4721 HW 6

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Problem 1. *Common forms assumed for the momentum distributions of valence quarks in the proton are:*

$$F_u = xu(x) = a(1-x)^3,\ F_d(x) = xd(x) = b(1-x)^3.$$

If the valence quarks account for half of the proton's momentum—i.e.

$$\int_0^1 xu(x)dx + \int_0^1 xd(x)dx = \frac{1}{2},$$

find the values of a and b. Hint: the u quarks carry approximately twice as much momentum as the d quarks in the proton.

Solution. Some calculus:

$$\int_0^1 (1-x)^3 dx = \int_1^0 u^3 \cdot -du = \frac{u^4}{4}\bigg|_{u=0}^{u=1} = \frac{(1-x)^4}{4}\bigg|_{x=1}^{x=0} = \frac{1}{4}$$

Accordingly,

$$\int_0^1 xu(x)dx + \int_0^1 xd(x)dx = \frac{1}{2} \Leftrightarrow a\int_0^1 (1-x)^3 dx + b\int_0^1 (1-x)^3 dx = \frac{1}{2} \Leftrightarrow \frac{a}{4} + \frac{b}{4} = \frac{1}{2} \Leftrightarrow a + b = 2.$$

There are two valence up quarks, and one valence down quark, so one would expect the total momentum in the up quarks to be double that of the down quark—accordingly,

$$2b + b = 2 \Leftrightarrow b = \frac{2}{3} \Rightarrow a = \frac{4}{3}.$$

□

Problem 2. *What is the color wavefunction for mesons, in analogy to that for baryons of*

$$y_{baryon} = y_{space}y_{spin}(rgb + gbr + brg - rbg - bgr - grb)?$$

Explain your answer.

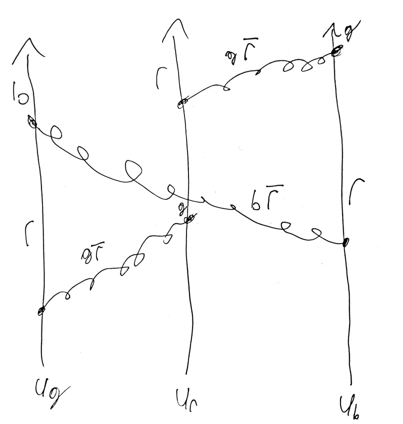
Solution. This baryon color wavefunction comes from noticing that baryons are made up of three quarks, all of different color charge (so as to produce a color-neutral baryon), and then requiring the resulting wavefunction to be antisymmetric under particle interchange, so as to produce a fermionic composite particle. By contrast, mesons are bosonic, and so the color wavefunction must stay the same under particle interchange:

$$y_{meson} = y_{space} y_{spin} \left(r\bar{r} + g\bar{g} + b\bar{b} \right).$$

□

Problem 3. *The diagram below shows the internal gluon interactions in a proton. Complete the diagram by labelling the color of the quarks and gluons.*

Solution. The governing principle is that the net color of the proton is always white.



Note that I accidentally copied down the quark types incorrectly; the rightmost trajectory should be a down quark, of course. □

Problem 4. *Which of the following processes are allowed? If not allowed, state why. If allowed, say whether the process is strong, weak, or electromagnetic.*

1. $\nu_e + p \to e^- + \pi^+ + p$
2. $e^+ + e^- \to \mu^+ + \mu^-$
3. $\Sigma^- \to n + \pi^-$
4. $\bar{\nu}_e + p \to e^- + n$
5. $e^- + p \to \nu_e + \pi^0$

Solution.

1. Allowed; weak.
2. Allowed; electromagnetic.
3. Allowed; strong.
4. Disallowed; charge changes.
5. Disallowed; baryon number changes.

□

Problem 5 (Double Points). *The differential cross section for $e^+ + e^- \to \mu^+ + \mu^-$ is given by*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(\hbar c)^2(1+\cos^2\theta)$$

in a collider experiment where $s = 4E_e$ and E_e is the electron/positron energy.

