

# Coxeter frieze patterns

Antoine de Saint Germain

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# Chapter 1

## Field-valued patterns

Throughout this document, we will study frieze patterns of finite *height*. This terminology (as compared to finite *width* or finite *order*) is unconventional but more convenient for formalisation. For us, the height of a frieze pattern corresponds to the number of rows, including the rows of ones but *excluding* the rows of zeros. Throughout this section, we fix an arbitrary field  $F$ .

**Definition 1.1.** Fix  $n \in \mathbb{N}^*$ . A map  $f : \{0, 1, \dots, n, n+1\} \times \mathbb{Z} \rightarrow F$  is called an  $F$ -valued pattern of height  $n$  if,

- 1) for all  $m \in \mathbb{Z}$ ,  $f(0, m) = f(n+1, m) = 0$ ,
- 2) for all  $m \in \mathbb{Z}$ ,  $f(1, m) = f(n, m) = 1$ , and
- 3) for all  $(i, m) \in \{1, 2, \dots, n\} \times \mathbb{Z}$ , we have

$$f(i, m)f(i, m+1) = 1 + f(i+1, m)f(i-1, m+1). \quad (1.1)$$

An  $F$ -valued pattern  $f$  of height  $n$  is said to be *nowhere zero* if  $f(i, m) \neq 0$ , for all  $i \in \{1, \dots, n\}$  and for all  $m \in \mathbb{Z}$ .

**Lemma 1.2.** Let  $f$  be a nowhere-zero  $F$ -valued pattern of height  $n$ . For all  $m$ , we have

$$\begin{aligned} f(i+2, m) &= f(2, m+i)f(i+1, m) - f(i, m), & i \in \{0, \dots, n-1\} \\ f(i, m) &= f(n-1, m)f(i+1, m-1) - f(i+2, m-2), & i \in \{0, n-1\}. \end{aligned}$$

*Proof.* We begin by proving the first statement. That is, we prove

$$P_i : \forall m \in \mathbb{Z}, f(i+2, m) = f(2, m+i+1)f(i+1, m) - f(i, m),$$

for  $i \in \{0, \dots, n-1\}$ . We do so by induction on  $i$ .

Base case  $P_0$ : We have that for all  $m \in \mathbb{Z}$ ,  $f(2, m)f(1, m) - f(0, m) = f(2, m+1) * 1 - 0 = f(2, m)$ .

Inductive hypothesis. Suppose that our claim holds for some  $i \in \{0, \dots, n-2\}$  fixed. Then,

$$\begin{aligned} f(i+3, m)f(i+1, m+1) &= f(i+2, m)f(i+2, m+1) - 1 \\ &= f(i+2, m)(f(2, m+i+1)f(i+1, m+1) - f(i, m+1)) - 1 \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - (f(i+2, m)f(i, m+1) + 1) \\ &= f(i+2, m)f(2, m+i+1)f(i+1, m+1) - f(i+1, m)f(i+1, m+1). \end{aligned}$$

Since  $f$  is nowhere-zero, we may divide both sides of the equation by  $f(i+1, m+1)$  to obtain the desired equality.

The second statement is proved almost identically. Namely, we prove

$$Q_i : \forall m \in \mathbb{Z}, f(i, m) = f(n-1, m)f(i+1, m-1) - f(i+2, m-2),$$

by induction on  $i$ , starting with  $i = n-1$  and proving the inductive step  $Q_i \Rightarrow Q_{i-1}$ .

Base case  $Q_{n-1}$ : for all  $m \in \mathbb{Z}$ ,  $f(n-1, m)f(n, m-1) - f(n+1, m-2) = f(n-1, m)*1 - 0 = f(n-1, m)$ .

