Dense Coding and Teleportation

Antonio Andrea Gargiulo - 1769185

Sapienza University of Rome

Difference dense coding vs. quantum teleportation

Dense Coding

Dense coding is a method enabling the transmission of two classical bits, with the utilization of a pre-shared entangled pair, to a recipient by utilizing only one qubit for the communication.

In other words, when the sender and receiver share an entangled state, two classical bits can be packed into only one qubit.

Quantum Teleportation

Quantum teleportation is a procedure through which the state of a qubit ($|\psi\rangle$) can be transmitted from one place to another, requiring two classical communication bits and an entangled pair.

In simpler terms, it can be described as a protocol that destroys the quantum state of a qubit at one location and reconstitutes it on a qubit situated at a remote location, utilizing entanglement.

Dense coding

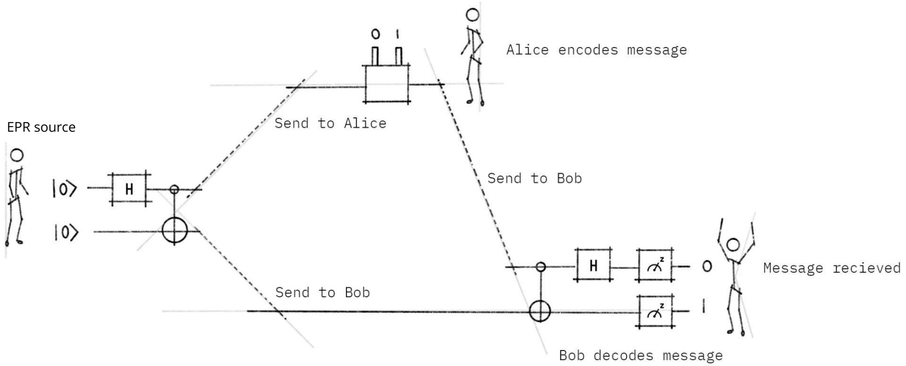
An EPR source shares two qubits in an entangled state with Alice and Bob:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice encode a number between 0 and 3 using only one of the Pauli Transformation on her qubit, acting on her part of the entangled pair

Message	Transformation	New state $(\frac{1}{\sqrt{2}}\cdot)$
00	$\lvert\Psi_0\rangle = (I \otimes I)\lvert\Psi_0\rangle$	$\lvert 00\rangle + \lvert 11\rangle$
01	$\lvert\Psi_0\rangle = (X \otimes I)\lvert\Psi_0\rangle$	$\lvert 10\rangle + \lvert 01\rangle$
10	$\lvert\Psi_0\rangle = (Z \otimes I)\lvert\Psi_0\rangle$	$\lvert 00\rangle - \lvert 11\rangle$
11	$\lvert\Psi_0\rangle = (Y \otimes I)\lvert\Psi_0\rangle$	$-\lvert 10\rangle + \lvert 01\rangle$

Figure is from IBM Qiskit courses

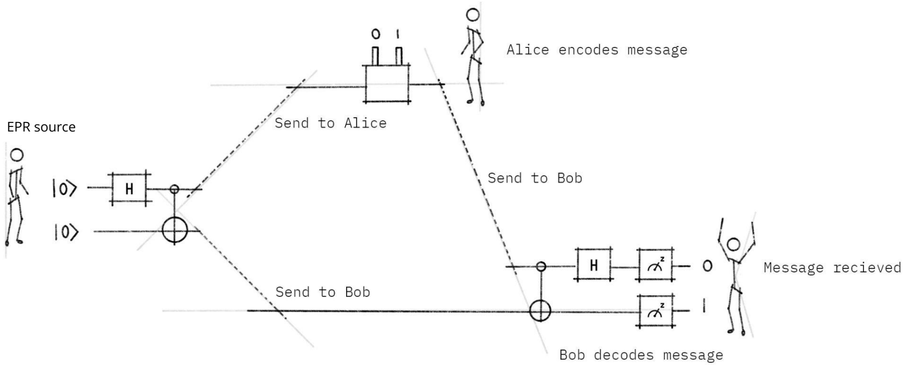


Then Alice send her qubit to Bob.

