7590 HW 1

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I occasionally swap between physics and math conventions for complex conjugation, the inner product notation, which arguments of the inner product are sesquillinear/linear, etc. It's an unfortunate consequence of learning the notations simultaneously from both departments.

1a

By the definition of the L^2 inner product and A, for any functions $f, g \in D(A)$ we have

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

Integrating by parts,

$$\overline{f'}g\Big|_0^1 - \int_0^1 \overline{f'(x)}g'(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

$$\Leftrightarrow \overline{f'}g\Big|_0^1 - \overline{f}g'\Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

the evaluation terms must both be zero at 0 and 1 since smooth compactly-supported functions on open sets vanish in the limit to the boundary of their domains. Therefore, this operator is symmetric. However, not all elements of $D(A^\dagger)$ are elements of $D(A)$: $g \in H$ is an element of $D(A^\dagger)$ iff there exists $h \in H$ such that $\forall f \in D(A)$

$$\int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Applying the same integration-by-parts argument as above, we may equivalently write this as

$$\Leftrightarrow \overline{f'}g\Big|_0^1 - \overline{f}g'\Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Since f is compactly supported, f' is as well, so the evaluation terms are zero by the same argument given above. Letting $g = x^2$, we then have

$$\int_0^1 \overline{f(x)} \cdot 2dx = \int_0^1 \overline{f(x)}h(x)dx$$

from which we can clearly see the $L^2([0,1])$ function $h = 2$ is the element adjoint to g with respect to A. g is therefore in $D(A^\dagger)$. It isn't in $D(A)$ though, since x^2 doesn't vanish at 1 and therefore isn't compactly supported on this interval. This implies $D(A^\dagger) \neq D(A)$, so $A \neq A^\dagger$, i.e. A isn't self-adjoint.

1b

Proceeding similarly,

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 (if'(x))^* g(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

$$\Leftrightarrow -if^*g\Big|_0^1 + \int_0^1 i(f(x))^* g'(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

By the same argument as above, the evaluation term is zero, in which case the equality follows immediately. This operator is symmetric. Once again, x^2 is in $D^\dagger(A)$ but not $D(A)$: from the formula derived for $\langle Af|g\rangle$ in the proof A is symmetric, the definition of membership in $D^\dagger(A)$ is

$$\int_0^1 (f(x))^* 2x dx = \int_0^1 (f(x))^* h(x)dx$$

which, choosing $h = 2x \in L^2([0,1])$, clearly holds. $2x$ isn't compactly supported on $(0,1)$ since it doesn't vanish in the limit to 1, so $D(A^\dagger) \neq D(A)$ and A isn't self-adjoint.

1c

The definition of a symmetric operator is that $\forall f, g \in D(A)$

$$\langle Af|g\rangle = \langle f|Ag\rangle$$

which in this case is

$$\int_\Omega \overline{[\partial_i(a_{ij}(x)\partial_j f)(x)]} g(x)dx = \int_\Omega \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g)(x)dx$$

$$\Leftrightarrow g(x)\overline{a_{ij}(x)\partial_j f(x)}\Big|_{\partial\Omega} - \int_\Omega \overline{[a_{ij}(x)\partial_j f(x)]}\partial_i g(x)dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

$$\Leftrightarrow \overline{-a_{ij}(x)f(x)}\partial_i g(x)\Big|_{\partial\Omega} + \int_\Omega \overline{f(x)}\partial_j\left(\overline{a_{ij}(x)}\partial_i g(x)\right)dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

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$$\Leftrightarrow \int_\Omega \overline{f(x)}\partial_i(\overline{a_{ji}(x)}\partial_j g(x))dx = \int_\Omega \overline{f(x)}\partial_i(a_{ij}(x)\partial_j g)(x)dx$$

where we have throughout used integration by parts and the same fact that functions of compact support vanish in the limit to their boundaries. Since $a_{ij}(x)$ is Hermitian, it is equal to $\overline{a_{ji}(x)}$, and so the two sides are equal and the operator is symmetric. Here, A is a bounded operator:

$$||Af|| \leq C||f|| \Leftrightarrow \int_\Omega |\partial_i(a_{ij}(x)\partial_j f)(x)|^2 dx \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \int_\Omega \partial_i a_{ij}(x)\partial_j f(x)dx \int_\Omega \overline{\partial_i a_{ij}(x)\partial_j f(x)}dx \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \left(f(x)\partial_i a_{ij}(x)\Big|_{\partial\Omega} - \int_\Omega f(x)\partial_j\partial_i a_{ij}(x)\right)\left(\overline{f(x)\partial_i a_{ij}(x)}\Big|_{\partial\Omega} - \int_\Omega \overline{f(x)\partial_j\partial_i a_{ij}(x)}dx\right) \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \int_\Omega \left(f(x)\overline{f(x)}\right)\left([\partial_i\partial_j a_{ij}(x)][\overline{\partial_i\partial_j a_{ij}(x)}]\right)dx \leq C\int_\Omega |f(x)|^2 dx$$

$$\Leftrightarrow \left|\int_\Omega f(x)\partial_i\partial_j a_{ij}(x)dx\right|^2 \leq C\int_\Omega |f(x)|^2 dx$$

From the Cauchy-Schwartz inequality, we have

$$\left|\int_\Omega f(x)\partial_i\partial_j a_{ij}(x)dx\right|^2 \leq \int_\Omega |f(x)|^2 dx \int_\Omega |\overline{\partial_i\partial_j a_{ij}(x)}|^2 dx = C\int_\Omega |f(x)|^2 dx$$

This proves the operator is bounded. Therefore, $D(A^\dagger) = H$, and there are certainly $L^2(\Omega)$ functions that aren't C^∞, so $D(A^\dagger) \not\subseteq D(A)$ implying A is not self-adjoint.

2a

Applying the definition of the infinitesimal generator,

$$Af(x) = -i\lim_{t\to 0}[f(x+vt) - f(x)]/t = -i\frac{\partial f}{\partial v}$$

where the last equality is valid where the limit exists, using the definition of the partial derivative. Since $V \in C^1$, and the above implies $D(A)$ is L^2 functions differentiable along v, we have $V \subseteq D(A)$, with action given above.

