2231 HW 5

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1a

Since there is no charge inside the arrangement of conductors, the potential follows the Laplace equation $\Delta V = 0$ with explicit boundary conditions $V(x, 0, z) = V(x, a, z) = V(0, y, z) = 0$ and $V(b, y, z) = V_0(y)$, and asymptotic boundary conditions $V \to 0$ as $z \to \infty$ and $z \to -\infty$. Physically, we expect that the problem is symmetric in z, so the problem reduces to two dimensions. We proceed by separation of variables: presuming that $V(x, y, z) = f(x)g(y)h(z)$,

$$\Delta V = 0 \Leftrightarrow f''(x)g(y) + f(x)g''(y) = 0 \Leftrightarrow \frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)}$$

Since both sides depend on a single variable, each must be constant—if this were not true, and say without loss of generality that the left side is nonconstant, any variation of the x varaible must produce a variation in the right side, or in other words the right side must depend on x, which is a contradiction. So, calling the constant each side is equal to ω, $f''(x) = \omega f(x)$ and $g''(y) = -\omega g(y)$. These are now linear, second-order, non-homogeneous ODEs of constant coefficients, and so are easily solved. The ansatze e^{mx} and e^{my} yield auxilliary equations $m^2 - \omega = 0$ and $m^2 + \omega = 0$, which correspond to solutions

$$f(x) = a_1 e^{\sqrt{\omega}x} + a_2 e^{-\sqrt{\omega}x}$$

$$g(y) = b_1 \sin(\sqrt{\omega}y) + b_2 \cos(\sqrt{\omega}y)$$

The solutions V are then

$$V = \left(a_1 e^{\sqrt{\omega}x} + a_2 e^{-\sqrt{\omega}x}\right)\left(b_1 \sin(\sqrt{\omega}y) + b_2 \cos(\sqrt{\omega}y)\right)$$

Applying the boundary conditions at $y = 0$,

$$\left(a_1 e^{\sqrt{\omega}x} + a_2 e^{-\sqrt{\omega}x}\right) b_2 = 0$$

Since the exponentials are always nonzero, this implies that either $a_1 = a_2 = 0$ or $b_2 = 0$. Applying the condition at $x = 0$,

$$(a_1 + a_2)\left(b_1 \sin(\sqrt{\omega}y) + b_2 \cos(\sqrt{\omega}y)\right) = 0$$

Since it is possible to choose y such that the trigonometric functions are both nonzero, this implies that either $a_1 + a_2 = 0$ or $b_1 = b_2 = 0$. The only combination of the logical or statements above which yields a nontrivial solution is $b_2 = 0$, and $a_1 = -a_2$ where both are nonzero. Call the magnitude of a_1 A, and rename b_1 to B. Then we have

$$V = AB\sin(\sqrt{\omega}y)\left(e^{\sqrt{\omega}x} - e^{-\sqrt{\omega}x}\right)$$

Applying the boundary condition at $y = b$,

$$AB\sin(\sqrt{\omega}b)\left(e^{\sqrt{\omega}x} - e^{-\sqrt{\omega}x}\right) = 0$$

which implies $\sqrt{\omega} = \frac{n\pi}{b}$ for integral n. The boundary condition at $x = b$ seems to require that

$$V_0(y) = AB\sin\left(\frac{n\pi}{b}y\right)\left(e^{n\pi} - e^{-n\pi}\right)$$

Since linear combinations of solutions of this form remain solutions, we can replace the right side by a linear combination $\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{b}y\right)$ with coefficients such that the solution is a Fourier series for $V_0(y)$. The form these coefficients take can be found by multiplying each side of the supposed equality by $\sin\left(\frac{n'\pi}{b}y\right)$ and integrating from 0 to b:

$$\sum_{n=1}^{\infty} C_n \int_0^b \sin\left(\frac{n'\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)dy = \int_0^b V_0(y)\sin\left(\frac{n'\pi}{b}y\right)dy$$

The integral on the left is zero if $n \neq n'$, and $b/2$ otherwise. Therefore, the coefficients are

$$C_n = \frac{2}{b}\int_0^b V_0(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

and the solution for the initial condition is of the form

$$V(b, y) = \sum_{n=1}^{\infty} \frac{2}{b}\sin\left(\frac{n\pi}{b}y\right)\int_0^b V_0(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

