Generating Random Graphs : models and applications

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We study methods for generating simple undirected random graphs with a prescribed degree distribution F, based on the reference [1]. In particular, we implement the models and study edge cases to explore the Molloy Reed criterion.

Contents

1 Introduction and mathematical framework

Graphs can model complex systems in a mulitutde of domains (physics, biology, social relations). Despite the diversity of such models, there exist some *universal* properties that suggest the self-organization of systems.

We consider a graph $(\mathcal{E}, \mathcal{V})$ being simple and undirected. The main mathematical object that enables the study of networks is the *degree distribution* $F : \mathbb{N} \mapsto [0,1]$. Each vertex has a probability $p_k = F(k)$ to be connected to exactly k among the n other vertices. The randomness of the graph is due to its construction. If we denote by $p_k^{(n)}$ the probability that a randomly chosen vertex of a constructed graph has a degree k, our goal is to provide an algorithm such that :

i) $\lim_{n\to\infty} p_k^{(n)} = F(k)$

ii) the resulting graph is simple and undirected.

We first present important mathematical results that will be of interest throughout this paper.

2 Some Mathematical results on degree distributions

2.1 Erdos-Renyi graph model

Consider a model with n vertices being connected independently at random with a probability p/n, with $p = \mathcal{O}(1)$. This is the Erdos-Renyi model. We are interested in the degree distribution $P(k)$ in the limit $n \to \infty$.

$$\begin{aligned} F(k) &= n-1 \left(\frac{p}{n}\right)^k \left(1-\frac{p}{n}\right)^{n-1-k} \\ &= p^k (e^{-p})^{(n-1-k)/n} \frac{1}{k!} \frac{(n-1)(n-2)...(n-k)}{n^k} \\ &\to p^k \frac{e^{-p}}{k!} \quad \text{as} \quad n \to \infty \end{aligned} \tag{2.1}$$

Thus the degree of a vertex follows a Poissonian distribution of parameter p. In particular,

$$\begin{aligned} \langle k \rangle &= p \\ \langle k^2 \rangle &= 2p^2 \end{aligned} \tag{2.2}$$

2.2 Zeta distribution

Consider a degree distribution given by the distribution (in the limit $n \to \infty$) :

$$F(k) \propto k^{-(1+\alpha)} \longrightarrow \frac{k^{-(1+\alpha)}}{\zeta(1+\alpha)}, \quad \zeta(1+\alpha) = \sum_{j=1}^{\infty} j^{-(1+\alpha)} \tag{2.3}$$

Networks satisfying such distributions of degree are called *scale free* and are very recurrent in complex system modeling. In particular, we can identity different regimes depending on the value of $\alpha > 0$.

$$
\begin{aligned}
\langle k\rangle &= \sum_{k=1}^{\infty}\frac{k^{-\alpha}}{\zeta(1+\alpha)} = \frac{\zeta(\alpha)}{\zeta(1+\alpha)} < \infty \quad \text{for} \quad \alpha > 1 \\
\langle k^2\rangle &= \sum_{k=1}^{\infty}\frac{k^{1-\alpha}}{\zeta(1+\alpha)} = \frac{\zeta(\alpha-1)}{\zeta(\alpha+1)} < \infty \quad \text{for} \quad \alpha > 2
\end{aligned} \tag{2.4}
$$

Another property of power law distribution is its heavy-tailed behaviour. Consider the probability $\mathbb{P}\{X \geq k\}$ for a sufficiently large k, so that we can make the integral approximation of the sum :

$$
\begin{aligned}
\mathbb{P}\{X \geq k\} &= \sum_{x=k}^{\infty}\mathbb{P}\{X = x\} \\
&\sim \int_k^{\infty} dx x^{-(1+\alpha)} \sim k^{-\alpha}
\end{aligned} \tag{2.5}
$$

2.3 Molloy-Reed criterion and existence of giant component

The Molloy-Reed criterion allows us to predict the existence of a giant cluster by looking at the moments of the degree distribution. We follow its derivation from [3]. We consider an infinite graph with a tree-like structure (such as Erdos-Renyi graphs) and define u the probability that a given edge does not connect to the giant component. Then u satisfies the self-consistent equation :

$$
u = \sum_{k=1}^{\infty} Q(k)u^{k-1} := f(u) \tag{2.6}
$$

where $Q(k)$ is the probability that the vertex at a given edge has degree k. In words, in order not to be connected to the giant cluster, the edge can either lead to a dead-end with probability $Q(1)$, or to a vertex with two edges where one is the edge that we consider and the second should neither lead to the giant component - this occurs with probability $Q(2) \times u$, etc. Clearly, $u^* = 1$ is always a fixed point of the equation. It is stable if $|f'(u^*)| < 1$, that is if

$$
\sum_k Q(k)(k-1)u^{k-2}|_{u=1} < 1 \iff \langle k^2\rangle < 2\langle k\rangle \tag{2.7}
$$

If $u^* = 1$ is unstable, there exist a second stable fixed point $0 \leq u^{**} < 1$ characterizing the existence of a giant component. We can therefore write the following criterion whose proof follows from the above :

Criterion 2.1 *A random graph with degree distribution $F(k)$ has a giant component if and only if*

$$
\langle k^2\rangle > 2\langle k\rangle \tag{2.8}
$$

Using this criterion we can study the existence of giant components by looking at the degree distribution. For Poisson distributions, Eq.2.2 allows us to predict the existence of a giant cluster for $p > 1$. For the power-law distributions, Eq.2.4 predicts the existence of a giant cluster for $\alpha \in]1, 2[$ since the second moment is infinite and the first is finite. In case $\alpha > 2$, the criterion can be solved using analytical expressions from Eq.2.4 and yields

$$
\zeta(\alpha-1) > 2\zeta(\alpha), \quad \alpha > 2 \tag{2.9}
$$

Thus providing a critical value of the exponent α for the presence of giant cluster:

$$
\alpha_c \approx 2.47875... \tag{2.10}
$$

These theoretical results can be tested by generating random graphs with prescribed degree distribution.

