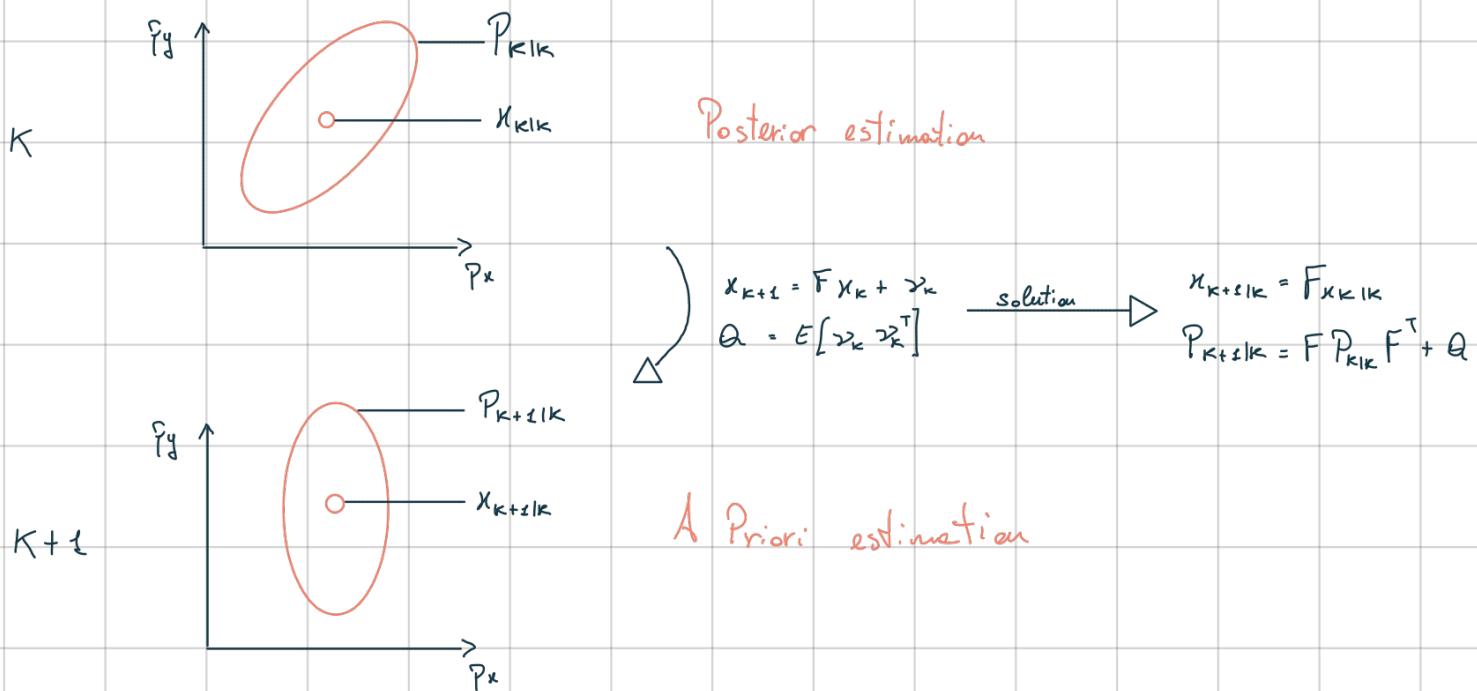
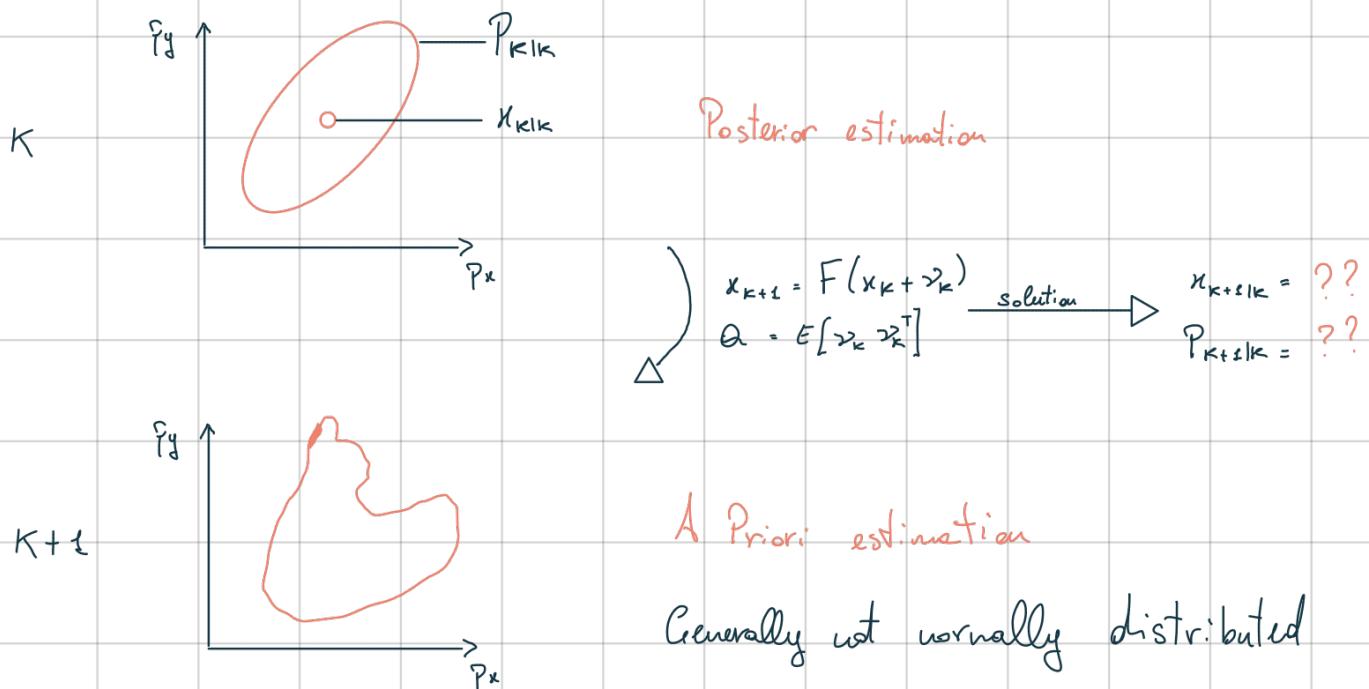


