4271 HW 1

Duncan Wilkie

1 February 2023

Problem 1. *Calculate the average binding energy for ^{40}K.*

Solution. Applying the binding energy formula,

$$B(19, 40) = \left[ZM(^1\text{H}) + (A - Z)M_n - M(A, Z) \right] \cdot c^2$$

$$= \left[(19)(938.783 \,\text{MeV}/\text{c}^2) + (40 - 19)(939.565 \,\text{MeV}/\text{c}^2) - (39.963\,998 \,\text{u})(931.494\,32 \,\text{MeV}/\text{c}^2\text{u}) \right] \cdot c^2$$

$$= 341.5 \,\text{MeV}.$$

$\square$

Problem 2. *The mass of ^{74}Ge is 73.921 177 u.*

1. *Calculate its mass defect in MeV/c^2.*

2. *Calculate its binding energy in MeV.*

3. *What is the binding energy per nucleon?*

Solution. The mass defect is

$$\Delta = (M(A, Z) - A)c^2 = (73.921\,177 \,\text{u} - 74 \,\text{u})(931.49 \,\text{MeV}/\text{u}) = -73.423 \,\text{MeV}.$$

The binding energy in terms of the mass defect is

$$B(32, 74) = \left[ZM(^1\text{H}) + (A - Z)M_n - \left(A + \frac{\Delta}{c^2} \right) \right] c^2$$

$$= 32(938.783 \,\text{MeV}) + (74 - 32)(939.565 \,\text{MeV}) - (74(931.49 \,\text{MeV}/\text{u}) - 73.423 \,\text{MeV}) = 645.95 \,\text{MeV}.$$
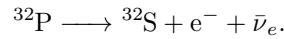
There are 74 nucleons, so the binding energy per nucleon is 8.729 MeV. $\square$

Problem 3.

1. *Calculate the energy released in the beta decay of ^{32}P.*

2. *If the resulting beta particle has energy 650 keV, how much energy does the antineutrino have?*

Solution. This is a β^- decay, because the nucleus has a neutron excess; its equation is

$$^{32}\text{P} \longrightarrow {}^{32}\text{S} + \text{e}^- + \bar{\nu}_e.$$

The energy released in the β-decay can be deduced from the difference in the atomic masses:

$$Q = m(15, 17)c^2 - m(16, 16)c^2 = (31.973\,907\,\text{u})(931.49\,\text{MeV/u}) - (31.972\,071\,\text{u})(931.49\,\text{MeV/u})$$

$$= 1.71\,\text{MeV}.$$

Because the mass of the nucleus is large, the energy imparted by the recoil from the ejection is negligible. Accordingly, the released energy is approximately the sum of that in the two ejected particles; the neutrino energy can then be estimated to be

$$E(\bar{\nu}_e) = Q - E(e^-) = 1.71\,\text{MeV} - 0.650\,\text{MeV} = 1.06\,\text{MeV}.$$

$\square$

Problem 4. *Use the semi-empirical mass formula to estimate the prompt energy released in spontaneous fission of ^{239}Pu as*

$$^{239}Pu \longrightarrow {}^{112}Pd + {}^{124}Cd + 3\,n$$
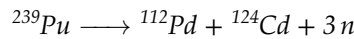
Solution. Applying the semi-empirical mass formula to the initial case, which has $f_5 = 0$ since the mass number is odd,

$$M(94, 239) = ZM(^1\text{H}) + (A - Z)(M_n) - a_v A + a_s A^{2/3} + a_C \frac{Z^2}{A^{1/3}} + a_a \frac{(Z - A/2)^2}{A}$$

$$= 94(938.783\,\text{MeV/c}^2) + 145(939.565\,\text{MeV/c}^2) - 15.56(239) + 17.23(239)^{2/3} + 0.697\frac{(94)^2}{(239)^{1/3}}$$

$$+93.14\frac{(94 - 239/2)^2}{239}$$

$$= 222.673\,\text{GeV/c}^2$$

The associated mass defect is

$$\Delta = [M(94, 239) - A]c^2 = 222.673\,\text{GeV} - (239\,\text{u})(931.49\,\text{MeV/u}) = 47.00\,\text{MeV}$$

