_I = \eta Z \left(P_{DC} \cos(\Omega t + \phi) + \left[J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega \Delta L / c) \right] \left[1 + \frac{1}{2} \sin(2\Omega t) \sin(2\phi) \right] \right)$$

The filtering out terms oscillating at Ω or above,

$$V_Q = \frac{1}{2} \eta Z J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega \Delta L / c) \sin(2\phi)$$

and

$$V_I = \eta Z J_0 J_1 \left(\frac{k\delta L}{c} \right) \sin(\Omega \Delta L / c)$$

Choosing $\phi = 0$ makes $V_Q = 0$ and $V_I = [\eta Z J_0 J_1 k \sin(\Omega \Delta L / c) / c] \delta L$; all quantities multiplying δL we know or set ourselves, so measuring V_I provides a measurement of δL . \square

Problem 4. Consider two mirrors with amplitude reflectivity and transmissivity r_1, t_1, r_2, t_2 at a fixed distance L , and an input electric field $E = E_0 e^{i\omega t}$ traveling in the z -direction, perpendicular to the mirrors, incident at mirror 1 which is at $z = 0$. Assume the “+” side of the mirrors are the sides facing each other.

1. Follow the single beam that is reflected from and transmitted through mirror 1: now there are two beams, one traveling forward in the cavity and another traveling back to the laser. The transmitted beam travels to mirror 2 (picking up a phase e^{ikL}) and is then reflected from and transmitted through mirror 2: now there are three beams: two traveling forward, one inside the cavity and one transmitted by the cavity, and two beams traveling back: the one promptly reflected by mirror 1 and the one reflected from mirror 2. The single beam reflected from mirror 2 is now reflected from and transmitted through mirror 2...you get the idea. Draw several of these beams and the expression for the fields for each one on each side of the mirror, and write the total transmitted beam, the total reflected beam, and the total field traveling forward in the cavity as a sum of exponentials.
2. Using the identity $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, obtain an expression for the total reflected beam, the total transmitted beam, and the total forward traveling beam in the cavity (compare these with the ones we derived in class using a steady state).
3. Plot the amplitude and phase of the cavity transmissivity, gain, and reflectivity as a function of the variable $f - f_0$ with $f_0 = c/\lambda$ for $L = 4$ km, $\lambda = 1064$ nm, and $f - f_0$ between -40 kHz and 40 kHz.

Solution. I'd write something up in TikZ, but I'm running close to the deadline. I have indeed drawn it out to come up with the following ansatz. It's simple to verify the following by induction on the term index, since the case for one or two bounces of a beam of known form is easy.

$$E_{back} = r_1 E_0 e^{i\omega t} + t_1^2 r_2 E_0 e^{i(\omega t - 2ikL)} + t_1^2 r_2^2 r_1 E_0 e^{i(\omega t + 4ikL)} + t_1^2 r_2^3 r_1^2 E_0 e^{i(\omega t - 6ikL)} + \dots + t_1^2 r_2^n r_1^{n-1} E_0 e^{i(\omega t - 2nikL)} + \dots$$

$$E_{in} = t_1 E_0 e^{i\omega t} + t_1 r_2 e^{i\omega t - ikL} + t_1 r_2 r_1 e^{i\omega t - 2ikL} + t_1 r_2 r_1^2 e^{i\omega t - 3ikL} + \dots + t_1 r_2^n r_1^{n-1} e^{i\omega t - nikL} + \dots$$

$$E_{out} = t_1 t_2 E_0 e^{i\omega t - ikL} + t_1 t_2 r_2 r_1 e^{i\omega t - 3ikL} + t_1 t_2 r_2^2 r_1^2 e^{i\omega t - 5ikL} + \dots + t_1 t_2 r_2^n r_1^n e^{i\omega t - (2n+1)ikL} + \dots$$

