Philosophically Consistent Gravitational-Wave Background Searches

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Paper

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Frequentist versus Bayesian analyses: Cross-correlation as an approximate sufficient statistic for LIGO-Virgo stochastic background searches

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Sufficient statistics are combinations of data in terms of which the likelihood function can be rewritten without loss of information. Depending on the data volume reduction, the use of sufficient statistics as a preliminary step in a Bayesian analysis can lead to significant increases in efficiency when sampling from posterior distributions of model parameters. Here we show that the frequency integrand of the cross-correlation statistic and its variance are approximate sufficient statistics for ground-based searches for stochastic gravitational-wave backgrounds. The sufficient statistics are approximate because one works in the weak-signal approximation and uses measured estimates of the autocorrelated power in each detector. We perform analytic and numerical calculations to show that, in this approximation, LIGO-Virgo's hybrid frequentist-Bayesian parameter estimation analysis is equivalent to a fully Bayesian analysis. This work closes a gap in the LIGO-Virgo literature and suggests directions for additional searches.

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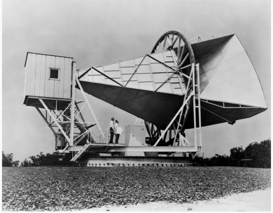
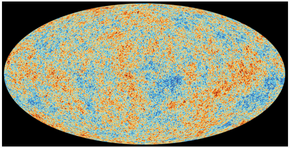
Stochastic Backgrounds?

- LIGO is optimized for transients.
- Stochastic $\Rightarrow$ "random stuff accumulating"
- Background $\Rightarrow$ "isotropic"



Idiopathic "noise" in this



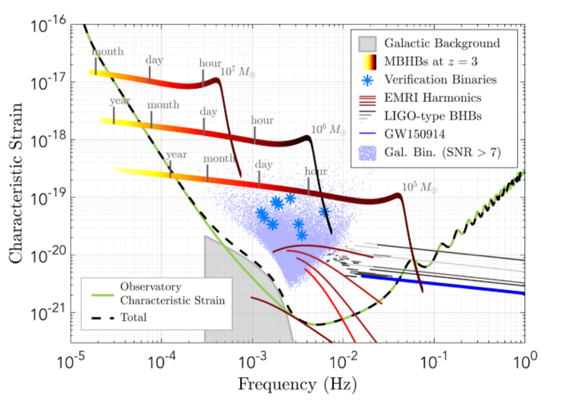
Was this

Stochastic Backgrounds?

What unresolved astrophysical events might accumulate to appear as detector noise?

- Inflation
- Singular points where cosmic strings/superstrings intersect
- Discontinuous phase transitions in the early universe
    - Strong-force disunity
    - Electroweak baryogenesis
    - Plasma bubble collisions and attendant shock waves
    - Magnetohydrodynamics
    - Most of the alternative electroweak models
- Pre-Bang stuff
- Lots of weak BBHs/BNSs
- Close compact binaries (e.g. two orbiting white dwarfs in our galaxy)
- Supernovae
- Pulsars/Magnetars
- Big Bang itself

Stochastic Backgrounds?



Mostly at low frequency, so we need LISA (Nelson Christensen 2019 Rep. Prog. Phys. 82 016903)

Stochastic Backgrounds?

- This has been looked for in LIGO! (maybe the effect is massive)
- Nothing found...
- Established a technique though!
- This paper justifies the current hybrid technique from a Bayesian perspective.

What's the Problem?

The current method:

1. Calculate a frequentist statistic (frequency-totaled detector X-correlation and its variance) over small time bins,
2. Weight per unit noise, and
3. Feed this into Bayesian analysis that checks for correlated noise attributable to background.

This is concerning *prima facie*; probabilities mean *completely* different things for the two schools! It's not bad as it looks, because you can ask each approach what it thinks of the other, but it's still ugly.

Frequentist methods tend to have poor algorithmics too...much of the recent boom in Bayesianism is likely attributable to MCMC!

What's the Problem?

Many think the Bayesian philosophy is just *right,* too.

