4142 HW 1

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Problem 1. *Using only the basic commutator of x_i and p_j, evaluate the commutators $[L_i, x_j]$ and $[L_i, p_j]$ where L_i is the angular momentum operator.*

We have, taking addition in the subscripts modulo 3 and considering $x_2 = y$, $L_2 = L_y$ etc.,

$$[L_i, x_j] = [x_{i+1}p_{i+2} - x_{i+2}p_{i+1}, x_j] = [x_{i+1}p_{i+2}, x_j] - [x_{i+2}p_{i+1}, x_j]$$

$$= x_{i+1}[p_{i+2}, x_j] + [x_{i+1}, x_j]p_{i+2} - x_{i+1}[p_{i+1}, x_j] - [x_{i+2}, x_j]p_{i+1}$$

The basic commutation relations give us that any two space variables commute, so the second and fourth terms are zero. Further, they give us that a momentum and a position only don't commute when they're along the same axis, so the first term is only nonzero when $i + 2 \pmod 3 = j$ and the second is only nonzero when $i + 1 \pmod 3 = j$. These never occur simultaneously, but in both cases the commutator evaluates to $-i\hbar$.

So, if $i + 2 \pmod 3 = j$ (e.g. $i = x$, $j = z$; $i = y$, $j = x$, etc.) the commutator evaluates to $-i\hbar x_{i+1}$, if $i + 1 \pmod 3 = j$ (e.g. $i = x$, $j = y$; $i = z$, $j = x$, etc.) the commutator evaluates to $i\hbar x_{i+1}$, and otherwise it is zero.

Problem 2. *Prove, using only basic commutators, that in an eigenstate of L_z, the expectation values of L_x and L_y must be zero.*

By definition,

$$[L_z, L_x] = [xp_y - yp_x, yp_z - p_y z] = [xp_y, yp_z] - [xp_y, p_y z] - [yp_x, yp_z] + [yp_x, p_y z]$$

Each of the middle terms is zero, since $xp_yp_yz = zp_yp_yx$ and $yp_xyp_z = p_zyp_xy$ follow from the canonical commutators, so

$$= [xp_y, yp_z] + [yp_x, p_y z] = x[p_y, yp_z] + [x, p_y z]p_y + y[p_x, p_y z] + [y, p_y z]p_x$$

$$= x\left(y[p_y, p_z] + [p_y, y]p_z\right) + \left(p_y[x, z] + [x, p_y]z\right)p_y + y\left(p_y[p_x, z] + [p_x, p_y]z\right) + \left(p_y[y, z] + [y, p_y]z\right)p_x$$

Identifying the basic commutators,

$$= x[p_y, y]p_z + [y, p_y]zp_x = i\hbar(zp_x - xp_x) = i\hbar L_y$$

An identical argument shows $[L_z, L_y] = -i\hbar L_x$. Letting $|\ell\rangle$ be an eigenstate of L_z,

$$\langle L_x\rangle = \langle \ell|\frac{i}{\hbar}[L_y, L_z]|\ell\rangle = \frac{i}{\hbar}\langle \ell|L_yL_z - L_zL_y|\ell\rangle = \frac{i}{\hbar}\left(\langle \ell|L_yL_z|\ell\rangle - \langle \ell|L_zL_y|\ell\rangle\right)$$

Since $L_z|\ell\rangle = \ell|\ell\rangle$, and L_z is Hermitian,

$$= \frac{i}{\hbar}\left(\langle \ell|L_y L_z|\ell\rangle - \langle L_z \ell|L_y \ell\rangle\right) = \frac{i}{\hbar}\left(\ell\langle \ell|L_y|\ell\rangle - \ell\langle \ell|L_y|\ell\rangle\right) = 0$$

An identical argument holds for L_y.

Problem 3. *A D_2 molecule is known to be in the state $\psi(\theta,\phi) = (3Y_1^1 + 4Y_7^3 + Y_7^1)/\sqrt{26}$. What values of L^2 and L_z can be found on measurement, and with what probabilities? If a measurement of L_z yields $3\hbar$, what is the expectation of L_y on a subsequent measurement?*

Spherical harmonics are simultaneous eigenstates of L_z and L^2:

$$L_z Y_\ell^m = \hbar m Y_\ell^m, L^2 Y_\ell^m = \hbar^2 \ell(\ell+1) Y_\ell^m$$

Evaluating the eigenvalues for the components of the given wave function, the possible values of these operators are

$$L_z = \hbar,\, 3\hbar;\; L^2 = 2\hbar^2,\, 56\hbar^2$$

The corresponding probabilities are the sum of the squares of the coefficients of every term with a given eigenvalue:

$$P(\hbar) = \frac{9}{26} + \frac{1}{26} = \frac{10}{26} = \frac{5}{13}$$

$$P(3\hbar) = \frac{16}{26} = \frac{8}{13}$$

$$P(2\hbar^2) = \frac{9}{26}$$

$$P(56\hbar^2) = \frac{16}{26} + \frac{1}{26} = \frac{17}{26}$$

Once L_z is measured, the wave function snaps to the measured eigenstate, and the expectation of L_y for any eigenstate of L_z 0, by the result of the previous problem.

Problem 4. *A rigid rotor is in a state described by the wave function*

$$f(\theta,\phi) = \sqrt{15/2\pi}\left(\frac{1}{4}\sin^2\theta\cos 2\phi - \frac{1}{2}\sin\theta\cos\theta\sin\phi\right)$$

What values of L^2 and L_z will be found, and with what probabilities, if the measurements were to be made?

Looking at a table of the first few spherical harmonics, the two terms have forms closest to

$$Y_2^2 = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}$$

and

$$Y_2^1 = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$$

respectively. Using Euler's formula,

$$Y_2^2 = \sqrt{\frac{15}{32\pi}} \sin^2\theta(\cos 2\phi + i\sin 2\phi)$$

and

$$Y_2^1 = \sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta(\cos\phi + i\sin\phi)$$

so we can write

$$f(\theta,\phi) = \Re Y_2^2 - \Im Y_2^1 = \frac{1}{2}\left(Y_2^2 + \overline{Y_2^2}\right) - \frac{1}{2i}\left(Y_2^1 - \overline{Y_2^1}\right)$$

Using the fact that $\overline{Y_2^2} = Y_2^{-2}$ and $\overline{Y_2^1} = Y_2^{-1}$, we can write

$$f(\theta,\phi) = \frac{1}{2}Y_2^2 + \frac{1}{2}Y_2^{-2} - \frac{1}{2i}Y_2^1 + \frac{1}{2i}Y_2^{-1}$$

From this, we can directly read out the possible values of L^2 and L_z from the eigenvalue equation written in the previous solution:

$$L_z = 2\hbar,\ -2\hbar,\ \hbar,\ -\hbar$$

$$L^2 = 6\hbar^2$$

These have corresponding probabilities of $\frac{1}{4}$ for each of the cases for L_z and probability 1 for L^2.