1. *Integrate over the solid angle to obtain an expression for the total cross section.*
2. *If you use an electron beam energy of 4 GeV, what rate of production of $\mu^+\mu^-$ would you expect at a luminosity of 10^{33} Hz/cm^2?*
3. *Calculate the ratio of the hadronic production cross section to that for $\mu^+\mu^-$ at $E_e = 500\,\mathrm{GeV}$. If you use an electron beam energy of 500 GeV, what must the luminosity be to measure the hadronic cross section within 24 hours with 10% statistical uncertainty?*

Solution.

1. We have a total cross-section

$$\int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega = \int_0^{2\pi}\int_0^{\pi} \frac{\alpha^2}{4s}(\hbar c)^2\bigl(1+\cos^2\theta\bigr)\sin\theta d\theta d\phi = \frac{2\pi(\alpha\hbar c)^2}{4s}\int_0^{\pi} \sin\theta + \sin\theta\cos^2\theta d\theta$$

$$= \frac{\pi(\alpha\hbar c)^2}{2s}\left(2 - \int_1^{-1} u^2 du\right) = \frac{\pi(\alpha\hbar c)^2}{2s}\left(2 + \frac{u^3}{3}\Big|_{-1}^{1}\right) = \frac{\pi(\alpha\hbar c)^2}{2s}\left(2+\frac{2}{3}\right) = \frac{4\pi(\alpha\hbar c)^2}{3s}$$

2. If the electron beam energy is 4 GeV, $s = 4\cdot E_e = 16\,\mathrm{GeV}$, and at the given luminosity, the expected production rate is

$$L\cdot\sigma = 10^{33}\,\mathrm{Hz/cm^2}\cdot\frac{4\pi\bigl(\frac{1}{137}\cdot 3.16\times10^{-24}\,\mathrm{J\cdot cm}\bigr)^2}{3\cdot 16\,\mathrm{GeV}\cdot 1.6\times10^{-19}\,\mathrm{J/GeV}} = 8.71\times10^{-10}\,\mathrm{Hz}$$

3. The cross-section ratio is, if the Standard Model's accounting of quarks is correct, should be $\frac{11}{3}$—a 500 GeV beam is well beyond the Z production threshold, so all quark types will be produced (including the top quark). Accordingly, the expected hadronic cross section would be

$$\sigma_h = R\sigma_{\mu\mu} = \frac{15}{3}\cdot\frac{4\pi\bigl(\frac{1}{137}\cdot 3.16\times10^{-24}\,\mathrm{J\cdot cm}\bigr)^2}{3\cdot 500\,\mathrm{GeV}\cdot 1.6\times10^{-19}\,\mathrm{J/GeV}} = 1.39\times10^{-43}\,\mathrm{cm}^2$$

As this is a count data experiment, the observed hadron production rate is the parameter of the Poisson distribution associated with the Poisson process of the number of hadrons produced. The sampling distribution from a Poisson distribution has standard deviation $\sqrt{\frac{\lambda}{n}}$, for a sample of n counts. This estimates the uncertainty in a post-hoc parameter estimation of r_h, given our prediction above. The uncertainty in the entailing estimate of σ_h is, by error propagation on the formula $\sigma_h = \frac{r_h}{L}$,

$$\frac{1}{L}\sqrt{\frac{r_h}{n_h}}$$

However, n_h and r_h we can estimate in advance as $n_h = \sigma_h L t$ and $r_h = \sigma_h L$; we therefore have an estimated uncertainty

$$\frac{1}{L}\sqrt{\frac{\sigma_h L}{\sigma_h L t}} = \frac{1}{L\sqrt{t}}.$$

A 10% margin around the expected value of σ_h would be $\pm 1.39 \times 10^{-44}\,\mathrm{cm}^2$, so equating this to the above,

$$1.39 \times 10^{-44}\,\mathrm{cm}^2 = \frac{1}{L\sqrt{t}} \Leftrightarrow L = \frac{1}{1.39 \times 10^{-44}\,\mathrm{cm}^2 \cdot \sqrt{24\,\mathrm{hr}}} = 2.54 \times 10^{41}\,\mathrm{Hz/cm}^2$$