To calculate the final energy, we must calculate the energy of each of the components. Using the formula above,

$$M(46, 112) = 46(938.783\,\text{MeV/c}^2) + 66(939.565\,\text{MeV/c}^2) - 15.56(112) + 17.23(112)^{2/3} + 0.697\frac{(46)^2}{(112)^{1/3}}$$

$$+93.14\frac{(46 - 112/2)^2}{112} - \frac{11.2}{(112)^{1/2}}$$

$$= 104.241\,\text{GeV/c}^2 \Rightarrow \Delta = 104.241\,\text{GeV} - 112(931.49\,\text{MeV/u}) = -85.88\,\text{MeV}$$

$$M(48, 124) = 48(938.783\,\text{MeV}/c^2) + 76(939.565\,\text{MeV}/c^2) - 15.56(124) + 17.23(124)^{2/3} + 0.697\frac{(48)^2}{(124)^{1/3}}$$

$$+93.14\frac{(48 - 124/2)^2}{124} - \frac{11.2}{(124)^{1/2}}$$

$$= 124.831\,\text{GeV}/c^2 \Rightarrow \Delta = 115.436\,\text{GeV} - 124(931.49\,\text{MeV}/u) = -68.97\,\text{MeV}$$

$$3M_n = 3(939.565\,\text{MeV}/c^2) = 2.81\,\text{GeV}/c^2 \Rightarrow \Delta = 2.81\,\text{GeV}/c^2 - 3(931.49\,\text{MeV}/u) = 15.53\,\text{MeV}$$

The total yield is then

$$Q = \Delta(^{239}\text{Pu}) - \Delta(^{112}\text{Pd} + {}^{134}\text{Cd} + 3\,\text{n}) = 47.00\,\text{MeV} - (-85.88\,\text{MeV} - 68.97\,\text{MeV} + 15.53\,\text{MeV})$$

$$= 186\,\text{MeV}$$

$\square$

Problem 5. *Use accurate masses from the 2021 atomic mass evaluation to calculate the actual prompt energy released in problem 4. Compare the answers. How different are they?*
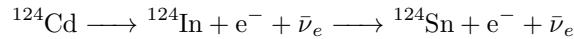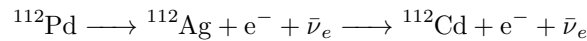
Solution. According to the tabulated mass defects,

$$Q = \Delta(^{239}\text{Pu}) - \Delta(^{112}\text{Pd} + {}^{124}\text{Cd} + 3\,\text{n}) = 48.59\,\text{MeV} - (-86.32\,\text{MeV} - 76.70\,\text{MeV} + 15.53\,\text{MeV})$$

$$= 196\,\text{MeV}$$

These are different by 10 MeV or 5%. $\square$

Problem 6. *Following spontaneous fission from the last problem, a chain of beta decays converts the ^{112}Pd and ^{124}Cd to stable nuclei.*

1. *What are the final nuclei produced?*

2. *What is the total beta-delayed energy released in the beta-decay chains? What is the ratio of beta-delayed to prompt energy released?*

3. *(Bonus) About how long does it take (on average) for all the energy to be released?*

Solution. The decay chains are

$$^{112}\text{Pd} \longrightarrow {}^{112}\text{Ag} + \text{e}^- + \bar{\nu}_e \longrightarrow {}^{112}\text{Cd} + \text{e}^- + \bar{\nu}_e$$

$$^{124}\text{Cd} \longrightarrow {}^{124}\text{In} + \text{e}^- + \bar{\nu}_e \longrightarrow {}^{124}\text{Sn} + \text{e}^- + \bar{\nu}_e$$

The beta-delayed energies of these chains are simply calculated from tabulated mass defects:

$$Q = \Delta(^{112}\text{Pd}) - \Delta(^{112}Cd) = -86.321\,\text{MeV} - -90.574\,\text{MeV} = 4.25\,\text{MeV}$$

and

$$Q = \Delta(^{124}\text{Cd}) - \Delta(^{124}\text{Sn}) = -76.699\,\text{MeV} - -88.231\,\text{MeV} = 11.53\,\text{MeV}$$

The total beta-delayed energy is then 15.79 MeV. This is 8.1% of the prompt energy. $\square$

Problem 7. *The isotope ^{252}Cf is commonly used to study the geology around underground wells due to the high number of neutrons emitted. Each spontaneous fission of ^{252}Cf produces an average of 3.77 neutrons. However, alpha decay dominates, with only 3.09% of the decays being fission.*

