4132 HW 7

Duncan Wilkie

5 April 2022

1a

Taking the Lorenz gauge,

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho \Rightarrow \rho = 0$$

$$\Box^2 \vec{A} - -\mu_0 \vec{J} \Rightarrow \nabla^2 \left(\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r}\right) - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\left(\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\right)\hat{r} = -\mu_0 \vec{J}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{-1}{2\pi\epsilon_0}\frac{qt}{r^3}\hat{r}\right) - 0 = -\mu_0 \vec{J}$$

$$\Rightarrow -\frac{qt}{2\pi\epsilon_0 r^2}\frac{-1}{r^2}\hat{r} = -\mu_0 \vec{J}$$

$$\Rightarrow \vec{J} = -\frac{qt}{2\pi\epsilon_0\mu_0 r^4}\hat{r}$$

1b

Under the given gauge,

$$V' = V - \frac{\partial \lambda}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

and

$$\vec{A}' = \vec{A} + \nabla\lambda = \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} + \frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\hat{r} = \frac{1}{2\pi\epsilon_0}\frac{qt}{r^2}\hat{r}$$

The first is simply the scalar potential due to a stationary point charge of magnitude q, and the second is radially outward and therefore curl-free, resulting in a zero magnetic field, also consistent with a stationary point charge.

2

Applying the forced wave equation ansatz given in the text,

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi}\int_{\mathbb{R}} \frac{J(\vec{r}', t)}{\mathscr{r}} dV$$

$$= \frac{\mu_0}{4\pi}\left(\int_{-b}^{-a}\frac{k\left(t-\frac{\eta}{c}\right)}{\eta}\hat{x}dx + \int_a^b \frac{k\left(t-\frac{\eta}{c}\right)}{\eta}\hat{x}dx + a\int_0^\pi \frac{k(t-\frac{\eta}{c})}{\eta}(-\hat{\theta})d\theta + b\int_0^\pi \frac{k(t-\frac{\eta}{c})}{\eta}(\hat{\theta})d\theta\right)$$

The x and θ dependence of η is, since the potential is being calculated at the origin, $\eta = x/\cos\theta$. Therefore,

$$= \frac{\mu_0}{4\pi}\bigg(\int_{-b}^{-a}\frac{k\cos\theta(t-\frac{cx}{\cos\theta})}{x}\hat{x}dx + \int_a^b \frac{k\cos\theta(t-\frac{cx}{\cos\theta})}{x}\hat{x}dx + a\int_0^\pi \frac{k(t-\frac{a}{c})}{a}(-\hat{\theta})d\theta$$

$$+b\int_0^\pi \frac{k(t-\frac{b}{c})}{b}\hat{\theta}d\theta\bigg)$$

$$= \frac{\mu_0}{4\pi}\left(\int_{-b}^{-a}\left(\frac{-kt}{x}-ck\right)\hat{x}dx + \int_a^b\left(\frac{kt}{x}-ck\right)\hat{x}dx + a\int_0^\pi \frac{k(t-\frac{a}{c})}{a}(-\hat{\theta})d\theta + b\int_0^\pi \frac{k(t-\frac{b}{c})}{b}\hat{\theta}d\theta\right)$$

$$\frac{\mu_0}{4\pi}\left(-kt\ln(x)\bigg|_{-b}^{-a}\hat{x} - ckx\bigg|_{-a}^{-b}\hat{x} + kt\ln(x)\bigg|_a^b\hat{x} - ckx\bigg|_a^b\hat{x} - kt\pi\hat{\theta} + \frac{ak}{c}\pi\hat{\theta} + kt\pi\hat{\theta} - \frac{bk}{c}\pi\hat{\theta}\right)$$

$$\frac{\mu_0}{4\pi}\left(2kt\ln\frac{b}{a}\hat{x} + \pi\frac{k}{c}\left[a-b\right]\hat{\theta}\right)$$

Its derivative with respect to time is nonzero at

$$\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 k}{2\pi}\ln\frac{b}{a}\hat{x}$$

Since the potential and therefore its gradient is zero, the electric field is the negative of the above. The magnetic field is incalculable because the curl of $\vec{A}$ cannot be determined from a single point.

