7550 HW 6

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Problem 1 (Volume form on a sphere)**.** *Let $S^n(r)$ be the sphere of radius r in $\mathbb{R}^{n+1}$, given by $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2$,*

$$\omega = \frac{1}{r}\sum_{i=1}^{n+1}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

1. *Compute the integral $\int_{S^n(1)} \omega$ and conclude that ω is not exact.*
2. *Viewing r as a function on $\mathbb{R}^{n+1} \setminus \{0\}$, show that $dr \wedge \omega = dx_1 \wedge \cdots \wedge dx_{n+1}$.*

Proof. By definition, since the sphere is compact and so ω is compactly supported,

$$\int_{S^n(1)} \omega = \sum_{i=1}^{n+1}(-1)^{i-1}\int_{S^n(1)} \frac{x_i}{r} dx_1 \cdots \widehat{dx_i} \cdots dx_{n+1}.$$

$$= \sum_{i=1}^{n+1}(-1)^{i-1}\frac{x_i}{r}A_{S^n(1)},$$

which, since the surface area of a nondegenerate hypersphere is nonzero, is nonzero, so ω is not exact.

$$dr \wedge \omega = d(r \wedge \omega) + r \wedge d\omega$$

□

Problem 2. *If $f : M \to N$ is a submersion of smooth manifolds, show that $f^* : \Omega^*(N) \to \Omega^*(M)$ is injective.*

Proof. Submersion means $f_* : T_pM \to T_{f(p)}N$ is onto at all p. The dual is a contravariant functor, so the image of f_* is the map $f^* : T^*_{f(p)}N \to T^*_pM$; epics in C^{op} are monic in C. Are monics preserved under dualization? Sufficient is right-adjointness; the internal hom (bi)functor is right-adjoint to the tensor product (bi)functor, so this is indeed satisfied (restricting the second argument to the bifunctors to $\mathbb{R}$), and f^* is injective.

Extending this map "algebra-homomorphically" will preserve injectivity, as it is such an extension by which Ω^* is defined. □

Problem 3. *Find the dimension of the vector space $H^1_\Omega(\mathbb{R}^2 \setminus \{n \text{ points}\})$. Can you find differential forms representing basis elements? Can you describe the ring structure of $H^*_\Omega(\mathbb{R}^2 \setminus \{n \text{ points}\})$?*

Proof. First of all, the punctured plane is homotopic to a circle. The circle can be covered by intersecting arcs; these are homotopic to line segments A and B, each of which (as it is contractible) has de Rahm cohomology $H^*_\Omega = \begin{cases} \mathbb{R}, & * = 0 \\ 0 & \text{else} \end{cases}$. The intersection is homotopic to the disjoint union of two line segments. This yields an initial segment of the Mayer-Vietoris sequence (surpressing the $_\Omega$ for visual clarity)

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R} \oplus \mathbb{R} \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}^2 \to H^1(A \cup B),$$

from which we observe that the pentultimate map must have kernel isomorphic to $\mathbb{R}$—and, accordingly, its image is isomorphic to $\mathbb{R}^2/\mathbb{R} \cong \mathbb{R} \cong H^1(A \cup B)$. The next map must have kernel $\mathbb{R}$; the sequence is trivial beyond this point.

We may take $A = \mathbb{R}^2 \setminus \{p_1\}$ and $B = \mathbb{R}^2 \setminus \{p_2, \dots p_n\}$, so that their union is the plane and their intersection is the set of interest. Then, the Mayer-Vietoris sequence is, initially,

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \to H^1(A \cup B) \cong 0 \to H^1(A) \oplus H^1(B) \to H^1(A \cap B) \to H^2(A \cup B$$

At this point, we may suppose for induction that the n-punctured plane has $H^1 \cong \mathbb{R}^n$. The once-punctured plane (circle) computation above forms a base case; evaluating the Mayer-Vietoris sequence above yields the segment

$$0 \to H^1(A) \oplus H^1(B) \cong \mathbb{R} \oplus \mathbb{R}^n \cong \mathbb{R}^{n+1} \to H^1(A \cap B) \to 0.$$

