List 2 report

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Exercise 1

Description of problem:

The point of this exercise is to check what effects have little change in data.

Results:

For disrupted data, the correct result is equal to -0.004296343192495245.

	result	error	relative error
(A) Float32	-0.4999443	0.49994429944939167	4.967e10
(B) Float32	-0.4543457	0.4543457031149343	4.514e10
(C) Float32	-0.5	0.4999999999899343	4.967e10
(D) Float32	-0.5	0.4999999999899343	4.967e10
(A) Float64	1.0251881368296672e-10	1.1258452438296671e-10	11.18
(B) Float64	-1.5643308870494366e-10	1.4636737800494365e-10	14.54
(C) Float64	0.0	1.0065710699999998e-11	1.0
(D) Float64	0.0	1.0065710699999998e-11	1.0
distorted (A) Float32	-0.4999443	0.4956479562669622	115.4
distorted (B) Float32	-0.4543457	0.4500493599325048	104.8
distorted (C) Float32	-0.5	0.4957036568075048	115.4
distorted (D) Float32	-0.5	0.4957036568075048	115.4
distorted (A) Float64	-0.004296342739891585	4.526036594121319e-10	1.053e-7
distorted (B) Float64	-0.004296342998713953	1.9378129136049527e-10	4.510e-8
distorted (C) Float64	-0.004296342842280865	3.502143800654389e-10	8.151e-8
distorted (D) Float64	-0.004296342842280865	3.502143800654389e-10	8.151e-8

The distortion that was made resulted in a much better approximation of the correct result. A possible explanation is that because of the change in data, new vectors are less perpendicular, and as a result condition number is smaller.

QA:

What impact do small changes in data have on results?

If we treat distortion as an error in input data and compare results to -1.00657107000000e-11 we will see that distored algorithms gave the same results for Float32, and worse results for Float64.

	result	error	relative error
(A) Float32	-0.4999443	0.49994429944939167	4.967e10
(B) Float32	-0.4543457	0.4543457031149343	4.514e10
(C) Float32	-0.5	0.4999999999899343	4.967e10
(D) Float32	-0.5	0.4999999999899343	4.967e10
(A) Float64	1.0251881368296672e-10	1.1258452438296671e-10	11.18
(B) Float64	-1.5643308870494366e-10	1.4636737800494365e-10	14.54
(C) Float64	0.0	1.0065710699999998e-11	1.0
(D) Float64	0.0	1.0065710699999998e-11	1.0
distorted (A) Float32	-0.4999443	0.49994429944939167	4.967e10
distorted (B) Float32	-0.4543457	0.4543457031149343	4.514e10
distorted (C) Float32	-0.5	0.4999999999899343	4.967e10
distorted (D) Float32	-0.5	0.4999999999899343	4.967e10
distorted (A) Float64	-0.004296342739891585	0.004296342729825875	4.268e8
distorted (B) Float64	-0.004296342998713953	0.004296342988648243	4.268e8
distorted (C) Float64	-0.004296342842280865	0.004296342832215154	4.268e8
distorted (D) Float64	-0.004296342842280865	0.004296342832215154	4.268e8

introduced error is too small to have any effects on Float32.

Interpretation and conclusions:

This task is badly conditioned because small changes produce big relative disruptions.

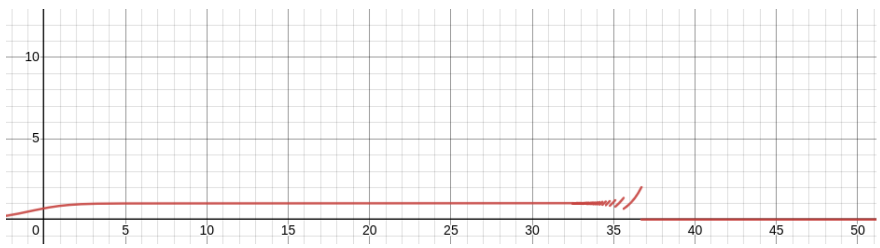
Exercise 2

Description of problem:

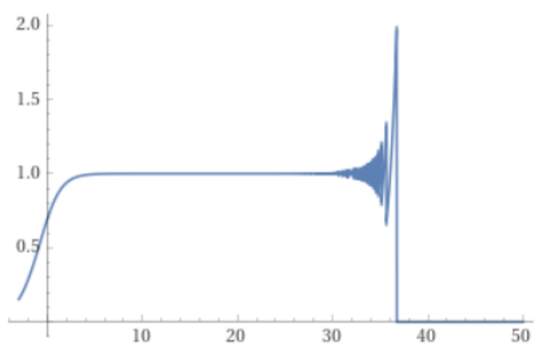
In this exercise, we will learn why we shouldn't have full trust in visualization software.

