7380 HW 2

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5.5

Suppose there existed a continuous f that satisfied the δ-function property, that

$$(f, \phi) = \int_\Omega f(x)\phi(x)dx = \phi(0)$$

for all $\phi \in D(\Omega)$. Consider two such test functions ϕ_1, ϕ_2 with the same value at 0 (such functions of course exist—choose x and x^2, for example). Then

$$\int_\Omega f(x)\phi_1(x)dx - \int_\Omega f(x)\phi_2(x)dx = \phi_1(0) - \phi_2(0) \Rightarrow \int_\Omega f(x)(\phi_1(x) - \phi_2(x))dx = 0$$

We consider the slight modification of the fundamental lemma of the calculus of variations result proven on the previous homework where instead of requiring that the varying factor do so over all C^∞, it does so over $\{\phi \in C^\infty | \phi(0) = 0\}$, and implying that $f(x) = 0$ for all $x \neq 0$, instead of for all x. The proof is much the same: if a continuous $f(x) \neq 0$ for some $x \neq 0$, then there exists an interval $[a, b]$ where it is is nonzero, and choosing a ϕ supported in this interval yields a nonzero integral. The difference lies in the constructability of such ϕ, but noting that it is possible to strengthen the existence of the support interval of f to the existence of a support interval not containing zero allows us to use the construction from the other proof, since the ϕ above are only different as $x \to 0$ (choose ϕ of the form of the new lemma to be linear from $x = 0$ to $x = a \neq 0$, and set its value at all points of the interval to be the same as in the construction from the other proof). Therefore, $f(x) = 0$ for all $x \in \Omega$ where $x \neq 0$. The only way for such a function to be continuous is for $f(0) = 0$. However, this contradicts the presumption of the δ-function property, since the original inner product's integrand and therefore the integral becomes zero, but not all test functions ϕ are zero at the origin. Therefore, f cannot be continuous.

5.11

For $\sin(nx)$,

$$\sin(nx) \to 0 \Leftrightarrow \int_\mathbb{R} \sin(nx)\phi(x)dx \to \int_\mathbb{R} 0 \cdot \phi(x)dx = 0$$

Much like on the previous homework, since all ϕ are compactly supported (i.e. region of being nonzero is bounded) there exists some M for each of them such that $\phi(v) = \phi(-v) = 0$ for all

$v \geq M$. The integral may the be rewritten and integration by parts applied as

$$\int_{-M}^{M} \sin(nx)\phi(x)dx = \frac{\cos(nx)}{n}\phi(x)\bigg|_{-M}^{M} - \int_{-M}^{M} \frac{\cos(nx)}{n}\phi'(x)dx$$

$$= -\int_{-M}^{M} \frac{\cos(nx)}{n}\phi'(x)dx$$

We now take the limit as $n \to \infty$. Since $|\cos(nx)| \leq 1$, $|\frac{\cos(nx)}{n}\phi'(x)| < \phi'(x)$, we may apply the dominated convergence theorem to see above integral is equal to

$$\int_{-M}^{M} \phi'(x)dx = \phi(x)\bigg|_{-M}^{M}$$