Prediction Problem IFF the Process model is Linear



Prediction Problem IFF the Process model is non Linear



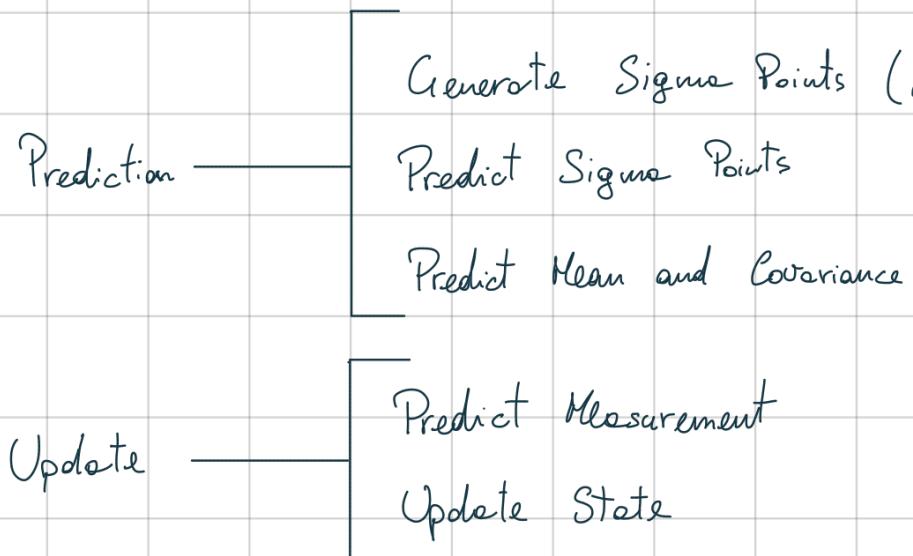
We want to find the normal distribution that represents the real predicted distribution as close as possible.

