7210 HW 4

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20 September 2022

Lemma 1. *If $H \trianglelefteq G$ is cyclic, then every $K \leq H$ has $K \trianglelefteq G$.*

Proof. Write $H = \langle a \rangle$. Normality of H in G is $ga^k g^{-1} = a^{k'}$ for all $g \in G$ Subgroups of cyclic groups are also cyclic and so $H = \langle (a^k)^i \rangle$; normality of H is then

$$ga^{ki}g^{-1} = (ga^k g^{-1})^i = a^{k'i} = (a^i)^{k'}$$

which is in $\langle a^i \rangle$ by closure. $\square$

3.4.8. *For a finite group G, the following are equivalent:*

1. *G is solvable*
2. *G has a chain of subgroups $1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_s = G$ such that H_{i+1}/H_i is always cyclic*
3. *all composition factors of G are of prime order*
4. *G has a chain of subgroups $1 = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N_i is always Abelian.*

Proof. It's very evident that $3 \Rightarrow 2$, $3 \Rightarrow 4$, $3 \Rightarrow 1$, $2 \Rightarrow 1$, and $4 \Rightarrow 1$. The observations necessary for each are, in order:

- simple groups of prime order are cyclic,
- simple groups of prime order are cyclic, and every subgroup of a normal cyclic subgroup is normal in the parent group by Lemma 1.
- simple groups of prime order are cyclic and therefore Abelian,
- cyclic groups are Abelian,
- the only difference is an extra condition in 4.

It therefore suffices to prove $1 \Rightarrow 3$, as 3 implies every other statement, and every other statement implies 1. This certainly holds when $|G| = 1$, since $1 = G_0 = G_s = G$ is a solution chain, and $1/1 = 1$ has no normal subgroups other than itself and the trivial group, so it is also a composition series. Presume for induction again that this holds for solvable groups with order less than $|G|$. By assumption, G has a solution chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_t = G$$

such that G_{i+1}/G_i is Abelian. G_{t-1} has order less than G, so it has composition factors all of prime order. We may then write

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k \trianglelefteq G_{t-1} \trianglelefteq G_t = G$$

where H_{i+1}/H_i is a simple group of prime order. Additionally, G_t/G_{t-1} has order less than G, and so has a composition series

$$1 = K_0 \trianglelefteq K_1 \trianglelefteq \cdots \trianglelefteq K_m = G_t/G_{t-1},$$

where K_{i+1}/K_i is a simple group of prime order, and therefore cyclic. By the lattice isomorphism theorem, the elements of this composition series correspond bijectively to subgroups of G_t containing G_{t-1}; this bijection preserves normality by the text of the theorem, but it also preserves the cyclic nature of subgroups and their order, since the bijection sends a coset to the set of its representatives and a set of representatives to the coset they mutually represent, and the order of a coset is the order of any of its representatives. Call the images of K_i under this bijection K_i'; we then have a chain

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_k \trianglelefteq G_{t-1} \trianglelefteq K_0' \trianglelefteq \cdots \trianglelefteq K_m' \trianglelefteq G_t = G$$

such that H_{i+1}/H_i, G_{t-1}/H_k, K_0'/G_{t-1}, K_{i+1}'/K_i', and G_t/K_m' are simple of prime order.

□

3.5.10. *Write a composition series for A_4 and deduce that A_4 is solvable.*

Proof. Looking at the lattice of subgroups of A_4 in the text, we're inspired to consider $G = \langle (1\ 2)(3\ 4), (1\ 3)(2\ 4) \rangle$, since it appears to have a unique position in the lattice. The generated subgroup is defined as all words of arbitrary powers of generator elements; since disjoint two-cycles commute, the powers of generator elements may be distributed into powers of all two-cycles appearing in the generators. These powers are the same two-cycles, which again commute, so the words of arbitrary powers of generator elements may be rearranged into the form $g = [(1\ 2)(3\ 4)]^n[(1\ 3)(2\ 4)]^k$ for some $n, k \in \mathbb{Z}$. For all $n, k \neq 0$, the power is the same as the underlying element, so we can restrict to $n, k \in \{0, 1\}$. The elements of G are then