Results:

Desoms:



WolframAlpha:



As we can see in both examples function converges to 0, but that's not true. The function should converge to 1:

$$\lim_{x\to\infty} e^x ln(1+e^{-x}) = \lim_{x\to\infty} \frac{ln(1+e^{-x})}{e^{-x}} = \lim_{x\to\infty} \frac{f(x)}{g(x)}$$

Both functions are differentiable for real numbers, both f and g are converging to 0 with x going to infinity and the derivative of a function g is non-zero. From L'Hôpital's rule we have got:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{-\frac{1}{e^x+1}}{-e^{-x}} = \lim_{x\to\infty} \frac{1}{1+e^{-x}} = 1$$

I did some experiments. For $x = 1, 2, ..., 40$ I have computed $e^{-x} + 1$:

x	result
1	1.3678794411714423
2	1.1353352832366128
...	...
34	1.0000000000000018
35	1.0000000000000007
36	1.0000000000000002
37	1.0
38	1.0
39	1.0
40	1.0

For $x = 37$ $e^{-x} + 1 = 1$, that means $\ln(e^{-x} + 1) = 0$. What explains why on visualisation function converged to 0. Behavior near $x = 35$ is caused by adding a small number to 1, which causes a loss of important information.

Interpretation and conclusions:

We can not trust visualization software to compute how the function is converging.

Exercise 3

Description of problem:

In this exercise, we test how different matrixes will affect the result of simple computation.

Results:

matrix	rank	cond	relative error A\b	relative error $A^{-1}b$
hilb(2)	2	19.28	5.661e-16	1.124e-15
hilb(3)	3	524.1	8.023e-15	9.826e-15
hilb(4)	4	15510.0	4.452e-13	2.95e-13
hilb(5)	5	476600.0	1.683e-12	8.5e-12
hilb(6)	6	1.495e7	2.619e-10	3.347e-10
hilb(7)	7	4.754e8	1.261e-8	5.164e-9
hilb(8)	8	1.526e10	1.027e-7	2.699e-7
hilb(9)	9	4.932e11	4.832e-6	9.176e-6
hilb(10)	10	1.602e13	0.0006329	0.0004552
hilb(11)	10	5.223e14	0.01154	0.008044
hilb(12)	11	1.752e16	0.2976	0.3439
hilb(13)	11	3.188e18	2.375	5.586
hilb(14)	11	6.201e17	5.281	4.801
hilb(15)	12	3.676e17	1.177	4.827
hilb(16)	12	7.046e17	20.56	31.74
hilb(17)	12	1.249e18	17.74	15.91
hilb(18)	12	2.248e18	41.48	33.88
hilb(19)	13	6.473e18	101.7	94.41
hilb(20)	13	1.148e18	6.505	71.95
matcond(5, 1.0)	5	1.0	4.965e-17	1.57e-16
matcond(5, 10.0)	5	10.0	1.986e-16	4.578e-16
matcond(5, 1000.0)	5	1000.0	3.186e-14	2.493e-14
matcond(5, 1.0e7)	5	1.0e7	2.154e-11	6.403e-11
matcond(5, 1.0e12)	5	1.0e12	7.469e-6	3.394e-6
matcond(5, 1.0e16)	4	9.577e15	0.02785	0.04316
matcond(10, 1.0)	10	1.0	1.955e-16	3.237e-16
matcond(10, 10.0)	10	10.0	3.458e-16	2.164e-16
matcond(10, 1000.0)	10	1000.0	1.862e-14	1.674e-14
matcond(10, 1.0e7)	10	1.0e7	2.411e-10	2.251e-10
matcond(10, 1.0e12)	10	1.0e12	1.352e-5	1.492e-5
matcond(10, 1.0e16)	9	1.353e16	0.4888	0.5268
matcond(20, 1.0)	20	1.0	3.933e-16	4.041e-16
matcond(20, 10.0)	20	10.0	3.169e-16	3.312e-16
matcond(20, 1000.0)	20	1000.0	1.252e-14	8.917e-15
matcond(20, 1.0e7)	20	1.0e7	2.076e-10	1.879e-10
matcond(20, 1.0e12)	20	1.0e12	5.835e-5	5.734e-5
matcond(20, 1.0e16)	19	2.025e16	0.133	0.163

Interpretation and conclusions:

The relative error is highly correlated with the condition number of the matrix, and the rank computed in float arithmetic isn't always equal mathematical rank.

