7550 HW 5

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Problem 1. *On an open set $U \subseteq \mathbb{R}^n$, show that the exterior derivative is the only operator $d : \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying:*

1. $d(\omega + \eta) = d\omega + d\eta$;
2. $\omega \in \Omega^p(U), \eta \in \Omega^q(U) \Rightarrow d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$;
3. $f \in \Omega^0(U) \Rightarrow df(X) = X(f)$; *and*
4. $f \in \Omega^0(U) \Rightarrow d(df) = 0$.

Deduce that d is independent of the coordinate system used to define it.

Proof. This smells a lot like it'd follow from uniqueness of some categorical construction, but I don't want to mess around with that. Accordingly, I'll have to do it the normal way.

Suppose that d' satisfies each of the properties above. We merely must show that d' satisfies the defining characteristic of d, namely, that if X is a smooth vector field on U, that $d'f(X) = X(f)$, and, letting $\omega = f dx_1 \wedge \cdots \wedge dx_p$, that $d'\omega = d'f dx_1 \wedge \cdots \wedge x_p$ (which can be extended linearly to all forms).

Let X be a smooth vector field on U. In local coordinates, $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, where a_i are smooth functions on U. Property 3 yields $d'f(X) = X(f)$, as zero-forms are exactly smooth functions.

If $\omega = f dx_1 \wedge \cdots \wedge dx_p = f \wedge dx_1 \wedge \cdots \wedge dx_p$, then by property 2 applied with $p = 0$ we can induct on p (the index in the wedges). For $p = 0$, it's immediate that the pure zero-form $\omega = f$ has $d'\omega = d'f$. Suppose that the property holds for all pure wedges with $p - 1$ terms. Then

$$d'\omega = d'f \wedge dx_1 \wedge \cdots \wedge dx_p + (-1)^0 f \wedge d'(dx_1 \wedge \cdots \wedge dx_p)$$

$$= d'f \wedge dx_1 \wedge \cdots \wedge dx_p + (-1)^0 f \wedge \left(\sum_i dx_1 \wedge \cdots d'dx_i \wedge \cdots \wedge dx_p \right)$$

By property 2, which is $d'f = \sum_i \frac{\partial f}{\partial x_i} dx_i$ in local coordinates, we can take $f = x_i$ (the coordinate function) to get

$$d'x_i = \sum_j \frac{\partial x_i}{\partial x_j} dx_j = dx_i.$$

Accordingly, $dx_i = d'(x_i)$, so we can apply property 4 to get $d'dx_i = 0$—all the terms on the right vanish, and we get what we want:

$$d'\omega = d'f \wedge dx_1 \wedge \cdots \wedge dx_p.$$

By assumption in property 1, d' is additive, but is it linear? Well, let $c \in \Omega^0(U)$ be a constant and η be a pure form of the form of ω above. By property 2 and the definition of the wedge product and scalar product in the exterior algebra,

$$d'(c\eta) = d'(c \wedge \eta) = d'c \wedge \eta + (-1)^0 c \wedge d'\eta$$

Trivially by the property proven above, $d'c = 0$—so $d'(c\eta) = cd'\eta$. This means that d' is indeed linear on forms that look like ω, and so can be honest-to-goodnes linearly extended to all forms.

In any particular coordinate system, the text proves that the d defined by that coordinate system has all four properties. Accordingly, since none of these properties (including the forms themselves) are coordinate-dependent, any two coordinate systems must define the same d operator. $\square$

Let G be a Lie group, and $g \in G$. Recall that left-translation by g, $L_g : G \to G$, is given by $L_g(h) = gh$, and recall also the definition and importance of left-invariant vector fields. A differential form ω on G is left-invariant if $L_g^*\omega = \omega$ for each $g \in G$. Let $E^p(G)$ denote the vector space of left-invariant p-forms on G, and $E^*(G) = \bigoplus_{p=0}^{\dim G} E^p(G)$. Here are some of their properties to establish. Several of them are analogs of properties of left-invariant vector fields we've seen.

Problem 2. *$E^*(G)$ is a subalgebra of the algebra $\Omega^*(G)$ of all smooth differential forms on G. If e denotes the identity element of G, the map $\omega \mapsto \omega_e$ is an algebra isomorphism of $E^*(G)$ and the exterior algebra $\Lambda((T_eG)^*)$. Note that this map gives an isomorphism of $E^1(G)$ with $(T_eG)^*$, that is, with the dual space of the Lie algebra $\mathfrak{g}$ of G.*

Proof. First, the subalgebra condition. Clearly, the zero-form is left-invariant. Let ω, ν be left-invariant p-forms (smooth functions). Letting these act on a vector fields $X_1, \dots, X_p$, which in turn act on a smooth function f,

$$\begin{aligned}
&L_g^*(\omega(X_1(f), \dots, X_p(f)) + \nu(X_1(f), \dots, X_p(f))) \\
&= \omega(X_1(f \circ L_g), \dots, X_p(f \circ L_g)) + \nu(X_1(f \circ L_g), \dots, X_p(f \circ L_g)) \\
&= L_g^*(\omega(X_1(f), \dots, X_p(f))) + L_g^*(\nu(X_1(f), \dots, X_p(f))) \\
&= \omega(X_1(f), \dots, X_p(f)) + \nu(X_1(f), \dots, X_p(f)),
\end{aligned}$$

and, if k is a constant,

$$\begin{aligned}
L_g^*(k\omega(X_1(f), \dots, X_p(f)) = k\omega(X_1(f \circ L_g), \dots, X_p(f \circ L_g)) = k\big(L_g^*(\omega(X_1(f), \dots, X_p(f)))\big); \\
= k\omega(X_1(f), \dots, X_p(f))
\end{aligned}$$

This shows that $E^*(G)$ is a vector subspace.

