7590 HW 1

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?

I interchangeably use the $\overline{z}$ and z^* notation for the complex conjugate. A thousand apologies.

1a

By the definition of the L^2 inner product and A, for any functions $f, g \in D(A)$ we have

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

Integrating by parts,

$$\overline{f'}g\Big|_0^1 - \int_0^1 \overline{f'(x)}g'(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

$$\Leftrightarrow \overline{f'}g\Big|_0^1 - \overline{f}g'\Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

the evaluation terms must both be zero at 0 and 1 since smooth compactly-supported functions on open sets vanish in the limit to the boundary of their domains. Therefore, this operator is symmetric. However, not all elements of $D(A^\dagger)$ are elements of $D(A)$: $g \in H$ is an element of $D(A^\dagger)$ iff there exists $h \in H$ such that $\forall f \in D(A)$

$$\int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Applying the same integration-by-parts argument as above, we may equivalently write this as

$$\Leftrightarrow \overline{f'}g\Big|_0^1 - \overline{f}g'\Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Since f is compactly supported, f' is as well, so the evaluation terms are zero by the same argument given above. Letting $g = x^2$, we then have

$$\int_0^1 \overline{f(x)} \cdot 2dx = \int_0^1 \overline{f(x)}h(x)dx$$

from which we can clearly see the $L^2([0,1])$ function $h = 2$ is the element adjoint to g with respect to A. g is therefore in $D(A^\dagger)$. It isn't in $D(A)$ though, since x^2 doesn't vanish at 1 and therefore isn't compactly supported on this interval. This implies $D(A^\dagger) \neq D(A)$, so $A \neq A^\dagger$, i.e. A isn't self-adjoint.

1b

Proceeding similarly,

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 (if'(x))^* g(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

$$\Leftrightarrow -if^*g\Big|_0^1 + \int_0^1 i(f(x))^* g'(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

By the same argument as above, the evaluation term is zero, in which case the equality follows immediately. This operator is symmetric. Once again, x^2 is in $D^\dagger(A)$ but not $D(A)$: from the formula derived for $\langle Af|g\rangle$ in the proof A is symmetric, the definition of membership in $D^\dagger(A)$ is

$$\int_0^1 (f(x))^* 2x dx = \int_0^1 (f(x))^* h(x)dx$$

which, choosing $h = 2x \in L^2([0,1])$, clearly holds. $2x$ isn't compactly supported on $(0,1)$ since it doesn't vanish in the limit to 1, so $D(A^\dagger) \neq D(A)$ and A isn't self-adjoint.

1c

The definition of a symmetric operator is that $\forall f, g \in D(A)$

$$\langle Af|g\rangle = \langle f|Ag\rangle$$

which in this case is

$$\int_\Omega \overline{[\partial_i(a_{ij}(x)\partial_j f)(x)]} g(x)dx = \int_\Omega \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g)(x)dx$$

$$\Leftrightarrow g(x)\overline{a_{ij}(x)\partial_j f(x)}\Big|_{\partial\Omega} - \int_\Omega \overline{[a_{ij}(x)\partial_j f(x)]}\partial_i g(x)dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

$$\Leftrightarrow \overline{-a_{ij}(x)f(x)}\partial_i g(x)\Big|_{\partial\Omega} + \int_\Omega \overline{f(x)}\partial_j\left(\overline{a_{ij}(x)}\partial_i g(x)\right)dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

$$\Leftrightarrow \int_\Omega \overline{f(x)}\partial_i(\overline{a_{ji}(x)}\partial_j g(x))dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

where we have throughout used integration by parts and the same fact that functions of compact support vanish in the limit to their boundaries. Since $a_{ij}(x)$ is Hermitian, it is equal to $\overline{a_{ji}(x)}$, and so the two sides are equal and the operator is symmetric. Here, A is a bounded operator:

$$||Af|| \leq C||f|| \Leftrightarrow \int_\Omega |\partial_i(a_{ij}(x)\partial_j f)(x)|^2 dx \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \int_\Omega \partial_i a_{ij}(x)\partial_j f(x)dx \int_\Omega \overline{\partial_i a_{ij}(x)\partial_j f(x)}dx \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \left(f(x)\partial_i a_{ij}(x)\Big|_{\partial\Omega} - \int_\Omega f(x)\partial_j\partial_i a_{ij}(x)\right)\left(\overline{f(x)\partial_i a_{ij}(x)}\Big|_{\partial\Omega} - \int_\Omega \overline{f(x)\partial_j\partial_i a_{ij}(x)}dx\right) \le C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \int_\Omega \left(f(x)\overline{f(x)}\right)\left([\partial_i\partial_j a_{ij}(x)]\overline{[\partial_i\partial_j a_{ij}(x)]}\right) dx \le C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \left|\int_\Omega f(x)\partial_i\partial_j a_{ij}(x)dx\right|^2 \le C\int_\Omega |f(x)|^2 dx$$

From the Cauchy-Schwartz inequality, we have

$$\left|\int_\Omega f(x)\partial_i\partial_j a_{ij}(x)dx\right|^2 \le \int_\Omega |f(x)|^2 dx \int_\Omega |\overline{\partial_i\partial_j a_{ij}(x)}|^2 dx = C\int_\Omega |f(x)|^2 dx$$

This proves the operator is bounded. Therefore, $D(A^\dagger) = H$, and there are certainly $L^2(\Omega)$ functions that aren't C^∞, so $D(A^\dagger) \not\subseteq D(A)$ implying A is not self-adjoint.

