A Based Derivative for Arbitrary Topological Spaces

Duncan Wilkie

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1 Abstract

There are many contexts throughout mathematics where a notion of a derivative is used. We present a universal characterization of the derivative that depends only on a notion of continuity, prove analogs of the usual derivative theorems in the general formalism, and recover myriad examples of the construction in mathematical practice. We also explore the question of the integral cognate to this derivative through the fundamental theorem of calculus.

2 Motivation

Ordinarily, a function $f : \mathbb{R} \to \mathbb{R}$ is called *differentiable at* c if the two-sided limit

$$L = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists. The number L is then called the *derivative* of f at c, denoted $f'(c)$.

This work is motivated by alternative definition that has been used for pedagogical purposes but, to our knowledge, has never appeared in the literature: a function $f : \mathbb{R} \to \mathbb{R}$ is called differentiable at c if there exists a function $u_c(x)$ continuous at c such that, for all x,

$$f(x) = f(c) + u_c(x)[x - c]$$

The derivative of f at c is then defined to be $f'(c) = u_c(c)$.

We now prove two illuminating theorems about this new derivative that provide schemas for similar results to come.

Theorem 1. *If the function $u_c(x)$ exists in the sense of the second definition, it is unique.*

Proof. Suppose there were $u_c(x)$ and $u'_c(x)$ satisfying the conditions of the second definition. Then

$$f(x) = f(c) + u_c(x)[x - c] = f(c) + u'_c(x)[x - c]$$

The last equality is algebraically equivalent to $u_c(x) = u'_c(x)$ for $x - c \neq 0$, and continuity means the limits as $x \to c$ are equal. $\square$

Theorem 2. *The above two notions of the derivative agree, i.e. if a function is differentiable somewhere in one sense, it is differentiable there in the other, and the values of the derivatives are equal.*

Proof. Consider the substitution into the limit definition $h = x - c$, yielding

$$L = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The function $u_c(x) = \frac{f(x) - f(c)}{x - c}$ is continuous at c iff L exists and equals $u_c(c)$ by the limit definition of continuity and satisfies the form of $u_c(x)$ in the definition. $\qquad\square$

Intuitively, the function $u_c(x)$ is the slope of the secant line between $f(x)$ and $f(c)$. This is quite similar to the geometric derivative that Leibniz knew. However, a critical feature of this definition is that it's almost purely *topological*: the key is the existence of a continuous map.

The remainder of the paper defines this derivative in a (demonstrably maximally) general context, derives some basic properties, and investigates applications.

3   The Based Derivative

Our task is now to inspect the above notion and strip away anything field-theoretic until only topology remains. One might say that a better view of $u_c(x)$ is as *scaling* a change in the domain to a corresponding change in the codomain under the function; left-multiplying by $u_c(x)$ can be viewed as a *function* that maps $x - c$ to $f(x) - f(c)$. The notion of "change" and "distance" provided by subtraction can also be lifted into an abstract topological space. The expression $x - c$ represents an *element* of $\mathbb{R}$; it also represents a *continuous function* $\mathbb{R} \to \mathbb{R}$. Such a function has a nice property that it's a homeomorphism; we want our definition to work for codomains nonhomeomorphic to domains, so we'll relax this slightly to an embedding.

Summarizing,

Definition. *A map $f : X \to Y$ between topological spaces, given some topology τ on the space of all such maps, is called **based g-differentiable** with respect to τ at c relative to a continuous map $g : X \to Y$ containing $f(c)$ in its range if there exists a function $u_c : X \to (Y \to Y)$, continuous at c in its first argument, such that for all x the diagram*

$$X \xrightarrow{\;g\;} Y$$
$$\begin{array}{c} f \searrow \quad \downarrow u_c[x] \\ Y \end{array}$$

*commutes. The corresponding map $u_c[c] : Y \to Y$ is called the **based g-derivative** of f at c.*

Intuitively, function $u_c[x]$ maps the point $g(x)$, which is *estimated value* of $f(x)$ according to the template g, to the actual point $f(x)$. There is some freedom in the choice of topology on the set of continuous functions between spaces; a common choice is the compact-open topology, which is generated by the subbasis of sets of continuous functions that map a given compact subspace to a given open subset. There is a theorem on this kind of topology, from its use in homotopy theory:

Theorem 3. *For topological spaces X, Y, Z, the map $H : Z \to (X \to Y)$ is continuous iff the map $\tilde{H} : X \times Z \to Y$ given by $\tilde{H}(x, z) = H[z](x)$ is continuous. Analogously, the map $H : Z \to (X \to Y)$ is continuous at $z \in Z$ iff the map*

Proof. $\qquad\square$

Theorem 4. *If a based derivative exists, it is unique.*

Proof. Suppose $f : X \to Y$ admits two based derivatives u_c, u'_c at c. The diagram says that for all continuous g that $f = u_c[x] \circ g = u'_c[x] \circ g$; everywhere the expressions make sense, we have $u_c[x] = u'_c[x]$.

Since $f(c)$ is in the range of g, the expressions make sense there, and if u_c is continuous at c, then $u_c[c]$ exists and equals $u'_c[c]$. $\qquad\qquad\square$

Definition. *The based id_X-derivative on endomorphisms $f : X \to X$ in the compact-open topology is simply the **based derivative** of f.*

Theorem 5. *The based derivative of a map f between normed vector spaces V, W and its Fréchet derivative agree.*

Proof. If f is Fréchet differentiable at c, then there exists a bounded linear operator $A : V \to W$ such that
$$\lim_{\|h\|_V \to 0} \frac{\|f(c+h) - f(c) - Ah\|_W}{\|h\|_V} = 0$$

The obvious question is what map $V \to W$ is the canonical one with respect to which to take the based derivative. It turns out that *any* bounded linear map will do; fix an arbitrary one G. The function $u_c[x](y) = \frac{f(x) - f(c)}{\|x - c\|_V} + y$ is continuous in general, and so certainly at $x = c$. $\qquad\qquad\square$

Theorem 6. *The compact-open topology is the unique topology on functions $\mathbb{R} \to \mathbb{R}$ that agrees with*