4271 HW 4

Duncan Wilkie

8 March 2023

Problem 1. *For the following gamma transitions, give all permitted multipoles and indicate which might be the most intense:*

1. $\frac{9}{2}^- \mapsto \frac{7}{2}^+$
2. $\frac{1}{2}^- \mapsto \frac{7}{2}^-$
3. $1^- \mapsto 2^+$
4. $4^+ \mapsto 2^+$
5. $3^+ \mapsto 3^+$

Solution. In the first case, the vector diference yields possible L of $1, 2, 3, 4, 5, 6, 7$. The parity is $(-1)^L$ for an electric transition, and $-(-1)^L$ for a magnetic transition, so these correspond to an electric dipole, magnetic quadrupole, electric octupole, magnetic 16-pole, etc. on up to electric 128-pole. In general, the lower-L transitions tend to be more intense.

Proceeding similarly, but in less detail for the other cases, with the most intese guess underlined,

$$3 \leq J_f - J_i \leq 4 \Leftrightarrow L = 3, 4 \Rightarrow \underline{\text{magnetic octupole}} \text{ and electric 16-pole transitions.}$$

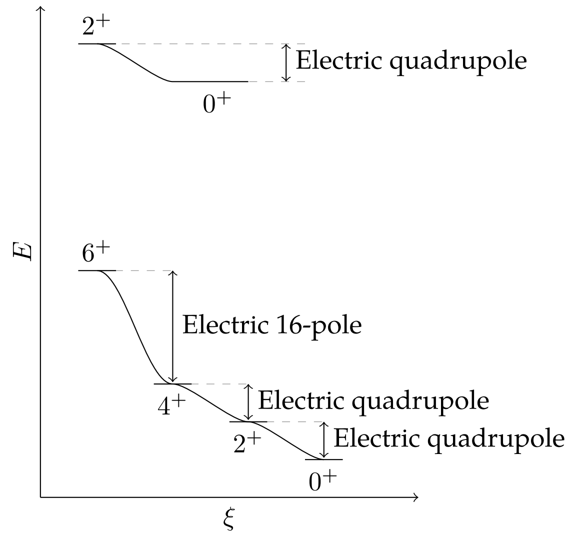
$$1 \leq J_f - J_i \leq 3 \Rightarrow L = 1, 2, 3 \Rightarrow \underline{\text{electric dipole}}\text{, magnetic quadrupole, and electric octupole transitions.}$$

$$2 \leq J_f - J_i \leq 6 \Rightarrow L = 2, 3, 4, 5, 6 \Rightarrow \text{electric } \underline{\text{quadrupole}}\text{, magnetic octupole, electic 16-pole,}$$
$$\text{magnetic 32-pole, and electric 64-pole transitions.}$$

In the last case, there can be no gamma transition, as there is no change in angular momentum. □

Problem 2. *An even-Z, even-N nucleus has the following sequence of levels:* $0+$ *(ground state)*, $2+$ *(89 keV)*, $4+$ *(288 keV)*, $6+$ *(585 keV)*, $0+$ *(1050 keV)*, $2+$ *(1129 keV)*. *Drawn an energy level diagram and show all reasonbably probable gamma-ray transitions and their dominant multipole assignments.*

Solution. Possible transitions:

