# 3355 Exam 4

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## 1a

The PDF is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$= \begin{cases} \frac{e^{-x/4}}{4} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

## 1b

The expectation is

$$E(X) = \int_{\mathbb{R}} xf(x)dx = \int_{\mathbb{R}} x\frac{e^{-x/4}}{4}dx = -xe^{-x/4}\Big|_0^\infty + \int_{\mathbb{R}} e^{-x/4}dx = -4e^{-x/4}\Big|_0^\infty = 4$$

## 1c

The CDF for the random variable is

$$F(x) = \begin{cases} 1 - e^{-x/4} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The probability in question is

$$P(X > 2E) = 1 - F(2E) = 1 - F(8) = 1 - e^{-2} \approx 0.865$$

## 2

The PDF is the normal distribution with these parameters:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] = \frac{1}{100\sqrt{2\pi}} \exp\left[-\frac{(x-82)^2}{20,000}\right]$$

The probabilities, since for continuous distributions the left and right limits are equal, are given in terms of the normal CDF as

$$P(X \geq 90) = 1 - F(90)$$

$$P(80 \leq X < 90) = F(90) - F(80)$$

If we standardize these values, we obtain $Z$-scores $90 \mapsto \frac{90-82}{100} = .08$ and $80 \mapsto \frac{80-82}{100} = -.02$. Going to the standard normal table, we find $F(90) = 0.5319$ and $F(80) = 0.4920$. The desired probabilities become then

$$P(X \geq 90) = 1 - 0.5319 = 0.4681 = 46.81\%$$

and

$$P(80 \leq X < 90) = 0.5391 - 0.4920 = 0.0471 = 4.71\%$$

## 3a

By definition, the integral over all variables must be one, so

$$\int_0^1 \int_0^1 c(x+2y)dxdy = \int_0^1 \left(\frac{c}{2} + 2cy\right) dy = \frac{c}{2} + c = \frac{3c}{2} = 1 \Rightarrow c = \frac{2}{3}$$

## 3b

$$E(XY) = \int_0^1 \int_0^1 xy\frac{2}{3}\left(x+2y\right) dxdy = \frac{2}{3}\int_0^1 \left(\frac{y}{3} + y\right) dy = \frac{4}{9}$$

## 3c

$$f_X(x) = \frac{2}{3}\int_0^1 \left(x+2y\right) dy = \frac{2}{3}\left(x+1\right)$$

and

$$f_Y(y) = \frac{2}{3}\int_0^1 \left(x+2y\right) dx = \frac{2}{3}\left(\frac{1}{2} + 2y\right)$$

## 3d

We first calculate the expectations of each variable from the marginal densities above:

$$E(X) = \int_0^1 xf_X(x)dx = \frac{2}{3}\int_0^1 x\left(x+1\right) dx = \frac{2}{9} + \frac{1}{3} = \frac{5}{9}$$

$$E(Y) = \int_0^1 yf_Y(y)dy = \frac{2}{3}\int_0^1 y\left(\frac{1}{2} + 2y\right) dy = \frac{1}{6} + \frac{4}{9} = \frac{11}{18}$$

We now can compute

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \frac{5}{9}\frac{11}{18} = \frac{17}{162}$$

## 4a

We first need the marginal density for $y$:

$$f_Y(y) = \int_0^\infty f(x,y)dx = -3e^{-3x}\Big|_0^\infty = 3$$

The conditional PDF is by definition

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} 3e^{-3x} & x, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

## 4b

The expectation is that of the above (univariate) PDF:

$$E(X|Y=y) = \int_0^\infty 3xe^{-3x}dx = -xe^{-3x}\Big|_0^\infty + \int_0^\infty e^{-3x}dx = -\frac{e^{-3x}}{3}\Big|_0^\infty = \frac{1}{3}$$

This is ordinarily a function of $y$, but there was no such dependence in the joint PDF.