4123 HW 5

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1

The charged particle Lagrangian is, in terms of the generalized coordinate r and potentials $\vec{A}$ and V,

$$L = \frac{1}{2}m\dot{q_i}^2 + Q\dot{q_i}A_i - QV$$

The generalized momentum is

$$p_i = \frac{\partial L}{\partial \dot{q_i}} = m\dot{q_i} + QA_i \Leftrightarrow \dot{q_i} = \frac{p_i - QA_i}{m}$$

Plugging this in to the definition of the Hamiltonian, we obtain

$$\mathcal{H}(p_i, q_i) = p_i\dot{q_i} - L = p_i\frac{p_i - QA_i}{m} - \frac{(p_i - QA_i)^2}{2m} - QA_i\frac{p_i - QA_i}{m} + QV$$

$$= (p_i - QA_i)\frac{p_i - QA_i}{m} - \frac{(p_i - QA_i)^2}{2m} + QV$$

$$= \frac{(p_i - QA_i)^2}{2m} + QV$$

From this, we obtain two Hamilton's equations of motion:

$$\dot{q_i} = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i - QA_i}{m}$$

$$\dot{p_i} = -\frac{\partial \mathcal{H}}{\partial q_i} = 0$$

These ought to imply the Lorentz force law.

2

The Lagrangian for such a system is

$$L = \frac{1}{2}m\dot{q}^2 - mgq$$

The generalized momentum is

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Leftrightarrow \dot{q} = \frac{p}{m}$$

Plugging this in to the definition of the Hamiltonian,

$$\mathcal{H}(p,q) = p\dot{q} - L = \frac{p^2}{2m} + mgq$$

The corresponding Hamilton's equations are

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -mg$$

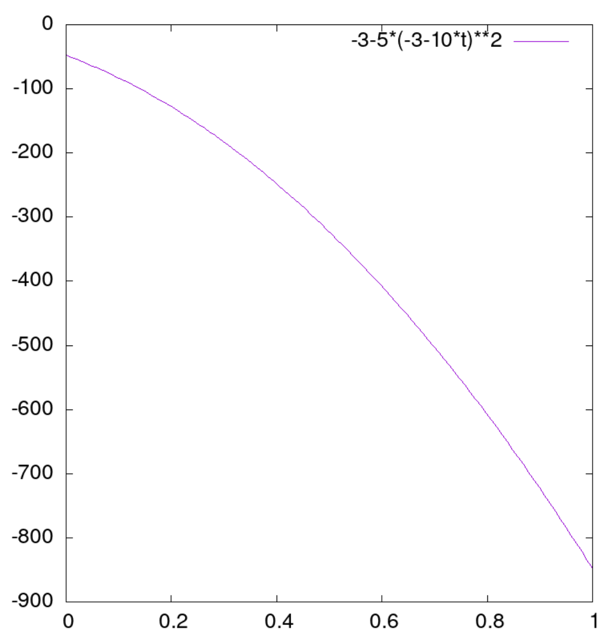
Integrating the second equation with respect to t,

$$p = -mgt + p_0 \Rightarrow \dot{q} = -gt + \frac{p_0}{m} \Rightarrow q = -\frac{gt^2}{2} + \frac{p_0}{m} + q_0$$

The phase-space vector of the system is then

$$\vec{z} = \left(p_0 - mgt, q_0 + \frac{p_0}{m} - \frac{1}{2}gt^2\right)$$

If we take $q_0 = 0$ and p_0 negative, consistent with throwing an object upward from the ground, the plot of the parametric curve in phase space is just a parabola, since the first component being linear is just a rescaling of the horizontal axis (i.e. this is equivalent to the second component under a coordinate transformation $t \mapsto (p_0 - q)/mg$). A plot appears below with some test values:



3

In cylindrical coordinates, the Lagrangian of a free particle is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

The generalized momenta are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} \end{aligned}$$

Therefore, we may write the Lagrangian as

$$L = \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + p_z^2\right]$$

The Hamiltonian is by definition

$$\begin{aligned} \mathcal{H} = \sum_i p_i\dot{q}_i - L &= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_z^2}{m} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{p_z^2}{2m} \\ &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) \end{aligned}$$

In spherical coordinates, the Lagrangian of a free particle is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2)$$

The generalized momenta are

$$\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ p_\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2\theta\dot{\varphi} \end{aligned}$$

Therefore, we may write the Lagrangian as

$$L = \frac{1}{2m}\left[p_r^2 + \left(\frac{p_\theta}{r}\right)^2 + \left(\frac{p_\varphi}{r\sin\theta}\right)^2\right]$$

The Hamiltonian is by definition

$$\begin{aligned} \mathcal{H} = \sum_i p_i\dot{q}_i - L &= \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_\varphi^2}{mr^2\sin^2\theta} - \frac{p_r^2}{2m} - \frac{p_\theta^2}{2mr^2} - \frac{p_\varphi^2}{2mr^2\sin^2\theta} \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\varphi^2}{2mr^2\sin^2\theta} \end{aligned}$$

This suggests a general method for converting Hamiltonians between (some subset of) coordinate systems.

4

We consider the problem in spherical coordinates. If $F = -\nabla U = -k\vec{r}$, we have a potential

$$U = \frac{k}{2}\left(x^2 + y^2 + z^2\right) = \frac{k}{2}\left(r^2\cos^2\theta + r^2\sin^2\theta + z^2\right) = \frac{k}{2}\left(r^2 + z^2\right)$$

The Lagrangian, including the constraint $r = R$ and its Lagrange multiplier, is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - \frac{k}{2}\left(r^2 + z^2\right)$$

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$
$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

The Hamiltonian is, in terms of these quantities,

$$\mathcal{H} = \sum_i p_i\dot{q}_i - L = \frac{p_r^2}{m} + \frac{p_\theta^2}{mr^2} + \frac{p_z^2}{m} + \frac{k}{2}\left(r^2 + z^2\right) - \frac{1}{2}m\left(\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4} + \frac{p_z^2}{m^2}\right)$$

Since $r = R$, this becomes

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{k}{2}\left(R^2 + z^2\right)$$

The Hamilton's equations are

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}$$
$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mR^2}$$
$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_z} = \frac{p_z}{m}$$
$$\dot{p_r} = -\frac{\partial \mathcal{H}}{\partial r} = 0$$
$$\dot{p_\theta} = -\frac{\partial \mathcal{H}}{\partial \theta} = 0$$
$$\dot{p_z} = -\frac{\partial \mathcal{H}}{\partial z} = -kz$$

From this, $\dot{r}$ and $\dot{\theta}$ are constant (the former being actually zero from the constraint), and the interesting equations of motion are

$$\dot{z} = \frac{p_z}{m}$$
$$\dot{p_z} = -kz$$