4142 HW 2

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Problem 1. *Given* $Y_2^1(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$*, use the raising operator* L_+ *to find* $Y_2^2(\theta,\phi)$*. What happens if* L_+ *is applied again?*

The raising operator has the form

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$$

Applying this to the given spherical harmonic,

$$L_+Y_2^1 = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}\right)$$

$$= -\hbar e^{i\phi}\sqrt{\frac{15}{8\pi}}\left[e^{i\phi}\frac{\partial}{\partial\theta}\frac{1}{2}\sin 2\theta + i\cot\theta\sin\theta\cos\theta\frac{\partial}{\partial\phi}e^{i\phi}\right]$$

$$= -\hbar e^{2i\phi}\sqrt{\frac{15}{8\pi}}\left(\cos 2\theta - \cos^2\theta\right)$$

$$= \hbar\sqrt{\frac{15}{8\pi}}\sin^2\theta e^{2i\phi}$$

The raising operator satisfies $L_+f_\ell^m = \hbar\sqrt{(\ell-m)(\ell+m+1)}f_\ell^{m+1}$; dividing by $\hbar\sqrt{(2-1)(2+1+1)} = 2\hbar$,

$$Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}$$

Since $m = \ell$, there are no more states with larger m, since $-\ell \leq m \leq \ell$. L_+ then maps Y_2^2 to zero.

Problem 2. *A particle is in an angular momentum state of* $\ell = 2, m = 1$*. What is the probability of finding it at the position* $\theta = \pi/4,\ \phi = \pi/4$ *to within* $d\theta = d\phi = 0.01\,\mathrm{rad}$*?*

We integrate the probability amplitude over this region:

$$P = \int_{\pi/4-0.005}^{\pi/4+0.005}\int_{\pi/4-0.005}^{\pi/4+0.005} Y_2^1 Y_2^{1*}d\phi d\theta = \int_{\pi/4-0.005}^{\pi/4+0.005}\int_{\pi/4-0.005}^{\pi/4+0.005}\frac{15}{8\pi}\frac{1}{4}\sin^2 2\theta e^{i\phi-i\phi}d\phi d\theta$$

$$= 0.01\frac{15}{32\pi}\int_{\pi/4-0.005}^{\pi/4+0.005}\sin^2 2\theta d\theta = 0.01\frac{15}{32\pi}\int_{\pi/4-0.005}^{\pi/4+0.005}\frac{1}{2}\left(1-\cos 4\theta\right)d\theta$$

$$= 0.01\frac{15}{64\pi}\left(\theta\Big|_{\pi/4-0.005}^{\pi/4+0.005} - \frac{1}{4}\sin 4\theta\Big|_{\pi/4-0.005}^{\pi/4+0.005}\right)$$

$$= 0.01\frac{15}{64\pi}\left(0.01 - \frac{1}{4}\sin(\pi+0.02) + \frac{1}{4}\sin(\pi-0.02)\right) = 7.33\times 10^{-6}$$

Problem 3. *A particle moving in a potential is described by the wave function*

$$\psi(x,y,z) = (xy+yz+zx)e^{-\alpha(x^2+y^2+z^2)}.$$

What is the probability that a measurement of L^2 and L_z gives $6\hbar^2$ and h, respectively?

In spherical coordinates, the wave function is

$$\psi(r,\phi,\theta) = r^2(\cos\phi\sin\phi + \sin\phi\cos\theta + \cos\theta\cos\phi)e^{-\alpha r^2}$$

The coefficients in the expansion of ψ in terms of the simultaneous eigenstates of L^2 and L_z are given by, denoting the angular part of ψ by $\psi_{ang} = \cos\phi\sin\phi + \sin\phi\cos\theta + \cos\theta\cos\phi$,

$$c_{\ell m} = \langle Y_\ell^m|\psi_{ang}\rangle = \int_0^\pi\int_0^{2\pi} Y_l^{m*}\psi_{ang}d\phi d\theta$$

$$= \int_0^\pi\int_0^{2\pi}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)}}e^{-im\phi}P_\ell^m(\cos\theta)(\cos\phi\sin\phi+\sin\phi\cos\theta+\cos\theta\cos\phi)d\phi d\theta$$

$$= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)}}\left(\int_0^\pi\cos\phi\sin\phi e^{-im\phi}d\phi\int_0^{2\pi}P_\ell^m(\cos\theta)d\theta\right.$$

$$+\int_0^\pi\sin\phi e^{-im\phi}d\phi\int_0^{2\pi}\cos\theta P_\ell^m(\cos\theta)d\theta$$

$$\left.+\int_0^\pi\cos\phi e^{-im\phi}d\phi\int_0^{2\pi}\cos\theta P_\ell^m(\cos\theta)d\theta\right)$$

$6\hbar^2$ corresponds to $\ell = 2$; $\hbar$ corresponds to $m = 1$, so we're looking for the coefficient of the Y_2^1 component. The associated Legendre function $P_2^1(\cos\theta) = -3\sin\theta\cos\theta$ is odd, and it's multiplied by an even function in the second factor in the last two terms, so each of the second factors is the integral of an odd periodic function over its period, which is zero in general (any two integrals of a periodic function over two period-long intervals are equal, so use the interval centered on 0).

The coefficient is therefore zero, i.e. the probability of getting $6\hbar^2$ and $\hbar$ is zero.

Problem 4. *An asymmetric rotor with two moments of inertia is described by the Hamiltonian $H = (L_x^2 + L_y^2)/2I_1 + L_z^2/2I_2$ with $I_1 > I_2$. By identifying two suitable commuting operators, describe the energy eigenstates and their eigenvalues.*

For this problem alone, denote $L_x^2 + L_y^2 = L^2$. This observable is compatible with L_z:

$$[L^2, L_z] = [L_x^2, L_z] + [L_y^2, L_z] = L_x[L_x, L_z] + [L_x, L_z]L_x + L_y[L_y, L_z] + [L_y, L_z]L_y$$

$$= -i\hbar L_x L_y - i\hbar L_y L_x + i\hbar L_y L_x + i\hbar L_x L_y = 0$$

Since these commute, they are simultaneously diagonalizable. Consider H acting on a simultaneous eigenstate f of L^2 and L_z, with eigenvalues a and b, respectively:

$$\hat{H}f = (L^2 f)/2I_1 + L_z^2 f/2I_2 = \left(\frac{a}{2I_1} + \frac{b^2}{2I_2}\right) f$$

so the energy eigenstates are precisely the simultaneous eigenstates of L^2 and L_z. The corresponding eigenvalues depend on the eigenvalues of L^2 and L_z as above.