3 Algorithms for generating random graphs

3.1 Configuration model

Consider a set of n vertices $\{v_i\}_{i=1}^n$ and a degree distribution F. For each vertex, we sample a degree d_i from the distribution F and create d_i "stubs" attached to the vertex. Then we join the stubs of all vertices pairwise. If the total number of stubs $\sum_{i=1}^n d_i$ is odd, we simply drop the extra stub. To ensure the simpleness of the generated graph, we can either remove the self loops and multiples edges (Erased configuration model) or re-sample the graph until it satisfies the property (Repeated configuration model).

3.1.1 Erased configuration model

We define $\{F_n\}_{n\geq 1}$ a sequence of function $\mathbb{N} \mapsto [0,1]$ such that $F_n(k) = p_k^{(n)}$, and $N_k^{(n)}$ the number of vertices having degree k, that both inherit from an erased configuration model construction. A theorem that is provided in [1] is the following :

Theorem 1 *If F has finite mean, $N_k^{(n)}/n \longrightarrow F(k)$ as $n \to \infty$. In particular, using the dominated convergence theorem, we have that $F_n \longrightarrow F$.*

The proof of this theorem is found in section A of the Appendix. The main argument for the necessary condition of F having a finite mean can be explained as follows : if the mean degree is infinite, the erasure procedure will affect the degree distribution. Indeed, it is known that for such distributions with infinite mean, the total number of degrees $\sum_{i=1}^n d_i$ is of the order of $\max_{1\leq i\leq n} d_i$. Consider the maximum degree vertex. For any other node with degree ≥ 2 there is a positive probability (bounded away from 0) to have initially more than one stub connected to it. Erasing the multiple edges, the degrees of the nodes will "non-neglectably" be less than the ones sampled from the distribution F.

3.1.2 Repeated configuration model

This algorithm consists in generating a random graph repeatedly until we obtain a simple one. We then can ask ourselves is the probability of solving the task is strictly positive. From [1] we have the following theorem :

Theorem 2 *If F has finite second moment, $N_k^{(n)}/n \longrightarrow F(k)$ as $n \to \infty$. In particular, using the dominated convergence theorem, we have that $F_n \longrightarrow F$.*

As opposed to the previous model, the necessary condition for convergence now depends on the convergence of the second moment. A proof of the theorem is provided in section B of the Appendix.

3.2 Generalized random graph

Consider a Erdos-Renyi graph with edge construction defined as independent Bernouilli random variables $\{X_{ij}\}_{i<j}$ with a value 1 signifying the presence of an edge. By definition, $X_{ji} = X_{ij}$ for $i < j$ and $X_{ii} = 0 \quad \forall i$. We consider a model where the probability p_{ij} of edge between vertices v_i and v_j depends on the value of i and j. In the same way we define

$q_{ij} = 1 - p_{ij}$ the probability of not creating an edge. By independence of X_{ij}'s, a graph configuration $X = \{X_{ij}\}_{i<j}$ has a probability mass function given by

$$\begin{aligned}\mathbb{P}(X = x) &= \prod_{i<j} p_{ij}^{x_{ij}} q_{ij}^{1-x_{ij}} \\ &= \prod_{i<j}(1 + r_{ij})^{-1} \prod_{i<j} r_{ij}^{x_{ij}}, \quad \text{with } r_{ij} := p_{ij}/q_{ij}\end{aligned} \tag{3.1}$$

Consider $r_{ij} = u_i u_j$ for some non-negative i.i.d. random variables $\{u_i\}_{i=1}^n$ and define $G(u) := \prod_{i<j}(1 + u_i u_j)$, we have the conditional probability on u

$$\begin{aligned}\mathbb{P}_u(X = x) &= G^{-1}(u) \prod_{i<j} (u_i u_j)^{x_{ij}} \\ &= G^{-1}(u) \prod_{i=1}^{n} u_i^{d_{i(x)}}\end{aligned} \tag{3.2}$$

where $d_i(x)$ is the degree of the vertex v_i in the configuration x and we naturally define $D_i := d_i(X) = d_i(X(u))$ its associated random variable. By considering $u_i = W_i/\sqrt{n}$ we have the following theorem from [1] :

Theorem 3 *Consider a generalized random graph on n vertices with edge probability defined by $p_{ij}/q_{ij} = W_i W_j/n$ where $\{W_i\}_{i=1}^n$ are non negative i.i.d. random variables with finite mean μ_W and finite moment of order $1 + \epsilon$, $\quad \epsilon > 0$. Then for any m, $D_1, ..., D_m$ are asymptotically independent and the limiting distribution of a degree variable D_j as $n \to \infty$ is mixed Poisson with parameter $W_j \mu_W$.*

The proof of this theorem can be found in section C of the Appendix.

In our study, we want to generate a random graph with a prescribed degree distribution F. If we set this distribution to be mixed-Poisson of parameter Q, we must sample independently the W_i from a a rescaled distribution of Q by a factor $\sqrt{\mu_Q}$. As an example, consider an Erdos-Renyi graph with probability of edge p/n, independent of the labels. According to equation Eq.2.2, the limiting degree distribution is Poisson of parameter p, corresponding to a mixed Poissonian of parameter $W_k \mu_W$ with deterministic $W_k = p^{1/2}$.