□

Problem 6 (Double Points)**.** *In an e^+e^- collider experiment, a resonance R is observed at $E_{cm} = 10\,\mathrm{GeV}$ in both the $\mu^+\mu^-$ and hadronic final states. The integrated cross sections are*

$$\int \sigma_{\mu\mu}(E)dE = 10\,\mathrm{nb}\cdot\mathrm{GeV}$$

and

$$\int \sigma_h(E)dE = 300\,\mathrm{nb}\cdot\mathrm{GeV}.$$

Use a Breit-Wigner form for the resonance production to deduce the partial widths $\Gamma_{\mu\mu}$ and Γ_h in MeV for the decays $R \to \mu^+\mu^-$ and $R \to$ hadrons. Assume the integral

$$\int_{resonance} \frac{dE}{(E - Mc^2) + \Gamma^2/4} dE \approx \frac{2\pi}{\Gamma}.$$

Solution. The Breit-Wigner resonance formula for a particular decay channel gives

$$\sigma_f = \frac{3\pi\lambda^2}{4} \cdot \frac{\Gamma_i \Gamma_f}{(E - E_r)^2 + \Gamma_T^2/4}$$

Accordingly, the integrated cross-section is expected to be

$$\int \sigma_f dE = \frac{3\pi\lambda^2}{4} \cdot \frac{2\pi\Gamma_i\Gamma_f}{\Gamma_T} = \frac{3\pi^2\lambda^2}{2} \cdot \frac{\Gamma_i\Gamma_f}{\Gamma_T} = \frac{3\lambda^2}{8} \cdot \frac{\Gamma_i\Gamma_f}{\Gamma_T}$$

For the two decays in question, presuming they are the only relevant channels,

$$\frac{3\lambda^2}{8} \cdot \frac{\Gamma_{e^+e^-}\Gamma_{\mu\mu}}{\Gamma_T} = 10\,\mathrm{nb}\cdot\mathrm{GeV}$$

$$\frac{3\lambda^2}{8} \cdot \frac{\Gamma_{e^+e^-}\Gamma_h}{\Gamma_T} = 300\,\mathrm{nb}\cdot\mathrm{GeV}$$

$$\Gamma_T = \Gamma_{\mu\mu} + \Gamma_h$$

We can compute, in the relativistic approximation,

$$\lambda = \frac{h}{p} = \frac{h}{E/c} = \frac{4.13\times 10^{-15}\,\mathrm{eV}\cdot\mathrm{s}}{10\,\mathrm{GeV}/3\times 10^{8}\,\mathrm{m/s}} = 1.239\times 10^{-16}\,\mathrm{m} \Rightarrow \frac{3\lambda^2}{8} = 5.76\times 10^{-33}\,\mathrm{m}^2 = 57.6\,\mu\mathrm{b}$$

Three equations, four unknowns; simplifying and substituting,

$$\frac{\Gamma_{e^+e^-}\Gamma_{\mu\mu}}{\Gamma_{\mu\mu}+\Gamma_h} = 17.4\,\mathrm{MeV}$$

$$\frac{\Gamma_{e^+e^-}\Gamma_h}{\Gamma_{\mu\mu}+\Gamma_h} = 5.21\,\mathrm{MeV}.$$

Taking the ratio,

$$\frac{\Gamma_{\mu\mu}}{\Gamma_h} = \frac{17.4\,\mathrm{MeV}}{5.21\,\mathrm{MeV}} = 3.34 \Leftrightarrow \Gamma_{\mu\mu} = 3.34\Gamma_h.$$

Substituting,

$$\Gamma_{e^+e^-} = (1+3.34)5.21\,\mathrm{MeV} = 22.61\,\mathrm{MeV}.$$

Try as I might, I cannot reduce further; all I get is different forms of the linewidth ratio expressed above (which is just the expected $10/3$). □

Problem 7. *Find the threshold kinetic energy for each of the following reactions, assuming the first particle to be incident on the second particle at rest:*

1. $K^- + p \to \Xi^- + K^+$
2. $\bar{p} + p \to \Upsilon$
3. $\pi^- + p \to \omega + n$

Solution. For all these calculations, we transform into the center-of-mass frame and apply the conservation of energy equation; in each case, the velocity of the initial particles is the same, and the velocity of the produced particles is zero in this frame. Then, we perform velocity addition on the particle that's supposed to move in the lab frame to obtain its velocity; the lab-frame beam kinetic energy is then immediate. Masses are read from the Particle Data Group tables.

