Question 4

Two players flip a fair coin alternately. The game ends when one gets heads

1.

Let head and tail denote the head and tail flip outcome respectively, then we have the probability space:

$$egin{aligned} \Omega &= \{ ext{head}, ext{tail}\} \ \mathcal{F} &= \mathcal{P}(\Omega) \ \\ \mathbb{P} : & \mathbb{P}(\{ ext{head}\}) = \mathbb{P}(\{ ext{tail}\}) = rac{1}{2} \end{aligned}$$

Let $T_i \subset \mathcal{F}$ be the i-th flip outcome:

$$T_i = \{ ext{tail}\}, \quad i \in \{1, 2, ..., n-1\}$$

And let $H_n\subset \mathcal{F}$ be the n-th flip outcome:

$$H_n = \{\text{head}\}$$

Then we have:

$$egin{aligned} \mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-1}}(H_n) &= \mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-2}}(T_{n-1}) \mathbb{P}(H_n) \ &= \mathbb{P}_{T_1 \cap T_2 \cap ... \cap T_{n-3}}(T_{n-2}) \mathbb{P}(T_{n-1}) \mathbb{P}(H_n) \ &= ... \ &= \mathbb{P}(T_1) \mathbb{P}(T_2) ... \mathbb{P}(T_{n-1}) \mathbb{P}(H_n) \ &= rac{1}{2^n} \end{aligned}$$

Answer

ullet $\mathbb{P}_{T_1\cap T_2\cap...\cap T_{n-1}}(H_n)=rac{1}{2^n}$

Let ${T'}_{i'}\subset \mathcal{F}$ be:

$$T'{}_{i'}=\{ ail\},\quad i'\in\{1,2,...,2k\},\quad k\geq 1\land k\in\mathbb{N}$$

And let $H'\subset \mathcal{F}$ be:

$${H'}_1={H'}_{2k+1}=\{\mathrm{head}\},\quad k\geq 1\land k\in\mathbb{N}$$

Let $F\subset \mathcal{F}$ be the event that contains first player wins, then we have:

$$\begin{split} \mathbb{P}(F) &= \mathbb{P}(H'_1) + \sum_{k=1}^{\infty} \mathbb{P}_{T'_1 \cap T'_2 \cap ... \cap T'_{2k}}(H'_{2k+1}) \\ &= \mathbb{P}(H'_1) + \sum_{k=1}^{\infty} \mathbb{P}(T'_1) \mathbb{P}(T'_2) ... \mathbb{P}(T'_{2k}) \mathbb{P}(H'_{2k+1}) \\ &= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2^{2k+1}} \\ &= \sum_{k=0}^{\infty} (\frac{1}{2})^{2k+1} \\ &= \frac{1}{2} \sum_{k=0}^{\infty} (\frac{1}{4})^k \\ &= \frac{1}{2} (\frac{1}{1 - \frac{1}{4}}) \\ &= \frac{2}{3} \end{split}$$

Answer

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$$\mathbb{P}(F) = \frac{2}{3}$$