Exercise 4

Description of problem:

In this exercise, we are testing the Wilkinson polynomial.

Results:

k	z	|P(z)|	|p(z)|	|z - k|
1	0.9999999999996989	35700.0	36630.0	3.011e-13
2	2.0000000000283182	176300.0	181300.0	2.832e-11
3	2.9999999995920965	279200.0	290200.0	4.079e-10
4	3.9999999837375317	3.027e6	2.042e6	1.626e-8
5	5.000000665769791	2.292e7	2.089e7	6.658e-7
6	5.999989245824773	1.29e8	1.125e8	1.075e-5
7	7.000102002793008	4.805e8	4.573e8	0.000102
8	7.999355829607762	1.638e9	1.556e9	0.0006442
9	9.002915294362053	4.877e9	4.688e9	0.002915
10	9.990413042481725	1.364e10	1.263e10	0.009587
11	11.025022932909318	3.586e10	3.3e10	0.02502
12	11.953283253846857	7.533e10	7.389e10	0.04672
13	13.07431403244734	1.961e11	1.848e11	0.07431
14	13.914755591802127	3.575e11	3.551e11	0.08524
15	15.075493799699476	8.216e11	8.423e11	0.07549
16	15.946286716607972	1.551e12	1.571e12	0.05371
17	17.025427146237412	3.695e12	3.317e12	0.02543
18	17.99092135271648	7.65e12	6.345e12	0.009079
19	19.00190981829944	1.144e13	1.229e13	0.00191
20	19.999809291236637	2.792e13	2.318e13	0.0001907

QA:

Can you explain discrepancies?

In the formula, we have got big powers, which cause loss of information, and because this task is badly conditioned small changes in data will cause huge changes in result.

Can you repeat Wilkinson experiment?

In this experiment, Wilkinson subtracted 2^{-23} from -210 in the formula and showed that the new polynomial has complex roots.

k	z	|P(z)|	|p(z)|	|z - k|
1	0.9999999999998357 + 0.0im	20260.0	19990.0	1.643e-13
2	2.0000000000550373 + 0.0im	346500.0	352400.0	5.504e-11
3	2.99999999660342 + 0.0im	2.366e6	2.416e6	3.397e-9
4	4.000000089724362 + 0.0im	1.002e7	1.126e7	8.972e-8
5	4.99999857388791 + 0.0im	4.625e7	4.476e7	1.426e-6
6	6.000020476673031 + 0.0im	2.024e8	2.142e8	2.048e-5
7	6.99960207042242 + 0.0im	1.711e9	1.785e9	0.0003979
8	8.007772029099446 + 0.0im	1.87e10	1.869e10	0.007772
9	8.915816367932559 + 0.0im	1.376e11	1.375e11	0.08418
10	10.095455630535774 - 0.6449328236240688im	1.491e12	1.49e12	0.652
11	10.095455630535774 + 0.6449328236240688im	1.491e12	1.49e12	1.111
12	11.793890586174369 - 1.6524771364075785im	3.297e13	3.296e13	1.665
13	11.793890586174369 + 1.6524771364075785im	3.297e13	3.296e13	2.046
14	13.992406684487216 - 2.5188244257108443im	9.546e14	9.546e14	2.519
15	13.992406684487216 + 2.5188244257108443im	9.546e14	9.546e14	2.713
16	16.73074487979267 - 2.812624896721978im	2.742e16	2.742e16	2.906
17	16.73074487979267 + 2.812624896721978im	2.742e16	2.742e16	2.825
18	19.5024423688181 - 1.940331978642903im	4.253e17	4.252e17	2.454
19	19.5024423688181 + 1.940331978642903im	4.253e17	4.252e17	2.004
20	20.84691021519479 + 0.0im	1.374e18	1.374e18	0.8469

As we can see there are complex roots, what's more 10 and 11 are only different on the imaginary part of a number. That is also true for 12 and 13, 14 and 15, 15 and 17, 18 and 19. It has also resulted in worse results.