Bob receives Alice's qubit and uses his qubit to decode Alice's message.

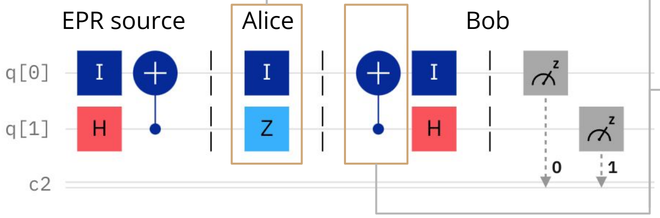
Notice: Bob does not need to have knowledge of the state in order to decode it, he simply apply a CNOT gate and then an Hadamard gate to the Alice’s qubit. Finally Bob measures the two qubits revealing the message of Alice.

Dense coding



Bob’s steps:

Bob receives $(\frac{1}{\sqrt{2}}\cdot)$	After CNOT $(\frac{1}{\sqrt{2}}\cdot)$	After $H \otimes I$ $(\frac{1}{\sqrt{2}}\cdot)$
$\|00\rangle + \|11\rangle$	$\|00\rangle + \|10\rangle$	$\|00\rangle$
$\|10\rangle + \|01\rangle$	$\|11\rangle + \|01\rangle$	$\|01\rangle$
$\boxed{\|00\rangle - \|11\rangle}$	$\boxed{\|00\rangle - \|10\rangle}$	$\boxed{\|10\rangle}$
$-\|10\rangle + \|01\rangle$	$-\|11\rangle + \|01\rangle$	$\|11\rangle$



After CNOT $(\frac{1}{\sqrt{2}}\cdot)$
$\|0\rangle + \|1\rangle \otimes \|0\rangle$
$\|1\rangle + \|0\rangle \otimes \|1\rangle$
$\boxed{\|0\rangle - \|1\rangle \otimes \|0\rangle}$
$-\|1\rangle + \|0\rangle \otimes \|1\rangle$

Note:

The CNOT disentagles the two qubits, so Bob can measure the second qubit without disturbing the first an obtain a partial result:

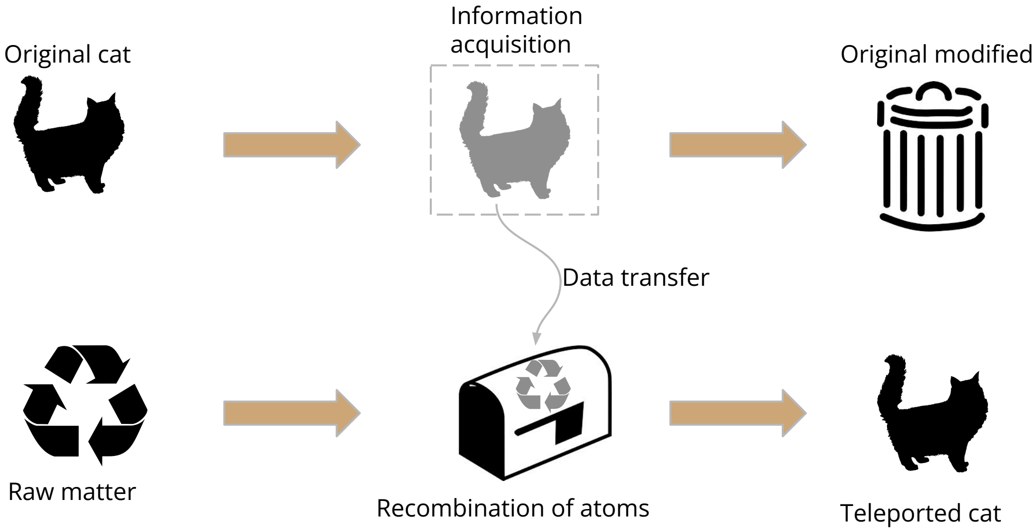
If he measure $|0\rangle$ the encoded message can be 0 or 2, and if he measure $|1\rangle$ the encoded message can be 1 or 3.