2b

The adjoint of U is defined by

$$\langle U(t)f(x)|g(x)\rangle = \langle f(x)|\left(U(t)\right)^\dagger g(x)\rangle$$

The left hand side is, applying the definition of the L^2 inner product and U,

$$\langle U(t)f(x)|g(x)\rangle = \int_{\mathbb{R}^3} \overline{f(e^{-tB}x)}g(x)dx$$

We can make the substitution $y = e^{-tB}x$, in which case $x = e^{tB}y$. The component functions of this change of variables take the form

$$h_i = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!}\left[(B^n)_{i1}x_1 + (B^n)_{i2}x_2 + (B^n)_{i3}x_3\right]$$

The Jacobian of this change of coordinates is then

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} & \frac{\partial h_3}{\partial x_3} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \begin{pmatrix} (B^n)_{11} & (B^n)_{12} & (B^n)_{13} \\ (B^n)_{21} & (B^n)_{22} & (B^n)_{23} \\ (B^n)_{31} & (B^n)_{32} & (B^n)_{33} \end{pmatrix} = e^{-tB}$$

Using $\det e^A = e^{\operatorname{tr} A}$, the Jacobian determinant is

$$\det J = e^{-t \operatorname{tr}(B)} = 1$$

since the trace of skew-symmetric matrices is zero. We can now finally rewrite the integral as

$$\int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB}y) |\det J| dy = \int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB}y) dy = \langle f(x) | U^\dagger(t) g(x) \rangle$$

which identifies $U^\dagger(t) : g(x) \mapsto g(e^{tB}x)$. Clearly, this is unitary:

$$UU^\dagger f(x) = f(e^{-tB}e^{tB}x) = If(x) = f(e^{tB}e^{-tB}x) = UU^\dagger f(x)$$

For $f \in C^1(\mathbb{R}^3)$, the infinitesimal generator acts as

$$Af(x) = -i \lim_{t\to 0} [f(e^{-tB}x) - f(x)]/t$$

The numerator limits to zero, since $e^{-at} \sim 1$ as $t \to 0$. Applying L'Hôpital's rule,

$$= -i \lim_{t\to 0} \frac{\frac{d}{dt} f(e^{-tB}x)}{1} = -i \lim_{t\to 0} \left(\frac{d}{dt} e^{-tB} x \right) \cdot \nabla f(e^{-tB}x) = -i \lim_{t\to 0} \left(-Be^{-tB}x \right) \cdot \nabla f(e^{-tB}x)$$

$$= iBx\nabla f(x)$$

This shows that the limit exists for every $f \in C^1(\mathbb{R})$ (so $V \subseteq D(A)$) and gives its action.

2c

Notice first that the definition of n and r give the number s "modulo" 2π in the sense that if one divides the real number line into partitions by integer multiples of 2π, $n(s)$ gives the multiple of 2π corresponding to the rightmost partition boundary which lies to the left of s and $r(s)$ gives the rightward displacement of s from the partition boundary. If n were further left, adding $0 \le r(s) < 2\pi$ couldn't equal s, and if it were to the right of s, the same is true because $r(s)$ is positive. We prove first that U_α is a continuous symmetry.
Unitarity: It preserves the inner product

$$\langle Uf|Ug \rangle = \int_0^{2\pi} \overline{\alpha^{n(x+t)} f(r(x+t))} \alpha^{n(x+t)} g(r(x+t)) dx = \int_0^{2\pi} |\alpha^{n(x+t)}|^2 \overline{f(r(x+t))} g(r(x+t)) dx$$

Since $|\alpha| = 1$, we may write $\alpha = e^{i\theta}$ in which case it is immediately clear $|\alpha^{n(x+t)}|^2 = 1$. We now make the substitution $y = r(x+t)$, under which $dy = r'(x+t)dx \Leftrightarrow dx = \frac{dy}{r'(x+t)}$. Differentiating the definition of n and r with respect to s yields $1 = n'(s)2\pi + r'(s) = r'(s)$ since n is a step function. Strictly speaking, it is possible there is one point in the integration interval where this derivative is not defined, but since this is a set of measure zero it won't contribute to the integral. We therefore have, applying the substitution,

$$= \int_{r(t)}^{r(2\pi+t)} \overline{f(y)} g(y) dy = \int_0^{2\pi} \overline{f(y)} g(y) dy = \langle f|g \rangle$$

where we have used for the penultimate equality the fact that y (and therefore the entire integrand) has periodicity 2π, so the integral over any two intervals of length 2π will be the same. It also is surjective, as if given a function $h \in L^2([0, 2\pi])$, we may write since $\alpha = e^{i\theta} \Rightarrow f(x)\alpha^{g(x)} = f(x)e^{i\theta g(x)}$

$$h(x) = f(x)e^{ig'(x)} = f(x)\alpha^{g(x)}$$

where $f : [0, 2\pi] \to \mathbb{R}$ and $g : [0, 2\pi] \to [0, 2\pi]$. For any given h (which we may rewrite in the form above), α, and t, one may construct the function $k \in L^2([0, 2\pi])$ given by $k(x) = f(r(x - t))\alpha^{g(r(x-t))-n(r(x-t)+t)}$. Noting that for $x \in [0, 2\pi)$

$$r(r(x + t) - t) = r(x + t - 2\pi n(x + t) - t) = r(x - 2\pi n(x + t)) = x$$

since

$$r(x + 2k\pi) = x + 2k\pi - n(x + 2k\pi) = x + 2k\pi - 2k\pi = x$$

by the characterization of $n(s)$ above, we have

$$U_\alpha k = \alpha^{n(x+t)} f[r(r(x+t)-t)]\alpha^{g[r(r(x+t)-t)]-n[r(r(x+t)-t)+t]} = \alpha^{n(x+t)} f(x)\alpha^{g(x)-n(x+t)} = f(x)\alpha^{g(x)}$$

Therefore, U_α is surjective for every α and t, and in conjunction with the above result this proves U_α is unitary. $U_\alpha(0) = I$, using $x \in [0, 2\pi]$:

$$U_\alpha(0)f(x) = \alpha^{n(x)} f(r(x)) = \alpha^0 f(x) = f(x)$$

The operator behaves properly under addition in t:

$$U_\alpha(t + s)f(x) = \alpha^{n(x+t+s)} f(r(x + t + s)) = U_\alpha(s) \left(\alpha^{n+t} f(r(x + t))\right) = U_\alpha(s)U_\alpha(t)f(x)$$

Lastly, using $x \in [0, 2\pi)$,

$$\lim_{t\to 0} U_\alpha(t)x = \lim_{t\to 0} \alpha^{n(x+t)} r(x + t) = \alpha^{n(x)} r(x) = \alpha^0 x = x$$

This argument becomes slightly more subtle at $x = 2\pi$ due to the discontinuities in n and $r(x)$, but since the limit must be taken from inside $[0, 2\pi]$ one may use left-hand limits $n(2\pi^-) = 0$ and $r(2\pi^-) = 2\pi$ to obtain the same result above.
The infinitesimal generator of U_α is by definition

$$A_\alpha f = -i \lim_{t\to 0}[U_\alpha(t)f - f]/t = -i \lim_{t\to 0}[\alpha^{n(x+t)} f(r(x + t)) - f(x)]/t$$

$$= -i \lim_{t\to 0} \frac{\frac{d}{dt}\alpha^{n(x+t)} f(r(x + t))}{1} = -i \lim_{t\to 0} \frac{d}{dt}\alpha^0 f(x + t) = -if'(x)$$

where we have used L'Hôpital's rule and $x \in [0, 2\pi)$ (in which case the limit will become at some point exclusively through t close enough to x that $0 \leq x + t < 2\pi$, yielding $r(x + t) = x + t$ and $n(x + t) = 0$). The special case $x = 2\pi$ is much the same, only the limit is taken inside $[0, 2\pi]$ so only the left-hand limit is taken, which coincides with the result above. Functions in the given V_α make the limit above exist since they are C^1, implying $f'(x)$ is well-defined. Therefore, $V_\alpha \in D(A_\alpha)$ and the action is as given above.

