4123 HW 2

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1a

From the reference frame of the attatchment point, we take θ to be the angle between the negative vertical axis and the first bob and ϕ to be the angle between the same axis and the second. The change of variables is $x_1 = l_1 \sin(\theta)$, $y_1 = l_1 - l_1 \cos(\theta)$, $x_2 = x_1 + l_2 \sin(\phi) = l_1 \sin(\theta) + l_2 \sin(\phi)$, and $y_2 = y_1 + l_2 + l_2 \cos(\phi) = l_1(1 - \cos(\theta) + l_2(1 - \cos(\phi))$. The potential as a function of these varaibles, measured of course with respect to the attatchment point, is therefore

$$U = -m_1 g y_1 - m_2 g(y_1 + y_2) = -(m_1 + m_2) g l_1 (1 - \cos(\theta)) - m_2 g l_2 (1 - \cos(\phi))$$

and the kiniteic energy is

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2} m_2 (\dot{x_2} + \dot{y_2}^2)$$

$$= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}^2 + m_2 l_1 l_2 \dot{\theta}\dot{\phi} \cos(\theta - \phi) + l_2^2 \dot{\phi}^2$$

The overall Lagrangian is therefore

$$L = T - U = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}^2 + m_2 l_1 l_2 \dot{\theta}\dot{\phi} \cos(\theta - \phi) + \frac{1}{2} m_2 l_2^2 \dot{\phi}^2 + (m_1 + m_2) g l_1 (1 - \cos(\theta)) + m_2 g l_2 (1 - \cos(\phi))$$

Applying the Euler-Lagrange equation,

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} \Leftrightarrow -m_2 l_1 l_2 \dot{\phi}\dot{\theta} \sin(\theta - \phi) + (m_1 + m_2) g l_1 \cos(\theta)$$

$$= \frac{d}{dt}[(m_1 + m_2) l_1^2 \dot{\theta} + m_2 l_1 l_2 \dot{\phi} \cos(\theta - \phi)] = (m_1 + m_2) l_1^2 \ddot{\theta} + m_2 l_1 l_2 \dot{\phi}[\sin(\theta - \phi)\dot{\phi} - \sin(\theta - \phi)\dot{\theta}] + m_2 l_1 l_2 \ddot{\phi} \cos(\theta - \phi)$$

$$\Leftrightarrow -m_2 l_1 l_2 \dot{\theta}\dot{\phi} \sin(\theta - \phi) + (m_1 + m_2) g l_1 \cos(\theta) = (m_1 + m_2) l_1^2 \ddot{\theta} + m_2 l_1 l_2 (\dot{\phi}[\sin(\theta - \phi)\dot{\phi} - \sin(\theta - \phi)\dot{\theta}] + \ddot{\phi} \cos(\theta - \phi))$$

and

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} \Leftrightarrow m_2 l_1 l_2 \dot{\theta}\dot{\phi} \sin(\theta - \phi) + m_2 g l_2 \sin(\phi)$$

$$= \frac{d}{dt}[m_2 l_1 l_2 \dot{\theta} \cos(\theta - \phi) + m_2 l_2^2 \dot{\phi} = m_2 l_1 l_2 (\ddot{\theta} \cos(\theta - \phi) + \dot{\theta}[\dot{\phi} \sin(\theta - \phi) - \dot{\theta} \sin(\theta - \phi)]) + m_2 l_2^2 \ddot{\phi}$$

$$\Leftrightarrow m_2 l_1 l_2 \dot{\theta}\dot{\phi} \sin(\theta - \phi) + m_2 g l_2 \sin(\phi) = m_2 l_1 l_2 (\ddot{\theta} \cos(\theta - \phi) + \dot{\theta}[\dot{\phi} \sin(\theta - \phi) - \dot{\theta} \sin(\theta - \phi)]) + m_2 l_2^2 \ddot{\phi}$$

1b

Applying the small-angle approximation to the Lagrangian, where $\cos(x) \approx 1 - x^2/2$. We also approximate $\dot\theta\dot\phi\cos(\theta-\phi)$ as $\dot\theta\dot\phi$, since the amplitude factor should remain small when the oscillations are small, and so the lower-order approximation isn't terrible. This yields, in terms of the new constants,

$$L = ml^2\dot\theta^2 + ml^2\dot\theta\dot\phi + ml^2\dot\phi^2 - mgl\theta^2 - mgl\phi^2/2$$