Here C_n contains a factor of $AB(e^{n\pi x} - e^{-n\pi x})$, so the overall solution is

$$V = \frac{1}{AB}\sum_{n=1}^{\infty} \frac{1}{(e^{n\pi x} - e^{-n\pi x})}\frac{2}{b}\sin\left(\frac{n\pi}{b}y\right)\int_0^b V_0(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

1b

If V_0 is a constant, then

$$C_n = \frac{2V_0}{b}\left(\frac{-b}{n\pi}\cos\left(\frac{n\pi}{b}y\right)\right)\Bigg|_0^b = \frac{2V_0}{n\pi}\left(1 - \cos(n\pi)\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4V_0}{n\pi} & n \text{ odd} \end{cases}$$

The solution is then

$$V = \frac{1}{AB}\sum_{k=0}^{\infty} \frac{1}{\left(e^{(2k+1)\pi x} - e^{-(2k+1)\pi x}\right)}\frac{4V_0}{(2k+1)\pi}\sin\left(\frac{(2k+1)\pi}{b}y\right)$$

2

The general solution to Laplace's equation in the case that there is no dependence on ϕ (which there should not be in this case, since the boundary value is only dependent on θ) is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta))$$

where P_l is the lth Legendre polynomial. For the inside of the sphere, if B_l were nonzero the potential at the origin would be infinite. Given there is no charge there, this is unphysical, so B_l is zero. At $r = R$, we must have $V = V_0$, i.e.

$$k\cos(2\theta) = 2k\cos^2(\theta) - k = \sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta))$$

By comparison of coefficients of the above as an equation of polynomials in $\cos(\theta)$, $A_l = 0$ for $l > 2$. For the $l = 2$ term, the leading coefficient of $P_2(\cos(\theta))$ is $\frac{3}{2}$, so $A_2 = \frac{4k}{3R^2}$. For the $l = 1$ term, the leading coefficient of $P_1(\cos(\theta))$ is 0 and there is no x term in P_2, so $A_1 = 0$. For the $l = 0$ term, the coefficient of P_0 is 1 and there is a $\frac{2}{3}$ term from $l = 2$ that needs to be accounted for, so $A_0 = -\frac{k}{3}$. These imply

$$V(r,\theta) = \frac{2kr^2}{3R^2}(3\cos^2(\theta) - 1) - \frac{k}{3}$$

For the case outside the sphere, we must have $A_l = 0$ in order for $V \to 0$ as $r \to \infty$. Once again,

$$k\cos(2\theta) = 2k\cos^2(\theta) - k = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos(\theta))$$

Again, we compare coefficients, having $B_l = 0$ for $l > 2$. The leading coefficient of P_2 is $\frac{3}{2}$, so $B_2 = \frac{4kR^3}{3}$. $l = 1$ is again absent, and for $l = 0$ we must account for the $\frac{2}{3}$ term in the same way, so $A_0 = -\frac{k}{3}$ These imply

$$V(r,\theta) = \frac{2kR^3}{3r^3}(3\cos^2(\theta) - 1) - \frac{k}{3}$$

3

In cylindrical coordinates,

$$\Delta f = 0 \Leftrightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} = 0 \Leftrightarrow \frac{1}{\rho}\left(\rho \frac{\partial^2 f}{\partial \rho^2} + \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} = 0$$

Presuming $V = R(\rho)\Phi(\phi)$,

$$\frac{\Phi(\phi)}{\rho}\left(\rho R''(\rho) + R'(\rho)\right) + \frac{R(\rho)}{\rho^2}\Phi''(\phi) = 0$$

$$\Leftrightarrow -\frac{\rho^2 R''(\rho) + \rho R'(\rho)}{R(\rho)} = \frac{\Phi''(\phi)}{\Phi(\phi)}$$

Since both sides depend solely on a single variable, they are constant. We then write

$$\Phi''(\phi) = \omega^2 \Phi(\phi)$$

and

$$\rho^2 R''(\rho) + \rho R'(\rho) + \omega^2 R(\rho) = 0$$

The first was shown in the first problem to result in $\Phi(\phi) = c_1 e^{\omega\phi} + c_2 e^{-\omega\phi}$. The second we may write as

$$R''(\rho) + \frac{1}{\rho} R'(\rho) + \frac{\omega^2}{\rho^2} R(\rho) = 0$$

One may check via differentiation that $R = b_1 \rho^{\omega} + b_2 \rho^{-\omega}$ are solutions to this equation. The overall solution is then

$$V = [b_1 \rho^{\omega} + b_2 \rho^{-\omega}] \left[c_1 e^{\omega\phi} + c_2 e^{-\omega\phi}\right]$$

