4123 HW 2

Duncan Wilkie

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1a

The Lagrangian is, from the reference frame of the attatchment point,

$$L = T - U = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - m_1 g y_1 - m_2 g y_2$$

where y_1 and y_2 are the (signed) vertical components of the distance from that point to each mass. The two constraints of this system are the lengths l_1 and l_2; mathematically,

$$\vec{r}_1 \cdot \vec{r}_1 = l_1^2$$

and

$$(\vec{r}_2 - \vec{r}_1) \cdot (\vec{r}_2 - \vec{r}_1) = l_2^2$$

where $\vec{r}_1, \vec{r}_2$ are the vectors from the attatchment point of the double pendulum to each of the masses. In the absence of constraints in two dimensions, this would have $2N = 4$ degrees of freedom. These constraints reduce that to 2, so that we have two generalized coordinates (the system is holonomic, as the constraints are expressible as $f = 0$ for some function f). We choose as our generalized coordinates θ, the angle between the positive x direction and $\vec{r}_1$, and ϕ, the angle between $\vec{r}_1$and $\vec{r}_2$. In terms of the new coordinates,

$$x_1 = l_1 \cos(\theta), y_1 = -l_1 \sin(\theta)$$

$$v_1 = \sqrt{\dot{x_1}^2 + \dot{y_1}^2} = \sqrt{l_1^2 \sin^2(\theta)\dot{\theta}^2 + l_1^2 \cos^2(\theta)\dot{\theta}^2} = l_1\dot{\theta}$$

$$x_2 = x_1 + l_2 \cos(\phi) = l_1 \cos(\theta) + l_2 \cos(\phi), y_2 = y_1 - l_2 \sin(\phi) = -l_1 \sin(\theta) - l_2 \sin(\phi)$$

$$v_2 = \sqrt{\dot{x_2}^2 + \dot{y_2}^2} = \sqrt{(l_1 \sin(\theta)\dot{\theta} + l_2 \sin(\phi)\dot{\phi})^2 + (l_1 \cos(\theta)\dot{\theta} + l_2 \cos(\phi)\dot{\phi})^2}$$

Expanding the binomials and getting creative with product-to-sum identities, this is

$$= \sqrt{l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1 l_2 \dot{\theta}\dot{\phi}\cos(\theta - \phi)}$$

Therefore,

$$L = \frac{1}{2}m_1 l_1^2 \dot{\theta}^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1 l_2 \dot{\theta}\dot{\phi}\cos(\theta - \phi)] + (m_1 + m_2)g l_1 \sin(\theta) + m_2 g l_2 \sin(\phi)$$

The two Euler-Lagrange equations for these generalized coordinates are

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Leftrightarrow (m_1+m_2)gl_1\cos(\theta) - m_2 l_1 l_2 \dot{\theta}\dot{\phi}\sin(\theta-\phi) = \frac{d}{dt}[(m_1+m_2)l_1^2\dot{\theta} + 2l_1 l_2\dot{\phi}\cos(\theta-\phi)]$$

$$\Leftrightarrow (m_1+m_2)gl_1\cos(\theta) - m_2 l_1 l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi)$$
$$= (m_1+m_2)l_1^2\ddot{\theta} + 2l_1 l_2[\ddot{\phi}\cos(\theta-\phi) - \dot{\phi}\dot{\theta}\sin(\theta-\phi) + \dot{\phi}^2\sin(\theta-\phi)]$$

and

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \Leftrightarrow m_2 g l_2\cos(\phi) + 2l_1 l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) = \frac{d}{dt}[m_2 l_2^2\dot{\phi} + 2l_1 l_2\dot{\theta}\cos(\theta-\phi)]$$

$$\Leftrightarrow m_2 g l_2\cos(\phi) + 2l_1 l_2\dot{\theta}\dot{\phi}\sin(\theta-\phi) = m_2 l_2^2\ddot{\phi} + 2l_1 l_2[\ddot{\theta}\cos(\theta-\phi) + \dot{\phi}\dot{\theta}\sin(\theta-\phi) - \dot{\theta}^2\sin(\theta-\phi)]$$