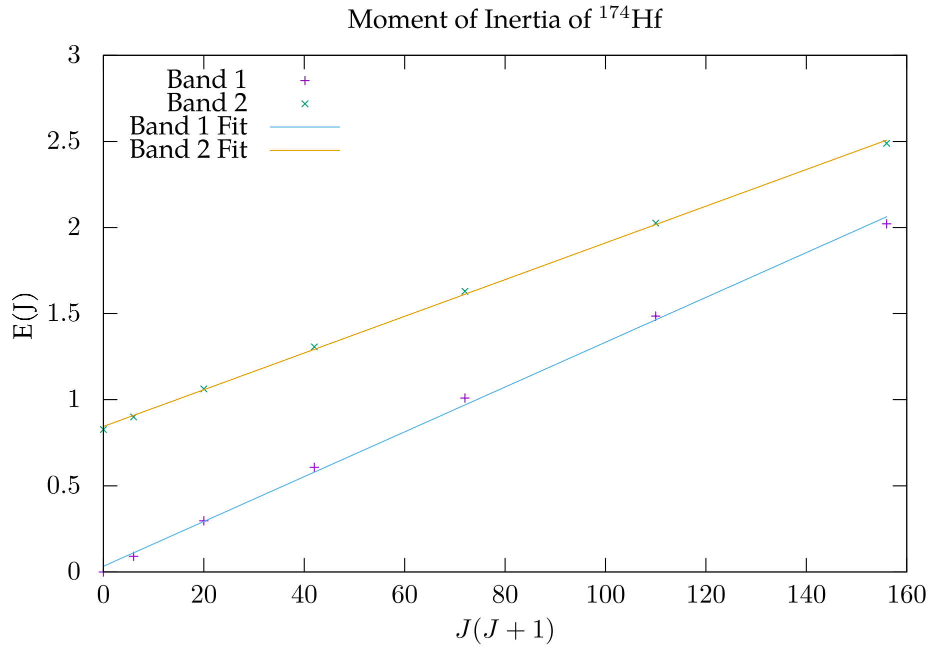


There is also an electric quadrupole transition from the highest 2^+ state to the lowest 0^+; the chemistry package I yoinked this drawing code from doesn't support that kind of a thing so easily. □

Problem 3. *The excited states of* $^{174}\mathrm{Hf}$ *have two similar rotational bands, with energies* (*in MeV*) *given in the following table. Calculate the moments of inertia for these two bands and comment on the difference.*

	$E(0^+)$	$E(2^+)$	$E(4^+)$	$E(6^+)$	$E(8^+)$	$E(10^+)$	$E(12^+)$
Band 1	0	0.091	0.297	0.608	1.010	1.486	2.021
Band 2	0.827	0.900	1.063	1.307	1.630	2.026	2.489

Solution. A quick linear fit yields:



The exact fit parameters extracted from gnuplot yield slopes of 0.0130 ± 0.0002 and 0.0106 ± 0.0001, for bands 1 and 2 respectively; the rule for rotational kinetic energy is that

$$E_{rot} = \frac{\hbar^2}{2I}[J(J+1)] + E_k,$$

implying that the moment of inertia in terms of the slope m is, considering J in natural units of $\hbar$,

$$I = \frac{\hbar^2}{2m} = \frac{1\,\hbar^2}{2(0.013)} = 38.5$$

for the first band and

$$I = \frac{\hbar^2}{2m} = \frac{1\,\hbar^2}{2(0.0106)} = 47.2$$

for the second. □

Problem 4 (Bonus)**.** *Show explicitly that a uniformly-charged ellipsoid at rest with a total charge of Ze and semi-axes a and b has a quadrupole moment*

$$Q = \frac{2}{5}Z\left(a^2 - b^2\right)$$

Solution. First, the volume of an ellipsoid with semi-axes a, b, c: in angular ellipsoidal coordinates, in which the angular coordinates from spherical coordinates remain unchanged, but the remaining variable parameterizes larger ellipsoidal constant surfaces with semi-axes a, b, c,

$$V = \int_0^{\pi} \int_0^{2\pi} \int_0^1 abcs^2 \sin\theta ds d\phi d\theta = 4\pi abc \left(\frac{s^3}{3} \bigg|_{s=0}^{1} \right) = \frac{4}{3}\pi abc.$$

The quadrupole moment is computed by

$$Q_{ij} = \int \rho(\vec{r})\big(3r_ir_j - |\vec{r}|^2\delta_{ij}\big)dV$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^c \frac{Ze}{V}\big(3r_ir_j - s^2(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta)\delta_{ij}\big)abcs^2\sin\theta ds d\phi d\theta$$

$$= \int_0^\pi \int_0^{2\pi} \int_0^c \frac{3Ze}{V} r_ir_j abcs^2\sin\theta ds d\phi d\theta$$

$$-\delta_{ij}\frac{Zec^5abc}{5V}\left(a^2\int_0^\pi \sin^3\theta d\theta \int_0^{2\pi}\cos^2\phi d\phi + b^2\int_0^\pi \sin^3\theta d\theta\int_0^{2\pi}\sin^2\phi d\phi + 2\pi c^2\int_0^\pi \sin\theta\cos^2\theta d\theta\right)$$

With some trig identities, we can compute antiderivatives:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta \Rightarrow \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta) \Rightarrow \text{antideriv.} = \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) + c$$

$$\cos^2\theta = \sin^2(\theta + \frac{\pi}{2}) \Rightarrow \text{antideriv} = \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + c.$$