- Frequentist statistics $\longleftrightarrow$ objective characteristics of repeated, well-defined random experiments, but QM means repeated experiments are *purely imaginary.*
- What about deterministic phenomena that are too hard to compute?
- Bayesian statistics $\longleftrightarrow$ scientific reasoning.
- Bayesian statistics $\longleftrightarrow$ inductive reasoning (+ transitive corollary).
- Minimizes nuisance variables (frequentists can't even see them).
- Frequentism is hard to get right (Bertrand's paradoxes; measure theory is an error unto itself; "The Null Ritual").
- Frequentism is just less powerful (cf. constructive vs classical logic);
- Very Platonic (clean line between objects and their shadows).
- Notable involvement from physicists! (Cox, Jeffreys, Jaynes).

Counterclaims: "What the heck is the Born rule then?" "Priors are hard to decide."

The Bayesian Process

The fundamental inferential principle is Bayes' theorem:

$$P(H \mid E) = \frac{P(E \mid H) \cdot P(H)}{P(E)},$$

$$\text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Evidence}},$$

or "the probability of a hypothesis given some new evidence is the probability of the evidence, given the hypothesis, times the probability of the hypothesis before the new evidence, divided by the probability of the evidence before it was observed."

This paper proves that, to an approximation, a current frequentist statistic losslessly compresses the information in the likelihood function, meaning the method is correct from a Bayesian perspective.

The Bayesian Process

The simplest sufficient statistic: the sample mean for a constant signal in standardized Gaussian noise of known variance, $d_i = a + n_i$. For N data points, the likelihood of a datum d is

$$p(d \mid a) = \frac{1}{(2\pi)^{N/2}\sigma^N} \exp\left[-\frac{1}{2\sigma^2}\sum_i (d_i - a)^2\right]$$

The average of the d_i, $\hat{a}$, is an unbiased, maximum-likelihood estimator; applying a little identity,

$$p(d \mid a) = \frac{1}{(2\pi)^{N/2}\sigma^N} \exp\left[-\frac{1}{2\sigma^2}\sum_i d_i^2\right] \exp\left[\frac{\hat{a}^2}{2\sigma_{\hat{a}}^2}\right] \exp\left[-\frac{(\hat{a}-a)^2}{2\sigma_{\hat{a}}^2}\right]$$

Sufficiency of the statistic $\hat{a} \longleftrightarrow$ likelihood $= f(\text{anything but } a)g(\hat{a})$.

The Result

- The paper does an analysis for a strong, white signal and a colored, but weak signal.
- The former result is just a weaker, illustrative result, so we can look at the latter.
- A result for arbitrarily large signals is proved in an appendix, but it's not manifestly different.