1. *Calculate the activity (in Curies) of ^{252}Cf that would be requried to produce 5×10^8 neutrons/s.*

2. *If we make such a source, the neutron flux will decrease over time as the ^{252}Cf decays. How many neutrons/s will be produced 4 years later? $(t_{1/2} = 2.645 \, \mathrm{yr})$*

Solution. We want

$$3.77 \cdot .0309 \cdot A = 5 \times 10^8 \Leftrightarrow A = \frac{5 \times 10^8}{3.77 \cdot .0309} = 4.292 \times 10^9 \, \mathrm{Bq} = 0.116 \, \mathrm{Ci}.$$

The proportion of californium remaining is $e^{-t \ln(2)/t_{1/2}} = e^{-(4) \ln(2)/(2.645)} = 0.351$; accordingly, since everything else stays the same, the production is $0.351(5 \times 10^8) = 175.2 \times 10^6$ neutrons/s. □

Problem 8. *The oldest formerly living objects that can be measured by ^{14}C dating are about 50 000 yo.*

1. *What is the ratio of ^{14}C/^{12}C in such a sample?*

2. *Assume the natural abundance ratio in the atmosphere is 1.5×10^{-12}. If you have to count 25 atoms of ^{14}C to make a measurement (20% accuracy), how much mass of carbon would you need to date a 50 000 yo object?*

Solution. Given the half-life of ^{14}C being 5700 y, the proportion remaining after 50 000 y is

$$\frac{N}{N_0} = e^{-t \ln(2)/t_{1/2}} = e^{-(50\,000\,\mathrm{y}) \ln(2)/(5700\,\mathrm{y})} = 0.00229.$$

We want

$$0.00229(1.5 \times 10^{-12})n_C = 25 \Leftrightarrow n_C = \frac{25}{(0.00229)(1.5 \times 10^{-12})} = 7.278 \times 10^{15} \, \mathrm{atoms}$$

$$= 1 \, \mathrm{g/mol} \cdot \frac{7.278 \times 10^{15} \, \mathrm{atoms}}{6.02 \times 10^{23} \, \mathrm{atoms/mol}} = 12.1 \, \mathrm{ng}$$

□

Problem 9. *The Cassini spacecraft went into orbit around the planet Saturn in July 2004, after a nearly seven-year journey from Earth. On-board electrical systems were powered by heat from three radioisotope thermoelectric generators, which together utilized a total of 32.7 kg of ^{238}Pu, encapsulated as PuO_2. The isotope has a half-life of 86.4 y and emits an alpha particle with an average energy of 5.49 MeV.*

1. *How much total thermal power is generated in the spacecraft?*

2. *To see why RTGs are so attractive, calculate the solar panel area needed to produce the same amount of wattage at the distance of Saturn. Assume your solar panel is 10% efficient and that the Sun's luminosity is 3.8×10^{26} W.*

Solution. The total number of atoms of ^{238}Pu intially on the spacecraft is

$$N = \frac{3 \cdot 32.7\,\text{kg}}{0.001\,\text{kg/mol}} \cdot 6.02 \times 10^{23}\,\text{atoms/mol} = 59.1 \times 10^{27}\,\text{atoms},$$

leading to an activity of

$$A = \lambda N = \frac{\ln 2}{t_{1/2}} N = \frac{\ln 2}{86.4\,\text{y} \cdot 365\,\text{d/y} \cdot 24\,\text{h/d} \cdot 3600\,\text{s/h}} \cdot 59.1 \times 10^{27}\,\text{atoms} = 8.622 \times 10^{17}\,\text{Bq}.$$

Each decay liberates $5.49\,\text{MeV}$, so the total thermal power is

$$P = (5.49\,\text{MeV})(8.622 \times 10^{17}\,\text{Bq}) = 4.733 \times 10^{18}\,\text{MeV/s} = 757.3\,\text{kW}.$$

Saturn's orbit has semi-major axis $1.434 \times 10^{12}\,\text{m}$, and, accordingly, the power density at such a distance is

$$P = \frac{3.8 \times 10^{26}\,\text{W}}{4\pi(1.434 \times 10^{12}\,\text{m})^2} = 14.71\,\text{W/m}^2.$$

Ten percent of this is $1.471\,\text{W/m}^2$—roughly six orders of magnitude smaller than the RTGs. $\qquad\square$