7.2a

The voltage drop across a capacitor is $V = \frac{q}{C}$, and the voltage drop across the resistor is $V = IR = \frac{dq}{dt}R$. Tracking the voltage drops around the loop,

$$\frac{q}{C} + \frac{dq}{dt}R = 0 \Leftrightarrow -\frac{dt}{RC} = \frac{dq}{q} \Leftrightarrow \ln q = -\frac{t}{RC} + c_1 \Leftrightarrow q(t) = c_2 e^{-t/RC}$$

Applying the boundary condition $q(0) = CV_0$,

$$q(t) = CV_0 e^{-t/RC}$$

The current is

$$I(t) = \frac{dq}{dt} = \frac{V_0}{R} e^{-t/RC}$$

7.2b

The initial energy in the capacitor is $E = \frac{1}{2}CV_0^2$. Integrating,

$$P = I^2 R = \frac{V_0^2}{R} e^{-2t/RC}$$

$$\Rightarrow E = \int_0^\infty \frac{V_0^2}{R} e^{-2t/RC} dt = \frac{V_0^2}{R} \left(-\frac{RC}{2} e^{-2t/RC} \bigg|_0^\infty \right) = \frac{1}{2} CV_0^2$$

7.2c

Similarly to above,

$$V_0 - \frac{q}{C} - \frac{dq}{dt}R = 0$$

Using non-homogeneous constant coefficients,

$$q(t) = c_1 e^{-t/RC} + CV_0$$

Applying the boundary condition that $q(0) = 0$,

$$q(t) = CV_0(1 - e^{-t/RC})$$

Differentiating,

$$I(t) = \frac{V_0}{R} e^{-t/RC}$$

7.2d

The total energy output of the battery is

$$E = \int_0^\infty V_0 I(t) dt = \int_0^\infty \frac{V_0^2}{R} e^{-t/RC} dt = \frac{V_0^2}{R} \left(-RCe^{-t/RC} \bigg|_0^\infty \right) = CV_0^2$$

The energy dissipated by the resistor is found by

$$P = I^2 R = \frac{V_0^2}{R} e^{-2t/RC} \Rightarrow E = \int_0^\infty \frac{V_0^2}{R} e^{-2t/RC} dt = \frac{1}{2} C V_0^2$$

The energy left on the capacitor at the end is the difference between the two:

$$E_o = E_R + E_C \Rightarrow E_C = C V_0^2 - \frac{1}{2} C V_0^2 = \frac{1}{2} C V_0^2$$

This half the total work done by the battery, and confirms what is obvious; as $t \to \infty$, the current becomes negligible, so the voltage across the capacitor is V_0, so the energy stored in it must be the above.