# Coxeter frieze patterns

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# Chapter 1

# Background on frieze patterns

Throughout this document, we will study frieze patterns of finite *height*. This terminology (as compared to finite *width* or finite *order*) is unconventional but more convenient for formalisation. For us, the height of a frieze pattern corresponds to the number of rows, including the rows of ones but *excluding* the rows of zeros.

## 1.1 Field-valued patterns

Throughout this section, we fix an arbitrary field $F$.

**Definition 1.1.** *Fix $n \in \mathbb{N}^*$. A map $f : \{0, 1, \dots, n, n+1\} \times \mathbb{Z} \longrightarrow F$ is called an $F$-valued pattern of height $n$ if,*

*1) for all $m \in \mathbb{Z}$, $f(0, m) = f(n+1, m) = 0$,*
*2) for all $m \in \mathbb{Z}$, $f(1, m) = f(n, m) = 1$, and*
*3) for all $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$, we have*

$$f(i, m)f(i, m+1) = 1 + f(i+1, m)f(i-1, m+1). \tag{1.1}$$

An $F$-valued pattern $f$ of height $n$ is said to be *nowhere zero* if $f(i, m) \neq 0$, for all $i \in \{1, \dots, n\}$ and for all $m \in \mathbb{Z}$.

**Lemma 1.2.** *Let $f$ be a nowhere-zero $F$-valued pattern of height $n$. For all $m$, we have*

$$f(i+2, m) = f(2, m+i)f(i+1, m) - f(i, m), \qquad i \in \{0, \dots, n-1\}$$
$$f(i, m) = f(n-1, m)f(i+1, m-1) - f(i+2, m-2), \qquad i \in \{0, n-1\}.$$

*Proof.* We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i+2, m) = f(2, m+i+1)f(i+1, m) - f(i, m),$$

for $i \in \{0, \dots, n-1\}$. We do so by induction on $i$.

Base case $P_0$: We have that for all $m \in \mathbb{Z}$, $f(2, m)f(1, m) - f(0, m) = f(2, m+1) * 1 - 0 = f(2, m)$.

Inductive hypothesis. Suppose that our claim holds for some $i \in \{0, \dots, n-2\}$ fixed. Then,

$$\begin{aligned}
f(i+3, m)f(i+1, m+1) &= f(i+2, m)f(i+2, m+1) - 1 \\
&= f(i+2, m)(f(2, m+i+1)f(i+1, m+1) - f(i, m+1)) - 1 \\
&= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - (f(i+2, m)f(i, m+1) + 1) \\
&= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - f(i+1, m)f(i+1, m+1).
\end{aligned}$$

Since $f$ is nowhere-zero, we may divide both sides of the equation by $f(i+1, m+1)$ to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i,m) = f(n-1,m)f(i+1,m-1) - f(i+2,m-2),$$

by induction on $i$, starting with $i = n-1$ and proving the inductive step $Q_i \Rightarrow Q_{i-1}$.

Base case $Q_{n-1}$: for all $m \in \mathbb{Z}$, $f(n-1,m)f(n,m-1) - f(n+1,m-2) = f(n-1,m) * 1 - 0 = f(n-1,m)$.

Inductive hypothesis. Suppose that $Q_{i+1}$ holds for some fixed $i \in \{0, \dots, n-2\}$. Then,

$$\begin{aligned} f(i,m)f(i+2,m-1) &= f(i+1,m-1)f(i+1,m) - 1 \\ &= f(i+1,m-1)(f(i+2,m-1)f(n-1,m) - f(i+3,m-2)) - 1 \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - (f(i+1,m-1)f(i+3,m-2) + 1) \\ &= f(i+1,m-1)f(n-1,m)f(i+2,m-1) - f(i+2,m-2)f(i+2,m-1). \end{aligned}$$

Again since $f$ is nowhere-zero, dividing by $f(i+2,m-1)$ on both sides we obtain $Q_i$. $\square$

