4142 HW 5

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Problem 1. *Consider the energy matrix*

$$\begin{pmatrix} a & 0 & 0 & \text{-}b \\ 0 & c & id & 0 \\ 0 & \text{-}id & \text{-}c & 0 \\ \text{-}b & 0 & 0 & \text{-}a \end{pmatrix}$$

Find the eigenvalues and eigenvectors of this Hamiltonian. It will help to not approach this as a direct diagonalization but regard the elements above as coupling between four states.

Solution. Looking at what will happen when the basis vectors, it's clear this is actually is two non-interacting problems: vectors in combinations of the first and last basis vectors get mapped to each other, and likewise for the second and third. The individual problems are

$$\begin{pmatrix} a & -b \\ -b & -a \end{pmatrix} \Rightarrow \det \begin{vmatrix} a-\lambda & -b \\ -b & -a-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - a^2 - b^2 = 0 \Rightarrow \lambda = \pm\sqrt{a^2+b^2}$$

and

$$\begin{pmatrix} c & id \\ -id & -c \end{pmatrix} \Rightarrow \det \begin{vmatrix} c-\lambda & id \\ -id & -c-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - c^2 - d^2 \Rightarrow \lambda = \pm\sqrt{c^2+d^2}$$

The corresponding eigenvectors are

$$\begin{pmatrix} \frac{-a\pm\sqrt{a^2+b^2}}{b} \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{id}{c} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

□

Problem 2. *At time $t = 0$, a spin-1/2 particle is in the state $|S_z = +\rangle$.*

1. *If S_x is measured at $t = 0$, what is the probability of getting a value $\hbar/2$?*
2. *Instead, with no measurement at $t = 0$, suppose the system evolves in a magnetic field $\vec{B} = B_0\hat{e}_y$. Use the S_z basis to calculate the state of the system at time t.*
3. *Suppose you now measure S_x at t. What is the probability of getting the value $\hbar/2$?*

Solution. In general,

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)},$$

so in the state $S_z = + \Rightarrow a = 1, b = 0$, so the probability that a measurement of S_x yields $\hbar/2$ is

$$P(S_x = +) = \frac{1}{\sqrt{2}}.$$

The Hamiltonian in the presence of the magnetic field is

$$H = \gamma B_0 S_y = -\frac{\gamma B_0 \hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The eigenstates of this are the usual eigenstates of S_y: in the S_z basis,

$$\chi_+^{(y)} = \begin{pmatrix} -i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},\ \chi_-^{(y)} = \begin{pmatrix} i/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

and accordingly

$$\chi = \frac{b+ia}{\sqrt{2}}\chi_+^{(y)} + \frac{b-ia}{\sqrt{2}}\chi_-^{(y)}$$

Since this is time-independent, the time-dependent solution expressed in the S_y basis is

$$\chi(t) = \begin{pmatrix} ae^{-iE_+t/\hbar} \\ be^{-iE_-t/\hbar} \end{pmatrix}$$

where a, b are specified by the initial condition (in this case, $\chi(0) = |S_z = +\rangle \Rightarrow a = \frac{i}{\sqrt{2}}, b = \frac{-i}{\sqrt{2}}$) and $E_\pm$ are the energies of the $\chi_\pm^{(y)}$ eigenstates, which are $\mp\frac{\gamma B_0 \hbar}{2}$ by the eigenvalues of the states:

$$\chi(t) = \begin{pmatrix} \frac{i}{\sqrt{2}}e^{i\gamma B_0 t/2} \\ \frac{-i}{\sqrt{2}}e^{-i\gamma B_0 t/2} \end{pmatrix}.$$

Changing back to the S_z basis,

$$\begin{aligned}\chi(t) &= \left(\frac{i}{\sqrt{2}}e^{i\gamma B_0 t/2}\right)\left(\frac{-i}{\sqrt{2}}\chi_+ + \frac{1}{\sqrt{2}}\chi_-\right) + \left(\frac{-i}{\sqrt{2}}e^{-i\gamma B_0 t/2}\right)\left(\frac{i}{\sqrt{2}}\chi_+ + \frac{1}{\sqrt{2}}\chi_-\right) \\ &= \frac{e^{i\gamma B_0 t/2}}{2}\chi_+ + \frac{ie^{i\gamma B_0 t/2}}{2}\chi_- + \frac{e^{-i\gamma B_0 t/2}}{2}\chi_+ - \frac{ie^{-i\gamma B_0 t/2}}{2}\chi_- \\ &= \begin{pmatrix} \cos(\gamma B_0 t/2) \\ -\sin(\gamma B_0 t/2) \end{pmatrix}\end{aligned}$$

This looks a lot like a clockwise parameterization of a circle.

The change of basis from S_z to S_x is

$$\chi = \frac{a+b}{\sqrt{2}}\chi_+^{(x)} + \frac{a-b}{\sqrt{2}}\chi_-^{(x)},$$

so the coefficient of $\chi_+^{(x)}$ using the above expression for the S_z basis is

$$\frac{\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)}{\sqrt{2}}$$

The corresponding probability of a measurement resulting in $\chi_+^{(x)}$ (which has eigenvalue $\hbar/2$) is

$$\left(\frac{\cos(\gamma B_0 t/2) - \sin(\gamma B_0 t/2)}{\sqrt{2}}\right)^2 = \frac{1}{2}(1 - 2\sin(\gamma B_0 t))$$

□

Problem 3. *Three quarks, each of spin 1/2, form a baryon. What are the allowed values of baryon spin?*

