4750 HW 2

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Problem 1. *Find an expression for the amplitude of gravitational waves emitted by a binary system for an observer in the plane of the orbit (but far away from the system). Is that smaller or larger than the amplitude of gravitational waves emitted along an axis perpendicular to the orbit?*

Solution. Far away from the source, one may choose the transverse-traceless gauge, in which case the Einstein field equations reduce to a simple computation for the first-order perturbation to the Minkowski metric due to some mass:

$$\bar{h}_{ij}^{TT} = \frac{2G}{rc^4}\ddot{Q}_{ij}^{TT}(t - r/c),$$

where Q_{ij}^{TT} is the quadrupole moment of the mass distribution

$$Q_{ij}^{TT}(t - r/c) = \int_{\mathbb{R}^3} \rho(\mathbf{x}, t - r/c)\left(x_i x_j - \frac{r^2}{3}\delta_{ij}\right)d\mathbf{x},$$

where r denotes the (constant, for the sake of the integral) distance from the source to the point where the waves are measured. I have switched the expression to covariant indices, because contravariant ones are needlessly confused with exponents when the distinction is moot. If the mass is two identical point masses orbiting with angular frequency ω a distance r_0 from their mutual center, which we place at the origin,

$$\rho(\mathbf{x}, t) = \delta(z)[m\delta(x - r_0\cos(\omega t))\delta(y - r_0\sin(\omega t)) + m\delta(x + r_0\cos(\omega t))\delta(x + r_0\sin(\omega t))].$$

Denoting the retarded time by t_r (also independent of the integral variables) and substituting into the integral,

$$Q_{ij}^{TT}(t_r) = m\Bigg[\int_{\mathbb{R}^3} \delta(z)\delta(x - r_0\cos(\omega t_r))\delta(y - r_0\sin(\omega t_r))\left[x_i x_j - \frac{r^2}{3}\delta_{ij}\right]dxdydz$$
$$+ \int_{\mathbb{R}^3} \delta(z)\delta(x + r_0\cos(\omega t_r))\delta(y + r_0\sin(\omega t_r))\left[x_i x_j - \frac{r^2}{3}\delta_{ij}\right]dxdydz\Bigg]$$

One can't go much further without computing specific components, but thankfully it's pretty routine via *a priori* properties of Q_{ij}:

$$Q_{xx} = m\left[\int_{\mathbb{R}} x^2\delta(x - r_0\cos(\omega t_r))dx\int_{\mathbb{R}}\delta(y - r_0\sin(\omega t_r))dy - \frac{r^2}{3}\int_{\mathbb{R}^2}\delta(x - r_0\cos\omega t_r)\delta(y - r_0\sin\omega t_r)dxdy\right]$$

$$+m\left[\int_{\mathbb{R}} x^2\delta(x+r_0\cos(\omega t_r))dx\int_{\mathbb{R}}\delta(y+r_0\cos(\omega t_r))dy-\frac{r^2}{3}\int_{\mathbb{R}^2}\delta(x+r_0\cos\omega t_r)\delta(y+r_0\cos\omega t_r)dxdy\right]$$

$$=m\left[r_0^2\cos^2(\omega t_r)-\frac{r^2}{3}+r_0^2\cos^2(\omega t_r)-\frac{r^2}{3}\right]=2m\left[r_0^2\cos^2(\omega t_r)-\frac{r^2}{3}\right]$$

Q_{yy} will proceed identically, except the y^2 will pair with the delta distribution containing y, leaving

$$Q_{yy}=2m\left[r_0^2\sin^2(\omega t_r)-\frac{r^2}{3}\right]$$

Last for the diagonal entries, $Q_{zz}=-\frac{2mr^2}{3}$, since the x_ix_j term pairs with $\delta(z)$, and the integral evaluates to 0^2.

Things are more interesting for the off-diagonal terms; symmetry of the tensor means we need not calculate reflected components, but

□

Problem 2. *Consider a binary black hole system, with black hole mass* $30\,\mathrm{M}_\odot$ *when their centers are 2.5 Schwarzschild radii away (that is, about to merge). If the system is 400 Mpc away, what is the amplitude of the wave on Earth? How close would this system would have to be if the strain on Earth is 1 ppm?*

Solution. The amplitude from a binary black hole system is given by

$$h\approx\frac{r_s^2}{Rr}$$

where r_s is the Schwarzschild radius, R is the radius of orbit, and r is the distance between observer and merger event. The Schwarzschild radius of the black holes is

$$r_s=\frac{2GM}{c^2}=\frac{2(6.67\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})(30\cdot2\times10^{30}\,\mathrm{kg})}{(3\times10^8\,\mathrm{m/s})^2}=89\,\mathrm{km}$$