1b

For small oscillations, $\sin(x) \approx x$ and $\cos(x) \approx 1 - \frac{x^2}{2}$. Going back to the calculation of v_2, by this cosine approximation we can obtain in terms of l the following:

$$v_2 \approx l\sqrt{\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(1 - \frac{(\theta-\phi)^2}{2})} = l\sqrt{\dot{\theta}^2 - \dot{\theta}\dot{\phi}\theta^2 + 2\dot{\theta}\dot{\phi}\theta\phi - \dot{\theta}\dot{\phi}\phi^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2}$$

The Lagrangian is then

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}ml^2[\dot{\theta}^2 - \dot{\theta}\dot{\phi}\theta^2 + 2\dot{\theta}\dot{\phi}\theta\phi - \dot{\theta}\dot{\phi}\phi^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2] + 2mgl\theta + mgl\phi$$

and the Euler-Lagrange equations are

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Leftrightarrow ml^2\dot{\theta}\dot{\phi}\phi - ml^2\dot{\theta}\dot{\phi}\theta + 2mgl$$

$$= \frac{d}{dt}\left[ml^2\dot{\theta} + ml^2\dot{\theta} - \frac{1}{2}ml^2\dot{\phi}\theta^2 + ml^2\dot{\phi}\theta\phi - \frac{1}{2}ml^2\dot{\theta}\phi^2\right]$$

$$= ml^2\left(2\ddot{\theta} - \dot{\phi}\theta\dot{\theta} - \frac{1}{2}\ddot{\phi}\theta^2 + \dot{\phi}(\theta\dot{\phi} + \dot{\theta}\phi) + \ddot{\phi}\theta\phi - \dot{\theta}\phi\dot{\phi} - \frac{1}{2}\ddot{\theta}\phi^2\right)$$

$$\Leftrightarrow \dot{\theta}\dot{\phi}(\phi-\theta) + \frac{2g}{l} = \dot{\theta}\ddot{\theta} + \ddot{\theta} - \dot{\phi}\theta\dot{\theta} - \frac{1}{2}\ddot{\phi}\theta^2 + \dot{\phi}(\theta\dot{\phi} + \dot{\theta}\phi) + \ddot{\phi}\theta\phi - \dot{\theta}\phi\dot{\phi} - \frac{1}{2}\ddot{\theta}\phi^2$$

and

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \Leftrightarrow ml^2\dot{\theta}\dot{\phi}\theta - ml^2\dot{\theta}\dot{\phi}\phi + mgl$$

$$= ml^2\frac{d}{dt}\left[-\frac{1}{2}\dot{\theta}\theta^2 + \dot{\theta}\theta\phi - \frac{1}{2}\dot{\theta}\phi^2 + \dot{\theta} + \dot{\phi}\right]$$

$$= ml^2\left(-\dot{\theta}^2\theta - \frac{1}{2}\ddot{\theta}\theta^2 + \dot{\theta}(\theta\dot{\phi} + \dot{\theta}\phi) + \ddot{\theta}\theta\phi - \dot{\theta}\phi\dot{\phi} - \frac{1}{2}\ddot{\theta}\phi^2 + \dot{\theta} + \dot{\phi}\right)$$

2

The Lorentz force is given by $\vec{F} = qE + qv \times B$. Writing the fields in terms of some potential, $-\nabla U = -q\nabla\phi + qv \times (\nabla \times \vec{A})$. We can integrate this along an arbitrary path γ between two points γ_0 and γ_1 and take the zero point of the potential to be at γ_0, yielding

$$U = q\phi + q\int_\gamma v \times (\nabla \times \vec{A})$$

We write out

$$v \times (\nabla \times \vec{A}) = v \times \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$

$$= \left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\hat{\imath} - \left[v_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]\hat{\jmath}$$

$$+ \left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]\hat{k}$$

The line integral of this is, up to a constant,

$$\int_\gamma \left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]dx - \left[v_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]dy$$

