4271 HW 3

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Problem 1. *Find the angle between the angular momentum vector ℓ and the z-axis for all possible orientations when $\ell = 3$.*

Solution. The z-component of ℓ in angular momentum space, ℓ_z, is determined by m_ℓ, with eigenvalues $\hbar m_\ell$. The magnitude of ℓ is given by $\sqrt{\ell^2}$, which has eigenvalue $\hbar\sqrt{\ell(\ell+1)}$. A vector of magnitude $\hbar\sqrt{\ell(\ell+1)}$ and z-component $\hbar m_\ell$ has elevation angle

$$\theta = \cos^{-1}\left(\frac{m_\ell}{\sqrt{\ell(\ell+1)}}\right).$$

For $\ell = 3$, $\sqrt{\ell(\ell+1)} = \sqrt{12} = 2\sqrt{3}$, and m_ℓ ranges from -3 to 3. Accordingly, the possible angles are

$$\theta_{m_\ell=0} = 90\,\mathrm{deg}$$
$$\theta_{m_\ell=1} = 73.2\,\mathrm{deg}$$
$$\theta_{m_\ell=-1} = 106.8\,\mathrm{deg}$$
$$\theta_{m_\ell=2} = 54.7\,\mathrm{deg}$$
$$\theta_{m_\ell=-2} = 125.3\,\mathrm{deg}$$
$$\theta_{m_\ell=3} = 30\,\mathrm{deg}$$
$$\theta_{m_\ell=-3} = 150\,\mathrm{deg}$$

□

Problem 2. *Calculate the binding energy and binding energy per nucleon of the deuteron.*

Solution. The atomic mass of the deuteron is

$$BE(Z,N) = (Zm_H + Nm_n - M_A)c^2$$

$$\Rightarrow BE(1,1) = (1\cdot 1.007\,825\,\mathrm{amu} + 1\cdot 1.008\,665\,\mathrm{amu} - 2.014\,102\,\mathrm{amu})\cdot(931.49\,\mathrm{MeV/amu}) = 2.22\,\mathrm{MeV}$$

$$\Rightarrow BE(1,1)/A = 2.22\,\mathrm{MeV}/2 = 1.11\,\mathrm{MeV}$$

□

Problem 3. *At what energy in the laboratory system does a proton beam scattering off a proton target become inelastic—i.e. at what proton beam energy can pions be produced?*

Solution. This is just conservation of energy: at this threshold, all of the kinetic energy of the beam goes into creating the pion, and the protons are at rest, so

$$E = \frac{m_p c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = 2m_p c^2 + m_\pi c^2 \Leftrightarrow v = c\sqrt{1 - \frac{1}{(2 + \frac{m_\pi}{m_p})^2}}$$

$$= c\sqrt{1 - \frac{1}{(2 + \frac{135\,\mathrm{MeV}/c^2}{938\,\mathrm{MeV}/c^2})^2}} = 0.885c$$

This corresponds to a beam energy of

$$K_b = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1.007\,825\,\mathrm{amu} \cdot 931.49\,\mathrm{MeV/amu}}{\sqrt{1 - \frac{(0.885c)^2}{c^2}}} = 2.016\,\mathrm{GeV}.$$

□

Problem 4. *In the spherical shell model, what are the expected ground state spins and parities for:* $^{11}\mathrm{B}$, $^{15}\mathrm{C}$, $^{17}\mathrm{F}$, $^{31}\mathrm{P}$, $^{141}\mathrm{Pr}$, *and* $^{207}\mathrm{Pb}$. *Look up the experimental values. Do they agree?*

Solution. $^{11}\mathrm{B}$ has 5 protons and 6 neutrons, and so is even-odd; the neutrons contribute 0^+, and, looking at the spin-orbit coupling chart, the unpaired proton is in a $1p_{3/2}$ state. Accordingly, the atom will have spin equal to the angular momentum of this unpaired electron, or $J^\pi = \frac{3}{2}^{(-1)^1} = \frac{3}{2}^-$.

