7550 HW 3

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Problem 1 (Bredon II.5.4)**.** *If $M^m \subseteq \mathbb{R}^n$ is a smoothly embedded manifold and f is a smooth real valued function defined on a neighborhood of $p \in M^m$ in $\mathbb{R}^n$ and which is constant on M, show that $\boldsymbol{\nabla} f$ is perpendicular to T_pM at p.*

Proof. By the problem prior to this one in the text, it would suffice to show $D_v(f) = 0$ for all $v \in T_p(M)$, since $D_v(f) = \langle \boldsymbol{\nabla} f, v\rangle = 0$ is the definition of perpendicularity.

Let ϕ be the embedding in question. View p in M, and consider coordinate functions $y_1, \ldots, y_m$ for one of its neighborhoods. The vectors $\frac{\partial}{\partial y_i}$ are the standard basis for T_pM, and, since ϕ_* is a monomorphism, are also a basis for $\phi_*(T_pM)$ in $T_p\mathbb{R}^n$. This basis can be extended by $n-m$ linearly independent vectors $\frac{\partial}{\partial y_i'}$ not in $\phi_*(T_pM)$ to get a basis for $T_p\mathbb{R}^n$.

This extended basis corresponds also to a basis for $\mathbb{R}^n$: letting A be the change-of-basis matrix (isomorphism) out of $\frac{\partial}{\partial x_i}$ and the isomorphism betwen $T_p\mathbb{R}^n$ and $\mathbb{R}^n$ be ψ, the images of $\frac{\partial}{\partial x_i}$ under the isomorphism $\psi \circ A$ are a basis for $\mathbb{R}^n$.

Since the argument in Example 5.4 is basis-independent, we may apply its result in our bases:

$$D_v = \sum_{i=1}^{m} v_i \frac{\partial f}{\partial y_i} + \sum_{i=m+1}^{n} v_i \frac{\partial f}{\partial y_i'}.$$

However, since f is constant on M, and so also for projections of $f \circ y^{-1}$, $\frac{\partial f}{\partial y_i} = 0$. Furthermore, vectors in T_pM are by construction precisely those whose $\frac{\partial}{\partial y_i'}$ components are zero, for every i; therefore $D_{\gamma_v} = 0$, for all $v \in T_pM$, precisely what was desired. □

Problem 2 (Bredon II.7.1 et. al.)**.** *Consider the real-valued function $f(x,y,z) = (2-(x^2+y^2)^{1/2})^2 + z^2$ on $\mathbb{R}^3 - \{(0,0,z)\}$.*

1. *Show that 1 is a regular value of f. Identify the manifold $M = f^{-1}(1)$.*
2. *Show that M is transverse to $N_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 4\}$. Identify the manifold $M \pitchfork N_1$.*
3. *Show that M is not transverse to $N_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 1\}$. Is $M \cap N_2$ a manifold?*
4. *Show that M is not transverse to $N_3 = \{(x,y,z) \in \mathbb{R}^3 \mid x = 1\}$. Is $M \cap N_3$ a manifold?*

Proof. Let $N = \mathbb{R}^3 - \{(0,0,z)\}$, and let p be any element of $f^{-1}(1)$. Consider local coordinates in a neighborhood of p (in N), and a smooth path γ through p at 0. One has

$$df(D_\gamma)g = D_{f(\gamma)}g = \frac{\mathrm{d}}{\mathrm{d}t} g(f(\gamma_1(t), \gamma_2(t), \gamma_3(t)))\bigg|_{t=0} = g'(f(\gamma_1(t), \gamma_2(t), \gamma_3(t)))\left(\sum_{i=1}^{3} \frac{\partial f}{\partial x_i}\frac{\mathrm{d}\gamma_i}{\mathrm{d}t}\right)\bigg|_{t=0}$$

The quantity in the brackets is certainly never zero, given that one is free to choose $\frac{\mathrm{d}\gamma_i}{\mathrm{d}t}$ at will. Accordingly, this operator behaves like some multiple of $\frac{\mathrm{d}}{\mathrm{d}t}$ to functions $g : \mathbb{R} \to \mathbb{R}$—which definitely spans the tangent space; df is onto for all points in $f^{-1}(1)$, so 1 is a regular value