$$+ \left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]dz$$

$$= \left(v_yA_y - v_y\int_\gamma \frac{\partial A_x}{\partial y}dx - v_z\int_\gamma \frac{\partial A_z}{\partial z}dx + v_zA_z\right) - \left(v_x\int_\gamma \frac{\partial A_y}{\partial x}dy - v_xA_x - v_zA_z + v_z\int_\gamma \frac{\partial A_y}{\partial z}dy\right)$$

$$+ \left(v_xA_x - v_x\int_\gamma \frac{\partial A_z}{\partial x}dz - v_y\int_\gamma \frac{\partial A_z}{\partial y}dz + v_yA_y\right)$$

$$= 2v \cdot A - \int_\gamma \left(v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}\right)dx + \left(v_x\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_y}{\partial z}\right)dy + \left(v_x\frac{\partial A_z}{\partial x} + v_y\frac{\partial A_z}{\partial y}\right)dz$$

3

The Lagrangian is $L = T - V = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m(\dot{x} + \dot{y})^2 - mgy$. The constraint for this system is $\vec{r} \cdot \vec{r} = R \Leftrightarrow x^2 + y^2 = R$. We obtain two constrained Euler-Lagrange equations:

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow -mg + 2\lambda y = \frac{d}{dt}(m\dot{y}) \Leftrightarrow 2\lambda y - mg = m\ddot{y}$$

$$\frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = \frac{d}{dt}(m\dot{x}) \Leftrightarrow 2\lambda x = m\ddot{x}$$

The equations of motion above are in the form of Newton's second law, and so we see the constraint force is precisely $2\lambda x\hat{x} + 2\lambda y\hat{y}$

4

For this problem, we have the same Lagrangian, but take spherical instead of polar coordinates:

$$L = \frac{1}{2}mv^2 - mgy \Leftrightarrow L = \frac{1}{2}mv^2 - mgr\sin(\phi)\sin(\theta)$$

Our constraint is $\vec{r}\cdot\vec{r} = l^2 \Leftrightarrow r = l$, so this becomes

$$L = \frac{1}{2}ml^2(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2(\theta)) - mgl\sin(\theta)\sin(\phi)$$

There are then two Euler-Lagrange equations:

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \Leftrightarrow -mgl\sin(\theta)\cos(\phi) = \frac{d}{dt}\left[ml^4\dot{\phi}\sin^2(\theta)\right]$$

$$\Leftrightarrow -mgl\sin(\theta)\cos(\theta) = ml^4\ddot{\phi}\sin^2(\theta) + ml^4\dot{\phi}\sin(2\theta)\dot{\theta}$$

and

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Leftrightarrow ml^2\dot{\phi}^2\sin(2\theta) - mgl\cos(\theta)\sin(\phi) = \frac{d}{dt}\left[ml^4\dot{\theta}\right]$$

$$\Leftrightarrow ml^2\dot{\phi}^2\sin(2\theta) - mgl\cos(\theta)\sin(\phi) = ml^4\ddot{\theta}$$

For small angles, these become

$$-g(1-\theta/2) = l^3\ddot{\phi}\theta + 2l^3\dot{\phi}\dot{\theta}$$

and

$$2l\dot{\phi}^2\theta - g(1-\theta^2/2)\phi = l^3\ddot{\theta}$$

5

We have the same inital Lagrangian as before, but keep the problem in rectangular coordinates:

$$L = \frac{1}{2}mv^2 - mgy \Leftrightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgy$$

with constraint $f(x,y,z) = c \Leftrightarrow \vec{r}\cdot\vec{r} = l^2 \Leftrightarrow x^2+y^2+z^2 = l^2$. This yields three constrained Euler-Lagrange equations

$$\frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = m\ddot{x}$$

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow 2\lambda y - mg = m\ddot{z}$$

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} \Leftrightarrow 2\lambda z = m\ddot{z}$$

which are all equations of motion in the form of Newton's second law, so the constraint force (equal in magnitude to the tension in the string) is $2\lambda x\hat{x} + 2\lambda y\hat{y} + 2\lambda z\hat{z}$