2231 HW 6

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The multipole expansion is given in general by

$$V(r) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\alpha)\rho(r')dV$$

where r is the distance from the origin to the point of consideration, r' is the distance from the origin to the differential volume element of consideration, α is the angle between r and r', and $\rho(r')$ is the charge density at r'. The first three terms of this sum are, since $\rho(r') = \delta(\rho - R)\delta(z)\lambda$ where ρ and z are the radial and z components of r' respectively in cylindrical coordinates.

$$V_0(r) = \frac{1}{4\pi\epsilon_0 r}\int \rho(r')dV = \frac{\lambda R}{2\epsilon_0 r}$$

$$V_1(r) = \frac{1}{4\pi\epsilon_0 r^2}\int r'\cos\alpha\rho(r')dV = \frac{1}{4\pi\epsilon_0 r^2}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty} r'\cos\alpha\delta(r'-R)\delta(z)\lambda r'dr'd\theta'dz'$$

$$= \frac{\lambda R^2}{8\pi\epsilon_0 r^2}\int_0^{2\pi}\cos\alpha d\theta$$

Noting

$$\vec{r} = r\cos\theta\hat{i} + r\sin\theta\hat{j} + z\hat{k}, \vec{r}' = R\cos\theta'\hat{i} + R\sin\theta'\hat{j}$$

$$\Rightarrow \vec{r}\cdot\vec{r}' = rR\cos\alpha = rR\cos\theta\cos\theta' + rR\sin\theta\sin\theta'$$

$$\Rightarrow V_1(r) = \frac{\lambda R^2}{8\pi\epsilon_0 r^2}\int_0^{2\pi}\cos\theta\cos\theta' + \sin\theta\sin\theta' d\theta' = 0$$

$$V_2(r) = \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2 P_2(\cos\alpha)\rho(r')dV = \frac{1}{4\pi\epsilon_0 r^3}\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}(r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\delta(r'-R)\delta(z)\lambda r'dr'd\theta'dz$$

$$= \frac{\lambda R^3}{8\pi\epsilon_0 r^3}\int_0^{2\pi}\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\theta' = \frac{\lambda R^3}{8\pi\epsilon_0 r^3}\int_0^{2\pi}\left(\frac{3}{2}(\cos\theta\cos\theta' + \sin\theta\sin\theta')^2 - \frac{1}{2}\right)d\theta'$$

$$= \frac{\lambda R^3}{8\pi\epsilon_0 r^3}\int_0^{2\pi}\left(\frac{3}{2}[\cos^2\theta\cos^2\theta' + 2\cos\theta\cos\theta'\sin\theta\sin\theta' + \sin^2\theta\sin^2\theta'] - \frac{1}{2}\right)d\theta'$$

$$= \frac{\lambda R^3}{8\pi\epsilon_0 r^3}\left[\frac{3}{2}\left(\cos^2\theta\int_0^{2\pi}\cos^2\theta' d\theta' + \frac{\sin(2\theta)}{2}\int_0^{2\pi}\sin(2\theta')d\theta' + \sin^2\theta\int_0^{2\pi}\sin^2\theta' d\theta'\right) - \int_0^{2\pi}\frac{1}{2}\right]$$

$$= \frac{\lambda R^3}{8\pi\epsilon_0 r^3}\left[\frac{3}{2}(\pi) - \pi\right] = \frac{\lambda R^3}{16\epsilon_0 r^3}$$

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By symmetry, $\vec{p}$ is in the $\hat{z}$ direction, so

$$|\vec{p}| = \int \vec{r}\rho(\vec{r})dV = \int zk\cos\theta dA = k\int (R\cos\theta)\cos\theta R^2\sin\theta d\theta d\phi = 2\pi kR^3\int \cos^2\theta\sin\theta d\theta$$

$$= -2\pi kR^3\frac{\cos^3(\theta)}{3}\bigg|_0^\pi = -\frac{4k\pi R^3}{3}$$

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The dipole moment of this arrangement should be in the $\hat{z}$ direction once again. This allows computation of

$$|\vec{p}| = \int \vec{r}\rho(\vec{r})dV = \int_N z\rho_0 dV - \int_S z\rho_0 dV$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^R \rho_0 R\cos\theta R^2\sin\theta dr d\phi d\theta - \int_{\frac{\pi}{2}}^{\pi}\int_0^{2\pi}\int_0^R \rho_0 R\cos\theta R^2\sin\theta dr d\phi d\theta$$

$$= 2\pi\rho_0 R^3\left(-\frac{\cos(2\theta)}{4}\bigg|_0^{\frac{\pi}{2}} + \frac{\cos(2\theta)}{4}\bigg|_{\frac{\pi}{2}}^{\pi}\right) = 2\pi\rho_0 R^3$$

The corresponding electric field is

$$E_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}) = \frac{\rho_0 R^3}{2\epsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

The force is attractive.

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The dipole moment of the atom due to the field of the charge is $\vec{p} = \alpha\frac{kq}{r^2}\hat{r}$. The force of attraction is

$$F = (\vec{p}\cdot\nabla)\vec{E} = \frac{\alpha kq}{r^2}\frac{\partial}{\partial r}\frac{kq}{r^2} = -\frac{2\alpha k^2 q^2}{r^5}$$

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We apply the method of images. A dipole $-\vec{p}$ at $-z$ instead of the plane is equivalent to this problem. The field due to this image dipole is

$$\vec{E} = \frac{p}{4\pi\epsilon_0(2z)^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$

and the associated torque is, with the origin at the image dipole,

$$\tau = \vec{p}\times\vec{E} = (p\cos\theta\hat{r} + p\sin\theta\hat{\theta})\times\left(\frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\vec{r} + \sin\theta\vec{\theta})\right)$$

$$= \frac{p^2}{4\pi\epsilon_0 r^3}\left(-\frac{\sin(2\theta)}{2}\hat{\phi}\right) = -\frac{p^2\sin(2\theta)}{8\pi\epsilon_0 r^3}\hat{\phi}$$