4142 HW 5

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Problem 1. *Consider the energy matrix*

$$\begin{pmatrix} a & 0 & 0 & \text{-}b \\ 0 & c & id & 0 \\ 0 & \text{-}id & \text{-}c & 0 \\ \text{-}b & 0 & 0 & \text{-}a \end{pmatrix}$$

Find the eigenvalues and eigenvectors of this Hamiltonian. It will help to not approach this as a direct diagonalization but regard the elements above as coupling between four states.

Solution. Looking at what will happen when the basis vectors, it's clear this is actually is two non-interacting problems: vectors in combinations of the first and last basis vectors get mapped to each other, and likewise for the second and third. The individual problems are

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} \Rightarrow \det \begin{vmatrix} a-\lambda & -b \\ -b & -a-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - a^2 - b^2 = 0 \Rightarrow \lambda = \pm\sqrt{a^2+b^2}$$

and

$$\begin{pmatrix} c & id \\ -id & -c \end{pmatrix} \Rightarrow \det \begin{vmatrix} c-\lambda & id \\ -id & -c-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - c^2 - d^2 \Rightarrow \lambda = \pm\sqrt{c^2+d^2}$$

The corresponding eigenvectors are

$$\begin{pmatrix} \frac{-a\pm\sqrt{a^2+b^2}}{b} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{id}{c} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

□

Problem 2. *At time $t = 0$, a spin-1/2 particle is in the state $|S_z = +\rangle$.*

1. *If S_x is measured at $t = 0$, what is the probability of getting a value $\hbar/2$?*
2. *Instead, with no measurement at $t = 0$, suppose the system evolves in a magnetic field $\vec{B} = B_0\hat{e}_y$. Use the S_z basis to calculate the state of the system at time t.*

3. *Suppose you now measure S_x at t. What is the probability of getting the value $\hbar/2$?*

Solution. In general,

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)},$$

so in the state $S_z = + \Rightarrow a = 1, b = 0$, so the probability that a measurement of S_x yields $\hbar/2$ is

$$P(S_x = +) = \frac{1}{\sqrt{2}}.$$

The Hamiltonian in the presence of the magnetic field is

$$H = \gamma B_0 S_y = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The eigenstates of this are the usual eigenstates of S_y: in the S_z basis,

$$\chi_+^{(y)} = \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},\ \chi_-^{(y)} = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

and accordingly

$$\chi = \frac{b+ia}{\sqrt{2}}\chi_+^{(y)} + \frac{b-ia}{\sqrt{2}}\chi_-^{(y)}$$

Since this is time-independent, the time-dependent solution expressed in the S_y basis is

$$\chi(t) = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

where a, b are specified by the initial condition (in this case, $\chi(0) = |S_z = +\rangle \Rightarrow a = \frac{i}{\sqrt{2}}, b = \frac{-i}{\sqrt{2}}$) and $E_\pm$ are the energies of the $\chi_\pm^{(y)}$ eigenstates, which are $\mp\frac{\gamma B_0 \hbar}{2}$ by the eigenvalues of the states:

$$\chi(t) = \begin{pmatrix} \frac{i}{\sqrt{2}}e^{i\gamma B_0 t/2} \\ \frac{-i}{\sqrt{2}}e^{-i\gamma B_0 t/2} \end{pmatrix}.$$

Changing back to the S_z basis,

$$\begin{aligned}
\chi(t) &= \left(\frac{i}{\sqrt{2}}e^{i\gamma B_0 t/2}\right)\left(\frac{-i}{\sqrt{2}}\chi_+ + \frac{1}{\sqrt{2}}\chi_-\right) + \left(\frac{-i}{\sqrt{2}}e^{-i\gamma B_0 t/2}\right)\left(\frac{i}{\sqrt{2}}\chi_+ + \frac{1}{\sqrt{2}}\chi_-\right) \\
&= \frac{e^{i\gamma B_0 t/2}}{2}\chi_+ + \frac{ie^{i\gamma B_0 t/2}}{2}\chi_- + \frac{e^{-i\gamma B_0 t/2}}{2}\chi_+ - \frac{ie^{-i\gamma B_0 t/2}}{2}\chi_- \\
&= \begin{pmatrix} \cos(\gamma B_0 t/2) \\ -\sin(\gamma B_0 t/2) \end{pmatrix}
\end{aligned}$$

This looks a lot like a clockwise parameterization of a circle.

