4271 HW 7

Duncan Wilkie

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Problem 1. *Suppose you have a KamLAND-like experiment where a detector 200 m from a reactor detects $\bar{\nu}_e$ at 90% of the flux expected with no oscillations. Assuming a 2-component model with maximal mixing ($\theta = 45°$) and an average neutrino energy of 3 MeV, what is the squared mass difference of the $\bar{\nu}_e$ and its oscillating partner?*

Solution. The two-component oscillation formula states

$$P(\bar{\nu}_e \to \bar{\nu}') = \sin^2 2\theta \sin^2\left(1.27 \cdot \frac{\Delta m_{21}^2 L}{E}\right) \Leftrightarrow \Delta m_{21}^2 = \frac{E}{1.27L}\sin^{-1}\left(\frac{\sqrt{P}}{\sin 2\theta}\right)$$

$$= \frac{0.003\,\mathrm{GeV}}{1.27 \cdot 0.2\,\mathrm{km}} \cdot \sin^{-1}\left(\frac{\sqrt{0.1}}{\sin(2\cdot 45°)}\right) = 0.218\,\mathrm{eV}^2$$

□

Problem 2. *A neutron star is nearly pure neutron matter with a density near that of nuclear matter (0.17 nucleons/fm^3).*

1. *Calculate the radius of a $1.0\,\mathrm{M}_\odot$ neutron star and compare to that of the sun.*
2. *A neutron star is formed from a supernova explosion when the core of the star (mostly iron and nickel) collapses. Assuming the neutron star is made by converting all of the protons in nickel into neutrons, calculate the number of neutrinos produced by the supernova explosion leaving behind a one-solar-mass neutron star.*
3. *The Super-K solar neutrino detector contains 500 000 t pure water. Using your answer from the previous part, calculate the maximum distance at which Super-K can observe supernovae. Assume the cross-section for neutrino interactions is 10^{-44} cm^2.*

Solution.

1. A solar mass is
$$M_\odot = \frac{2\times 10^{30}\,\mathrm{kg}}{1.6\times 10^{-27}\,\mathrm{kg}} = 1.25\times 10^{57}\,\text{neutron masses.}$$
Accordingly, presuming the given density of a neutron star,
$$\frac{M_\odot}{\rho} = V = \frac{4}{3}\pi R^3 \Rightarrow R = \sqrt[3]{\frac{3M_\odot}{4\pi\rho}} = \sqrt[3]{\frac{3\cdot 1.25\times 10^{57}\,\text{neutrons}}{4\pi\cdot 0.17\,\text{neutrons/fm}^3}} = 1.21\times 10^{19}\,\mathrm{fm} = 12.1\,\mathrm{km}$$

2. The number of neutrons necessary to make a solar mass was computed above. Using this, there are

$$\frac{1.25 \times 10^{57}\,\text{neutrons}}{56\,\text{nucleons/nickel nucleus}} = 2.23 \times 10^{55}\,\text{nickel nuclei}$$

that need to be converted for an all-nickel star to become an all-neutron star. For each of these nuclei, there are 28 protons that must be converted to neutrons, and for each conversion, a neutrino, so there are 6.25×10^{56} neutrinos produced over the entire conversion process.

3. Presuming isotropic emissions of these neutrinos, they will be evenly spread over a spherical surface of radius equal to the distance to the supernova. This allows us to relate the luminosity at the detector over the entire supernova event to the distance, via

$$L = \Phi N_b \Rightarrow L_{int} = F N_b = \frac{6.25 \times 10^{56}\,\text{neutrinos}}{4\pi R^2} N_b$$

where F is the neutrino fluence at Earth and N_b is the number of target water particles. In order to detect a supernova, one needs at least one neutrino interaction (presuming, unrealistically, that there is no background so as to get an upper bound). Accordingly, by $N = \sigma L_{int}$, we can obtain an upper bound on the detectable distance of supernovae as

$$R = \sqrt{\frac{\sigma \cdot 6.25 \times 10^{56}\,\text{neutrinos}}{4\pi} N_b} = \sqrt{\frac{(10^{-44}\,\text{cm}^2) \cdot 6.25 \times 10^{56}\,\text{neutrinos}}{4\pi}}$$

$$\cdot \sqrt{50\,000\,\text{t} \cdot 5.46 \times 10^{29}\,\text{amu/t} \cdot 18\,\text{water molecules/amu} \cdot 10\,\text{electrons/water molecule}}$$

$$= 1.56 \times 10^{24}\,\text{cm} = 0.51\,\text{Mpc}$$

□

Problem 3. *Calculate the energy of the Coulomb barrier for:*

1. d + t,

2. p + ^{12}C,

3. ^{4}He + ^{12}C,

4. ^{28}Si + ^{28}Si.

Solution.

