

# 7590 HW 1

Duncan Wilkie

?

I interchangeably use the  $\bar{z}$  and  $z^*$  notation for the complex conjugate. A thousand apologies.

## 1a

By the definition of the  $L^2$  inner product and  $A$ , for any functions  $f, g \in D(A)$  we have

$$\langle Af|g \rangle = \langle f|Ag \rangle \Leftrightarrow \int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}g''(x)dx$$

Integrating by parts,

$$\begin{aligned} \overline{f'}g \Big|_0^1 - \int_0^1 \overline{f'(x)}g'(x)dx &= \int_0^1 \overline{f(x)}g''(x)dx \\ \Leftrightarrow \overline{f'}g \Big|_0^1 - \overline{f}g' \Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx &= \int_0^1 \overline{f(x)}g''(x)dx \end{aligned}$$

the evaluation terms must both be zero at 0 and 1 since smooth compactly-supported functions on open sets vanish in the limit to the boundary of their domains. Therefore, this operator is symmetric. However, not all elements of  $D(A^\dagger)$  are elements of  $D(A)$ :  $g \in H$  is an element of  $D(A^\dagger)$  iff there exists  $h \in H$  such that  $\forall f \in D(A)$

$$\int_0^1 \overline{f''(x)}g(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Applying the same integration-by-parts argument as above, we may equivalently write this as

$$\Leftrightarrow \overline{f'}g \Big|_0^1 - \overline{f}g' \Big|_0^1 + \int_0^1 \overline{f(x)}g''(x)dx = \int_0^1 \overline{f(x)}h(x)dx$$

Since  $f$  is compactly supported,  $f'$  is as well, so the evaluation terms are zero by the same argument given above. Letting  $g = x^2$ , we then have

$$\int_0^1 \overline{f(x)} \cdot 2dx = \int_0^1 \overline{f(x)}h(x)dx$$

from which we can clearly see the  $L^2([0, 1])$  function  $h = 2$  is the element adjoint to  $g$  with respect to  $A$ .  $g$  is therefore in  $D(A^\dagger)$ . It isn't in  $D(A)$  though, since  $x^2$  doesn't vanish at 1 and therefore isn't compactly supported on this interval. This implies  $D(A^\dagger) \neq D(A)$ , so  $A \neq A^\dagger$ , i.e.  $A$  isn't self-adjoint.

## 1b

Proceeding similarly,

$$\begin{aligned}\langle Af|g\rangle &= \langle f|Ag\rangle \Leftrightarrow \int_0^1 (if'(x))^* g(x) dx = \int_0^1 (f(x))^* ig'(x) dx \\ &\Leftrightarrow -if^*g \Big|_0^1 + \int_0^1 i(f(x))^* g'(x) dx = \int_0^1 (f(x))^* ig'(x) dx\end{aligned}$$

By the same argument as above, the evaluation term is zero, in which case the equality follows immediately. This operator is symmetric. Once again,  $x^2$  is in  $D^\dagger(A)$  but not  $D(A)$ : from the formula derived for  $\langle Af|g\rangle$  in the proof  $A$  is symmetric, the definition of membership in  $D^\dagger(A)$  is

$$\int_0^1 (f(x))^* 2x dx = \int_0^1 (f(x))^* h(x) dx$$

which, choosing  $h = 2x \in L^2([0, 1])$ , clearly holds.  $2x$  isn't compactly supported on  $(0, 1)$  since it doesn't vanish in the limit to 1, so  $D(A^\dagger) \neq D(A)$  and  $A$  isn't self-adjoint.

## 1c

The definition of a symmetric operator is that  $\forall f, g \in D(A)$

$$\langle Af|g\rangle = \langle f|Ag\rangle$$

which in this case is

$$\begin{aligned}\int_{\Omega} [\overline{\partial_i(a_{ij}(x)\partial_j f(x))}] g(x) dx &= \int_{\Omega} \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g(x)) dx \\ \Leftrightarrow g(x) \overline{a_{ij}(x)\partial_j f(x)} \Big|_{\partial\Omega} - \int_{\Omega} [\overline{a_{ij}(x)\partial_j f(x)}] \partial_i g(x) dx &= \int_{\Omega} \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g(x)) dx \\ \Leftrightarrow -\overline{a_{ij}(x)f(x)} \partial_i g(x) \Big|_{\partial\Omega} + \int_{\Omega} \overline{f(x)} \partial_j (\overline{a_{ij}(x)} \partial_i g(x)) dx &= \int_{\Omega} \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g(x)) dx \\ \Leftrightarrow \int_{\Omega} \overline{f(x)} \partial_i(\overline{a_{ji}(x)} \partial_j g(x)) dx &= \int_{\Omega} \overline{f(x)} \partial_i(a_{ij}(x)\partial_j g(x)) dx\end{aligned}$$

