4141 HW 9

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1

Presuming an induction hypothesis $[A^{n-1}, C] = (n-1)A^{n-2}C$,

$[A^n, B] = A[A^{n-1}, B] + [A, B]A^{n-1} = A(n-1)A^{n-2}C + CA^{n-1} = (n-1)A^{n-1}C + A^{n-1}C = nA^{n-1}C$

$$\frac{d}{d\lambda}e^{\lambda(A+B)} = \frac{d}{d\lambda}\left(1 + \lambda(A+B) + \frac{\lambda^2}{2}(A+B)^2 + ...\right) = (A+B) + \lambda(A+B)^2 + \frac{\lambda^2}{2}(A+B)^3 + ...$$
$$= (A+B)e^{\lambda(A+B)}$$

Also,

$$\frac{d}{d\lambda}(e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}) = Ae^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}Be^{\lambda B}e^{-\lambda^2 C/2} + e^{\lambda A}e^{\lambda B}\left(-\lambda Ce^{-\lambda^2 C/2}\right)$$
$$= e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}(A + B + \lambda C - \lambda C) = (A+B)e^{\lambda A}e^{\lambda B}e^{-\lambda^2 C/2}$$

Since these functions satisfy the same differential equation and have the same value at $\lambda = 0$, they must be the same function, so in particular for $\lambda = 1$ we obtain the desired result.

2a

The system is in state ψ_1 immediately after the measurement due to wave function collapse.

2b

Prior to the measurement, the wave function in ϕ space may be written

$$\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$$

The two possible results are then b_1 and b_2, with probabilities $|3/5|^2 = \frac{9}{25}$ and $|4/5|^2 = \frac{16}{25}$ respectively.

2c

We can solve for ϕ_1 and ϕ_2 in the eigenstate relation:

$$\begin{cases} 5\psi_1 = 3\phi_1 + 4\phi_2 \\ 5\psi_2 = 4\phi_1 - 3\phi_2 \end{cases} \Leftrightarrow \begin{cases} 20\psi_1 = 12\phi_1 + 16\phi_2 \\ 15\psi_2 = 12\phi_1 - 9\phi_2 \end{cases} \Rightarrow 5(4\psi_1 - 3\psi_2) = 25\phi_2 \Leftrightarrow \phi_2 = (4\psi_1 - 3\psi_2)/5$$

$$\Rightarrow 3\phi_1 = 5\psi_1 - \frac{4}{5}(4\psi_1 - 3\psi_2) \Leftrightarrow \phi_1 = (3\psi_1 + 4\psi_2)/5$$

There are two possibilities after the measurement of B: the particle is in state ϕ_1, or it is in state ϕ_2. If it's in state ϕ_1, its probability of being in state ψ_1 on a subsequent measurement is $|3/5|^2 = 9/25$; if it's in state ϕ_2, this probability is $|4/5|^2 = 16/25$. Weighting these values by the probabilities of being in ϕ_1 and ϕ_2 and adding (by the law of total probability),

$$P(a_1) = \frac{9}{25}\frac{9}{25} + \frac{16}{25}\frac{16}{25} = \frac{337}{625} = 0.5392$$

3

By definition,

$$\langle x \rangle = \int_{\mathbb{R}} \left(\sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} \right)^* x \left(\sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} \right) dx$$

$$= \int_{\mathbb{R}} x \sum_m \sum_n c_n^* c_m \psi_n(x) \psi_m(x) e^{-it(E_n - E_m)/\hbar} dx = \sum_m \sum_n c_n^* c_m e^{-it(E_n - E_m)/\hbar} \int_{\mathbb{R}} x \psi_n(x) \psi_m(x) dx$$

Using the hint for problem 3.39 in the book,

$$= \sum_m \sum_n c_n^* c_m e^{-it(E_n - E_m)/\hbar} \left(\sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{m}\delta_{n,m-1} + \sqrt{n}\delta_{m,n-1} \right] \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sum_n c_n^* c_{n+1} e^{-it(E_n - E_{n+1})/\hbar} \sqrt{n+1} + \sum_n c_n^* c_{n-1} e^{-it(E_n - E_{n-1})/\hbar} \sqrt{n} \right)$$

The energy eigenvalues for the simple harmonic oscillator are $E_n = \hbar\omega(n + \frac{1}{2})$, so the difference between successive terms is $E_n - E_{n-1} = -(E_n - E_{n+1}) = \hbar\omega$. Substituting this,