Interpretation and conclusions:

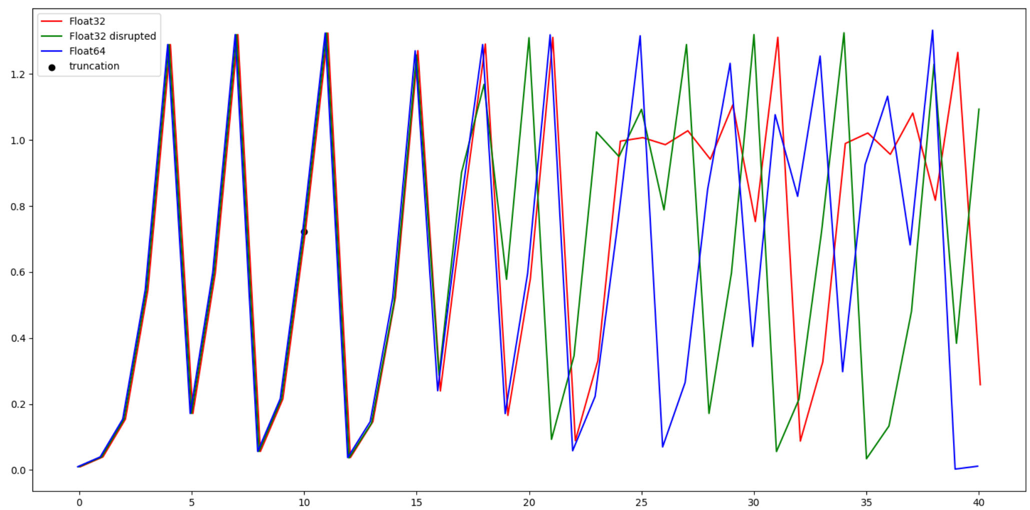
We have to be careful when dealing with Wilkinson polynomials. It is really sensitive to little changes.

Exercise 5

Description of problem:

In this exercise, we are experimenting with recursive formula, by computing it in Float32, Float64 and Float32 but with truncation to the 3rd digit after the decimal point on the 10th iteration.

Results:



Although at the beginning they are all close to each other, small errors sum up causing big changes in late iterations.

Interpretation and conclusions:

Prediction of far future is really hard, small meaningless errors at the beginning lead to chaos after some time.

Exercise 6

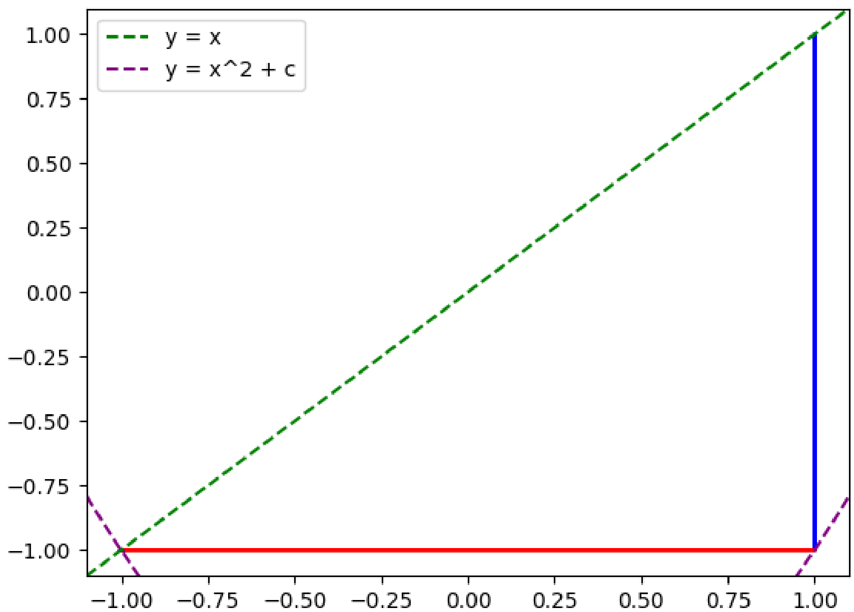
Description of problem:

