

Coxeter frieze patterns

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Chapter 1

Field-valued patterns

Throughout this document, we will study frieze patterns of finite *height*. This terminology (as compared to finite *width* or finite *order*) is unconventional but more convenient for formalisation. For us, the height of a frieze pattern corresponds to the number of rows, including the rows of ones but *excluding* the rows of zeros. Throughout this section, we fix an arbitrary field F .

Definition 1.1. Fix $n \in \mathbb{N}^*$. A map $f : \{0, 1, \dots, n, n+1\} \times \mathbb{Z} \rightarrow F$ is called an F -valued pattern of height n if,

- 1) for all $m \in \mathbb{Z}$, $f(0, m) = f(n+1, m) = 0$,
- 2) for all $m \in \mathbb{Z}$, $f(1, m) = f(n, m) = 1$, and
- 3) for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have

$$f(i, m)f(i, m+1) = 1 + f(i+1, m)f(i-1, m+1). \quad (1.1)$$

An F -valued pattern f of height n is said to be *nowhere zero* if $f(i, m) \neq 0$, for all $i \in \{1, \dots, n\}$ and for all $m \in \mathbb{Z}$.

Lemma 1.2. Let f be a nowhere-zero F -valued pattern of height n . For all m , we have

$$\begin{aligned} f(i+2, m) &= f(2, m+i)f(i+1, m) - f(i, m), & i \in \{0, \dots, n-1\} \\ f(i, m) &= f(n-1, m)f(i+1, m-1) - f(i+2, m-2), & i \in \{0, n-1\}. \end{aligned}$$

Proof. We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i+2, m) = f(2, m+i+1)f(i+1, m) - f(i, m),$$

for $i \in \{0, \dots, n-1\}$. We do so by induction on i .

Base case P_0 : We have that for all $m \in \mathbb{Z}$, $f(2, m)f(1, m) - f(0, m) = f(2, m+1) * 1 - 0 = f(2, m)$.

Inductive hypothesis. Suppose that our claim holds for some $i \in \{0, \dots, n-2\}$ fixed. Then,

$$\begin{aligned} f(i+3, m)f(i+1, m+1) &= f(i+2, m)f(i+2, m+1) - 1 \\ &= f(i+2, m)(f(2, m+i+1)f(i+1, m+1) - f(i, m+1)) - 1 \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - (f(i+2, m)f(i, m+1) + 1) \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - f(i+1, m)f(i+1, m+1). \end{aligned}$$

Since f is nowhere-zero, we may divide both sides of the equation by $f(i+1, m+1)$ to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i, m) = f(n-1, m)f(i+1, m-1) - f(i+2, m-2),$$

by induction on i , starting with $i = n-1$ and proving the inductive step $Q_i \Rightarrow Q_{i-1}$.

Base case Q_{n-1} : for all $m \in \mathbb{Z}$, $f(n-1, m)f(n, m-1) - f(n+1, m-2) = f(n-1, m)*1 - 0 = f(n-1, m)$.

Inductive hypothesis. Suppose that Q_{i+1} holds for some fixed $i \in \{0, \dots, n-2\}$. Then,

$$\begin{aligned} f(i, m)f(i+2, m-1) &= f(i+1, m-1)f(i+1, m) - 1 \\ &= f(i+1, m-1)(f(i+2, m-1)f(n-1, m) - f(i+3, m-2)) - 1 \\ &= f(i+1, m-1)f(n-1, m)f(i+2, m-1) - (f(i+1, m-1)f(i+3, m-2) + 1) \\ &= f(i+1, m-1)f(n-1, m)f(i+2, m-1) - f(i+2, m-2)f(i+2, m-1). \end{aligned}$$

Again since f is nowhere-zero, dividing by $f(i+2, m-1)$ on both sides we obtain Q_i . \square

Proposition 1.3. *Let f be a nowhere-zero F -valued pattern of height n . Then, for all $m \in \mathbb{Z}$ and all $i \in \{0, \dots, n+1\}$, we have*

$$f(i, m) = f(i, m+n+1).$$

Proof. We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map $\rho_n : \{0, 1, \dots, n+1\} \times \mathbb{Z} \longrightarrow \{0, 1, \dots, n+1\} \times \mathbb{Z}$ given by

$$\rho_n(i, m) = (n+1-i, m+i). \quad (1.2)$$

We show that every nowhere-zero F -valued pattern of height n is ρ_n -invariant, i.e. satisfies

$$f(\rho_n(i, m)) = f(i, m), \quad \forall (i, m) \in \{0, 1, \dots, n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that $\rho_n^2 : (i, m) \mapsto (i, m+n+1)$. Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i, m) = f(n+1-i, m+i),$$

where $i \in \{0, \dots, n+1\}$. To prove that P_i holds for all i , it is sufficient to prove that P_0, P_1 hold, and that $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$.

P_0 : for all $m \in \mathbb{Z}$, $f(0, m) = 0 = f(n+1, m)$.

P_1 : for all $m \in \mathbb{Z}$, $f(1, m) = 1 = f(n, m+1)$.

