Planar Graphs - 5 Marks Questions & Answers

Q: Define a planar graph and a plane graph with an example.

Ans: A planar graph is a graph that can be drawn on a plane without any edges crossing except at their common vertices. A plane graph is a particular drawing of a planar graph without crossings. Example: K_4 is planar; when drawn without crossings, it is a plane graph.

Q: State the theorem regarding $K_{3,3}$ and K_5 being non-planar.

Ans: Theorem: The complete bipartite graph $K_{3,3}$ and the complete graph K_5 are non-planar, i.e., they cannot be drawn in a plane without edge crossings.

Q: Define the crossing number of a graph. Give the crossing number of $K_{3,3}$ and K_5.

Ans: The crossing number cr(G) of a graph G is the minimum number of edge crossings in any plane drawing of G. $cr(K_{3,3}) = 1$ and $cr(K_5) = 1$.

Q: Define a face in a planar graph. What is the infinite face?

Ans: In a planar graph, the plane is divided into regions by the edges, and each region is called a face. The unbounded outer region is called the infinite face.

Q: State and explain Euler's formula for a connected planar graph.

Ans: Euler's Formula: For any connected planar graph, $n - m + f = 2$ where n = vertices, m = edges, and f = faces.

Q: State the corollary of Euler's formula for a graph with k components.

Ans: Corollary: For a plane graph with n vertices, m edges, f faces, and k components: $n - m + f = k + 1$.

Q: Prove that for a connected simple planar graph with $n \geq 3$ vertices and m edges, we have $m \leq 3n - 6$.

Ans: In a plane drawing, each face is bounded by at least 3 edges. So, $3f \leq 2m$. From Euler's formula: $n - m + f = 2 \Rightarrow f = m - n + 2$. Substituting: $3(m - n + 2) \leq 2m \Rightarrow m \leq 3n - 6$.

Q: If a connected simple planar graph has no triangles, prove that $m \leq 2n - 4$.

Ans: Each face is bounded by at least 4 edges. So, $4f \leq 2m \Rightarrow 2f \leq m$. From Euler's formula: $f = m - n + 2$. Substituting: $2(m - n + 2) \leq m \Rightarrow m \leq 2n - 4$.

Q: Prove that every simple planar graph contains a vertex of degree at most 5.

Ans: Assume each vertex has degree ≥ 6. Then, $2m \geq 6n \Rightarrow m \geq 3n$. But by corollary: $m \leq 3n - 6$. Contradiction! Hence, there must exist at least one vertex of degree ≤ 5.

Q: Define the thickness of a graph with example. What is the thickness of K_5 and $K_{3,3}$?

Ans: Thickness t(G): The minimum number of planar subgraphs whose union is G. For planar graphs: $t(G) = 1$. Thickness of $K_5 = 2$ and $K_{3,3} = 2$.

Q: Define a dual graph. Explain the construction steps.

Ans: A dual graph G^* is obtained from a plane drawing of G: 1. Place a vertex inside each face of G. 2. For each edge of G, draw an edge between the corresponding face-vertices of G^*.

Q: State and prove Lemma 15.1.

Ans: If G has n vertices, m edges, and f faces, then its dual G^* has: $n^* = f$, $m^* = m$, $f^* = n$. Proof: By construction, each face $\rightarrow$ vertex, each edge $\rightarrow$ edge, and applying Euler's formula confirms the relations.

Q: State Theorem 15.3.

Ans: Theorem: In a planar graph G, a set of edges forms a cycle in G if and only if the corresponding set forms a cutset in G*.