3a

The given $D(A)$ is a subset of the V_α given in problem 2c, since C^1 compactly-supported functions on $(0, 2\pi)$ are a subset of $C^1([0, 2\pi])$ functions that vanish in the limits to 0 and 2π, which are a subset of $C^1([0, 2\pi])$ functions where $f(2\pi) = f(0) = 0$, which are a subset of $C^1([0, 2\pi])$ functions where $f(2\pi) = \alpha f(0)$. Since $V_\alpha \subseteq D(A_\alpha)$ was proven in 2c, we have $D(A) \subseteq D(A_\alpha)$. Further, $A = A_\alpha$ on $D(A)$ is immediate from the symbolic expression of each being identical. This proves $A \subset A_\alpha$.

3b

$A \subset A_\alpha$ implies that $e^{-itA_\alpha} = e^{-itA}$ on $D(A)$. The theorem gives the result that

$$u(x,t) = e^{-itA_\alpha} u_0 = U_\alpha(t) u_0(x)$$

is the unique solution to $\dot{u}(t) = iAu(t)$ where A is the infinitesimal generator of $U(t)$. Applying this, the result follows immediately:

$$u(x,t) = U(t)u_0(x) = U_1(t)u_0(x) = 1^{n(x+t)} u_0(r(x+t)) = u_0(x+t)$$

4a

We have, choosing a basis for which A is diagonal (which exists by positive-definiteness)

$$Z = \int_{\mathbb{R}} e^{-\langle Ax,x\rangle/2} dx = \int_{\mathbb{R}} e^{-\sum_{i=1}^n \lambda_i x_i^2/2} dx = \int_{\mathbb{R}} \prod_{i=1}^n e^{-\lambda_i x_i^2/2} dx = \prod_{i=1}^n \int_{\mathbb{R}^n} e^{-\lambda_i x_i^2/2} dx_i$$

$$= \prod_{i=1}^n \sqrt{\frac{2\pi}{\lambda_i}} = \sqrt{\frac{2\pi}{\det A}}$$

4b

Noting that since A (and therefore G) are Hermitian,

$$\langle p, Gp\rangle/2 - \langle x - Gp, A(x - Gp)/2\rangle = \langle p, Gp\rangle/2 - \langle x, Ax - p\rangle/2 + \langle Gp, Ax - p\rangle/2$$

$$= \langle p, Gp\rangle/2 - \langle x, Ax\rangle/2 + \langle x, p\rangle/2 + \langle Gp, Ax\rangle/2 - \langle Gp, p\rangle/2$$

$$= \langle p, Gp\rangle/2 - \langle x, Ax\rangle/2 + \langle x, p\rangle - \langle Gp, p\rangle/2$$

$$= \langle p, x\rangle - \langle Ax, x\rangle/2$$

We can then write

$$\int_{\mathbb{R}^n} e^{\langle p,x\rangle} d\mu(x) = \int_{\mathbb{R}^n} e^{\langle p,x\rangle} \frac{1}{Z} e^{-\langle Ax,x\rangle/2} dx = \frac{1}{Z} \int_{\mathbb{R}^n} e^{\langle p,x\rangle - \langle Ax,x\rangle/2} dx$$

as

$$\frac{1}{Z}\int_{\mathbb{R}^n} e^{\langle p,Gp\rangle/2-\langle x-Gp,A(x-Gp)/2\rangle}dx$$

Making the substitution $y = x - Gp$, $dy = dx$ we have

$$= \frac{1}{Z}\int_{\mathbb{R}^n} e^{\langle Gp,p\rangle/2-\langle y,Ay\rangle/2}dy = e^{\langle Gp,p\rangle/2}\int_{\mathbb{R}^n}\frac{1}{Z}e^{-\langle Ay,y\rangle/2}dy$$

The integral is one by construction, proving the identity.

4c

We find a similar identity:

$$-\langle p,Gp\rangle/2 - \langle x-iGp,A(x-iGp)/2\rangle = -\langle p,Gp\rangle/2 - \langle x,Ax-ip\rangle/2 + \langle iGp,Ax-ip\rangle/2$$

$$= -\langle p,Gp\rangle/2 - \langle x,Ax\rangle/2 + \langle x,ip\rangle/2 + \langle iGp,Ax\rangle/2 - \langle iGp,ip\rangle/2$$

$$= -\langle p,Gp\rangle/2 - \langle x,Ax\rangle/2 + i\langle x,p\rangle + \langle Gp,p\rangle/2$$

$$= -i\langle p,x\rangle - \langle Ax,x\rangle/2$$

As before, this yields

$$\int_{\mathbb{R}^n} e^{i\langle p,x\rangle}d\mu(x) = \int_{\mathbb{R}^n}\frac{1}{Z}e^{-\langle p,x\rangle-\langle Ax,x\rangle/2}dx = \int_{\mathbb{R}^n}\frac{1}{Z}e^{-\langle p,Gp\rangle/2}e^{-\langle x-iGp,A(x-iGp)\rangle/2}dx$$

$$= e^{-\langle p,Gp\rangle}\int_{\mathbb{R}^n}\frac{1}{Z}e^{-\langle y,Ay\rangle/2}dx = e^{-\langle p,Gp\rangle}$$

4d

We want to compute

$$\int_{\mathbb{R}^n}\langle v,\nabla f\rangle\frac{1}{Z}e^{-\langle Ax,x\rangle/2}dx$$

There is an integration-by-parts rule that follows from the product rule for divergence the same as in one dimension:

$$\int_\Omega f(x)\nabla\cdot\vec{V}(x)dx = \int_{\partial\Omega} f(x)\vec{V}\cdot\hat{n}dx' - \int_\Omega \nabla f(x)\cdot\vec{V}dx$$

We may take $\vec{V} = \frac{v}{Z}e^{-\langle Ax,x\rangle/2}$ and note our integral is the last term; if f is "nice" enough so the limit of the boundary integral towards infinity vanishes, we have

$$\int_{\mathbb{R}^n}\langle v,\nabla f\rangle\frac{1}{Z}e^{-\langle Ax,x\rangle/2}dx = \frac{1}{Z}\int_{\mathbb{R}^n} f(x)\nabla\cdot\left(ve^{-\langle Ax,x\rangle/2}\right)dx$$

Since we may write $e^{-\langle Ax,x\rangle/2} = ve^{-\sum_i\lambda_i x_i^2/2}$,

$$\nabla\cdot ve^{-\langle Ax,x\rangle/2} = \sum_i -\lambda_i x_i v_i e^{-\sum_i\lambda_i x_i^2/2} = -\langle Ax,v\rangle e^{-\langle Ax,x\rangle/2}$$

yielding

$$\int_{\mathbb{R}^n} \langle v, \nabla f\rangle d\mu(x) = \int_{\mathbb{R}^n} f(x)\langle Ax, v\rangle d\mu(x)$$