- 1
- (1 2)(3 4)
- (1 3)(2 4)
- (1 2)(3 4)(1 3)(2 4) = (1 4)(2 3)

Note now that conjugation preserves the decomposition signature. Consider $\mu = \tau\sigma\tau^{-1}$ and the cycle $a_1 \overset{\sigma}{\mapsto} a_2 \overset{\sigma}{\mapsto} \cdots \overset{\sigma}{\mapsto} a_k \overset{\sigma}{\mapsto} a_1$. Notice that just by expanding the definition

$$\mu\tau(a_i) = \tau\sigma(a_i) = \tau(a_{i+1}).$$

This implies that $\tau(a_1) \overset{\mu}{\mapsto} \tau(a_2) \overset{\mu}{\mapsto} \cdots \overset{\mu}{\mapsto} \tau(a_k) \overset{\mu}{\mapsto} \tau(a_1)$, i.e. k-cycles of σ bijectively correspond with k-cycles of $\tau\sigma\tau^{-1}$ by applying τ to each element of the cycle of σ. Of course, conjugation of a cycle decomposition is conjugation of each of the cycles:

$$\tau\sigma\tau^{-1} = \tau\sigma_1\tau^{-1}\tau\sigma_2\tau^{-1}\cdots\tau\sigma_n\tau^{-1}.$$

Since permutations are bijective, the disjointness of the initial cycle decomposition implies disjointness of the product of the conjugated cycles, so the "decomposition signature" of a permutation is preserved under conjugation, i.e. the number of cycles in the decomposition and their lengths.

G is exactly all twofold products of 2-cycles, and the above result implies conjugation yields another twofold product of 2-cycles, so $G \trianglelefteq A_4$.

Further, by Lagrange's theorem, $|A_4/G| = |A_4|/|G| = 12/4 = 3$. There is only one group of order 3: a general such group has distinct elements $\{1, a, b\}$, and $ab = 1$, as $ab = a \Rightarrow b = 1$ and vice versa. Correspondingly, $a = b^{-1} \Leftrightarrow a^{-1} = b$. Further, since every finite group has an element of order p for every prime p dividing $|G|$, one of a or b cubes to 1. Suppose WLOG it's a; then $a^2 = a^3 a^{-1} = 1b = b$. This implies a generates the group, since $a^1, a^2 = b, a^3 = 1$ enumerates every element. It is therefore isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

This proves we have a composition series, since groups of prime order have no nontrivial subgroups and so no nontrivial normal subgroups by Lagrange's theorem. Fully, we've demonstrated

$$1 = H_0 \trianglelefteq G \trianglelefteq H_1 = A_4$$

such that $A_4/H_1 \cong \mathbb{Z}/3\mathbb{Z}$, which is simple.

Additionally, the composition factor is Abelian, so the composition series serves to show A_4 is solvable. □

4.2.7. *Let Q_8 be the quaternion group of order 8. Q_8 is isomorphic to a subgroup of S_8, but not to any subgroup of S_n for $n \leq 7$.*

Proof. We can follow the outline given in the proof of Cayley's theorem. Q_8 as a list is

$$1, -1, i, j, k, -i, -j, -k.$$

This list represents a bijection ϕ between Q_8 and $9 = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Let Q_8 act on itself by left-multiplication (corresponding to $H = 1$ in Theorem 3, just like in Cauchy's theorem). The map $\phi \circ (g\cdot) \circ \phi^{-1}$ is an element of S_8 that takes a number between 1 and 8, represents it as an element of Q_8, does multiplication by $g \in Q_8$, and then uses the representation to find what number from 1 to 8 this corresponds to. The permutation representation of each element of Q_8 in 1-line notation on Q_8 and then in cycle notation on 9, in order according to the order given by the list, is