**Proposition 1.3.** *Let $f$ be a nowhere-zero $F$-valued pattern of height $n$. Then, for all $m \in \mathbb{Z}$ and all $i \in \{0, \dots, n+1\}$, we have*

$$f(i,m) = f(i, m+n+1).$$

*Proof.* We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map $\rho_n : \{0, 1, \dots, n+1\} \times \mathbb{Z} \longrightarrow \{0, 1, \dots, n+1\} \times \mathbb{Z}$ given by

$$\rho_n(i,m) = (n+1-i, m+i). \tag{1.2}$$

We show that every nowhere-zero $F$-valued pattern of height $n$ is $\rho_n$-invariant, i.e. satisfies

$$f(\rho_n(i,m)) = f(i,m), \qquad \forall (i,m) \in \{0, 1, \dots, n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that $\rho_n^2 : (i,m) \mapsto (i, m+n+1)$. Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i,m) = f(n+1-i, m+i),$$

where $i \in \{0, \dots, n+1\}$. To prove that $P_i$ holds for all $i$, it is sufficient to prove that $P_0, P_1$ hold, and that $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$.

$P_0$ : for all $m \in \mathbb{Z}, f(0,m) = 0 = f(n+1,m)$.

$P_1$ : for all $m \in \mathbb{Z}, f(1,m) = 1 = f(n,m+1)$.

Now suppose we are given $i \in \{0, 1, \dots, n-1\}$ such that $P_i$ and $P_{i+1}$ hold. Then, for any fixed $m \in \mathbb{Z}$, we have

$$\begin{aligned} f(i+2,m) &= f(2,m+i)f(i+1,m) - f(i,m) \\ &= f(n-1,m+i+2)f(n-i,m+i+1) - f(n+1-i,m+i) \\ &= f(n-i-1,m+i+2). \end{aligned}$$

$\square$

**Corollary 1.4.** *Let $f$ be a nowhere-zero $F$-valued pattern of height $n$. Then,* $\mathrm{Im}(f) := \{f(i,m) : i \in \{1, \dots, n\}, m \in \mathbb{Z}\}$ *is a finite set.*

*Proof.* Consider the finite set $\mathcal{D} = \{(i,m) : i \in \{1, \dots, n\}, m \in \{0, \dots, n\}\}$. By Proposition 1.3,

$$\mathrm{Im}(f) = \{f(i,m) : (i,m) \in \mathcal{D}\},$$

and the right-hand side is obviously finite. $\square$

## 1.2 Arithmetic frieze patterns

We now turn to arithmetic frieze patterns. We begin with a preliminary definition.

**Definition 1.5.** *A $\mathbb{Q}$-valued pattern of height $n$ is said to be positive if, for all $m \in \mathbb{Z}$ and all $i \in \{1, \dots, n\}$, we have $f(i, m) \in \mathbb{Q}_{>0}$.*

**Definition 1.6.** *A $\mathbb{Q}$-valued pattern of height $n$ is said to be an arithmetic frieze pattern (of height $n$) if it takes values in $\mathbb{Z}_{>0}$. We denote by Frieze$(n)$ the set of arithmetic frieze patterns of height $n$.*

We now introduce an auxiliary object called an *$n$-diagonal*.

**Definition 1.7.** *Let $n$ be a positive integer. By an $n$-diagonal, we mean any $n$-tuple $(a_1, \dots, a_n)$ of positive integers such that $a_1 = a_n$ and*

$$a_i \mid a_{i-1} + a_{i+1}, \qquad i \in \{2, \dots, n-1\}.$$

*We denote by $\mathrm{Diag}(n)$ the set of $n$-diagonals.*

**Proposition 1.8.** *Let $f$ be an arithmetic frieze pattern of height $n$. For all $m \in \mathbb{Z}$, the $n$-tuple*

$$(f(1, m), f(2, m), \dots, f(n, m))$$

*is an $n$-diagonal.*

*Proof.* □

We prove a couple of key properties about $n$-diagonals.