where we have throughout used integration by parts and the same fact that functions of compact support vanish in the limit to their boundaries. Since  $a_{ij}(x)$  is Hermitian, it is equal to  $\overline{a_{ji}(x)}$ , and so the two sides are equal and the operator is symmetric. Here,  $A$  is a bounded operator:

$$\begin{aligned}\|Af\| \leq C\|f\| &\Leftrightarrow \int_{\Omega} |\partial_i(a_{ij}(x)\partial_j f(x))|^2 dx \leq C \int_{\Omega} |f(x)|^2 dx \\ \Leftrightarrow \int_{\Omega} \partial_i a_{ij}(x) \partial_j f(x) dx \int_{\Omega} \overline{\partial_i a_{ij}(x) \partial_j f(x)} dx &\leq C \int_{\Omega} |f(x)|^2 dx\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \left( f(x) \partial_i a_{ij}(x) \Big|_{\partial\Omega} - \int_{\Omega} f(x) \partial_j \partial_i a_{ij}(x) \right) \left( \overline{f(x) \partial_i a_{ij}(x)} \Big|_{\partial\Omega} - \int_{\Omega} \overline{f(x) \partial_j \partial_i a_{ij}(x)} dx \right) \leq C \int_{\Omega} |f(x)|^2 dx \\
&\Leftrightarrow \int_{\Omega} \left( f(x) \overline{f(x)} \right) \left( [\partial_i \partial_j a_{ij}(x)] [\overline{\partial_i \partial_j a_{ij}(x)}] \right) dx \leq C \int_{\Omega} |f(x)|^2 dx \\
&\Leftrightarrow \left| \int_{\Omega} f(x) \partial_i \partial_j a_{ij}(x) dx \right|^2 \leq C \int_{\Omega} |f(x)|^2 dx
\end{aligned}$$

From the Cauchy-Schwartz inequality, we have

$$\left| \int_{\Omega} f(x) \partial_i \partial_j a_{ij}(x) dx \right|^2 \leq \int_{\Omega} |f(x)|^2 dx \int_{\Omega} |\overline{\partial_i \partial_j a_{ij}(x)}|^2 dx = C \int_{\Omega} |f(x)|^2 dx$$

This proves the operator is bounded. Therefore,  $D(A^\dagger) = H$ , and there are certainly  $L^2(\Omega)$  functions that aren't  $C^\infty$ , so  $D(A^\dagger) \not\subseteq D(A)$  implying  $A$  is not self-adjoint.

## 2a

Applying the definition of the infinitesimal generator,

$$Af(x) = -i \lim_{t \rightarrow 0} [f(x + vt) - f(x)]/t = -i \frac{\partial f}{\partial v}$$

where the last equality is valid where the limit exists, using the definition of the partial derivative. Since  $V \in C^1$ , and the above implies  $D(A)$  is  $L^2$  functions differentiable along  $v$ , we have  $V \subseteq D(A)$ , with action given above.

## 2b

The adjoint of  $U$  is defined by

$$\langle U(t)f(x)|g(x) \rangle = \langle f(x) | (U(t))^\dagger g(x) \rangle$$

The left hand side is, applying the definition of the  $L^2$  inner product and  $U$ ,

$$\langle U(t)f(x)|g(x) \rangle = \int_{\mathbb{R}^3} \overline{f(e^{-tB}x)} g(x) dx$$

We can make the substitution  $y = e^{-tB}x$ , in which case  $x = e^{tB}y$ . The component functions of this change of variables take the form

$$h_i = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} [(B^n)_{i1} x_1 + (B^n)_{i2} x_2 + (B^n)_{i3} x_3]$$

The Jacobian of this change of coordinates is then

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \\ \frac{\partial h_3}{\partial x_1} & \frac{\partial h_3}{\partial x_2} & \frac{\partial h_3}{\partial x_3} \end{pmatrix} = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \begin{pmatrix} (B^n)_{11} & (B^n)_{12} & (B^n)_{13} \\ (B^n)_{21} & (B^n)_{22} & (B^n)_{23} \\ (B^n)_{31} & (B^n)_{32} & (B^n)_{33} \end{pmatrix} = e^{-tB}$$

