X, T, θ

	Discr.	Gen.	LVM
Freq.	$p(T \mid X, \theta)$	$p(X, T \mid \theta)$	$p(X \mid \theta) = \int p(X, Z \mid \theta) dZ$
Bayes.	$p(T, \theta \mid X) = p(T \mid X, \theta) p(\theta)$	$p(X, T, \theta) = p(X, T \mid \theta) p(\theta)$	$p(X, \theta) = p(X \mid \theta) p(\theta)$

$\vdots$

$$p(T \mid X, \theta) \longrightarrow p(T, \theta \mid X) \longrightarrow$$

Обучение	$(X_{tr}, T_{tr}) = \{(x_i, t_i)\}_{i=1}^{n}$	$\theta_{ML} = \arg\max p(T_{tr} \mid X_{tr}, \theta)$	$p(\theta \mid X_{tr}, T_{tr}) = \dfrac{p(T_{tr} \mid X_{tr}, \theta)\, p(\theta)}{\int p(T_{tr} \mid X_{tr}, \theta)\, p(\theta)\, d\theta}$
Тестирование	x	$p(t \mid x, \theta_{ML})$	$\int p(t \mid x, \theta)\, p(\theta \mid X_{tr}, T_{tr})\, d\theta = p(t \mid x, X_{tr}, T_{tr})$

$$p(X|\theta) = \int p(X,z|\theta)\,dz = \int p(X|z,\theta)\,p(z|\theta)\,dz$$

z

$$\theta_{ML} = \arg\max_{\theta} p(X_{tr}|\theta) = \arg\max_{\theta} \log p(X_{tr}|\theta)$$

$$\log p(X_{tr}|\theta) = \log \int p(X_{tr},z|\theta)\,dz = \log \underbrace{\int p(X_{tr}|z,\theta)\,p(z|\theta)\,dz} \quad \theta$$

$$\geq \int p(z|\theta)\log p(X_{tr}|z,\theta)\,dz \approx \log p(X_{tr}|\hat{z},\theta)$$

$$\hat{z} \sim p(z|\theta)$$

$$\log p(X_{tr}|\theta) = \int q(z)\log p(X_{tr}|\theta)\,dz = \left\{ p(z|x,\theta) = \frac{p(x,z|\theta)}{p(x|\theta)} \right\} =$$

$$= \int q(z)\log\frac{p(X_{tr},z|\theta)}{p(z|X_{tr},\theta)}\,dz = \int q(z)\log\frac{p(X_{tr},z|\theta)\,q(z)}{p(z|X_{tr},\theta)\,q(z)}\,dz =$$

$$= \int q(z)\log\frac{p(X_{tr},z|\theta)}{q(z)}\,dz + \underbrace{\int q(z)\log\frac{q(z)}{p(z|X_{tr},\theta)}\,dz}_{KL \geq 0} \geq$$

$$\geq \underbrace{\boxed{\int q(z)\log\frac{p(X_{tr},z|\theta)}{q(z)}\,dz}}_{\mathcal{L}(q,\theta)} \to \max_{\theta,\,q(z)}$$

$$\underline{\text{E-step}} \quad q(z) = \arg\max_q \mathcal{L}(q,\theta_n) = \arg\min_q KL(q(z)\,\|\,p(z|X_{tr},\theta)) = p(z|X_{tr},\theta)$$

$$\underline{\text{M-step}} \quad \theta_{n+1} = \arg\max_\theta \mathcal{L}(q,\theta) = \arg\max_\theta \mathbb{E}_{q(z)}\log p(X_{tr},z|\theta)$$

$$X_{tr} = (x_1 \ldots x_n) \quad x_i \in \mathbb{R}^D \quad \rightarrow \quad Z = (z_1 \ldots z_n) \quad z_i \in \mathbb{R}^d \quad D > d$$

$$p(X, Z|\theta) = \prod_{i=1}^{n} p(x_i, z_i|\theta) = \prod_{i=1}^{n} p(x_i|z_i, \theta)\, p(z_i|\theta) =$$

$$= \prod_{i=1}^{n} \underbrace{\mathcal{N}(x_i | \mu + W z_i, \sigma^2 I)\, \mathcal{N}(z_i|0, I)} \qquad \theta = \{W, \mu, \sigma\}$$

$$W \in \mathbb{R}^{D\times d}, \; \mu \in \mathbb{R}^D, \; \sigma \in \mathbb{R}^+$$

$$\theta_{ML} = \arg\max_\theta p(X_{tr}|\theta) = \arg\max_\theta \prod_{i=1}^{n} p(x_i|\theta) =$$