2a

Applying the definition of the infinitesimal generator,

$$Af(x) = -i\lim_{t\to 0}[f(x+vt) - f(x)]/t = -i\frac{\partial f}{\partial v}$$

where the last equality is valid where the limit exists, using the definition of the partial derivative. Since $V \in C^1$, and the above implies $D(A)$ is L^2 functions differentiable along v, we have $V \subseteq D(A)$, with action given above.

2b

The adjoint of U is defined by

$$\langle U(t)f(x)|g(x)\rangle = \langle f(x)|\left(U(t)\right)^\dagger g(x)\rangle$$

The left hand side is, applying the definition of the L^2 inner product and U,

$$\langle U(t)f(x)|g(x)\rangle = \int_{\mathbb{R}^3} \overline{f(e^{-tB}x)}g(x)dx$$

We can make the substitution $y = e^{-tB}x$, in which case $x = e^{tB}y$. The component functions of this change of variables take the form

$$h_i = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\left[(B^n)_{i1}x_1 + (B^n)_{i2}x_2 + (B^n)_{i3}x_3\right]$$

The Jacobian of this change of coordinates is then

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} & \frac{\partial h_3}{\partial x_3} \end{pmatrix} = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\begin{pmatrix} (B^n)_{11} & (B^n)_{12} & (B^n)_{13} \\ (B^n)_{21} & (B^n)_{22} & (B^n)_{23} \\ (B^n)_{31} & (B^n)_{32} & (B^n)_{33} \end{pmatrix} = e^{-tB}$$

Using $\det e^A = e^{\operatorname{tr} A}$, the Jacobian determinant is

$$\det J = e^{-t\operatorname{tr}(B)} = 1$$

since the trace of skew-symmetric matrices is zero. We can now finally rewrite the integral as

$$\int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB}y) |\det J| dy = \int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB}y) dy = \langle f(x) | U^\dagger(t) g(x) \rangle$$

which identifies $U^\dagger(t) : g(x) \mapsto g(e^{tB}x)$. Clearly, this is unitary:

$$UU^\dagger f(x) = f(e^{-tB}e^{tB}x) = If(x) = f(e^{tB}e^{-tB}x) = UU^\dagger f(x)$$

For $f \in C^1(\mathbb{R}^3)$, the infinitesimal generator acts as

$$Af(x) = -i \lim_{t\to 0} [f(e^{-tB}x) - f(x)]/t$$

The numerator limits to zero, since $e^{-at} \sim 1$ as $t \to 0$. Applying L'Hôpital's rule,

$$= -i \lim_{t\to 0} \frac{\frac{d}{dt} f(e^{-tB}x)}{1} = -i \lim_{t\to 0} \left(\frac{d}{dt} e^{-tB} x \right) \cdot \nabla f(e^{-tB}x) = -i \lim_{t\to 0} \left(-Be^{-tB}x \right) \cdot \nabla f(e^{-tB}x)$$

$$= iBx\nabla f(x)$$

This shows that the limit exists for every $f \in C^1(\mathbb{R})$ (so $V \subseteq D(A)$) and gives its action.

2c

Notice first that the definition of n and r give the number s "modulo" 2π in the sense that if one divides the real number line into partitions by integer multiples of 2π, $n(s)$ gives the multiple of 2π corresponding to the rightmost partition boundary which lies to the left of s and $r(s)$ gives the rightward displacement of s from the partition boundary. If n were further left, adding $0 \le r(s) < 2\pi$ couldn't equal s, and if it were to the right of s, the same is true because $r(s)$ is positive. We prove first that U_α is a continuous symmetry.
Unitarity: It preserves the inner product

$$\langle Uf | Ug \rangle = \int_0^{2\pi} \overline{\alpha^{n(x+t)} f(r(x+t))} \alpha^{n(x+t)} g(r(x+t)) dx = \int_0^{2\pi} |\alpha^{n(x+t)}|^2 \overline{f(r(x+t))} g(r(x+t)) dx$$