Since the potential must go to zero at infinity, $b_1 = 0 \vee c_1 = 0$ and $b_1 = 0 \vee c_2 = 0$. If $b_1 \neq 0$, then $c_1 = c_2 = 0$ yields the trivial solution, so the general solution is

$$V = b_2 \rho^{-\omega} \left[c_1 e^{\omega\phi} + c_2 e^{-\omega\phi}\right]$$

All solutions in spherical coordinates are linear combinations of this.

4

In this case, there is azimuthal symmetry, so the general solution to the Lagrange equations with such a symmetry has the form

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos(\theta))$$

In the interior of the sphere, there is no charge, so there can be no discontinuities in the potential. This implies $B_l = 0$, and

$$V(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

In the exterior of the sphere, the potential goes to zero at infinity, so A_l must all be zero, and

$$V(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta))$$

These two must have continuity on the boundary, so

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta)) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos(\theta))$$

Comparing coefficients,

$$A_l R^l = \frac{B_l}{R^{l+1}} \Leftrightarrow B_l = A_l R^{2l+1}$$

There is a discontinuity in the derivative at the surface however, of magnitude

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\bigg|_{r=R} = -\frac{\sigma(\theta)}{\epsilon_0}$$

$$\Leftrightarrow -\sum_{l=0}^{\infty}(l+1)\frac{B_l}{R^{l+2}}P_l(\cos(\theta)) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos(\theta)) = -\frac{\sigma(\theta)}{\epsilon_0}$$

$$\Leftrightarrow \sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos(\theta)) = \frac{\sigma(\theta)}{\epsilon_0}$$

Multiplying both sides by $P_{l'}(\cos(\theta))\sin(\theta)$, since the Legendre polynomials are orthogonal on $[-1, 1]$, and integrating from 0 to π,

$$\sum_{l=0}^{\infty}(2l+1)A_k R^{l-1}\int_0^{\pi} P_l(\cos(\theta))P_{l'}(\cos(\theta))\sin(\theta)d\theta = \frac{1}{\epsilon_0}\int_0^{\pi}\sigma(\theta)P_{l'}(\cos(\theta))\sin(\theta)d\theta$$

Since

$$\int_{-1}^{1} P_l(x)P_{l'}(x)dx = \int_0^{\pi} P_l(\cos(\theta))P_{l'}(\cos(\theta))\sin(\theta)d\theta = \begin{cases} 0, & l' \neq l \\ \frac{2}{2l+1}, & l' = l \end{cases}$$

this becomes

$$2A_l R^{l-1} = \frac{1}{\epsilon_0}\int_0^{\pi}\sigma(\theta)P_{l'}(\cos(\theta))\sin(\theta)d\theta$$

$$\Leftrightarrow A_l = \frac{1}{2R^{l-1}\epsilon_0}\left(\int_0^{\pi/2}\sigma_0 P_{l'}(\cos(\theta))\sin(\theta)d\theta - \int_{\pi/2}^{\pi}\sigma_0 P_{l'}(\cos(\theta))\sin(\theta)d\theta\right)$$

Since cos is odd and sin is even, and any function of an even function is even, the integrand is odd. Introducing a change of variables $\phi = \theta - \pi/2$ turns this to

$$A_l = \frac{\sigma_0}{\epsilon_0 R^{l-1}}\int_0^{\pi/2} P_{l'}(\cos(\theta))\sin(\theta)d\theta$$

Evaluating these up to $l = 4$,

$$A_0 = \frac{\sigma_0}{\epsilon_0 R^{l-1}}$$

$$A_1 = \frac{\sigma_0}{2\epsilon_0 R^{l-1}}$$

$$A_2 = 0$$

$$A_3 = -\frac{\sigma_0}{8\epsilon_0 R^{l-1}}$$

$$A_4 = 0$$

That the even ls all evaluate to zero could likely be proven by an appeal to π periodicity of those terms, however, that is not necessary for this problem.