The change of basis from S_z to S_x is

$$\chi = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)},$$

so the coefficient of $\chi_+^{(x)}$ using the above expression for the S_z basis is

$$\frac{\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)}{\sqrt{2}}$$

The corresponding probability of a measurement resulting in $\chi_+^{(x)}$ (which has eigenvalue $\hbar/2$) is

$$\left(\frac{\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)}{\sqrt{2}}\right)^2 = \frac{1}{2}(1 - 2\sin(\gamma B_0 t))$$

□

Problem 3. *Three quarks, each of spin 1/2, form a baryon. What are the allowed values of baryon spin?*

Solution. Distinguish one pair of quarks. This combination has a spin of either 1 or 0, as those are the only integers between $\frac{1}{2} + \frac{1}{2}$ and $\frac{1}{2} - \frac{1}{2}$. The spin of the whole system is a combination of the spin of the pair with the spin of the remaining quark; it's either $1 + \frac{1}{2} \Rightarrow \frac{3}{2}, \frac{1}{2}$ or $0 + \frac{1}{2} \Rightarrow \frac{1}{2}, -\frac{1}{2}$. Accordingly, the only possibly baryon spins are $\frac{1}{2}$ and $\frac{3}{2}$. □

Problem 4. *Suppose an electron is in potentials $\vec{A} = B_0(x\hat{\jmath} - y\hat{\imath})$, $\Phi = Kz^2$.*

1. *Find the electric and magnetic fields.*
2. *Find the allowed energies of the electron.*

Solution. Since Φ is time-independent, $\vec{B} = \boldsymbol{\nabla} \times \vec{A}$ and $\vec{E} = \boldsymbol{\nabla}\Phi$:

$$\boldsymbol{\nabla} \times \vec{A} = \det\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -B_0 y & B_0 x & 0 \end{vmatrix} = 2B_0\hat{k},$$

$$\boldsymbol{\nabla}\Phi = 2Kz\hat{k}.$$

The minimal coupling Hamiltonian is

$$H = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} - e\vec{A})^2 + e\Phi = \frac{1}{2m}(-i\hbar\boldsymbol{\nabla} + eB_0(x\hat{\jmath} - y\hat{\imath}))^2 + eKz^2$$

$$= \frac{1}{2m}(i\hbar\boldsymbol{\nabla} + eB_0(x\hat{\jmath} - y\hat{\imath})) \cdot (i\hbar\boldsymbol{\nabla} + eB_0(x\hat{\jmath} - y\hat{\imath})) + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\boldsymbol{\nabla}^2 + i\hbar eB_0\Big(\boldsymbol{\nabla} \cdot (x\hat{\jmath} - y\hat{\imath}) + (x\hat{\jmath} - y\hat{\imath}) \cdot \boldsymbol{\nabla}\Big) + e^2B_0^2(x^2 + y^2)\right] + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 + i\hbar eB_0\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + e^2B_0^2(x^2 + y^2)\right] + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 - eB_0L_z + e^2B_0^2(x^2 + y^2)\right] + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 - eB_0L_z\right] + \left[\frac{e^2B_0^2}{2m}\hat{\imath} + \frac{e^2B_0^2}{2m}\hat{\jmath} + eK\hat{z}\right] \cdot \left[x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}\right]$$

Applying this to a wave function ψ,

$$H\psi = \frac{1}{2m}\left[-\hbar^2\nabla^2\psi - eB_0L_z\psi\right] + \left[\frac{e^2B_0^2}{2m}\hat{\imath} + \frac{e^2B_0^2}{2m}\hat{\jmath} + eK\hat{z}\right]\cdot\left[x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}\right]\psi = E\psi$$

It's clear that this problem has an angular component, from the presence of L_z, and that the z-axis is distinguished, from the dot-product term. Accordingly, we move to cylindrical coordinates, in which $L_z = -i\hbar\frac{\partial\psi}{\partial\theta}$.

$$\frac{1}{2m}\left[-\hbar^2\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}\right) + ie\hbar B_0\frac{\partial\psi}{\partial\theta}\right] + \frac{e^2B_0^2}{2m}r^2\psi + eKz^2\psi = E\psi.$$

$$\frac{-\hbar^2}{2m}\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\psi}{\partial r}\right] - \frac{e^2B_0^2}{\hbar^2}r^2\psi\right) + \left(\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{ieB_0}{\hbar}\frac{\partial\psi}{\partial\theta}\right) + \left(\frac{\partial^2\psi}{\partial z^2} - \frac{2meKz^2}{\hbar^2}\psi\right)\right] = E\psi.$$

