2231 HW 7

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1

The bound surface charge density is $\sigma_b = P \cdot \hat{n}$ and the bound volume charge density is $\rho_b = -\nabla \cdot \vec{P}$. The sum of the totals of these two charges must be zero, i.e.

$$q = \int_{\partial V} \sigma_b dA + \int_V \rho_b dV = \int_{\partial V} 1\vec{P} \cdot \hat{n} dA - \int_V 1\nabla \cdot \vec{P} dV$$

This is the right half of an integration by parts, with the ones inserted to make correspondence with the cannonical statement obvious. The associated left side is

$$\int_V \vec{P} \cdot \nabla 1 dV$$

which is clearly zero, since $\nabla 1 = 0$.

2

The electric field inside the first configuration without the dielectric is $\frac{\sigma}{\epsilon_0}$. In the dielectric, it is $\frac{\sigma}{\epsilon}$, and integrating piecewise to find the potential we obtain $V = \frac{\sigma d}{2\epsilon_0} + \frac{\sigma d}{2\epsilon}$. Writing out the charge density as $\frac{Q}{A}$ and factoring out $2\epsilon_0$, this becomes $V = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$; the corresponding capacitance is $C = \frac{Q}{V} = \frac{2\epsilon_0 A}{d(1+\epsilon_0/\epsilon)}$. This differs from the vacuum value of $C = \frac{A\epsilon_0}{d}$ by a factor of $\frac{2}{1+1/\epsilon_r}$.

For the second configuration, we instead express Q in terms of V. In the region without the dielectric, $E = \frac{\sigma}{\epsilon_0} \Leftrightarrow \sigma = \frac{\epsilon_0 V}{d}$. In the other region, $\sigma' = \frac{\epsilon V}{d}$. The total charge is then $Q = \frac{A}{2}\sigma + \frac{A}{2}\sigma' = \frac{\epsilon_0 AV}{2d} + \frac{\epsilon AV}{2d}$. Dividing by V and factoring out ϵ_0, $C = \frac{\epsilon_0 A}{2d}\left(1 + \frac{\epsilon}{\epsilon_0}\right)$. This differs from the vacuum capacitance by a factor of $\frac{1+\epsilon/\epsilon_0}{2}$.

The calculations for each of the values of interest have been all but done already, so we present them without comment:

$$E_a = \begin{cases} \frac{-2(\epsilon+\epsilon_0)V}{\epsilon d}\hat{z} & \text{in the dielectric} \\ \frac{-2(\epsilon+\epsilon_0)V}{\epsilon_0 d}\hat{z} & \text{in the air} \end{cases}$$

$$E_b = \begin{cases} -\frac{V}{d}\hat{z} & \text{in the air} \\ -\frac{V}{d}\hat{z} & \text{in the dielectric} \end{cases}$$

$$P_a = \begin{cases} 0 & \text{in the air} \\ \epsilon(\epsilon_r - 1)\left(\frac{-2(\epsilon+\epsilon_0)V}{\epsilon d}\hat{z}\right) & \text{in the dielectric} \end{cases}$$

$$P_b = \begin{cases} 0 & \text{in the air} \\ \epsilon(\epsilon_r - 1)\left(-\frac{V}{d}\hat{z}\right) & \text{in the dielectric} \end{cases}$$

$$D_a = \begin{cases} \epsilon_0\frac{-2(\epsilon+\epsilon_0)V}{\epsilon d}\hat{z} & \text{in the air} \\ \epsilon_0\frac{-2(\epsilon+\epsilon_0)V}{\epsilon_0 d} + \epsilon(\epsilon_r - 1)\frac{-2(\epsilon+\epsilon_0)V}{\epsilon_0 d}\hat{z} & \text{in the dielectric} \end{cases}$$

$$D_b = \begin{cases} -\frac{V}{b}\hat{z} & \text{in the air} \\ -\frac{V}{b}\hat{z} + \epsilon(\epsilon_r - 1)\left(-\frac{V}{d}\right)\hat{z} & \text{in the dielectric} \end{cases}$$

$$\sigma_{fa} = -\nabla \cdot P_a = 0,$$

$$\sigma_{ba} = P_a \cdot \hat{n} = \epsilon(\epsilon_r - 1)\left(\frac{-2(\epsilon+\epsilon_0)V}{\epsilon d}\right)$$

$$\sigma_{fb} = -\nabla \cdot P_b = 0$$

$$\sigma_{bb} = P_b \cdot \hat{n} = \epsilon(\epsilon_r - 1)\left(-\frac{V}{d}\right)$$

3

Since the displacement follows Gauss's law with the free charge, we have

$$4\pi r^2 D = q \Leftrightarrow D = \frac{q}{4\pi r^2}$$

Since $D = \epsilon E$, we can calculate E as

$$\vec{E} = \begin{cases} \frac{q}{4\pi\epsilon r^2}\hat{r}, & r \leq R \\ \frac{q}{4\pi\epsilon_0 r^2}\hat{r} & r > R \end{cases}$$

We can write $P = \epsilon_0\chi_e E$, so

$$P = \begin{cases} \frac{\epsilon_0\chi_e q}{4\pi\epsilon r^2}\hat{r} & r \leq R \\ \frac{\chi_e q}{4\pi r^2}\hat{r} & r > R \end{cases}$$

For the bound charge, we have since P is radial

$$\sigma_b = P \cdot \hat{n} = P$$

Totalling this density over the surface area of the sphere,

$$q_b = 4\pi R^2\left(\frac{\epsilon_0\chi_e q}{4\pi\epsilon R^2}\right) = \frac{\chi_e}{1+\chi_e}q$$

The corresponding negative charge is clustered at the center of the dielectric near the point charge.

4

Like above, we may calculate the electric field via Gauss's law on the displacement.

$$D = \begin{cases} \frac{Q}{4\pi r^2}\hat{r}, & r > b \\ \frac{Q}{4\pi r^2}\hat{r}, & a < r < b \\ 0, & r < a \end{cases}$$

The electric field is then, employing $\epsilon = \epsilon_0(1+\chi_e)$

$$E = \frac{D}{\epsilon} = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, & r > b \\ \frac{Q}{4\pi\epsilon r^2}\hat{r}, & a \leq r \leq b \\ 0, & r < a \end{cases}$$

The energy of the configuration is

$$W = \frac{1}{2}\int_{\mathbb{R}^3} E \cdot D dV = \frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\left(\int_0^a + \int_a^b + \int_b^{\infty}\right) E \cdot D r^2 \sin\theta dr d\varphi d\theta$$

$$= 2\pi\left(\int_a^b \frac{Q^2}{16\pi^2\epsilon r^2}dr + \int_b^{\infty}\frac{Q^2}{16\pi^2\epsilon_0 r^2}dr\right)$$

$$= \frac{Q^2}{8\pi}\left(\frac{1}{\epsilon a} - \frac{1}{\epsilon b} + \frac{1}{\epsilon_0 b}\right)$$

5

The potential in the absence of the dielectric is, according to the book,

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

Therefore the field is the negative gradient of this potential:

$$E = E_0\left[\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta\hat{r} - \left(1 - \frac{R^3}{r^3}\right)\sin\theta\hat{\theta}\right]$$