Inductive hypothesis. Suppose that  $Q_{i+1}$  holds for some fixed  $i \in \{0, \dots, n-2\}$ . Then,

$$\begin{aligned} f(i, m)f(i+2, m-1) &= f(i+1, m-1)f(i+1, m) - 1 \\ &= f(i+1, m-1)(f(i+2, m-1)f(n-1, m) - f(i+3, m-2)) - 1 \\ &= f(i+1, m-1)f(n-1, m)f(i+2, m-1) - (f(i+1, m-1)f(i+3, m-2) + 1) \\ &= f(i+1, m-1)f(n-1, m)f(i+2, m-1) - f(i+2, m-2)f(i+2, m-1). \end{aligned}$$

Again since  $f$  is nowhere-zero, dividing by  $f(i+2, m-1)$  on both sides we obtain  $Q_i$ .  $\square$

**Proposition 1.3.** *Let  $f$  be a nowhere-zero  $F$ -valued pattern of height  $n$ . Then, for all  $m \in \mathbb{Z}$  and all  $i \in \{0, \dots, n+1\}$ , we have*

$$f(i, m) = f(i, m+n+1).$$

*Proof.* We prove a stronger statement, called the *glide symmetry* of frieze patterns. First, consider the map  $\rho_n : \{0, 1, \dots, n+1\} \times \mathbb{Z} \rightarrow \{0, 1, \dots, n+1\} \times \mathbb{Z}$  given by

$$\rho_n(i, m) = (n+1-i, m+i). \quad (1.2)$$

We show that every nowhere-zero  $F$ -valued pattern of height  $n$  is  $\rho_n$ -invariant, i.e. satisfies

$$f(\rho_n(i, m)) = f(i, m), \quad \forall (i, m) \in \{0, 1, \dots, n+1\} \times \mathbb{Z}.$$

The proposition will then follow by observing that  $\rho_n^2 : (i, m) \mapsto (i, m+n+1)$ . Thus, consider the statement

$$P_i : \forall m \in \mathbb{Z}, f(i, m) = f(n+1-i, m+i),$$

where  $i \in \{0, \dots, n+1\}$ . To prove that  $P_i$  holds for all  $i$ , it is sufficient to prove that  $P_0, P_1$  hold, and that  $P_i \wedge P_{i+1} \Rightarrow P_{i+2}$ .

$P_0$  : for all  $m \in \mathbb{Z}$ ,  $f(0, m) = 0 = f(n+1, m)$ .

$P_1$  : for all  $m \in \mathbb{Z}$ ,  $f(1, m) = 1 = f(n, m+1)$ .

Now suppose we are given  $i \in \{0, 1, \dots, n-1\}$  such that  $P_i$  and  $P_{i+1}$  hold. Then, for any fixed  $m \in \mathbb{Z}$ , we have

$$\begin{aligned} f(i+2, m) &= f(2, m+i)f(i+1, m) - f(i, m) \\ &= f(n-1, m+i+2)f(n-i, m+i+1) - f(n+1-i, m+i) \\ &= f(n-i-1, m+i+2). \end{aligned}$$

$\square$

**Corollary 1.4.** *Let  $f$  be a nowhere-zero  $F$ -valued pattern of height  $n$ . Then,  $\text{Im}(f) := \{f(i, m) : i \in \{1, \dots, n\}, m \in \mathbb{Z}\}$  is a finite set.*

*Proof.* Consider the finite set  $\mathcal{D} = \{(i, m) : i \in \{1, \dots, n\}, m \in \{0, \dots, n\}\}$ . By Proposition 1.3,

$$\text{Im}(f) = \{f(i, m) : (i, m) \in \mathcal{D}\},$$

and the right-hand side is obviously finite.  $\square$

## Chapter 2

# n-Diagonals

**Definition 2.1.** Let  $n$  be a positive integer. By an  $n$ -diagonal, we mean any  $n$ -tuple  $(a_1, \dots, a_n)$  of positive integers such that  $a_1 = a_n$  and

$$a_i \mid a_{i-1} + a_{i+1}, \quad i \in \{2, \dots, n-1\}.$$

We denote by  $\text{Diag}(n)$  the set of  $n$ -diagonals.

