4132 HW 3

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7.28a

The energy in the solenoid is

$$E = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2 l\pi R^2 I^2$$

7.28b

The vector potential of the solenoid was found in the book to be

$$\vec{A} = \begin{cases} \frac{1}{2}\mu_0 nIs\hat{\phi} & s \leq R \\ \frac{1}{2}\mu_0 nI\frac{R^2}{s}\hat{\phi} & s > R \end{cases}$$

By the formulae mentioned in the problem for work where the integral is over a loop at the surface of the solenoid,

$$W_0 = \frac{1}{2}I\int \vec{A}\cdot d\vec{l} = \frac{1}{2}I\int \frac{\mu_0 nI}{2}R\hat{\phi}\cdot d\vec{l} = \frac{\pi}{2}\mu_0 nI^2R^2$$

is the work due to one turn. Over l, there are nl turns, so the total work is

$$W = \frac{1}{2}\mu_0 n^2 l\pi R^2 I^2$$

7.28c

The magnetic field inside the solenoid is

$$\vec{B} = \mu_0 nI\hat{z}$$

and zero outside, so the work (approximating the field to be zero outside the length of consideration) is

$$W = \frac{1}{2\mu_0}\int_0^l \int_0^{2\pi} \int_0^R (\mu_0 nI)^2 \hat{z} r dr d\theta dz = \frac{1}{2}\mu_0 n^2 l\pi R^2 I^2$$

7.28d

The vector potential and magnetic fields are given by their formulae in the previous two problems, so

$$\vec{A} \times \vec{B} = \begin{cases} \frac{1}{2}\mu_0^2 n^2 I^2 s\hat{r} & s \leq R \\ 0 & s > r \end{cases}$$

The energy stored is then

$$\begin{aligned} W &= \frac{1}{2\mu_0}\left[\int_0^l \int_0^{2\pi} \int_a^R \mu_0^2 n^2 I^2 r dr d\theta dz - \left(-\int_0^l \int_0^{2\pi} \frac{1}{2}\mu_0^2 n^2 I^2 a (a d\theta dz)\right)\right] \\ &= \frac{1}{2\mu_0}\left[\pi l \mu_0^2 n^2 I^2 (R^2 - a^2) - \left(-\pi l \mu_0^2 n^2 I^2 a^2\right)\right] = \frac{1}{2}\mu_0 n^2 l \pi R^2 I^2 \end{aligned}$$

7.30

The magnetic field inside the cable is by Ampère's law

$$\int_C \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \Leftrightarrow 2\pi s B = \mu_0 \frac{\pi s^2}{\pi R^2} I \Leftrightarrow B = \frac{\mu_0 I s}{2\pi R^2}$$

Outside the cable, the field is zero since the net enclosed current is zero. The energy stored in the magnetic field of the cable over a length l is then

$$W = \frac{1}{2\mu_0}\int_V B^2 dV = \frac{1}{2\mu_0}\int_0^l \int_0^{2\pi} \int_0^R \frac{\mu_0^2 I^2 s^2}{4\pi^2 R^4} s ds d\theta dz = \frac{\mu_0 l I^2}{4\pi R^4}\left(\frac{s^4}{4}\bigg|_0^R\right) = \frac{\mu_0 l I^2}{16\pi}$$

This is equal to $\frac{1}{2}LI^2$, so the inductance is

$$L = \frac{\mu_0 l}{8\pi}$$

The inductance per unit length is

$$L' = \frac{\mu_0}{8\pi}$$

7.34

There is no current flowing through the gap, so any magnetic field must be due to changes in the electric field. The capacitance of a parallel-plate capacitor yields

$$C = \frac{q}{V} = \frac{\epsilon_0 A}{d} \Leftrightarrow V = \frac{wq}{\epsilon_0 A} \Rightarrow \vec{E} = \frac{q}{\pi \epsilon_0 a^2}$$

directed along $\vec{I}$. Differentiating, the displacement current is

$$\vec{J} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\vec{I}}{\pi a^2} \Rightarrow \vec{\nabla} \times \vec{B} = \frac{\mu_0 \vec{I}}{\pi a^2}$$

This is the form of Ampère's law, and so we may transition to an integral form where we integrate over a loop of radius s concentric with and parallel to the plates of the capacitor. The magnetic field will be in the $\hat{\phi}$ direction and should be constant over the loop by rotational symmetry, so

$$\int_C \vec{B} \cdot d\vec{l} = \mu_0 I_{denc} \Leftrightarrow 2\pi s B = \mu_0 \frac{I}{a^2} s^2 \Leftrightarrow B = \frac{\mu_0 I}{2\pi} \frac{s}{a^2}$$

directed along $\hat{\phi}$.

7.36a

The displacement current for the given field is

$$\vec{J}_d = \epsilon_0 \frac{\partial E}{\partial t} = \frac{\mu_0 \epsilon_0 I_0 \omega^2}{2\pi} \cos(\omega t) \ln\left(\frac{a}{s}\right) \hat{z}$$

7.36b

The total displacement current is

$$I_d = \int \vec{J}_d d\vec{a} = \int_0^{2\pi} \int_0^a \frac{\mu_0 \epsilon_0 I_0 \omega^2}{2\pi} \cos(\omega t) \ln\left(\frac{a}{s}\right) s ds d\theta = \mu_0 \epsilon_0 I_0 \omega^2 \cos(\omega t) \left(\frac{s^2}{2} \ln\left(\frac{a}{s}\right) \Big|_0^a + \int_0^a \frac{s}{2} \right)$$

$$= \mu_0 \epsilon_0 I_0 \omega^2 \cos(\omega t) \left(0 - \frac{s^2}{2} \ln(a) \Big|_0 + \lim_{s \to 0} \frac{s^2}{2/\ln(s)} + \frac{s^2}{4} \Big|_0^a \right) = \mu_0 \epsilon_0 I_0 \omega^2 \cos(\omega t) \frac{a^2}{4}$$

7.36c

The current is $I_0 \cos(\omega t)$, so

$$\frac{I_d}{I} = \mu_0 \epsilon_0 \omega^2 \frac{a^2}{4}$$

Solving for frequency and using the given values,

$$\Leftrightarrow \omega = \sqrt{\frac{4 I_d}{\mu_0 \epsilon_0 a^2 I}} = \sqrt{\frac{4}{(1.26 \times 10^{-6}\,\mathrm{H/m})(8.85 \times 10^{-12}\,\mathrm{F/m})(2 \times 10^{-3}\,\mathrm{m})}(0.01)} = 1.34\,\mathrm{GHz}$$

The non-angular frequency is only a factor of 2π less. So, unless you're designing CPUs the displacement current is nearly negligible.

7.38

For the symmetry to be maintained between "electric things" and "magnetic things," we go to the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

and swap the two concepts:

$$\vec{F} = q_m \left(\vec{B} + \vec{v} \times \vec{E} \right)$$

However, notice that in the Maxwell equations everything that involves “moving magnetic things” (the magnetic current and the change in the magnetic field) has opposite sign compared to the corresponding electric things. Therefore, it would make sense to modify the sign of the second term of the above equation to get

$$\vec{F} = q_m \left(\vec{B} - \vec{v} \times \vec{E} \right)$$