The Result

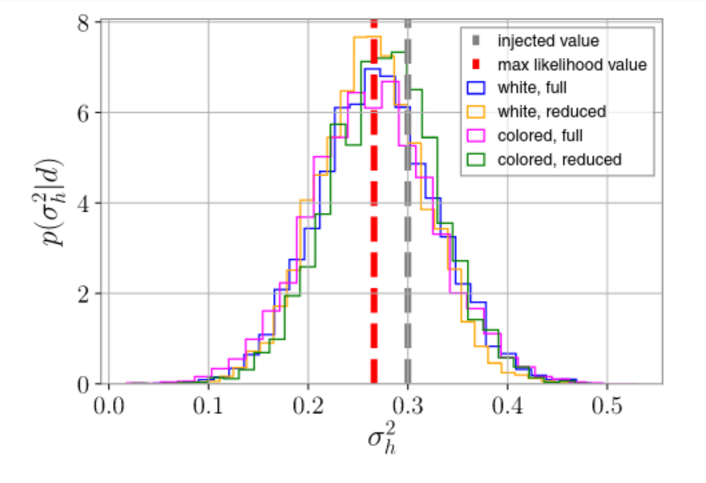


Fig 2

The Result

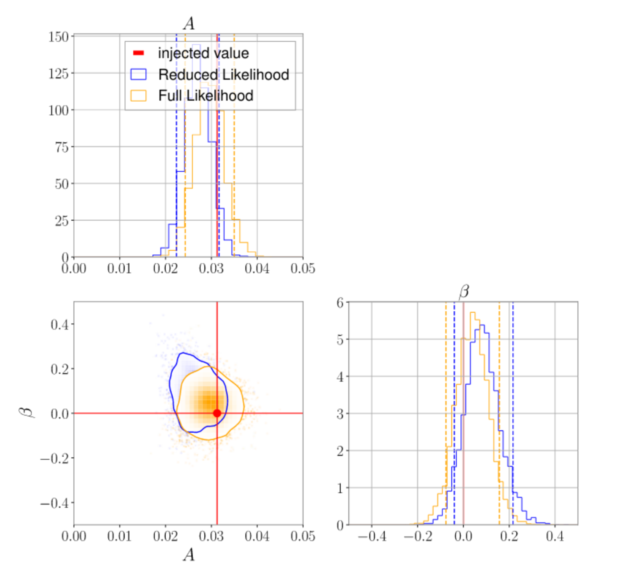


Fig 3

The Result

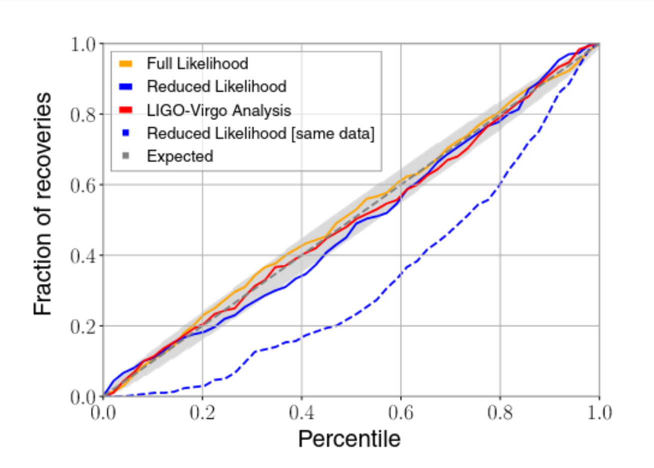
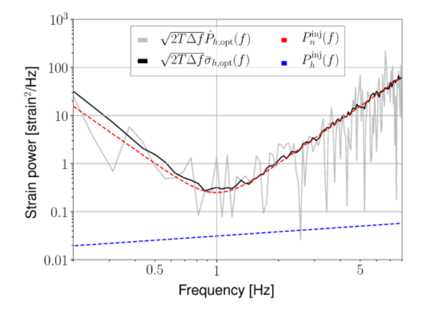
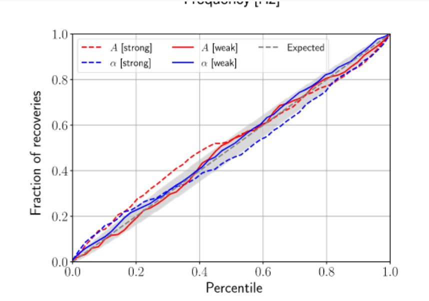


Fig 4

The Result


10^3
100
10
1
0.1
0.01
Strain power [strain^2/Hz]
$\sqrt{2T\Delta f}\hat{P}_{h,\mathrm{opt}}(f)$
$\sqrt{2T\Delta f}\bar{\sigma}_{h,\mathrm{opt}}(f)$
$P_n^{\mathrm{inj}}(f)$
$P_h^{\mathrm{inj}}(f)$
0.5
1
5
Frequency [Hz]


A [strong]
α [strong]
A [weak]
α [weak]
Expected
Fraction of recoveries
0.0
0.2
0.4
0.6
0.8
1.0
Percentile

Critical Analysis

- Admirable exposition overall; the distance between computational steps was sensible, and the typesetting clear and consistent.
- One illustrative example is sufficient. If results really supersede each other, just go through the pain to derive the general one, then deduce the simplifications, and explain that the simplifications came first (Research-order, not textbook-order! No one reads derivations unless they have reason to).
- Multiple overlapping histograms are hard to read. Use cubic-spline interpolations, and if you're concerned about misinterpretation, indicate the histogram midpoints; it's easier on the reader.