Proceeding similarly (writing less painstaking detail, since it's all the same),

$$^{15}\mathrm{C} = 6p + 9n \Rightarrow n_{unp} \in 1d_{5/2} \Rightarrow J^\pi = \frac{5}{2}^+$$

$$^{17}\mathrm{F} = 9p + 8n \Rightarrow p_{unp} \in 1d_{5/2} \Rightarrow J^\pi = \frac{5}{2}^+$$

$$^{31}\mathrm{P} = 15p + 16n \Rightarrow p_{unp} \in 2s_{1/2} \Rightarrow J^\pi = \frac{1}{2}^+$$

$$^{141}\mathrm{Pr} = 59p + 82n \Rightarrow p_{unp} \in 2d_{5/2} \Rightarrow J^\pi = \frac{5}{2}^+$$

$$^{207}\mathrm{Pb} = 82p + 125n \Rightarrow n_{unp} \in 1i_{13/2} \Rightarrow J^\pi = \frac{13}{2}^+$$

Comparing with JAEA's nuclide charts, $^{15}\mathrm{C}$ is given as $\frac{1}{2}^+$ and $^{207}\mathrm{Pb}$ as $\frac{1}{2}^-$ (however, there is an excited state with spin-parity $\frac{13}{2}^+$). The rest agree. This is possibly attributable to $^{15}\mathrm{C}$ being unstable and $^{207}\mathrm{Pb}$ being large and non-spherical. □

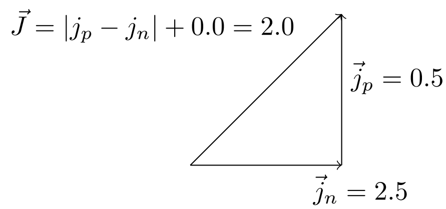
Problem 5. *The ground state of* $^{17}\mathrm{F}$ *has spin-parity of* $\frac{5}{2}^+$ *and the first excited state has a spin-parity of* $\frac{1}{2}^+$. *Using Povh Fig. 18.7, suggest two possible configurations for this excited state.*

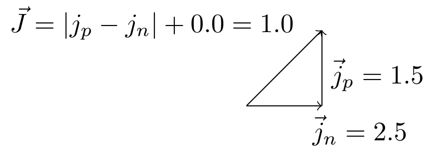
Solution. If the unpaired proton elevates into the $2s_{1/2}$ state just above $1d_{5/2}$, the excited state will have $J^\pi = \frac{1}{2}^+$. Since this destination state has a degeneracy of 2, there are two microstates corresponding to this energy.e □

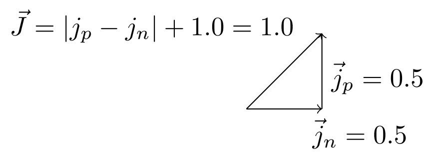
Problem 6. *The ground state of a nucleus with an odd proton and an odd neutron (aka. an "odd-odd" nucleus) is determined from the angular momentum coupling of the odd proton and neutron:* $\vec{J} = \vec{j}_p + \vec{j}_n$. *Consider the following nuclei:* ${}^{16}\mathrm{N}(2^-)$, ${}^{12}\mathrm{B}(1^+)$, ${}^{34}\mathrm{P}(1^+)$, ${}^{28}\mathrm{Al}(3^+)$.

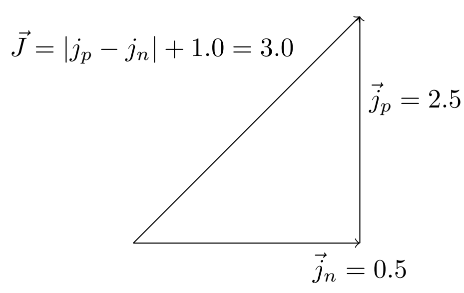
1. *Draw simple vector diagrams illustrating these couplings—i.e.* $\vec{J} = \vec{j}_p + \vec{j}_n$.
2. *Replace* $\vec{j}_p$ *and* $\vec{j}_n$*, respectively, by* $\vec{\ell}_p + \vec{s}_p$ *and* $\vec{\ell}_n + \vec{s}_n$*, illustrating the two vectors* $\vec{\ell}$ *and* $\vec{s}$*.*
3. *Examine your four diagrams and deduce an empirical rule for the relative orientation of* $\vec{s}_p$ *and* $\vec{s}_n$ *in the ground state.*
4. *Use this empirical rule to predict the* $\vec{J}^\pi$ *assignments of* ${}^{26}\mathrm{Na}$ *and* ${}^{28}\mathrm{Na}$*.*

