Question 4

> Two players flip a fair coin alternately. The game ends when one gets heads

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1.

Let head and tail denote the head and tail flip outcome respectively, then we have the probability space:

$$\begin{aligned}
&\Omega = \{\mathrm{head}, \mathrm{tail}\} \\
&\mathcal{F} = \mathcal{P}(\Omega) \\
&\mathbb{P}: \quad \mathbb{P}(\{\mathrm{head}\}) = \mathbb{P}(\{\mathrm{tail}\}) = \frac{1}{2}
\end{aligned}$$

Let $T_i \subset \mathcal{F}$ be the i-th flip outcome:

$$T_i = \{\mathrm{tail}\}, \quad i \in \{1, 2, ..., n-1\}$$

And let $H_n \subset \mathcal{F}$ be the n-th flip outcome:

$$H_n = \{\mathrm{head}\}$$

Then we have:

$$\begin{aligned}
\mathbb{P}(T_1 \cap T_2 \cap ... \cap T_{n-1} \cap H_n) &= \mathbb{P}(T_1 \cap T_2 \cap ... \cap T_{n-2} \cap T_{n-1})\mathbb{P}(H_n) \\
&= \mathbb{P}(T_1 \cap T_2 \cap ... \cap T_{n-3} \cap T_{n-2})\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\
&= ... \\
&= \mathbb{P}(T_1)\mathbb{P}(T_2)...\mathbb{P}(T_{n-1})\mathbb{P}(H_n) \\
&= \frac{1}{2^n}
\end{aligned}$$

Answer

- $\mathbb{P}(T_1 \cap T_2 \cap ... \cap T_{n-1} \cap H_n) = \frac{1}{2^n}$

2.

Let ${T'}_{i'} \subset \mathcal{F}$ be:

$${T'}_{i'} = \{\text{tail}\}, \quad i' \in \{1, 2, ..., 2k\}, \quad k \geq 1 \land k \in \mathbb{N}$$

And let $H' \subset \mathcal{F}$ be:

$${H'}_1 = {H'}_{2k+1} = \{\text{head}\}, \quad k \geq 1 \land k \in \mathbb{N}$$

Let $F \subset \mathcal{F}$ be the event that contains first player wins, then we have:

$$\begin{aligned}
\mathbb{P}(F) &= \mathbb{P}({H'}_1) + \sum_{k=1}^{\infty} \mathbb{P}({T'}_1 \cap {T'}_2 \cap ... \cap {T'}_{2k} \cap {H'}_{2k+1}) \\
&= \mathbb{P}({H'}_1) + \sum_{k=1}^{\infty} \mathbb{P}({T'}_1)\mathbb{P}({T'}_2)...\mathbb{P}({T'}_{2k})\mathbb{P}({H'}_{2k+1}) \\
&= \frac{1}{2} \qquad + \sum_{k=1}^{\infty} \frac{1}{2^{2k+1}} \\
&= \quad \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2k+1} \\
&= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^{k} \\
&= \frac{1}{2} \left(\frac{1}{1 - \frac{1}{4}}\right) \\
&= \frac{2}{3}
\end{aligned}$$

Answer

- $\mathbb{P}(F) = \frac{2}{3}$