One can take slices along the z-axis to get an idea of the contour diagram. At $z = 0$ one has

$$1 = (2 - \sqrt{x^2 + y^2})^2 \Leftrightarrow |2 - \sqrt{x^2 + y^2}| = 1 \Leftrightarrow x^2 + y^2 = 1 \vee x^2 + y^2 = 9;$$

these are two circles of radius 1 and 3. If $|z| > 1$, the plane doesn't intersect the shape, as $(2 - \sqrt{x^2 - y^2}) > 0$, and so the sum of it with something greater than 1 can never equal 1. At $|z| = 1$, one has $(2 - \sqrt{x^2 + y^2})^2 = 0 \Leftrightarrow x^2 + y^2 = 4$—a single circle of radius 2. As $|z|$ increases towards 1, the value of $\sqrt{1 - z^2}$ decreases, so $|2 - \sqrt{x^2 + y^2}|$ must also. This means that the circle of smaller radius, corresponding to the positive branch of the absolute value, gets larger; the larger circle gets smaller. This feels a lot like a torus! Sure enough, taking cross-sections in the $x - z$ plane yields

$$1 = (2 - \sqrt{x^2})^2 + z^2 \Leftrightarrow (|x| - 2)^2 + z^2 = 1.$$

This is two copies of a circle, with a center shifted two units to the left and right of the z-axis, respectively.

N_1 is a cylinder concentric with the z-axis of radius 2. It intersects M at those points of M satisfying

$$1 = (2 - \sqrt{4})^2 + z^2 \Leftrightarrow z^2 = 1 \Leftrightarrow z = \pm 1.$$

As discussed above, these are two circles in the topmost and bottommost $x - y$ planes intersecting the torus. Pick any point p in these intersections, and coordinates on $\mathbb{R}^3$ near it x, y, z. The space $T_p\mathbb{R}^3$ then has basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$. Exhibiting each of these as a linear combination of vectors in T_pN_1 or T_pM suffices to show transversality; if one can do so, then anything in $T_p\mathbb{R}^3$ (a linear combination of the basis) can be easily rearranged such that closure of the subspaces under linear combination implies the result is a sum of a vector from each subspace.

Let $p = (x_p, y_p, z_p)$. One, $\frac{\partial}{\partial z}$, can be realized completely in N_1 as corresponding to the unit-speed curve passing through p parallel to the axis of the cylinder. Let $\gamma(t) = (x_p, y_p, t \pm 1)$, according to whether $z_p = \pm 1$. This lies completely within N_1, as $p \in N^1 \Rightarrow x_p^2 + y_p^2 = 4$. In $\mathbb{R}^3$,

$$D_\gamma g = \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma_x(t), \gamma_y(t), \gamma_z(t))\bigg|_{t=0} = \frac{\partial g}{\partial x}\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}\gamma_y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}\gamma_z}{\mathrm{d}t}\bigg|_{t=0} = \frac{\partial g}{\partial z},$$

so, indeed, $D_\gamma \in T_pN_1$ realizes $\frac{\partial}{\partial z} \in T_p\mathbb{R}^3$.

The other two are realizable completely within M, as curves in slices of the torus parallel to the axes not appearing in the basis vector. Let

$$\gamma_x = \left(t - x_p, y_p, \pm\sqrt{1 - \left(2 - \sqrt{y_p^2 + (x_p - t)^2}\right)^2}\right)$$

and

$$\gamma_y = \left(x_p, t - y_p, \pm\sqrt{1 - \left(2 - \sqrt{x_p^2 + (y_p - t)^2}\right)^2}\right),$$

with the $\pm$, as before, chosen according to the value of z at p. These are both in M for all t:

$$(2-\sqrt{x(\gamma_x(t))^2+y(\gamma_x(t))^2})^2+z(\gamma_x(t))^2=(2-\sqrt{(x_p-t)^2+y_p^2})^2+\left(\pm\sqrt{1-(2-\sqrt{y_p^2+(x_p-t)^2})^2}\right)^2=1$$

and

$$(2-\sqrt{x(\gamma_x(t))^2+y(\gamma_y(t))^2})^2+z(\gamma_z(t))^2=(2-\sqrt{x_p^2+(y_p-t)^2})^2+\left(\sqrt{1-(2-\sqrt{x_p^2+(y_p-t)^2})^2}\right)^2=1$$

both hold immediately (as constructed). Analogously, $\gamma_x(0)=\gamma_y(0)=p$. We may compute