Using  $\det e^A = e^{\text{tr } A}$ , the Jacobian determinant is

$$\det J = e^{-t \text{tr}(B)} = 1$$

since the trace of skew-symmetric matrices is zero. We can now finally rewrite the integral as

$$\int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB} y) |\det J| dy = \int_{\mathbb{R}^3} \overline{f(y)} g(e^{tB} y) dy = \langle f(x) | U^\dagger(t) g(x) \rangle$$

which identifies  $U^\dagger(t) : g(x) \mapsto g(e^{tB} x)$ . Clearly, this is unitary:

$$UU^\dagger f(x) = f(e^{-tB} e^{tB} x) = If(x) = f(e^{tB} e^{-tB} x) = UU^\dagger f(x)$$

For  $f \in C^1(\mathbb{R}^3)$ , the infinitesimal generator acts as

$$Af(x) = -i \lim_{t \rightarrow 0} [f(e^{-tB} x) - f(x)]/t$$

The numerator limits to zero, since  $e^{-at} \sim 1$  as  $t \rightarrow 0$ . Applying L'Hôpital's rule,

$$\begin{aligned} &= -i \lim_{t \rightarrow 0} \frac{\frac{d}{dt} f(e^{-tB} x)}{1} = -i \lim_{t \rightarrow 0} \left( \frac{d}{dt} e^{-tB} x \right) \cdot \nabla f(e^{-tB} x) = -i \lim_{t \rightarrow 0} (-B e^{-tB} x) \cdot \nabla f(e^{-tB} x) \\ &= i B x \nabla f(x) \end{aligned}$$

This shows that the limit exists for every  $f \in C^1(\mathbb{R})$  (so  $V \subseteq D(A)$ ) and gives its action.

## 2c

Notice first that the definition of  $n$  and  $r$  give the number  $s$  “modulo”  $2\pi$  in the sense that if one divides the real number line into partitions by integer multiples of  $2\pi$ ,  $n(s)$  gives the multiple of  $2\pi$  corresponding to the rightmost partition boundary which lies to the left of  $s$  and  $r(s)$  gives the rightward displacement of  $s$  from the partition boundary. If  $n$  were further left, adding  $0 \leq r(s) < 2\pi$  couldn't equal  $s$ , and if it were to the right of  $s$ , the same is true because  $r(s)$  is positive. We prove first that  $U_\alpha$  is a continuous symmetry.

*Unitarity:* It preserves the inner product

$$\langle Uf | Ug \rangle = \int_0^{2\pi} \overline{\alpha^{n(x+t)} f(r(x+t))} \alpha^{n(x+t)} g(r(x+t)) dx = \int_0^{2\pi} |\alpha^{n(x+t)}|^2 \overline{f(r(x+t))} g(r(x+t)) dx$$

Since  $|\alpha| = 1$ , we may write  $\alpha = e^{i\theta}$  in which case it is immediately clear  $|\alpha^{n(x+t)}|^2 = 1$ . We now make the substitution  $y = r(x+t)$ , under which  $dy = r'(x+t) dx \Leftrightarrow dx = \frac{dy}{r'(x+t)}$ . Differentiating the definition of  $n$  and  $r$  with respect to  $s$  yields  $1 = n'(s)2\pi + r'(s) = r'(s)$  since  $n$  is a step function. Strictly speaking, it is possible there is one point in the integration interval where this derivative is not defined, but since this is a set of measure zero it won't contribute to the integral. We therefore have, applying the substitution,

$$= \int_{r(t)}^{r(2\pi+t)} \overline{f(y)} g(y) dy = \int_0^{2\pi} \overline{f(y)} g(y) dy = \langle f | g \rangle$$

where we have used for the penultimate equality the fact that  $y$  (and therefore the entire integrand) has periodicity  $2\pi$ , so the integral over any two intervals of length  $2\pi$  will be the same. It also is surjective, as if given a function  $h \in L^2([0, 2\pi])$ , we may write since  $\alpha = e^{i\theta} \Rightarrow f(x)\alpha^{g(x)} = f(x)e^{i\theta g(x)}$

