4123 HW 4

Duncan Wilkie

9 November 2021

1a

The Lagrangian for this system is, taking down to be the $+y$ direction,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

The modified Euler-Lagrange equations are

$$\frac{\partial L}{\partial x} + \lambda \frac{\partial C}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \Leftrightarrow 2\lambda x = m\ddot{x}$$

and

$$\frac{\partial L}{\partial y} + \lambda \frac{\partial C}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} \Leftrightarrow -mg + 2\lambda y = m\ddot{y}$$

Solving for x and y,

$$x = \frac{m\ddot{x}}{2\lambda}, y = \frac{m\ddot{y} + mg}{2\lambda}$$

Plugging this in to the constraint equation,

$$\left(\frac{m\ddot{x}}{2\lambda}\right)^2 + \left(\frac{m\ddot{y} + mg}{2\lambda}\right)^2 = l^2$$

$$\Leftrightarrow \lambda^2 = \frac{m^2}{4l^2}\left(\ddot{x}^2 + \ddot{y}^2 + 2\ddot{y}g + g^2\right)$$

1b

Multiplying by $1 = \frac{l^2}{l^2}$,

$$\lambda^2 = \frac{m^2}{4l^4}\left(x^2 + y^2\right)\left(\ddot{x}^2 + \ddot{y}^2 + 2\ddot{y}g + g^2\right)$$

$$= \frac{m^2}{4l^4}\left(x^2\ddot{x}^2 + x^2\ddot{y}^2 + 2x^2\ddot{y}g + g^2x^2 + g^2y^2 + y^2\ddot{x}^2 + y^2\ddot{y}^2 + 2y^2\ddot{y}g + g^2y^2\right)$$

Differentiating the constraint twice,

$$C = 0 \Rightarrow x\dot{x} + y\dot{y} = 0 \Rightarrow x\ddot{x} + \dot{x}^2 + y\ddot{y} + \dot{y}^2 = 0$$

$$\Leftrightarrow \dot{x}^2 + \dot{y}^2 = -x\ddot{x} - y\ddot{y}$$

Plugging this in to the target equation, we obtain

$$\lambda = \frac{m}{2l^2}(x\ddot{x} + y\ddot{y} + gy)$$

$$\Leftrightarrow \lambda^2 = \frac{m^2}{4l^4}\left(x^2\ddot{x}^2 + 2x\ddot{x}y\ddot{y} + 2x\ddot{x}gy + y^2\ddot{y}^2 + 2y^2\ddot{y}g + g^2y^2\right)$$

Equating the two expressions for λ^2, we must have

$$x^2\ddot{y}^2 - 2x\ddot{x}y\ddot{y} + 2x^2\ddot{y}g - 2x\ddot{x}gy + y^2\ddot{x}^2 + g^2x^2 + g^2y^2 = 0$$

$$\Leftrightarrow (\ddot{x}y - x\ddot{y})^2 - 2xg(y\ddot{x} - x\ddot{y}) + g^2l^2 = 0$$

$$\Leftrightarrow y\ddot{x} - x\ddot{y} = gx \pm g\sqrt{l^2 - x^2} = gx \pm gy$$

$$\Leftrightarrow x(g - \ddot{y}) = \pm y(g - \ddot{x})$$

By physical intuition, we always expect $g > \ddot{y}$ and $g > \ddot{x}$. The branch therefore changes to make the signs of the two sides agree, and we may write this as

$$|x|(g - \ddot{y}) = |y|(g - \ddot{x})$$

$$\Leftrightarrow |\tan\theta| = \frac{g - \ddot{y}}{g - \ddot{x}}$$

By itself, this is a nice way to describe the motion of a pendulum. We must prove it holds. Multiply the right side by $1 = \frac{m}{m}$ and move the denominator over. Breaking up the tangent as well, his becomes a force equation

$$mg|\sin\theta| - m\ddot{x}|\sin\theta| = mg|\cos\theta| - m\ddot{y}|\cos\theta|$$

Since we have Newton's third law, and the above is a weakening (by adding absolute values) of the sum of the equations for the two components, the equality holds.