Since $|\alpha| = 1$, we may write $\alpha = e^{i\theta}$ in which case it is immediately clear $|\alpha^{n(x+t)}|^2 = 1$. We now make the substitution $y = r(x+t)$, under which $dy = r'(x+t)dx \Leftrightarrow dx = \frac{dy}{r'(x+t)}$. Differentiating the definition of n and r with respect to s yields $1 = n'(s)2\pi + r'(s) = r'(s)$ since n is a step function. Strictly speaking, it is possible there is one point in the integration interval where this derivative is not defined, but since this is a set of measure zero it won't contribute to the integral. We therefore have, applying the substitution,

$$= \int_{r(t)}^{r(2\pi+t)} \overline{f(y)} g(y) dy = \int_0^{2\pi} \overline{f(y)} g(y) dy = \langle f | g \rangle$$

where we have used for the penultimate equality the fact that y (and therefore the entire integrand) has periodicity 2π, so the integral over any two intervals of length 2π will be the same. It also is surjective, as if given a function $h \in L^2([0,2\pi])$, we may write since $\alpha = e^{i\theta} \Rightarrow f(x)\alpha^{g(x)} = f(x)e^{i\theta g(x)}$

$$h(x) = f(x)e^{ig'(x)} = f(x)\alpha^{g(x)}$$

where $f : [0,2\pi] \to \mathbb{R}$ and $g : [0,2\pi] \to [0,2\pi]$. For any given h (which we may rewrite in the form above), α, and t, one may construct the function $k \in L^2([0,2\pi])$ given by $k(x) = f(r(x-t))\alpha^{g(r(x-t))-n(r(x-t)+t)}$. Noting that for $x \in [0,2\pi)$

$$r(r(x+t)-t) = r(x+t-2\pi n(x+t)-t) = r(x-2\pi n(x+t)) = x$$

since

$$r(x+2k\pi) = x+2k\pi - n(x+2k\pi) = x+2k\pi-2k\pi = x$$

by the characterization of $n(s)$ above, we have

$$U_\alpha k = \alpha^{n(x+t)} f[r(r(x+t)-t)]\alpha^{g[r(r(x+t)-t)]-n[r(r(x+t)-t)+t]} = \alpha^{n(x+t)}f(x)\alpha^{g(x)-n(x+t)} = f(x)\alpha^{g(x)}$$

Therefore, U_α is surjective for every α and t, and in conjunction with the above result this proves U_α is unitary. $U_\alpha(0) = I$, using $x \in [0,2\pi]$:

$$U_\alpha(0)f(x) = \alpha^{n(x)}f(r(x)) = \alpha^0 f(x) = f(x)$$

The operator behaves properly under addition in t:

$$U_\alpha(t+s)f(x) = \alpha^{n(x+t+s)}f(r(x+t+s)) = U_\alpha(s)\left(\alpha^{n+t}f(r(x+t))\right) = U_\alpha(s)U_\alpha(t)f(x)$$

Lastly, using $x \in [0,2\pi)$,

$$\lim_{t\to 0} U_\alpha(t)x = \lim_{t\to 0}\alpha^{n(x+t)}r(x+t) = \alpha^{n(x)}r(x) = \alpha^0 x = x$$

This argument becomes slightly more subtle at $x = 2\pi$ due to the discontinuities in n and $r(x)$, but since the limit must be taken from inside $[0,2\pi]$ one may use left-hand limits $n(2\pi^-) = 0$ and $r(2\pi^-) = 2\pi$ to obtain the same result above.
The infinitesimal generator of U_α is by definition

$$A_\alpha f = -i\lim_{t\to 0}[U_\alpha(t)f - f]/t = -i\lim_{t\to 0}[\alpha^{n(x+t)}f(r(x+t)) - f(x)]/t$$

$$= -i\lim_{t\to 0}\frac{\frac{d}{dt}\alpha^{n(x+t)}f(r(x+t))}{1} = -i\lim_{t\to 0}\frac{d}{dt}\alpha^0 f(x+t) = -if'(x)$$

where we have used L'Hôpital's rule and $x \in [0,2\pi)$ (in which case the limit will become at some point exclusively through t close enough to x that $0 \le x+t < 2\pi$, yielding $r(x+t) = x+t$ and $n(x+t) = 0$). The special case $x = 2\pi$ is much the same, only the limit is taken inside $[0,2\pi]$ so only the left-hand limit is taken, which coincides with the result above. Functions in the given V_α make the limit above exist since they are C^1, implying $f'(x)$ is well-defined. Therefore, $V_\alpha \in D(A_\alpha)$ and the action is as given above.

3a

The given $D(A)$ is a subset of the V_α given in problem 2c, since C^1 compactly-supported functions on $(0, 2\pi)$ are a subset of $C^1([0, 2\pi])$ functions that vanish in the limits to 0 and 2π, which are a subset of $C^1([0, 2\pi])$ functions where $f(2\pi) = f(0) = 0$, which are a subset of $C^1([0, 2\pi])$ functions where $f(2\pi) = \alpha f(0)$. Since $V_\alpha \subseteq D(A_\alpha)$ was proven in 2c, we have $D(A) \subseteq D(A_\alpha)$. Further, $A = A_\alpha$ on $D(A)$ is immediate from the symbolic expression of each being identical. This proves $A \subset A_\alpha$.

