4123 HW 4

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1a

The Lagrangian for this system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy$$

The modified Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} + \lambda\frac{\partial C}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = m\ddot{x}$$

and

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial C}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow mg + 2\lambda y = m\ddot{y}$$

Adding these together and solving for λ, we obtain

$$mg + 2\lambda(x+y) = m(\ddot{x} + \ddot{y}) \Leftrightarrow \lambda = \frac{m}{2}\frac{\ddot{x} + \ddot{y} - g}{x+y}$$

1b

Differentiating the constraint twice,

$$C = 0 \Rightarrow x\dot{x} + y\dot{y} = 0 \Rightarrow x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0$$

Solving for $\ddot{x}$ and $\ddot{y}$,

$$\ddot{x} = -\frac{y\ddot{y} + \dot{x}^2 + \dot{y}^2}{x}$$
$$\ddot{y} = -\frac{x\ddot{x} + \dot{x}^2 + \dot{y}^2}{y}$$

Adding,

$$\begin{aligned}\ddot{x} + \ddot{y} &= -\left(\frac{y^2\ddot{y} + y\dot{x}^2 + y\dot{y}^2 + x^2\ddot{x} + x\dot{x}^2 + x\dot{y}^2}{xy}\right) \\ &= \frac{-1}{xy}\left(x^2\ddot{x} + y^2\ddot{y} + (x+y)(\dot{x}^2 + \dot{y}^2)\right)\end{aligned}$$

Plugging this in to the equation for λ,

$$\lambda = -\frac{m}{2}\left(\frac{x^2\ddot{x} + y^2\ddot{y}}{x^2y + y^2x} + \frac{\dot{x}^2 + \dot{y}^2}{xy} + \frac{g}{x+y}\right)$$

Multiplying by $1 = \frac{l^2}{l^2}$ and noting that $l^2 = x^2 + y^2 = (x+y)^2 - 2xy$,

$$=$$

2a

Since there is no "real" potential, there is no force on m_2 and it must move with constant velocity.

2b

The resulting Lagrangian in the CM frame (which is approximately the m_2 frame for the same reason) is

$$L = \frac{1}{2}m_2\dot{r}^2 - \frac{l^2}{2m_2r^2}$$

The resulting Euler-Lagrange equation of motion is

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Leftrightarrow \frac{l^2}{m_2r^3} = m_2\ddot{r} \Leftrightarrow \frac{l^2}{m_2^2} = r^3\ddot{r}$$

There is a centrifugal force term present. This is a second-order autonomous equation with no first-derivative dependence, and so can be solved as

$$t(r) + c_1 = \pm\int\frac{dr}{\sqrt{2\int\frac{l^2}{m_2^2r^3}dr + c_2}} = \pm\frac{m_2}{l}\int\frac{dr}{\sqrt{c_2 - \frac{1}{r^2}}} = \pm\frac{m_2}{l}\frac{r\sqrt{c_2 - \frac{1}{r^2}}}{c_2}$$

$$\Rightarrow \sqrt{r^2c_2 - 1} = \pm\left(c_2t\frac{l}{m_2} + c_1c_2\right)$$

$$\Leftrightarrow r^2 = 1 + \left(t\frac{l}{m_2} + c\right)^2$$

This is a hyperbola, translated in the t-axis by c. This is not consistent with a constant-velocity trajectory, but since hyperbolas have oblique asymptotes it becomes approximately a constant-velocity trajectory. Therefore, the deviance we see is a breakdown of the approximation of the CM frame by the m_1 frame.

3

The ordinary central-force radial Lagrangian is

$$L = \frac{1}{2}\mu\dot{r}^2 - \frac{l^2}{2\mu r^2} - V(r)$$

The corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Leftrightarrow -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} = \mu\ddot{r} \Leftrightarrow F(r) + \frac{l^2}{\mu r^3} = \mu\ddot{r}$$

Taking r to be a function of θ alone, $\frac{d^2s}{d\theta^2} = \frac{\partial s}{\partial r}r'' + \frac{\partial^2 s}{\partial\theta\partial r}r' = \frac{-r''}{r^2}$ (since s is not directly dependent on θ). The target equation may be written in terms of r as

$$\frac{-r''}{r^2} + \frac{1}{r} = \frac{-\mu r^2}{l^2}F(r) \Leftrightarrow \frac{r''-r}{r^2}\frac{l^2}{\mu r^2} = F(r)$$

$$\Leftrightarrow \frac{l^2 r''}{\mu r^4} - \frac{l^2}{\mu r^3} = F(r)$$

Noting that by the chain rule $\ddot{r} = r'\ddot{\theta} + r''\dot{\theta}^2 = r''\dot{\theta}^2 = r''\left(\frac{l}{\mu r^2}\right)^2$ (constant angular momentum $\Rightarrow$ no angular acceleration), $\frac{l^2 r''}{\mu r^4} = \mu\ddot{r}$. Therefore, the target equation is

$$\mu\ddot{r} - \frac{l^2}{\mu r^3} = F(r)$$

which is clearly equivalent to the the Euler-Lagrange equation found above.

4

We have $s = \frac{1}{k\theta^2}$, so $\frac{d^2s}{d\theta^2} = \frac{6}{k\theta^4}$. The above equation is then

$$\frac{6}{k\theta^4} + \frac{1}{k\theta^2} = \frac{-\mu k^2\theta^4}{l^2}F(r)$$

$$\Leftrightarrow F(r) = \frac{-l^2}{\mu k^2\theta^4}\left(\frac{6}{k\theta^4} + \frac{1}{k\theta^2}\right)$$