

Question 4

We are given a bowl that contains 6 balls, numbered from 1 to 6. We extract two balls and denote X and Y the numbers on the ball obtained at the first (second) extraction, and $W = \max(X, Y)$ the maximum value obtained. In all scenarios, describe the probability distribution of W .

Part 1: In the first scenario, assume that the extractions are made with replacement.

Solution

By applying the principle of symmetry, easy to define the probability space as

$$\begin{aligned}\Omega &= \{1, 2, \dots, 6\} \\ \mathcal{F} &= \mathcal{P}(\Omega) \\ \mathbb{P}: \quad \mathbb{P}(\{1\}) &= \mathbb{P}(\{2\}) = \dots = \mathbb{P}(\{6\}) = \frac{1}{6}\end{aligned}$$

Then as given, we have

$$\begin{aligned}X, Y &: \Omega \rightarrow R \\ W &= \max(X, Y) \\ R &= \{1, 2, \dots, 6\}\end{aligned}$$

Then we have the probability distribution of W in scenario #1 (denoted as W_1 and p_1)

$$p_1(w_1) = \begin{cases} \mathbb{P}(W_1 = 1) = \mathbb{P}(\{(1, 1)\}) & = \frac{1}{6^2} = \frac{1}{36}, & w_1 = 1 \\ \mathbb{P}(W_1 = 2) = \mathbb{P}(\{(1, 2), (2, 1), (2, 2)\}) & = \frac{3}{36}, & w_1 = 2 \\ \mathbb{P}(W_1 = 3) = \mathbb{P}(\{(1, 3), (2, 3), \dots, (3, 3)\}) & = \frac{5}{36}, & w_1 = 3 \\ \mathbb{P}(W_1 = 4) = \mathbb{P}(\{(1, 4), (2, 4), \dots, (4, 4)\}) & = \frac{7}{36}, & w_1 = 4 \\ \mathbb{P}(W_1 = 5) = \mathbb{P}(\{(1, 5), (2, 5), \dots, (5, 5)\}) & = \frac{9}{36}, & w_1 = 5 \\ \mathbb{P}(W_1 = 6) = \mathbb{P}(\{(1, 6), (2, 6), \dots, (6, 6)\}) & = \frac{11}{36}, & w_1 = 6 \end{cases}$$

Or a general formula without cases

$$p_1(w_1) = \frac{2w_1 - 1}{6^2} = \frac{2w_1 - 1}{36}, \quad w_1 \in \{1, 2, \dots, 6\}$$

Answer

$$p_1(w_1) = \begin{cases} \frac{1}{36}, & w_1 = 1 \\ \frac{3}{36}, & w_1 = 2 \\ \frac{5}{36}, & w_1 = 3 \\ \frac{7}{36}, & w_1 = 4 \\ \frac{9}{36}, & w_1 = 5 \\ \frac{11}{36}, & w_1 = 6 \end{cases}$$

Part 2: In the second scenario, assume that the extractions are performed without replacement.

Solution

Easy to see that

$$p_2(w_2) = \begin{cases} \mathbb{P}(W_2 = 2) = \mathbb{P}(\{(1, 2), (2, 1)\}) & = \frac{2}{6 \times 5} = \frac{1}{15}, & w_2 = 2 \\ \mathbb{P}(W_2 = 3) = \mathbb{P}(\{(1, 3), (2, 3), \dots, (3, 2)\}) & = \frac{2}{15}, & w_2 = 3 \\ \mathbb{P}(W_2 = 4) = \mathbb{P}(\{(1, 4), (2, 4), \dots, (4, 3)\}) & = \frac{3}{15}, & w_2 = 4 \\ \mathbb{P}(W_2 = 5) = \mathbb{P}(\{(1, 5), (2, 5), \dots, (5, 4)\}) & = \frac{4}{15}, & w_2 = 5 \\ \mathbb{P}(W_2 = 6) = \mathbb{P}(\{(1, 6), (2, 6), \dots, (6, 5)\}) & = \frac{5}{15}, & w_2 = 6 \end{cases}$$

Or a general formula without cases

$$p_2(w_2) = \frac{2w_2 - 2}{6 \times 5} = \frac{w_2 - 1}{15}, \quad w_2 \in \{2, 3, \dots, 6\}$$

Answer

$$p_2(w_2) = \begin{cases} \frac{1}{15}, & w_2 = 2 \\ \frac{2}{15}, & w_2 = 3 \\ \frac{3}{15}, & w_2 = 4 \\ \frac{4}{15}, & w_2 = 5 \\ \frac{5}{15}, & w_2 = 6 \end{cases}$$

Part 3: In the third scenario, assume that after the first extraction, we replace the ball in the urn, together with another one with the same number.

Solution

Easy to see that

$$p_3(w_3) = \begin{cases} \mathbb{P}(W_3 = 1) = \mathbb{P}(\{(1, 1)\}) & = \frac{1}{6} \times \frac{2}{7} = \frac{1}{21}, & w_3 = 1 \\ \mathbb{P}(W_3 = 2) = \mathbb{P}(\{(1, 2), (2, 1), (2, 2)\}) & = \frac{2}{42} + 2 \left(\frac{1}{6} \times \frac{1}{7} \right) = \frac{2}{21}, & w_3 = 2 \\ \mathbb{P}(W_3 = 3) = \mathbb{P}(\{(1, 3), (2, 3), \dots, (3, 3)\}) & = \frac{2}{42} + 4 \left(\frac{1}{42} \right) = \frac{3}{21}, & w_3 = 3 \\ \mathbb{P}(W_3 = 4) = \mathbb{P}(\{(1, 4), (2, 4), \dots, (4, 4)\}) & = \frac{2}{42} + 6 \left(\frac{1}{42} \right) = \frac{4}{21}, & w_3 = 4 \\ \mathbb{P}(W_3 = 5) = \mathbb{P}(\{(1, 5), (2, 5), \dots, (5, 5)\}) & = \frac{2}{42} + 8 \left(\frac{1}{42} \right) = \frac{5}{21}, & w_3 = 5 \\ \mathbb{P}(W_3 = 6) = \mathbb{P}(\{(1, 6), (2, 6), \dots, (6, 6)\}) & = \frac{2}{42} + 10 \left(\frac{1}{42} \right) = \frac{6}{21}, & w_3 = 6 \end{cases}$$

Or a general formula without cases

$$p_3(w_3) = \frac{w_3}{21}, \quad w_3 \in \{1, 2, \dots, 6\}$$

Answer

$$p_3(w_3) = \begin{cases} \frac{1}{21}, & w_3 = 1 \\ \frac{2}{21}, & w_3 = 2 \\ \frac{3}{21}, & w_3 = 3 \\ \frac{4}{21}, & w_3 = 4 \\ \frac{5}{21}, & w_3 = 5 \\ \frac{6}{21}, & w_3 = 6 \end{cases}$$