3b

$A \subset A_\alpha$ implies that $e^{-itA_\alpha} = e^{-itA}$ on $D(A)$. The theorem gives the result that

$$u(x,t) = e^{-itA_\alpha} u_0 = U_\alpha(t) u_0(x)$$

is the unique solution to $\dot{u}(t) = iAu(t)$ where A is the infinitesimal generator of $U(t)$. Applying this, the result follows immediately:

$$u(x,t) = U(t)u_0(x) = U_1(t)u_0(x) = 1^{n(x+t)} u_0(r(x+t)) = u_0(x+t)$$

4a

4b

Noting that since A (and therefore G) are Hermitian,

$$\begin{aligned}
\langle p, Gp \rangle/2 - \langle x - Gp, A(x - Gp)/2 \rangle &= \langle p, Gp \rangle/2 - \langle x, Ax - p \rangle/2 + \langle Gp, Ax - p \rangle/2 \\
&= \langle p, Gp \rangle/2 - \langle x, Ax \rangle/2 + \langle x, p \rangle/2 + \langle Gp, Ax \rangle/2 - \langle Gp, p \rangle/2 \\
&= \langle p, Gp \rangle/2 - \langle x, Ax \rangle/2 + \langle x, p \rangle - \langle Gp, p \rangle/2 \\
&= \langle p, x \rangle - \langle Ax, x \rangle/2
\end{aligned}$$

We can then write

$$\int_{\mathbb{R}^n} e^{\langle p, x \rangle} d\mu(x) = \int_{\mathbb{R}^n} e^{\langle p, x \rangle} \frac{1}{Z} e^{-\langle Ax, x \rangle/2} dx = \frac{1}{Z} \int_{\mathbb{R}^n} e^{\langle p, x \rangle - \langle Ax, x \rangle/2} dx$$

as

$$\frac{1}{Z} \int_{\mathbb{R}^n} e^{\langle p, Gp \rangle/2 - \langle x - Gp, A(x - Gp)/2 \rangle} dx$$

Making the substitution $y = x - Gp$, $dy = dx$ we have

$$= \frac{1}{Z} \int_{\mathbb{R}^n} e^{\langle Gp, p \rangle/2 - \langle y, Ay \rangle/2} dy = e^{\langle Gp, p \rangle/2} \int_{\mathbb{R}^n} \frac{1}{Z} e^{-\langle Ay, y \rangle/2} dy$$

The integral is one by construction, proving the identity.

4c

We find a similar identity:

$$-\langle p, Gp\rangle/2 - \langle x - iGp, A(x - iGp)/2\rangle = -\langle p, Gp\rangle/2 - \langle x, Ax - ip\rangle/2 + \langle iGp, Ax - ip\rangle/2$$
$$= -\langle p, Gp\rangle/2 - \langle x, Ax\rangle/2 + \langle x, ip\rangle/2 + \langle iGp, Ax\rangle/2 - \langle iGp, ip\rangle/2$$
$$= -\langle p, Gp\rangle/2 - \langle x, Ax\rangle/2 + i\langle x, p\rangle + \langle Gp, p\rangle/2$$
$$= -i\langle p, x\rangle - \langle Ax, x\rangle/2$$

As before, this yields

$$\int_{\mathbb{R}^n} e^{i\langle p,x\rangle} d\mu(x) = \int_{\mathbb{R}^n} \frac{1}{Z} e^{-\langle p,x\rangle - \langle Ax,x\rangle/2} dx = \int_{\mathbb{R}^n} \frac{1}{Z} e^{-\langle p,Gp\rangle/2} e^{-\langle x-iGp, A(x-iGp)\rangle/2} dx$$
$$= e^{-\langle p,Gp\rangle} \int_{\mathbb{R}^n} \frac{1}{Z} e^{-\langle y,Ay\rangle/2} dx = e^{-\langle p,Gp\rangle}$$

4d

The notation $\partial^\dagger = -\partial_v + \langle Av, \cdot\rangle$ makes no sense to me; I would read it as "substitute the argument of the operator as the second entry of the inner product" but the inner product of a vector and a scalar (since $f : \mathbb{R}^n \to \mathbb{R}$) isn't a comprehensible notion. I will presume it means "take the argument times the argument's argument as the second entry of the inner product," because this gives consistent results. We want to compute

$$\int_{\mathbb{R}^n} \langle v, \nabla f\rangle \frac{1}{Z} e^{-\langle Ax,x\rangle/2} dx$$

There is an integration-by-parts rule that follows from the product rule for divergence the same as in one dimension:

$$\int_\Omega f(x) \nabla \cdot \vec{V}(x) dx = \int_{\partial\Omega} f(x) \vec{V} \cdot \hat{n} dx' - \int_\Omega \nabla f(x) \cdot \vec{V} dx$$