Applying the Euler-Lagrange equation,

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} \Leftrightarrow -2mgl\theta = \frac{d}{dt}[2ml^2\dot\theta + ml^2\dot\phi] = 2ml^2\ddot\theta + ml^2\ddot\phi$$

$$\Leftrightarrow -\frac{2g\theta}{l} = 2\ddot\theta + \ddot\phi$$

and

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot\phi} \Leftrightarrow -mgl\phi = \frac{d}{dt}[ml^2\dot\theta + 2ml^2\dot\phi] = ml^2\ddot\theta + 2ml^2\ddot\phi$$

$$\Leftrightarrow -\frac{g\phi}{l} = \ddot\theta + 2\ddot\phi$$

As a matrix equation, this system can be written

$$A\ddot x = -Kx$$

where $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $K = \begin{bmatrix} \frac{2g}{l} & 0 \\ 0 & \frac{g}{l} \end{bmatrix}$, $\ddot x = \begin{bmatrix} \ddot\theta \\ \ddot\phi \end{bmatrix}$, and $x = \begin{bmatrix} \theta \\ \phi \end{bmatrix}$. The oscillatory solutions for this system will have freqency f and initial condtions a satisfying

$$(K - f^2A)a = 0 \Leftrightarrow 2a_1g/l - 2a_1f^2 = 0, 2a_2g/l - 2a_2f^2 = 0 \Leftrightarrow f = \sqrt{\frac{g}{l}}$$

and all allowed frequencies will be multiples of this.

2

With $L = \frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) - q\phi + q(A_x\dot x + A_y\dot y + A_z\dot z)$,

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot x} \Leftrightarrow -q\frac{\partial \phi}{\partial x} + q\left(\frac{\partial A_x}{\partial x}\dot x + \frac{\partial A_y}{\partial x}\dot y + \frac{\partial A_z}{\partial x}\dot z\right) = \frac{d}{dt}\left[m\dot x + qA_x\right] = m\ddot x + q\frac{dA_x}{dt}$$

$$= m\ddot x + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot x + \frac{\partial A_x}{\partial y}\dot y + \frac{\partial A_x}{\partial z}\dot z\right)$$

$$\Leftrightarrow m\ddot x = -q\frac{\partial \phi}{\partial x} - q\frac{\partial A_x}{\partial t} + q\left[\dot y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot z\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right]$$

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot y} \Leftrightarrow -q\frac{\partial \phi}{\partial y} + q\left(\frac{\partial A_x}{\partial y}\dot x + \frac{\partial A_y}{\partial y}\dot y + \frac{\partial A_z}{\partial y}\dot z\right) = \frac{d}{dt}[m\dot y + qA_y] = m\ddot y + q\frac{dA_y}{dt}$$

$$= m\ddot{y} + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_y}{\partial x}\dot{x} + \frac{\partial A_y}{\partial y}\dot{y} + \frac{\partial A_y}{\partial z}\dot{z}\right)$$

$$\Leftrightarrow m\ddot{y} = -q\frac{\partial \phi}{\partial y} - q\frac{\partial A_y}{\partial t} + q\left[\dot{x}\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) + \dot{z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]$$

$$\frac{\partial L}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} \Leftrightarrow -q\frac{\partial \phi}{\partial z} + q\left(\frac{\partial A_x}{\partial z}\dot{x} + \frac{\partial A_y}{\partial z}\dot{y} + \frac{\partial A_z}{\partial z}\dot{z}\right) = \frac{d}{dt}[m\dot{z} + qA_z] = m\ddot{z} + q\frac{dA_z}{dt}$$

$$= m\ddot{z} + q\left(\frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x}\dot{x} + \frac{\partial A_z}{\partial y}\dot{y} + \frac{\partial A_z}{\partial z}\dot{z}\right)$$

$$\Leftrightarrow m\ddot{z} = -q\frac{\partial \phi}{\partial z} - q\frac{\partial A_z}{\partial t} + q\left[\dot{x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\right]$$

The Lorentz force is given by $\vec{F} = qE + qv \times B$. Writing the fields in terms of the corresponding potentials, $F = -q\nabla\phi + q\frac{\partial A}{\partial t} + qv \times (\nabla \times A)$. We write out

$$v \times (\nabla \times \vec{A}) = v \times \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$

$$= \left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\hat{i} - \left[v_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]\hat{j}$$

$$+ \left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]\hat{k}$$

Equating the components of this vector equation,

$$F_x = q\frac{\partial \phi}{\partial x} - q\frac{\partial A_x}{\partial t} + q\left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_Z}{\partial x}\right)\right]$$

$$F_y = q\frac{\partial \phi}{\partial y} - q\frac{\partial A_y}{\partial t} + q\left[v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - v_x\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]$$

$$F_z = q\frac{\partial \phi}{\partial z} - q\frac{\partial A_z}{\partial t} + q\left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]$$

These are exactly the values obtained for $F_i = m\ddot{x}_i$ in the application of the Euler-Lagrange equations to the target Lagrangian, it is the correct one.