2a

Since there is no "real" potential, there is no force on m_2 and it must move with constant velocity.

2b

The resulting Lagrangian in the CM frame (which is approximately the m_2 frame for the same reason) is

$$L = \frac{1}{2}m_2\dot{r}^2 - \frac{l^2}{2m_2r^2}$$

The resulting Euler-Lagrange equation of motion is

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Leftrightarrow \frac{l^2}{m_2 r^3} = m_2 \ddot{r} \Leftrightarrow \frac{l^2}{m_2^2} = r^3 \ddot{r}$$

There is a centrifugal force term present. This is a second-order autonomous equation with no first-derivative dependence, and so can be solved as

$$t(r) + c_1 = \pm \int \frac{dr}{\sqrt{2\int \frac{l^2}{m_2^2 r^3} dr + c_2}} = \pm \frac{m_2}{l} \int \frac{dr}{\sqrt{c_2 - \frac{1}{r^2}}} = \pm \frac{m_2}{l} \frac{r\sqrt{c_2 - \frac{1}{r^2}}}{c_2}$$

$$\Rightarrow \sqrt{r^2 c_2 - 1} = \pm \left(c_2 t \frac{l}{m_2} + c_1 c_2 \right)$$

$$\Leftrightarrow r^2 = 1 + \left(t\frac{l}{m_2} + c \right)^2$$

This is a hyperbola, translated in the t-axis by c. This is not consistent with a constant-velocity trajectory, but since hyperbolas have oblique asymptotes it becomes approximately a constant-velocity trajectory. Therefore, the deviance we see is a breakdown of the approximation of the CM frame by the m_1 frame.

3

The ordinary central-force radial Lagrangian is

$$L = \frac{1}{2}\mu \dot{r}^2 - \frac{l^2}{2\mu r^2} - V(r)$$

The corresponding Euler-Lagrange equation is

$$\frac{\partial L}{\partial r} = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} \Leftrightarrow -\frac{\partial V}{\partial r} + \frac{l^2}{\mu r^3} = \mu \ddot{r} \Leftrightarrow F(r) + \frac{l^2}{\mu r^3} = \mu \ddot{r}$$

Taking r to be a function of θ alone, $\frac{d^2 s}{d\theta^2} = \frac{\partial s}{\partial r} r'' + \frac{\partial^2 s}{\partial \theta \partial r} r' = \frac{-r''}{r^2}$ (since s is not directly dependent on θ). The target equation may be written in terms of r as

$$\frac{-r''}{r^2} + \frac{1}{r} = \frac{-\mu r^2}{l^2} F(r) \Leftrightarrow \frac{r'' - r}{r^2} \frac{l^2}{\mu r^2} = F(r)$$

$$\Leftrightarrow \frac{l^2 r''}{\mu r^4} - \frac{l^2}{\mu r^3} = F(r)$$

Noting that by the chain rule $\ddot{r} = r'\ddot{\theta} + r''\dot{\theta}^2 = r''\dot{\theta}^2 = r'' \left(\frac{l}{\mu r^2}\right)^2$ (constant angular momentum $\Rightarrow$ no angular acceleration), $\frac{l^2 r''}{\mu r^4} = \mu \ddot{r}$. Therefore, the target equation is

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} = F(r)$$

which is clearly equivalent to the the Euler-Lagrange equation found above.

4

We have $s = \frac{1}{k\theta^2}$, so $\frac{d^2s}{d\theta^2} = \frac{6}{k\theta^4}$. The above equation is then

$$\frac{6}{k\theta^4} + \frac{1}{k\theta^2} = \frac{-\mu k^2\theta^4}{l^2} F(r)$$

$$\Leftrightarrow F(r) = \frac{-l^2}{\mu k^2\theta^4}\left(\frac{6}{k\theta^4} + \frac{1}{k\theta^2}\right)$$