Coxeter frieze patterns

Antoine de Saint Germain

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Chapter 1

Basics on frieze patterns

1.1 Closed frieze patterns

Definition 1.1.1. *Fix $n \in \mathbb{N}$. A map $f : \{1, 2, \dots, n\} \times \mathbb{Z} \longrightarrow \mathbb{R}$ is called a frieze pattern of width n if, for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have*

$$f(i, m)f(i, m + 1) = 1 + f(i + 1, m)f(i - 1, m + 1),$$

where by convention we set $f(0, m) = f(n + 1, m) = 1$ for all $m \in \mathbb{Z}$.

Consider the map $\rho_n : \{1, 2, \dots, n\} \times \mathbb{Z} \longrightarrow \{1, 2, \dots, n\} \times \mathbb{Z}$ given by

$$\rho_n(i, m) = (n + 1 - i, m + i + 1). \tag{1.1}$$

Note that $\rho_n^2 : (i, m) \mapsto (i, m + n + 3)$, and therefore ρ_n is bijective. The main result in this section is the so-called *glide symmetry* of frieze patterns.

Proposition 1.1.2. *Every frieze pattern of width n is ρ_n-invariant, i.e. satisfies*

$$f(\rho_n(i, m)) = f(i, m), \qquad \forall (i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}.$$

The remainder of this section is dedicated to the proof of Proposition 1.1.2.

1.2 Arithmetic frieze patterns

Definition 1.2.1. *A frieze pattern of width n is said to be arithmetic if it takes values in $\mathbb{Z}_{>0}$.*

We denote by $\mathrm{Frieze}(n)$ the set of all arithmetic frieze patterns of width n. The main result in this section is Conway and Coxeter's enumeration of arithmetic frieze patterns.

Theorem 1.2.2. *Fix $n \in \mathbb{N}$. There is a set-theoretic bijection from the set of triangulations of the regular $n + 3$-gon in the plane and the set $\mathit{Frieze}(n)$. In particular we have*

$$|\mathit{Frieze}(n)| = C_{n+1},$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n^{th} Catalan number.

The remainder of this section is dedicated to the proof of Theorem 1.2.2.