Figure is from IBM Qiskit courses, circuit is autoproduced

A bit of Science Fiction

Science Fiction Teleportation

Teleportation as we know from film (not feasible at the moment, moreover impossible for quantum mechanics).



A bit of Scientific Background

Unitary Transformations Property

A linear transformation operator M representing the time evolution of a system must be unitary:

$$M^\dagger M = I \implies M^\dagger = M^{-1}$$

Consequence: every quantum state transformation must be reversible.

So transformation operators are never measurement operators!

No Cloning Principle

Quantum states cannot be copied or cloned because of the unitary property.

Proof: If U is a unitary transformation that clones, then U(|a⟩|0⟩)=|a⟩|a⟩ for all state |a⟩. Let |a⟩ and |b⟩ be two ortogonal quantum states. Consider |c⟩ =(1/√2)(|a⟩ +|b⟩).   By Linearity:

$$U(|c\rangle|0\rangle) = \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) = \boxed{\frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle)}$$

But if U is a cloning transformation,then:

$$U(|c\rangle|0\rangle) = |c\rangle|c\rangle = \boxed{\frac{1}{2}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle)}$$

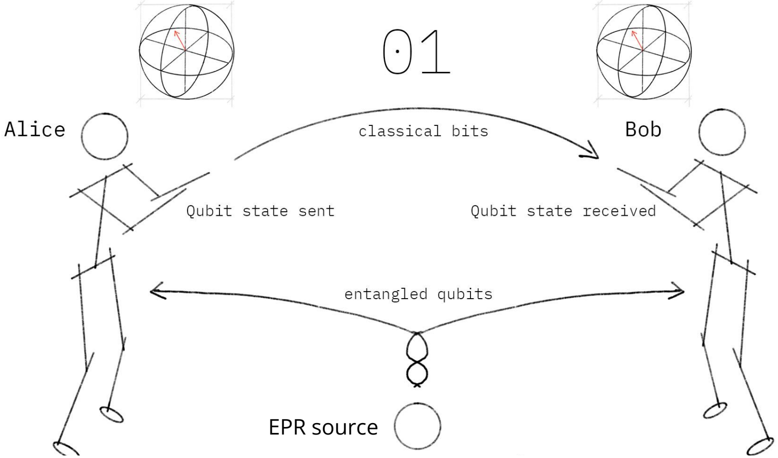
q.e.d. $\neq$

This tell us that there isn't a unitary operation that can reliably clone unknown quantum states.

Notice: it is possible to clone a known quantum state. What the no cloning principle tells us is that it is impossible to reliably clone an unknown quantum state.

A bit of Science

Quantum Teleportation



An EPR source shares two qubits in an entangled state with Alice and Bob:

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Alice ha a qubit in an unknown state:

$$|\phi\rangle = a|0\rangle + b|1\rangle$$

With: $$|a|^2 + |b|^2 = 1$$

She want to transmit it to Bob using a classical channel. The starting state is:

$$|\phi\rangle \otimes |\Psi\rangle = \frac{1}{\sqrt{2}}(a|0\rangle \otimes (|00\rangle + |11\rangle) + b|1\rangle \otimes (|00\rangle + |11\rangle))$$
$$= \frac{1}{\sqrt{2}}(a|0_A 0_A 0_B\rangle + a|0_A 1_A 1_B\rangle + b|1_A 0_A 0_B\rangle + b|1_A 1_A 1_B\rangle)$$

Then Alice applies a CNOT on her entangled qubit with the control on the qubit she want to send, after she applies an Hadamard to the qubit she want to send and finally she measure her qubits.