We are now looking for a way to generate random graphs with mixed-Poissonian degree distribution that behave as a power law. This can be achieved by considering $\{W_i\}$ being sampled from a power law distribution. The above theorem can be extended in the case W_i have an infinite mean, which is the case if $\mathbb{P}\{W_i = w\} \propto w^{-(1+\alpha)}$ and $\alpha \in]0, 1[$ (see Eq.2.4. The main reference provides the following theorem :

Theorem 4 *Suppose that $\{W_i\}$ are i.i.d. with $\mathbb{P}\{W_i \geq w\} \sim cw^{-\alpha}$ with $\alpha \in]0, 1[$ and $c > 0$, and consider the generalized random graph with $p_{ij}/q_{ij} = W_i W_j/n^{1/\alpha}$. Then for any m, $D_1, ..., D_m$ are asymptotically independent and the limiting distribution of a degree variable D_j as $n \to \infty$ is mixed Poisson with parameter γW_j^α with $\gamma = c \int_0^\infty (1 + x)^{-2} x^{-\alpha}$.*

The last theorem of this section provides the result on empirical distribution convergence of N_k^n/n that enables to verify numerically the two previous theorems.

Theorem 5 *Consider a generalized random graph in the limit $n \to \infty$.*

- *a) If $\{W_i\}$ have finite moment of order $1 + \epsilon$ ($\epsilon \geq 0$), then $N_k^{(n)} \to F(k)$ in probability for all k with F a mixed Poisson distribution of parameter $W \mu_W$*
- *b) If $\mathbb{P}\{W_i \geq w\} \sim cw^{-\alpha}$ ($\alpha \in]0, 1[$) then $N_k^{(n)} \to F^\alpha(k)$ in probability for all k with F^α a mixed Poisson distribution of parameter γW^α*

4 Simulations

Simulations of the three models are made in the limits of the theorems. We study for each case the Possonian distribution and some Zeta distributions with various parameters α that allow us to explore the cases of infinite mean and/or second moment. Each simulation is conducted with 5 independent run and we plot the mean and standard deviation bars of the empirical distributions.

4.1 Erased configuration model

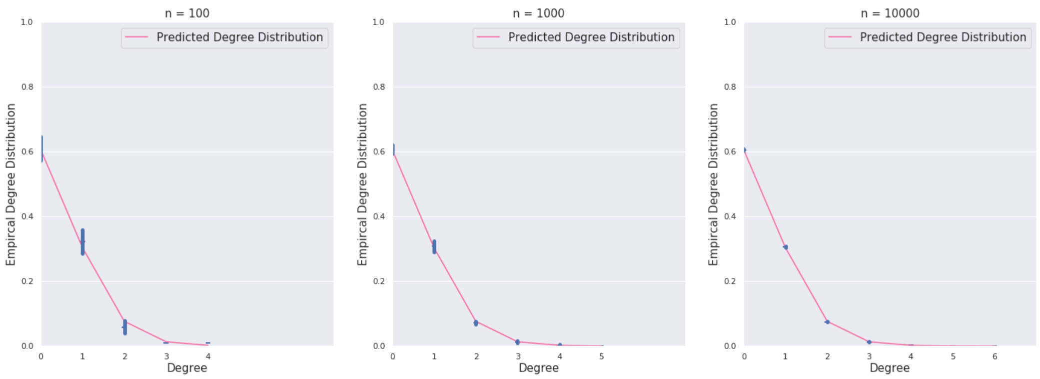

Figure 1: Emprirical distribution of degree using erased configuration model for with a prescribed Poisson distribution of parameter $p = 0.5$. The empirical distribution converge to the intended one as $n \to \infty$

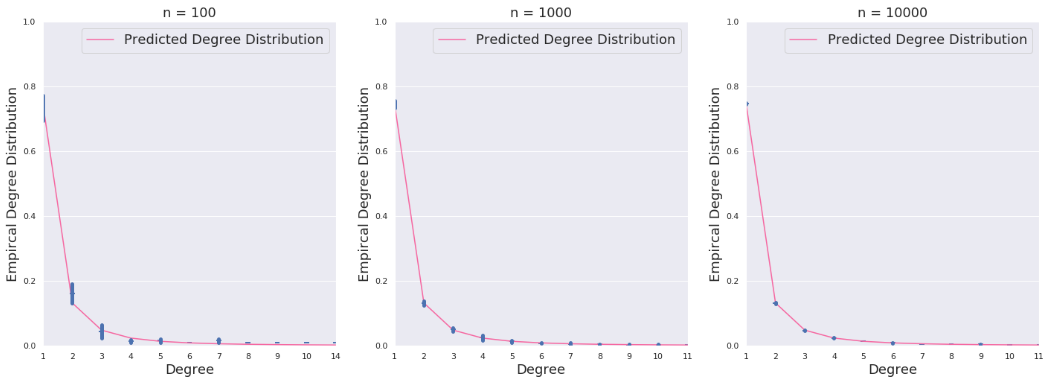

Figure 2: Emprirical distribution of degree using erased configuration model with a prescribed Zeta distribution of parameter $\alpha = 1.5$. The empirical distribution converge to the intended one as $n \to \infty$.

By giant component, we consider a set of connected vertices of size $\Omega(n)$. The presence of a giant cluster is easily checked since if it exists, a randomly chosen vertex has a non neglectable probability of belonging to it. We use the configuration model to explore the extreme case of the Molloy-Reed condition by taking a power-law distribution as defined in Eq.2.3 with $\alpha \in]1, 2[$, thus with finite mean and infinite second moment.