We may take $\vec{V} = \frac{v}{Z} e^{-\langle Ax,x\rangle/2}$ and note our integral is the last term; if f is "nice" enough so the limit of the boundary integral towards infinity vanishes, we have

$$\int_{\mathbb{R}^n} \langle v, \nabla f\rangle \frac{1}{Z} e^{-\langle Ax,x\rangle/2} dx = \frac{1}{Z} \int_{\mathbb{R}^n} f(x) \nabla \cdot \left(e^{-\langle Ax,x\rangle/2}\right) dx$$

Since we may write $e^{-\langle Ax,x\rangle/2} = ve^{-\sum_i \lambda_i x_i^2/2}$,

$$\nabla \cdot e^{-\langle Ax,x\rangle/2} = \sum_i -\lambda_i x_i v_i e^{-\sum_i \lambda_i x_i^2/2} = -\langle Ax, v\rangle e^{-\langle Ax,x\rangle/2}$$

yielding

$$\int_{\mathbb{R}^n} \langle v, \nabla f\rangle d\mu(x) = \int_{\mathbb{R}^n} f(x) \langle Ax, v\rangle d\mu(x)$$

One can very quickly construct a counterexample to the stated theorem (the given right side is a sum of the product of one-dimensional integrals of odd functions, implying it is equal to zero for all f), so I will presume this is the intended result. Under the L^2 inner product,

$$\int_{\mathbb{R}^n} [\partial_v f(x)]g(x)d\mu(x) = \int_{\mathbb{R}^n} -f(x)\partial_v g(x) + f(x)\langle Av, g(x)x\rangle d\mu(x)$$

$$\Leftrightarrow \int_{\mathbb{R}^n} [\partial_v f(x)]g(x)d\mu(x) = -\int_{\mathbb{R}^n} f(x)\partial_v g(x)d\mu(x) + \int_{\mathbb{R}^n} f(x)g(x)\langle Av, x\rangle d\mu(x)$$

By the product rule for the gradient operator,

$$\Leftrightarrow \int_{\mathbb{R}^n} \partial_v(f(x)g(x))d\mu(x) = \int_{\mathbb{R}^n} f(x)g(x)\langle Av, x\rangle d\mu(x)$$

By the first part of this problem, the two are equal.

4e

$$\int_{\mathbb{R}^n} \langle v, x\rangle\langle w, x\rangle d\mu(x) = \int_{\mathbb{R}^n} \left(\sum_{i=1}^n v_i x_i\right)\left(\sum_{k=1}^n w_k x_k\right)\frac{1}{Z}e^{\sum_{j=1}^n -\lambda_j x_j^2/2}dx = \sum_{i=1}^n\sum_{k=1}^n \int_{\mathbb{R}^n} v_i w_k x_i x_k \frac{1}{Z}e^{\sum_{j=1}^n -\lambda_j x_j^2/2}dx$$

$$= \sum_{i=1}^n\sum_{k=1}^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i w_k x_k x_i e^{-(\lambda_i x_i^2+\lambda_k x_k^2)/2}dx_i dx_k \left(\int_{\mathbb{R}^{n-2}} \frac{1}{Z}e^{\sum_{j\neq i,k}\lambda_j x_j^2/2}\right)$$

When $i \neq k$, the inner integrand is odd, implying the term will be zero. We may therefore write

$$= \sum_{i=1}^n \int_{\mathbb{R}^n} v_i w_i x_i^2 \frac{1}{Z}e^{-\langle Ax, x\rangle/2}dx$$

$$= \sum_{i=1}^n \prod_{k=i}^n \int_{-\infty}^{\infty} v_i w_i x_i^2 \frac{1}{Z}e^{-\lambda_k x_k^2/2}dx_k$$

For $i \neq k$, the multiplicand is $v_i w_i x_i^2$; when $i = k$, it is equal to

$$v_i w_i \frac{1}{Z}\int_{-\infty}^{\infty} x_i^2 e^{-\lambda_i x_i/2}dx_i = v_i w_i \sqrt{\frac{\det A}{2\pi}}\sqrt{\frac{2\pi}{\lambda_i}} = v_i w_i \sqrt{\prod_{k\neq i}\lambda_k}$$

Thus, each term of the sum will be of the form

$$v_i^n w_i^n x_i^{2n-2}\sqrt{\frac{\det A}{\lambda_i}}$$

I'm unsure if there's a similar typographical error as I presumed above, but I don't think this is reducible to $\langle Gv, w\rangle$

