Coxeter frieze patterns

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Chapter 1

Field-valued patterns

Throughout this document, we will study frieze patterns of finite *height*. This terminology (as compared to finite *width* or finite *order*) is unconventional but more convenient for formalisation. For us, the height of a frieze pattern corresponds to the number of rows, including the rows of ones but *excluding* the rows of zeros. Throughout this section, we fix an arbitrary field F.

Definition 1.1. *Fix $n \in \mathbb{N}^*$. A map $f : \{0, 1, \dots, n, n+1\} \times \mathbb{Z} \longrightarrow F$ is called an F-valued pattern of height n if,*
1) for all $m \in \mathbb{Z}$, $f(0, m) = f(n + 1, m) = 0$,
2) for all $m \in \mathbb{Z}$, $f(1, m) = f(n, m) = 1$, and
3) for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have

$$f(i, m)f(i, m + 1) = 1 + f(i + 1, m)f(i - 1, m + 1). \tag{1.1}$$

An F-valued pattern f of height n is said to be *nowhere zero* if $f(i, m) \neq 0$, for all $i \in \{1, \dots, n\}$ and for all $m \in \mathbb{Z}$.

Lemma 1.2. *Let f be a nowhere-zero F-valued pattern of height n. For all m, we have*

$$f(i + 2, m) = f(2, m + i)f(i + 1, m) - f(i, m), \qquad i \in \{0, \dots, n - 1\}$$
$$f(i, m) = f(n - 1, m)f(i + 1, m - 1) - f(i + 2, m - 2), \qquad i \in \{0, n - 1\}.$$

Proof. We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i + 2, m) = f(2, m + i + 1)f(i + 1, m) - f(i, m),$$

for $i \in \{0, \dots, n - 1\}$. We do so by induction on i.

Base case P_0: We have that for all $m \in \mathbb{Z}$, $f(2, m)f(1, m) - f(0, m) = f(2, m + 1) * 1 - 0 = f(2, m)$.

Inductive hypothesis. Suppose that our claim holds for some $i \in \{0, \dots, n - 2\}$ fixed. Then,

$$\begin{aligned} f(i + 3, m)f(i + 1, m + 1) &= f(i + 2, m)f(i + 2, m + 1) - 1 \\ &= f(i + 2, m)(f(2, m + i + 1)f(i + 1, m + 1) - f(i, m + 1)) - 1 \\ &= f(i + 2, m)f(2, m + i + 1)f(i + 1, m + 1) - (f(i + 2, m)f(i, m + 1) + 1) \\ &= f(i + 2, m)f(2, m + i + 1)f(i + 1, m + 1) - f(i + 1, m)f(i + 1, m + 1). \end{aligned}$$

Since f is nowhere-zero, we may divide both sides of the equation by $f(i + 1, m + 1)$ to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i,m) = f(n-1,m)f(i+1,m-1) - f(i+2,m-2),$$

by induction on i, starting with $i = n-1$ and proving the inductive step $Q_i \Rightarrow Q_{i-1}$.

Base case Q_{n-1}: for all $m \in \mathbb{Z}$, $f(n-1,m)f(n,m-1)-f(n+1,m-2) = f(n-1,m)*1-0 = f(n-1,m)$.

Inductive hypothesis. Suppose that Q_{i+1} holds for some fixed $i \in \{0,\dots,n-2\}$. Then,

$$\begin{aligned} f(i,m)f(i+2,m-1) &= f(i+1,m-1)f(i+1,m) - 1 \\ &= f(i+1,m-1)(f(i+2,m-1)f(n-1,m) - f(i+3,m-2)) - 1 \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - (f(i+1,m-1)f(i+3,m-2)+1) \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - f(i+2,m-2)f(i+2,m-1). \end{aligned}$$

Again since f is nowhere-zero, dividing by $f(i+2,m-1)$ on both sides we obtain Q_i. □

Proposition 1.3. *Let f be a nowhere-zero F-valued pattern of height n. Then, for all $m \in \mathbb{Z}$ and all $i \in \{0,\dots,n+1\}$, we have*

$$f(i,m) = f(i,m+n+1).$$

Proof. We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map $\rho_n : \{0,1,\dots,n+1\} \times \mathbb{Z} \longrightarrow \{0,1,\dots,n+1\} \times \mathbb{Z}$ given by

$$\rho_n(i,m) = (n+1-i, m+i). \tag{1.2}$$

We show that every nowhere-zero F-valued pattern of height n is ρ_n-invariant, i.e. satisfies

$$f(\rho_n(i,m)) = f(i,m), \qquad \forall (i,m) \in \{0,1,\dots,n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that $\rho_n^2 : (i,m) \mapsto (i, m+n+1)$. Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i,m) = f(n+1-i, m+i),$$

where $i \in \{0,\dots,n+1\}$. To prove that P_i holds for all i, it is sufficient to prove that P_0, P_1 hold, and that $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$.

P_0 : for all $m \in \mathbb{Z}$, $f(0,m) = 0 = f(n+1,m)$.