$$h(x) = f(x)e^{ig'(x)} = f(x)\alpha^{g(x)}$$

where  $f : [0, 2\pi] \rightarrow \mathbb{R}$  and  $g : [0, 2\pi] \rightarrow [0, 2\pi]$ . For any given  $h$  (which we may rewrite in the form above),  $\alpha$ , and  $t$ , one may construct the function  $k \in L^2([0, 2\pi])$  given by  $k(x) = f(r(x-t))\alpha^{g(r(x-t))-n(r(x-t)+t)}$ . Noting that for  $x \in [0, 2\pi]$

$$r(r(x+t)-t) = r(x+t-2\pi n(x+t)-t) = r(x-2\pi n(x+t)) = x$$

since

$$r(x+2k\pi) = x+2k\pi - n(x+2k\pi) = x+2k\pi - 2k\pi = x$$

by the characterization of  $n(s)$  above, we have

$$U_\alpha k = \alpha^{n(x+t)} f[r(r(x+t)-t)] \alpha^{g[r(r(x+t)-t)]-n[r(r(x+t)-t)+t]} = \alpha^{n(x+t)} f(x) \alpha^{g(x)-n(x+t)} = f(x) \alpha^{g(x)}$$

Therefore,  $U_\alpha$  is surjective for every  $\alpha$  and  $t$ , and in conjunction with the above result this proves  $U_\alpha$  is unitary.  $U_\alpha(0) = I$ , using  $x \in [0, 2\pi]$ :

$$U_\alpha(0)f(x) = \alpha^{n(x)} f(r(x)) = \alpha^0 f(x) = f(x)$$

The operator behaves properly under addition in  $t$ :

$$U_\alpha(t+s)f(x) = \alpha^{n(x+t+s)} f(r(x+t+s)) = U_\alpha(s) (\alpha^{n+t} f(r(x+t))) = U_\alpha(s) U_\alpha(t) f(x)$$

Lastly, using  $x \in [0, 2\pi]$ ,

$$\lim_{t \rightarrow 0} U_\alpha(t)x = \lim_{t \rightarrow 0} \alpha^{n(x+t)} r(x+t) = \alpha^{n(x)} r(x) = \alpha^0 x = x$$

The infinitesimal generator of  $U_\alpha$  is by definition

$$\begin{aligned} A_\alpha f &= -i \lim_{t \rightarrow 0} [U_\alpha(t)f - f]/t = -i \lim_{t \rightarrow 0} [\alpha^{n(x+t)} f(r(x+t)) - f(x)]/t \\ &= -i \lim_{t \rightarrow 0} \frac{\frac{d}{dt} \alpha^{n(x+t)} f(r(x+t))}{1} = -i \lim_{t \rightarrow 0} \frac{d}{dt} \alpha^0 f(x+t) = -i f'(x) \end{aligned}$$

where we have used L'Hôpital's rule and  $x \in (0, 2\pi)$  (in which case the limit will become at some point exclusively through  $t$  close enough to  $x$  that  $0 \leq x+t < 2\pi$ , yielding  $r(x+t) = x+t$  and  $n(x+t) = 0$ ). The special cases  $x = 0$  and  $x = 2\pi$  must be treated differently. In each case, the limit from the interior of  $[0, 2\pi]$  yields the above result, but the limit from outside will have  $n(x+t) = 1$ ,  $r(x+t) = x+t-2\pi = t$ , yielding  $A_\alpha f(2\pi) = -i \lim_{t \rightarrow 0} \frac{d}{dt} \alpha^1 f(t) = -i \alpha f'(0)$ . Functions in the given  $V_\alpha$  make this exist for all  $x \in [0, 2\pi]$ , since they are  $C^1$  so the derivative exists and the condition at  $2\pi$  makes the two sides of the limit at  $2\pi$  agree. Therefore,  $V_\alpha \in D(A_\alpha)$  and the action is as given above.

### 3a

The given  $D(A)$  is a subset of the  $V_\alpha$  given in problem 2c, since  $C^1$  compactly-supported functions on  $(0, 2\pi)$  are a subset of  $C^1([0, 2\pi])$  functions that vanish in the limits to 0 and  $2\pi$ , which are a subset of  $C^1([0, 2\pi])$  functions where  $f(2\pi) = f(0) = 0$ , which are a subset of  $C^1([0, 2\pi])$  functions where  $f(2\pi) = \alpha f(0)$ . Since  $V_\alpha \subseteq D(A_\alpha)$  was proven in 2c, we have  $D(A) \subseteq D(A_\alpha)$ . Further,  $A = A_\alpha$  on  $D(A)$ , since the only difference in their action occurs when  $x = 2\pi$  for  $A_\alpha$ ; this is outside  $D(A)$ . This proves  $A \subset A_\alpha$ .

