Question 4

We are given a bowl that contains 6 balls, numbered from 1 to 6. We extract two balls and denote X and Y the numbers on the ball obtained at the first (second) extraction, and $W = \max(X, Y)$ the maximum value obtained. In all scenarios, describe the probability distribution of W.

Part 1: In the first scenario, assume that the extractions are made with replacement.

Solution

By applying the principle of symmetry, easy to define the probability space as

$$\begin{aligned} &\Omega = \{1, 2, \dots, 6\} \\ &\mathcal{F} = \mathcal{P}(\Omega) \\ &\mathbb{P}: \ \mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{1}{6} \end{aligned}$$

Then as given, we have

$$\begin{aligned} X, Y:&\ \ \Omega \to R \\ W &= \max(X, Y) \\ R &= \{1, 2, \dots, 6\} \end{aligned}$$

Then we have the probability distribution of W in scenario #1 (denoted as W_1 and p_1)

$$\begin{aligned} p_1(1) = \mathbb{P}(W_1 = 1) &= \mathbb{P}(\{(1,1)\}) = \frac{1}{6^2} = \frac{1}{36} \\ p_1(2) = \mathbb{P}(W_1 = 2) &= \mathbb{P}(\{(1,2), (2,1), (2,2)\}) = \frac{3}{36} \\ p_1(3) = \mathbb{P}(W_1 = 3) &= \mathbb{P}(\{(1,3), (2,3), \dots, (3,3)\}) = \frac{5}{36} \\ p_1(4) = \mathbb{P}(W_1 = 4) &= \mathbb{P}(\{(1,4), (2,4), \dots, (4,4)\}) = \frac{7}{36} \\ p_1(5) = \mathbb{P}(W_1 = 5) &= \mathbb{P}(\{(1,5), (2,5), \dots, (5,5)\}) = \frac{9}{36} \\ p_1(6) = \mathbb{P}(W_1 = 6) &= \mathbb{P}(\{(1,6), (2,6), \dots, (6,6)\}) = \frac{11}{36} \end{aligned}$$

Or a general formula without cases

$$p_1(w_1) = \frac{2w_1 - 1}{6^2} = \frac{2w_1 - 1}{36}, \quad w_1 \in \{1, 2, \dots, 6\}$$

Answer

$$\boxed{\text{See above.}}$$

Part 2: In the second scenario, assume that the extractions are performed without replacement.

Solution

Easy to see that

$$\begin{aligned}
p_2(2) = \mathbb{P}(W_2 = 2) &= \mathbb{P}(\{(1,2),(2,1)\}) = \frac{2}{6 \times 5} = \frac{2}{30} \\
p_2(3) = \mathbb{P}(W_2 = 3) &= \mathbb{P}(\{(1,3),(2,3),\dots,(3,2)\}) = \frac{4}{30} \\
p_2(4) = \mathbb{P}(W_2 = 4) &= \mathbb{P}(\{(1,4),(2,4),\dots,(4,3)\}) = \frac{6}{30} \\
p_2(5) = \mathbb{P}(W_2 = 5) &= \mathbb{P}(\{(1,5),(2,5),\dots,(5,4)\}) = \frac{8}{30} \\
p_2(6) = \mathbb{P}(W_2 = 6) &= \mathbb{P}(\{(1,6),(2,6),\dots,(6,5)\}) = \frac{10}{30}
\end{aligned}$$

Or a general formula without cases

$$p_2(w_2) = \frac{2w_2 - 2}{6 \times 5} = \frac{2w_2 - 2}{30}, \quad w_2 \in \{2, 3, \dots, 6\}$$

Answer

$$\boxed{\text{See above.}}$$

Part 3: In the third scenario, assume that after the first extraction, we replace the ball in the urn, together with another one with the same number.

Solution

We know that

$$p_{3_X}(x) = \mathbb{P}(X = x) = \frac{1}{6}, \quad x \in \{1, 2, \dots, 6\}$$

Then we have

$$
\begin{aligned}
p_{3_Y}(y) &= \mathbb{P}(Y = y) \\
&= \mathbb{P}(X \neq y)\mathbb{P}_{X \neq y}(Y = y) + \mathbb{P}(X = y)\mathbb{P}_{X=y}(Y = y) \\
&= \frac{5}{6} \times \frac{1}{6 - 1 + 2} + \frac{1}{6} \times \frac{1 + 1}{6 - 1 + 2} \\
&= \frac{1}{6}
\end{aligned}
$$

$$
y \in \{1, 2, \dots, 6\}
$$

With $p_{3_X} = p_{3_Y} = \frac{1}{6}$, easy to notice that scenario #3 is identical as scenario #1, which implies

$$
\begin{aligned}
p_3(1) &= \frac{1}{36} \\
p_3(2) &= \frac{3}{36} \\
p_3(3) &= \frac{5}{36} \\
p_3(4) &= \frac{7}{36} \\
p_3(5) &= \frac{9}{36} \\
p_3(6) &= \frac{11}{36}
\end{aligned}
$$

Or a general formula without cases

$$
p_3(w_3) = \frac{2w_3 - 1}{36}, \quad w_3 \in \{1, 2, \dots, 6\}
$$

Answer

$$
\boxed{\text{See above.}}
$$