Coxeter frieze patterns

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Chapter 1

Background on frieze patterns

1.1 Field-valued patterns

Throughout this section, we fix an arbitrary field F.

Definition 1.1. *Fix $n \in \mathbb{N}$. A map $f : \{1, 2, \dots, n\} \times \mathbb{Z} \longrightarrow F$ is called an F-valued pattern of width n if, for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have*

$$f(i, m)f(i, m + 1) = 1 + f(i + 1, m)f(i - 1, m + 1), \tag{1.1}$$

where by convention we set $f(0, m) = f(n + 1, m) = 1$ for all $m \in \mathbb{Z}$.

In a frieze pattern of width n, we may assume without loss of generality that (1.1) also holds for $i = 0$ and $i = n + 1$, by setting $f(-1, m) = f(n + 2, m) = 0$ for all $m \in \mathbb{Z}$.

An F-valued pattern f of width n is said to be *nowhere zero* if $f(i, m) \neq 0$, for all $i \in \{1, \dots, n\}$ and for all $m \in \mathbb{Z}$.

Lemma 1.2. *Let f be a nowhere-zero F-valued pattern of width n. For all m, we have*

$$\begin{aligned} f(i, m) &= f(1, m + i - 1)f(i - 1, m) - f(i - 2, m), \qquad i \in \{1, \dots, n + 2\} \\ f(i, m) &= f(n, m)f(i + 1, m - 1) - f(i + 2, m - 2), \qquad i \in \{-1, n\}. \end{aligned}$$

Proof. We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i, m) = f(1, m + i - 1)f(i - 1, m) - f(i - 2, m),$$

for $i \in \{1, \dots, n + 2\}$. We do so by induction on i.

Base case P_1: We have that for all $m \in \mathbb{Z}$, $f(1 - 1, m)f(1, m + 1 - 1) - f(1 - 2, m) = 1 * f(1, m) - 0 = f(1, m)$.

Inductive hypothesis. Suppose that our claim holds for some $i \in \{1, \dots, n + 1\}$ fixed. Then,

$$\begin{aligned} f(i + 1, m)f(i - 1, m + 1) &= f(i, m)f(i, m + 1) - 1 \\ &= f(i, m)(f(i - 1, m + 1)f(1, m + i) - f(i - 2, m + 1)) - 1 \\ &= f(i, m)f(1, m + (i + 1) - 1)f(i - 1, m + 1) - (f(i, m)f(i - 2, m + 1) + 1) \\ &= f(i, m)f(1, m + (i + 1) - 1)f(i - 1, m + 1) - f(i - 1, m)f(i - 1, m + 1). \end{aligned}$$

Since f is nowhere-zero, we may divide both sides of the equation by $f(i - 1, m + 1)$ to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i,m) = f(n,m)f(i+1,m-1) - f(i+2,m-2),$$

by induction on i, starting with $i = n$ and proving the inductive step $Q_i \Rightarrow Q_{i-1}$.

Base case Q_n: for all $m \in \mathbb{Z}$, $f(n,m) = 1 * f(n,m) - 0 = f(n,m)f(n+1,m-1) - f(n+2,m-2)$.

Inductive hypothesis. Suppose that Q_{i+1} holds for some fixed $i \in \{-1, \dots, n-1\}$. Then,

$$\begin{aligned} f(i,m)f(i+2,m-1) &= f(i+1,m-1)f(i+1,m) - 1 \\ &= f(i+1,m-1)(f(i+2,m-1)f(n,m) - f(i+3,m-2)) - 1 \\ &= f(i+1,m-1)f(n,m)f(i+2,m-1) - (f(i+1,m-1)f(i+3,m-2) + 1) \\ &= f(i+1,m-1)f(n,m)f(i+2,m-1) - f(i+2,m-2)f(i+2,m-1). \end{aligned}$$

Again since f is nowhere-zero, dividing by $f(i+2,m-1)$ on both sides we obtain Q_i. □