### 3b

The theorem gives the result that  $u(t) = U(t)u_0$  is the unique solution to  $\dot{u}(t) = iAu(t)$  where  $A$  is the infinitesimal generator of  $U(t)$ . Applying this, the result follows immediately:

$$u(t, x) = U(t)u_0(x) = U_1(t)u_0(x) = 1^{n(x+t)}u_0(r(x+t)) = u_0(x+t)$$

where we have used the fact that  $A = A_\alpha$  on  $(\pi/2, \pi)$

### 4a

First note that  $\langle Ax, x \rangle = \sum_{i,j=1}^n A_{ij}x_i x_j$  where  $x_i$  denotes the  $i$ th component of  $x$ . Since  $A$  is positive-definite, there exists a basis with respect to which  $A$  is diagonal with positive eigenvalues. The value of the inner product remains the same, since the length of an  $\mathbb{R}^n$  vector is basis-independent; writing the eigenvalues as  $\lambda_i$  and choosing our vector components to be in this basis, the inner product becomes  $\sum_{i=1}^n \lambda_i x_i^2$ . Substituting this in to the formula for the measure,

$$d\mu(x) = Z^{-1} e^{-\sum_{i=1}^n \lambda_i x_i^2 / 2} = Z^{-1} \prod_i e^{-\lambda_i x_i^2 / 2}$$

$$\Rightarrow 1 = \mu(\mathbb{R}^n) = \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\lambda_i x_i^2 / 2} dx_i$$

$$\Rightarrow Z = \prod_{i=1}^n \left( \sqrt{\frac{2\pi}{\lambda_i}} \right) = \sqrt{\frac{2\pi}{\det A}}$$

### 4b

Since  $G = A^{-1}$ , the eigenvalues of  $G$  are  $\lambda_i^{-1}$ , and  $G$  is simultaneously diagonalizable with  $A$ , so noting

$$e^{\langle Gp, p \rangle / 2} = e^{\sum_{i=1}^n p_i^2 / 2\lambda_i} = \prod_{i=1}^n e^{p_i^2 / 2\lambda_i}$$

and

$$e^{\langle p, x \rangle - \langle Ax, x \rangle / 2} = e^{\langle p - Ax/2, x \rangle} = \prod_{i=1}^n e^{p_i x_i - \lambda_i x_i^2 / 2}$$

we show

$$\begin{aligned}
e^{\langle Gp,p \rangle/2} &= e^{\langle Gp,p \rangle/2} \int_{\mathbb{R}^n} Z^{-1} e^{-\langle Ax,x \rangle/2} dx = \left( \prod_{i=1}^n e^{p_i^2/2\lambda_i} \right) \left( \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\lambda_i x_i^2/2} dx_i \right) \\
&= \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{p_i^2/2\lambda_i - \lambda_i x_i^2/2} dx_i = \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{p_i^2/2\lambda_i - x_i p_i + p_i x_i - \lambda_i x_i^2/2} dx_i \\
&= \frac{1}{Z} \int_{\mathbb{R}^n} \left( \prod_{i=1}^n e^{p_i^2/2\lambda_i - x_i p_i} \right) e^{\langle p,x \rangle - \langle Ax,x \rangle/2} dx = \frac{1}{Z} \int_{\mathbb{R}^n} e^{\langle Gp,p \rangle - \langle x,p \rangle} e^{\langle p,x \rangle - \langle Ax,x \rangle/2} dx \\
&= \int_{\mathbb{R}} e^{\langle Gp-x,0 \rangle} e^{\langle p,x \rangle} \frac{1}{Z} e^{-\langle Ax,x \rangle/2} dx = \int_{\mathbb{R}} e^{\langle p,x \rangle} d\mu(x)
\end{aligned}$$

as desired.

