7550 HW 4

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Problem 1. *Let $f : V \to W$ be linear. Show that if f is surjective, then the induced map $f^* : W^* \to V^*$ defined by $(f^*(g))(v) = g(f(v))$ is injective.*

Solution. Suppose $f^*(g_1) = f^*(g_2)$. By definition, this is equivalent to $g_1 \circ f = g_2 \circ f$; since surjections are epic, one may right-cancel f to get $g_1 = g_2$. □

Problem 2. *Let f be a linear operator on V (finite-dimensional), and let $\mathcal{B}$ and $\mathcal{B}^*$ be dual bases for V and V^*. If A and B are the matrices of f and f^* with respect to $\mathcal{B}$ and $\mathcal{B}^*$, show that B is the transpose of A.*

Solution. The matrix of a linear operator f with respect to a basis $\mathcal{B}$ is defined by

$$a_{ij} = f(\mathcal{B}_i)_j$$

where $\mathcal{B}_i$ is the ith basis vector and v_i is the coefficient of the ith basis vector in the (unique) expression of v as a linear combination of $\mathcal{B}$. By definition of the transpose, we must show the entries a_{ij} of A and b_{ij} of B satisfy $b_{ij} = a_{ji}$, or $f^*(\mathcal{B}_i^*)_j = f(\mathcal{B}_j)_i$. The dual basis vector $\mathcal{B}_i^*$ is the functional that evaluates to 1 on $\mathcal{B}_i$ and 0 on the other basis vectors; that is, it picks out the coefficient of the ith basis vector. Accordingly,

$$f^*(\mathcal{B}_i^*) = [v \mapsto \mathcal{B}_i^*(f(v))] = \left[\sum_k c_k \mathcal{B}_k \mapsto \sum_k c_k \mathcal{B}_i^*(f(\mathcal{B}_k))\right]$$

The action of this functional on the vector $\mathcal{B}_j$ is its jth component in the dual space, and so $f^*(\mathcal{B}_i^*)_j = B_i^* f(B_j) = f(B_j)_i$. □

Problem 3. *If $f : V_1 \to V_2$ is a surjective linear map, show that, for any W, the induced map $V_1 \otimes W \to V_2 \otimes W$ defined by $v_1 \otimes w \mapsto f(v_1) \otimes w$, is also surjective.*

Solution. The tensor product is constructed as a quotient of the (ordinary categorical) product. Therefore, if it can be shown that the corresponding map on the products (which takes $(v, w) \to (f(v), w)$ before quotienting) is surjective, then the induced map on the quotient (sending further each element to the coset containing it) is obviously also surjective: the map from element to coset is surjective, since every coset is nonempty, and the composition of surjective maps is surjective. Indeed, for any element $(v_2, w) \in V_2 \times W$, by surjectivity of f there must be v_1 such that $v_2 = f(v_1)$, and, accordingly, $(v_2, w) = (f(v_1), w)$. The identity is trivially surjective, and product maps are surjective iff their components are. □

Problem 4. *Show that $\{v_1, \ldots, v_r\}$ is a linearly independent set in V iff $v_1 \wedge \cdots \wedge v_r \neq 0$.*

Solution. We show this for the case $r = 2$, and generalize by trivial induction.

The fact that the exterior product is anticommuting means that $v_1 \wedge v_2 = -v_2 \wedge v_1$, so it suffices to show that the exterior product is commutative iff there are a_1, a_2 such that $a_1 v_1 + a_2 v_2 = 0$.

In order for $v_1 \wedge v_2 = v_2 \wedge v_1$ in $\Lambda(V)$, one must have in $T(V)$, since the exterior product is that induced by the tensor product under the quotient by the ideal I consisting of finite linear combinations of elements of the form $x \otimes x$, $v_1 \otimes v_2 - v_2 \otimes v_1 \in I \Leftrightarrow v_1 \otimes v_2 - v_2 \otimes v_1 = \sum_i a_i x_i \otimes x_i$ for some collection of scalars a_i and tensors x_i. $v_1 \otimes v_2$ is of rank 2, and since each grading of the tensor algebra is a vector subspace, each x_i must be an element of V.

Suppose $a_1 v_1 + a_2 v_2 = 0$ for some scalars a_1, a_2. By linearity, $v_1 \otimes v_2 = \frac{-1}{a_1 a_2} \cdot (a_1 v_1 \otimes -a_2 v_2)$, and the bracketed term is of the form $x \otimes x$ for some x by the linear dependence assumption; accordingly, when this tensor product is quotiented to become $v_1 \wedge v_2$, it vanishes, as this is the very definition for elements of the ideal by which one quotients.