Under the L^2 inner product,

$$\langle \partial_v f, g\rangle = \int_{\mathbb{R}^n} [\partial_v f(x)] g(x) d\mu(x) = \int_{\mathbb{R}^n} \langle v, g(x)\nabla f\rangle d\mu(x) = \int_{\mathbb{R}^n} \langle v, \nabla[f(x)g(x)]\rangle d\mu(x) - \int_{\mathbb{R}^n} \langle v, f(x)\nabla g\rangle d\mu(x)$$

By the result above,

$$= \int_{\mathbb{R}^n} f(x)g(x)\langle Ax, v\rangle d\mu(x) - \int_{\mathbb{R}^n} f(x)\langle v, \nabla g\rangle d\mu(x)$$

$$= -\int_{\mathbb{R}^n} f(x)[\partial_v g(x)] d\mu(x) + \int_{\mathbb{R}^n} f(x)[g(x)\langle Ax, v\rangle] d\mu(x) = \langle f, -\partial_v g + g\langle A(\cdot), v\rangle\rangle$$

Since A is positive-definite and therefore self-adjoint, and the real inner product is symmetric, this identifies $\partial^\dagger[f(x)] = -\partial_v f(x) + f(x)\langle Av, x\rangle$

4e

$$\int_{\mathbb{R}^n} \langle v, x\rangle\langle w, x\rangle d\mu(x) = \int_{\mathbb{R}^n} \left(\sum_{i=1}^n v_i x_i\right)\left(\sum_{k=1}^n w_k x_k\right) \frac{1}{Z} e^{\sum_{j=1}^n -\lambda_j x_j^2/2} dx = \sum_{i=1}^n \sum_{k=1}^n \int_{\mathbb{R}^n} v_i w_k x_i x_k \frac{1}{Z} e^{\sum_{j=1}^n -\lambda_j x_j^2/2} dx$$

$$= \sum_{i=1}^n \sum_{k=1}^n \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_i w_k x_k x_i e^{-(\lambda_i x_i^2 + \lambda_k x_k^2)/2} dx_i dx_k \left(\int_{\mathbb{R}^{n-2}} \frac{1}{Z} e^{\sum_{j\neq i,k} \lambda_j x_j^2/2}\right)$$

When $i \neq k$, the left integrand is odd, implying the term will be zero. We may therefore write

$$= \sum_{i=1}^n \int_{\mathbb{R}^n} v_i w_i x_i^2 \frac{1}{Z} e^{-\langle Ax, x\rangle/2} dx$$

$$= \sum_{i=1}^n \frac{1}{Z} \prod_{k=i}^n \int_{-\infty}^{\infty} v_i w_i x_i^2 e^{-\lambda_k x_k^2/2} dx_k$$

For each i, there is one $k = i$; we may reorder the product until the $k = i$ integral is the innermost. This integral is then equal to $v_i w_i \sqrt{\frac{2\pi}{\lambda_i^3}}$, and the subsequent integrals each contribute $\sqrt{\frac{2\pi}{\lambda_k}}$; each multiplicand is then equal to $v_i w_i \frac{1}{\lambda_i}\sqrt{\frac{2\pi}{\det A}} = v_i w_i \frac{1}{\lambda_i} Z$. We then have

$$= \sum_i v_i w_i/\lambda_i = \langle Gv, w\rangle$$

4f

$$\int_{\mathbb{R}^n} e^{\langle p,x\rangle} d\mu(x) = \sum_{i=0}^{\infty} \frac{1}{i!} \int_{\mathbb{R}^n} \langle p,x\rangle^i d\mu(x)$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \langle Gp,p\rangle/2$$

$$\int_{\mathbb{R}^m} \left(\sum_{i_1=1}^{m} (v_1)_{i_1} x_{i_1}\right) \cdots \left(\sum_{i_{2n}=1}^{m} (v_{2n})_{i_1} x_{i_{2n}}\right) d\mu(x)$$

$$= \sum_{i_1=1}^{m} \cdots \sum_{i_{2n}=1}^{m} \int_{\mathbb{R}^m} (v_1)_j x_{i_1} (v_2)_j x_{i_2} \cdots (v_{2n})_j x_{i_{2n}} d\mu(x)$$

$$= \sum \langle v_{i_1}, v_{j_1}\rangle \langle v_{i_2}, v_{j_2}\rangle \cdots \langle v_{i_n}, v_{j_n}\rangle \int_{\mathbb{R}^m} \sum_{i_1=1}^{m} \cdots \sum_{i_{2n}=1}^{m} x_{i^1} \cdots x_{i^{2n}} d\mu(x)$$

$$= \sum \langle v_{i_1}, v_{j_1}\rangle \langle v_{i_2}, v_{j_2}\rangle \cdots \langle v_{i_n}, v_{j_n}\rangle \int_{\mathbb{R}^m} \langle x,x\rangle^n d\mu(x)$$

$$= \sum \langle v_{i_1}, v_{j_1}\rangle \langle v_{i_2}, v_{j_2}\rangle \cdots \langle v_{i_n}, v_{j_n}\rangle \int_{\mathbb{R}^n}$$

4g

If $F(x) = \sum_{l=1}^{a} \beta_l e^{i \sum_k \alpha_k \langle v_k, x\rangle}$ and $H(x) = \sum_{j=1}^{b} \gamma_j e^{i \sum_m \epsilon_m \langle w_m, x\rangle}$, then

$$\int_{\mathbb{R}^n} F(x)H(x) d\mu(x) = \sum_{l=1}^{a} \sum_{j=1}^{b} \beta_l \gamma_j \int_{\mathbb{R}^n} e^{i \sum_{k,m=0}^{a,b} \alpha_k \langle v_k, x\rangle + \epsilon_m \langle w_m, x\rangle} d\mu(x)$$

Using

$$\sum_{k,m} \alpha_k \langle v_k x\rangle + \epsilon_m \langle w_m, x\rangle = \sum_{k,m} \langle \alpha_k v_k + \epsilon_m w_m, x\rangle = \langle \sum_{k,m} \alpha_k v_k + \epsilon_m w_m, x\rangle$$

and part c,

$$= \sum_l \sum_j \beta_l \gamma_j e^{-\langle G \sum_{k,m} \alpha_k v_k + \epsilon_m w_m, \sum_{k,m} \alpha_k v_k + \epsilon_m w_m\rangle/2}$$

$$= \sum_l \sum_j \beta_l \gamma_j e^{-\langle \sum_{k,m} \alpha_k G v_k + \epsilon_m G w_m, \sum_{p,q} \alpha_p v_p + \epsilon_q w_q\rangle/2}$$

$$= \sum_l \sum_j \beta_l \gamma_j e^{-(\sum_{k,m} \sum_{p,q} \alpha_k \alpha_p \langle G v_k, v_p\rangle + \epsilon_m \alpha_p \langle G w_m, v_p\rangle + \epsilon_m \epsilon_q \langle G w_m, w_q\rangle)/2}$$

