4125 HW 4

Duncan Wilkie

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3.10a

The heat transfer in this process is, using $334\,\mathrm{J/g}$ as the latent heat of fusion of water,

$$Q = (334\,\mathrm{J/g})(30\,\mathrm{g}) = 10\,\mathrm{kJ}$$

The change in entropy is therefore

$$\Delta S = \frac{Q}{T} = \frac{10\,\mathrm{kJ}}{273\,\mathrm{K}} = 36.7\,\mathrm{J/K}$$

3.10b

The heat capacity of the melted ice cube is $C_V = (4.2\,\mathrm{J/g \cdot K})(30\,\mathrm{g}) = 126\,\mathrm{J/K}$ The change in entropy is therefore, presuming (justifiably) a constant heat capacity,

$$\Delta S = C_V \ln\left(\frac{T_f}{T_i}\right) = (126\,\mathrm{J/K}) \ln\left(\frac{298\,\mathrm{K}}{273\,\mathrm{K}}\right) = 137.5\,\mathrm{J/K}$$

3.10c

The heat required to melt the ice cube is $C_V(25\,\mathrm{K}) = 3150\,\mathrm{J}$ The temoerature of the kitchen doesn't change much, so the change in its entropy is

$$\Delta S = -\frac{Q}{T} = -\frac{3150\,\mathrm{J}}{298\,\mathrm{K}} = -10.6\,\mathrm{J/K}$$

where the negative sign is added because Q is flowing out of the kitchen.

3.10d

Adding the changes in entropy,

$$\Delta S_{total} = 137.5\,\mathrm{J/K} - 10.6\,\mathrm{J/K} = 126.9\,\mathrm{J/K}$$

3.16a

When the memory was initialized, it is in one of $2^{2^{33}}$ microstates. For each final state of the system, the previous state could have been any of those initial microstates, so the multiplicity is the above number. The entropy is then

$$S = k \ln \Omega = (1.38 \times 10^{-23}\,\mathrm{J/K}) \ln \left(2^{2^{33}}\right) = 8.22 \times 10^{-14}\,\mathrm{J/K}$$

3.16b

The corresponding heat would be

$$Q = ST = (8.22 \times 10^{-14}\,\mathrm{J/K})(300\,\mathrm{K}) = 2.56 \times 10^{-11}\,\mathrm{J}$$

which is a likely immesurable amount of additional energy.

3.23

The expression for the temperature of a two-state paramagnet is

$$\frac{1}{T} = \frac{k}{2\mu B} \ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right) \Leftrightarrow x = \frac{\mu B}{kT} = \frac{1}{2} \ln\left(\frac{N - U/\mu B}{N + U/\mu B}\right)$$

Writing $U/\mu B = N - 2N_\uparrow$ from the definition of U,

$$x = \frac{1}{2} \ln\left(\frac{N_\uparrow}{N - N_\uparrow}\right) = \frac{1}{2} \ln\left(\frac{N_\uparrow}{N_\downarrow}\right)$$

Plugging this in to the given expression,

$$S = Nk \left[\ln\left(2\cosh x\right) - x \tanh x\right]$$

$$Nk\left\{\ln\left(2\cosh\left[\frac{1}{2}\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\right]\right) - \frac{1}{2}\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\tanh\left[\frac{1}{2}\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\right]\right\}$$

$$= Nk\Bigg\{ \ln\left(2 \cdot \frac{1}{2}\left[\frac{\exp\left(\ln\left[\frac{N_\uparrow}{N_\downarrow}\right]\right) + 1}{\exp\left(\frac{1}{2}\ln\left[\frac{N_\uparrow}{N_\downarrow}\right]\right)}\right]\right)$$

$$-\frac{1}{2}\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\left[\frac{\exp\left(\ln\left[\frac{N_\uparrow}{N_\downarrow}\right]\right) - 1}{\exp\left(\ln\left[\frac{N_\uparrow}{N_\downarrow}\right]\right) + 1}\right]\Bigg\}$$

$$= Nk\left\{\ln\left(\frac{\frac{N_\uparrow}{N_\downarrow} + 1}{\frac{N_\uparrow}{N_\downarrow}\sqrt{e}}\right) - \frac{1}{2}\ln\left(\frac{N_\uparrow}{N_\downarrow}\right)\left[\frac{\frac{N_\uparrow}{N_\downarrow} - 1}{\frac{N_\uparrow}{N_\downarrow} + 1}\right]\right\}$$