Solution. Noting ${}^{16}\mathrm{N} = 7p + 9n$, ${}^{12}\mathrm{B} = 5p + 9n$, ${}^{34}\mathrm{P} = 15p + 19n$, and ${}^{28}\mathrm{Al} = 13p + 15n$, we can use the shell model to compute the $\vec{j}$ for each proton and neutron, and verify that the given J is in the vector sum:

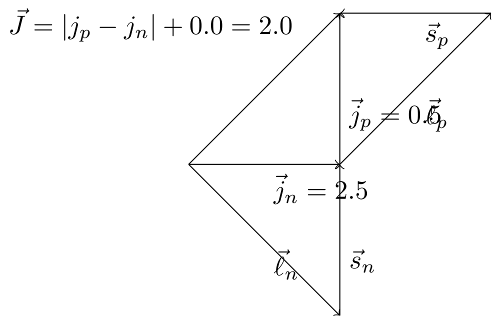


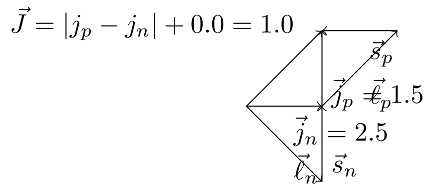


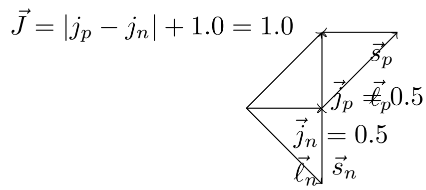


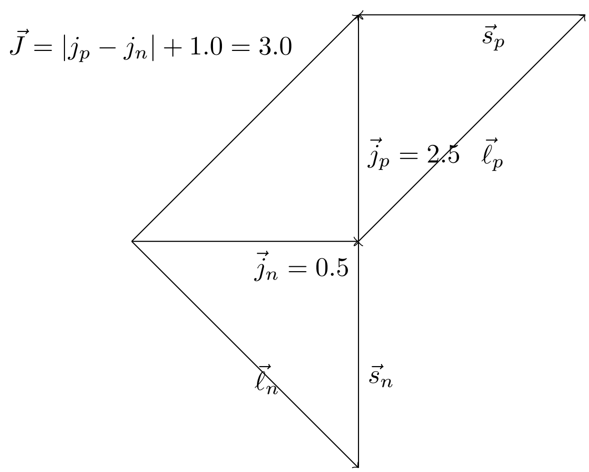


Adding on the spin dependence (please excuse the shoddy TikZing; I don't quite have time to work out the scaling properly),









My conjecture is that the spins of the unpaired nucleons must be aligned and pointing in the direction indicated by the parity. I am not quite sure how this, or any alternative conjecture, could possibly disambiguate things enough to determine $\vec{J}$. At this point, I'm not even sure the vector sum is associative. $\square$

Problem 7 (Bonus)**.** *Let's suppose we can form* ^{3}He *or* ^{3}H *by adding a proton or a neutron* (*respectively*) *to* ^{2}H, *which has* $\vec{J}^\pi = 1^+$.

1. *What are the possile values of the total angular momentum for* $^3\mathrm{He}$ *and* $^3\mathrm{H}$*, given an orbital angular momentum* ℓ *for the added nucleon?*
2. *Given that* $^3\mathrm{He}$ *and* $^3\mathrm{H}$ *have positive parity, which of these is still possible?*
3. *What is the most likely value for the ground-state orbital angular momentum of* $^3\mathrm{He}$ *and* $^3\mathrm{H}$.

No attempt. □