7380 HW 1

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Presuming ϵ is everywhere nonzero, we may rewrite (2) as $\frac{i}{\omega\epsilon}(\nabla \times H) = E$. Taking the curl of both sides and substituting the right side of (1) in for $\nabla \times E$ yields

$$\nabla \times \left(\frac{i}{\omega\epsilon}(\nabla \times H)\right) = i\omega\mu H$$

Since H is always perpendicular to the (x_1, x_2) plane, it only has a component in the x_3 direction, and so its curl is computed as $\left(\frac{\partial H}{\partial x_2}, -\frac{\partial H}{\partial x_1}, 0\right)$ using the abusive notation $H = |H|$. Moving the constants to the other side, the subsequent curl is

$$\left(-\frac{1}{\epsilon}\frac{\partial^2 H}{\partial x_3 \partial x_1} - \frac{1}{\epsilon^2}\frac{\partial \epsilon}{\partial x_3}, \frac{1}{\epsilon^2}\frac{\partial \epsilon}{\partial x_3}\frac{\partial H}{\partial x_2} - \frac{1}{\epsilon}\frac{\partial^2 H}{\partial x_3 \partial x_2}, \frac{1}{\epsilon^2}\left(\frac{\partial \epsilon}{\partial x_1}\frac{\partial H}{\partial x_1} + \frac{\partial \epsilon}{\partial x_2}\frac{\partial H}{\partial x_2}\right) + \frac{1}{\epsilon}\left(\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2}\right)\right)$$

Since neither H nor ϵ depend on x_3, all the terms containing those partials are zero. This expression then reduces to

$$\left(0, 0, \frac{1}{\epsilon^2}\left(\frac{\partial \epsilon}{\partial x_1}\frac{\partial H}{\partial x_1} + \frac{\partial \epsilon}{\partial x_2}\frac{\partial H}{\partial x_2}\right) + \frac{1}{\epsilon}\left(\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2}\right)\right)$$

On the right hand side, we similarly have a nonzero component only in the x_3 direction since H is perpendicular to the (x_1, x_2) plane. Thus, we obtain a single scalar equation. Applying the same abusive notation to the right side, this equation is

$$\frac{1}{\epsilon^2}\left(\frac{\partial \epsilon}{\partial x_1}\frac{\partial H}{\partial x_1} + \frac{\partial \epsilon}{\partial x_2}\frac{\partial H}{\partial x_2}\right) + \frac{1}{\epsilon}\left(\frac{\partial^2 H}{\partial x_1^2} + \frac{\partial^2 H}{\partial x_2^2}\right) = \omega^2\mu H$$

This is clearly a scalar second-order PDE for H, as desired.

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For the forward implication, let $\nabla \times F = A$. We then write

$$\nabla \times A = J \Leftrightarrow (\nabla \times A) \cdot \Phi = J \cdot \Phi$$

There is a vector identity for the cross product

$$\nabla \cdot (A \times B) = (\nabla \times A) \cdot B - A \cdot (\nabla \times B)$$

$$\Leftrightarrow (\nabla \times A) \cdot B = \nabla \cdot (A \times B) + A \cdot (\nabla \times B)$$

which we may apply to the previous expression to obtain

$$\nabla \cdot (A \times \Phi) + A \cdot (\nabla \times \Phi) = J \cdot \Phi$$

Expanding the expression for A,

$$\Leftrightarrow \nabla \cdot ((\nabla \times F) \times \Phi) + (\nabla \times F) \cdot (\nabla \times \Phi) = J \cdot \Phi$$

Integrating both sides over $\mathbb{R}^3$, set $B = (\nabla \times F) \times \Phi$ and use the linearity of the integral to obtain

$$\int \nabla \cdot B dV + \int (\nabla \times F) \cdot (\nabla \times \Phi) dV = \int J \cdot \Phi dV$$