Proposition 1.3. *Let f be a nowhere-zero F-valued pattern of width n. Then, for all i, m, we have*

$$f(i,m) = f(i, m+n+3).$$

Proof. We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map $\rho_n : \{0, 1, \dots, n+1\} \times \mathbb{Z} \longrightarrow \{0, 1, \dots, n+1\} \times \mathbb{Z}$ given by

$$\rho_n(i,m) = (n+1-i, m+i+1). \tag{1.2}$$

We show that every nowhere-zero F-valued pattern of width n is ρ_n-invariant, i.e. satisfies

$$f(\rho_n(i,m)) = f(i,m), \qquad \forall (i,m) \in \{0,1,\dots,n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that $\rho_n^2 : (i,m) \mapsto (i, m+n+3)$. Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i,m) = f(n+1-i, m+i+1),$$

where $i \in \{0, \dots, n+1\}$. To prove that P_i holds for all i, it is sufficient to prove that P_0, P_1 hold, and that $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$.

P_0 : for all $m \in \mathbb{Z}$, $f(0,m) = 1 = f(n+1, m+1)$.

P_1 : Fix an arbitrary $m \in \mathbb{Z}$. By the second equation in Lemma 1.2, we know that $f(-1, m+2) = f(n, m+2)f(0, m+1) - f(1,m)$. By re-arranging, we get that $f(1,m) = f(n, m+2)$.

Now suppose we are given $i \in \{0, 1, \dots, n-1\}$ such that P_i and P_{i+1} hold. Then, for any fixed $m \in \mathbb{Z}$, we have

$$\begin{aligned} f(i+2,m) &= f(1, m+i+1)f(i+1,m) - f(i,m) \\ &= f(n, m+i+3)f(n-i, m+i+2) - f(n-i+1, m+i+1) \\ &= f(n-i-1, m+i+3). \end{aligned}$$

□

1.2 Arithmetic frieze patterns

Throughout this section, we will consider the so-called arithmetic frieze patterns.

Definition 1.4. *A $\mathbb{Q}$-valued frieze pattern of width n is said to be arithmetic if it takes values in $\mathbb{Z}_{>0}$. We denote by Frieze(n) the set of arithmetic frieze patterns of width n.*

The following lemma will be important in what follows.

Lemma 1.5. *Let n be a fixed positive integer.*

1) If f is an arithmetic frieze pattern of width n, then

$$f(i,0) \mid f(i-1,0)+f(i+1,0), \qquad i \in \{1,\dots,n\}.$$

2) If an $(n+2)$-tuple $(a_0,\dots,a_{n+1})$ of positive integers satisfies $a_0=a_{n+1}=1$ and

$$a_i \mid a_{i-1}+a_{i+1}, \qquad i \in \{1,\dots,n\},$$

then there exists a (necessarily unique) arithmetic frieze pattern of width n such that

$$f(i,0)=a_i, \qquad i \in \{1,\dots,n\}$$

Proof. 1) follows from the first equation in Lemma 1.2. It remains to show 2). □

It is not clear from the definitions that, for a given positive integer n, the set $\mathrm{Frieze}(n)$ is non empty.

Lemma 1.6. *Fix a positive integer n. The set Frieze(n) is non-empty.*

Proof. According to Lemma 1.5, there exists an arithmetic frieze pattern of width n, denoted f, such that $f(i,0)=1$ for $i \in \{1,\dots,n\}$. □

Proposition 1.7. *Fix a positive integer n. The set Frieze(n) is finite.*

Proof. □

Chapter 2

Sequence of maxima of frieze patterns

In §1, we introduced the notion of an (arithmetic) frieze pattern of width n, where n is a positive integer.

In particular, we showed that

1) Each (arithmetic) frieze pattern of width n takes on finitely many values (Proposition 1.3). In other words, for each n and each $f \in \text{Frieze}(n)$, there is a well-defined positive integer

$$u_n(f) := \max(f(i,m) : (i,m) \in \{1, \dots, n\} \times \mathbb{Z}).$$

2) For each n there are finitely many arithmetic frieze patterns of width n (Lemma 1.6 and Proposition 1.7). In particular, for each n, there is a well-defined positive integer

$$u_n := \max(u_n(f) : f \in \text{Frieze}(n)).$$

We are now able to formulate and prove the main theorem of this section.

Theorem 2.1. *For all $n \geq 1$, we have*

$$u_n = F_{n+2},$$

where $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence, i.e. the sequence recursively defined by $F_0 = 0, F_1 = 1$ and the recursive formula

$$F_n = F_{n-1} + F_{n-2}.$$

Proof. □