4132 HW 9

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1

If the light from the object at point a originates at time t_0, it will arrive at Earth at time $t_a = t_0 + d_a/c$; the light from the object at point b, originating at time t_f will arrive at time $t_b = t_f + d_b/c$. The time interval is then

$$\Delta t = t_b - t_a = t_f - t_0 + \frac{d_b - d_a}{c}$$

The difference in distances to the Earth $d_b - d_a$ is just the negative length of the adjacent side to angle θ:

$$d_b - d_a = v(t_0 - t_f)\cos\theta$$

The apparent displacement is also easily computed trigonometrically:

$$\Delta s = v(t_f - t_0)\sin\theta$$

The apparent velocity is then

$$u = \frac{\Delta s}{\Delta t} = \frac{v(t_f - t_0)\sin\theta}{t_f - t_0 + \frac{v(t_0 - t_f)\cos\theta}{c}} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}$$

This is maximized if

$$\frac{du}{d\theta} = 0 \Leftrightarrow \frac{\left(1 - \frac{v}{c}\cos\theta\right) v\cos\theta - v\sin\theta\left(\frac{v}{c}\sin\theta\right)}{\left(1 - \frac{v}{c}\cos\theta\right)^2} = 0 \Leftrightarrow \left(1 - \frac{v}{c}\cos\theta\right) v\cos\theta = v\sin\theta\left(\frac{v}{c}\sin\theta\right)$$

$$\Leftrightarrow \cos\theta = \frac{v}{c}\sin^2\theta + \frac{v}{c}\cos^2\theta \Leftrightarrow \theta = \cos^{-1}\left(\frac{v}{c}\right)$$

Evaluating u at this maximum,

$$u = \frac{v\sin\left(\cos^{-1}\frac{v}{c}\right)}{1 - \frac{v^2}{c^2}}$$

By the definition of cosine, $\cos^{-1}(\frac{v}{c})$ is the angle of a triangle whose adjacent is v and hypotenuse is c. The opposite therefore has length $\sqrt{c^2 - v^2}$, and the sine is the opposite over the hypotenuse, so

$$u = \frac{\frac{v}{c}\sqrt{c^2 - v^2}}{1 - \frac{v^2}{c^2}} = \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This becomes arbitrarily large as $v \to c$ (even through $v < c$), since the denominator diverges.

2

From the length contraction formula, the problem is to find the free velocity such that

$$L'_{VW} = L_{VW}\sqrt{1 - \frac{v_{VW}^2}{c^2}}$$

is equal to

$$L'_{Lincoln} = L_{Lincoln}\sqrt{1 - \frac{v_{Lincoln}^2}{c^2}}$$

We are given $v_{VW} = .5c$; accordingly, $L'_{VW} = L_{VW}\sqrt{3}/2$. We additionally know $L_{Lincoln} = 2L_{VW}$, so we can set up

$$L_{VW}\sqrt{3}/2 = 2L_{VW}\sqrt{1 - \frac{v_{Lincoln}^2}{c^2}} \Leftrightarrow \frac{3}{16} = 1 - \frac{v_{Lincoln}^2}{c^2} \Leftrightarrow v_{Lincoln} = \frac{\sqrt{13}}{4}c \approx 0.9c$$

3

Simultaneity of A and B in another reference frame C at velocity v is the statement that the zeroth coordinate of

$$\tilde{A} = \Lambda^\mu_\nu A^{nu}$$

equals the zeroth coordinate of

$$\tilde{B} = \Lambda^\mu_\nu B^\nu$$

From the matrix multiplication, these coordinates are

$$\tilde{A}^0 = \gamma A^0 - \gamma\beta A^1 = \gamma c t_A - \gamma\beta x_A$$

and

$$\tilde{B}^0 = \gamma B^0 - \gamma\beta B^1 = \gamma c t_B - \gamma\beta x_B$$

Equating these,

$$t_A\gamma c - x_A\gamma\beta = \gamma c t_B - x_B\gamma\beta$$

$$\Leftrightarrow t_A - t_B = (x_A - x_B)\frac{\gamma\beta}{\gamma c} \Leftrightarrow v = c^2\frac{t_A - t_B}{x_A - x_B}$$

4

The proper velocity in a single dimension may be written in a form solvable for the ordinary velocity:

$$\eta = \frac{u}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1/u^2 - 1/c^2}} \Leftrightarrow \frac{1}{u^2} = \frac{1}{\eta^2} + \frac{1}{c^2} \Leftrightarrow u = \frac{1}{\sqrt{\frac{1}{\eta^2} + \frac{1}{c^2}}} = \frac{\eta}{\sqrt{1 + \frac{\eta^2}{c^2}}}$$

$$= \frac{4 \times 10^8 \text{ m/s}}{\sqrt{1 + \frac{(4c/3)^2}{c^2}}} = 2.4 \times 10^8 \text{ m/s}$$

This is under the ordinary velocity limit, so no law was violated.

5

The relativistic kinetic energy of such a particle is

$$E_{kin} = E - mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

which, by assumption, equals nmc^2. Solving for the ordinary velocity,

$$nmc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \Leftrightarrow n + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \Leftrightarrow \frac{1}{(n+1)^2} = 1 - \frac{v^2}{c^2}$$

$$\Leftrightarrow v^2 = c^2 \left(1 - \frac{1}{(n+1)^2}\right) \Leftrightarrow v = c\sqrt{1 - \frac{1}{(n+1)^2}}$$