This expresses an isomorphism on the middle arrow; accordingly, the inductive hypothesis holds for all n; namely,

$$H^*_\Omega(\mathbb{R}^2 \setminus \{n \text{ points}\}) \cong \begin{cases} \mathbb{R} & * = 0 \\ \mathbb{R}^n & * = 1 \\ 0 & \text{otherwise} \end{cases}.$$

There are no higher terms because we've shown the $* = 2$ case and higher terms simply have no forms. So, $\dim H^1_\Omega(\mathbb{R}^2 \setminus \{n \text{ points}\}) = n$.

One can make several coordinate systems on the n-punctured plane by placing the origin at one of the punctures and describing all of the non-punctured points via polar coordinates. Letting the angular coordinate in each case be θ_i, $d\theta_i$ are obviously closed. However, despite appearances, these $d\theta_i$ are *not* exact: θ_i themselves are not smooth functions, as they're irreparably multi-valued (just like ordinary polar coordinates—their local smoothness makes them fine charts though). $\square$

plm For the following subsets X of $\mathbb{R}^3$, find the dimension of the vector space $H^1_\Omega(\mathbb{R}^3 \setminus X)$.

1. $X =$ one point.
2. $X =$ two distinct points.
3. $X =$ one line.
4. $X =$ two lines which do not intersect.
5. $X =$ two lines which intersect in a point.

Proof. We omit the zeroth cohomology when possible, as it is identically $\mathbb{R}$ in each case (they're all connected spaces).

1. This scenario is cohomologous to that arising from a radial straight-line homotopy: a 2-sphere. Let A and B be open hemispheres whose boundaries lie in the interior of the other. The intersection is then a 1-sphere (cohomology computed above); the pieces are contractable. Mayer-Vietoris:

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \cong 0$$

   The kernel of the map going into the cohomology we're looking for is isomorphic to $\mathbb{R}$, so the cohomology is trivial.

2. This is homotopic to $S^2 \wedge S^2$—let A and B be these spheres; their intersection is homeomorphic to a disc. Mayer-Vietoris, using the computation above for the last group:

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \cong 0;$$

   the cohomology is trivial because it's the same sequence.

3. This is homotopic to a torus, by a simultaneous stereographic projection of the line onto a circle and a straight-line radial homotopy. To compute this, do "interpenetrating toroidal hemispheres" (which flatten out to annuli, and ultimately circles):

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}^2$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \cong \mathbb{R}^2 \to H^1(A \cap B) \cong \mathbb{R}^2$$

   The kernel going into the critical one is again isomorphic to $\mathbb{R}$, so $H^1(A \cup b) \cong \mathbb{R}^2/\mathbb{R}. \cong \mathbb{R}$

4. The space may be homotopically deformed until the lines are parallel; a cross-section with the lines out of the page can be reduced to a figure-8; and a stereographic-style projection made to produce a two-celled torus: $(S^1 \wedge S^1) \times S^1$. We can let A and B be toroids, and their intersection be a circle along which they're glued to form the above.

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \cong \mathbb{R}^2 \to H^1(A \cap B) \cong \mathbb{R}$$

   This will be trivial by the arguments made for the first two.

5. Straight-line homotopies down the barrel of the lines followed by stereographic projection of the arms in opposite directions results in $T^2 T^2$, which can be divided into two punctured tori (homotopic to figure-8's) with intersection homotopic to a circle.

$$0 \to H^0(A \cup B) \cong \mathbb{R} \to H^0(A) \oplus H^0(B) \cong \mathbb{R}^2 \to H^0(A \cap B) \cong \mathbb{R}$$
$$\to H^1(A \cup B) \to H^1(A) \oplus H^1(B) \cong \mathbb{R}^4 \to H^1(A \cap B) \cong \mathbb{R}$$

   which is trivial by the arguments made for the first three.

□