- $1, -1, i, j, k, -i, -j, -k \mapsto 1$,
- $-1, 1, -i, -j, -k, i, j, k \mapsto (1\ 2)(3\ 6)(4\ 7)(5\ 8)$,
- $i, -i, -1, k, j, 1, -k, -j \mapsto (1\ 6\ 2\ 3)(4\ 5)(7\ 8)$,
- $j, -j, -k, -1, i, k, 1, -i \mapsto (1\ 7\ 2\ 4)(3\ 5\ 6\ 8)$,
- $k, -k, -j, -i, -1, j, i, 1 \mapsto (1\ 8\ 2\ 5)(3\ 7)(4\ 6)$,
- $-i, i, 1, -k, -j, -1, k, j \mapsto (1\ 3\ 2\ 6)(4\ 8)(5\ 7)$,
- $-j, j, k, 1, -i, -k, -1, i \mapsto (1\ 4\ 2\ 7)(3\ 8\ 6\ 5)$,
- $-k, k, j, i, 1, -j, -i, -1 \mapsto (1\ 5\ 2\ 8)(3\ 4)(6\ 7)$.

If $Q_8 \cong K \leq S_7$, then K is a permutation representation of Q_8 acting on a 7 element set. Accordingly, the action of K on the 8-element set Q_8 must fix some element of Q_8; this element is in the kernel of the action. However, the action of Q_8 on itself must be transitive, and if it fixes some element of Q_8, that is a separate orbit. Since isomorphisms preserve representation-independent properties of groups, such as transitivity of particular actions, this is a contradiction. □

4.2.8. *If $H \leq G$ has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$*

Proof. Let G act on the left cosets of H by left-multiplication. There's an associated permutation representation induced by this action; by Theorem 3, the kernel of this action K is a normal subgroup of G that is the largest such subgroup contained in H. Applying the first isomorphism theorem to the representation homomorphism, G/K is isomorphic to a subgroup of $S_{|G:H|} = S_n$, since the image of the representation is the set on which G acts. The cardinality of S_n is $n!$, and so $|G : K| \leq n!$. □

4.2.10. *Every non-Abelian group of order 6 has a non-normal subgroup of order 2. Classify groups of order 6.*

Proof. By Sylow's theorem, there exists subgroups of any group of order 6 with orders 2 and 3. There's only one group of order 2 and of order 3 up to isomorphism: $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/3\mathbb{Z}$ (for the latter cf. above). Let a be the generator of the subgroup with order 2, and b the generator of that with order 3. If $ab = ba$, then G is cyclic, since distributivity of exponentiation over commuting products means $(ab)^6 = (a^2)^3(b^3)^2 = 1$; this implies G is cyclic, generated by ab (it has 6 elements and is a subgroup of G), but it must be non-Abelian.

Therefore, $b\langle a\rangle = \{ba^k \mid k = 0, 1\} = \{b, ba\}$ is not equal to $\langle a\rangle b = \{a^k b \mid k = 0, 1\} = \{b, ab\}$, so the group of order 2 is not a normal subgroup.

Every non-Abelian group of order 6 is isomorphic to S_3: the permutation representation of the action of G by left-multiplication on the cosets of $\langle a\rangle$ has kernel contained in $\langle a\rangle$, since no elements of $\langle b\rangle$ commute with those of $\langle a\rangle$ as we've proven their generators don't commute and each is cyclic. The kernel is therefore trivial, since $\langle a\rangle$ has 2 elements and if the kernel included the nonidentity one it'd show $\langle a\rangle$ was be normal. Therefore, the representation homomorphism is an injective function between two groups of the same order, and so it's a bijection, showing $G \cong S_3$. All non-Abelian groups of order 6 were shown to be Abelian in the proof of the first part, so groups of order 6 have isomorphism class of either S_3 or $\mathbb{Z}/6\mathbb{Z}$. □