3

The Lagrangian is $L = T - V = \frac{1}{2}mv^2 - mgy = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$. The constraint for this system is $\vec{r} \cdot \vec{r} = R^2 \Leftrightarrow x^2 + y^2 = R^2$. We obtain two constrained Euler-Lagrange equations:

$$\frac{\partial L}{\partial y} + \lambda\frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow -mg + 2\lambda y = \frac{d}{dt}(m\dot{y}) \Leftrightarrow 2\lambda y - mg = m\ddot{y}$$

$$\frac{\partial L}{\partial x} + \lambda\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = \frac{d}{dt}(m\dot{x}) \Leftrightarrow 2\lambda x = m\ddot{x}$$

Solving each of these equations of motion for the un-differentiated variable, substituting into the constraint, and solving for λ, we obtain

$$\left(\frac{m\ddot{x}}{2\lambda}\right)^2 + \left(\frac{m\ddot{y}+mg}{2\lambda}\right)^2 = R^2$$

$$\Leftrightarrow \lambda = \frac{m}{2R}\sqrt{\ddot{x}^2 + (\ddot{y}+g)^2}$$

Since the aboce Euler-Lagrange equations have the form of Newton's second law, it follows the tension has components

$$F_{tx} = 2\lambda x = \frac{mx}{R}\sqrt{\ddot{x}^2 + (\ddot{y}+g)^2}$$

$$F_{ty} = 2\lambda y = \frac{my}{R}\sqrt{\ddot{x}^2 + (\ddot{y}+g)^2}$$

4

For this problem, we have the same Lagrangian, but take spherical instead of polar coordinates:

$$L = \frac{1}{2}mv^2 - mgy \Leftrightarrow L = \frac{1}{2}mv^2 - mgr\sin(\phi)\sin(\theta)$$

Our constraint is $\vec{r}\cdot\vec{r} = l^2 \Leftrightarrow r = l$, so this becomes

$$L = \frac{1}{2}ml^2(l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2(\theta)) - mgl\sin(\theta)\sin(\phi)$$

There are then two Euler-Lagrange equations:

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} \Leftrightarrow -mgl\sin(\theta)\cos(\phi) = \frac{d}{dt}\left[ml^4\dot{\phi}\sin^2(\theta)\right]$$

$$\Leftrightarrow -mgl\sin(\theta)\cos(\theta) = ml^4\ddot{\phi}\sin^2(\theta) + ml^4\dot{\phi}\sin(2\theta)\dot{\theta}$$

and

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \Leftrightarrow ml^2\dot{\phi}^2\sin(2\theta) - mgl\cos(\theta)\sin(\phi) = \frac{d}{dt}\left[ml^4\dot{\theta}\right]$$

$$\Leftrightarrow ml^2\dot{\phi}^2\sin(2\theta) - mgl\cos(\theta)\sin(\phi) = ml^4\ddot{\theta}$$

For small angles, these become

$$-g(1-\theta/2) = l^3\ddot{\phi}\theta + 2l^3\dot{\phi}\dot{\theta}$$

and

$$2l\dot{\phi}^2\theta - g(1-\theta^2/2)\phi = l^3\ddot{\theta}$$

5

We have the same inital Lagrangian as before, but keep the problem in rectangular coordinates:

$$L = \frac{1}{2}mv^2 - mgy \Leftrightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgy$$

with constraint $f(x, y, z) = c \Leftrightarrow \vec{r} \cdot \vec{r} = l^2 \Leftrightarrow x^2 + y^2 + z^2 = l^2$. This yields three constrained Euler-Lagrange equations

$$\frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = m\ddot{x}$$

$$\frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow 2\lambda y - mg = m\ddot{y}$$

$$\frac{\partial L}{\partial y} + \lambda \frac{\partial f}{\partial z} = \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} \Leftrightarrow 2\lambda z = m\ddot{z}$$

Solving each of these for the un-differentiated variable, substituting into the constraint, and solving for λ, we obtain

$$\left(\frac{m\ddot{x}}{2\lambda}\right)^2 + \left(\frac{m\ddot{y} + mg}{2\lambda}\right)^2 + \left(\frac{m\ddot{z}}{2\lambda}\right)^2 = l^2$$

$$\Leftrightarrow \lambda = \frac{m}{2l}\sqrt{\ddot{x}^2 + (\ddot{y} + g)^2 + \ddot{z}^2}$$

Since the Euler-Lagrange equations had the form of Newton's second law, we can read the components of the tension force from there as

$$F_{tx} = \frac{mx}{2l}\sqrt{\ddot{x}^2 + (\ddot{y} + g)^2 + \ddot{z}^2}$$

$$F_{ty} = \frac{my}{2l}\sqrt{\ddot{x}^2 + (\ddot{y} + g)^2 + \ddot{z}^2}$$

$$Ftz = \frac{mz}{2l}\sqrt{\ddot{x}^2 + (\ddot{y} + g)^2 + \ddot{z}^2}$$