Question 3

Alice proposes to Bob the following bet. Alice tosses a fair coin n times, and computes the number of heads X. Bob tosses the coin $n+1$ times, and obtain Y heads. Bob wins the bet if $Y > X$.

Part 1: Is the bet fair?

Solution

As given, we can define the Bernoulli distribution with sample space $\Omega = \{\text{head}, \text{tail}\}$ (*head* denotes *drop headed* and *tail* denotes *drop tailed*) and with parameter $p = 0.5$ (by applying the principle of symmetry), then we can define the Binomial distribution of both X and Y as

$$\begin{aligned} X \sim & \text{Bin}(n, p) \\ Y \sim & \text{Bin}(n+1, p) \end{aligned}$$

Let as define $Z = Y - X$, then we have

$$Z \sim \text{Bin}(1, p)$$

Then we have the distribution of Z

$$p(z) = \begin{cases} 0.5, & z = 0 \\ 0.5, & z = 1 \end{cases}$$

That is

$$\begin{aligned} \mathbb{P}(Y > X) &= \mathbb{P}(Z > 0) \\ &= \frac{p(1)}{p(0) + p(1)} \\ &= \frac{1}{2} \end{aligned}$$

Answer

$$\boxed{\text{Fair.}}$$

Part 2: Compute the answer for a general coin.

Solution

Here we let p_g denote a random variable in $(0,1)$, then we have the Z in general case

$$Z_g \sim \mathrm{Bin}(1, p_g)$$

Then we have the distribution of Z_g

$$p_g(z) = \begin{cases} 1 - p_g, & z = 0 \\ p_g, & z = 1 \end{cases}$$

By denoting X and Y in general case as X_g and Y_g, we have

$$\begin{aligned} \mathbb{P}(Y_g > X_g) &= \mathbb{P}(Z_g > 0) \\ &= \frac{p_g(1)}{p_g(0) + p_g(1)} \\ &= p_g \end{aligned}$$

That is

$$\text{Bet is} \begin{cases} \text{Unfair, Alice tends to win more,} & p_g < 0.5 \\ \text{Fair,} & p_g = 0.5 \\ \text{Unfair, Bob tends to win more,} & p_g > 0.5 \end{cases}$$

Answer

$$\boxed{\text{Bet is} \begin{cases} \text{Unfair, Alice tends to win more,} & p_g < 0.5 \\ \text{Fair,} & p_g = 0.5 \\ \text{Unfair, Bob tends to win more,} & p_g > 0.5 \end{cases}}$$