

Question 4

Two players flip a fair coin alternately. The game ends when one gets heads

1.

Let head and tail denote the head and tail flip outcome respectively, then we have the probability space:

$$\Omega = \{\text{head}, \text{tail}\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

$$\mathbb{P} : \mathbb{P}(\{\text{head}\}) = \mathbb{P}(\{\text{tail}\}) = \frac{1}{2}$$

Let $T_i \subset \mathcal{F}$ be the i -th flip outcome:

$$T_i = \{\text{tail}\}, \quad i \in \{1, 2, \dots, n-1\}$$

And let $H_n \subset \mathcal{F}$ be the n -th flip outcome:

$$H_n = \{\text{head}\}$$

Then we have:

$$\begin{aligned} \mathbb{P}_{T_1 \cap T_2 \cap \dots \cap T_{n-1}}(H_n) &= \mathbb{P}_{T_1 \cap T_2 \cap \dots \cap T_{n-2}}(T_{n-1})\mathbb{P}(H_n) \\ &= \mathbb{P}_{T_1 \cap T_2 \cap \dots \cap T_{n-3}}(T_{n-2})\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\ &= \dots \\ &= \mathbb{P}(T_1)\mathbb{P}(T_2)\dots\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\ &= \frac{1}{2^n} \end{aligned}$$

Answer

- $\mathbb{P}_{T_1 \cap T_2 \cap \dots \cap T_{n-1}}(H_n) = \frac{1}{2^n}$

2.

Let $T'_{i'} \subset \mathcal{F}$ be the i' -th flip outcome:

$$T'_{i'} = \{\text{tail}\}, \quad i' \in \{1, 2, \dots, 2k\}, \quad k \in \mathbb{N}$$

And let $H'_{2k+1} \subset \mathcal{F}$ be the $(2k+1)$ -th flip outcome, where the game ends:

$$H'_{2k+1} = \{\text{head}\}$$

Let $F \subset \mathcal{F}$ be the event that contains first player wins, then we have:

$$\begin{aligned} \mathbb{P}(F) &= \sum_{k=1}^{\infty} \mathbb{P}_{T'_1 \cap T'_2 \cap \dots \cap T'_{2k}}(H'_{2k+1}) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(T'_1) \mathbb{P}(T'_2) \dots \mathbb{P}(T'_{2k}) \mathbb{P}(H'_{2k+1}) \\ &= \sum_{k=1}^{\infty} \frac{1}{2^{2k+1}} \\ &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k \\ &= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}}\right) \\ &= \frac{2}{3} \end{aligned}$$

Answer

- $\mathbb{P}(F) = \frac{2}{3}$
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