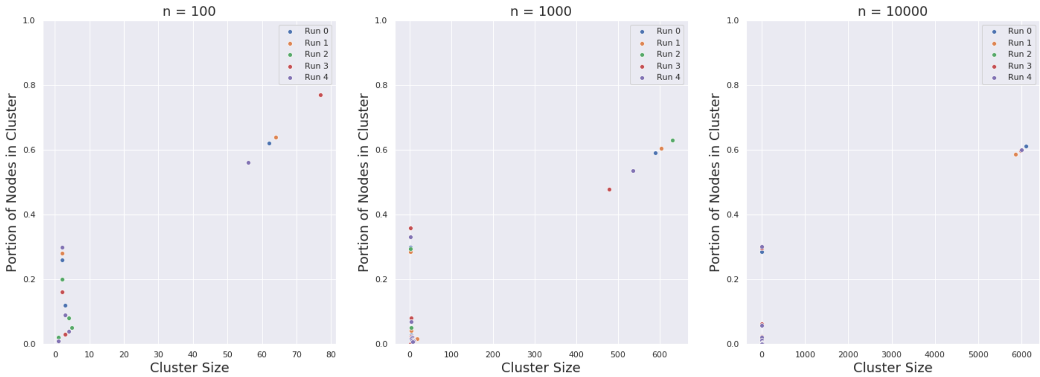

Figure 3: Emprirical probability of belonging to cluster of given size for a Zeta distribution of parameter $\alpha = 1.5$. The existence of a giant cluster is shown as a non neglectable proportion of nodes belong to it, thus validating the Molloy Reed criterion for a finite mean and infinite second moment.

Using this model, we explore the critical value of exponent $\alpha \approx 2.48$ for the appearance of giant cluster.

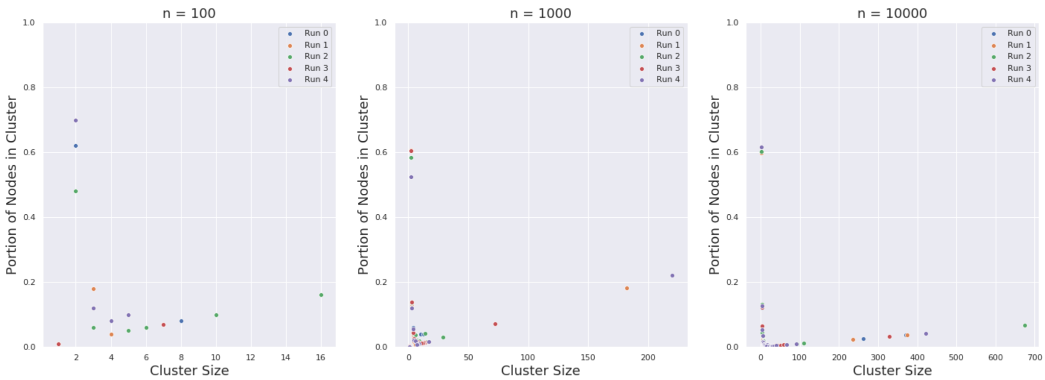

Figure 4: Emprirical probability of belonging to cluster of given size, for a prescribed power-law degree distribution of parameter $\alpha = 2.3 < \alpha_c$. The existence of a giant cluster is shown as a non neglectable proportion of nodes belong to it.

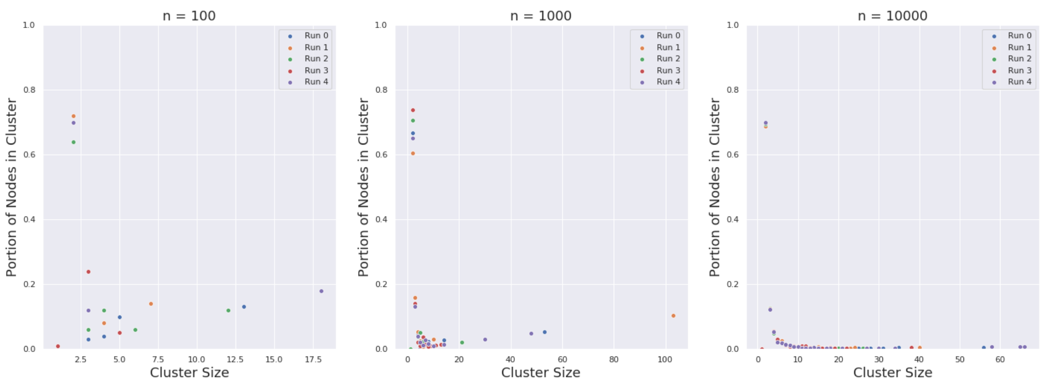

Figure 5: Emprirical probability of belonging to cluster of given size, for a prescribed power-law degree distribution of parameter $\alpha = 2.6 > \alpha_c$. The existence of a giant cluster is disproved as the maximum cluster size is $\ll n$ as $n \to \infty$, and a neglectable portion of nodes connect to it.

4.2 Repeated configuration model

We implement the repeated configuration model for 2 Poisson distributions with parameters $p = 0.5$ and $p = 1.5$. As in previous section, we show the convergence of the empirical distributions and the presence or absence of infinite cluster depending if the parameter of the distribution is less or greater than $p_c = 1$, using the Molloy Reed criterion (see Fig.7 and 12).

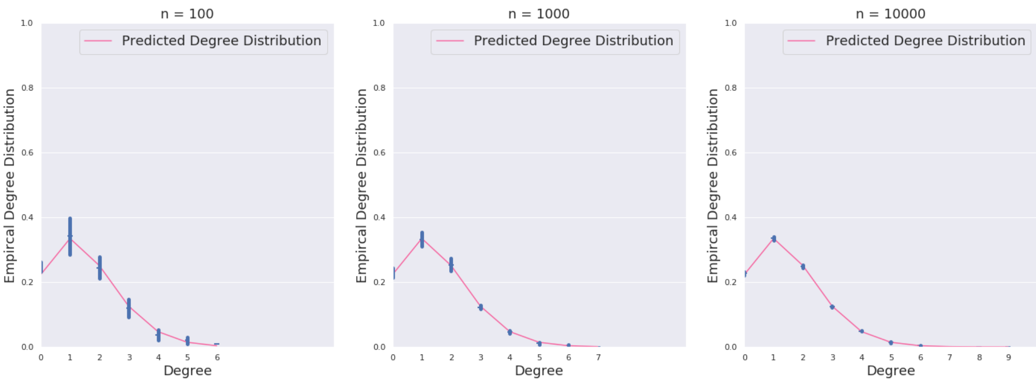

Figure 6: Emprirical distribution of degree using repeated configuration model for with a prescribed Poisson distribution of parameter $p = 1.5$. The empirical distribution converge to the intended one as $n \to \infty$, as predicted by Theorem 2 since the variance is finite.