$$= Nk\left\{\ln(N_\uparrow + N_\downarrow) - \ln(N_\downarrow) - \ln(N_\uparrow) + \ln(N_\downarrow) - \frac{1}{2} - \frac{1}{2}\frac{N_\uparrow - N_\downarrow}{N_\uparrow + N_\downarrow}\left(\ln(N_\uparrow) - \ln(N_\downarrow)\right)\right\}$$

$$= Nk\left\{\ln N - \ln N_\uparrow - \frac{1}{2} - \frac{1}{2}\frac{N_\uparrow}{N}\ln N_\uparrow + \frac{1}{2}\frac{N-N_\uparrow}{N}\ln(N-N_\uparrow)\right\}$$
$$= Nk\ln N - Nk\ln N_\uparrow - \frac{N_\uparrow}{2}k\ln N_\uparrow + \frac{Nk}{2}\ln(N-N_\uparrow) - \frac{N_\uparrow k}{2}\ln(N-N_\uparrow) - \frac{Nk}{2}$$

Neglecting the merely large terms,

3.25a

$$S = k\ln\Omega = k\ln\left[\left(\frac{q+N}{q}\right)^q\left(\frac{q+N}{N}\right)^N\right] = k\left[q\ln(q+N) - q\ln q + N\ln(q+N) - N\ln N\right]$$
$$= k[N\ln N - q\ln q + (q+N)\ln(q+N)]$$

The neglected term is a division of the multiplicity expression by $\sqrt{2\pi q(q+N)/N}$ which, after a logarithm, is asymptotic to $-\log N + \log q + \log(q+N)$ as $q, N \to \infty$ which is $o(N\log N + q\log q + (q+N)\log(q+N))$ in the same limit. It therefore is acceptable to neglect this for large q and N.

3.25b

Since $U = \epsilon q \Leftrightarrow q = \frac{U}{\epsilon}$, the entropy may be written

$$S = k\left[N\ln N - \frac{U}{\epsilon}(\ln U - \ln\epsilon) + \left(\frac{U}{\epsilon} + N\right)\ln\left(\frac{U}{\epsilon} + N\right)\right]$$

Differentiating with respect to internal energy at constant N and V,

$$\frac{1}{kT} = \frac{1}{k}\left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{\ln\epsilon}{\epsilon} - \left(\frac{1}{\epsilon} + \frac{\ln U}{\epsilon}\right) + \left(\frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\left[\frac{U}{\epsilon} + N\right]\right)$$
$$\Leftrightarrow T = \frac{\epsilon}{k\ln\left(\frac{U+\epsilon N}{U}\right)} = \frac{\epsilon}{k\ln\left(1 + \frac{\epsilon N}{U}\right)}$$

3.25c

Solving for U,

$$1 + \frac{\epsilon N}{U} = e^{\epsilon/kT} \Leftrightarrow U = \frac{\epsilon N}{e^{\epsilon/kT} - 1}$$

Differentiating with respect to temperature,

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{N,V} = \frac{\epsilon^2 N e^{\epsilon/kT}}{kT^2(e^{\epsilon/kT} - 1)^2}$$