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(e^{-it\omega} \left[\sum_n c_n^* c_{n+1} \sqrt{n+1} \right] + e^{it\omega} \left[\sum_n c_n^* c_{n-1} \sqrt{n} \right] \right)$$

To get this in the desired form, notice that in the second term the $n = 0$ term is zero, so relabeling n to $n+1$ yields

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(e^{-it\omega} \left[\sum_n c_n^* c_{n+1} \sqrt{n+1} \right] + e^{it\omega} \left[\sum_n c_{n+1}^* c_n \sqrt{n+1} \right] \right)$$

Since $(a^*b)^* = ab^*$, the two sum factors are merely some complex numbers which are conjugate to each other, which implies one may distribute the initial constant and write it times the sums as $Ce^{i\phi}$ and $Ce^{-i\phi}$. Multiplying and dividing by a factor of two,

$$= \frac{1}{2}\left(e^{-it\omega}Ce^{i\phi} + e^{it\omega}Ce^{-i\phi}\right) = C\cos(\omega t - \phi)$$

with constants of the given form.

4a

The eigenvalues may be found via $\det(M - \lambda I) = 0$. We can ignore the leading coefficients for now and multiply the resulting eigenvalue by them due to homogeneity of the determinant (i.e. $\det(aM - \lambda I) = 0 \Leftrightarrow a\det(M - \frac{\lambda}{a}I) = 0$). For H,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2 = 0$$

clearly has solutions 1 and 2 (with multiplicity 2 for the latter), corresponding to eigenvalues $\hbar\omega$ and $2\hbar\omega$. For A,

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = \lambda^2(2-\lambda) + \lambda - 2 = (2-\lambda)(\lambda^2-1) = (2-\lambda)(\lambda-1)(\lambda+1)$$

so the eigenvalues are 2λ, λ, and $-\lambda$. For B,

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = (2-\lambda)\lambda^2 - (2-\lambda) = (2-\lambda)(\lambda^2-1) = (2-\lambda)(\lambda+1)(\lambda-1)$$

which is the same equation as for A, yielding eigenvalues 2μ, μ, and $-\mu$. The corresponding eigenvectors are the basis vectors in the case of H, since it's already diagonalized. For A, we have

$$\lambda\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} b \\ a \\ 2c \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \Rightarrow a = 0, b = 0, c = 1$$

$$\lambda\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} b \\ a \\ 2c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = 1/\sqrt{2}, b = 1/\sqrt{2}, c = 0$$

$$\lambda\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\lambda\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} b \\ a \\ 2c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} \Rightarrow a = 1/\sqrt{2}, b = -1/\sqrt{2}, c = 0$$

where we have chosen normalized results. For B, we similarly have

$$\mu\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2\mu\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2a \\ c \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \\ 2c \end{pmatrix} \Rightarrow a = 1, b = 0, c = 0$$

$$\mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2a \\ c \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow a = 0, b = 1/\sqrt{2}, c = 1/\sqrt{2}$$

$$\mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\mu \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2a \\ c \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} \Rightarrow a = 0, b = -1/\sqrt{2}, c = 1/\sqrt{2}$$

4b

We have

$$\langle H \rangle = \sum_n E_n |c_n|^2 = \hbar\omega \left(|c_1|^2 + 2|c_2|^2 + 3|c_3|^2 \right) = \hbar\omega(2 - |c_1|^2),$$

$$\langle A \rangle = \langle S(0)|A|S(0)\rangle = \lambda \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \lambda \left(c_1^* c_2 + c_2^* c_1 + 2|c_3|^2 \right)$$

$$\langle B \rangle = \langle S(0)|A|S(0)\rangle = \mu \begin{pmatrix} c_1^* & c_2^* & c_3^* \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \mu \left(2|c_1|^2 + c_2^* c_3 + c_3^* c_2 \right)$$

4c

We write

$$|S(t)\rangle = \begin{pmatrix} c_1 e^{-itE_1/\hbar} \\ c_2 e^{-itE_2/\hbar} \\ c_3 e^{-itE_3/\hbar} \end{pmatrix} = \begin{pmatrix} c_1 e^{-it\omega} \\ c_2 e^{-2it\omega} \\ c_3 e^{-2it\omega} \end{pmatrix}$$

A measurement of the energy would yield $\hbar\omega$ with probability $|c_1 e^{-it\omega}|^2 = |c_1|^2$, and $2\hbar\omega$ with probability $|c_2 e^{-2it\omega}|^2 + |c_3 e^{-2it\omega}|^2 = |c_2|^2 + |c_3|^2$