Math 7550 HW 2

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Problem 1. *If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map and we identify $T_p(\mathbb{R}^k)$ with $\mathbb{R}^k$ by identifying $\frac{\partial}{\partial x}$ with the ith standard basis vector, show that ϕ_* is just ϕ.*

In other words, using the $(d\phi)_p$ notation for ϕ_* at p, for any $p \in \mathbb{R}^m$, show that $(d\phi)_p = \phi$. (Note that you are implicitly showing that ϕ is differentiable)

Proof. The identification is given by considering paths through p given by $\gamma_v = p + tv$, and noticing that $D_{\gamma_v} = v$. By definition,

$$\phi_*(D_{\gamma_v}) = D_{\phi\circ\gamma_v} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(p+tv)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p) + t\phi(v)]\bigg|_{t=0} = \phi(v).$$

□

Problem 2. *Generalize Problem 1: if $\phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^k$ is a bilinear map, show that ϕ is differentiable, and that, for any $(p,q) \in \mathbb{R}^m \times \mathbb{R}^n$, we have $(d\phi)_{(p,q)}(x,y) = \phi(p,y) + \phi(x,q)$.*

Proof. Writing elements of the domain (u,v), we can define a natural vector space structure from the isomorphic space $\mathbb{R}^{n+m}$: $a(u,v)+(u',v') = (au+u', av+v')$. With respect to this, we can define a path through (p,q) analogous to that above by $\gamma_{(u,v)}(t) = (p,q) + t(u,v)$. Note that this vector space structure is used strictly to define $\gamma_{(u,v)}$; the result does not depend on it. Similarly, one can immediately see that $D_{\gamma_{(u,v)}} = (u,v)$ and

$$\phi_*(D_{\gamma_{(u,v)}}) = D_{\phi\circ\gamma_{(u,v)}} = \frac{\mathrm{d}}{\mathrm{d}t}\phi[(p,q)+t(u,v)]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\phi(p+tu, q+tv)\bigg|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p,q+tv) + \phi(tu,q+tv)]\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(p,q) + \phi(p,tv) + \phi(tu,q) + \phi(tu,tv)]\bigg|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left[\phi(p,q) + t\phi(p,v) + t\phi(u,q) + t^2\phi(u,v)\right]\bigg|_{t=0} = \phi(p,v) + \phi(u,q) + 2t\phi(u,v)\bigg|_{t=0}$$

$$= \phi(p,v) + \phi(u,q)$$

which is of the desired form (despite my using different variables). □

Problem 3. *Let $M = M_{m,n}(\mathbb{R})$ be the set of $m \times n$ matrices with real entries. Fix $k \leq \min m, n$, and let $U_k = \{A \in M \mid \operatorname{rank}(A) \geq k\}$. Describe a C^∞ structure for U_k.*

Proof. The topology on the set of matrices is not specified, so I'll presume it's the one borrowed from Euclidean space by considering matrices as vectors in $\mathbb{R}^{mn}$. One can apply row- or column-reduction to matrices in U_k (according to whichever dimension is the smallest). Since there are k linearly independent rows or columns, the reduced echelon form has a $k \times k$ identity submatrix in the initial part, and an arbitrary matrix with k entries one way and $\max m, n-k$ in the other left over. Interpreting this left-over matrix as a vector in $\mathbb{R}^{k(\max m, n-k)}$, one has a map from U_k to $\mathbb{R}^{k(\max m, n-k)}$: reduce, and reinterpret. The desired functional structure can then be the functional structure induced by this map. $\square$

Problem 4. *The Grassman manifold or Grassmannian $G_{k,n}$: let $G_{k,n}$ be the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. Show that $G_{k,n}$ is a smooth manifold of dimension $k(n-k)$.*

Proof. There is an onto correspondence of lists of k linearly-independent vectors in $\mathbb{R}^n$ and its k-dimensional vector subspaces, since such lists generate subspaces by spanning and every subspace can be so-generated since every subspace has a basis. Such lists can be viewed as $k \times n$ matrices with real entries by just concatenating the column vectors wrt. the standard basis; the condition that the list is linearly independent translates to these matrices having rank k, by definition of rank. Noticing that this means that the space of such lists is U_k from the previous problem, we can borrow the C^∞ structure defined there. Defining two elements of U_k to be equivalent via $\sim$ if their column space is the same (or, equivalently, if they have the same reduced column echelon form), the functional structure in the previous problem becomes a smooth manifold on the quotient. The resulting quotient is obviously in bijective correspondence with $G_{k,n}$, so we'll define everything on $G_{k,n}$ via this correspondence, forgetting the difference and writing $G_{k,n} = U_k / \sim$.

It remains to prove that the map $f : U_k \to \mathbb{R}^{k(n-k)}$ actually gives local homeomorphisms. We'll proceed by interpreting the map $f : U_k \to \mathbb{R}^{k(n-k)}$ given in the previous problem as a chart. Applying f and then mapping to the equivalence class containing that representative is precisely the quotient map for the equivalence we've defined, with $\tau \sim \sigma$ iff their images are equal, so the quotient topological space $G_{k,n}$ and $\mathbb{R}^{k(n-k)}$ are homeomorphic. Of course, any restriction of a homeomorphism is also a homeomorphism, so the local homeomorphism condition is satisfied. Certainly, every element of $G_{k,n}$ is in the domain of this chart. The transition condition follows since these charts are invertible linear maps, making the transition functions smooth. $\square$