The sums of these series are

$$E_{back} = \frac{t_1^2 E_0 e^{i\omega t} / r_1}{1 - r_1 r_2 e^{-2ikL}}$$

$$E_{in} = \frac{t_1 E_0 e^{i\omega t} / r_1}{1 - r_1 r_2 e^{-ikL}}$$

$$E_{out} = \frac{t_1 t_2 E_0 e^{i\omega t - ikL}}{1 - r_1 r_2 e^{-2ikL}}$$

These are analogous to the steady-state expressions, but the first two have a divisor r_1 here not appearing otherwise. Since they're mirrors, this divisor will be close to one, making the steady-state approximation excellent. The reflectivity, gain, and transmissivity of the cavities are the three multipliers of $E_0 e^{i\omega t}$ in the above equations, respectively. Using gnuplot,

```
set terminal tikz
set output 'p4.tex'

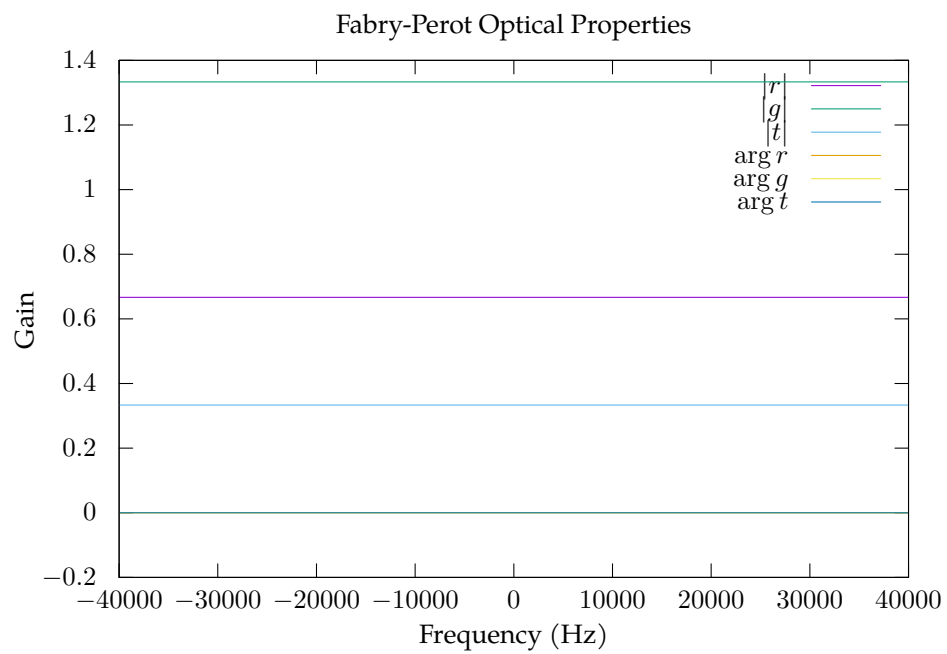
c = 3e8
L = 4e3
l = 1064e-9
f0 = c / l
i = {0, 1}

w(fmf0) = 2 * pi * (fmf0 + f0)
k(fmf0) = 1 / w(fmf0)
Eb(fmf0) = (0.5**2 / 0.5) / (1 - 0.5**2 * exp(-2 * i * k(fmf0) * L))
Ei(fmf0) = 1 / (1 - 0.5**2 * exp(-1 * i * k(fmf0) * L))
Eo(fmf0) = 0.5 **2 * exp(-1 * i * k(fmf0) * L) / (1 - 0.5**2 * exp(-2 * i * k(fmf0) * L))

absB(fmf0) = abs(Eb(fmf0))
absI(fmf0) = abs(Ei(fmf0))
absO(fmf0) = abs(Eo(fmf0))

phiB(fmf0) = arg(Eb(fmf0))
phiI(fmf0) = arg(Ei(fmf0))
phiO(fmf0) = arg(Eo(fmf0))

set title 'Fabry-Perot Optical Properties'
set xlabel 'Frequency (Hz)'
set ylabel 'Gain'
# set logscale xy
plot [fmf0=-40e3:40e3] absB(fmf0) title '$|r|$', absI(fmf0) title '$|g|$', absO(fmf0) title '$|t|$',
    phiB(fmf0) title '$\arg r$', phiI(fmf0) title '$\arg g$', phiO(fmf0) title '$\arg t$'
```



□