Physics 4271 HW 2

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10 February 2023

Problem 1. *Calculate the minimum energy required to be over the Coulomb barrier for:*

1. $\mathrm{p} + \mathrm{p}$,
2. $\mathrm{p} + {}^{12}\mathrm{C}$,
3. ${}^{4}\mathrm{He} + {}^{208}\mathrm{Pb}$.

Solution. The Coulomb potential is

$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}.$$

Whenever the reactants are close enough to make "physical" contact, the strong force kicks in; the Coulomb potential at this radius is roughly the peak of the Coulomb barrier. The radius of nuclei can be estimated as $r = 1.2\,\mathrm{fm}\cdot A^{1/3}$.

This gives us sufficient information to compute:

$$r_{p+p} = 2r_p = 2(1.2\,\mathrm{fm})(1)^{1/3} = 2.4\,\mathrm{fm} \Rightarrow E_{p+p} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r_{p+p}} = \frac{1}{4\pi(8.85\times 10^{-12}\,\mathrm{F/m})}\frac{1\cdot 1\cdot(1.6\times 10^{-19}\,\mathrm{C})^2}{2.4\,\mathrm{fm}}$$

$$= 9.6\times 10^{-14}\,\mathrm{J} = 559\,\mathrm{keV}$$

$$r_{\mathrm{p}+{}^{12}\mathrm{C}} = r_p + r_{{}^{12}\mathrm{C}} = 1.2\,\mathrm{fm} + 1.2\,\mathrm{fm}\cdot(12)^{1/3} = 3.95\,\mathrm{fm} \Rightarrow E_{\mathrm{p}+{}^{12}\mathrm{C}} = \frac{1}{4\pi\epsilon_0}\frac{Ze^2}{r_{\mathrm{p}+{}^{12}\mathrm{C}}}$$

$$= \frac{1}{4\pi(8.85\times 10^{-12}\,\mathrm{F/m})}\frac{1\cdot 6\cdot(1.6\times 10^{-19}\,\mathrm{C})^2}{3.95\,\mathrm{fm}} = 2.19\,\mathrm{MeV}$$

$$r_{{}^{4}\mathrm{He}+{}^{208}\mathrm{Pb}} = r_{{}^{4}\mathrm{He}} + r_{{}^{208}\mathrm{Pb}} = 1.2\,\mathrm{fm}\cdot 4^{1/3} + 1.2\,\mathrm{fm}\cdot 208^{1/3} = 9.01\,\mathrm{fm}$$

$$\Rightarrow E_{{}^{4}\mathrm{He}+{}^{208}\mathrm{Pb}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} = \frac{1}{4\pi(8.85\times 10^{-12}\,\mathrm{F/m})}\frac{4\cdot 208\cdot(1.6\times 10^{-19}\,\mathrm{C})^2}{9.01\,\mathrm{fm}} = 133\,\mathrm{GeV}$$

□

Problem 2. *The cross section for charged-particle reactions is proportional to the probability of tunneling through the Coulomb barrier given by the Gamow factor, which has a convenient approximation:*

$$e^{-2\pi\eta} = e^{-2\pi Z_1 Z_2 e^2/\hbar\nu} = e^{-31.287 Z_1 Z_2\sqrt{\mu/E}}$$

where μ is the reduced mass in amu and E is the center-of-mass energy in keV. For the 3 cases you considered above, calculate the Gamow factor for an energy that is one-quarter the barrier energy you found in problem 1.

Solution. Plug-and-chug:

$$\mu_{\mathrm{p+p}} = \frac{m_p m_p}{m_p + m_p} = \frac{m_p}{2} = 0.5\,\mathrm{amu} \Rightarrow G_{\mathrm{p+p}} = \exp\left(-31.287 \cdot 1 \cdot 1 \sqrt{\frac{0.5\,\mathrm{amu}}{559\,\mathrm{keV}/4}}\right) = 0.94$$