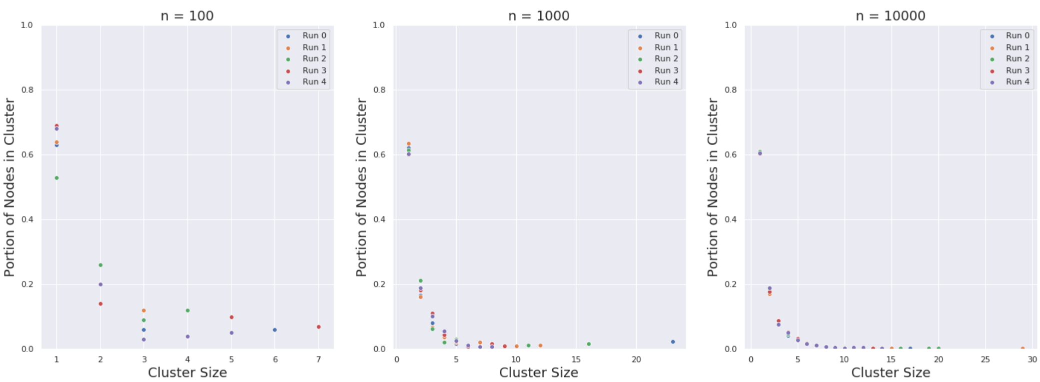

Figure 7: Emprirical probability of belonging to cluster of given size, for a prescribed Poisson degree distribution of parameter $p = 0.5$. The existence of giant cluster is clearly disproved.

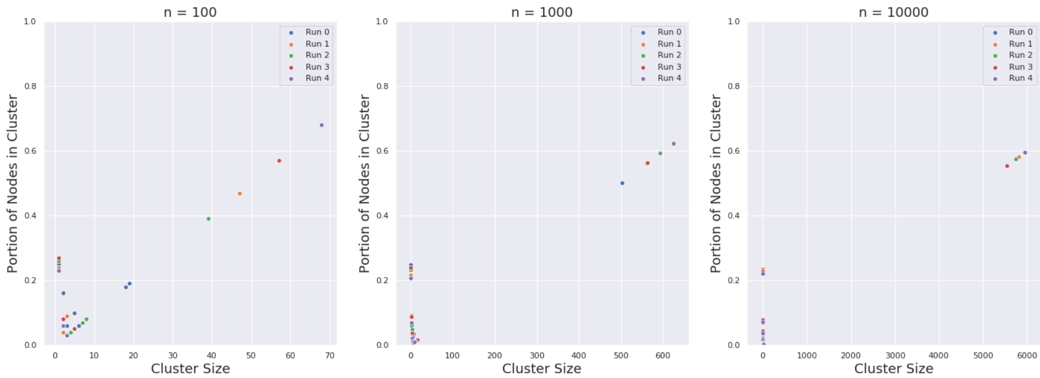

Figure 8: Emprirical probability of belonging to cluster of given size, for a prescribed Poisson distribution of parameter $p = 1.5$. The existence of a giant cluster is clear as the proportion of nodes connecting to it is significantly large.

Same observation can be made on convergence of empirical distributions and giant cluster existence when taking power-law distributions (for $\alpha > 2$ to assure finite second moment).

4.3 Generalized random graph

We first recover the construction of an Erdos Renyi graph by fitting the emprirical distributions obtained with a Poisson distribution, thus verifying Theorem 3 and 5. To this end we consider a deterministic W fixed at a value $\lambda^{1/2}$. Thus

$$p = p_{ij} = \frac{W_i W_j / n}{1 + W_i W_j / n} = \frac{W_i W_j}{n + W_i W_j} \approx \frac{\lambda}{n} \tag{4.1}$$

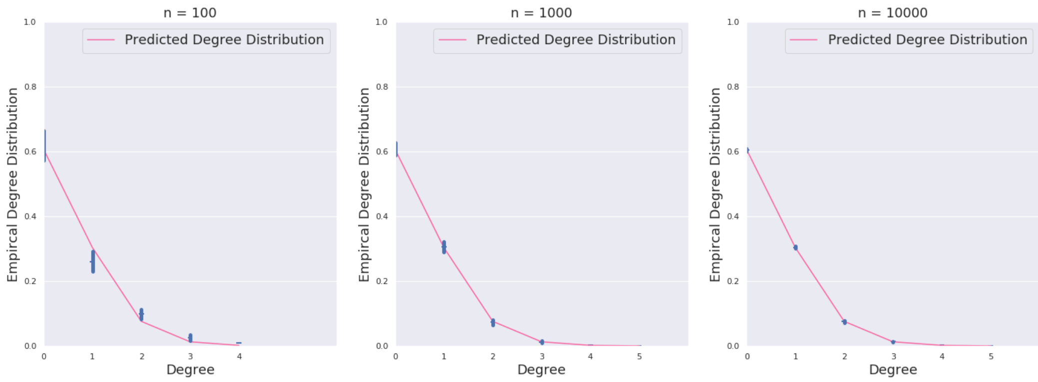

Figure 9: Emprirical distribution of degree using repeated configuration model with $W_i = \lambda^{1/2} \quad \forall i$. The resulting distribution is Poissonian with parameter $\lambda = 0.5$. The study of giant component is not shown here but its absence in the case $\lambda < 1$ is once again verified.

We can then explore the case where $\{W_i\}$ are i.i.d. sampled from a distribution with mean $\mu_W = \lambda^{1/2}$. We show that the resulting empirical distribution is on average (conducted over W values) a Poisson distribution of parameter λ.