Further, if ω, ν are as above, then

$$\begin{aligned}
&L_g^*(\omega(X_1(f), \dots, X_p(f)) \wedge \nu(X_1(f), \dots, X_p(f))) \\
&= \omega(X_1(f \circ L_g), \dots, X_p(f \circ L_g)) \wedge \nu(X_1(f \circ L_g), \dots, X_p(f \circ L_g)) \\
&= L_g^*(\omega(X_1(f), \dots, X_p(f))) \wedge L_g^*(\nu(X_1(f), \dots, X_p(f))) \\
&= \omega(X_1(f), \dots, X_p(f)) \wedge (X_1(f), \dots, X_p(f)),
\end{aligned}$$

so it is also a subalgebra. Note that all of these proofs are by way of showing that L_g^* has a homomorphism property with respect to each operation checked, and accordingly is an algebra homomorphism.

Since (the operations on) global forms are defined by gluing together forms at points, the map restricting to the identity is trivially a homomorphism. However, for all left-invariant ω, the value ω_g is the same as $L_g^*\omega_e$, i.e. ω is determined solely by its value at ω_e; this means that there exists a map from ν_e to ν for all forms at the identity ν. This is, in particular, given pointwise by $\nu_g = L_g^*\nu_e$; as L_g^* has the homomorphism property, this inverse is a homomorphism also.

□

Problem 3. *If ω is a left-invariant form and X is a left-invariant vector field, then $\omega(X)$ is a constant function on G.*

Proof. At a point g,

$$\omega_g(X_g) = L_g^*\omega_e(L_g^* X_e) = \omega_e(X_e),$$

i.e. for every g, $\omega(X)$ takes on the same value. □

Problem 4. *Let $\{X_1, \ldots, X_n\}$ and $\{\omega_1, \ldots, \omega_n\}$ be dual bases for $\mathfrak{g}$ and $E^1(G)$. Then there are constants c_{ijk} so that $[X_i, X_j] = \sum_{k=1}^n c_{ijk}X_k$. These structure constants of G with respect to the specified basis of $\mathfrak{g}$ satisfy $c_{ijk}+c_{jik} = 0$ and $\sum_r(c_{ijr}c_{rks}+c_{jkr}c_{ris}+c_{kir}c_{rjs}) = 0$. Use the invariant formula for the exterior derivative to show that the exterior derivatives of the form ω_i are given by the Maurer-Cartan equations*

$$d\omega_i = \sum_{j<k} c_{jki}\omega_k \wedge \omega_j.$$

Proof. Applying that invariant form,

$$(d\omega_i)(X_m, X_n) = X_m(\omega_i(X_n)) - X_n(\omega_i(X_m)) - \omega_i([X_m, X_n]).$$

$$= X_m(\omega_i(X_n)) - X_n(\omega_i(X_m)) - \omega_i\left(\sum_{l=1}^n c_{mnl}X_l\right) = X_m(\omega_i(X_n)) - X_n(\omega_i(X_m)) - \sum_{l=1}^n c_{mnl}\omega_i(X_l).$$

On the other hand,

$$\left(\sum_{j<k} c_{jki}\omega_k \wedge \omega_j\right)(X_m, X_n) = \sum_{j<k} c_{jki}(\omega_k(X_m)\omega_j(X_n) - \omega_k(X_n)\omega_j(X_m))) = \sum$$

Since ω_i is left-invariant, □

Problem 5. *Show that a Lie group G is orientable. Hint: can you use (a basis for) $E^1(G)$ to produce a nowhere-vanishing n-form, where $n = \dim G$?*

Proof. Let ω_i be a basis for $E^1(G)$. The isomorphism in Problem 3 yields, when restricted to the 1-forms, $E^1(G) \cong \Lambda^1(T_e^*G)$; the cotangent space T_e^*G has the same dimension as T_eG, i.e. n—and the dimension of the pth grading of the exterior algebra is $\binom{n}{p}$, implying this basis has $\binom{n}{1} = n$ elements. We can choose the form given by $\nu = \omega_1 \wedge \cdots \wedge \omega_n$. If there were some g such that $\nu_g = 0$, then by left-invariance $\nu_e = 0$. However, this means that for all $X_1, \ldots, X_n$

$$\omega_1 \wedge \cdots \wedge \omega_n(X_1, \ldots, X_n) = \det[\omega_i(X_j)] = 0;$$

Choosing a basis for the tangent space, this expresses a linear dependence between ω_i, contradicting that they're a basis, so this form must not vanish. □