### 4c

$e^{i\langle p,x \rangle} = \prod_{k=1}^n e^{ip_k x_k}$  Proceeding similarly, this time

$$e^{-\langle Gp,p \rangle/2} = e^{\sum_{i=1}^n -p_i^2/2\lambda_i} = \prod_{i=1}^n e^{-p_i^2/2\lambda_i}$$

so

$$\begin{aligned}
e^{-\langle Gp,p \rangle/2} &= e^{-\langle Gp,p \rangle/2} \int_{\mathbb{R}^n} Z^{-1} e^{-\langle Ax,x \rangle/2} dx = \left( \prod_{i=1}^n e^{-p_i^2/2\lambda_i} \right) \left( \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\lambda_i x_i^2/2} dx_i \right) \\
&= \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-p_i^2/2\lambda_i - \lambda_i x_i^2/2} dx_i = \frac{1}{Z} \prod_{i=1}^n \int_{-\infty}^{\infty} e^{-p_i^2/2\lambda_i - x_i p_i + p_i x_i - \lambda_i x_i^2/2} dx_i \\
&= \frac{1}{Z} \int_{\mathbb{R}^n} \left( \prod_{i=1}^n e^{-p_i^2/2\lambda_i - x_i p_i} \right) e^{\langle p,x \rangle - \langle Ax,x \rangle/2} dx = \frac{1}{Z} \int_{\mathbb{R}^n} e^{-\langle Gp,p \rangle - \langle x,p \rangle} e^{\langle p,x \rangle - \langle Ax,x \rangle/2} dx \\
&= \int_{\mathbb{R}} e^{-\langle Gp+x,2p \rangle} e^{\langle p,x \rangle} \frac{1}{Z} e^{-\langle Ax,x \rangle/2} dx = \int_{\mathbb{R}} e^{-\langle Gp+2x,3p \rangle} d\mu(x)
\end{aligned}$$

We note  $\langle Gp+2x,3p \rangle = \sum_i 3p_i^2/\lambda_i + 6x_i p_i$

### 11a

$$\begin{aligned}
\det \Lambda &= \begin{vmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{vmatrix} = \gamma^2 - \gamma^2\beta^2 \\
&= \gamma^2(1 - \beta^2) = (1 - \beta^2)^{-1}(1 - \beta^2) = 1
\end{aligned}$$

so this is a proper Lorentz transformation. The proposed inverse satisfies

$$\begin{aligned}\Lambda^{-1}\Lambda &= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I\end{aligned}$$

and

$$\begin{aligned}\Lambda\Lambda^{-1} &= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & \gamma^2\beta - \gamma^2\beta & 0 & 0 \\ -\gamma^2\beta + \gamma^2\beta & -\gamma^2\beta^2 + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I\end{aligned}$$

so it is, in fact, the inverse. The action of the Lorentz transformation on the unit vectors along coordinate axes yields

$$\hat{x}' = \gamma\hat{x}$$

$$\hat{y}' = \hat{y}$$

$$\hat{z}' = \hat{z}$$

and so the new axes are parallel to the old. The origin of the primed system is at  $\vec{x}' = 0$ ; this occurs at  $y = 0$  and  $z = 0$  trivially, but the  $x$  variable has

$$-\gamma\beta t + \gamma x = 0 \Leftrightarrow x = \beta t$$

which is exactly the origin moving along the positive  $x$  axis with velocity  $\beta$ .

## 11b

Taking the events to happen at the origin of the primed system,

$$\Lambda^{-1}\vec{x}'_2 - \Lambda^{-1}\vec{x}'_1 = \begin{pmatrix} \gamma t'_2 \\ \gamma\beta t'_2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma t'_1 \\ \gamma\beta t'_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(t'_2 - t'_1) \\ \gamma\beta(t'_1 - t'_2) \\ 0 \\ 0 \end{pmatrix}$$

The first component is  $\gamma T$ ; since this is the computation of the difference between the two events in the unprimed frame, this proves the time dilation formula.



## 11c

A similar computation to the above result applies:

$$\Lambda^{-1} \vec{x}_2' - \Lambda^{-1} \vec{x}_1' = \begin{pmatrix} \gamma\beta x_2' \\ \gamma x_2' \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \gamma\beta x_1' \\ \gamma x_1' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\beta L \\ \gamma L \\ 0 \\ 0 \end{pmatrix}$$