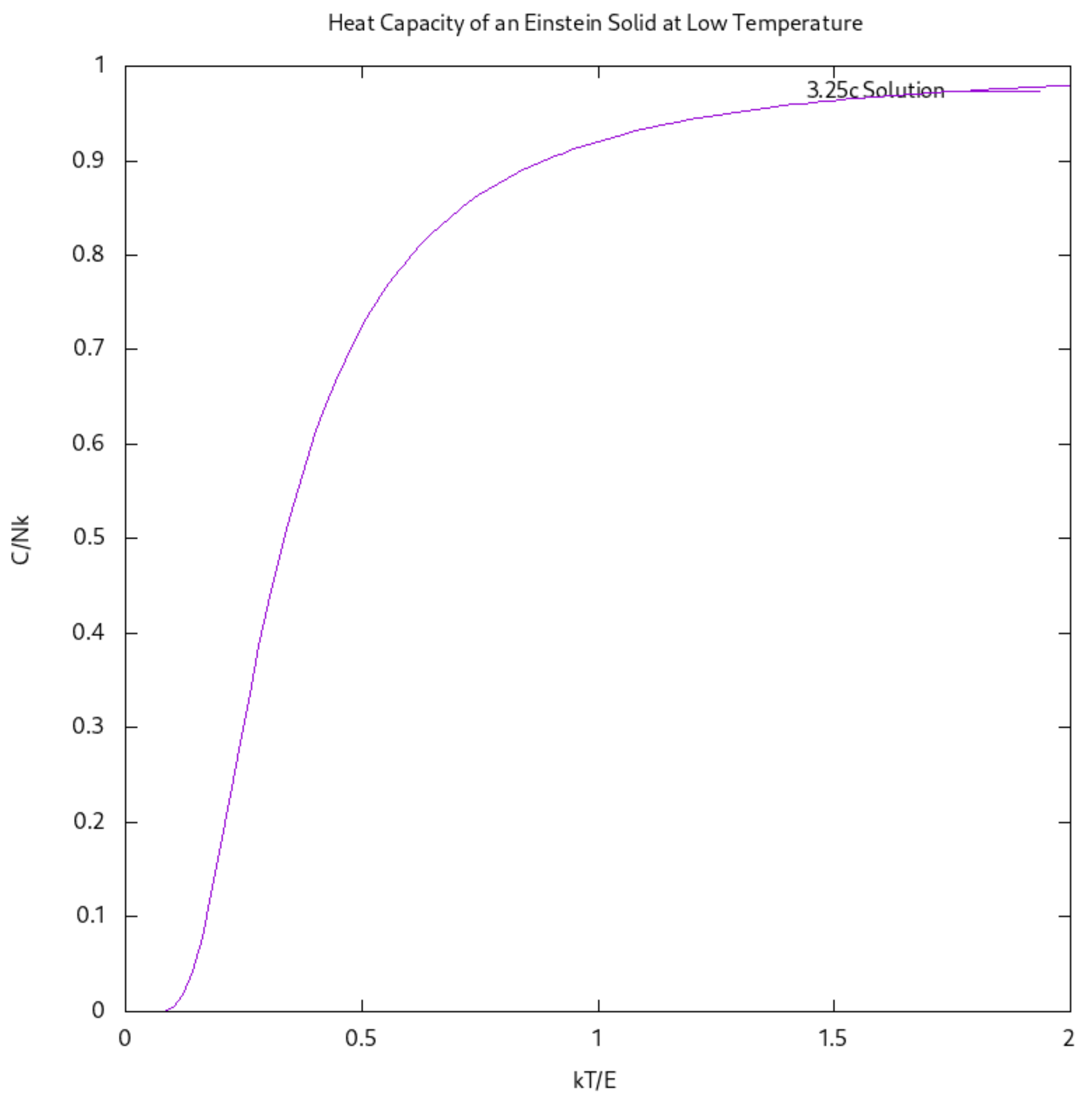
3.25d

Using $e^x = 1 + x + O(x^2)$ as $x \to 0$ and $\frac{1}{T} \to 0$ as $T \to \infty$,

$$C_V \approx \frac{\epsilon^2 N}{k}\frac{1 + \frac{\epsilon}{kT}}{T^2\left(1 + \frac{\epsilon}{kT} - 1\right)^2} = k\epsilon^2 N\left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon kT}\right) \approx Nk$$

3.25e

Using gnuplot,



This predicts an exponential falloff of the heat capacity at low temperature, which is corroborated qualitatively by the experimental data presented in figure 1.14. From this plot, $\frac{kT}{\epsilon} = 0.5$ is a little past the inflection point of the curve. Looking at the experimental data, the analogous point is around 40 K for lead, 110 K for aluminum, and 400 K for diamond (the x range of the graph is too short to really tell, but it appears the slope is decreasing at 400 K, so it's a decent estimate). Solving for ϵ in the initial equation,

$$\epsilon_{Pb} = 2kT_{Pb} = 2(1.38 \times 10^{-23}\,\mathrm{J/K})(40\,\mathrm{K}) = 1.1 \times 10^{-21}\,\mathrm{J} = 0.0069\,\mathrm{eV}$$

$$\epsilon_{Al} = 2kT_{Al} = 2(1.38 \times 10^{-23}\,\mathrm{J/K})(110\,\mathrm{K}) = 3.04 \times 10^{-21}\,\mathrm{J} = 0.019\,\mathrm{eV}$$

$$\epsilon_{Dia} = 2kT_{Dia} = 2(1.38 \times 10^{-23}\,\mathrm{J/K})(400\,\mathrm{K}) = 1.10 \times 10^{-20}\,\mathrm{J} = 0.069\,\mathrm{eV}$$

Couldn't progress further due to the odd deadline of 6pm instead of the standard midnight time.