7590 HW 1

Duncan Wilkie

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1a

By the definition of the L^2 inner product and A, for any functions $f, g \in D(A)$ we have

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 f''(x)g(x)dx = \int_0^1 f(x)g''(x)dx$$

Integrating by parts,

$$f'g\Big|_0^1 - \int_0^1 f'(x)g'(x)dx = \int_0^1 f(x)g''(x)dx$$

$$\Leftrightarrow f'g\Big|_0^1 - fg'\Big|_0^1 + \int_0^1 f(x)g''(x)dx = \int_0^1 f(x)g''(x)dx$$

the evaluation terms must both be zero at 0 and 1 since smooth compactly-supported functions on open sets vanish in the limit to the boundary of their domains. Therefore, this operator is symmetric. However, not all elements of $D(A^\dagger)$ are elements of $D(A)$: $g \in H$ is an element of $D(A^\dagger)$ iff there exists $h \in H$ such that $\forall f \in D(A)$

$$\int_0^1 f''(x)g(x)dx = \int_0^1 f(x)h(x)dx$$

Applying the same integration-by-parts argument as above, we may equivalently write this as

$$\Leftrightarrow f'g\Big|_0^1 - fg'\Big|_0^1 + \int_0^1 f(x)g''(x)dx = \int_0^1 f(x)h(x)dx$$

Since f is compactly supported, f' is as well, so the evaluation terms are zero by the same argument given above. Letting $g = x^2$, we then have

$$\int_0^1 f(x) \cdot 2dx = \int_0^1 f(x)h(x)dx$$

from which we can clearly see the $L^2([0,1])$ function $h = 2$ is the element adjoint to g with respect to A. g is therefore in $D(A^\dagger)$. It isn't in $D(A)$ though, since x^2 doesn't vanish at 1 and therefore isn't compactly supported on this interval. This implies $D(A^\dagger) \neq D(A)$, so $A \neq A^\dagger$, i.e. A isn't self-adjoint.

1b

Proceeding similarly,

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_0^1 (if'(x))^* g(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

$$\Leftrightarrow -if^*g\Big|_0^1 + \int_0^1 i(f(x))^* g'(x)dx = \int_0^1 (f(x))^* ig'(x)dx$$

By the same argument as above, the evaluation term is zero, in which case the equality follows immediately. This operator is symmetric. Once again, x^2 is in $D^\dagger(A)$ but not $D(A)$: from the formula derived for $\langle Af|g\rangle$ in the proof A is symmetric, the definition of membership in $D^\dagger(A)$ is

$$\int_0^1 (f(x))^* 2xdx = \int_0^1 (f(x))^* h(x)dx$$

which, choosing $h = 2x \in L^2([0,1])$, clearly holds. $2x$ isn't compactly supported on $(0,1)$ since it doesn't vanish in the limit to 1, so $D(A^\dagger) \neq D(A)$ and A isn't self-adjoint.

1c

One may write $\partial_i(a_{ij}(x)\partial_j f) = (\partial_i a_{ij})\partial_j f + a_{ij}\partial_i\partial_j f$. Applying the definition,

$$\langle Af|g\rangle = \langle f|Ag\rangle \Leftrightarrow \int_\Omega [\partial_i a_{ij}(x)][\partial_j f(x)]g(x)dx + \int_\Omega a_{ij}(x)[\partial_i\partial_j f(x)]g(x)dx$$

$$= \int_\Omega f(x)\partial_i(a_{ij}(x))\partial_j g(x)dx + \int_\Omega f(x)a_{ij}(x)\partial_i\partial_j g(x)dx$$

Integrating by parts,

$$a_{ij}(x)\partial_j f(x)g(x)\Big|_{\partial\Omega} - \int_\Omega a_{ij}(x)[\partial_i\partial_j f(x)]g(x)dx - \int_\Omega a_{ij}(x)\partial_j f(x)\partial_i g(x)dx + \int_\Omega a_{ij}(x)[\partial_i\partial_j f(x)]g(x)dx$$

$$= a_{ij}(x)f(x)\partial_j g(x)\Big|_{\partial\Omega} - \int_\Omega a_{ij}(x)\partial_i f(x)\partial_j g(x)dx - \int_\Omega a_{ij}(x)f(x)\partial_i\partial_j g(x)dx + \int_\Omega f(x)a_{ij}(x)\partial_i\partial_j g(x)dx$$

Since the functions are compactly supported, the evaluation at the boundary vanishes like above, so

$$\int_\Omega a_{ij}(x)\partial_j f(x)\partial_i g(x)dx = \int_\Omega a_{ij}(x)\partial_i f(x)\partial_j g(x)dx$$

2a

Applying the definition of the infinitesimal generator,

$$Af(x) = -i\lim_{t\to 0}[f(x+vt) - f(x)]/t = -i\frac{\partial f}{\partial v}$$

by the definition of the partial derivative. Since $V = C^1$, and the above implies $D(A)$ is L^2 functions differentiable along v, we have $V \subseteq D(A)$, with action given above.

2b

We compute

$$U^\dagger =$$