Linearization with Taylor expansion (EKF)

Two possible methods:

→ Use unscented transformation to eval  $\bar{x}_k$   $P_k$  (UKF)

## Unscented Kalman Filter



### Generate Sigma points

State dimension:  $n_x$ , in our case  $\bar{x} = [p_x \ p_y \ \psi \ \ddot{\psi}]^T$

Number of  $\sigma$  points:  $n_f = 2n_x + 1$

Lambdale, design parameter:  $\lambda = 3 - n_x$

Matrix with Sigma points:  $\chi_{k|k} = \begin{bmatrix} \bar{x}_{k|k} & \bar{x}_{k|k} + \sqrt{(\lambda + n_x) P_{k|k}} & \bar{x}_{k|k} - \sqrt{(\lambda + n_x) P_{k|k}} \\ \vdots & \uparrow & \downarrow \\ & \text{mean state estimate} & \\ & \text{This term give } n_x \text{ sigma points} & \end{bmatrix}$

## State Augmentation

It is used to consider the process noise vector (if it is part of the process function)

So, in our case,  $f(x_k, \varphi_k)$ , we have to include it.

The process noise, in our case, is  $\varphi_k = \begin{bmatrix} \varphi_{e,k} \\ \varphi_{\dot{p},k} \end{bmatrix}$ , this representation does not express effects on state vector and it is independent from  $\Delta t$ .

$$\text{In terms of covariance, we assume } Q = E\{\varphi_k \cdot \varphi_k^T\} = \begin{bmatrix} \sigma_e^2 & 0 \\ 0 & \sigma_{\dot{p}}^2 \end{bmatrix}$$

$$\text{State augmentation } x_{a,k} = [p_x \ p_y \ \psi \ \dot{\psi} \ \varphi_e \ \varphi_{\dot{p}}]$$

$$\text{Covariance augmentation } P_{a,k|k} = \begin{bmatrix} P_{kk} & 0 \\ 0 & Q \end{bmatrix}$$

We can compute the Augmented Sigma points with these new variables.

## Predict Sigma points

$$x_{a,k|k} \quad x_{k+1} = f(x_k, \varphi_k) \quad x_{k+1|k}$$

$$\text{Dimensions: } [n_a, 2n_a + 1]$$

$$7 \times 15$$

$$[p_x \ p_y \ \psi \ \dot{\psi} \ \varphi_e \ \varphi_{\dot{p}}]^T$$

$$[p_x \ p_y \ \psi \ \dot{\psi}]^T$$

## Predict Mean and Covariance (exploiting the predicted sigma points)

Predicted mean

$$x_{k+1|k} = \sum_{i=0}^{2n_a+1} w_i x_{k+1|k,i}$$

Predicted covariance

$$P_{k+1|k} = \sum_{i=0}^{2n_a+1} w_i (x_{k+1|k,i} - x_{k+1|k})(x_{k+1|k,i} - x_{k+1|k})^T$$

The weights are calculated as:

$$w_i = \begin{cases} \frac{\lambda}{\lambda + n_a}, & i = 0 \\ \frac{1}{2(\lambda + n_a)}, & i = 1, \dots, n_a \end{cases}$$

## Predict Measurement

$$\text{Measurement model: } z_{k+1} = h(x_{k+1}) + w_{k+1}$$

We can apply exactly the same unscented transformation approach as we did in the state prediction, however we can take 2 shortcuts.

- ① We already have the sigma points from the prediction step, so we can skip this step

2) We can also skip the augmentation step. Previously, the process noise had a non linear effect on the state ( $f(x_k, \nu_k)$ ), but in this example the measurement noise has a pure additive effect.

$$Z_{k+1} = h(X_{k+1}) + w_{k+1}$$

Dimensions:  $[n_x, 2n_a + 1]$

$5 \times 15$

$[p_x \ p_y \ \psi \ \dot{\psi}]^T$

$Z_{k+1|k}$

$Z_{k+1}$

$n_o: \text{radar measurement dimension}$

$[n_o, 2n_a + 1]$

$3 \times 15$

$[p \ q \ \dot{p}]$

Predicted measurement mean

$$Z_{k+1|k} = \sum_{i=0}^{2n_a+1} w_i Z_{k+1|k,i}$$

Predicted measurement covariance

$$S_{k+1|k} = \sum_{i=0}^{2n_a+1} w_i (Z_{k+1|k,i} - Z_{k+1|k})(Z_{k+1|k,i} - Z_{k+1|k})^T + R$$

to consider  $w_k$   
additive contribution

## Update State

We need to compute the cross-correlation between sigma points in state space and measurement space

$$T_{k+1|k} = \sum_{i=0}^{2n_a} w_i (X_{k+1|k,i} - X_{k+1|k})(Z_{k+1|k,i} - Z_{k+1|k})^T$$

Kalman Gain :  $K_{k+1|k} = T_{k+1|k} S_{k+1|k}^{-1}$

State update :  $X_{k+1|k+1} = X_{k+1|k} + K_{k+1|k}(Z_{k+1} - Z_{k+1|k})$

Covariance matrix update :  $P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k} S_{k+1|k} K_{k+1|k}^T$

# Parameters and Consistency

Process noise covariance, how to evaluate these values?