Appendix: Captions

FIG. 2. Recovered posterior distributions for the amplitude of the GWB as obtained from the full and reduced versions of the Bayesian likelihood functions appropriate for both white and colored data. The simulated data consisted of a white signal and white detector noise. (As with Fig. 1, this plot is based on a single realization of the data and cannot be used to infer the presence or absence of a systematic bias. See Figs. 4 and 5 for further discussion.)

FIG. 3. Recovered posterior distributions obtained from the full and reduced likelihood functions for a colored signal + noise model. The simulated data consisted of a white GWB signal injected into power-law detector noise. The solid red lines show the injected values of the GWB amplitude and spectral index.

FIG. 4. White signal + noise analysis. The pp plot compares recoveries of the amplitude by the full likelihood, reduced likelihood, and LIGO-Virgo stochastic analyses, showing that they both have good Bayesian coverage. The latter two analyses are not identical because of different choices made in conditioning the data. The dotted line shows the bias obtained when using the reduced likelihood analysis if the same data segment as the analysis segment is used to estimate the autocorrelated power in the two detectors.

Appendix: Captions

FIG. 5. Colored signal + noise analysis. In the top panel, we show the injected noise power spectrum $P_n(f)$ (the same for both detectors) and the injected GWB power spectrum $P_h(f)$, along with the optimal estimator $\hat{P}_{h,\mathrm{opt}}(f)$ and its uncertainty $\bar{\sigma}_{h,\mathrm{opt}}(f)$ for one segment in one realization. We rescale the optimal estimator and its uncertainty by a factor of $\sqrt{2T\Delta f}$, where T is the observation time and Δf is the frequency resolution, so that they can be directly compared with the injected autopower and cross power. In the bottom panel, we show a pp plot generated by performing 300 strong-signal and 300 weak-signal injections and recoveries. We see that when the weak-signal approximation is satisfied, the LIGO-Virgo stochastic analysis has excellent Bayesian coverage. Outside of the weak-signal approximation, the coverage is less good, as expected.

Appendix: Equations

$$p_I(d_I|P_{n_1 I}, P_{n_2 I}, P_h) = \prod_\ell \frac{1}{(\pi T/2)^{2M}(P_{1I}(f_\ell)P_{2I}(f_\ell) - \gamma^2(f_\ell)P_h^2(f_\ell))^M}$$
$$\times \exp\left\{-\frac{M}{(1-\gamma^2(f_\ell)P_h^2(f_\ell)/(P_{1I}(f_\ell)P_{2I}(f_\ell)))}\left[\frac{\hat{P}_{1I}(f_\ell)}{P_{1I}(f_\ell)} + \frac{\hat{P}_{2I}(f_\ell)}{P_{2I}(f_\ell)} - 2\gamma^2(f_\ell)\frac{P_h(f_\ell)\hat{P}_{hI}(f_\ell)}{P_{1I}(f_\ell)P_{2I}(f_\ell)}\right]\right\},$$

Full, colored, moving-signal likelihood

$$p(d|\{\bar{P}_{1I}\}, \{\bar{P}_{2I}\}, P_h) = \prod_\ell \frac{e^{-2MN_{\text{seg}}}}{(\pi T/2)^{2MN_{\text{seg}}}\prod_I(\bar{P}_{1I}(f_\ell)\bar{P}_{2I}(f_\ell))^M}\exp\left[\frac{\hat{P}^2_{h,\text{opt}}(f_\ell)}{2\bar{\sigma}^2_{h,\text{opt}}(f_\ell)}\right]\exp\left[-\frac{(\hat{P}_{h,\text{opt}}(f_\ell) - P_h(f_\ell))^2}{2\bar{\sigma}^2_{h,\text{opt}}(f_\ell)}\right].$$

Reduced, colored, moving-signal likelihood

$$p(P_h(f_\ell)|d, \{\bar{P}_{1I}(f_\ell)\}, \{\bar{P}_{2I}(f_\ell)\}) \propto \exp\left[-\frac{(\hat{P}_{h,\text{opt}}(f_\ell) - P_h(f_\ell))^2}{2\bar{\sigma}^2_{h,\text{opt}}(f_\ell)}\right]p(P_h(f_\ell)).$$

Reduced likelihood is sufficient