**Lemma 1.9.** *Fix a positive integer $n$. The set Diag$(n)$ is non-empty.*

*Proof.* The $n$-tuple $(1, 1, \dots, 1)$ consisting entirely of 1s is clearly an $n$-diagonal. □

We conclude this section with a discussion of upper bounds on the entries of an $n$-diagonal.

**Definition 1.10.** *The Fibonacci sequence $(F_n)_{n \in \mathbb{N}}$ is defined by $F_0 = 0, F_1 = 1$ and the recursive formula*

$$F_n = F_{n-1} + F_{n-2}.$$

**Lemma 1.11.** *In an $n$-diagonal $(a_1, \dots, a_n)$, there exists an index $i \in \{2, \dots, n-1\}$ such that $a_i = a_{i-1} + a_{i+1}$.*

*Proof.* □

**Proposition 1.12.** *Fix a positive integer $n$, and let $(a_1, \dots, a_n) \in \mathrm{Diag}(n)$. For any $i \in \{1, \dots, n\}$, we have $a_i \leq F_n$.*

*Proof.* By induction on $n$, and using Lemma 1.11. □

# Chapter 2

# Sequence of maxima of frieze patterns

In §1, we introduced the notion of an (arithmetic) frieze pattern of height $n$, where $n$ is a positive integer.

In particular, we showed that each (arithmetic) frieze pattern of height $n$ takes on finitely many values (Corollary 1.4).

We now show that there are finitely many arithmetic frieze patterns of height $n$.

**Lemma 2.1.** *Fix a positive integer $n$. The set Frieze$(n)$ is non-empty.*

*Proof.* The proof of Lemma 1.9 showed that $(1, 1, \dots, 1)$ is an $n$-diagonal. By Proposition **??**, $(FR_n)^{-1}(1, 1, \dots, 1)$ is an arithmetic frieze pattern of height $n$. □

**Proposition 2.2.** *Fix a positive integer $n$. The set Frieze$(n)$ is finite.*

*Proof.* □

This leads to the following two definitions.

**Definition 2.3.** *For each $n \in \mathbb{N}^*$ and each $f \in \mathrm{Frieze}(n)$, there is a well-defined positive integer*

$$u_n(f) := \max(f(i, m) : (i, m) \in \{1, \dots, n\} \times \mathbb{Z}).$$

**Definition 2.4.** *For each $n$, there is a well-defined positive integer, called the maximum value among arithmetic frieze patterns of height $n$, which we write as*

$$u_n := \max(u_n(f) : f \in \mathrm{Frieze}(n)).$$

We are now able to formulate and prove the main theorem of this section.

**Theorem 2.5.** *For all $n \geq 1$, we have*

$$u_n = F_n.$$

We break down the proof into two lemmas.

**Lemma 2.6.** *For all $n \geq 1$, we have*

$$u_n \geq F_n.$$

*Proof.* We construct, for each $n$, a frieze pattern containing the value $F_{n+2}$. By Proposition **??**, it is sufficient to specify an $n$-tuple $(a_1, \dots, a_n)$ of positive integers such that $a_1 = a_n = 1$ and

$$a_i \mid a_{i-1} + a_{i+1}, \qquad i \in \{2, \dots, n-1\}. \tag{2.1}$$

1) If $n$ is odd, one sees by a direct computation that the $n$-tuple

$$(F_2, F_4, F_6, \dots, F_{n-1}, F_n, F_{n-2}, F_{n-4}, \dots, F_5, F_3, F_1),$$

satisfies the conditions of (2.1).

2) If $n$ is even, the $n$-tuple

$$(F_2, F_4, F_6, \dots, F_{n-2}, F_n, F_{n-1}, F_{n-3}, \dots, F_5, F_3, F_1),$$

satisfies the conditions of (2.1). □

We now turn to the reverse equality.

**Proposition 2.7.** *For all $n \geq 1$, we have*

$$u_n \leq F_n.$$

*Proof.* □