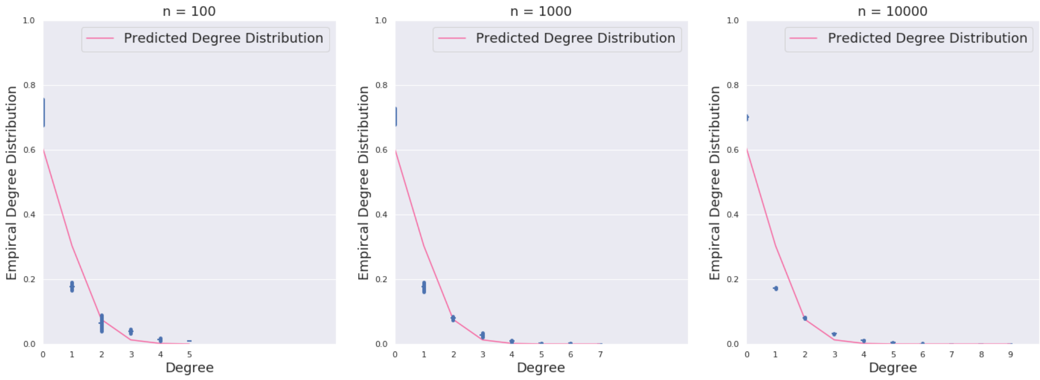

Figure 10: Emprirical distribution of degree using generalized graph model for with $W_i \sim Poi(\sqrt{\lambda}) \quad \forall i$. The resulting distribution, averaged over the different runs, is Poissonian with parameter $\lambda = 0.5$.

The last part of our study focuses on the case $\{W_i\}$ are sampled from a distribution with infinite mean, e.g. $W_i \sim Zeta(1+\alpha)$ with $\alpha \in]0,1[$. The expected degree distribution function, to which the empirical distribution N_k/n must converge to, is computed as follows :

$$
\begin{aligned}
\frac{1}{n}\mathbb{E}[N_k] &= \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[\mathbb{1}\{D_i = k\}] \\
&= \mathbb{P}\{D_1 = k\} \\
&= \int_0^\infty P(D = k|W = w)P(W = w) \\
&\approx \int_0^\infty c\frac{\lambda^k e^{-\lambda}}{k!}w^{-(1+\alpha)}dw, \quad c = 1/\zeta(1+\alpha) \text{ and } \lambda = \gamma w^\alpha \\
&= \frac{c\alpha\gamma^k}{k!}\int_0^\infty e^{\gamma w^\alpha}w^{\alpha(k-1)-1}dw \\
&= \frac{\gamma c}{k!}\gamma^{-k+1}\left[\Gamma(k-1, w^\alpha\gamma\right]_0^\infty \\
&= \frac{\gamma c}{k!}\Gamma(k-1) \\
&= \frac{\gamma c}{k(k-1)}
\end{aligned}
\tag{4.2}
$$

Where we have used the results from Theorem 4 in the first line. We can then fit our empirical distributions running multiple simulations.

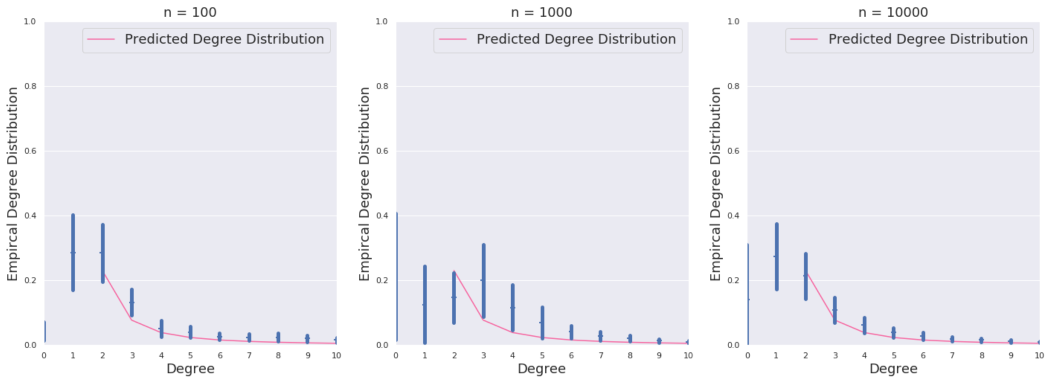

Figure 11: Emprirical distribution of degree using generalized graph model with $W_i \sim Zeta(1+\alpha)$ and $\alpha = 0.5$. 20 independent runs were made. We can see that the distribution for large degrees is on average asymptotically the expression found in 4.2.

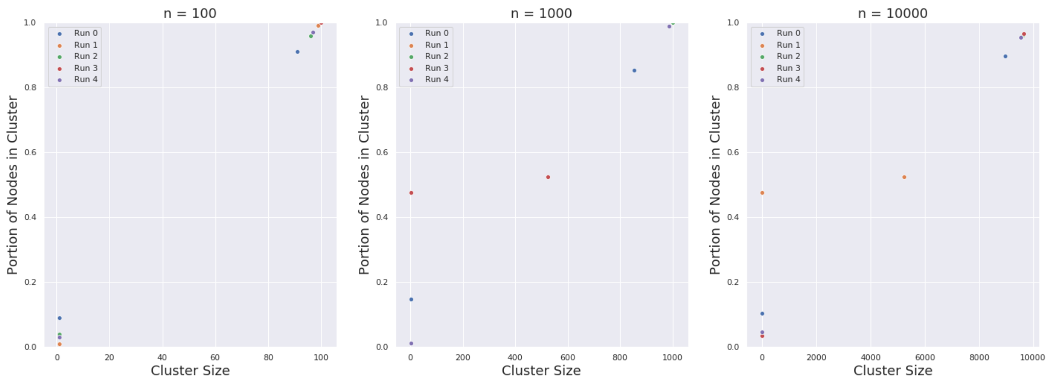

Figure 12: Emprirical probability of belonging to cluster of given size using generalized random graph model, with $W_i \sim Zeta(1+\alpha)$ with $\alpha = 0.5$. The existence of a giant cluster is observed.