Now suppose we are given $i \in \{0, 1, \dots, n-1\}$ such that P_i and P_{i+1} hold. Then, for any fixed $m \in \mathbb{Z}$, we have

$$\begin{aligned} f(i+2, m) &= f(2, m+i)f(i+1, m) - f(i, m) \\ &= f(n-1, m+i+2)f(n-i, m+i+1) - f(n+1-i, m+i) \\ &= f(n-i-1, m+i+2). \end{aligned}$$

\square

Corollary 1.4. *Let f be a nowhere-zero F -valued pattern of height n . Then, $\text{Im}(f) := \{f(i, m) : i \in \{1, \dots, n\}, m \in \mathbb{Z}\}$ is a finite set.*

Proof. Consider the finite set $\mathcal{D} = \{(i, m) : i \in \{1, \dots, n\}, m \in \{0, \dots, n\}\}$. By Proposition 1.3,

$$\text{Im}(f) = \{f(i, m) : (i, m) \in \mathcal{D}\},$$

and the right-hand side is obviously finite. \square

Chapter 2

Pandean sequences, flutes and the Fibonacci sequence

Definition 2.1. A sequence $(a_n)_{n \in \mathbb{N}}$ of positive integers is called pandean if $a_0 = 1$ and if, for every $n \geq 1$, we have

$$a_n \mid a_{n-1} + a_{n+1}.$$

Given a pandean sequence (a_n) , if there exists a positive integer k such that $a_n = a_{n+k}$ for all $n \in \mathbb{N}$, the tuple (a_0, a_1, \dots, a_k) is called a Pan flute, or simply a flute, of height k . The set of all flutes of a given height k is denoted $\text{Flute}(k)$.

It is clear that the constant sequence consisting entirely of ones is pandean, and such a pandean sequence gives rise to a flute of height k for any k . In other words, we have the following.

Lemma 2.2. For any positive integer k , the set $\text{Flute}(k)$ is non-empty.

Definition 2.3. The Fibonacci sequence $(F_n)_{n \in \mathbb{N}}$ is defined by $F_0 = 0, F_1 = 1$ and the recursive formula

$$F_n = F_{n-1} + F_{n-2}.$$

Lemma 2.4. 1) If n is odd, the n -tuple

$$(F_2, F_4, F_6, \dots, F_{n-1}, F_n, F_{n-2}, F_{n-4}, \dots, F_5, F_3, F_1),$$

is a Pan flute of height $n - 1$.

2) If n is even, the n -tuple

$$(F_2, F_4, F_6, \dots, F_{n-2}, F_n, F_{n-1}, F_{n-3}, \dots, F_5, F_3, F_1),$$

is a Pan flute of height $n - 1$.

Proof. These are a tedious but straightforward calculation. □

Lemma 2.5. In a flute (a_0, \dots, a_n) , there exists an index $i \in \{1, \dots, n - 1\}$ such that $a_i = a_{i-1} + a_{i+1}$.

Proof. □

Proposition 2.6. Fix a positive integer n , and let $(a_0, \dots, a_n) \in \text{Flute}(n)$. For any $i \in \{1, \dots, n\}$, we have $a_i \leq F_{n+1}$.

Proof. By induction on n , and using Lemma ?? □

Chapter 3

Maximal values of arithmetic frieze patterns

Definition 3.1. A \mathbb{Q} -valued pattern of height n is said to be an arithmetic frieze pattern (of height n) if it takes values in $\mathbb{Z}_{>0}$. We denote by $\text{Frieze}(n)$ the set of arithmetic frieze patterns of height n .

The following proposition is a key result connecting arithmetic frieze patterns to n -diagonals.

Proposition 3.2. 1) Let f be an arithmetic frieze pattern of height n . For all $m \in \mathbb{Z}$, the n -tuple

$$(f(1, m), f(2, m), \dots, f(n, m))$$

is a flute (of height $n - 1$).

2) Given a flute (a_0, \dots, a_n) , there exists a arithmetic frieze pattern f of height $n + 1$ such that

$$(f(1, 0), \dots, f(n, 0)) = (a_0, \dots, a_n).$$

Proof.

□

Corollary 3.3. Fix a positive integer n . The set $\text{Frieze}(n)$ is non-empty.

Proof. The proof of Lemma 2.2 showed that $(1, 1, \dots, 1)$ is a flute. The claim then follows from 2) of Proposition 3.2. □

Corollary 3.4. For each n , there is a well-defined positive integer, called the maximum value among arithmetic frieze patterns of height n , defined by

$$u_n := \max(f(i, m) : f \in \text{Frieze}(n), i \in \{1, \dots, n\}, m \in \mathbb{Z}).$$

Proof.

□

We are now able to formulate and prove the main theorem of this section.

Theorem 3.5. For all $n \geq 1$, we have

$$u_n = F_n.$$

Proof. By Propositions 3.2 and 2.6, $u_n \leq F_n$ for all n . On the other hand, Lemma 2.4 and 2) of Proposition 3.2 show that there exists an arithmetic frieze pattern of height n containing F_n as a value. □