7210 HW 3

Duncan Wilkie

13 September 2022

3.1.25. *A subgroup N of a group G is normal iff $gNg^{-1} \subseteq N$ for all $g \in G$.*

Proof. The definition of $N \trianglelefteq G$ is that $gNg^{-1} = N$ for all $g \in G$. The forward equivalence is therefore trivial. Conversely, suppose $gNg^{-1} \subseteq N$ for all g.

$$N = \{n \in N \mid n \in N\} = \{n \in N \mid gg^{-1}ngg^{-1} \in N\} = \{n \in N \mid g\left(g^{-1}n^{-1}g\right)^{-1} g^{-1} \in N\}$$

Take any element $n \in N$;

$$n = gg^{-1}ngg^{-1} = g(g^{-1}n^{-1}g)^{-1}g^{-1} = g(g'n'(g')^{-1})^{-1}g^{-1} = gn''g^{-1}$$

where $g' = g^{-1}$, $n' = n^{-1} \in N$ by closure, and $n'' \in N$ is that which must then exist to satisfy $g'n'(g')^{-1} \subseteq N$. This demonstrates that every element of N equals an element of gNg^{-1}, so that $N \subseteq gNg^{-1}$ and consequently $gNg^{-1} = N$. $\square$

3.1.36. *If G is a group such that $G/Z(G)$ is cyclic, then G is Abelian.*

Proof. If $G/Z(G)$ is cyclic, it has a single generator $xZ(G)$. This means any element of $G/Z(G)$ may be written $x^aZ(G)$ for some $a \in \mathbb{Z}$.

Any element $g \in G$ generates a coset $gZ(G)$. Since the factor group is cyclic, this coset is $x^aZ(G)$ for some integer a. Symbolically,

$$gZ(G) = x^aZ(G) \Leftrightarrow (x^aZ(G))^{-1}(gZ(G)) = 1 \cdot Z(G) \Leftrightarrow x^{-a}gZ(G) = Z(G) \Leftrightarrow x^{-a}g = z \in Z(G).$$

Thus all $g \in G$ can be written x^az for some integer a, some central element z, and x a representative of the generator coset. Therefore, since central z commute with all elements of G,

$$ab = x^{a_1}z_1x^{a_2}z_2 = z_2x^{a_1}x^{a_2}z_1 = z_2x^{a_1+a_2}z_1 = z_2x^{a_2}x^{a_1}z_1 = x^{a_2}z_2x^{a_1}z_1 = ba$$

for any $a, b \in G$. $\square$

3.1.41. *Let G be a group. Then $N = \langle x^{-1}y^{-1}xy \mid x, y \in G\rangle$ is a normal subgroup of G and G/N is Abelian.*

Proof. Consider $g \in G$ and $n \in N$.

$$gng^{-1} = gng^{-1}n^{-1}n = (gng^{-1}n^{-1})n$$

The left factor is in N by taking $x = g^{-1}$ and $y = n^{-1}$. The right factor is in N by assumption, and N is a (sub)group, and therefore closed under multiplication. This proves this subgroup is closed under conjugation, and is therefore normal.

In G/N, consider the coset $ghN = \{ghn \mid n \in N\}$. The element of ghN corresponding to the element of N with $x = h$ and $y = g$ is $ghh^{-1}g^{-1}hg = hg$. This implies that gh and hg generate the same coset, as cosets by partition G. In other words,

$$ghN = (gN)(hN) = hgN = (hN)(gN),$$

therefore G/N is Abelian. $\square$

3.2.16. *If p is prime then $a^p \equiv a \pmod p$ for all $a \in \mathbb{Z}$.*

Proof. Consider $\frac{(\mathbb{Z}/p\mathbb{Z})^\times}{\langle a \rangle}$, where the generator is multiplicative, of course. The order of the numerator group is $p - 1$; the order of the denominator group is $|a|$. Since the order of the factor group must be integral, $|a| \mid p-1$, implying $a^{p-1} \equiv 1 \pmod p \Leftrightarrow a^p = a \pmod p$. $\square$