7a

$$[a(v), a^\dagger(w)]f = \partial_{Gv}\partial^\dagger_{Gw} f - \partial^\dagger_{Gw}\partial_{Gv} f$$
$$= \langle Gv, \nabla(\cdot)\rangle \circ (-\langle Gw, \nabla(\cdot)\rangle + \langle AGw, \cdot\rangle)\, f - (-\langle Gw, \nabla(\cdot)\rangle + \langle AGw, \cdot\rangle) \circ \langle Gv, \nabla(\cdot)\rangle f$$
$$= \langle Gv, \nabla\left(-\langle Gw, \nabla f\rangle + \langle w, f\rangle\right)\rangle - \left(-\langle Gw, \nabla\left(\langle Gv, \nabla f\rangle\right)\rangle + \langle w, \langle Gv, \nabla f\rangle\rangle\right)$$
$$= \langle Gv, \nabla(-\langle Gw, \nabla f\rangle)\rangle + \langle Gv, \langle w, f\rangle\rangle + \langle Gw, \nabla(\langle Gv, \nabla f\rangle)\rangle - \langle w, \langle Gv, \nabla f\rangle\rangle$$
$$= \sum_i \frac{v_i}{\lambda_i}\frac{\partial}{\partial x_i}\sum_k -\frac{w_k}{\lambda_k}\frac{\partial f}{\partial x_k} + \sum_i \frac{v_i}{\lambda_i}\sum_k w_k f_k + \sum_i \frac{w_i}{\lambda_i}\frac{\partial}{\partial x_i}\sum_k \frac{v_k}{\lambda_k}\frac{\partial f}{\partial x_k} - \sum_i w_i \left(\sum_k \frac{v_k}{\lambda_k}\frac{\partial f}{\partial x_k}\right)_i$$
$$= -\sum_i\sum_k \frac{v_i w_k}{\lambda_i\lambda_k}\frac{\partial^2 f}{\partial x_i \partial x_k} + \sum_i\sum_k \frac{v_i w_k}{\lambda_i} f_k + \sum_i\sum_k \frac{w_i v_k}{\lambda_i\lambda_k}\frac{\partial^2 f}{\partial x_i\partial x_k} - \sum_i \frac{w_i v_i}{\lambda_i}\frac{\partial f}{\partial x_i}$$
$$= \sum_i\sum_k \frac{v_i w_k}{\lambda_i} f_k - \sum_i \frac{w_i v_i}{\lambda_i}\frac{\partial f}{\partial x_i}$$
$$= \sum_i \frac{v_i w_i}{\lambda_i} f_i - \sum_i \frac{v_i}{\lambda_i}\sum_{k\neq i} w_k f_k$$

7b

We can obtain a characterization of $[A^n, B]$ as follows: presuming an induction hypothesis $[A^{n-1}, B] = (n-1)A^{n-2}C$, the commutator formula $[AB, C] = A[B, C] + [A, C]B$ gives

$$[A^n, B] = A[A^{n-1}, B] + [A, B]A^{n-1} = A(n-1)A^{n-2}C + CA^{n-1} = (n-1)A^{n-1}C + A^{n-1}C = nA^{n-1}C$$

This implies

$$e^{\lambda A}B = B + \lambda AB + \lambda^2 A^2 B/2 + ... = B + \lambda\left(BA + [A, B]\right) + \lambda^2(BA^2 + [A^2, B])/2 + ...$$
$$= B + \lambda\left(BA + C\right) + \lambda^2\left(BA^2 + 2CA\right)/2 + ...$$
$$= Be^{\lambda A} + \left(\lambda C + \lambda^2 CA + \lambda^3 CA^2/2 + ...\right)$$
$$= (Be^{\lambda A} + \lambda Ce^{\lambda A}) = (B + \lambda C)e^{\lambda A}$$

We then compute the derivatives

$$\frac{d}{d\lambda}e^{\lambda(A+B)} = \frac{d}{d\lambda}\left(1 + \lambda(A+B) + \frac{\lambda^2}{2}(A+B)^2 + ...\right) = (A+B) + \lambda(A+B)^2 + \frac{\lambda^2}{2}(A+B)^3 + ...$$
$$= (A+B)e^{\lambda(A+B)}$$

and

$$\frac{d}{d\lambda}(e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}) = Ae^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}Be^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}e^{\lambda B}\left(-\lambda Ce^{-\lambda^2 C/2}\right)$$
$$= e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}\left(A + B + \lambda C - \lambda C\right) = (A+B)e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}$$

Since these two functions of λ satisfy the same differential equation and have the same value at $\lambda = 0$, they must be the same function, so for $\lambda = 1$ we obtain the desired result.

7c

Notice that

$M_f = e^{\phi(v)} \Rightarrow: e^{\phi(v)} :=: e^{a(v)+a(v)^\dagger} :=: e^{a(v)^\dagger + a(v)} =: e^{a(v)^\dagger} e^{a(v)} e^{-[a(v),a(v)^\dagger]/2} := e^{-\langle Gv,v\rangle/2} e^{a(v)^\dagger} e^{a(v)}$

by the result of part b.