Accordingly,

$$h\approx\frac{(89\,\mathrm{km})^2}{(2.5\cdot89\,\mathrm{km})(400\,\mathrm{Mpc}\cdot3.1\times10^{22}\,\mathrm{m/Mpc})}=2.9\times10^{-21}$$

We can rearrange the approximate formula for h for r:

$$r=\frac{r_s^2}{Rh}$$

If the strain is $1\,\mathrm{ppm}$, or, equivalently, if $h=1\times10^{-6}$,

$$r=\frac{r_s^2}{Rh}=\frac{(89\,\mathrm{km})^2}{(2.5\cdot89\,\mathrm{km})(1\times10^{-6})}=3.56\times10^{10}\,\mathrm{m}=0.24\,\mathrm{AU}$$

□

Problem 3. *The Crab pulsar has a rotation frequency of* $30\,\mathrm{Hz}$ *and is approximately 6,400 light years away. The period is observed to decrease about* $40\,\mathrm{ns}$ *per day, probably due to energy carried away by pulsar wind. If we assume all the energy is instead lost to gravitational waves, this provides allows a "spin-down" upper* *limit on the ellipticity ϵ of the neutron star at about a part in a thousand. Assume the radius of the Crab pulsar (a neutron star) is $10\,\mathrm{km}$ and its mass is $1.4\,\mathrm{M}_\odot$. Using the formula for power emitted by such systems $P = \frac{G}{5c^5}\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle$, derive the upper limit to ϵ. What would be the amplitude of gravitational waves emitted by the Crab pulsar, assuming $\epsilon = 10^{-3}$? Compare this estimate with results in Table 3 of Phys. Rev. D 105, 022002 (2022).*

Solution. In the transverse-traceless gauge,

$$\ddot{Q} = \ddot{I} = 2\epsilon I_{zz}\omega \begin{pmatrix} -\cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so

$$\dddot{Q} = 4\epsilon I_{zz}\omega^2 \begin{pmatrix} \sin 2\omega t & \cos 2\omega t & 0 \\ \cos 2\omega t & -\sin 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The expectation value is that of the sum of the squares of each entry:

$$\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\rangle = 16\epsilon^2 I_{zz}^2\omega^4\langle \sin^2 2\omega t + \cos^2 2\omega t + \cos^2 2\omega t + \sin^2 2\omega t\rangle = 16\epsilon^2 I_{zz}^2\omega^4\langle 2\rangle = 32\epsilon^2 I_{zz}^2\omega^4$$

We then get the expression for the gravitational wave radiated power as

$$P = \frac{32G}{5c^5}\epsilon^2 I_{zz}^2\omega^4$$

The rotational kinetic energy of the star is

$$E = \frac{1}{2}\boldsymbol{\omega}\cdot I\boldsymbol{\omega} = \frac{1}{2}I_{zz}\omega^2$$

where $\boldsymbol{\omega} = \omega\hat{z}$ is the angular velocity. The corresponding radiated power is

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\left(\dot{I}_{zz}\omega^2 + 2I_{zz}\omega\dot{\omega}\right)$$

Equating the two expressions for power and solving for ϵ,

$$\epsilon^2 = \frac{5c^5}{64GI_{zz}^2\omega^4}\left(\dot{I}_{zz}\omega^2 + 2I_{zz}\omega\dot{\omega}\right)$$

$$\Leftrightarrow \epsilon = \frac{c^2}{8I_{zz}\omega^2}\sqrt{\frac{5c}{G}\left(\dot{I}_{zz}\omega^2 + 2I_{zz}\omega\dot{\omega}\right)}$$

The moment of inertia changes negligibly, so

$$\epsilon = \frac{c^2}{8I_{zz}\omega^2}\sqrt{\frac{10c}{G}I_{zz}\omega\dot{\omega}}$$

The derivative of the angular velocity in terms of the derivative of period is computed by

$$\omega = \frac{2\pi}{T} \Rightarrow \dot{\omega} = -\frac{2\pi\dot{T}}{T^2} = -2\pi f^2\frac{\Delta T}{\Delta t} = -2\pi(30\,\mathrm{Hz})^2\frac{-40\times 10^{-9}\,\mathrm{s}}{24\cdot 60\cdot 60\,\mathrm{s}} = 2.6\times 10^{-9}\,\mathrm{Hz}^2$$