$$D_{\gamma_x}g=\frac{\mathrm{d}}{\mathrm{d}t}g(\gamma_{xx}(t),\gamma_{xy}(t),\gamma_{xz}(t))\bigg|_{t=0}=\frac{\partial g}{\partial x}\frac{\mathrm{d}\gamma_{xx}}{\mathrm{d}t}+\frac{\partial g}{\partial y}\frac{\mathrm{d}\gamma_{xy}}{\mathrm{d}t}+\frac{\partial g}{\partial z}\frac{\mathrm{d}\gamma_{xz}}{\mathrm{d}t}\bigg|_{t=0}$$

$$=\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\cdot 0+\frac{\partial g}{\partial z}\cdot\frac{\mp\left(2-\sqrt{y_p^2+(x_p-t)^2}\right)}{\sqrt{1-\left(2-\sqrt{y_p^2+(x_p-t)^2}\right)^2}}\cdot\frac{x_p-t}{\sqrt{y_p^2+(x_p-t)^2}}\bigg|_{t=0}$$

Of course, since p is one of the points such that $|z|=1$, in order for the defining equation of M to be satisfied one must have $1=\left(2-\sqrt{x_p^2+y_p^2}\right)^2+1^2\Leftrightarrow 2-\sqrt{x_p^2+y_p^2}=0$, so the above is

$$=\frac{\partial g}{\partial x}.$$

Since addition is commutative, renaming x_p to y_p and vice versa doesn't change the calculus computation above, so

$$D_{\gamma_y}g=\frac{\partial g}{\partial y}.$$

We therefore may say $\frac{\partial}{\partial x}=D_{\gamma_x}\in T_pM$, $\frac{\partial}{\partial y}=D_{\gamma_y}\in T_pM$, and $\frac{\partial}{\partial z}=D_{\gamma_x}\in T_pN_1$, and conclude $T_p\mathbb{R}^3=T_pM+T_pN_1$, i.e. $M\pitchfork N_1$.

N_2 is another cylinder concentric with the the z-axis, but this time of radius 1. One sees

$$1=(2-\sqrt{1})^2+z^2\Leftrightarrow z=0;$$

this identifies the intersection as the circle centered on the origin of radius 1 in the $x-y$ plane with $z=0$ (indeed, a manifold).

We merely need to exhibit any point p such that there's a vector in $T_p\mathbb{R}^3$ that cannot be written as an element of $T_pM+T_pN_2$. Our candidate is $\frac{\partial}{\partial x}$ at $p=(1,0,0)$.

Take any smooth path $\gamma(t)$ in M through p at $t=0$. Then

$$D_\gamma g=\frac{\mathrm{d}}{\mathrm{d}t}g(\gamma_x(t),\gamma_y(t),\gamma_z(t))\bigg|_{t=0}=\frac{\partial g}{\partial x}\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}+\frac{\partial g}{\partial y}\frac{\mathrm{d}\gamma_y}{\mathrm{d}t}+\frac{\partial g}{\partial z}\frac{\mathrm{d}\gamma_z}{\mathrm{d}t}.$$

If $D_\gamma=\frac{\partial}{\partial x}+\frac{\partial}{\partial y}$, then $\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}\big|_{t=0}=\frac{\mathrm{d}\gamma_y}{\mathrm{d}t}\big|_{t=0}=1$ and $\frac{\mathrm{d}\gamma_z}{\mathrm{d}t}\big|_{t=0}=0$. However, for paths in M one must have

$$1=\left(2-\sqrt{\gamma_x(t)^2+\gamma_y(t)^2}\right)^2+\gamma_z(t)^2\Rightarrow 2\gamma_z(t)\frac{\partial\gamma_z}{\partial t}=\left(4-2\sqrt{\gamma_x(t)^2+\gamma_y(t)^2}\right)\cdot\frac{\gamma_x(t)\frac{\mathrm{d}\gamma_x}{\mathrm{d}t}+\gamma_y(t)\frac{\mathrm{d}\gamma_y}{\mathrm{d}t}}{\sqrt{\gamma_x(t)^2+\gamma_y(t)^2}}$$

Evaluating this at $t = 0$ so that $\gamma(t) = p$, one obtains $\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} = 0$, so T_pM spans the $\frac{\partial}{\partial y}$-$\frac{\partial}{\partial z}$ plane and nothing more.