11a

$$\det\Lambda = \begin{vmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{vmatrix} = \gamma^2 - \gamma^2\beta^2$$
$$= \gamma^2(1-\beta^2) = (1-\beta^2)^{-1}(1-\beta^2) = 1$$

so this is a proper Lorentz transformation. The proposed inverse satisfies

$$\Lambda^{-1}\Lambda = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

and

$$\Lambda\Lambda^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

so it is, in fact, the inverse. The action of the Lorentz transformation on the unit vectors along coordinate axes yields

$$\hat{x}' = \gamma\hat{x}$$
$$\hat{y}' = \hat{y}$$
$$\hat{z}' = \hat{z}$$

and so the new axes are parallel to the old. The origin of the primed system is at $\vec{x'} = 0$; this occurs at $y = 0$ and $z = 0$ trivially, but the x variable has

$$-\gamma\beta t + \gamma x = 0 \Leftrightarrow x = \beta t$$

which is exactly the origin moving along the positive x axis with velocity β.

11b

Taking the events to happen at the origin of the primed system,

$$\Lambda^{-1}\vec{x'}_2 - \Lambda^{-1}\vec{x'}_1 = \begin{pmatrix} \gamma t'_2 \\ \gamma\beta t'_2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma t_1 \\ \gamma\beta t'_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(t'_2 - t'_1) \\ \gamma\beta(t'_1 - t'_2) \\ 0 \\ 0 \end{pmatrix}$$

The first component is γT; since this is the computation of the difference between the two events in the unprimed frame, this proves the time dilation formula.

11c

A similar computation to the above result applies:

$$\Lambda^{-1}\vec{x'_2} - \Lambda^{-1}\vec{x'_1} = \begin{pmatrix} \gamma\beta x'_2 \\ \gamma x'_2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma\beta x'_1 \\ \gamma x'_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\beta L \\ \gamma L \\ 0 \\ 0 \end{pmatrix}$$

Reading off the first component confirms the result. Whichever occurs first along the x axis occurs first in time, since L is a distance and therefore positive. For spacelike separated events,

$$\Lambda\vec{x_2} - \Lambda\vec{x_1} = \begin{pmatrix} \gamma t_2 - \gamma\beta x_2 - (\gamma t_1 - \gamma\beta x_1) \\ \gamma x_2 - \gamma\beta t_2 - (\gamma x_1 - \gamma\beta t_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T - \gamma\beta L \\ \gamma L - \gamma\beta T \\ 0 \\ 0 \end{pmatrix}$$

For the time component to be zero,

$$\gamma T = \gamma\beta L \Leftrightarrow \beta = \frac{T}{L}$$

The spacelike condition ensures conformance with $|\beta| < 1$:

$$T^2 - L^2 < 0 \Leftrightarrow T^2 < L^2 \Leftrightarrow |T| < |L| \Leftrightarrow \frac{|T|}{|L|} < 1 \Rightarrow |\beta| < 1$$

11d

Once again,

$$\Lambda^{-1}\vec{x'_2} - \Lambda^{-1}\vec{x'_1} = \begin{pmatrix} \gamma t_2 + \gamma\beta x_2 - (\gamma t_1 + \gamma\beta x_1) \\ \gamma\beta t_2 + \gamma x_2 - (\gamma\beta t_1 + \gamma x_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T + \gamma\beta L \\ \gamma\beta T + \gamma L \\ 0 \\ 0 \end{pmatrix}$$

The measurements are simultaneous in the unprimed frame, i.e. the first component of the above displacement is zero, yielding $T = -\beta L$. Plugging this in to the second component, one obtains

$$\gamma L - \gamma\beta^2 L = L\gamma[1 - \beta^2] = L\gamma(1/\gamma^2) = L/\gamma$$

as desired.

12

For convenience of notation, define $f : SL(2,\mathbb{C}) \to SO^+(1,3) :: A \mapsto (X \mapsto AXA^\dagger)$ where X is of the form given in the problem. This map is indeed a homomorphism: using $(AB)^\dagger = B^\dagger A^\dagger$,

$$f(A)f(B) = (X \mapsto AXA^\dagger) \circ (X \mapsto BXB^\dagger) = X \mapsto A(BXB^\dagger)A^\dagger = (AB)x(B^\dagger A^\dagger) = f(AB)$$

The kernel of f are those elements $A \in SL(2,\mathbb{C})$ such that $X = AXA^\dagger \Leftrightarrow A^{-1}X = XA^\dagger$ for all X of the given form. Since $A \in SL(2,\mathbb{C}) \Rightarrow \det A = 1$, A is unitary, i.e. $AA^\dagger = A^\dagger A = I$. We then can write

$$X = AXA^\dagger \Leftrightarrow XA = AXA^\dagger A \Leftrightarrow XA = AX$$

Elements of the center of $GL(n,\mathbb{F})$ are c^*I where c^* is any unit of $\mathbb{F}$ and I is the identity matrix, and since the units in $\mathbb{R}$ are ± 1, the kernel of the homomorphism is $\pm I$. By the isomorphism theorem, $\operatorname{im} f \cong SL(2,\mathbb{C})/\ker f$, and $\ker f$ is discrete. Since $\operatorname{im} f$ is a subgroup of the Lorentz group, and the Lorentz group is connected, $\operatorname{im} f$ is isomorphic to the whole group, implying f is surjective.