(see calculation on the following slide)

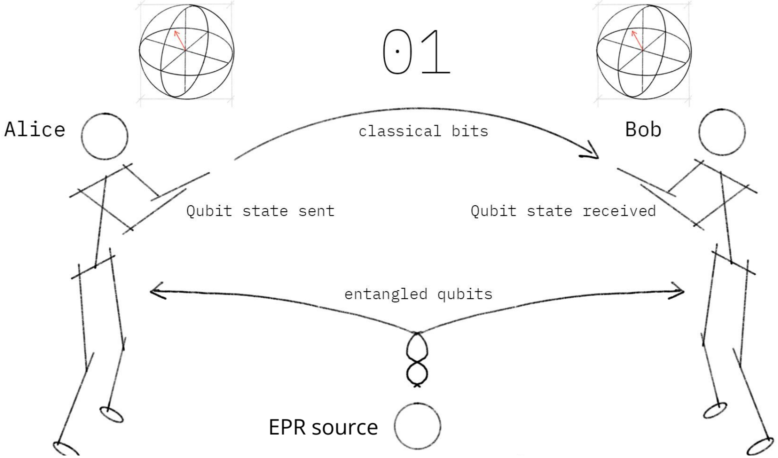
Figure is from IBM Qiskit courses

Quantum Teleportation

Alice applies CNOT on her entangled qubit with the control on the qubit she want to send, after she applies an Hadamard to the qubit she want to send and finally she measure her qubits:

$$(H \otimes I \otimes I)(CNOT \otimes I)(|\phi\rangle \otimes |\Psi_0\rangle) =$$
$$= (H \otimes I \otimes I)\frac{1}{\sqrt{2}}(a|000\rangle + a|011\rangle + b|110\rangle + b|101\rangle) =$$
$$= \frac{1}{\sqrt{2}}(|00\rangle(a|0\rangle + b|1\rangle) + |01\rangle(a|1\rangle + b|0\rangle) +$$
$$+|10\rangle(a|0\rangle - b|1\rangle) + |11\rangle(a|1\rangle - b|0\rangle))$$

Regardless of the unknown state $|\phi\rangle$ the four measurement outcomes of Alice are equally likely with P = 1/4.



Then Bob's qubit have been projected into one of the four superposed states of the last eq. So, to reconstruct the original state of Alice's qubit $|\phi\rangle$, Bob must wait a classical communication from Alice that tells him what are the two bits from her measurement and then he can knows the transformation to apply to obtain the same information of the original Alice's qubit.

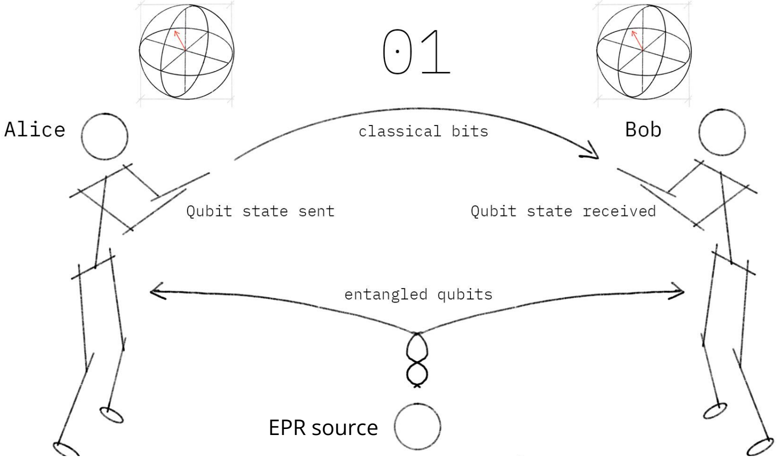
Figure is from IBM Qiskit courses

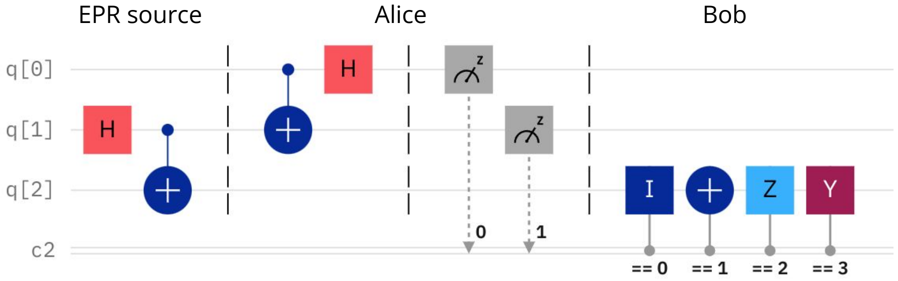
Quantum Teleportation

Bob must do:

Bob receives	Transformation	Final State
00	I	$a\|0\rangle + b\|1\rangle$
01	X	$a\|1\rangle + b\|0\rangle$
10	Z	$a\|0\rangle - b\|1\rangle$
11	Y	$a\|1\rangle - b\|0\rangle$

After receiving the classical message and applied the correct transformation Bob has successfully reconstructed Alice's state $|\phi\rangle$.