Try to estimate the maximum value for that state that you expect in your environment. Choose half of the maximum value you expect

In our case:

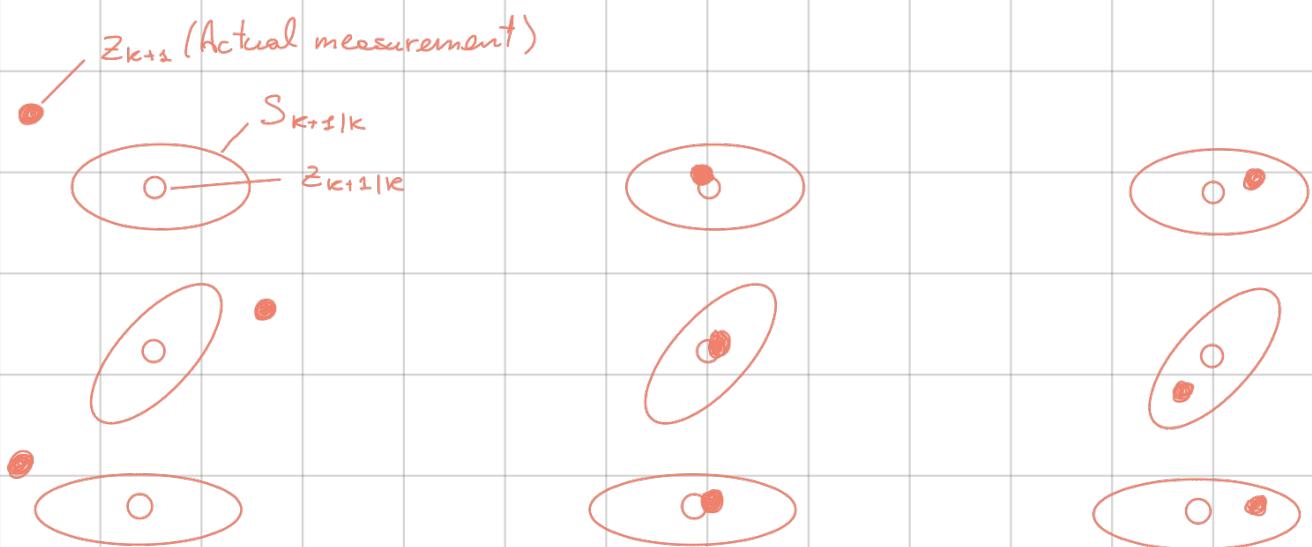
$$Q = \begin{bmatrix} \bar{\sigma}_\alpha^2 & 0 \\ 0 & \bar{\sigma}_\psi^2 \end{bmatrix}$$
$$\bar{\sigma}_\alpha = 3 \frac{m}{s^2}$$
$$a_{max} \approx 6 \text{ m/s}^2$$

Is it important for your application to react fast on changes?  
Then choose a value slightly higher.

It is important to provide smooth accelerations?  
Then choose a value lower.

## Consistency check via Normalized Innovation Squared (NIS)

You need to check if your estimate is consistent



Underestimating  
the uncertainty

Overestimating  
the uncertainty

Consistent!

## Normalized Innovation Squared (NIS)

$$\epsilon = (z_{k+s} - z_{k+s|k})^T \cdot S_{k+s|k}^{-1} \cdot (z_{k+s} - z_{k+s|k})$$

The NIS follows a distribution called Chi Squared distribution  $\chi^2$

Through a table, we can know which number to expect from  $\epsilon$

e.g.

$\epsilon \sim \chi^2$

df →

df	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$
1	0.004	0.016	2.706	3.841
2	0.103	0.211	4.605	5.991
3	0.352	0.584	6.251	7.815
4	0.711	1.064	7.779	9.488
5	1.145	1.610	9.236	11.070

In 5% of your cases,  
the NIS will be higher  
than this value