13

Using the fact $\tilde{x}$ is the difference of two observables and therefore Hermitian,

$$(\Delta x)^2(\Delta p)^2 = \langle\psi|\tilde{x}^2|\psi\rangle\langle\psi|\tilde{p}^2|\psi\rangle = \langle\psi|\tilde{x}^2\psi\rangle\langle\psi|\tilde{p}^2\psi\rangle = \langle\tilde{x}\psi|\tilde{x}\psi\rangle\langle\tilde{p}\psi|\tilde{p}\psi\rangle \geq |\langle\tilde{x}\psi|\tilde{p}\psi\rangle|^2 = |\langle\psi|\tilde{x}\tilde{p}\psi\rangle|^2 = |\langle\psi|\tilde{x}\tilde{p}|\psi\rangle|^2$$

We write

$$\langle\psi|\tilde{x}\tilde{p}|\psi\rangle = \langle\psi|\left(x - \langle\psi|x|\psi\rangle\right)\left(p - \langle\psi|p|\psi\rangle\right)|\psi\rangle = \langle\psi|xp|\psi\rangle - \langle\psi|x\langle\psi|p|\psi\rangle|\psi\rangle - \langle\psi|p\langle\psi|x|\psi\rangle|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\langle\psi|\psi\rangle$$

$$= \langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle$$

Complex numbers have the property

$$|z|^2 = (\Re z)^2 + (\Im z)^2 \geq (\Im z)^2 = \left(\frac{1}{2i}(z - z^*)\right)^2$$

Applying this to $z = \langle\psi|\tilde{x}\tilde{p}|\psi\rangle$,

$$|\langle\psi|\tilde{x}\tilde{p}|\psi\rangle|^2 \geq \left(\frac{1}{2i}\left[\left(\langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\right) - \left(\langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\right)^*\right]\right)^2$$

$$= \left(\frac{1}{2i}\left[\langle\psi|xp|\psi\rangle - \langle\psi|px|\psi\rangle\right]\right)^2 = \left(\frac{1}{2i}\langle\psi|[x,p]|\psi\rangle\right)^2 = \frac{1}{4} = \frac{\hbar^2}{4}$$

where in the last equality we have returned from natural units. The ground state of the simple harmonic oscillator is $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$, from which we can compute

$$\langle\psi_0|x|\psi_0\rangle = 0 \text{ (odd function, symmetric interval)}$$

$$\langle\psi_0|p|\psi_0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}\left(-i\hbar\frac{\partial}{\partial x}e^{-m\omega x^2/2\hbar}\right)dx = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}\left(im\omega x e^{-m\omega x^2/2\hbar}\right)dx$$

$$= 0 \text{ (odd function, symmetric interval)}$$

$$\Rightarrow \tilde{x} = x, \tilde{p} = p$$

$$\Rightarrow (\Delta x)^2 = \langle \psi_0 | \tilde{x}^2 | \psi_0 \rangle = \langle \psi_0 | x^2 | \psi_0 \rangle = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} x^2 e^{-m\omega x^2/2\hbar} dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} = \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} = \sqrt{\frac{\hbar^2}{4m^2\omega^2}} = \frac{\hbar}{2m\omega}$$

$$\Rightarrow (\Delta p)^2 = \langle \psi_0 | \tilde{p}^2 | \psi_0 \rangle = \langle \psi_0 | p^2 | \psi_0 \rangle = -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \frac{\partial^2}{\partial x^2} e^{-m\omega x^2/2\hbar} dx$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \left(\frac{1}{\hbar^2} m\omega e^{-m\omega x^2/2\hbar} (m\omega x^2 - \hbar) \right) dx$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \left(m^2\omega^2 \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx - \hbar m\omega \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} \right)$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \left(m^2\omega^2 \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} - \hbar m\omega \sqrt{\frac{\pi\hbar}{m\omega}} \right) = -(\hbar m\omega/2 - \hbar m\omega) = m\omega\frac{\hbar}{2}$$

where we have used a table for the nasty Gaussian-type integrals. Multiplying the results, we indeed confirm this is a minimum-uncertainty state:

$$(\Delta x)^2 (\Delta p)^2 = \frac{\hbar}{2m\omega} \left(m\omega \frac{\hbar}{2} \right) = \frac{\hbar^2}{4}$$

14

We have

$$\langle 0 | T x(t) x(t') | 0 \rangle = -\frac{\delta^2}{\delta J(t) \delta J(t')} Z(J) \Big|_{J=0}$$

where

$$Z(J) = \frac{\langle x't' | x, t \rangle^J}{\langle x' | 0 \rangle \langle 0 | x \rangle}$$