Since Φ is compactly supported, there exists some bounded region $W \subseteq \mathbb{R}^3$ whose interior encloses the support of Φ. We may integrate over the interior of W instead, since Φ being zero elsewhere implies the integrand and associated contributions to the integral are zero elsewhere. The first summand on the left side is by the divergence theorem equal to $\int_{\partial W} n \cdot B dS$, which, since we have established that the integrand is zero outside the support of Φ, must be zero. Therefore, we obtain the desired result, that

$$\int (\nabla \times F) \cdot (\nabla \times \Phi) dV = \int J \cdot \Phi dV$$

For the reverse direction, we may apply much of the same logic in reverse: we may add 0 to the left hand side in the form of $\int_{\partial W} n \cdot ((\nabla \times F) \times \Phi) = \int_W \nabla \cdot ((\nabla \times F) \times \Phi) dV$; merging the integrals using linearity yields an integrand equal to the right hand side of the cross product identity used above. Applying it yields

$$\int (\nabla \times \nabla \times F) \cdot \Phi = \int J \cdot \Phi \Leftrightarrow \int (\nabla \times \nabla \times F) \cdot \Phi - J \cdot \Phi = 0$$

$$\Leftrightarrow \int (\nabla \times \nabla \times F - J) \cdot \Phi = 0$$

using the linearity of the dot product. The problem then reduces, setting $C = \nabla \times \nabla \times F - J$, to showing that if $\forall \Phi$, $I = \int C \cdot \Phi = 0$, then $C = 0$. We prove the contrapositive: let $C \neq 0$, then there exists some Φ such that $I \neq 0$. Since F is smooth, all its derivatives are smooth, and so its curl and its curl's curl are both smooth. Since J is also smooth, this implies C is smooth and therefore continuous. Let $\Phi = C$. Then $C \cdot \Phi = C \cdot C = |C|^2$, which is also continuous since the Euclidean norm and squaring both are. Further, for a point x such that $C(x) \neq 0$, $|C(x)|^2 \neq 0$. Since $|C|^2$ is continuous and nonzero at a point, there is a region of positive measure where it is nonzero. Since the function is also non-negative everywhere, this implies its integral is nonzero.

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There is a vector calculus theorem that for $W \in \mathbb{R}^3$

$$\int_{\partial W} A \times n dS = -\int_W \nabla \times A dV$$

or, by anticommutativity of the cross product,

$$\int_{\partial W} n \times A dS = \int_W \nabla \times A dV$$

Over each of W_1 and W_2, we construct a preliminary estimate of the surface integral, using m for a general normal vector to a surface (i.e. sometimes meaning different things in a single equation corresponding to the surface being integrated over), as

$$\int_{\partial W_1} m \times F_1 dS + \int_{\partial W_2} m \times F_2 dS$$

This over-counts the regions $\partial W_1 \cap \Gamma$ and $\partial W_2 \cap \Gamma$, so we subtract them out to obtain

$$\int_{\partial W} m \times F dS = \int_{\partial W_1} m \times F_1 dS + \int_{\partial W_2} m \times F_2 dS - \int_{\partial W_1 \cap \Gamma} m \times F_1 dS - \int_{\partial W_2 \cap \Gamma} m \times F_2 dS$$

We may apply the theorem from above and use that $F_1 = F_2 = F$ on $W \setminus \Gamma$ to get

$$\int_{\partial W} m \times F dS = \int_{W_1 \cup W_2} \nabla \times F dV - \int_{\partial W_1 \cap \Gamma} m \times F_1 dS - \int_{\partial W_2 \cap \Gamma} m \times F_2 dS$$

The rightmost two terms may be rewritten in terms of n, since in the first integral $m = n$ and in the second $m = -n$, to yield

$$\int_{\partial W} m \times F dS = \int_{W_1 \cup W_2} \nabla \times F dV + \int_{\partial W_2 \cap \Gamma} n \times F_2 dS - \int_{\partial W_1 \cap \Gamma} n \times F_1 dS$$

By the linearity of the integral and cross product, and by definition of $[F]$, this is exactly the equation desired:

$$\int_{\partial W} m \times F dS = \int_{W_1 \cup W_2} \nabla \times F dV + \int_{W \cap \Gamma} n \times [F] dS$$

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Let $\tau = t - t_0$. Then $\tilde{E}(x,t) = E(x,\tau)$, $\tilde{H}(x,t) = H(x,\tau)$ and subsequently

$$\nabla \times \tilde{E}(x,t) = -\frac{\partial}{\partial t}(\mu * \tilde{H}(x,t)) \Leftrightarrow \nabla \times E(x,\tau) = -\left(\frac{\partial}{\partial \tau}(\mu * H(x,\tau))\right)\frac{\partial \tau}{\partial t}$$

$$\Leftrightarrow \nabla \times E(x,\tau) = -\frac{\partial}{\partial \tau}(\mu * H(x,\tau))$$

This is identical to the assumption that E and H satisfy this equation in t. An identical argument holds for the second equation of the system.