These entail, alongside integration by parts and u-substitution, that the original integral is

$$= \int_0^\pi \int_0^{2\pi} \int_0^c \frac{3Ze}{V} r_ir_j abcs^2\sin\theta ds d\phi d\theta$$

$$-\delta_{ij}\frac{Zec^5abc}{5V}\left(a^2\left(\frac{4}{3}\right)(\pi) + b^2\left(\frac{4}{3}\right)(\pi) + 2\pi c^2\left(\frac{2}{3}\right)\right)$$

$$= \frac{3Ze}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^c r_ir_js^2\sin\theta ds d\phi d\theta - \frac{Zec^5\delta_{ij}}{5}(a^2 + b^2 + c^2)$$

which in particular is

$$Q_{xx} = \frac{3Ze}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^1 a^2s^4\sin^3\theta\cos^2\phi ds d\phi d\theta - \frac{Zec^5}{5}(a^2 + b^2 + c^2) = \frac{Ze}{5}(2a^2 - b^2 - c^2)$$

$$Q_{yy} = \frac{3Ze}{4\pi}\int_0^\pi \int_0^{2\pi} \int_0^1 b^2s^4\sin^3\theta\sin^2\phi ds d\phi d\theta - \frac{Zec^5}{5}(a^2 + b^2 + c^2) = \frac{Ze}{5}(2b^2 - a^2 - c^2)$$

$$Q_{zz} = \frac{3Ze}{4\pi}\int_0^\pi \int_0^{2\pi} \phi^2 \int_0^1 c^2s^4\cos^2\theta\sin\theta ds d\phi d\theta - \frac{Zec^5}{5}(a^2 + b^2 + c^2) = \frac{Ze}{5}(2c^2 - a^2 - b^2)$$

I caved and used a table (Gradshteyn-Ryzhik), but these little trig integrals should be pretty quick with integration-by-parts. The quadrupole tensor is symmetric, because multiplication and the Kronecker delta are commutative. Furthermore, the symmetry of the problem would indicate that the non-diagonal components are zero; indeed, if one writes them out, one finds immediate functional parity arguments to enforce this, so this is the full quadrupole moment. From the form of the solution, I presume it is intended that this is an ellipsoid of revolution, i.e. circular in some projection; say WLOG $b = c$.

Accordingly, the tensor becomes

$$\frac{Ze}{5}\begin{pmatrix} 2a^2-2b^2 & 0 & 0 \\ 0 & b^2-a^2 & 0 \\ 0 & 0 & b^2-a^2 \end{pmatrix}$$

Were I to privelege an axis along which to compute, it'd be the axis of revolution of the ellipsoid, which is the x-axis (given our choice of which semi-axes are identified). Accordingly, the 1-1 component of the tensor is a sensible choice for a scalar to be called the "quadrupole moment:"

$$Q = \frac{2Ze}{5}(a^2-b^2),$$

which matches the form given closely enough that I'm not too worried about it. □

Problem 5. *Use the answer to Problem 4 to determine the sizes of the semi-major and semi-minor axes of* $^{165}\mathrm{Ho}$*, which has a quadrupole moment of* $Q = 3.5\,\mathrm{b}$*.*

Solution. My result derived above has units of $\mathrm{C}\cdot\mathrm{m}^2$; to get areal units, I'll suppose that I've missed a convention somewhere, and that the formula given, with e divided out, is the correct one. If the "average" radius obeys the phenomenological rule $r = 1.2\,\mathrm{fm}A^{1/3} = 6.58\,\mathrm{fm}$, but the nucleus is truly spherical, then $r = \frac{a+b}{2} \Leftrightarrow a = 2r-b$. This gives us a second equation to solve simultaneously with that given by the quadrupole moment formula:

$$Q = \frac{2Z}{5}((2r-b)^2-b^2) = \frac{2Z}{5}(b^2-4rb+4r^2-b^2) \Leftrightarrow b = \frac{1}{4r}\left[4r^2-\frac{5Q}{2Z}\right] = r-\frac{5Q}{8Zr}$$

$$= 6.58\,\mathrm{fm} - \frac{5\cdot 3.5\times 10^{-28}\,\mathrm{m}}{8\cdot 67\cdot 6.58\,\mathrm{fm}} = 6.08\,\mathrm{fm}$$
$$\Rightarrow a = 2\cdot 6.58\,\mathrm{fm} - 6.08\,\mathrm{fm} = 7.08\,\mathrm{fm}$$

□