P_1 : for all $m \in \mathbb{Z}$, $f(1,m) = 1 = f(n,m+1)$.

Now suppose we are given $i \in \{0,1,\dots,n-1\}$ such that P_i and P_{i+1} hold. Then, for any fixed $m \in \mathbb{Z}$, we have

$$\begin{aligned} f(i+2,m) &= f(2,m+i)f(i+1,m) - f(i,m) \\ &= f(n-1,m+i+2)f(n-i,m+i+1) - f(n+1-i,m+i) \\ &= f(n-i-1,m+i+2). \end{aligned}$$

□

Corollary 1.4. *Let f be a nowhere-zero F-valued pattern of height n. Then,* $\mathrm{Im}(f) := \{f(i,m) : i \in \{1,\dots,n\}, m \in \mathbb{Z}\}$ *is a finite set.*

Proof. Consider the finite set $\mathcal{D} = \{(i,m) : i \in \{1,\dots,n\}, m \in \{0,\dots,n\}\}$. By Proposition 1.3,

$$\mathrm{Im}(f) = \{f(i,m) : (i,m) \in \mathcal{D}\},$$

and the right-hand side is obviously finite. □

Chapter 2

n-Diagonals

Definition 2.1. *Let n be a positive integer. By an n-diagonal, we mean any n-tuple $(a_1, \dots, a_n)$ of positive integers such that $a_1 = a_n$ and*

$$a_i \mid a_{i-1} + a_{i+1}, \qquad i \in \{2, \dots, n-1\}.$$

We denote by $\mathrm{Diag}(n)$ *the set of n-diagonals.*

We prove a couple of key properties about n-diagonals.

Lemma 2.2. *Fix a positive integer n. The set Diag(n) is non-empty.*

Proof. The n-tuple $(1, 1, \dots, 1)$ consisting entirely of 1s is clearly an n-diagonal. □

Definition 2.3. *The Fibonacci sequence $(F_n)_{n\in\mathbb{N}}$ is defined by $F_0 = 0, F_1 = 1$ and the recursive formula*

$$F_n = F_{n-1} + F_{n-2}.$$

The following n-diagonals will play an important role for us.

Lemma 2.4. *1) If n is odd, the n-tuple*

$$(F_2, F_4, F_6, \dots, F_{n-1}, F_n, F_{n-2}, F_{n-4}, \dots, F_5, F_3, F_1),$$

is an n-diagonal.

2) If n is even, the n-tuple

$$(F_2, F_4, F_6, \dots, F_{n-2}, F_n, F_{n-1}, F_{n-3}, \dots, F_5, F_3, F_1),$$

is an n-diagonal.

Proof. These are a tedious but straightforward calculation. □

Lemma 2.5. *In an n-diagonal $(a_1, \dots, a_n)$, there exists an index $i \in \{2, \dots, n-1\}$ such that $a_i = a_{i-1} + a_{i+1}$.*

Proof. □

Proposition 2.6. *Fix a positive integer n, and let $(a_1, \dots, a_n) \in \mathrm{Diag}(n)$. For any $i \in \{1, \dots, n\}$, we have $a_i \leq F_n$.*

Proof. By induction on n, and using Lemma 1.9. □

Chapter 3

Maximal values of arithmetic frieze patterns

Definition 3.1. *A $\mathbb{Q}$-valued pattern of height n is said to be an arithmetic frieze pattern (of height n) if it takes values in $\mathbb{Z}_{>0}$. We denote by Frieze(n) the set of arithmetic frieze patterns of height n.*

The following proposition is a key result connecting arithmetic frieze patterns to n-diagonals.

Proposition 3.2. *1) Let f be an arithmetic frieze pattern of height n. For all $m \in \mathbb{Z}$, the n-tuple*

$$(f(1, m), f(2, m), \dots, f(n, m))$$

is an n-diagonal.

2) Given an n-diagonal $(a_1, \dots, a_n)$, there exists a arithmetic frieze pattern f of height n such that

$$(f(1, 0), \dots, f(n, 0)) = (a_1, \dots, a_n).$$

Proof. □

Corollary 3.3. *Fix a positive integer n. The set Frieze(n) is non-empty.*

Proof. The proof of Lemma 1.6 showed that $(1, 1, \dots, 1)$ is an n-diagonal. The claim then follows from 2) of Proposition 2.2. □

Corollary 3.4. *For each n, there is a well-defined positive integer, called the maximum value among arithmetic frieze patterns of height n, defined by*

$$u_n := \max(f(i, m) : f \in \mathrm{Frieze}(n), i \in \{1, \dots, n\}, m \in \mathbb{Z}).$$

Proof. □

We are now able to formulate and prove the main theorem of this section.

Theorem 3.5. *For all $n \geq 1$, we have*

$$u_n = F_n.$$

Proof. By Propositions 2.2 and 1.10, $u_n \leq F_n$ for all n. On the other hand, Lemma 1.8 and 2) of Proposition 2.2 show that there exists an arithmetic frieze pattern of height n containing F_n as a value. □