To explain the existence of giant cluster of the obtained mixed-Poisson distribution, we shall note that the heavy tail behavior of W_i translates to the obtained distribution. Indeed, from [1], if Y is a mixed-Poisson variable of parameter γW^α, then we have

$$\begin{aligned} P(Y \geq y) &\approx P(W^\alpha \geq y) \\ &= P(W \geq y^{1/\alpha}) \quad \sim cy^{-1} \end{aligned} \tag{4.3}$$

Thus the resulting distribution is heavytailed in the limit of large degrees with parameter $\alpha_Y = 1$. In this limit, the Molloy Reed criterion predicts the existence of a giant cluster, independently of the value of initial parameters $\alpha_W \in]0, 1[$.

5 Conclusion

After deriving the Molloy Reed criterion, we were able to verify it using the results from [1]. We implemented three models for generating random simple graphs, each with its own limitations. For the erased configuration model, in the case the mean of the degree distribution was finite and the second moment infinite, we were able to show the existence of a giant cluster. For the repeated configuration model, we explored the case of finite first and second moment, and showed the existence of critical parameter value α_c for the Zeta distribution, from which the cluster is not present, as predicted by the calculations. Finally, using generalized random graph model, we were able to reconstruct the Erods-Renyi graph and explored the case of infinite mean and variance.

References

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A Proof of Theorem 1

This proof follows the one presented in [1]. Consider $\tilde{N}_k^{(n)}$ the number of vertices with degree k before stub erasure, and $\{D_i\}_{i=1}^n$ a set of i.i.d. random variables following the prescribed distribution F i . Clearly, using the law of large numbers we have :

$$\frac{\tilde{N}_k^{(n)}}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{D_i = k\} = \mathbb{E}[\mathbb{1}\{D_1 = k\}] = F(k) \tag{A.1}$$

Thus if we show $(\tilde{N}_k^{(n)} - N_k^{(n)})/n \longrightarrow 0$ in probability we are done. Define $M^{(n)}$ the number of vertices where at least 1 stub is removed. Since $\tilde{N}_k^{(n)} - N_k^{(n)} \leq M^{(n)}$, using the first moment method, showing $\mathbb{E}[M^{(n)}]/n \longrightarrow 0$ proves the theorem.
Let E_i be the number of removed stubs attached to vertex v_i. we can then define $M^{(n)} = \sum_{i=1}^n M_i^{(n)}$ with $M_i^{(n)} := \mathbb{1}\{E_i \geq 1\}$. Using the independence and identical distribution of $\{M_i^{(n)}\}$, we have :

$$\frac{1}{n}\mathbb{E}[M^{(n)}] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[M_i^{(n)}] = \mathbb{E}[M_1^{(n)}] = \mathbb{P}\{E_1 \geq 1\} \tag{A.2}$$

Showing the latter $\longrightarrow 0$ allows us to conclude. To show that almost surely no stub is being removed, we define the event A_k that a chosen stub of a vertex with initial degree k is not being removed. We then have to show $P(A_k) \longrightarrow 1, \quad \forall k = 1, ..., n$. Indeed, if the first randomly chosen out of k stubs is not erased, the probability that any other stub is not removed is asymptotically $P(A_{k-1})$. For completeness, we consider the erasure configuration model where we could erase a stub due to an odd number of stubs in total, a self loop or multiple edges connecting 2 vertices. We define the associated events of not removing the stub due to these reasons by A_k^{odd}, A_k^{loop} and A_k^{mult}. Conditioning on the total of stubs being m, and on the number of stubs j being connected to the chosen vertex, using the derivations from [1], we have :

$$\begin{aligned}
P_m(A_k^{odd}) &\geq 1 - \frac{1}{m} \\
P_m(A_k^{loop}) &= 1 - \frac{k-1}{m-1} \\
P_{m,k}(A_k^{mult}) &\geq \left(\frac{m-k-2j}{m}\right)_+^{k-1} \\
P_{m,k}(A_k) &= P_{m,k}(A_k^{odd} \cup A_k^{loop} \cup A_{k,m}^{mult})
\end{aligned} \tag{A.3}$$

Removing the conditioning with respect to m, j by introducing the random variables L_n (number of total stubs) and J_n (number of stubs attached to the vertex we are considering) and using Bonferoni inequality, we have :

$$P(A_k) \geq \mathbb{E}\left[\left(\frac{L_n - J_n - 2k}{L_n}\right)^{k-1} - \frac{1}{L_n} - \frac{k-1}{L_n - 1}\right] \tag{A.4}$$

The total number of stubs is the sum of i.i.d. random variables D_i with distribution F and finite mean μ_F by assumption. Hence by the law of large numbers it implies $L_n/n \longrightarrow \mu_F$ a.a.s. On the other hand, K_n converges in distribution to a random variable K with distribution $P(j) = jp_j/\mu_F$ and $K_n/n \longrightarrow 0$ almost surely. Using these results we have $P(A_k) \longrightarrow 1$ as $n \to \infty$, finishing the proof.

Second part of theorem This proof is common to Theorem 1 and Theorem 2. We show that the convergence in probability of the empirical distribution $N_k^{(n)}/n \longrightarrow F(k)$ implies the convergence $p_k^{(n)} = F_n(k) \longrightarrow F(k)$. Clearly $0 \leq N_k^{(n)}/n \leq 1$. Hence

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}\left[\frac{N_k^{(n)}}{n}\right] &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} \mathbb{1}\left\{d_i^{(n)} = k\right\}\right] = \lim_{n\to\infty} \frac{np_k^{(n)}}{n} \\ &= \mathbb{E}\left[\lim_{n\to\infty} \frac{N_k^{(n)}}{n}\right] = F(k) \end{aligned} \tag{A.5}$$

where we used the dominated convergence theorem and the assumption in the last line.