$$\mu_{\mathrm{p+}^{12}\mathrm{C}} = \frac{m_p m_{^{12}\mathrm{C}}}{m_p + m_{^{12}\mathrm{C}}} = \frac{1\,\mathrm{amu} \cdot 12\,\mathrm{amu}}{1\,\mathrm{amu} + 12\,\mathrm{amu}} = 0.92\,\mathrm{amu} \Rightarrow G_{\mathrm{p+}^{12}\mathrm{C}} = \exp\left(-31.287 \cdot 1 \cdot 6 \sqrt{\frac{0.92\,\mathrm{amu}}{2.91\,\mathrm{MeV}/4}}\right)$$
$$= 0.81$$

$$\mu_{^4\mathrm{He}+^{208}\mathrm{Pb}} = fracm_{^4\mathrm{He}} m_{^{208}\mathrm{Pb}} m_{^4\mathrm{He}} + m_{^{208}\mathrm{Pb}} = \frac{4\,\mathrm{amu} \cdot 208\,\mathrm{amu}}{4\,\mathrm{amu} + 208\,\mathrm{amu}} = 3.92\,\mathrm{amu}$$
$$\Rightarrow G_{^4\mathrm{He}+^{208}\mathrm{Pb}} = \exp\left(-31.287 \cdot 4 \cdot 208 \sqrt{\frac{3.92\,\mathrm{amu}}{133\,\mathrm{GeV}/4}}\right) = 0.75$$

□

Problem 3. *A 2 MeV beam of protons bombards a ^{16}O target and the differential cross section is measured to be 0.094 b/sr at a lab angle of $167°$.*

1. *What is the expected cross-section if you assume Rutherford scattering?*
2. *What is the calculated Mott cross-section?*
3. *How do your answers to (a) and (b) differ from the measured cross section and why might they be different?*

Solution. Using the nonrelativistic Rutherford scattering formula (2 MeV is pretty slow),

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} = \frac{(zZe^2)^2}{(4\pi\epsilon_0)^2 \cdot (4E_{kin})^2 \sin^4\frac{\theta}{2}} = \frac{(1 \cdot 8(1.6 \times 10^{-19}\,\mathrm{C})^2)^2}{(4\pi \cdot 8.85 \times 10^{-12}\,\mathrm{F/m})^2 \cdot (4 \cdot 2\,\mathrm{MeV})^2 \sin^4\frac{167}{2}} = 0.0212\,\mathrm{b/sr}.$$

The Mott cross-section can be computed by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Mott}} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{Rutherford}} \cdot \cos^2\frac{\theta}{2} = (0.0212\,\mathrm{b/sr}) \cdot \cos^2\frac{167}{2} = 2.72 \times 10^{-4}\,\mathrm{b/sr}.$$

The Rutherford cross-section is within an order of magnitude, whereas the Mott cross-section is pretty far off. The large discrepancy in the Mott cross-section is expected, since it's designed to account for spin effects in relativistic fermions, and this is a nonrelativistic boson scattering. The Ruterford cross-section doesn't take into account any spin effects, so that may be the source of its discrepancy. □

Problem 4. *Assume that ^{197}Au is made from a solid, uniform sphere of nuclear material with a radius of $R = 1.2\,\mathrm{fm} \cdot A^{1/3}$. Calculate the form factor $F(q)$.*

Solution. The charge distribution is

$$\rho(r) = \begin{cases} \frac{Ze}{4\pi R^3/3} & r \leq R \\ 0 & r > R \end{cases}$$

The nonzero density part can be computed to be $\frac{87 \cdot 1.6 \times 10^{-19}\,\mathrm{C}}{4\pi(1.2\,\mathrm{fm} \cdot 197^{1/3})/3} = 4.76 \times 10^{-4}\,\mathrm{C/m^3}$ □

Problem 5. *Show that the mean-square charge radius of a uniformly charged sphere is* $\langle r^2 \rangle = 3R^2/5$*.*

Problem 6. *A nuclear charge distribution more realistic than the uniformly charged distribution is the Fermi distribution,* $\rho(r) = \frac{\rho_0}{1+\exp[(r-c)/a]}$*. Find the value of* a *if* $t = 2.3\,\mathrm{fm}$

Problem 7 (Bonus)**.** *Evaluate* $\langle r^2 \rangle$ *for the Fermi distribution in Problem 6.*