We prove a couple of key properties about  $n$ -diagonals.

**Lemma 2.2.** Fix a positive integer  $n$ . The set  $\text{Diag}(n)$  is non-empty.

*Proof.* The  $n$ -tuple  $(1, 1, \dots, 1)$  consisting entirely of 1s is clearly an  $n$ -diagonal.  $\square$

**Definition 2.3.** The Fibonacci sequence  $(F_n)_{n \in \mathbb{N}}$  is defined by  $F_0 = 0, F_1 = 1$  and the recursive formula

$$F_n = F_{n-1} + F_{n-2}.$$

The following  $n$ -diagonals will play an important role for us.

**Lemma 2.4.** 1) If  $n$  is odd, the  $n$ -tuple

$$(F_2, F_4, F_6, \dots, F_{n-1}, F_n, F_{n-2}, F_{n-4}, \dots, F_5, F_3, F_1),$$

is an  $n$ -diagonal.

2) If  $n$  is even, the  $n$ -tuple

$$(F_2, F_4, F_6, \dots, F_{n-2}, F_n, F_{n-1}, F_{n-3}, \dots, F_5, F_3, F_1),$$

is an  $n$ -diagonal.

*Proof.* These are a tedious but straightforward calculation.  $\square$

**Lemma 2.5.** In an  $n$ -diagonal  $(a_1, \dots, a_n)$ , there exists an index  $i \in \{2, \dots, n-1\}$  such that  $a_i = a_{i-1} + a_{i+1}$ .

*Proof.*  $\square$

**Proposition 2.6.** Fix a positive integer  $n$ , and let  $(a_1, \dots, a_n) \in \text{Diag}(n)$ . For any  $i \in \{1, \dots, n\}$ , we have  $a_i \leq F_n$ .

*Proof.* By induction on  $n$ , and using Lemma 2.5.  $\square$

## Chapter 3

# Maximal values of arithmetic frieze patterns

**Definition 3.1.** A  $\mathbb{Q}$ -valued pattern of height  $n$  is said to be an arithmetic frieze pattern (of height  $n$ ) if it takes values in  $\mathbb{Z}_{>0}$ . We denote by  $\text{Frieze}(n)$  the set of arithmetic frieze patterns of height  $n$ .

The following proposition is a key result connecting arithmetic frieze patterns to  $n$ -diagonals.

**Proposition 3.2.** 1) Let  $f$  be an arithmetic frieze pattern of height  $n$ . For all  $m \in \mathbb{Z}$ , the  $n$ -tuple

$$(f(1, m), f(2, m), \dots, f(n, m))$$

is an  $n$ -diagonal.

2) Given an  $n$ -diagonal  $(a_1, \dots, a_n)$ , there exists a arithmetic frieze pattern  $f$  of height  $n$  such that

$$(f(1, 0), \dots, f(n, 0)) = (a_1, \dots, a_n).$$

*Proof.*

□

**Corollary 3.3.** Fix a positive integer  $n$ . The set  $\text{Frieze}(n)$  is non-empty.

*Proof.* The proof of Lemma 2.2 showed that  $(1, 1, \dots, 1)$  is an  $n$ -diagonal. The claim then follows from 2) of Proposition 3.2. □

**Corollary 3.4.** For each  $n$ , there is a well-defined positive integer, called the maximum value among arithmetic frieze patterns of height  $n$ , defined by

$$u_n := \max(f(i, m) : f \in \text{Frieze}(n), i \in \{1, \dots, n\}, m \in \mathbb{Z}).$$

*Proof.*

□

We are now able to formulate and prove the main theorem of this section.

**Theorem 3.5.** For all  $n \geq 1$ , we have

$$u_n = F_n.$$

*Proof.* By Propositions 3.2 and 2.6,  $u_n \leq F_n$  for all  $n$ . On the other hand, Lemma 2.4 and 2) of Proposition 3.2 show that there exists an arithmetic frieze pattern of height  $n$  containing  $F_n$  as a value. □