4141 HW 8

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1

Call the operator G. By the definition of hermicity,

$$\langle Gf|g\rangle = \langle f|Gg\rangle \Leftrightarrow \int_{\mathbb{R}} (Gf(\phi))^* g(\phi)d\phi = \int_{\mathbb{R}} f(\phi)^* Gg(\phi)d\phi \Leftrightarrow \frac{\hbar}{i}\int_{\mathbb{R}} f'(\phi)^* g(\phi)d\phi = \frac{\hbar}{i}\int_{\mathbb{R}} f(\phi)^* g'(\phi)d\phi$$

$$\Leftrightarrow \frac{\hbar}{i}\left(f(\phi)^* g(\phi)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(\phi)^* g'(\phi)d\phi\right) = \frac{\hbar}{i}\left(f(\phi)^* g(\phi)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} f'(\phi)^* g'(\phi)d\phi\right)$$

Therefore, G is Hermitian. Taking the inverse Fourier transform of Gf,

$$\mathcal{F}^{-1}(Gf) = \int_{\mathbb{R}} e^{2\pi i x\phi}\frac{\hbar}{i}f'(\phi)d\phi = \frac{\hbar}{i}f(\phi)e^{2\pi i x\phi}\Big|_{-\infty}^{\infty} - \frac{\hbar}{i}\int_{\mathbb{R}} f(\phi)2\pi i x e^{2\pi i x\phi}d\phi$$

Because f is periodic, the first term is zero, since if one splits the evaluation into a sum of periodic sub-intervals, one obtains zero on each sub-interval. This then becomes

$$= 2\pi\hbar x\int_{\mathbb{R}} f(\phi)e^{2\pi i x\phi}d\phi = 2\pi\hbar x f(x)$$

The expectation value of G is therefore, in position space, the expectation value of the position operator $4\pi^2\hbar^2 x^2$, so I presumably used a heterogeneous Fourier transform definition and the operator is identified with x^2.

2

It's easy to represent H as a matrix if one compares how it acts on $\psi = a|1\rangle + b|2\rangle$ to the action of a matrix on the vector (a, b).

$$H = \epsilon\begin{pmatrix}\langle 1|1\rangle + \langle 2|1\rangle & \langle 1|2\rangle + \langle 2|2\rangle \\ \langle 1|1\rangle - \langle 2|1\rangle & \langle 1|2\rangle - \langle 2|2\rangle\end{pmatrix}$$

By orthonormality, we may write this

$$= \epsilon\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}$$

This corresponds to an eigenvalue equation

$$\det(H - \lambda I) = \det \begin{vmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 1 - 1 = \lambda^2 - 2 = 0 \Rightarrow \lambda = \pm\sqrt{2}$$

with multiplicity 2. The eigenvectors may then be immediately computed as $(1-\sqrt{2})|1\rangle + |2\rangle$ and $(1+\sqrt{2})|1\rangle + |2\rangle$, which one may normalize by dividing by the norm if desired.

3

$$\hat{O} = |\psi\rangle\langle\phi|$$

$$\frac{\partial}{\partial x} f(x) = |h\rangle\langle h|g\rangle$$

4

The variance of these observable quantities are

$$\Delta E = \left\langle \left(\hat{E} - \langle E\rangle\right)\Psi \middle| \left(\hat{E} - \langle E\rangle\right)\Psi \right\rangle$$

and

$$\Delta x = \left\langle \left(\hat{x}\langle x\rangle\Psi \middle| \left(\hat{x} - \langle x\rangle\right)\Psi\right) \right\rangle$$

By Cauchy-Schwarz, By the generalized uncertainty principle,

$$\sigma_E^2\sigma_x^2 \geq \left(\frac{1}{2i}\left\langle\left[\hat{H}, \hat{x}\right]\right\rangle\right)^2 = \left(\frac{1}{2i}\left\langle\frac{\hat{p}^2}{2m}\hat{x} + \hat{V}\hat{x} - \hat{x}\frac{\hat{p}^2}{2m} - \hat{x}\hat{V}\right\rangle\right)^2 = \left(\frac{1}{2i}\left\langle\frac{\hat{p}^2}{2m}\hat{x} - \hat{x}\frac{\hat{p}^2}{2m}\right\rangle\right)^2$$

$$= \left(\frac{1}{2i}\left\langle\frac{\hat{p}}{2m}\left[\hat{p}\hat{x} - \hat{x}\hat{p}\right]\right\rangle\right)^2 = \left(\frac{1}{2i}\left\langle\frac{\hat{p}}{2m}i\hbar\right\rangle\right)^2 = \left(\frac{\hbar}{4m}\langle\hat{p}\rangle\right)^2$$

Taking the square root,

$$\sigma_E\sigma_x \geq \frac{\hbar|\langle p\rangle|}{4m}$$

5

The definition of the ladder operators for the harmonic oscillator is

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x - i\hat{p}\right)$$

$$\hat{a}_- = \frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x + i\hat{p}\right)$$

Adding them together, one eliminates $\hat{p}$ to obtain

$$\hat{a}_+ + \hat{a}_- = \frac{2m\omega x}{\sqrt{2\hbar m\omega}} \Leftrightarrow x = (\hat{a}_+ + \hat{a}_-)\sqrt{\frac{\hbar}{2m\omega}}$$

Subtracting them, one eliminates $\hat{x}$ to obtain

$$\hat{a}_+ - \hat{a}_- = \frac{-2i\hat{p}}{\sqrt{2\hbar m\omega}} \Leftrightarrow \hat{p} = (\hat{a}_+ - \hat{a}_-)i\sqrt{\frac{\hbar m\omega}{2}}$$

One can now square (in the sense of composition) these operators to obtain

$$\hat{x}^2 = \frac{\hbar}{2m\omega}(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2)$$

$$\hat{p}^2 = \frac{\hbar m\omega}{2}(\hat{a}_-^2 - \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- + \hat{a}_+^2)$$

The kinetic energy is then

$$T = \frac{p^2}{2m} = \frac{\hbar\omega}{4}(\hat{a}_-^2 - \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- + \hat{a}_+^2)$$

and the potential is

$$V = \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{4}(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2)$$

In calculating the expectation values for both these quantities, the squared terms drop out, because they result in something proportional to the state with respect to which the expectation is taken with n plus or minus 2, which, since the eigenvector basis is orthonormal, results in zero. We then need only calculate, since the other term is the negation of this,

$$\langle \hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_- \rangle = \int_{\mathbb{R}} \psi_n^*(x)(\hat{a}_-\hat{a}_+ + \hat{a}_+\hat{a}_-)\psi_n(x)dx$$