In this exercise we are making an experiment testing the recursive formula:

$$x_{\mathrm{n}\,+\,1} := {x_{\mathrm{n}}}^2 + c$$

Results:

$$\mathbf{c = \text{-}2,\ x_0 = 1}$$



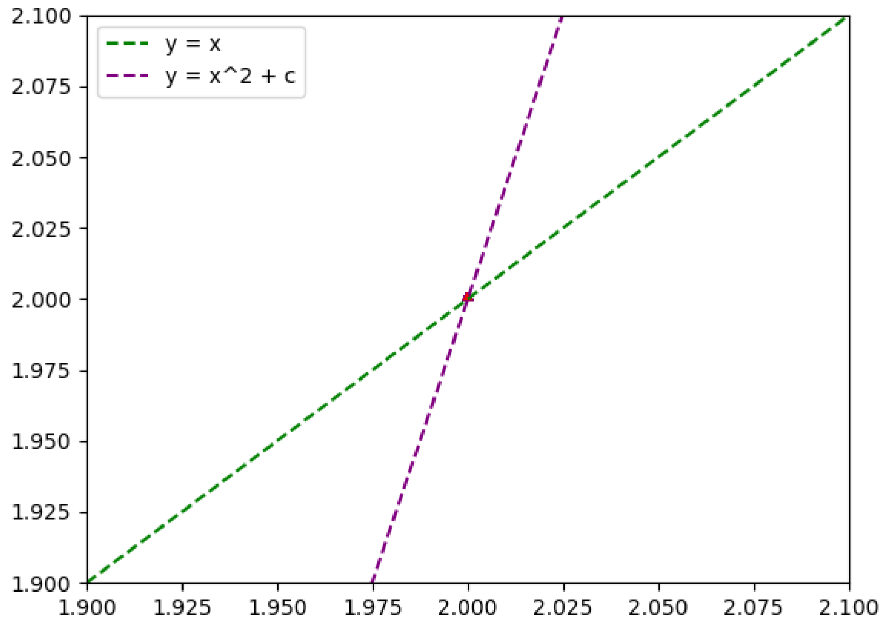
n	x
0	1
1	-1
2	-1
...	...
39	-1
40	-1

In the first iteration formula reached a stable point:

$$f(1) = 1^2 - 2 = 1 - 2 = -1$$

$$f(-1) = -1^2 - 2 = 1 - 2 = -1$$

$\mathbf{c = -2, x_0 = 2}$

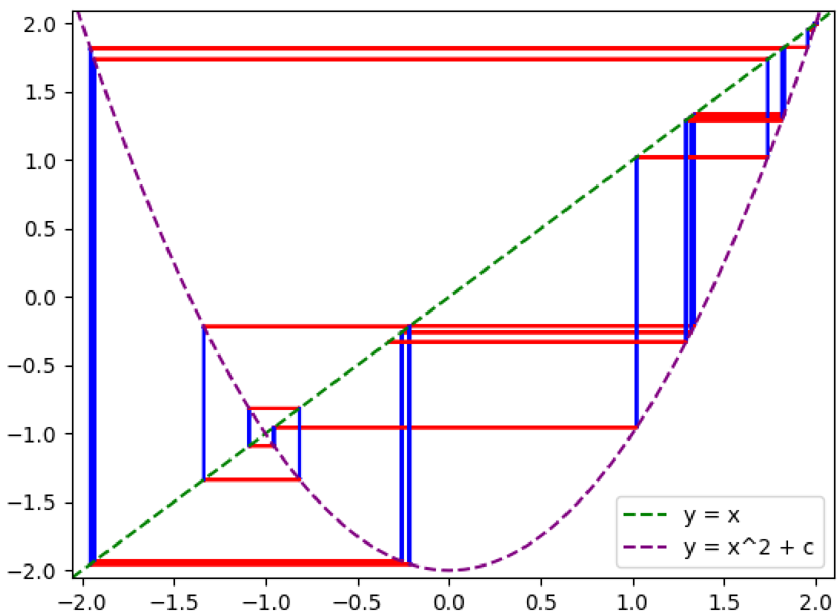


n	x
0	2
1	2
2	2
...	...
39	2
40	2

The formula starts from a stable point:

