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1 Analyse

1.1 Cas général de la formule du binome de Newton

Proposition.
Pour $x, \alpha, \beta \in \mathbf{R}$ et $n \in \mathbf{N}$:

$$(x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$$

Preuve.
Pour $n \in \mathbf{N}$, on définit $P_n : \forall x, \alpha, \beta \in \mathbf{R}$ et $n \in \mathbf{N}, (x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$
P_0 est vraie
Soit $n \in \mathbf{N}$ tel que P_n soit vraie
$x, \alpha, \beta \in \mathbf{R}$
$(x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$
$(n+1)(x+\alpha)^n = \sum_{\mu=0}^{n} (n+1)\binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$
$(n+1)(x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu}(n+1-\mu) \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n-\mu}$
$(x+\alpha)^{n+1} = \sum_{\mu=0}^{n} \binom{n+1}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n+1-\mu} + C$
Avec $x = -(n+1)\beta$:
$(\alpha-(n+1)\beta)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}((\mu-(n+1))\beta)^{n-\mu}$
$(\alpha-(n+1)\beta)^{n+1} = \sum_{\mu=0}^{n} \binom{n+1}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}((\mu-(n+1))\beta)^{n+1-\mu} + C$
En multipliant la première par $(n+1)\beta$ et ajoutant à la deuxième :
$(\alpha-(n+1)\beta)^{n+1} + (n+1)\beta(\alpha-(n+1)\beta)^n = (n+1)\beta \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}((\mu-(n+1))\beta)^{n-\mu} + \sum_{\mu=0}^{n} \binom{n+1}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}((\mu-(n+1))\beta)^{n+1-\mu} + C$
$= (-1)^n \sum_{\mu=0}^{n} (-1)^\mu (n+1)\binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}\beta((n+1-\mu)\beta)^{n-\mu} - (-1)^n \sum_{\mu=0}^{n} (-1)^\mu \frac{n+1}{n+1-\mu}\binom{n}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}((n+1-\mu)\beta)^{n+1-\mu} + C$
$= (-1)^n \sum_{\mu=0}^{n} (-1)^\mu (n+1)\binom{\mu}{n} \alpha(\alpha-\mu\beta)^{\mu-1}\beta^{n+1-\mu}[(n+1-\mu)^{n-\mu} - \frac{1}{n+1-\mu}(n+1-\mu)^{n+1-\mu}] + C$
$= C$
$C = (\alpha-(n+1)\beta)^{n+1} + (n+1)\beta(\alpha-(n+1)\beta)^n$
$\quad = \alpha(\alpha-(n+1)\beta)^n$
$(x+\alpha)^{n+1} = \sum_{\mu=0}^{n} \binom{n+1}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n+1-\mu} + \alpha(\alpha-(n+1)\beta)^n$
$\quad = \sum_{\mu=0}^{n+1} \binom{n+1}{\mu} \alpha(\alpha-\mu\beta)^{\mu-1}(x+\mu\beta)^{n+1-\mu}$
Donc P_{n+1} est vraie.

Remarque.
Avec $\beta = 0$:
$(x+\alpha)^n = \sum_{\mu=0}^{n} \binom{n}{\mu} \alpha^\mu x^{n-\mu}$ *on retrouve la formule du binome de Newton*