By the given orthogonality data,

$$= \sum_l \sum_j \beta_l \gamma_j e^{-(\sum_{k,m} \sum_{p,q} \alpha_k \alpha_p \langle G v_k, v_p\rangle + \epsilon_m \epsilon_q \langle G w_m, w_q\rangle)/2}$$

Similarly,

$$\int_{\mathbb{R}^n} F(x)d\mu(x)\int_{\mathbb{R}^n} H(x)d\mu(x) = \int_{\mathbb{R}^n}\sum_l \beta_l e^{i\sum_k \alpha_k\langle v_k,x\rangle}d\mu(x)\int_{\mathbb{R}^n}\sum_j \gamma_j e^{i\sum_m \epsilon_m\langle w_m,x\rangle}$$

$$= \sum_l\sum_j \beta_l e^{-\langle G\sum_k \alpha_k v_k,\sum_p \alpha_p v_p\rangle/2}\gamma_j e^{-\langle G\sum_m \epsilon_m w_m,\sum_q \epsilon_q w_q\rangle/2}$$

$$= \sum_l\sum_j \beta_l\gamma_j e^{-\sum_{km}\sum_{pq}\alpha_k\alpha_p\langle Gv_k,v_p\rangle/2-\epsilon_m\epsilon_q\langle Gw_m,w_q\rangle/2}$$

Therefore, these expressions are equal.

5

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^n} f(x+t^{1/2}y)d\mu(y) = \int_{\mathbb{R}^n}\frac{1}{2}\frac{y}{\sqrt{t}}\cdot\nabla f d\mu(y)$$

Integrating by parts,

$$= \int_{\mathbb{R}^n} f(x+t^{1/2}y)\nabla\cdot\left(\frac{1}{2}\frac{y}{\sqrt{t}}\frac{1}{Z}e^{-\langle Ay,y\rangle/2}\right)dy = \frac{1}{2Z\sqrt{t}}\int_{\mathbb{R}^n}\sum_{i=1}^n f(x+t^{1/2}y)\frac{\partial}{\partial y_i}y_i e^{-\sum_{j=1}^n\lambda_j y_j^2/2}dy$$

$$= \frac{1}{2Z\sqrt{t}}\sum_{i=1}^n\int_{\mathbb{R}^n} f(x+t^{1/2}y)\left(e^{-\sum_{j=1}^n\lambda_j y_j^2/2}-\lambda_i y_i^2 e^{-\sum_{j=1}^n\lambda_j y_j^2/2}\right)dy$$

$$= \frac{1}{2\sqrt{t}}\left(\int_{\mathbb{R}^n} f(x+t^{1/2}y)d\mu(y)-\int_{\mathbb{R}^n} f(x+t^{1/2}y)\langle Ay,y\rangle d\mu(y)\right)$$

We also have, choosing a basis in which G_{ij} is diagonal as we've been doing the whole section,

$$L\int_{\mathbb{R}^n} f(x+t^{1/2}y)d\mu(y) = \frac{1}{2}\sum_{i=1}^n\frac{1}{\lambda_i}\partial_i^2\int_{\mathbb{R}^n} f(x+t^{1/2}y)d\mu(y)$$

$$= \frac{1}{2}\sum_{i=1}^n\frac{1}{\lambda_i}\int_{\mathbb{R}^n}\frac{\partial^2}{\partial x_i^2}f(x+t^{1/2}y)\frac{1}{Z}e^{-\sum_{j=1}^n\lambda_j y_j^2}dy = \frac{1}{2}\sum_{i=1}^n\frac{1}{\lambda_i}\int_{\mathbb{R}^n} f_{x_ix_i}(x+t^{1/2}y)\frac{1}{Z}e^{-\sum_{j=1}^n\lambda_j y_j^2}dy$$

I presume from this point there will be some change of variables (with a Jacobian that gives the factor of $\frac{1}{\sqrt{t}}$) that allows one to integrate out the partial derivatives by parts, which will pull down $\lambda_i^2 y_i^2$ from the exponent and recover the expression above.

6a

$$e^{-L}p(x) = \int_{\mathbb{R}^n} p(x+iy)d\mu(y) = \int_{\mathbb{R}^n}\sum_{j=1}^m a_j(x+iy)^j d\mu(y)$$

$$= \sum_{j=1}^m a_j\int_{\mathbb{R}^n}(x+iy)^j d\mu(y) = \sum_{j=1}^m a_j\int_{\mathbb{R}^n}\sum_{k=0}^j\binom{j}{k}x^k(iy)^{j-k}d\mu(y)$$

$$= \sum_{j=1}^{m} \sum_{k=0}^{j} a_j \binom{j}{k} x^k (i)^{j-k} \int_{\mathbb{R}^n} y^{j-k} d\mu(y)$$

$$= \sum_{j=1}^{m} \sum_{k=0}^{j} a_j \binom{j}{k} x^k (i)^{j-k} \frac{1}{Z} \prod_{p=1}^{n} \int_{\mathbb{R}^n} y_p^{j-k} e^{-\lambda_p x_p^2/2} dx_p$$

For $j-k$ odd, the integral is zero (meaning the imaginary unit disappears, which is good). The integral certainly exists for each y^{j-k}, so we're done in principle: this formula gives a well-defined function $f : \mathbb{R}^n \to \mathbb{R}$ for arbitrary a_j determining any given polynomial. For completeness, a CAS computation of the integral yields

$$= \sum_{j=1}^{m} \sum_{k=0}^{j} (j-k+1 \mod 2) a_j \binom{j}{k} x^k (-1)^{j-k} \frac{1}{Z} \prod_{p=1}^{n} \left(\frac{\lambda_p}{2}\right)^{-(j-k+1)/2} \Gamma([j-k+1]/2)$$

$$= \sum_{j=1}^{m} \sum_{k=0}^{j} (j-k+1 \mod 2) a_j \binom{j}{k} x^k (-1)^{j-k} \frac{(\det A)^{-(j-k+1)/2}}{Z} \left(\frac{\Gamma([j-k+1]/2)}{2}\right)^{-(j-k+1)/2}$$

For the exponential, we may employ the analytic continuation from 4c:

$$: e^{\langle p,x\rangle} := \int_{\mathbb{R}^n} e^{\langle p,x+iy\rangle} d\mu(y) = e^{\langle p,x\rangle} \int_{\mathbb{R}^n} e^{i\langle p,y\rangle} d\mu(x) = e^{\langle p,x\rangle - \langle Gp,p\rangle/2}$$

As this is a function $f : \mathbb{R}^n \to \mathbb{R}$, the normal ordering exists and is equal to the above.