The other case proceeds similarly. Take any smooth path $\gamma(t)$ in N_2 through p at $t = 0$. Then

$$D_\gamma g = \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma_x(t), \gamma_y(t), \gamma_z(t))\bigg|_{t=0} = \frac{\partial g}{\partial x}\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}\gamma_y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}\gamma_z}{\mathrm{d}t}\bigg|_{t=0}.$$

For paths in N_2 one must have

$$\gamma_x(t)^2 + \gamma_y(t)^2 = 1 \Rightarrow \gamma_x(t)\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} + \gamma_y(t)\frac{\mathrm{d}\gamma_y}{\mathrm{d}t} = 0;$$

evaluating this at $t = 0$, one obtains $\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} = 0$, and, like above, this means T_pN_2 is the subspace of $T_p\mathbb{R}^3$ spanned by $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$. This proves that M is not transverse to N_2, as the sum of identical subspaces is just the subspace, and, for example, $\frac{\partial}{\partial x}$ is not in this subspace.

N_3 is a y-z plane with $x = 1$; it slices the torus so that the two circular regions of the cross-section intersect at a single point, whereupon their tangents coencide. As above, we need only exhibit some p and some element of $T_p\mathbb{R}^3$ at that point that isn't an element of $T_pN_3 + T_pM$. This point where the tangents heuristically coencide seems to be a good bet: let's again let $p = (1, 0, 0)$, and try to show that $\frac{\partial}{\partial x}$ isn't representable in the sum.

The M case is unchanged from the above—we're using the same point. Let γ be a path in N_3 through p. Then, as always,

$$D_\gamma g = \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma_x(t), \gamma_y(t), \gamma_z(t))\bigg|_{t=0} = \frac{\partial g}{\partial x}\frac{\mathrm{d}\gamma_x}{\mathrm{d}t} + \frac{\partial g}{\partial y}\frac{\mathrm{d}\gamma_y}{\mathrm{d}t} + \frac{\partial g}{\partial z}\frac{\mathrm{d}\gamma_z}{\mathrm{d}t}\bigg|_{t=0}.$$

For paths of N_3, $\gamma_x(t) = 1 \Rightarrow \frac{\mathrm{d}\gamma_x}{\mathrm{d}t} = 0$. This means that T_pN_3 also coincides with T_pM, and, consequently, that M is not transverse to N_3 □

Problem 3. *Let $f : M \to N$ be a smooth map.*

1. *Show that f induces a smooth map $T(f) : T(M) \to T(N)$ on tangent bundles.*
2. *Show that f is an immersion iff $T(f)$ is injective on each fiber of $\pi : T(M) \to M$.*

Proof. Let $T(f) : (x, \xi) \mapsto (f(x), df(\xi))$. Consider a chart ϕ for a point $p \in M$, and a chart ψ for its image $f(p) \in N$. These induce charts on the bundles $\phi \times d\phi$ and $\psi \times d\psi$. Smoothness of f tells us that $\psi \circ f \circ \phi^{-1}$ is smooth where it's defined, and we want the same for

$$[\psi \times d\psi] \circ T(f) \circ [\phi \times d\phi]^{-1} = (\psi \circ f \circ \phi^{-1}, d\psi \circ df \circ (d\phi)^{-1})$$

Note that the differential of a chart is immediately an isomorphism of vector spaces T_pM onto $T_p\mathbb{R}^m$, by the definition of the local coordinates, so the inverse appearing here really does make sense. As a map $\mathbb{R}^{2m} \to \mathbb{R}^{2n}$, this is presumed smooth in its first m components, and linear and therefore smooth (cf. last homework) also in its last m. Certainly in Euclidean space, the Cartesian product of smooth functions is smooth, so $T(f)$ so-defined is smooth.

If f is an immersion, then its differential is a monomorphism of vector spaces at all points, i.e. for all p, the map $df : T_pM \to T_pN$ is injective. The restriction of $T(f)$ to a fiber of the bundle projection is merely the statement that one is restricting the x in (x, ξ) in the domain to be a single point; calling that p, injectivity of $T(f)$ on that fiber is, by definition of $T(f)$, exactly injectivity of df.

Conversely, if $T(f)$ is injective on a fiber of the bundle projection is exactly the statement that df is injective, as whenever the first m coordinates are fixed, injectivity of the map is injectivity of the map comprising those that remain.

$\square$