# 2231 HW 1

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## 1

Let $\vec{A} = \hat{\imath}$, $\vec{B} = \hat{\jmath}$, and $\vec{C} = \hat{\jmath}$. We don't need to compute to see that this fails to be associative: in $\vec{A} \times (\vec{B} \times \vec{C})$, we observe that the first-evaluated product is zero, since the angle between a vector and itself is zero; the overall product is therefore zero as well. However, in $(\vec{A} \times \vec{B}) \times \vec{C}$ the first-evaluated product is a nonzero vector perpendicular to the $\hat{\imath}$-$\hat{\jmath}$ plane, and therefore the cross product of this vector with $\hat{\jmath}$ is nonzero.

## 2

We denote components as $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and so on. Then

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (B_yC_z - B_zC_y)\hat{\imath} - (B_xC_z - B_zC_x)\hat{\jmath} + (B_xC_y - B_yC_x)\hat{k}$$

$$\Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= (A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z)\hat{\imath}$$

$$-(A_xB_xC_y - A_xB_yC_x - A_zB_yC_z + A_zB_zC_y)\hat{\jmath}$$

$$+(A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y)\hat{k}$$

On the other hand,

$$\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) = \vec{B}(A_xC_x + A_yC_y + A_zC_z) - \vec{C}(A_xB_x + A_yB_y + A_zB_z)$$

$$= (A_xB_xC_x + A_yB_xC_y + A_zB_xC_z)\hat{\imath} - (A_xB_xC_x + A_yB_yC_x + A_zB_zC_x)\hat{\imath}$$

$$+(A_xB_yC_x + A_yB_yC_y + A_zB_yC_z)\hat{\jmath} - (A_xB_xC_y + A_yB_yC_y + A_zB_zC_y)\hat{\jmath}$$

$$+(A_xB_zC_x + A_yB_zC_y + A_zB_zC_z)\hat{k} + (A_xB_xC_z + A_yB_yC_z + A_zB_zC_z)\hat{k}$$

$$\begin{aligned}
&= (A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x)\hat{\imath} \\
&+(A_xB_yC_x + A_zB_yC_z - A_xB_xC_y - A_zB_zC_y)\hat{\jmath} \\
&+(A_xB_zC_x + A_yB_zC_y - A_xB_xC_z - A_yB_yC_z)\hat{k}
\end{aligned}$$

which is precisely what was obtained from the cross product, simply rearranged with a negative distributed. This proves the identity.

## 3a

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) xyz = yz\hat{\imath} + xz\hat{\jmath} + xy\hat{k}$$

## 3b

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) e^x y^3 \sin(z) = e^x y^3 \sin(z)\hat{\imath} + 3e^x y^2 \sin(z)\hat{\jmath} + e^x y^3 \cos(z)\hat{k}$$

## 3c

$$\nabla f = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) e^{-(x^2+y^2+z^2)} = -2xe^{-(x^2+y^2+z^2)}\hat{\imath} - 2ye^{-(x^2+y^2+z^2)}\hat{\jmath} - 2ze^{-(x^2+y^2+z^2)}\hat{k}$$

## 4

We write $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

The divergence of this is

$$\begin{aligned}
\nabla \cdot (\nabla \times \vec{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}
\end{aligned}$$

Presuming continuity of all second pure partials, it is justified to interchange the order of partial differentiation. Setting each term in the order $x$-$y$-$z$, it becomes evident that the expression is three pairs of cancelling terms, and the identity is thus proven.

## 5

Simply,

$$\vec{A} \cdot \nabla = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

$$\Rightarrow (\vec{A} \cdot \nabla)\vec{B} = \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) B_x + \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) B_y$$

$$+ \left( A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) B_z$$

$$= A_x \frac{\partial B_x}{\partial x} + A_y \frac{\partial B_x}{\partial y} + A_z \frac{\partial B_x}{\partial z} + A_x \frac{\partial B_y}{\partial x} + A_y \frac{\partial B_y}{\partial y} + A_z \frac{\partial B_y}{\partial z} + A_x \frac{\partial B_z}{\partial x} + A_y \frac{\partial B_z}{\partial y} + A_z \frac{\partial B_z}{\partial z}$$