Question 4

> Two players flip a fair coin alternately. The game ends when one gets heads

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1.

Let head and tail denote the head and tail flip outcome respectively, then we have the probability space:

$$\begin{aligned}
&\Omega = \{\text{head}, \text{tail}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \quad \mathbb{P}(\{\text{head}\}) = \mathbb{P}(\{\text{tail}\}) = \frac{1}{2}
\end{aligned}$$

Let $T_i \subset \mathcal{F}$ be the i-th flip outcome:

$$T_i = \{\text{tail}\}, \quad i \in \{1, 2, ..., n-1\}$$

And let $H_n \subset \mathcal{F}$ be the n-th flip outcome:

$$H_n = \{\text{head}\}$$

Then we have:

$$\begin{aligned}
\mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-1}}(H_n) &= \mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-2}}(T_{n-1})\mathbb{P}(H_n) \\
&= \mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-3}}(T_{n-2})\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\
&= ... \\
&= \mathbb{P}(T_1)\mathbb{P}(T_2)...\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\
&= \frac{1}{2^n}
\end{aligned}$$

Answer

- $\mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-1}}(H_n) = \frac{1}{2^n}$

2.

Let ${T'}_{i'} \subset \mathcal{F}$ be the i'-th flip outcome:

$$ {T'}_{i'} = \{\text{tail}\}, \quad i' \in \{1, 2, ..., 2k\}, \quad k \in \mathbb{N} $$

And let ${H'}_{2k+1} \subset \mathcal{F}$ be the $(2k+1)$-th flip outcome, where the game ends:

$$ {H'}_{2k+1} = \{\text{head}\} $$

Let $F \subset \mathcal{F}$ be the event that contains first player wins, then we have:

$$
\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}({H'}_1) + \sum_{k=1}^{\infty} \mathbb{P}_{{T'}_1 \cap {T'}_2 \cap ... \cap {T'}_{2k}}({H'}_{2k+1}) \\
&= \mathbb{P}({H'}_1) + \sum_{k=1}^{\infty} \mathbb{P}({T'}_1)\mathbb{P}({T'}_2)...\mathbb{P}({T'}_{2k})\mathbb{P}({H'}_{2k+1}) \\
&= \sum_{k=1}^{\infty} \frac{1}{2^{2k+1}} + \frac{1}{2} \\
&= \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} (\frac{1}{4})^k \\
&= \frac{1}{2}(\frac{1}{1 - \frac{1}{4}}) \\
&= \frac{2}{3}
\end{aligned}
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Answer

- $\mathbb{P}(F) = \frac{2}{3}$