Assuming the problem separates, i.e. $\psi = R(r)\Theta(\theta)Z(z)$ where each factor depends only on the indicated variable, this becomes

$$\begin{aligned}
&\frac{-\hbar^2}{2m}\left[\Theta(\theta)Z(z)\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] - \frac{e^2B_0^2}{\hbar^2}r^2R(r)\right) + R(r)Z(z)\left(\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} - \frac{ieB_0}{\hbar}\frac{\partial\Theta}{\partial\theta}\right)\right.\\
&\qquad\left. + R(r)\Theta(\theta)\left(\frac{\partial^2 Z}{\partial z^2} - \frac{2meKz^2}{\hbar^2}Z(z)\right)\right] = ER(r)\Theta(\theta)Z(z)\\
\Leftrightarrow\ &\frac{-\hbar^2}{2m}\left[\frac{1}{R(r)}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial R}{\partial r}\right] - \frac{e^2B_0^2}{\hbar^2}r^2R(r)\right) + \frac{1}{\Theta(\theta)}\left(\frac{1}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} - \frac{ieB_0}{\hbar}\frac{\partial\Theta}{\partial\theta}\right)\right.\\
&\qquad\left. + \frac{1}{Z(z)}\left(\frac{\partial^2 Z}{\partial z^2} - \frac{2meKz^2}{\hbar^2}Z(z)\right)\right] = E.
\end{aligned}$$

Each term must be equal to a constant, since the sum is: if any varied with respect to its variable, the others don't depend on that variable, and so can't vary to compensate and make the entire expression equal E. The problem then splits into three ODEs:

$$\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R}{\mathrm{d}r}\right) &= \left(c_1 + \frac{e^2B_0}{\hbar^2}r^2\right)R(r),\\
\frac{1}{r^2}\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} - \frac{ieB_0}{\hbar}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} - c_2\Theta(\theta) &= 0,\\
\frac{\mathrm{d}^2Z}{\mathrm{d}z^2} &= \left(c_3 + \frac{2meK}{\hbar^2}z^2\right)Z(z)
\end{aligned}$$

□

Problem 5. *Consider the observables $A = x^2$ and $B = L_z$.*

1. *Find the uncertainty principle governing A and B, that is, $\Delta A\Delta B$.*
2. *Evaluate ΔB for the hydrogenic $|n\ell m\rangle$ state.*
3. *Therefore, what can you conclude about $\langle xy\rangle$ in this state?*

Solution. The generalized uncertainty principle states

$$\Delta A \Delta B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|.$$

We first compute the commutator:

$$[A, B] = \left[x^2, L_z\right] = x[x, L_z] + [x, L_z]x = x[x, xp_y - yp_x] + [x, xp_y - yp_x]x$$

$$= x([x, xp_y] - [x, yp_x]) + ([x, xp_y] - [x, yp_x])x$$

$$= x([x, x]p_y + x[x, p_y] - [x, y]p_x - y[x, p_x]) + ([x, x]p_y + x[x, p_y] - [x, y]p_x - y[x, p_x])x$$

The commutator among any two position operators and that of a position and a momentum operator along different axes is zero; accordingly,

$$= -xy[x, p_x] - y[x, p_x]x = -xyi\hbar - yxi\hbar = -i\hbar(xy + yx)$$

$$= -2i\hbar xy$$

The uncertainty principle is therefore

$$\Delta A \Delta B \geq \hbar |\langle xy \rangle|$$

The variance of the B operator is given by $\langle L_z^2 \rangle - \langle L_z \rangle^2$; the former is by L_z Hermitian $\langle \psi | L_z^2 | \psi \rangle = \langle L_z \psi | L_z \psi \rangle$. The $|n \ell m_\ell\rangle$ state of hydrogen is in an eigenstate of L_z by assumption (corresponding to the eigenvalue m_ℓ). In particular, $L_z \psi = \hbar m_\ell \psi \Rightarrow \left\langle L_z^2 \right\rangle = \hbar^2 m_\ell^2$, and $\langle L_z \rangle$ is the eigenvalue of L_z corresponding to the eigenstate; of course, $\hbar m_\ell$. So, the variance of B is $\hbar^2 m_\ell^2 - \hbar^2 m_\ell^2 = 0$. This necessarily means $\langle xy \rangle = 0$. $\square$