4141 HW 7

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1

For a state $|\psi\rangle$, hermicity of A is by definition

$$A = A^\dagger \Leftrightarrow \langle A\psi|\psi\rangle = \langle\psi|A\psi\rangle$$

This implies

$$\langle A^2\rangle = \langle\psi|A^2|\psi\rangle = \langle\psi|A^2\psi\rangle = \langle A\psi|A\psi\rangle = ||A\psi||^2$$

where the norm is not the complex absolute value but the norm induced by the L^2 inner product. Since it is a norm, it is positive-definite, which implies the final result, that $\langle A^2\rangle \geq 0$ (with $\langle A^2\rangle = 0$ iff $A\psi = 0$).

2

For a wavefunction ψ,

$$\Pi\hat{p}\psi = \Pi\frac{\hbar}{i}\frac{\partial\psi}{\partial x} = \frac{\hbar}{i}\frac{\partial\psi}{\partial x}(-x)$$

$$\hat{p}\Pi\psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\psi(-x) = -\frac{\hbar}{i}\frac{\partial\psi}{\partial x}(-x)$$

so these operators anticommute. Therefore,

$$[\Pi, T] = \Pi\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\Pi = \frac{1}{2m}\left(-\hat{p}\Pi\hat{p} + \hat{p}\Pi\hat{p}\right) = 0$$

3

We know that in general $[\hat{x}, \hat{p}] = i\hbar \neq 0$, $\hat{H} = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + \hat{V}$, and $\Pi : x \mapsto -x$. For the case of the free particle, $\hat{V} = 0$, we can then write down commutation relations

$$[\hat{x}, \hat{p}] = i\hbar \neq 0$$

$$[\hat{x}, \hat{H}] = \hat{x}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{x} = \frac{\hat{p}}{2m}\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right) - \frac{\hat{p}\hat{x}\hat{p}}{2m} + \frac{x\hat{p}^2}{2m} = \frac{1}{2m}\left(\hat{p}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{p}\right)$$

$$= \frac{i\hbar}{2m}\left(\hat{p} + \hat{p}\right) = \frac{i\hbar}{m}\hat{p} \neq 0 \text{ if the particle is moving}$$

$$[\hat{x},\Pi]\psi = \hat{x}\Pi\psi - \Pi\hat{x}\psi = \int_{\mathbb{R}} \psi^*(-x)x\psi(-x)dx - \int_{\mathbb{R}} \psi^*(-x)(-x)\psi(-x) \neq 0$$

$$[\hat{p},\Pi] \neq 0 \text{ as shown above}$$

$$[\hat{p},\hat{H}] = \hat{p}\frac{\hat{p}^2}{2m} - \frac{\hat{p}^2}{2m}\hat{p} = \frac{1}{2m}\left(\hat{p}^3 - \hat{p}^3\right) = 0$$

$$[\Pi,\hat{H}] = [\Pi,T] = 0 \text{ as shown above}$$

Therefore, the subsets which are internally mutually commutative are

$$\{\hat{p},\hat{H}\}$$

and

$$\{\Pi,\hat{H}\}$$

4

We must apply a change of basis to the original wavefunction so that one of its basis vectors is $|\psi_f\rangle$.

$$|\psi_i\rangle = |\psi_f\rangle\langle\psi_f|\psi_i\rangle + |\beta\rangle\langle\beta|\psi_i\rangle + |\gamma\rangle\langle\gamma|\psi_i\rangle = ((i-1)/\sqrt{3} + \frac{1}{3})|\psi_f\rangle + \sqrt{2/3}|\beta\rangle$$

The probability is then the norm-squared of the coefficient of $|\psi_f\rangle$, so

$$\Rightarrow P(|\psi_f\rangle) = |i/3|^2 = \frac{1}{3}$$

This expression of $|\psi_i\rangle$ retains normalization, which is always good to check.

5

Normalization means $\langle\psi|\psi\rangle = 1$. If this holds for ψ, then for some unitary operator U we have

$$\langle U\psi|U\psi\rangle = \langle\psi|U^\dagger U\psi\rangle = \langle\psi|I\psi\rangle = \langle\psi|\psi\rangle = 1$$

so unitary operators preserve normalization.