4123 HW 1

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1a

Since we would like to extremize with respect to time, we must find a functional that expresses the travel time in terms of the variables given. We can write $dt = \frac{ds}{v(y)} = \frac{\eta(y)dx\sqrt{1+y'^2}}{c}$. The integral of this expression for dt is of course the total travel time,

$$J[y] = \int \frac{\eta(y)\sqrt{1+y'^2}}{c}dx$$

Since there is no direct dependence on x in this expression, the first integral of the Euler-Lagrange equations must be satisfied. We then have, letting κ being an arbitrary constant,

$$f - y'\frac{\partial f}{\partial y'} = \kappa \Leftrightarrow \frac{\eta(y)\sqrt{1+y'^2}}{c} - y'\frac{\partial}{\partial y'}\left(\frac{\eta(y)\sqrt{1+y'^2}}{c}\right) = \kappa$$

$$\Leftrightarrow \frac{\eta(y)\sqrt{1+y'^2}}{c} - \frac{\eta(y)y'^2}{c}\frac{1}{\sqrt{1+y'^2}} = \kappa \Leftrightarrow \frac{\eta(y)}{c}\sqrt{1+y'^2}\left(1 - \frac{y'^2}{1+y'^2}\right) = \kappa$$

$$\Leftrightarrow \frac{\eta(y)}{c}\sqrt{1+y'^2}\left(\frac{1}{1+y'^2}\right) = \kappa \Leftrightarrow \eta(y) = K\sqrt{1+y'^2}$$

1b

The Euler-Lagrange equation applied to the functional above yields

$$\frac{\partial}{\partial y}\left(\frac{\eta(y)\sqrt{1+y'^2}}{c}\right) - \frac{d}{dx}\frac{\partial}{\partial y'}\left(\frac{\eta(y)\sqrt{1+y'^2}}{c}\right) = 0$$

$$\Leftrightarrow \frac{\sqrt{1+y'^2}}{c}\eta'(y) - \frac{d}{dx}\left(\frac{\eta(y)}{c}\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Leftrightarrow \frac{\sqrt{1+y'^2}}{c}\eta'(y) - \frac{1}{c}\left(\frac{y'^2}{\sqrt{1+y'^2}}\eta'(y) + \eta(y)y''\left(\frac{-y'^2}{(1+y'^2)^{3/2}} + \frac{1}{\sqrt{1+y'^2}}\right)\right) = 0$$

$$\Leftrightarrow \eta'(y)(1+y'^2) = y'^2\eta'(y) + \eta(y)y''\left(\frac{-y'^2}{1+y'^2}+1\right)$$
$$\Leftrightarrow 1 = \frac{\eta(y)}{\eta'(y)}y''\left(\frac{-y^2}{1+y'^2}+1\right)$$
$$\Leftrightarrow 1 = \frac{\eta(y)}{\eta'(y)}y''\frac{1}{1+y'^2}$$
$$\Leftrightarrow y'' = \frac{\eta'(y)(1+y'^2)}{\eta(y)}$$

If the refractive index decreases, $\eta'(y) < 0$. $\eta(y)$ is positive (except for certain exotic occasions, according to my MATH 7380 professor), and of course $1+y'^2$ is as well. Therefore $y'' < 0$, i.e. the path concaves downwards.

2

We apply the Euler-Lagrange equation to get

$$1 - 2y'' = 0$$

Integrating twice,

$$\frac{x^2}{2} - 2y = c_1x + c_2, \text{ where } c_1, c_2 \in \mathbb{R}$$

Applying the first boundary condition,

$$c_2 = 0$$

Applying the second,

$$\frac{-7}{2} = \frac{c_1}{2} + c_2 \Rightarrow c_1 = -7$$

Substituting within the integrated form these values of c_1, c_2 yields

$$\frac{x^2}{2} - 2y = -7x \Leftrightarrow y = \frac{x^2}{4} + \frac{7}{2}x$$