1. We have

$$\gamma m_{K^-}c^2 + \gamma m_p c^2 = m_{\Xi^-}c^2 + m_{K^+}c^2 \Leftrightarrow \gamma = \frac{m_\Xi + m_K}{m_K + m_p} \Leftrightarrow \frac{v}{c} = \sqrt{1-\left(\frac{m_K+m_p}{m_\Xi+m_K}\right)^2}$$

$$= \sqrt{1-\left(\frac{494\,\mathrm{MeV/c^2}+938\,\mathrm{MeV/c^2}}{1321\,\mathrm{MeV/c^2}+494\,\mathrm{MeV/c^2}}\right)^2} = 0.61$$

Using velocity addition, one gets an incoming K^- velocity of

$$v = \frac{2v_{cm}}{1 + v_{cm}^2/c^2} = \frac{2 \cdot 0.61c}{1 + (0.61c)^2/c^2} = 0.896c$$

corresponding to an energy of

$$E = \gamma m_{K^-} c^2 = \frac{494\,\mathrm{MeV/c^2} \cdot c^2}{\sqrt{1 - (0.896c)^2/c^2}} = 2.5\,\mathrm{GeV},$$

or a kinetic energy of

$$K = E - m_{K^-} c^2 = 2.5\,\mathrm{GeV} - 494\,\mathrm{MeV} = 2\,\mathrm{GeV}$$

2. We have

$$\gamma m_{\bar{p}} c^2 + \gamma m_p c^2 = m_\Upsilon c^2 \Leftrightarrow \gamma = \frac{m_\Upsilon}{m_{\bar{p}} + m_p} \Leftrightarrow \frac{v}{c} = \sqrt{1 - \left(\frac{m_{\bar{p}} + m_p}{m_\Upsilon}\right)^2}$$

$$= \sqrt{1 - \left(\frac{938\,\mathrm{MeV/c^2} + 938\,\mathrm{MeV/c^2}}{9.46\,\mathrm{GeV/c^2}}\right)^2} = 0.98$$

Using velocity addition, one gets an incoming K^- velocity of

$$v = \frac{2v_{cm}}{1 + v_{cm}^2/c^2} = \frac{2 \cdot 0.98c}{1 + (0.98c)^2/c^2} = 0.9994c$$

corresponding to an energy of

$$E = \gamma m_{\bar{p}} c^2 = \frac{494\,\mathrm{MeV/c^2} \cdot c^2}{\sqrt{1 - (0.99994c)^2/c^2}} = 4\,\mathrm{TeV},$$

with respect to which the proton mass is a rounding error, so the kinetic energy will be the same.

3. We have

$$\gamma m_{\pi^-} c^2 + \gamma m_p c^2 = m_\omega c^2 + m_n c^2 \Leftrightarrow \gamma = \frac{m_\omega + m_n}{m_{\pi^-} + m_p} \Leftrightarrow \frac{v}{c} = \sqrt{1 - \left(\frac{m_{\pi^-} + m_p}{m_\omega + m_n}\right)^2}$$

$$= \sqrt{1 - \left(\frac{140\,\mathrm{MeV/c^2} + 938\,\mathrm{MeV/c^2}}{783\,\mathrm{MeV/c^2} + 940\,\mathrm{MeV/c^2}}\right)^2} = 0.76$$

Using velocity addition, one gets an incoming K^- velocity of

$$v = \frac{2v_{cm}}{1 + v_{cm}^2/c^2} = \frac{2 \cdot 0.76c}{1 + (0.76c)^2/c^2} = 0.96c$$

corresponding to an energy of

$$E = \gamma m_{\pi^-} c^2 = \frac{140\,\mathrm{MeV/c^2} \cdot c^2}{\sqrt{1 - (0.96c)^2/c^2}} = 1.85\,\mathrm{GeV},$$

or a kinetic energy of

$$K = E - m_{\pi^-} c^2 = 1.85\,\mathrm{GeV} - 140\,\mathrm{MeV} = 1.71\,\mathrm{MeV}.$$

□