B Proof of Theorem 2

This proof follows the one presented in [1]. Let $D_1, ..., D_n$ be i.i.d. random variables with distribution F and finite second moment and let $\tilde{p}_k^{(n)} = n^{-1}\sum_{i=1}^{n} \mathbb{1}\{D_i = k\}$. We define S_n the event that the configuration model produces a simple graph. The empirical distribution of the repeated configuration model is the same as the distribution $\tilde{p}_k^{(n)}$ conditioned on S_n and we have to show :

$$\mathbb{P}\{|\tilde{p}_k^{(n)} - p_k| > \epsilon | S_n\} \longrightarrow 0 \quad \text{as } n \to \infty \text{ for any } \epsilon > 0 \text{ and for any } k \tag{B.1}$$

We have however the following inequality :

$$\mathbb{P}\{|\tilde{p}_k^{(n)} - p_k| > \epsilon | S_n\} = \frac{\mathbb{P}\{|\tilde{p}_k^{(n)} - p_k| > \epsilon, S_n\}}{\mathbb{P}\{S_n\}} \leq \frac{\mathbb{P}\{|\tilde{p}_k^{(n)} - p_k| > \epsilon\}}{\mathbb{P}\{S_n\}} \tag{B.2}$$

Using the law of large numbers, (see proof of Theorem 1 in A), we have that the numerator converges to 0. We need to show that the denominator is greater than a constant $c > 0$, using the fact that the second moment of the distribution is finite. Some results from [4] :

$$\begin{aligned} \mathbb{E}[\#\text{ of multi edges}] &= \frac{1}{2}\left[\frac{\langle k^2\rangle - \langle k\rangle}{\langle k\rangle}\right]^2 \\ \mathbb{E}[\#\text{ of self loops}] &= \frac{\langle k^2\rangle - \langle k\rangle}{\langle 2k\rangle} \end{aligned} \tag{B.3}$$

allow us to conclude stating that there exist a non zero probability of generating a simple random graph since self-plops and multi edges are a vanishingly small fraction of all edges in the large n limit.

C Proof of Theorem 3

This proof follows the one presented in [1]

Consider a collection of random variables D_j corresponding to the degree of vertex v_j and recall that W_i are i.i.d. random variables with mean μ_W and finite moment of order $1 + \epsilon$. To show the convergence of the distribution of $\{D_j\}$, we will show that its moment genereating function converges to the one of a Poisson distribution. Using the result from [1] we have

$$\mathbb{E}\left[t_j^D\right] = \mathbb{E}\left[\prod_{i\neq j} \frac{1 + W_iW_jt/n}{1 + W_iW_j/n}\right] \tag{C.1}$$

For $n \to \infty$ and using the Taylor approximation $\ln(1 + x) = x + \mathcal{O}(x^2)$ we have

$$\prod_i \frac{1+W_iW_jt/n}{1+W_iW_j/n} \approx \exp\left\{\frac{W_j\sum_i W_i}{n}(t-1)\right\} \tag{C.2}$$

Using the law of large numbers,

$$\mathbb{E}\left[t^{D_j}\right] \longrightarrow \mathbb{E}\left[e^{W_j\mu_W(t-1)}\right] \quad \text{a.a.s.} \tag{C.3}$$

which is the moment generating function of a Poisson distribution of parameter $W_k\mu_W$. The independence of m random variable $D_1, ..., D_m$ is shown by independence of the W_i and using the above result.

D Proof of Theorem 4

This proof follows the one presented in [1]. As previously, we have

$$\mathbb{E}\left[t_j^D\right] = \mathbb{E}\left[\prod_{i\neq j} \frac{1+W_iW_jt/n^{1/\alpha}}{1+W_iW_j/n^{1/\alpha}}\right] \tag{D.1}$$

Conditioning over the value $W_j = w$ we can write $\phi(x) := (1+xwt)/(1+xw)$. By denoting $f(.)$ the probability density function of W_i we have, by i.i.d. of W_i

$$\begin{aligned}
\Phi_n(w) &= \left(\int_0^\infty \phi\left(\frac{x}{n^{1/\alpha}}\right) f(x)dx\right)^{n-1} \\
&= \left(1 + \int_0^\infty (\phi(x)-1) f(xn^{1/\alpha})n^{1/\alpha}dx\right)^{n-1} \\
&= \left(1 + \int_0^\infty \phi'(x)(1-f(xn^{1/\alpha})dx\right)^{n-1} \quad \text{using partial integration}
\end{aligned} \tag{D.2}$$

To show the limiting exponential character of Φ for $n \to \infty$, we show that the integral in the last expression is $\mathcal{O}(1/n)$, using the assumption that the limiting distribution $\{W_i\}$ is $\sim cw^{-\alpha}$, thus $y^\alpha(1-f(y))$ is a bounded function that converges to $c > 0$ as $y \to \infty$:

$$\begin{aligned}
\int_0^\infty \phi'(x)(1-f(xn^{1/\alpha})dx &= \frac{1}{n}\int_0^\infty \frac{\phi'(x)}{x\alpha}(n^{1/\alpha}x)^\alpha(1-f(xn^{1/\alpha})dx \\
&\longrightarrow \frac{c}{n}\int_0^\infty \frac{\phi'(x)}{x^\alpha}dx \\
&= \frac{1}{n}(t-1)w^\alpha\gamma
\end{aligned} \tag{D.3}$$

Thus $\lim\limits_{n\to\infty} \Phi_n(w) = e^{(t-1)\gamma w}$, which is the moment generating function of a Poisson distribution with parameter γw^α. The moment generating function of D_j if found by integrating over $w = W_j$ and yields

$$\mathbb{E}\left[t^{D_k}\right] = \mathbb{E}\left[e^{(t-1)W_k^\alpha}\right] \tag{D.4}$$