Question 3

You have two unfair coins, one with the probability of heads equal to p_1 and the other with the probability of heads equal to p_2, where $p_2 \neq p_1$. In strategy A, you choose one coin at random and toss it twice. In strategy B, you toss both coins. What is the best strategy to maximize the probability of the event $E =$ "the two tosses are both heads"?

Solution

Let *head* and *tail* denote the results of a drop. Define the probability space as

$$
\begin{aligned}
&\Omega = \{\text{head, tail}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \text{ the probability measure on } \mathcal{F}
\end{aligned}
$$

Let C_1 and C_2 denote the events of selecting the first mentioned coin and the second mentioned coin, respectively, then we have

$$
\begin{aligned}
\mathbb{P}_{C_1}(\{\text{head}\}) = p_1 \\
\mathbb{P}_{C_2}(\{\text{head}\}) = p_2
\end{aligned}
$$

Let the first drop and the second drop be denoted by subscripts 1 and 2, respectively, then we have

$$
\mathbb{P}_{C_{i_j}}(\{\text{head}\}_k) = p_i, \quad i, j, k \in \{1, 2\}
$$

We know that C_{1_1}, C_{2_1}, C_{1_2} and C_{2_2} are stochastically independent

$$
\mathbb{P}(C_{1_1} \cap C_{2_1} \cap C_{1_2} \cap C_{2_2}) = \left(\frac{1}{2}\right)^4 = \mathbb{P}(C_{1_1})\mathbb{P}(C_{2_1})\mathbb{P}(C_{1_2})\mathbb{P}(C_{2_2})
$$

And we know that $\{C_{1_1}, C_{2_1}\}$ and $\{C_{1_2}, C_{2_2}\}$ are two partitions of Ω, then we know that $\{head\}_1$ and $\{head\}_2$ are stochastically independent

$$
\begin{aligned}
&\mathbb{P}(\{head\}_1 \cap \{head\}_2) \\
=&\left(\sum_{j=1}^{2} \mathbb{P}_{C_{j_1}}(\{\text{head}\}_1)\mathbb{P}(C_{j_1})\right)\left(\sum_{j=1}^{2} \mathbb{P}_{C_{j_2}}(\{\text{head}\}_2)\mathbb{P}(C_{j_2})\right) \\
=&\mathbb{P}(\{\text{head}\}_1)\mathbb{P}(\{\text{head}\}_2)
\end{aligned}
$$

Strategy A: choose one coin at random and toss it twice

By applying the principle of symmetry, we have

$$
\begin{aligned}
&\mathbb{P}(C_{1_1}) = \mathbb{P}(C_{2_1}) = \mathbb{P}(C_{1_2}) = \mathbb{P}(C_{2_2}) \\
=&\mathbb{P}_{C_{1_1}}(C_{1_2}) = \mathbb{P}_{C_{2_1}}(C_{1_2}) = \mathbb{P}_{C_{1_2}}(C_{2_2}) = \mathbb{P}_{C_{2_2}}(C_{2_2}) \\
=&\frac{1}{2}
\end{aligned}
$$

We know that $\{C_{1_1}, C_{2_1}\}$ and $\{C_{1_2}, C_{2_2}\}$ are two partitions of Ω, then we have the probability of event E in strategy A

$$
\begin{aligned}
\mathbb{P}(E_A) &= \mathbb{P}(\{\text{head}\}_1 \cap \{\text{head}\}_2) \\
&= \mathbb{P}(\{\text{head}\}_1)\mathbb{P}(\{\text{head}\}_2) \\
&= \sum_{j=1}^{2} \big(\mathbb{P}(C_{j_1})\mathbb{P}_{C_{j_1}}(\{\text{head}\}_1)\mathbb{P}_{C_{j_2}}(\{\text{head}\}_2)\big) \\
&= \frac{{p_1}^2 + {p_2}^2}{2}
\end{aligned}
$$

Strategy B: toss both coins

Because the order does not matter here, one can assume to drop the coin C_1 first with the coin C_2 second, then we have the probability of event E in strategy B

$$
\begin{aligned}
\mathbb{P}(E_B) &= \mathbb{P}(\{\text{head}\}_1 \cap \{\text{head}\}_2) \\
&= \mathbb{P}(\{\text{head}\}_1)\mathbb{P}(\{\text{head}\}_2) \\
&= \mathbb{P}_{C_{1_1}}(\{\text{head}\}_1)\mathbb{P}_{C_{2_2}}(\{\text{head}\}_2) \\
&= p_1 p_2
\end{aligned}
$$

Conclusion

By applying AM–GM inequality, we know that

Answer

$\square$