3

The scalar potential is

$$V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{|\eta|c - \eta\cdot v}$$

The position of the charge at time t is, in cylindrical coordinates, $r' = a\hat{r} + \omega t\hat{\theta}$. We may then write

$$\eta = r - r' = (r-a)\hat{r} + (\theta - \omega t)\hat{\theta} + z\hat{z}$$

The velocity of the charge in cylindrical coordinates is

$$v = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z} = a\omega\hat{\theta}$$

Plugging this in to the scalar potential,

$$V(r,\theta,z,t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{a\omega(\omega t-\theta) + c\sqrt{(r-a)^2+(\theta-\omega t)^2+z^2}}$$

On the z-axis, $r = \theta = 0$, so

$$V(z,t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{at\omega^2 - c\sqrt{a^2+\omega^2t^2+z^2}}$$

The vector potential is

$$\vec{A}(\vec{r},t) = \frac{v}{c^2}V(\vec{r},t) = \frac{a\omega\hat{\theta}}{c^2}\frac{1}{4\pi\epsilon_0}\frac{qc}{a\omega(\omega t-\theta)+c\sqrt{(r-a)^2+(\theta-\omega t)^2+z^2}}$$

On the z-axis, this becomes

$$\vec{A}(z,t) = \frac{1}{4\pi\epsilon_0 c}\frac{a\omega q}{at\omega^2 - c\sqrt{a^2+\omega^2t^2+z^2}}\hat{\theta}$$

4

The expression for the electric field of a moving charge is, using the equation from the book,

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left[(c^2-v^2)\vec{u}+\vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]$$

where $\vec{u} = c\hat{\mathscr{r}} - \vec{v}$ and $\vec{a}$ is the acceleration of the particle. In this case, the cross products are zero since everything's confined to a line, and the dot products are the scalar product of signed magnitudes, so

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{[\mathscr{r}(c-v)]^3}(c^2-v^2)(c-v)\hat{x} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{c^2-v^2}{(c-v)^2}\hat{x} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)}{(c-v)^2}\hat{x} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{c+v}{c-v}\hat{x}$$

This is the given expression, save for the factor of q missing in the assignment which I presume is a copying error. The magnetic field is, using the equation given in the same chapter,

$$\vec{B} = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}$$

This cross product is zero, since $\hat{\mathscr{r}} = \hat{x}$ and $\vec{E}$ was found to lie along the x-axis.

5a

For a point distance d from the wire, the distance from a differential charge element $dq = \lambda R d\theta$ may be written in terms of angle as $R = d/\sin(\theta)$, as the distance from the wire forms a right triangle with opposite d and hypotenuse R. Substituting this result and integrating over all dq,

$$\vec{E} = \int_0^\pi \frac{\lambda R d\theta}{4\pi\epsilon_0}\frac{1-v^2/c^2}{(1-v^2\sin^2\theta/c^2)^{3/2}}\frac{\hat{R}}{R^2} = \frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0}\overline{\hat{R}}\int_0^\pi \frac{1}{(1-v^2\sin^2\theta/c^2)^{3/2}}\frac{1}{d/\sin\theta}d\theta$$

$$= \frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 d}\overline{\hat{R}}\int_0^\pi \frac{\sin\theta}{(1-v^2\sin^2\theta/c^2)^{3/2}}d\theta$$

Sympy says the nasty integral evaluates to $\frac{2}{1-v^2/c^2}$, so the electric field is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 d}\hat{r}$$

5b

The magnetic field is, in the constant-velocity case,

$$\vec{B} = \frac{1}{c^2}(\vec{v} \times \vec{E}) = \frac{\lambda v}{2\pi\epsilon_0 c^2 d}\hat{\theta} = \frac{\mu_0 I}{2\pi d}\hat{\theta}$$

where the velocity is taken in the $\hat{z}$ direction and we have used $c^2 = \frac{1}{\mu_0\epsilon_0}$. Both of these agree with the easy derivations from Coulomb's and Ampere's laws.