6b

$$\int_{\mathbb{R}^m} \langle v_1, x+iy\rangle \cdots \langle v_m, x+iy\rangle$$

$$\langle v_1, x\rangle \cdots \langle v_m, x\rangle = \prod_{i=1}^{m} \sum_{j=1}^{m} (v_i)_j x_j$$

$$\Rightarrow e^{-L} \langle v_1, x\rangle \cdots \langle v_m, x\rangle = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \left(\sum_{l=1}^{m} \frac{1}{\lambda_l} \partial_l^2\right)^k \prod_{i=1}^{m} \sum_{j=1}^{m} (v_i)_j x_j$$

For $k > 0$, this is zero, since a differential operator with degree two acting on x_j will always yield zero, so

$$= \prod_{i=1}^{m} \sum_{j=1}^{m} (v_i)_j x_j = \langle v_1, x\rangle \cdots \langle v_m, x\rangle$$

We have

$$(: e^{\alpha_i \langle v_i,x\rangle} :)_i = \int_{\mathbb{R}^n} e^{\alpha_i \langle v_i, x+iy\rangle} d\mu(y) = e^{\alpha_i \langle v_i,x\rangle} \int_{\mathbb{R}^n} e^{i\langle \alpha_i v_i, y\rangle} d\mu(y)$$

$$= e^{\langle \alpha_i v_i, x\rangle} e^{-\langle G\alpha_i v_i, \alpha_i v_i\rangle} = e^{\alpha_i \langle v_i,x\rangle - \alpha_i^2 \langle G v_i, v_i\rangle} = e^{\alpha_i \sum_{k=1}^{m} v_k x_k - \alpha_i^2 \sum_{j=1}^{m} v_j^2/\lambda_j}$$

$$\Rightarrow \partial_{\alpha_1} \cdots \partial_{\alpha_m} : e^{\alpha_i \langle v_i, x\rangle} := \prod_{i=1}^{m} \left(\sum_{k=1}^{m} v_k x_k - 2\alpha_i \sum_{j=1}^{m} v_j^2 \right) e^{\alpha_i \sum_{k=1}^m v_k x_k - \alpha_i^2 \sum_{j=1}^m v_j^2/\lambda_j}$$

Taking $\alpha_i = 0$,

$$= \prod_{i=1}^{m} \sum_{k=1}^{m} v_k x_k$$

It should be noted I am somewhat suspicious of this result as what the problem means by α isn't fully clear.

6c

Writing $f = e^L g$,

$$\partial_v : e^L g := \partial_v e^{-L} e^L g = \partial_v g = \langle v, \nabla g\rangle$$

$$: \partial_v e^L g := e^{-L} \partial_v e^L g = e^L \langle v, \nabla e^{-L} g\rangle = \langle v, \nabla e^L e^{-L} g\rangle = \langle v, \nabla g\rangle$$

where we have used commutativity of partial derivatives of analytic functions for the penultimate step.

6d

For clarity, we write

$$[e^{tL}, M_{\langle v,\cdot\rangle}] f(x) = \left(e^{tL} M_{\langle v,x\rangle} - M_{\langle v,x\rangle} e^{tL}\right) f(x) = e^{tL} \langle v, x\rangle f(x) - \langle v, x\rangle e^{tL} f(x)$$

Differentiating,

$$\frac{d}{dt}[e^{tL}, M_{\langle v,\cdot\rangle}] f(x) = L e^{tL} \langle v, x\rangle f(x) - \langle v, x\rangle L e^{tL} f(x) = e^{tL} \langle v, x\rangle L f(x) - \langle v, x\rangle e^{tL} L f(x) + e^{tL} |v| f(x)$$

$$= \left([e^{tL}, M_{\langle v,\cdot\rangle}] L + e^{tL}|v|\right) f(x)$$

$$\frac{d^2}{dt^2} = \left[e^{tL}, M_{\langle v,\cdot\rangle}\right] L^2 + 2L e^{tL} |v| f(x)$$

and

$$\frac{d}{dt} t e^{tL} \partial_{Gv} = tL e^{tL} \partial_{Gv} + e^{tL} \partial_{Gv}$$

$$\frac{d^2}{dt^2} t e^{tL} \partial_{Gv} = tL^2 e^{tL} \partial_{Gv} + 2L e^{tL} \partial_{Gv}$$

Both equations satisfy

$$\frac{d^2 O}{dt^2} = \left(2\frac{dO}{dt} - O\right) L$$

and are equal to zero at $t = 0$. Therefore, by uniqueness of ODE solutions, they are equal.

6e

We have by the last part

$$[e^{tL}, M_{\langle v,\cdot\rangle}]f(x) = e^{tL}\langle v,x\rangle f(x) - \langle v,x\rangle e^{tL}f(x) \Rightarrow e^{tL}\langle v,x\rangle f(x) = \langle v,x\rangle e^{tL}f(x) + te^{tL}\partial_{Gv}f(x)$$

Taking $t=-1$,

$$:\langle v,x\rangle f(x) := \langle v,x\rangle : f : - : \partial_{Gv}f(x) :$$

By part c,

$$:\langle v,x\rangle f(x) := \langle v,x\rangle : f : -\partial_{Gv} : f :$$

as desired.

6f

I suspect another typo here. This seems like a robust result to me, but it's wildly inconsistent with the rest of this problem, not to mention it making zero intuitive sense.

$$: f :=: \prod_{i=1}^{n+1}\langle v_i,x\rangle := \prod_{i=1}^{n+1}\partial^\dagger_{Gv_i} = \prod_{i=1}^{n+1} -\langle Gv_i, \nabla 1\rangle + \langle v_i,x\rangle = \prod_{i=1}^{n+1}\langle v_i,x\rangle$$

7a

$$\begin{aligned}
[a(v), a^\dagger(w)]f(x) &= \partial_{Gv}\partial^\dagger_{Gw}f(x) - \partial^\dagger_{Gw}\partial_{Gv}f(x) \\
&= \partial_{Gv}\left(-\partial_{Gw}f(x) + f(x)\langle AGw,x\rangle\right) + \partial_{Gw}[\partial_{Gv}f(x)] - [\partial_{Gv}f(x)]\langle AGw,x\rangle \\
&= -\partial_{Gv}\partial_{Gw}f(x) + \partial_{Gv}[f(x)\langle w,x\rangle] + \partial_{Gw}\partial_{Gv}f(x) - [\partial_{Gv}f(x)]\langle w,x\rangle \\
&= -\langle Gv, \nabla\langle Gw,\nabla f\rangle\rangle + \langle Gv, \nabla[f(x)\langle w,x\rangle]\rangle + \langle Gw, \nabla\langle Gv,\nabla f\rangle\rangle - \langle Gv,\nabla f(x)\rangle\langle w,x\rangle
\end{aligned}$$

Noting that $\nabla\langle v,\nabla f(x)\rangle = \nabla\sum_{i=1}^n v_i\frac{\partial f}{\partial x_i} = \sum_{i=1}^n v_i\frac{\partial^2 f}{\partial x_i^2}\hat{x}_i = \langle v,\nabla^2 f\rangle$,

$$= -\langle Gv, \langle Gw,\nabla^2 f\rangle\rangle + \langle Gv, \langle w,x\rangle\nabla f + f(x)\langle w,\nabla x\rangle\rangle + \langle Gw,\langle Gv,\nabla^2 f\rangle\rangle - \langle Gv,\nabla f(x)\rangle\langle w,x\rangle$$

$$\begin{aligned}
&= -\langle Gw, \langle Gv,\nabla^2 f\rangle\rangle + \langle Gv, \langle w,x\rangle\nabla f\rangle + \langle Gv, f(x)w\rangle + \langle Gw,\langle Gv,\nabla^2 f\rangle\rangle - \langle Gv,\nabla f(x)\rangle\langle w,x\rangle \\
&= \langle Gv, f(x)w\rangle \\
&= f(x)\langle Gv, w\rangle
\end{aligned}$$

as desired.