$$= \arg\max_\theta \prod_{i=1}^{n} \int p(x_i|z_i, \theta)\, p(z_i)\, dz_i = \arg\max_\theta \prod_{i=1}^{n} \mathcal{N}(x_i | \mu, WW^T + \sigma^2 I)$$

$$x_i = \mu + W z_i + \varepsilon \qquad \varepsilon \sim \mathcal{N}(\varepsilon|0, \sigma^2 I)$$

$$z_i \sim \mathcal{N}(z_i|0, I) \qquad W z_i \sim \mathcal{N}(W z_i | 0, WW^T)$$

$$\log p(X_{tr} \mid \theta) = \log \int p(X_{tr} \mid Z, \theta)\, p(Z)\, dZ$$

E-step $\quad q(Z) = p(Z \mid X_{tr}, \theta) = \dfrac{p(X_{tr} \mid Z, \theta)\, p(Z)}{\int p(X_{tr} \mid Z, \theta)\, p(Z)\, dZ} =$

$$= \frac{\prod_{i=1}^{n} p(x_i \mid z_i, \theta)\, p(z_i)}{\int \prod_{i=1}^{n} \underbrace{p(x_i \mid z_i, \theta)\, p(z_i)}\, dz_i} = \prod_{i=1}^{n} \frac{p(x_i \mid z_i, \theta)\, p(z_i)}{\underbrace{\int p(x_i \mid z_i, \theta)\, p(z_i)\, dz_i}_{\mathcal{N}(x \mid \mu,\, WW^T + \sigma^2 I)}} = \prod_{i=1}^{n} p(z_i \mid x_i, \theta) =$$

$$= \prod_{i=1}^{n} \mathcal{N}(z_i \mid m_i, S)$$

$$m_i = \overbrace{(\sigma^2 I + W^T W)^{-1}}^{\text{через } S} W^T (x_i - \mu)$$

$$S = (I + \sigma^{-2} W^T W)^{-1}$$

M-step $\quad \mathbb{E}_{q(Z)} \underbrace{\log p(X_{tr}|Z,\theta)\,p(Z)} \longrightarrow \max_{\theta}$ $\qquad \theta = \{W, \mu, \sigma\}$

$$= \mathbb{E}_{q(Z)}\left[\sum_{i=1}^{n} \underbrace{\log p(x_i|z_i,\theta)} + \sum_{i=1}^{n} \log p(z_i)\right] =$$

Const

$$= \sum_{i=1}^{n} \mathbb{E}_{q(Z)}\left(-\frac{D}{2}\log 2\pi\sigma - \frac{1}{2\sigma^2}\underbrace{\left[(x_i-\mu)-Wz_i\right]^T\left[(x_i-\mu)-Wz_i\right]}\right) + \text{Const}$$

w. r.t. of W

$$\sum_{i=1}^{n} \frac{\partial}{\partial W} \mathbb{E}_{q(Z)} -\frac{1}{2\sigma^2}\left(\underbrace{(x_i-\mu)^T(x_i-\mu)} - \underline{2(x_i-\mu)^T W z_i} + z_i^T W^T W z_i\right) =$$

$$\left[\frac{\partial}{\partial \mu}, \frac{\partial}{\partial \sigma}, \frac{\partial}{\partial W}\right] = 0$$

$$= -\sum_{i=1}^{n} \frac{1}{2\sigma^2} \mathbb{E}_{q(Z)}\left[-2(x_i-\mu)z_i^T + 2W \underbrace{z_i z_i^T}_{d\times d}\right] =$$

$$= -\sum_{i=1}^{n} \frac{1}{2\sigma^2}\left(-2(x_i-\mu)\mathbb{E}z_i^T + 2W\,\mathbb{E}z_i z_i^T\right) =$$

$$\ominus - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(-2(x_i-\mu)\,\mathbb{E}z_i^T + 2W\,\mathbb{E}z_iz_i^T\right) = \left\{\begin{array}{l}\mathbb{E}z_i = m_i\\ \mathbb{E}z_iz_i^T = S + m_im_i^T\end{array}\right\} = \xi$$

$$= \cancel{\frac{1}{\sigma^2}}\sum_{i=1}^{n}(x_i-\mu)m_i^T - \cancel{\frac{1}{\sigma^2}}W\left(\sum_{i=1}^{n} S + m_im_i^T\right) = 0 \qquad \Big|\,(\cdot)\times\left(\sum_{i=1}^{n} S + m_im_i^T\right)^{-1}$$

$$W = \underbrace{\left(\sum_{i=1}^{n}\underbrace{(x_i-\mu)}_{}m_i^T\right)}_{zg=1}\left(\sum_{i=1}^{n} S + m_im_i^T\right)^{-1}$$

μ, σ – аналит. ф-ла

PCA : $O(nD^2 + D^3) \overset{n > D}{=\!=\!=} O(nD^2)$

1 итерация EM : $O(d^3 + nDd) \overset{n > D \gg d}{=\!=\!=} O(nDd)$

100 итераций $\qquad D = 10^4 \qquad d = 10$

EM в 10 раз быстрее аналит. реш.