$$f(2) = 2^2 - 2 = 4 - 2 = 2$$

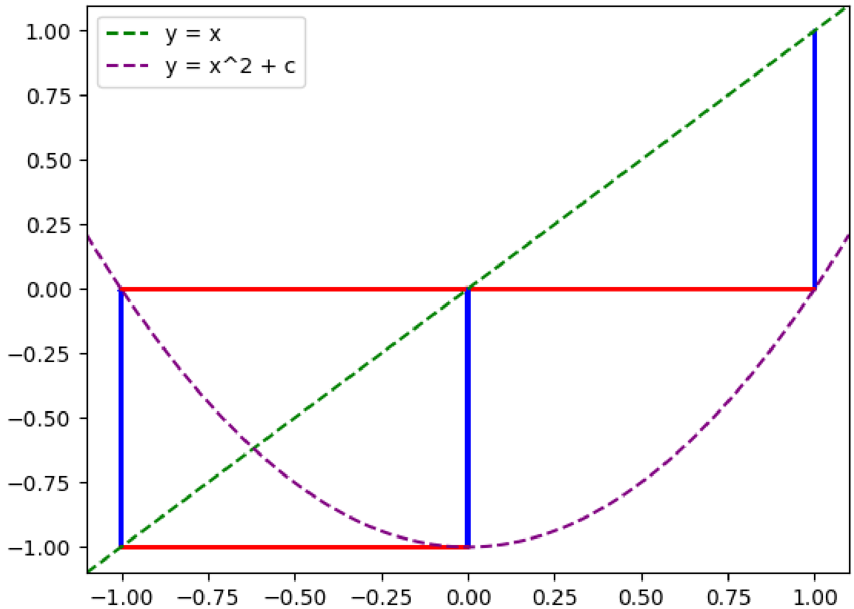
c = -2, x_0 = 1.99999999999999



n	x
0	1.99999999999999
1	1.99999999999996
...	...
21	1.9562153843260486
22	1.82677862987391
23	1.3371201625639997
24	-0.21210967086482313
25	-1.9550094875256163
26	1.822062096315173
27	1.319910282828443
28	-0.2578368452837396
29	-1.9335201612141288
30	1.7385002138215109
31	1.0223829934574389
32	-0.9547330146890065
...	...

For the first 20 iterations, it is slowly going away from x = 2, then it goes into a 4-point cycle (1.8, 1.3, -0.2, -1.9) just to go close to stable point -1 and spirals away from it and do one last 4-point cycle. There could be some more stable pattern that isn't possible due to floating point errors.

c = -1, $\mathbf{x_0}$ = 1



n	x
0	1
1	0
2	-1
3	0
...	...
39	0
40	-1

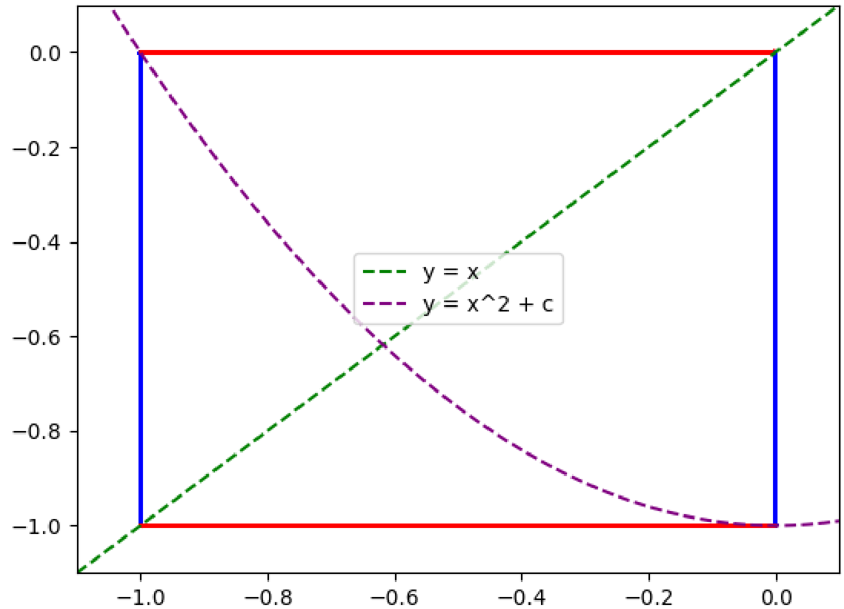
The formula ends up in a stable 2-point cycle:

$$f(1) = 1^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0^2 - 1 = 0 - 1 = -1$$

