7380 HW Corrections

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My confusion with this problem came down to the interpretation of "with a piecewise continuous derivative," which I took to mean that a derivative is defined at every point of the domain of f, in concert with the usual analytical meaning of existence of a derivative. Under that definition, f is vacuously piecewise continuous. I presume they mean instead "has a piecewise continuous derivative on the set where f is continuous," and proceed accordingly.

By definition,

$$\left\langle \frac{\partial f}{\partial x}, \phi \right\rangle = -\left\langle f, \frac{\partial \phi}{\partial x} \right\rangle = -\int_{\mathbb{R}} f \frac{\partial \phi}{\partial x} dx$$

It is sufficient for now to consider a function with a single jump discontinuity. Call the point of discontinuity x_0, and the value of the left- and right-hand limits $f^-(x_0)$ and $f^+(x_0)$, respectively. We have

$$\left\langle \frac{\partial f}{\partial x}, \phi \right\rangle = -\int_{-\infty}^{a} f \frac{\partial \phi}{\partial x} dx \int_{b}^{\infty} f \frac{\partial \phi}{\partial x} dx - \int_{a}^{b} f \frac{\partial \phi}{\partial x} = -f\phi\Big|_{-M}^{x_0} + \int_{-\infty}^{x_0} f'\phi dx - f\phi\Big|_{x_0}^{M} + \int_{x_0}^{\infty} f'\phi dx$$

$$= \int_{\mathbb{R}\setminus x_0} f'\phi dx - (f^+(x_0) - f^-(x_0))\phi(x_0)$$

Writing the integral in the form $f'\Big|_{\mathbb{R}\setminus x_0}$ and noting that the constant term is the action of a shifted delta distribution, this may be identified for multiple discontinuities in a finite set J with the distribution

$$f'\Big|_{\mathbb{R}\setminus J} + \sum_{x_i \in J} \delta(x - x_0)(f^-(x_i) + f^+(x_i))$$

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Substituting $v = x + t$, $w = x - t$

$$(u, \phi_{tt}) = \int_{\mathbb{R}^2} f(x+t)\phi_{tt}(x,t) = \int_{\mathbb{R}^2} f(v) \frac{\partial^2}{\partial t^2} \phi\left(\frac{v+w}{2}, \frac{v-w}{2}\right)(-2) dv dw$$

$$= -2\int_{\mathbb{R}^2} f(v) \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial t}\right) dv dw$$

$$= -2\int_{\mathbb{R}^2} f(v)\left[\frac{\partial^2\phi}{\partial w^2}\left(\frac{\partial w}{\partial t}\right)^2 - 2\frac{\partial^2\phi}{\partial v\partial w} + \frac{\partial^2\phi}{\partial v}\left(\frac{\partial v}{\partial t}\right)^2\right] dvdw = -2\int_{\mathbb{R}^2} f(v)\left[\frac{\partial^2\phi}{\partial w^2} - 2\frac{\partial^2\phi}{\partial v\partial w} + \frac{\partial^2\phi}{\partial v^2}\right] dvdw$$

Similarly for u_{xx},

$$(u, \phi_{tt}) = \int_{\mathbb{R}^2} f(x+t)\phi_{tt}(x,t) = \int_{\mathbb{R}^2} f(v)\frac{\partial^2}{\partial x^2}\phi\left(\frac{v+w}{2}, \frac{v-w}{2}\right)(-2)dvdw$$

$$= -2\int_{\mathbb{R}^2} f(v)\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}\right) = -2\int_{\mathbb{R}^2} f(v)\left[\frac{\partial^2\phi}{\partial w^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial^2\phi}{\partial w\partial v} + \frac{\partial^2\phi}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= -2\int_{\mathbb{R}^2} f(v)\left[\frac{\partial^2\phi}{\partial w^2} + 2\frac{\partial^2\phi}{\partial w\partial v} + \frac{\partial^2\phi}{\partial v^2}\right]$$

Taking $u_{xx} - u_{tt}$ and performing the integral with respect to w,

$$u_{xx} - u_{tt} = -2\int_{\mathbb{R}^2} f(v)\left[4\frac{\partial^2\phi}{\partial w\partial v}\right] dvdw = -8\int_{\mathbb{R}} f(v)\frac{\partial\phi}{\partial v}dv$$

$$= -8\int_{\mathbb{R}} f(x+t)\frac{\partial\phi}{\partial t}dt = 8\int_{\mathbb{R}} f(x+t)\frac{\partial\phi}{\partial x}dx$$

The last two equalities come from the two choices of variable to write v in terms of; $u_{xx} - u_{tt}$ is equal to them both. However, one is the negation of the other, since the difference between them is merely a change of symbol. In other words,

$$u_{xx} - u_{tt} = c = -c$$

which implies $c = 0$, i.e. $u_{xx} = u_{tt}$.