Conversely, suppose $v_1 \wedge v_2 = 0$. Then $v_1 \otimes v_2 = \sum_i a_i x_i \otimes x_i$, for some scalars a_i and vectors x_i. In terms of a basis $\mathcal{B}$ for V, which induces an obvious basis on $V \otimes V$, linearity lets one compute

$$\sum_{i,j} b_i c_j \mathcal{B}_i \otimes \mathcal{B}_j = v_1 \otimes v_2 = \sum_i a_i x_i \otimes x_i = \sum_{i,j,k} a_i d_j d_k \mathcal{B}_j \otimes \mathcal{B}_k$$

In particular, we may compare coefficients and see that $b_i c_j = (\sum_k a_k) d_i d_j$. Were v_1, v_2 linearly independent, we could take them as the first two elements of the basis. Then $b_i = \delta_{1i}$ and $c_j = \delta_{2j}$, implying $v_1 \otimes v_2 = 0$. But this necessarily means that one of the factors is 0, and therefore cannot be a basis vector—a contradiction.

This forms a base case. If the theorem holds for r vectors, then the $r + 1$ case equivalently characterizes linear independence of $\{v_1 \wedge \cdots \wedge v_r, v_{r+1}\}$ by $(v_1 \wedge \cdots \wedge v_r) \wedge v_{r+1} \neq 0$—a list is linearly independent iff all sublists are, and the wedge product is associative. Successive application of the inductive hypothesis and the base case produces the full result. □

Problem 5. *Show that two linearly independent sets $\{v_1, \ldots, v_r\}$ and $\{w_1, \ldots, w_r\}$ in V span the same r-dimensional subspace iff $v_1 \wedge \ldots \wedge v_r = c \cdot w_1 \wedge \ldots \wedge w_r$, where $c = \det(A)$, and $A = (a_{ij})$ is given by $v_i = \sum_{j=1}^r a_{ij} w_j$.*

Solution. Consider first the reverse implication. Linear independence means, by the above result, that each product term is nonzero. Accordingly, A must be nonsingular, as if the determinant were zero, the left side of the equality with the products would be zero. Furthermore, this is equivalent to A being an isomorphism of vector spaces. By definition of A, any element in the span of $\{v_1, \ldots, v_r\}$ is

$$\sum_i c_i v_i = \sum_i \sum_j c_i a_{ij} w_j,$$

establishing that every element of the span of $\{v_1, \ldots, v_r\}$ is a linear combination of $\{w_1 \ldots, w_r\}$, i.e. is in the span of $\{w_1 \ldots, w_r\}$. Since A is invertible, we can apply the same reasoning to $A^{-1} = b_{ij}$ for which $w_i = \sum_{j=1}^r b_{ij} v_j$, establishing bicontainment and therefore equality of the spans of these two sets.

Conversely, suppose the two sets span the same r-dimensional subspace. This means that there exists a change-of-basis matrix A from w_i to v_i of the above form—they are bases for the same subspace. We may simply chug along:

$$v_1 \wedge \cdots \wedge v_r = \left(\sum_{j_1=1}^r a_{1j_1} w_{j_1} \right) \wedge \cdots \wedge \left(\sum_{j_r=1}^r a_{rj_r} w_{j_r} \right)$$

$$= \sum_{j_1,\ldots,j_r=1}^{r} a_{1j_1} \cdots a_{rj_r} w_{j_1} \wedge \cdots \wedge w_{j_r}$$

If any two j_i are the same, then that summand goes to zero. Permuting the factors of a wedge picks up the sign of the permutation. Accordingly, we may choose our favorite canonical ordering to the w_i (I'll do ascending), put all wedges in this form, and sum instead over the permutations of the wedges, with a coefficient ensuring that the permuted summands correspond bijectively with summands in the above.

$$\sum_{\sigma \in S_r} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)} w_1 \wedge \cdots \wedge w_r = \left(\sum_{\sigma \in S_r} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{r\sigma(r)} \right) w_1 \wedge \cdots \wedge w_r$$

This is precisely the constructive definition of the determinant. $\square$

Problem 6. *Let $f : V \to V$ be linear, let $\mathcal{B}$ be a basis for V, and let A be the matrix of f with respect to $\mathcal{B}$.*

1. *Let $\phi : \Lambda^n V \to \Lambda^n V$ be the map induced by f, defined by $\phi(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$. Since $\Lambda^n V$ is 1-dimensional, ϕ corresponds to multiplication by some scalar, say c. Show that $c = \det(A)$.*
2. *Use the above to prove the product formula* $\det(AB) = \det(A) \cdot \det(B)$

