

Coxeter frieze patterns

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July 16, 2024

Chapter 1

Symmetries of frieze patterns

Definition 1.1. Fix $n \in \mathbb{N}$. A map $f : \{1, 2, \dots, n\} \times \mathbb{Z} \rightarrow \mathbb{R}$ is called a frieze pattern of width n if, for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have

$$f(i, m)f(i, m+1) = 1 + f(i+1, m)f(i-1, m+1),$$

where by convention we set $f(0, m) = f(n+1, m) = 1$ for all $m \in \mathbb{Z}$.

Consider the map $\rho_n : \{1, 2, \dots, n\} \times \mathbb{Z} \rightarrow \{1, 2, \dots, n\} \times \mathbb{Z}$ given by

$$\rho_n(i, m) = (n+1-i, m+i+1). \quad (1.1)$$

Note that $\rho_n^2 : (i, m) \mapsto (i, m+n+3)$, and therefore ρ_n is bijective. The main result in this section is the so-called *glide symmetry* of frieze patterns.

Proposition 1.2. Every frieze pattern of width n is ρ_n -invariant, i.e. satisfies

$$f(\rho_n(i, m)) = f(i, m), \quad \forall (i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}.$$

The remainder of this section is dedicated to the proof of Proposition [1.0.2](#).

Chapter 2

Classification of arithmetic frieze patterns

Definition 2.1. *A frieze pattern of width n is said to be arithmetic if it takes values in $\mathbb{Z}_{>0}$.*

We denote by $\text{Frieze}(n)$ the set of all arithmetic frieze patterns of width n . The main result in this section is Conway and Coxeter's enumeration of arithmetic frieze patterns.

Theorem 2.2. *Fix $n \in \mathbb{N}$. There is a set-theoretic bijection from the set of triangulations of the regular $n + 3$ -gon in the plane and the set $\text{Frieze}(n)$. In particular we have*

$$|\text{Frieze}(n)| = C_{n+1},$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n^{th} Catalan number.

The remainder of this section is dedicated to the proof of Theorem [2.0.2](#).