E-L Proof

Duncan Wilkie

10 October 2021

1

We wish to find a function f that extremizes the functional

$$J = \int_{x_0}^{x_1} K(x, f, f')dx$$

A small variation of the function f may be written as $\delta f = f(x) + \epsilon\eta(x)$, where $\eta(x_0) = \eta(x_1) = 0$. Parameterizing the functional with respect to ϵ, which controlls the "size" of the variation,

$$J_\epsilon = \int_{x_0}^{x_1} K(x, \delta f, \delta f')dx$$

If f extremizes J, the value of J is not changing with respect to an infintesimal variation in the function f, i.e.

$$\frac{dJ_\epsilon}{d\epsilon} = 0 \Leftrightarrow \int_{x_0}^{x_1} \frac{dK(x, \delta f, \delta f')}{d\epsilon}dx = 0$$

By the multivariable chain rule,

$$\Leftrightarrow \int_{x_0}^{x_1} \frac{\partial K}{\partial x}\frac{dx}{d\epsilon} + \frac{\partial K}{\partial f}\frac{df}{d\epsilon} + \frac{\partial K}{\partial f'}\frac{df'}{d\epsilon}dx = 0$$

$$\Leftrightarrow \int_{x_0}^{x_1} 0 + \frac{\partial K}{\partial f}\eta(x) + \frac{\partial K}{\partial f'}\eta'(x)dx$$

Breaking this into two integrals and applying integration by parts to the second,

$$\Leftrightarrow \int_{x_0}^{x_1} \frac{\partial K}{\partial f}\eta(x)dx + \frac{\partial K}{\partial f'}\eta(x)\bigg|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta(x)\frac{d}{dt}\left(\frac{\partial K}{\partial f'}\right)dx$$

Applying the boundary conditions on η,

$$\Leftrightarrow \int_{x_0}^{x_1} \left(\frac{\partial K}{\partial f} - \frac{d}{dt}\frac{\partial K}{\partial f'}\right)\eta(x)dx$$

Since this must hold for all $\eta(x)$, a couple extra real analytic steps we're expected to take for granted imply

$$\Leftrightarrow \frac{\partial K}{\partial f} - \frac{d}{dt}\frac{\partial K}{\partial f'} = 0$$