7550 HW 4

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Problem 1. *Let $f : V \to W$ be linear. Show that if f is surjective, then the induced map $f^* : W^* \to V^*$ defined by $(f^*(g))(v) = g(f(v))$ is injective.*

Problem 2. *Let $f : V \to V$ be a linear endomorphism, and let $\mathcal{B}$ and $\mathcal{B}^*$ be dual bases for V and V^*. If A and B are the matrices of f and $f^* : V^* \to V^*$ with respect to $\mathcal{B}$ and $\mathcal{B}^*$, show that B is the transpose of A.*

Problem 3. *If $f : V_1 \to V_2$ is a surjective linear map, show that, for any W, the induced map $V_1 \otimes W \to V_1 \otimes W$ defined by $v_1 \otimes w \mapsto f(v_1) \otimes w$, is also surjective.*

Problem 4. *Show that $\{v_1, \ldots, v_r\}$ is a linearly independent set in V iff $v_1 \wedge \cdots \wedge v_r \neq 0$.*

Problem 5. *Show that two linearly independent sets $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ in V span the same r-dimensional subspace iff $v_1 \wedge \ldots \wedge v_r = c \cdot w_1 \wedge \ldots \wedge w_r$, where $c = \det(A)$, and $A = (a_{ij})$ is given by $v_i = \sum_{j=1}^r a_{ij} w_j$.*

Problem 6. *Let $f : V \to V$ be linear, let $\mathcal{B}$ be a basis for V, and let A be the matrix of f with respect to $\mathcal{B}$.*

1. *Let $\phi : \Lambda^n V \to \Lambda^n V$ be the map induced by f, defined by $\phi(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$. Since $\Lambda^n V$ is 1-dimensional, ϕ corresponds to multiplication by some scalar, say c. Show that $c = \det(A)$.*

Problem 7. *Let V be a real inner product space, a real vector space equiped with a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, with $\langle v, v \rangle = 0 \Leftrightarrow v = 0$. Then $\langle \cdot, \cdot \rangle$ induces an inner product $\langle \cdot, \cdot \rangle : \Lambda V \times \Lambda V \to \mathbb{R}$, defined as follows: if $u = u_1 \wedge \cdots \wedge u_r$ and $v = v_1 \wedge \cdots \wedge v_s$ are pure wedges, set $\langle u, v \rangle = 0$ if $r \neq s$ and $\langle u, v \rangle = \det(\langle u_i, v_j \rangle)$ if $r = s$. This can be extended linearly in each argument. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V, a basis for which $\langle e_i, e_j \rangle = \delta_{ij}$. Show that the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq n, 0 \leq r \leq n\}$ is an orthonormal basis for ΛV. For $r = 0$, the empty wedge product is interpreted to be $1 \in \mathbb{R} = \Lambda^0 V$.*

Problem 8. *An endomorphism $\psi : \Lambda V \to \Lambda V$ is an anti-derivation if, for $u \in \Lambda^k V$ and $v \in \Lambda V$, $\psi(u \wedge v) = \psi(u) \wedge v + (-1)^k u \wedge \psi(v)$. Show that $\psi : \Lambda V \to \Lambda V$ is an anti-derivation iff*

$$\psi(v_1 \wedge \cdots \wedge v_r) = \sum_{k=1}^{r} (-1)^{k+1} v_1 \wedge \cdots \wedge \psi(v_k) \wedge \cdots \wedge v_r$$