4261 HW 1

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1a

The partition function given corresponds to

$$Z(\beta) = \frac{1}{(2\pi h)^3} \int \exp\left(-\beta \frac{\vec{p}^2}{2m}\right) d\vec{p} \int \exp\left(-\beta k \frac{\vec{x}^2}{2}\right) d\vec{x}$$

The integral in one dimension

$$I = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

may be calculated by first squaring the integral:

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-a(x^2+y^2)} dx dy$$

Transforming to polar coordinates, the Jacobian determinant of which you'll recall is $rdrd\theta$, the above becomes

$$I^2 = \int_0^{2\pi} \int_0^{\infty} re^{-ar^2} drd\theta$$

This can be solved trivially by a u-substitution $u = r^2$, $du = 2rdr$:

$$I^2 = 2\pi \left(\frac{1}{2} \int_0^{\infty} e^{-au} du\right) = \pi \left(-\frac{e^{-au}}{a}\bigg|_0^{\infty}\right) = \frac{\pi}{a}$$

Therefore, the integral is equal to $\sqrt{\frac{\pi}{a}}$.

The integral terms of the partition function are then a product of three such integrals with $a = \frac{\beta}{2m}$ for the three scalar variables of momentum and a product of three such integrals with $a = \frac{\beta k}{2}$ for the three scalar variables of position, i.e.

$$Z(\beta) = \frac{1}{(2\pi h)^3} \left(\sqrt{\frac{\pi}{\beta/2m}}\right)^3 \left(\sqrt{\frac{\pi}{\beta k/2}}\right)^3 = \frac{m^{3/2}}{h^3 \beta^3 k^{3/2}}$$

1b

The expectation of the energy based on the partition function is

$$\langle E\rangle = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{h^3\beta^3k^{3/2}}{m^{3/2}}\frac{3m^{3/2}}{h^3\beta^4k^{3/2}} = \frac{3}{\beta} = 3k_BT$$

Differentiating to find the heat capacity,

$$C = \frac{\partial\langle E\rangle}{\partial T} = 3k_B$$

as desired.