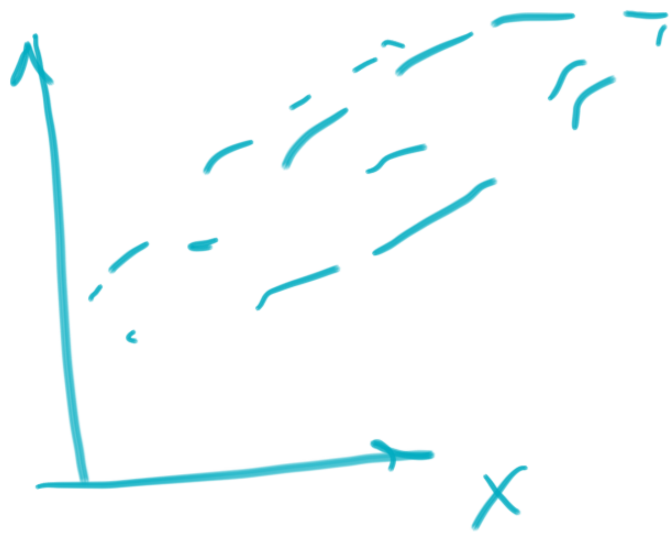
$$X = (X_{known} \cup X_{unk}) \qquad Z = (Z_{obs} \cup Z_{hid})$$

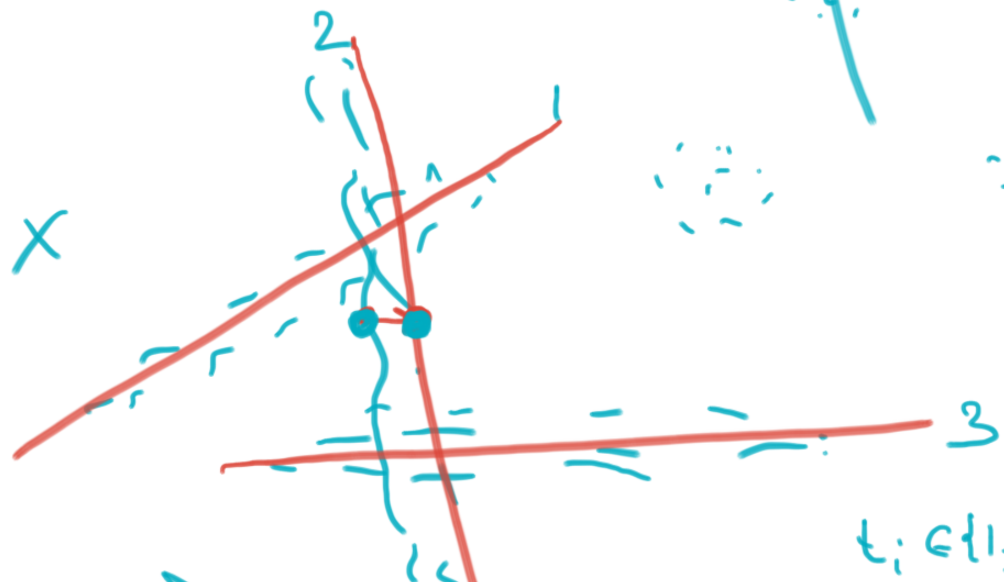
$$p(X, Z \mid \theta)$$

E-step $$q(X_{unk}, Z_{hid}) = \frac{p(X_{known}, X_{unk}, Z_{obs}, Z_{hid} \mid \theta)}{p(X_{known}, Z_{obs} \mid \theta)} =$$

$$= p(X_{unk}, Z_{hid} \mid X_{known}, Z_{obs}, \theta)$$

M-step $$\mathbb{E}_{q(X_{unk}, Z_{hid})} \log p(X_{known}, X_{unk}, Z_{obs}, Z_{hid} \mid \theta) \to \max_{\theta}$$





$p(X, Z, T \mid \Theta) =$

$t_i \in \{1, \dots, K\}$

$$= \prod_{i=1}^{n} \mathcal{N}(x_i \mid \mu_{t_i} + W_{t_i} z_i, \sigma^2 I)\, \mathcal{N}(z_i \mid 0, I)\, p(t_i \mid \pi)$$

$\Theta = \{\{W\}, \{\mu\}, \sigma^2, \pi\}$ $\quad \pi_k > 0 \quad \sum_{k=1} \pi_k = 1$