Question 3

Alice proposes to Bob the following bet. Alice tosses a fair coin n times, and computes the number of heads X. Bob tosses the coin $n+1$ times, and obtain Y heads. Bob wins the bet if $Y > X$.

Part 1: Is the bet fair?

Solution

Let *head* and *tail* denote the results of a drop. By applying the principle of symmetry, define the probability space as

$$\begin{aligned}
&\Omega = \{\text{head, tail}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \ \mathbb{P}(\{\text{head}\}) = \mathbb{P}(\{\text{tail}\}) = \frac{1}{2}
\end{aligned}$$

Then we have the expected values of both X (E[X]) and Y (E[Y])

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} \frac{1}{2} &&= \frac{n}{2} \\
E[Y] &= \sum_{i=1}^{n+1} \frac{1}{2} &&= \frac{n+1}{2}
\end{aligned}$$

Easy to see that

$$\begin{gathered}
E[Y] = E[X] + \frac{1}{2} \\
\downarrow \\
E[Y] > E[X] \\
\downarrow \\
\mathbb{P}(Y > X) > \frac{1}{2}
\end{gathered}$$

Then we say that the bet is unfair, with Bob has a higher probability to win.

Answer

$$\boxed{\text{Unfair, Bob tends to win more.}}$$

Part 2: Compute the answer for a general coin.

Solution

Similarly, by denoting both X and Y in general case as X_g and Y_g with $\mathbb{P}(\{\text{head}\}) = p$, we have

$$\begin{aligned} E[X_g] &= \sum_{i=1}^{n} p &&= np \\ E[Y_g] &= \sum_{i=1}^{n+1} p &&= (n+1)p \end{aligned}$$

Easy to see that

$$\begin{gathered} E[Y] = E[X] + p, \quad p > 0 \\ \downarrow \\ E[Y] > E[X] \\ \downarrow \\ \mathbb{P}(Y > X) > \frac{1}{2} \end{gathered}$$

Then we say that the bet is unfair, with Bob has a higher probability to win.

Answer

$$\boxed{\text{Unfair, Bob tends to win more.}}$$