Solution. Distinguish one pair of quarks. This combination has a spin of either 1 or 0, as those are the only integers between $\frac{1}{2}+\frac{1}{2}$ and $\frac{1}{2}-\frac{1}{2}$. The spin of the whole system is a combination of the spin of the pair with the spin of the remaining quark; it's either $1+\frac{1}{2} \Rightarrow \frac{3}{2}, \frac{1}{2}$ or $0+\frac{1}{2} \Rightarrow \frac{1}{2}, -\frac{1}{2}$. Accordingly, the only possibly baryon spins are $\frac{1}{2}$ and $\frac{3}{2}$. □

Problem 4. *Suppose an electron is in potentials $\vec{A} = B_0(x\hat{j} - y\hat{\imath})$, $\Phi = Kz^2$.*

1. *Find the electric and magnetic fields.*
2. *Find the allowed energies of the electron.*

Solution. Since Φ is time-independent, $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = \nabla\Phi$:

$$\nabla \times \vec{A} = \det\begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -B_0 y & B_0 x & 0 \end{vmatrix} = 2B_0\hat{k},$$

$$\nabla\Phi = 2Kz\hat{k}.$$

The minimal coupling Hamiltonian is

$$H = \frac{1}{2m}(-i\hbar\nabla - e\vec{A})^2 + e\Phi = \frac{1}{2m}(-i\hbar\nabla + eB_0(x\hat{j} - y\hat{\imath}))^2 + eKz^2$$

$$= \frac{1}{2m}(i\hbar\nabla + eB_0(x\hat{j} - y\hat{\imath})) \cdot (i\hbar\nabla + eB_0(x\hat{j} - y\hat{\imath})) + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 + i\hbar e B_0\Big(\nabla \cdot (x\hat{j} - y\hat{\imath}) + (x\hat{j} - y\hat{\imath}) \cdot \nabla\Big) + e^2B_0^2(x^2+y^2)\right] + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 + i\hbar e B_0\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + e^2B_0^2(x^2+y^2)\right] + eKz^2$$

$$= \frac{1}{2m}\left[-\hbar^2\nabla^2 - eB_0L_z + e^2B_0^2r^2\right] + eKz^2$$

$$= \frac{p^2}{2m} - \frac{eB_0}{2m}L_z + \frac{e^2B_0^2}{2m}r^2 + eKz^2$$

The L_z operator commutes with this Hamiltonian: it commutes with central-potential Hamiltonians, with itself, and with the z operator, so it must commute with a linear combination of these three. Therefore, it is simultaneously diagonalizable with the Hamiltonian; call its eigenvalue $\hbar m_\ell$ as usual. The right two potential terms are independent harmonic oscillator potentials with $k_r = \frac{e^2 B_0^2}{m} \Rightarrow \omega_r = \frac{eB_0}{m}$ and $k_z = 2eK \Rightarrow \omega_z = \sqrt{\frac{2eK}{m}}$. The energies of these two solutions are $\hbar\omega_r(n_r + \frac{1}{2})$ and $\hbar\omega_z(n_z + \frac{1}{2})$. When the Hamiltonian acts on a simultaneous eigenstate in the Schrödinger equation, the eigenvalues of these operators add to yield allowed energies

$$E(n_r, n_z, m_\ell) = \hbar\omega_r\left(n_r + \frac{1}{2}\right) + \hbar\omega_z\left(n_z + \frac{1}{2}\right) + \hbar m_\ell$$

□

Problem 5. *Consider the observables $A = x^2$ and $B = L_z$.*

1. *Find the uncertainty principle governing A and B, that is, $\Delta A \Delta B$.*
2. *Evaluate ΔB for the hydrogenic $|n\ell m\rangle$ state.*
3. *Therefore, what can you conclude about $\langle xy \rangle$ in this state?*

Solution. The generalized uncertainty principle states

$$\Delta A \Delta B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|.$$

We first compute the commutator:

$$\begin{aligned}
[A, B] = \left[x^2, L_z\right] &= x[x, L_z] + [x, L_z]x = x[x, xp_y - yp_x] + [x, xp_y - yp_x]x \\
&= x([x, xp_y] - [x, yp_x]) + ([x, xp_y] - [x, yp_x])x \\
&= x([x, x]p_y + x[x, p_y] - [x, y]p_x - y[x, p_x]) + ([x, x]p_y + x[x, p_y] - [x, y]p_x - y[x, p_x])x
\end{aligned}$$

The commutator among any two position operators and that of a position and a momentum operator along different axes is zero; accordingly,

$$\begin{aligned}
&= -xy[x, p_x] - y[x, p_x]x = -xyi\hbar - yxi\hbar = -i\hbar(xy + yx) \\
&= -2i\hbar xy
\end{aligned}$$

The uncertainty principle is therefore

$$\Delta A \Delta B \geq \hbar |\langle xy \rangle|$$

The variance of the B operator is given by $\langle L_z^2 \rangle - \langle L_z \rangle^2$; the former is by L_z Hermitian $\langle \psi | L_z^2 | \psi \rangle = \langle L_z \psi | L_z \psi \rangle$. The $|n\ell m_\ell\rangle$ state of hydrogen is in an eigenstate of L_z by assumption (corresponding to the eigenvalue m_ℓ). In particular, $L_z \psi = \hbar m_\ell \psi \Rightarrow \left\langle L_z^2 \right\rangle = \hbar^2 m_\ell^2$, and $\langle L_z \rangle$ is the eigenvalue of L_z corresponding to the eigenstate; of course, $\hbar m_\ell$. So, the variance of B is $\hbar^2 m_\ell^2 - \hbar^2 m_\ell^2 = 0$. This necessarily means $\langle xy \rangle = 0$. □