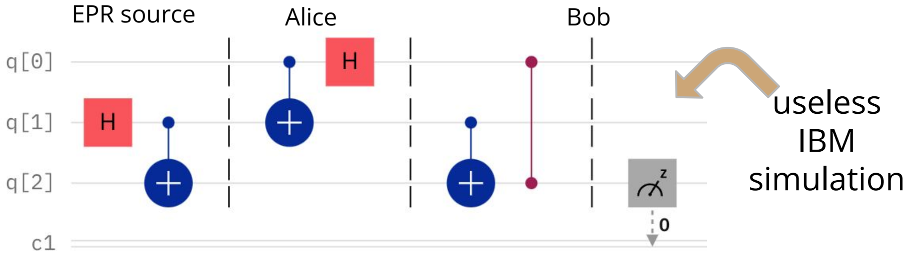
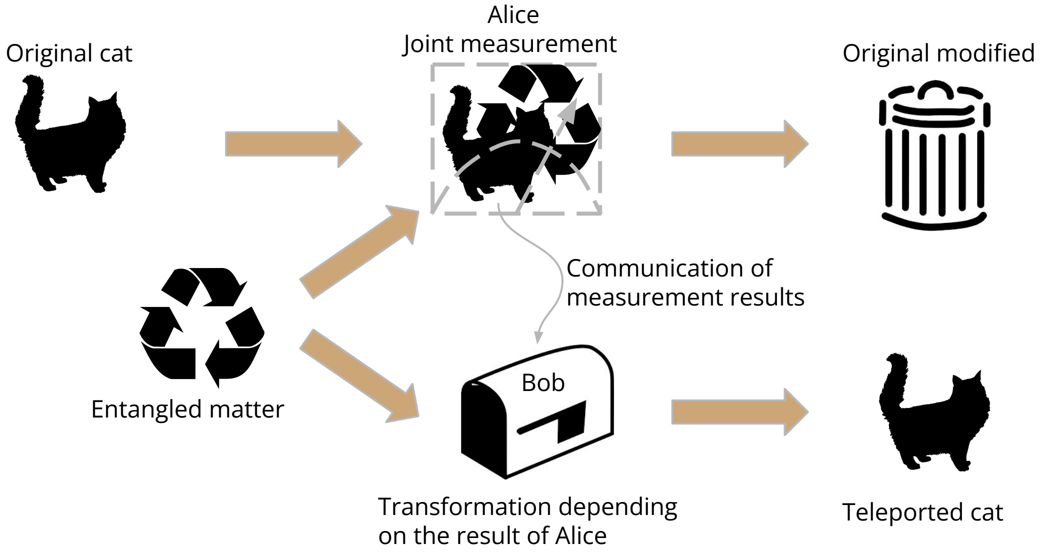


Figure is from IBM Qiskit courses, circuits are autoproduced

Quantum Teleportation

Since when Alice measure its qubits, she irrecoverably altered the state of her original qubit ϕ that she want to send to Bob. This loss of the original state is the motive teleportation does not violate the no cloning principle.



References

K. S. Krane - Modern Physics

P. Rapagnani - slides of the course Topics in Physics

R. Penrose - The Road to Reality

L. Susskind, A. Friedman - Quantum mechanics. The theoretical minimum

E. Rieffel, W. Polak - Introduction to Quantum Computing for Non Physicist

R.P. Feynman - Feynman Lectures on Computation

R.P. Feynman - Quantum mechanical computers. Found Phys 16, 507–531 (1986). https://doi.org/10.1007/BF01886518