Solution. First, the constant corresponding to the identity mapping is clearly 1, as $\phi(v_1 \wedge \cdots \wedge v_n) = v_1 \wedge \cdots \wedge v_n$ is the identity linear map on a one-dimensional vector space. Secondly, considered as a functional from all possible columns of A's to c's, ϕ is linear in each argument: multiplying by a and adding w to the jth column of a matrix results in $f(v) \mapsto af(v) + \sum_i v_i w_i \mathcal{B}_j$ by the definition of matrix multiplication, and so by linearity of the wedge product the scalar corresponding to ϕ changes linearly. It's also alternating. If two consective columns in A are the same, then the kernel of f is nontrivial by the rank-nullity theorem. Accordingly, there exists a nonzero vector v such that $f(v) = 0$; any element $\lambda \in \Lambda^n V$ that includes a factor of this vector will have $\phi(\lambda) = 0$. There exists a nonzero such λ: we can choose a basis for V containing v, and the wedge of this basis is nonzero by Problem 4 above. If a linear operator on a one-dimensional space has a nontrivial kernel, then it must be the zero operator, as any nonzero vector in the kernel spans the space—so if two columns in A are the same, then $c = 0$.

The determinant is the unique alternating multilinear functional on matrices of linear maps that sends the identity to 1, establishing the first result.

Letting g be a linear map whose matrix is B, the number $\det(AB)$ corresponds to the ϕ given by $f \circ g$, which acts as $\phi_{f \circ g}(v_1 \wedge \cdots \wedge v_n) = f \circ g(v_1) \wedge \cdots \wedge f \circ g(v_n) = \phi_f \circ \phi_g$. Composition of 1-dimensional linear operators is scalar multiplication, so this yields exactly

$$\det(AB) = \det(A) \cdot \det(B).$$

Interestingly, this establishes that this property of the determinant is *precisely* a functoriality result. $\square$

Problem 7. *Let V be a real inner product space, a real vector space equiped with a symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, with $\langle v, v \rangle = 0 \Leftrightarrow v = 0$. Then $\langle \cdot, \cdot \rangle$ induces an inner product $\langle \cdot, \cdot \rangle : \Lambda V \times \Lambda V \to \mathbb{R}$, defined as follows: if $u = u_1 \wedge \cdots \wedge u_r$ and $v = v_1 \wedge \cdots \wedge v_s$ are pure wedges, set $\langle u, v \rangle = 0$ if $r \neq s$ and $\langle u, v \rangle = \det(\langle u_i, v_j \rangle)$ if $r = s$. This can be extended linearly in each argument. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V, a basis for which $\langle e_i, e_j \rangle = \delta_{ij}$. Show that the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_r} \mid 1 \leq$*

$i_1 < \cdots < i_r \leq n, 0 \leq r \leq n\}$ is an orthonormal basis for ΛV. For $r = 0$, the empty wedge product is interpreted to be $1 \in \mathbb{R} = \Lambda^0 V$.

Solution. Given two basis elements of different lengths, their inner product is *a fortiori* zero. Suppose the two basis elements have the same length, but there's some wedge factor u_i (WLOG; the symmetry of the form implies that the matrix of dot products is symmetric) that doesn't equal any v_j. Then the column given by $\langle u_i, v_j \rangle$ is all zeroes, and the matrix is degenerate, so the dot product is zero. If, however, two basis elements are identical, then $\langle u_i, v_j \rangle = \delta_{ij}$ by the definition of the dual basis, i.e., the matrix is the identity matrix, which has determinant 1 (by definition). Accordingly, the orthonormal condition holds for all possible basis elements of this cannonical basis for ΛV.

Furthermore, it is indeed a basis for ΛV. A basis for a graded vector space is a basis for each grading, so we may restrict attention to $\Lambda^r V$. This space is a quotient of the space of r-fold tensor products; by definition, this in turn has basis of the form $e_{i_1} \otimes \cdots \otimes e_{i_r}$ for $1 \leq i_j \leq n$. Under the quotient, precisely the following linear dependences between these basis elements are introduced: as seen in Problem 5, any basis element with repeated factors goes to the zero vector, and vectors that are the same up to reordering are the same up to a sign (the sign of the permutation that reorders them). In the latter case, we can choose the basis vector whose indices are in ascending order to represent the associated class in the quotient. This recovers precisely our candidate basis: the wedge product of distinct basis elements for V with ascending indices. □

Problem 8. *An endomorphism $\psi : \Lambda V \to \Lambda V$ is an anti-derivation if, for $u \in \Lambda^k V$ and $v \in \Lambda V$, $\psi(u \wedge v) = \psi(u) \wedge v + (-1)^k u \wedge \psi(v)$. Show that $\psi : \Lambda V \to \Lambda V$ is an anti-derivation iff for all r*

$$\psi(v_1 \wedge \cdots \wedge v_r) = \sum_{k=1}^{r} (-1)^{k+1} v_1 \wedge \cdots \wedge \psi(v_k) \wedge \cdots \wedge v_r.$$