3.2.19. *If N is a normal subgroup of the finite group G and $|N|$ and $|G : N|$ are relatively prime then N is the unique subgroup of G of order $|N|$.*

Proof. If there exists some other subgroup H with $|H| = |N|$, $HN \leq G$ since N is normal (Corollary 15). Then we can take $|G : HN|$; $|HN| = |H||N|/|H \cap N|$ (Proposition 13), so

$$|G : HN| = \frac{|G|}{|HN|} = \frac{|G| \cdot |H \cap N|}{|H| \cdot |N|}$$

By assumption, $|G|/|N|$ is coprime to $|N| = |H|$, so $|H|$ must divide $|H \cap N|$ if $|G : HN|$ is to remain integral (no factor of $|H|$ divides $|G|/|N|$, so they must *all* divide $|H \cap N|$).

However, since H and N are of the same size, $|H \cap N|$ is at most $|H|$, so it must equal $|H|$. This implies that H and H are identical, since each of their elements is in the intersection. $\square$

3.2.21. *$\mathbb{Q}$ has no proper subgroups of finite index, as does $\mathbb{Q}/\mathbb{Z}$.*

Proof. The left cosets by N are of the form

$$\frac{p}{q} + N = \left\{ \frac{p}{q} + n \mid n \in N \right\}$$

For a fixed N, each $\frac{p}{q}$ generates a different coset, since elementwise addition by $\frac{p}{q}$ is a nonidentity (since $N \neq \mathbb{Q}$, so there exists a "hole" that gets moved) set automorphism on $\mathcal{P}(\mathbb{Q})$ with inverse elementwise subtraction by $\frac{p}{q}$. Therefore the cosets are in bijection with $\mathbb{Q}$, i.e. $|\mathbb{Q} : N|$ is infinite.

Using this, elements of $\mathbb{Q}/\mathbb{Z}$ are of the form $q + Z$, and each q generates a distinct coset. The left cosets by another N of this group are of the form

$$\frac{p}{q} + N = \left\{ (\frac{p}{q} + \mathbb{Z}) + (n + \mathbb{Z}) \mid n + \mathbb{Z} \in N \right\} = \left\{ (\frac{p}{q} + n + \mathbb{Z}) \mid n + \mathbb{Z} \in N \right\}$$

The representative is another element of $\mathbb{Q}$ that's distinct for every $\frac{p}{q}$, and since distinct such elements generate distinct cosets, the cosets are in bijection with $\mathbb{Q}$, i.e. $|\mathbb{Q}/\mathbb{Z} : N|$ is infinite. $\square$

3.3.3. *If H is a normal subgroup of G of prime index p then for all $K \leq G$ either $K \leq H$ or $G = HK$ and $|K : K \cap H| = p$.*

First of all, $HK \leq G$ since H is normal; equivalently, $HK = KH$. If $K \not\leq H$, then there's some element $k \in K$ that's not in H. Then kH generates G/H: by Lagrange's theorem, $|\langle kH \rangle|$ must divide $|G/H|$, but $|G/H| = |G : H|$ is prime, and $\langle kH \rangle$ is nontrivial since $k \cdot 1 \notin H \Rightarrow kH \neq H$.

Stating that another way, all cosets by H in G are of the form $k^i H$. This implies $HK = KH = \{k'h \mid k' \in K, h \in H\} \supseteq \cup_i k^i H$, and since cosets of the quotient group partition G, the last equals G. Since all elements are in G by default, $G = HK = KH$.

We can now apply the second isomorphism theorem with the knowledge that $KH = G$. Since H is normal, $K \leq N_G(H)$, since H normalizes to G. This implies $K \cap H \trianglelefteq K$ and $KH/H = G/H \cong K/K \cap H$ by the second isomorphism theorem. In particular,

$$|K : K \cap H| = |G : H| = p$$