4141 HW 9

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1

Presuming an induction hypothesis $[A^{n-1}, C] = (n-1)A^{n-2}C$,

$[A^n, B] = A[A^{n-1}, B] + [A, B]A^{n-1} = A(n-1)A^{n-2}C + CA^{n-1} = (n-1)A^{n-1}C + A^{n-1}C = nA^{n-1}C$

$$\frac{d}{d\lambda}e^{\lambda(A+B)} = \frac{d}{d\lambda}\left(1 + \lambda(A+B) + \frac{\lambda^2}{2}(A+B)^2 + ...\right) = (A+B) + \lambda(A+B)^2 + \frac{\lambda^2}{2}(A+B)^3 + ...$$
$$= (A+B)e^{\lambda(A+B)}$$

2a

The system is in state ψ_1 immediately after the measurement due to wave function collapse.

2b

Prior to the measurement, the wave function in ϕ space may be written

$$\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$$

The two possible results are then b_1 and b_2, with probabilities $|3/5|^2 = \frac{9}{25}$ and $|4/5|^2 = \frac{16}{25}$ respectively.

2c

We can solve for ϕ_1 and ϕ_2 in the eigenstate relation:

$$\begin{cases} 5\psi_1 = 3\phi_1 + 4\phi_2 \\ 5\psi_2 = 4\phi_1 - 3\phi_2 \end{cases} \Leftrightarrow \begin{cases} 20\psi_1 = 12\phi_1 + 16\phi_2 \\ 15\psi_2 = 12\phi_1 - 9\phi_2 \end{cases} \Rightarrow 5(4\psi_1 - 3\psi_2) = 25\phi_2 \Leftrightarrow \phi_2 = (4\psi_1 - 3\psi_2)/5$$

$$\Rightarrow 3\phi_1 = 5\psi_1 - \frac{4}{5}(4\psi_1 - 3\psi_2) \Leftrightarrow \phi_1 = (3\psi_1 + 4\psi_2)/5$$

There are two possibilities after the measurement of B: the particle is in state ϕ_1, or it is in state ϕ_2. If it's in state ϕ_1, its probability of being in state ψ_1 on a subsequent measurement is $|3/5|^2 = 9/25$; if it's in state ϕ_2, this probability is $|4/5|^2 = 16/25$. Weighting these values by the probabilities of being in ϕ_1 and ϕ_2 and adding (by the law of total probability),

$$P(a_1) = \frac{9}{25}\frac{9}{25} + \frac{16}{25}\frac{16}{25} = \frac{337}{625} = 0.5392$$

3

By definition,

$$\langle x \rangle = \int_{\mathbb{R}} \left(\sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} \right)^* x \left(\sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} \right) dx$$

$$= \int_{\mathbb{R}} x \sum_m \sum_n c_n^* c_m \psi_n(x) \psi_m(x) e^{it(E_n - E_m)/\hbar} dx = \sum_m \sum_n c_n^* c_m e^{it(E_n - E_m)/\hbar} \int_{\mathbb{R}} x \psi_n(x) \psi_m(x) dx$$

4a

The eigenvalues may be found via $\det(M - \lambda I) = 0$. We can ignore the leading coefficients for now and multiply the resulting eigenvalue by them due to homogeneity of the determinant (i.e. $\det(aM - \lambda I) = 0 \Leftrightarrow a \det(M - \frac{\lambda}{a} I) = 0$). For H,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2 = 0$$

clearly has solutions 1 and 2 (with multiplicity 2 for the latter), corresponding to eigenvalues $\hbar\omega$ and $2\hbar\omega$. For A,

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = \lambda^2(2-\lambda) + \lambda - 2 = (2-\lambda)(\lambda^2 - 1) = (2-\lambda)(\lambda-1)(\lambda+1)$$

so the eigenvalues are 2λ, λ, and $-\lambda$. For B,

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (2-\lambda)\lambda^2 - (2-\lambda) = (2-\lambda)(\lambda^2-1) = (2-\lambda)(\lambda+1)(\lambda-1)$$

which is the same equation as for A, yielding eigenvalues 2μ, μ, and $-\mu$.

4b

We have

$$\langle H \rangle = \sum_n E_n |c_n|^2 = \hbar\omega \left(|c_1|^2 + 2|c_2|^2 + 3|c_3|^2 \right) = \hbar\omega(2 - |c_1|^2),$$

$$\langle A \rangle = \langle S(0)|A|S(0)\rangle = \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$