Solution. Suppose that $\psi(v_1 \wedge \cdots \wedge v_r) = \sum_{k=1}^{r} (-1)^{k+1} v_1 \wedge \cdots \psi(v_k) \wedge \cdots \wedge v_r$. Then, given any $u \in \Lambda^k V$ and $v \in \Lambda V$, we may write u as a k-fold pure wedge and v as a sum of pure wedges, and apply linearity:

$$\psi(u \wedge v) = \psi\left(u_1 \wedge \cdots \wedge u_k \wedge \sum_i \bigwedge_{j=1}^{i} v_j\right) = \psi\left(\sum_i u_1 \wedge \cdots \wedge u_k \wedge v_1 \wedge \cdots \wedge v_i\right)$$

Representing the jth vector in the wedge above as w_j,

$$= \sum_i \psi(w_1 \wedge \cdots \wedge w_{k+i}) = \sum_i \sum_{j=1}^{k+i} (-1)^{j+1} w_1 \wedge \cdots \wedge \psi(w_j) \wedge \cdots \wedge w_{k+i}$$

$$= \left(\sum_i \sum_{j=1}^{k} (-1)^{j+1} w_1 \wedge \cdots \wedge \psi(w_j) \wedge \cdots \wedge w_{k+i}\right) + \left(\sum_{j=k+1}^{k+i} \sum_i (-1)^{j+1} w_1 \wedge \cdots \wedge \psi(w_j) \wedge \cdots \wedge w_{k+i}\right)$$

$$= \left(\sum_i \left(\sum_{j=1}^{k} (-1)^{j+1} w_1 \wedge \cdots \wedge \psi(w_j) \wedge w_k\right) \wedge w_{k+1} \wedge \cdots \wedge w_{k+i}\right)$$

$$+\left(\sum_i w_1 \wedge \cdots \wedge w_k \wedge \left(\sum_{j=k+1}^{k+i} (-1)^{j+1} w_{k+1} \wedge \cdots \wedge \psi(w_j) \wedge \cdots \wedge w_{k+i}\right)\right)$$

$$= \left(\sum_i \psi(u_1 \wedge \cdots \wedge u_k) \wedge v_1 \wedge \cdots \wedge v_i\right) + \left(u_1 \wedge \cdots \wedge u_k \wedge \left(\sum_i \sum_{j=1}^{i} (-1)^{j+1-k} v_1 \wedge \cdots \wedge \psi(v_j) \wedge \cdots \wedge v_i\right)\right)$$

Factoring out the scalar $(-1)^{-k} = (-1)^k$ from each term,

$$= \left(\psi(u) \wedge \sum_i v_1 \wedge \cdots \wedge v_i\right) + \left(u \wedge (-1)^k \sum_i \sum_{j=1}^{i} (-1)^{j+1} v_1 \wedge \cdots \wedge \psi(v_j) \wedge \cdots \wedge v_i\right)$$

$$= \psi(u) \wedge v + (-1)^k u \wedge \sum_i \psi(v_i) = \psi(u) \wedge v + (-1)^k u \wedge \psi(v)$$

Conversely, suppose that for all k, $u \in \Lambda^k V$, and $v \in \Lambda V$ that $\psi(u \wedge v) = \psi(u) \wedge v + (-1)^k u \wedge \psi(v)$. Proceeding by induction, it's clear that for $r = 0$ the conclusion is reflexive. Supposing that it holds for r, then by the supposition

$$\psi((v_1 \wedge \cdots \wedge v_r) \wedge v_{r+1}) = \psi(v_1 \wedge \cdots \wedge v_r) \wedge v_{r+1} + (-1)^r (v_1 \wedge \cdots \wedge v_r) \wedge \psi(v_{r+1}).$$

Applying the induction hypothesis,

$$= \left(\sum_{k=1}^{r} (-1)^{k+1} v_1 \wedge \cdots \wedge \psi(v_k) \wedge \cdots \wedge v_r\right) \wedge v_{r+1} + (-1)^r (v_1 \wedge \cdots \wedge v_r) \wedge \psi(v_{r+1}).$$

Since $(-1)^r = (-1)^{(r+1)+1}$, we can distribute and get

$$= \sum_{k=1}^{r+1} (-1)^{k+1} v_1 \wedge \cdots \wedge \psi(v_k) \wedge \cdots \wedge v_{r+1}.$$

$\square$