Reading off the first component confirms the result. Whichever occurs first along the  $x$  axis occurs first in time, since  $L$  is a distance and therefore positive. For spacelike separated events,

$$\Lambda \vec{x}_2 - \Lambda \vec{x}_1 = \begin{pmatrix} \gamma t_2 - \gamma\beta x_2 - (\gamma t_1 - \gamma\beta x_1) \\ \gamma x_2 - \gamma\beta t_2 - (\gamma x_1 - \gamma\beta t_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T - \gamma\beta L \\ \gamma L - \gamma\beta T \\ 0 \\ 0 \end{pmatrix}$$

For the time component to be zero,

$$\gamma T = \gamma\beta L \Leftrightarrow \beta = \frac{T}{L}$$

The spacelike condition ensures conformance with  $|\beta| < 1$ :

$$T^2 - L^2 < 0 \Leftrightarrow T^2 < L^2 \Leftrightarrow |T| < |L| \Leftrightarrow \frac{|T|}{|L|} < 1 \Rightarrow |\beta| < 1$$

## 11d

Once again,

$$\Lambda^{-1} \vec{x}_2' - \Lambda^{-1} \vec{x}_1' = \begin{pmatrix} \gamma t_2 + \gamma\beta x_2 - (\gamma t_1 + \gamma\beta x_1) \\ \gamma\beta t_2 + \gamma x_2 - (\gamma\beta t_1 + \gamma x_1) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma T + \gamma\beta L \\ \gamma\beta T + \gamma L \\ 0 \\ 0 \end{pmatrix}$$

The measurements are simultaneous in the unprimed frame, i.e. the first component of the above displacement is zero, yielding  $T = -\beta L$ . Plugging this in to the second component, one obtains

$$\gamma L - \gamma\beta^2 L = L\gamma[1 - \beta^2] = L\gamma(1/\gamma^2) = L/\gamma$$

as desired.

## 12

For convenience of notation, define  $f : SL(2, \mathbb{C}) \rightarrow SO^+(1, 3) :: A \mapsto (X \mapsto AXA^\dagger)$  where  $X$  is of the form given in the problem. This map is indeed a homomorphism: using  $(AB)^\dagger = B^\dagger A^\dagger$ ,

$$f(A)f(B) = (X \mapsto AXA^\dagger) \circ (X \mapsto BXB^\dagger) = X \mapsto A(BXB^\dagger)A^\dagger = (AB)x(B^\dagger A^\dagger) = f(AB)$$

The kernel of  $f$  are those elements  $A \in SL(2, \mathbb{C})$  such that  $X = AXA^\dagger \Leftrightarrow A^{-1}X = XA^\dagger$  for all  $X$  of the given form. Since  $A \in SL(2, \mathbb{C}) \Rightarrow \det A = 1$ ,  $A$  is unitary, i.e.  $AA^\dagger = A^\dagger A = I$ . We then can write

$$X = AXA^\dagger \Leftrightarrow XA = AXA^\dagger A \Leftrightarrow XA = AX$$

Elements of the center of  $GL(n, \mathbb{F})$  are  $c^*I$  where  $c^*$  is any unit of  $\mathbb{F}$  and  $I$  is the identity matrix, and since the units in  $\mathbb{R}$  are  $\pm 1$ , the kernel of the homomorphism is  $\pm I$ . By the isomorphism theorem,  $\text{im } f \cong SL(2, \mathbb{C})/\ker f$ , and  $\ker f$  is discrete. Since  $\text{im } f$  is a subgroup of the Lorentz group, and the Lorentz group is connected,  $\text{im } f$  is isomorphic to the whole group, implying  $f$  is surjective.

## 13

Using the fact  $\tilde{x}$  is the difference of two observables and therefore Hermitian,

$$(\Delta x)^2(\Delta p)^2 = \langle \psi | \tilde{x}^2 | \psi \rangle \langle \psi | \tilde{p}^2 | \psi \rangle = \langle \psi | \tilde{x}^2 | \psi \rangle \langle \psi | \tilde{p}^2 | \psi \rangle = \langle \tilde{x} \psi | \tilde{x} \psi \rangle \langle \tilde{p} \psi | \tilde{p} \psi \rangle \geq |\langle \tilde{x} \psi | \tilde{p} \psi \rangle|^2 = |\langle \psi | \tilde{x} \tilde{p} | \psi \rangle|^2 = |\langle \psi | \tilde{x} \tilde{p} | \psi \rangle|^2$$