7b

We can obtain a characterization of $[A^n, B]$ as follows: presuming an induction hypothesis $[A^{n-1}, B] = (n-1)A^{n-2}C$, the commutator formula $[AB, C] = A[B, C] + [A, C]B$ gives

$$[A^n, B] = A[A^{n-1}, B]+[A, B]A^{n-1} = A(n-1)A^{n-2}C+CA^{n-1} = (n-1)A^{n-1}C+A^{n-1}C = nA^{n-1}C$$

This implies

$$\begin{aligned} e^{\lambda A}B = B + \lambda AB + \lambda^2 A^2 B/2 + ... &= B + \lambda\left(BA + [A, B]\right) + \lambda^2(BA^2 + [A^2, B])/2 + ... \\ &= B + \lambda\left(BA + C\right) + \lambda^2\left(BA^2 + 2CA\right)/2 + ... \\ &= Be^{\lambda A} + \left(\lambda C + \lambda^2 CA + \lambda^3 CA^2/2 + ...\right) \\ &= (Be^{\lambda A} + \lambda Ce^{\lambda A}) = (B + \lambda C)e^{\lambda A} \end{aligned}$$

We then compute the derivatives

$$\begin{aligned} \frac{d}{d\lambda}e^{\lambda(A+B)} = \frac{d}{d\lambda}\left(1 + \lambda(A + B) + \frac{\lambda^2}{2}(A + B)^2 + ...\right) &= (A + B) + \lambda(A + B)^2 + \frac{\lambda^2}{2}(A + B)^3 + ... \\ &= (A + B)e^{\lambda(A+B)} \end{aligned}$$

and

$$\begin{aligned} \frac{d}{d\lambda}(e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}) &= Ae^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}Be^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}e^{\lambda B}\left(-\lambda Ce^{-\lambda^2 C/2}\right) \\ &= e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}\left(A + B + \lambda C - \lambda C\right) = (A + B)e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2} \end{aligned}$$

Since these two functions of λ satisfy the same differential equation and have the same value at $\lambda = 0$, they must be the same function, so for $\lambda = 1$ we obtain the desired result.

7c

Notice that by part a $[a(v), a(v)^\dagger]$ commutes with both a and $a^\dagger$ since it's a scalar. We then have

$$M_f = e^{\phi(v)} = e^{a(v)+a^\dagger(v)} = e^{a^\dagger(v)+a(v)} = e^{a^\dagger(v)}e^{a(v)}e^{-[a^\dagger(v),a(v)]/2} = e^{\langle Gv,v\rangle/2}e^{a^\dagger(v)}e^{a(v)}$$

The normal ordering $: M_f :$ is $e^{a^\dagger(v)}e^{a(v)}$, so

$$M_f / : M_f := e^{\langle Gv,v\rangle/2} \Rightarrow : M_f := e^{-\langle Gv,v\rangle/2}M_f$$

We also have from a previous problem the analytic normal ordering

$$: e^{\langle v,x\rangle} := e^{\langle p,x\rangle - \langle Gv,v\rangle/2}$$

which indeed confirms $: M_f := M_{:f:}$ in this case. If all f are sufficiently "nice," they may be represented by a $\mathbb{C}$-linear combination of exponential functions of this form. This result therefore generalizes, since $: M_f + M_g :=: M_f : + : M_g$ and $: \alpha M_f := \alpha : M_f$, implying $: \sum_i \alpha_i M_f := \sum_i \alpha_i : M_f :$ for $: M_f :$ of the form above.

11a

$$\det\Lambda = \begin{vmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{vmatrix} = \gamma^2 - \gamma^2\beta^2$$

$$= \gamma^2(1-\beta^2) = (1-\beta^2)^{-1}(1-\beta^2) = 1$$

so this is a proper Lorentz transformation. The proposed inverse satisfies

$$\Lambda^{-1}\Lambda = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

and

$$\Lambda\Lambda^{-1} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I$$

so it is, in fact, the inverse. The action of the Lorentz transformation on the unit vectors along coordinate axes yields

$$\hat{x}' = \gamma\hat{x}$$

$$\hat{y}' = \hat{y}$$

$$\hat{z}' = \hat{z}$$

and so the new axes are parallel to the old. The origin of the primed system is at $\vec{x'} = 0$; this occurs at $y = 0$ and $z = 0$ trivially, but the x variable has

$$-\gamma\beta t + \gamma x = 0 \Leftrightarrow x = \beta t$$

which is exactly the origin moving along the positive x axis with velocity β.

11b

Taking the events to happen at the origin of the primed system,

$$\Lambda^{-1}\vec{x'}_2 - \Lambda^{-1}\vec{x'}_1 = \begin{pmatrix} \gamma t'_2 \\ \gamma\beta t'_2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma t_1 \\ \gamma\beta t'_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(t'_2 - t'_1) \\ \gamma\beta(t'_1 - t'_2) \\ 0 \\ 0 \end{pmatrix}$$

The first component is γT; since this is the computation of the difference between the two events in the unprimed frame, this proves the time dilation formula.

11c

A similar computation to the above result applies:

$$\Lambda^{-1}\vec{x'_2} - \Lambda^{-1}\vec{x'_1} = \begin{pmatrix} \gamma\beta x'_2 \\ \gamma x'_2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma\beta x'_1 \\ \gamma x'_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\beta L \\ \gamma L \\ 0 \\ 0 \end{pmatrix}$$

Reading off the first component confirms the result. Whichever occurs first along the x axis occurs first in time, since L is a distance and therefore positive. For spacelike separated events,

$$\Lambda\vec{x_2} - \Lambda\vec{x_1} = \begin{pmatrix} \gamma t_2 - \gamma\beta x_2 - (\gamma t_1 - \gamma\beta x_1) \\ \gamma x_2 - \gamma\beta t_2 - (\gamma x_1 - \gamma\beta t_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T - \gamma\beta L \\ \gamma L - \gamma\beta T \\ 0 \\ 0 \end{pmatrix}$$

For the time component to be zero,

$$\gamma T = \gamma\beta L \Leftrightarrow \beta = \frac{T}{L}$$

The spacelike condition ensures conformance with $|\beta| < 1$:

$$T^2 - L^2 < 0 \Leftrightarrow T^2 < L^2 \Leftrightarrow |T| < |L| \Leftrightarrow \frac{|T|}{|L|} < 1 \Rightarrow |\beta| < 1$$

11d

Once again,

$$\Lambda^{-1}\vec{x'_2} - \Lambda^{-1}\vec{x'_1} = \begin{pmatrix} \gamma t_2 + \gamma\beta x_2 - (\gamma t_1 + \gamma\beta x_1) \\ \gamma\beta t_2 + \gamma x_2 - (\gamma\beta t_1 + \gamma x_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T + \gamma\beta L \\ \gamma\beta T + \gamma L \\ 0 \\ 0 \end{pmatrix}$$

The measurements are simultaneous in the unprimed frame, i.e. the first component of the above displacement is zero, yielding $T = -\beta L$. Plugging this in to the second component, one obtains

$$\gamma L - \gamma\beta^2 L = L\gamma[1 - \beta^2] = L\gamma(1/\gamma^2) = L/\gamma$$

as desired.