1. The strong force kicks in at a distance of $R_d + R_t \approx 1.2\,\text{fm} \cdot 2^{1/3} + 1.2\,\text{fm} \cdot 3^{1/3} = 3.24\,\text{fm}$. Accordingly, the Coulomb barrier energy is

$$E = \frac{ke^2}{R_d + R_t} = \frac{9 \times 10^9\,\text{N} \cdot \text{m}^2/\text{C}^2 \cdot (1.6 \times 10^{-19}\,\text{C})^2}{3.24 \times 10^{-15}\,\text{m}} = 7.11 \times 10^{-14}\,\text{J} = 444\,\text{keV}$$

2. As above, $R_p + R_{^{12}C} \approx 1.2\,\text{fm} \cdot 1^{1/3} + 1.2\,\text{fm} \cdot 12^{1/3} = 3.95\,\text{fm}$;

$$E = \frac{6ke^2}{R_p + R_{^{12}C}} = \frac{9 \times 10^9\,\text{N} \cdot \text{m}^2/\text{C}^2 \cdot 6(1.6 \times 10^{-19}\,\text{C})^2}{3.95 \times 10^{-15}\,\text{m}} = 3.5 \times 10^{-13}\,\text{J} = 2.19\,\text{MeV}$$

3. $R_{^4He} + R_{^{12}C} \approx 1.2\,\text{fm} \cdot 4^{1/3} + 1.2\,\text{fm} \cdot 12^{1/3} = 4.65\,\text{fm}$

$$E = \frac{12ke^2}{R_{^4He} + R_{^{12}C}} = \frac{12 \cdot 9 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2 \cdot (1.6 \times 10^{-19}\,\text{C})^2}{4.65 \times 10^{-15}\,\text{m}} = 5.95 \times 10^{-13}\,\text{J} = 3.72\,\text{MeV}$$

4. $R_{^{28}Si} + R_{^{28}Si} \approx 2 \cdot 1.2\,\text{fm} \cdot 28^{1/3} = 7.29\,\text{fm}$

$$E = \frac{196ke^2}{R_{^{28}Si} + R_{^{28}Si}} = \frac{196 \cdot 9 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2 \cdot (1.6 \times 10^{-19}\,\text{C})^2}{7.29 \times 10^{-15}\,\text{m}} = 6.19 \times 10^{-12}\,\text{J} = 38.7\,\text{MeV}$$

□

Problem 4. *Four hypothetical narrow s-wave resonances occur at low energies in the* $^{20}\text{Ne}(p,\gamma)^{21}\text{Na}$ *reaction at* $E_r = 10, 40, 60,$ *and 120 keV. The resonance strengths are 7.24* $\times$ *10*$^{-33}$ *eV, 3.81* $\times$ *10*$^{-15}$ *eV, 1.08* $\times$ *10*$^{-9}$ *eV, and 3.27* $\times$ *10*$^{-4}$ *eV, respectively.* ***Without*** *calculating reaction rates, which resonances do you expect to dominate the total reaction rates at* $T = 0.03\,\text{GK}$*? At* $T = 0.09\,\text{GK}$*?*

Solution. We can compute the Gamow peak for these two temperatures as

$$E_0 = 0.122\left[10^2 \cdot 11^2 \frac{20 \cdot 21}{20+21} \cdot 0.03^2\right]^{1/3} = 539\,\text{keV};$$

$$E_0 = 0.122\left[10^2 \cdot 11^2 \frac{20 \cdot 21}{20+21} \cdot 0.09^2\right]^{1/3} = 1.12\,\text{MeV}.$$

This is way off the end of the scale for the resonances, but this would lead one to conclude that the 120 keV resonance will dominate in both cases, with the lower-energy resonances being rarer. □

Problem 5. *For the hypothetical resonances given in Problem 4, calculate the reaction rates numerically for* $T = 0.03\,\text{GK}$ *and* $T = 0.09\,\text{GK}$*. Is the Gamow peak concept valid in this case?*

Solution. Scheme:

```
(define mu (/ (* 21 20) (+ 21 20)))
(define fac1 1.5399e11)
(define fac2 11.605)
(define resonances (map (lambda (x)
                          (list (/ (car x) 1e6)
                                (/ (cadr x) 1e6)))
                        '((10 7.24e-33) (40 3.81e-15) (60 1.08e-9) (120 3.27e-4))))


(define (reaction-rate T9)
  (* (/ fac1
        (expt (* mu T9)
              (/ 3 2)))
     (apply + (map (lambda (x)
                     (* (cadr x)
                        (exp (* -1
```

```
                              fac2
                              (car x)
                              (/ T9)))))
               resonances))))

(reaction-rate 0.03)
;; => 282.1641000008018
(reaction-rate 0.09)
;; => 56.009257991628544
```

Since the Gamow peak applies to non-resonant reactions, one wouldn't expect it to apply to a resonant reaction; indeed, this indicates a *larger* reaction rate for the *lower* temperature—opposite the prediction above. □