The moment of inertia of a sphere about any axis through its center is $\frac{2}{5}MR^2$, and the star is very closely spherical, so an upper bound on the eccentricity is

$$\epsilon = \frac{5c^2}{8MR\omega^2}\sqrt{\frac{Mc}{G}\omega\dot{\omega}}$$

$$= \frac{5(3\times 10^8\,\mathrm{m/s})^2}{8(1.4\cdot 2\times 10^{30}\,\mathrm{kg})(10\,\mathrm{km})(2\pi\cdot 30\,\mathrm{Hz})^2}\sqrt{\frac{(1.4\cdot 2\times 10^{30}\,\mathrm{kg})(3\times 10^8\,\mathrm{m/s})}{6.67\times 10^{-11}\,\mathrm{N\cdot m^2/kg^2}}(2\pi\cdot 30\,\mathrm{Hz})(2.6\times 10^{-9}\,\mathrm{Hz}^2)}$$
$$= 0.14$$

The amplitude of gravitational waves from a rotating ellipsoid is

$$h = \frac{4G\epsilon I_{zz}\omega^2}{c^4 r}.$$

If $\epsilon = 10^{-3}$, the amplitude is, using the spherical approximation to compute I_{zz},

$$h = \frac{4(6.67\times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(10^{-3})\frac{2}{5}(1.4\cdot 2\times 10^{30}\,\mathrm{kg})(10\,\mathrm{km})^2(2\pi\cdot 30\,\mathrm{Hz})^2}{(3\times 10^8\,\mathrm{m/s})^4(400\,\mathrm{Mpc})(3.1\times 10^{22}\,\mathrm{m/Mpc})} = 1.1\times 10^{-29}$$

The cited article lists likely detectable strains in a range from 5×10^{-26} to 11×10^{-26}, so this is well below the detectable level, by a factor of ten thousand. □

Problem 4. *Estimate the wavelength and order of magnitude of gravitational waves on Earth emitted by:*

- *J2322+0509, a white dwarf system with a* $1200\,\mathrm{s}$ *orbital period,* $0.76\,\mathrm{kpc}$ *from the Sun, with masses* $0.27\,\mathrm{M}_\odot$ *and* $0.24\,\mathrm{M}_\odot$ *(Warren R. Brown et al 2020 ApJL 892 L35).*
- *the Hulse-Taylor binary pulsar, PSR B1913+16, with an orbital period of 7.75 hours,* $6.6\,\mathrm{kpc}$ *away, and* $1.4\,\mathrm{M}_\odot$ *masses for each neutron star. The orbit is eccentric, but use a circular orbit with 2 million km radius.*
- *an ice skater, rotating about a vertical axis (assume the observer is a wavelength away).*
- *Estimate the energy emitted in a single revolution and compare it with $\hbar\omega$, where ω is the gravitational wave frequency—what can you conclude from this?*

Solution. For the first,

$$\lambda = \frac{c}{f} = cT = (3\times 10^8\,\mathrm{m/s})(1200\,\mathrm{s}) = 3.6\times 10^{11}\,\mathrm{m}$$

The radius of the orbit can be deduced from Kepler's third:

$$R \approx \sqrt[3]{\frac{G(M_1+M_2)T^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67\times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})[0.27\cdot 2\times 10^{30}\,\mathrm{kg} + 0.24\cdot 2\times 10^{30}\,\mathrm{kg}](1200\,\mathrm{s})^2}{4\pi^2}}$$
$$= 1.35\times 10^8\,\mathrm{m}$$

$$h \propto \frac{1}{r}\frac{GMR^2\omega^2}{c^4} = \frac{1}{(6.6\,\mathrm{kpc})(3.1\times 10^{19}\,\mathrm{m/kpc})}\frac{(6.67\times 10^{-11}\,\mathrm{N\cdot m^2/kg^2})(0.25\cdot 2\times 10^{30}\,\mathrm{kg})(1.35\times 10^8\,\mathrm{m})^2(2\pi/(1200\,\mathrm{s}))}{(3\times 10^8\,\mathrm{m/s})^4}$$

Proceeding similarly for the next,

$$\lambda = (3 \times 10^8\,\mathrm{m/s})(60\,\mathrm{s/hr} \cdot 7.75\,\mathrm{hr}) = 1.4 \times 10^{11}\,\mathrm{m}$$

$$h \propto \frac{1}{r}\frac{GMR^2\omega^2}{c^4} = \frac{1}{(6.6\,\mathrm{kpc})(3.1 \times 10^{19}\,\mathrm{m/kpc})}$$

$$\cdot \frac{(6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2})(1.4 \cdot 2 \times 10^{30}\,\mathrm{kg})(2 \times 10^9\,\mathrm{m})^2(2\pi/(60\,\mathrm{s/hr} \cdot 7.55\,\mathrm{hr}))^2}{(3 \times 10^8\,\mathrm{m/s})^4}$$

$$= 8.7 \times 10^{-20}$$

$\square$