

2231 HW 1

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1

Let $\vec{A} = \hat{i}$, $\vec{B} = \hat{j}$, and $\vec{C} = \hat{j}$. We don't need to compute to see that this fails to be associative: in $\vec{A} \times (\vec{B} \times \vec{C})$, we observe that the first-evaluated product is zero, since the angle between a vector and itself is zero; the overall product is therefore zero as well. However, in $(\vec{A} \times \vec{B}) \times \vec{C}$ the first-evaluated product is a nonzero vector perpendicular to the \hat{i} - \hat{j} plane, and therefore the cross product of this vector with \hat{j} is nonzero.

2

We denote components as $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and so on. Then

$$\begin{aligned} \vec{B} \times \vec{C} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (B_y C_z - B_z C_y) \hat{i} - (B_x C_z - B_z C_x) \hat{j} + (B_x C_y - B_y C_x) \hat{k} \\ \Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} \\ &= (A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z) \hat{i} \\ &\quad - (A_x B_x C_y - A_x B_y C_x - A_z B_y C_z + A_z B_z C_y) \hat{j} \\ &\quad + (A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y) \hat{k} \end{aligned}$$

On the other hand,

$$\begin{aligned} \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) &= \vec{B}(A_x C_x + A_y C_y + A_z C_z) - \vec{C}(A_x B_x + A_y B_y + A_z B_z) \\ &= (A_x B_x C_x + A_y B_x C_y + A_z B_x C_z) \hat{i} - (A_x B_x C_x + A_y B_y C_x + A_z B_z C_x) \hat{i} \\ &\quad + (A_x B_y C_x + A_y B_y C_y + A_z B_y C_z) \hat{j} - (A_x B_x C_y + A_y B_y C_y + A_z B_z C_y) \hat{j} \end{aligned}$$

$$\begin{aligned}
& +(A_x B_z C_x + A_y B_z C_y + A_z B_z C_z) \hat{k} + (A_x B_x C_z + A_y B_y C_z + A_z B_z C_z) \hat{k} \\
& = (A_y B_x C_y + A_z B_x C_z - A_y B_y C_x - A_z B_z C_x) \hat{i} \\
& + (A_x B_y C_x + A_z B_y C_z - A_x B_x C_y - A_z B_z C_y) \hat{j} \\
& + (A_x B_z C_x + A_y B_z C_y - A_x B_x C_z - A_y B_y C_z) \hat{k}
\end{aligned}$$

which is precisely what was obtained from the cross product, simply rearranged with a negative distributed. This proves the identity.

3a

$$\nabla f = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) xyz = yz \hat{i} + xz \hat{j} + xy \hat{k}$$

3b

$$\nabla f = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) e^x y^3 \sin(z) = e^x y^3 \sin(z) \hat{i} + 3e^x y^2 \sin(z) \hat{j} + e^x y^3 \cos(z) \hat{k}$$

3c

$$\nabla f = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) e^{-(x^2+y^2+z^2)} = -2xe^{-(x^2+y^2+z^2)} \hat{i} - 2ye^{-(x^2+y^2+z^2)} \hat{j} - 2ze^{-(x^2+y^2+z^2)} \hat{k}$$

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We write $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k}$$

The divergence of this is

$$\nabla \cdot (\nabla \times \vec{A}) =$$