We write

$$\begin{aligned} \langle \psi | \tilde{x} \tilde{p} | \psi \rangle &= \langle \psi | (x - \langle \psi | x | \psi \rangle) (p - \langle \psi | p | \psi \rangle) | \psi \rangle = \langle \psi | xp | \psi \rangle - \langle \psi | x \langle \psi | p | \psi \rangle | \psi \rangle - \langle \psi | p \langle \psi | x | \psi \rangle | \psi \rangle + \langle \psi | x | \psi \rangle \langle \psi | p | \psi \rangle \langle \psi | \psi \rangle \\ &= \langle \psi | xp | \psi \rangle + \langle \psi | x | \psi \rangle \langle \psi | p | \psi \rangle \end{aligned}$$

Complex numbers have the property

$$|z|^2 = (\Re z)^2 + (\Im z)^2 \geq (\Im z)^2 = \left( \frac{1}{2i} (z - z^*) \right)^2$$

Applying this to  $z = \langle \psi | \tilde{x} \tilde{p} | \psi \rangle$ ,

$$\begin{aligned} |\langle \psi | \tilde{x} \tilde{p} | \psi \rangle|^2 &\geq \left( \frac{1}{2i} [(\langle \psi | xp | \psi \rangle + \langle \psi | x | \psi \rangle \langle \psi | p | \psi \rangle) - (\langle \psi | xp | \psi \rangle + \langle \psi | x | \psi \rangle \langle \psi | p | \psi \rangle)^*] \right)^2 \\ &= \left( \frac{1}{2i} [\langle \psi | xp | \psi \rangle - \langle \psi | px | \psi \rangle] \right)^2 = \left( \frac{1}{2i} \langle \psi | [x, p] | \psi \rangle \right)^2 = \frac{1}{4} = \frac{\hbar^2}{4} \end{aligned}$$

where in the last equality we have returned from natural units. The ground state of the simple harmonic oscillator is  $\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-m\omega x^2/2\hbar}$ , from which we can compute

$$\begin{aligned} \langle \psi_0 | x | \psi_0 \rangle &= 0 \text{ (odd function, symmetric interval)} \\ \langle \psi_0 | p | \psi_0 \rangle &= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \left( -i\hbar \frac{\partial}{\partial x} e^{-m\omega x^2/2\hbar} \right) dx = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} (im\omega x e^{-m\omega x^2/2\hbar}) dx \\ &= 0 \text{ (odd function, symmetric interval)} \\ &\Rightarrow \tilde{x} = x, \tilde{p} = p \\ &\Rightarrow (\Delta x)^2 = \langle \psi_0 | \tilde{x}^2 | \psi_0 \rangle = \langle \psi_0 | x^2 | \psi_0 \rangle = \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} x^2 e^{-m\omega x^2/2\hbar} dx \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx = \sqrt{\frac{m\omega}{\pi\hbar}} \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} = \sqrt{\frac{\hbar^2}{4m^2\omega^2}} = \frac{\hbar}{2m\omega} \\
\Rightarrow (\Delta p)^2 &= \langle \psi_0 | \tilde{p}^2 | \psi_0 \rangle = \langle \psi_0 | p^2 | \psi_0 \rangle = -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \frac{\partial^2}{\partial x^2} e^{-m\omega x^2/2\hbar} dx \\
&= -\sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar} \hbar^2 \left( \frac{1}{\hbar^2} m\omega e^{-m\omega x^2/2\hbar} (m\omega x^2 - \hbar) \right) dx \\
&= -\sqrt{\frac{m\omega}{\pi\hbar}} \left( m^2\omega^2 \int_{-\infty}^{\infty} x^2 e^{-m\omega x^2/\hbar} dx - \hbar m\omega \int_{-\infty}^{\infty} e^{-m\omega x^2/\hbar} dx \right) \\
&= -\sqrt{\frac{m\omega}{\pi\hbar}} \left( m^2\omega^2 \sqrt{\frac{\pi\hbar^3}{4m^3\omega^3}} - \hbar m\omega \sqrt{\frac{\pi\hbar}{m\omega}} \right) = -(\hbar m\omega/2 - \hbar m\omega) = m\omega \frac{\hbar}{2}
\end{aligned}$$

where we have used a table for the nasty Gaussian-type integrals. Multiplying the results, we indeed confirm this is a minimum-uncertainty state:

$$(\Delta x)^2 (\Delta p)^2 = \frac{\hbar}{2m\omega} \left( m\omega \frac{\hbar}{2} \right) = \frac{\hbar^2}{4}$$

14