Question 4

We are given a bowl that contains 6 balls, numbered from 1 to 6. We extract two balls and denote X and Y the numbers on the ball obtained at the first (second) extraction, and $W = \max(X, Y)$ the maximum value obtained. In all scenarios, describe the probability distribution of W.

Part 1: In the first scenario, assume that the extractions are made with replacement.

Solution

By applying the principle of symmetry, easy to define the probability space as

$$
\begin{aligned}
&\Omega = \{1, 2, \dots, 6\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6}
\end{aligned}
$$

Then as given, we have

$$
\begin{aligned}
X, Y: \ &\Omega \to R \\
W &= \max(X, Y) \\
R &= \{1, 2, \dots, 6\}
\end{aligned}
$$

Then we have the probability distribution of W in scenario #1 (denoted as W_1 and p_1)

$$
p_1(w_1) = \begin{cases}
\mathbb{P}(W_1 = 1) = \mathbb{P}(\{(1,1)\}) & = \frac{1}{6^2} = \frac{1}{36}, \quad w_1 = 1 \\
\mathbb{P}(W_1 = 2) = \mathbb{P}(\{(1,2), (2,1), (2,2)\}) & = \frac{3}{36}, \quad w_1 = 2 \\
\mathbb{P}(W_1 = 3) = \mathbb{P}(\{(1,3), (2,3), \dots, (3,3)\}) & = \frac{5}{36}, \quad w_1 = 3 \\
\mathbb{P}(W_1 = 4) = \mathbb{P}(\{(1,4), (2,4), \dots, (4,4)\}) & = \frac{7}{36}, \quad w_1 = 4 \\
\mathbb{P}(W_1 = 5) = \mathbb{P}(\{(1,5), (2,5), \dots, (5,5)\}) & = \frac{9}{36}, \quad w_1 = 5 \\
\mathbb{P}(W_1 = 6) = \mathbb{P}(\{(1,6), (2,6), \dots, (6,6)\}) & = \frac{11}{36}, \quad w_1 = 6
\end{cases}
$$

Or a general formula without cases

$$
p_1(w_1) = \frac{2w_1 - 1}{6^2} = \frac{2w_1 - 1}{36}, \quad w_1 \in \{1, 2, \dots, 6\}
$$

Answer

$$\boxed{p_1(w_1) = \begin{cases} \frac{1}{36}, & w_1 = 1 \\ \frac{3}{36}, & w_1 = 2 \\ \frac{5}{36}, & w_1 = 3 \\ \frac{7}{36}, & w_1 = 4 \\ \frac{9}{36}, & w_1 = 5 \\ \frac{11}{36}, & w_1 = 6 \end{cases}}$$

Part 2: In the second scenario, assume that the extractions are performed without replacement.

Solution

Easy to see that

$$p_2(w_2) = \begin{cases} \mathbb{P}(W_2 = 2) = \mathbb{P}(\{(1,2),(2,1)\}) & = \frac{2}{6 \times 5} = \frac{2}{30}, \quad w_2 = 2 \\ \mathbb{P}(W_2 = 3) = \mathbb{P}(\{(1,3),(2,3),\ldots,(3,2)\}) & = \frac{4}{30}, \quad w_2 = 3 \\ \mathbb{P}(W_2 = 4) = \mathbb{P}(\{(1,4),(2,4),\ldots,(4,3)\}) & = \frac{6}{30}, \quad w_2 = 4 \\ \mathbb{P}(W_2 = 5) = \mathbb{P}(\{(1,5),(2,5),\ldots,(5,4)\}) & = \frac{8}{30}, \quad w_2 = 5 \\ \mathbb{P}(W_2 = 6) = \mathbb{P}(\{(1,6),(2,6),\ldots,(6,5)\}) & = \frac{10}{30}, \quad w_2 = 6 \end{cases}$$

Or a general formula without cases

$$p_2(w_2) = \frac{2w_2 - 2}{6 \times 5} = \frac{2w_2 - 2}{30}, \quad w_2 \in \{2, 3, \ldots, 6\}$$

Answer

$$\boxed{p_2(w_2) = \begin{cases} \dfrac{2}{30}, & w_2 = 2 \\ \dfrac{4}{30}, & w_2 = 3 \\ \dfrac{6}{30}, & w_2 = 4 \\ \dfrac{8}{30}, & w_2 = 5 \\ \dfrac{10}{30}, & w_2 = 6 \end{cases}}$$

Part 3: In the third scenario, assume that after the first extraction, we replace the ball in the urn, together with another one with the same number.

Solution

We know that

$$p_{3_X}(x) = \mathbb{P}(X = x) = \frac{1}{6}, \quad x \in \{1, 2, \ldots, 6\}$$

Then we have

$$\begin{aligned} p_{3_Y}(y) &= \mathbb{P}(Y = y) \\ &= \mathbb{P}(X \neq y)\mathbb{P}_{X \neq y}(Y = y) + \mathbb{P}(X = y)\mathbb{P}_{X=y}(Y = y) \\ &= \frac{5}{6} \times \frac{1}{6 - 1 + 2} + \frac{1}{6} \times \frac{1 + 1}{6 - 1 + 2} \\ &= \frac{1}{6} \end{aligned}$$

$$y \in \{1, 2, \ldots, 6\}$$

With $p_{3_X} = p_{3_Y} = \frac{1}{6}$, easy to notice that scenario #3 is identical as scenario #1, which implies

$$p_3(w_3) = \begin{cases} \mathbb{P}(W_3 = 1) = \mathbb{P}(W_1 = 1) & = \dfrac{1}{36}, \quad w_3 = 1 \\ \mathbb{P}(W_3 = 2) = \mathbb{P}(W_1 = 2) & = \dfrac{3}{36}, \quad w_3 = 2 \\ \mathbb{P}(W_3 = 3) = \mathbb{P}(W_1 = 3) & = \dfrac{5}{36}, \quad w_3 = 3 \\ \mathbb{P}(W_3 = 4) = \mathbb{P}(W_1 = 4) & = \dfrac{7}{36}, \quad w_3 = 4 \\ \mathbb{P}(W_3 = 5) = \mathbb{P}(W_1 = 5) & = \dfrac{9}{36}, \quad w_3 = 5 \\ \mathbb{P}(W_3 = 6) = \mathbb{P}(W_1 = 6) & = \dfrac{11}{36}, \quad w_3 = 6 \end{cases}$$

Or a general formula without cases

$$p_3(w_3) = \frac{2w_3 - 1}{36}, \quad w_3 \in \{1, 2, \ldots, 6\}$$

Answer

$$\boxed{p_3(w_3) = \begin{cases} \dfrac{1}{36}, & w_3 = 1 \\ \dfrac{3}{36}, & w_3 = 2 \\ \dfrac{5}{36}, & w_3 = 3 \\ \dfrac{7}{36}, & w_3 = 4 \\ \dfrac{9}{36}, & w_3 = 5 \\ \dfrac{11}{36}, & w_3 = 6 \end{cases}}$$