$$f(-1) = -1^2 - 1 = 1 - 1 = 0$$

$\mathbf{c = -1, x_0 = -1}$



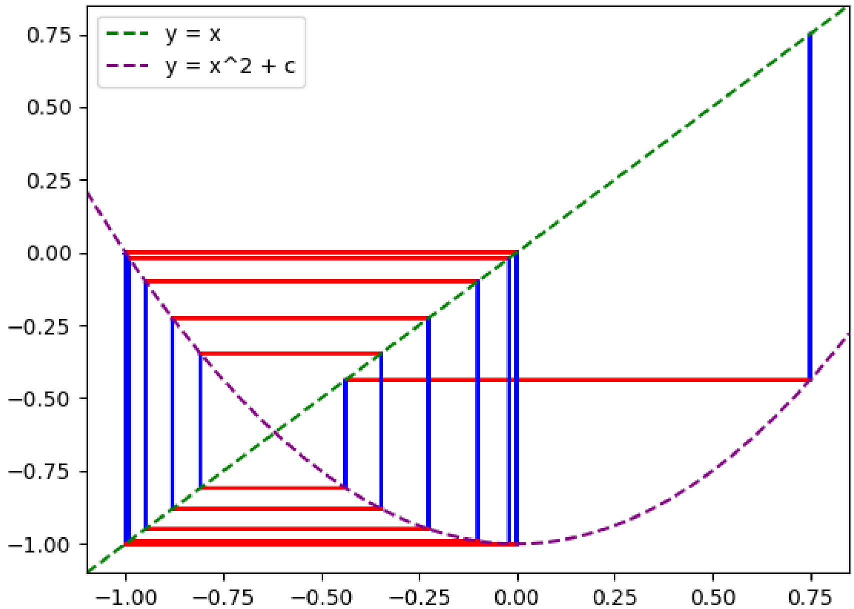
n	x
0	-1
1	0
2	-1
3	0
...	...
39	0
40	-1

The formula starts in a stable 2-point cycle:

$$f(-1) = -1^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0^2 - 1 = 0 - 1 = -1$$

c = -1, x_0 = 0.75



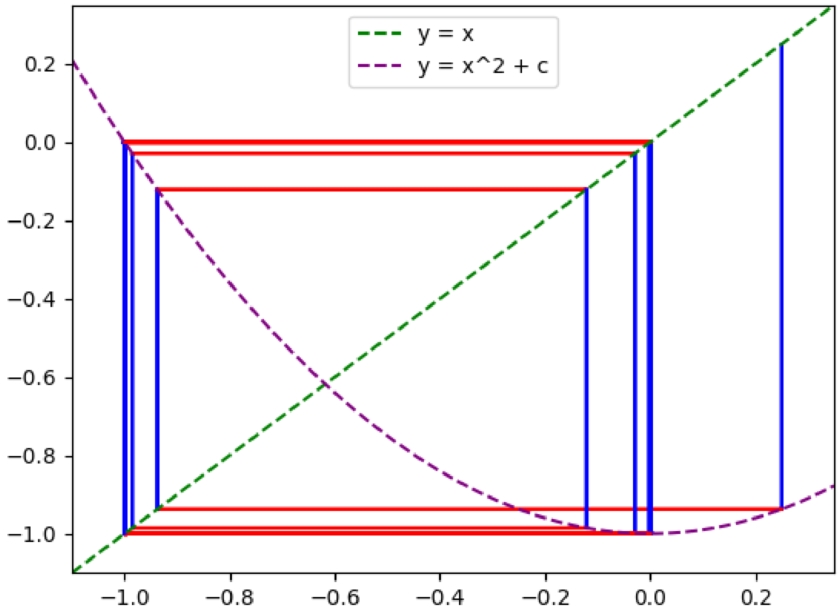
n	x
0	0.75
1	-0.4375
2	-0.80859375
3	-0.3461761474609375
...	...
37	0
38	-1
39	0
40	-1

The formula quickly goes into a spiral and makes its way out into a stable 2-point cycle:

$$f(-1) = -1^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0^2 - 1 = 0 - 1 = -1$$

c = -1, x_0 = 0.25



n	x
0	0.25
1	-0.9375
2	-0.12109375
3	-0.9853363037109375
...	...
37	-1
38	0
39	-1
40	0

The formula quickly goes into a spiral and makes its way out into a stable 2-point cycle:

$$f(-1) = -1^2 - 1 = 1 - 1 = 0$$

$$f(0) = 0^2 - 1 = 0 - 1 = -1$$

Interpretation and conclusions:

Recursive formulas can have a wide range of behaviors, like going into a stable point or in a stable cycle.