12

For convenience of notation, define $f : SL(2,\mathbb{C}) \to SO^+(1,3) :: A \mapsto (X \mapsto AXA^\dagger)$ where X is of the form given in the problem. This map is indeed a homomorphism: using $(AB)^\dagger = B^\dagger A^\dagger$,

$$f(A)f(B) = (X \mapsto AXA^\dagger) \circ (X \mapsto BXB^\dagger) = X \mapsto A(BXB^\dagger)A^\dagger = (AB)x(B^\dagger A^\dagger) = f(AB)$$

The kernel of f are those elements $A \in SL(2,\mathbb{C})$ such that $X = AXA^\dagger \Leftrightarrow A^{-1}X = XA^\dagger$ for all X of the given form. Since $A \in SL(2,\mathbb{C}) \Rightarrow \det A = 1$, A is unitary, i.e. $AA^\dagger = A^\dagger A = I$. We then can write

$$X = AXA^\dagger \Leftrightarrow XA = AXA^\dagger A \Leftrightarrow XA = AX$$

Elements of the center of $GL(n,\mathbb{F})$ are c^*I where c^* is any unit of $\mathbb{F}$ and I is the identity matrix, and since the units in $\mathbb{R}$ are ± 1, the kernel of the homomorphism is $\pm I$. By the isomorphism theorem, $\operatorname{im} f \cong SL(2,\mathbb{C})/\ker f$, and $\ker f$ is discrete. Since $\operatorname{im} f$ is isomorphic to a connected subgroup of the Lorentz group with the same dimension (since $\ker f$ is discrete), $\operatorname{im} f$ is isomorphic to the whole group, implying f is surjective.

13

Using the fact $\tilde{x}$ is the difference of two observables and therefore Hermitian,

$$(\Delta x)^2(\Delta p)^2 = \langle\psi|\tilde{x}^2|\psi\rangle\langle\psi|\tilde{p}^2|\psi\rangle = \langle\psi|\tilde{x}^2\psi\rangle\langle\psi|\tilde{p}^2\psi\rangle = \langle\tilde{x}\psi|\tilde{x}\psi\rangle\langle\tilde{p}\psi|\tilde{p}\psi\rangle \geq |\langle\tilde{x}\psi|\tilde{p}\psi\rangle|^2 = |\langle\psi|\tilde{x}\tilde{p}\psi\rangle|^2 = |\langle\psi|\tilde{x}\tilde{p}|\psi\rangle|^2$$

We write

$$\langle\psi|\tilde{x}\tilde{p}|\psi\rangle = \langle\psi|\left(x - \langle\psi|x|\psi\rangle\right)\left(p - \langle\psi|p|\psi\rangle\right)|\psi\rangle = \langle\psi|xp|\psi\rangle - \langle\psi|x\langle\psi|p|\psi\rangle|\psi\rangle - \langle\psi|p\langle\psi|x|\psi\rangle|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\langle\psi|\psi\rangle$$
$$= \langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle$$

Complex numbers have the property

$$|z|^2 = (\Re z)^2 + (\Im z)^2 \geq (\Im z)^2 = \left(\frac{1}{2i}(z - z^*)\right)^2$$

Applying this to $z = \langle\psi|\tilde{x}\tilde{p}|\psi\rangle$,

$$|\langle\psi|\tilde{x}\tilde{p}|\psi\rangle|^2 \geq \left(\frac{1}{2i}\left[\left(\langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\right) - \left(\langle\psi|xp|\psi\rangle + \langle\psi|x|\psi\rangle\langle\psi|p|\psi\rangle\right)^*\right]\right)^2$$
$$= \left(\frac{1}{2i}\left[\langle\psi|xp|\psi\rangle - \langle\psi|px|\psi\rangle\right]\right)^2 = \left(\frac{1}{2i}\langle\psi|[x,p]|\psi\rangle\right)^2 = \frac{1}{4} = \frac{\hbar^2}{4}$$

where in the last equality we have returned from natural units. The ground state of the simple harmonic oscillator is $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$, from which we can compute

$$\langle\psi_0|x|\psi_0\rangle = 0 \text{ (odd function, symmetric interval)}$$

$$\langle\psi_0|p|\psi_0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}\left(-i\hbar\frac{\partial}{\partial x}e^{-m\omega x^2/2\hbar}\right)dx = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}\left(im\omega x e^{-m\omega x^2/2\hbar}\right)dx$$

$$= 0 \text{ (odd function, symmetric interval)}$$

$$\Rightarrow \tilde{x} = x, \tilde{p} = p$$

$$\Rightarrow (\Delta x)^2 = \langle\psi_0|\tilde{x}^2|\psi_0\rangle = \langle\psi_0|x^2|\psi_0\rangle = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} x^2 e^{-m\omega x^2/2\hbar} dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} = \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} = \sqrt{\frac{\hbar^2}{4m^2\omega^2}} = \frac{\hbar}{2m\omega}$$

$$\Rightarrow (\Delta p)^2 = \langle\psi_0|\tilde{p}^2|\psi_0\rangle = \langle\psi_0|p^2|\psi_0\rangle = -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \frac{\partial^2}{\partial x^2} e^{-m\omega x^2/2\hbar} dx$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \left(\frac{1}{\hbar^2} m\omega e^{-m\omega x^2/2\hbar} (m\omega x^2 - \hbar) \right) dx$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \left(m^2\omega^2 \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx - \hbar m\omega \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} \right)$$

$$= -\sqrt{\frac{m\omega}{\pi\hbar}} \left(m^2\omega^2 \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} - \hbar m\omega \sqrt{\frac{\pi\hbar}{m\omega}} \right) = -(\hbar m\omega/2 - \hbar m\omega) = m\omega\frac{\hbar}{2}$$

where we have used a table for the nasty Gaussian-type integrals. Multiplying the results, we indeed confirm this is a minimum-uncertainty state:

$$(\Delta x)^2 (\Delta p)^2 = \frac{\hbar}{2m\omega} \left(m\omega \frac{\hbar}{2} \right) = \frac{\hbar^2}{4}$$