Math 7550 HW 2

Duncan Wilkie

10 February 2023

Problem 1. *If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map and we identify $T_p(\mathbb{R}^k)$ with $\mathbb{R}^k$ by identifying $\frac{\partial}{\partial x_i}$ with the ith standard basis vector, show that ϕ_* is just ϕ.*

In other words, using the $(d\phi)_p$ notation for ϕ_* at p, for any $p \in \mathbb{R}^m$, show that $(d\phi)_p = \phi$. (Note that you are implicitly showing that ϕ is differentiable)

Proof. The identification is given by considering paths through p given by $\gamma_v = p + tv$, and noticing that $D_{\gamma_v} = v$. This restricted set of paths is sufficient to determine the rule of ϕ_*, because the tangent vectors so-generated span a space of dimension n, and since they are all elements of T_pM, which is of dimension n, they must span T_pM. By definition,

$$\phi_*(D_{\gamma_v}) = D_{\phi\circ\gamma_v} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(p+tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p) + t\phi(v)]\bigg|_{t=0} = \phi(v).$$

□

Problem 2. *Generalize Problem 1: if $\phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ is a bilinear map, show that ϕ is differentiable, and that, for any $(p,q) \in \mathbb{R}^m \times \mathbb{R}^n$, we have $(d\phi)_{(p,q)}(x,y) = \phi(p,y) + \phi(x,q)$.*

Proof. Writing elements of the domain (u,v), we can define a natural vector space structure from the isomorphic space $\mathbb{R}^{n+m}$: $a(u,v) + (u',v') = (au + u', av + v')$. With respect to this, we can define a path through (p,q) analogous to that above by $\gamma_{(u,v)}(t) = (p,q) + t(u,v)$. Note that this vector space structure is used strictly to define $\gamma_{(u,v)}$; the result does not depend on it. For identical reasons as in the previous problem, such paths are sufficient to describe al germs of differential operators through (p,q). Similarly, one can immediately see that $D_{\gamma_{(u,v)}} = (u,v)$ and

$$\phi_*(D_{\gamma_{(u,v)}}) = D_{\phi\circ\gamma_{(u,v)}} = \frac{\mathrm{d}}{\mathrm{d}t}\phi[(p,q)+t(u,v)]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(p+tu, q+tv)\bigg|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p, q+tv) + \phi(tu, q+tv)]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p,q) + \phi(p,tv) + \phi(tu,q) + \phi(tu,tv)]\bigg|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\phi(p,q) + t\phi(p,v) + t\phi(u,q) + t^2\phi(u,v)\right]\bigg|_{t=0} = \phi(p,v) + \phi(u,q) + 2t\phi(u,v)\bigg|_{t=0}$$

$$= \phi(p,v) + \phi(u,q)$$

which is of the desired form (despite my using different variables). □

Problem 3. *Let $M = M_{m,n}(\mathbb{R})$ be the set of $m \times n$ matrices with real entries. Fix $k \leq \min(m, n)$, and let $U_k = \{A \in M \mid \operatorname{rank}(A) \geq k\}$. Describe a C^∞ structure for U_k.*

Proof. The topology on the set of matrices is not specified, so I'll presume it's the one borrowed from Euclidean space by considering matrices as vectors in $\mathbb{R}^{mn}$. The complement of U_k is the set of matrices A of rank less than k. A matrix has rank $< k$ iff for every $r \geq k$, the determinant of every $r \times r$ submatrix is zero. The number of such submatrices for a given r is computed by choosing which rows and columns to include, so $\sum_{r=k}^{\min(n,m)} \binom{n}{r}\binom{m}{r}$; call this number l. Consider the linear map $f : M \to \mathbb{R}^l$ that, with respect to some arbitrary ordering of the submatrices counted by l, maps a matrix to the vector whose ith entry is the determinant of the ith submatrix. This is continuous, because the determinant is a linear functional, and the fact that maps into products are continuous iff all their projections are continuous. Then the complement of U_k is the kernel of f; kernels of continuous maps are closed, and so U_k is open in M.

This allows us to immediately borrow the functional structure from M, identified with $\mathbb{R}^{mn}$, as U_k is accordingly identified with an open subset of $\mathbb{R}^{mn}$, and as such has a chart. □

Problem 4. *The Grassman manifold or Grassmannian $G_{k,n}$: let $G_{k,n}$ be the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. Show that $G_{k,n}$ is a smooth manifold of dimension $k(n-k)$.*

Proof. There is an onto correspondence of lists of k linearly-independent vectors in $\mathbb{R}^n$ and its k-dimensional vector subspaces, since such lists generate subspaces by spanning and every subspace can be so-generated since every subspace has a basis. Such lists, call them $F_{k,n}$, can be viewed as $k \times n$ matrices with real entries by just juxtaposing the column vectors expressed wrt. the standard basis; the condition that the list is linearly independent translates to these matrices having rank k, by definition of rank. Since $n \geq k$, and k is one dimension of the matrix, the space of such lists is U_k from the previous problem; we can borrow the C^∞ structure defined there. Letting two elements of $F_{k,n}$ be equivalent via $\sim$ if the lists span the same vector subspace, one recognizes $G_{k,n} = F_{k,n}/\sim$. In a just world, we'd have a way of constructing quotient manifolds already. But we don't yet.

The coordinate charts on $G_{k,n}$ may be given by taking some choice J of k elements of $\{1, \ldots, n\}$ and producing an open subset $U_{k,n} \subseteq F_{k,n}$ given by the set of all $k \times k$ submatrices with column indices in J with nonzero determinant. All matrices in such an open subset have reduced-row echelon form (i.e. are in the same equivalence class) with the same row and column interchange operations; since there is a $k \times k$ submatrix with nonzero determinant in each one, the $k \times k$ submatrix at the extreme left of the reduced matrix is the identity matrix, and the leftover submatrix has dimensions $k \times (n-k)$. The charts are obtained by mapping each element of the open set to the vector in $\mathbb{R}^{k(n-k)}$ obtained by flattening this leftover submatrix. There's a matrix B that's a part of the original $U_{k,n}$ given by performing column and row exchange operations on this reduced form until the submatrix given by J is the identity. Any matrix C in $U_{k,n}$ can be written as the product of its submatrix via J with B;

Certainly, every matrix in $F_{k,n}$ is in some $U_{k,n}$ by the definition of linear independence. The mapping is linear and invertible (as a linear map), since the reduced-row echelon form uniquely characterizes the column space. It is, accordingly, a local homeomorphism. Given two such mappings for two different open sets, the fact their inverses are linear maps imply the transition maps are linear, and accordingly smooth. □