1 Continuity equation in a periodic electrodynamic medium

We consider a periodic cell Z of a random heterogeneous material. The conductivity at $\mathbf{x} \in Z$ is $\boldsymbol{\sigma}(\mathbf{x})$ (symmetric, positive definite, second-order tensor); $\mathbf{E}(\mathbf{x})$, $\phi(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ denote the electric field, the electric potential and the volumic current, respectively, at point $\mathbf{x}$. The permittivity is noted $\boldsymbol{\epsilon}(\mathbf{x})$.

$$Z: \quad \operatorname{div}\left(\mathbf{j} + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}\right) = 0, \tag{1}$$

$$Z: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{2}$$

$$Z: \quad \mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}, \tag{3}$$

$$Z: \quad \mathbf{E} = \overline{\mathbf{E}} + \mathbf{grad}\, \phi^{\text{per}}, \tag{4}$$

where ϕ^{per} is a Z-periodic field, $\mathbf{j}$ is a Z-antiperiodic field, and $\overline{\mathbf{E}}$ is a prescribed constant vector. Eq. (4) ensures that the electric field is curl-free, with $\overline{\mathbf{E}} = \langle \mathbf{E} \rangle$, where angle brackets denote volume averages over the cell Z.

2 The time dependence

We consider the following electric field as a data:

$$\overline{\mathbf{E}} = t\overline{\mathbf{E}}_1 + \frac{t^2}{2}\overline{\mathbf{E}}_2 + \frac{t^3}{6}\overline{\mathbf{E}}_3 \tag{5}$$

and we assume the following development for ϕ^{per}

$$\phi^{\text{per}} = \sum_{i=1}^{\infty} \phi_i \frac{t^i}{i!} \tag{6}$$

and we note $\mathbf{grad}\, \phi^{\text{per}} = \mathbf{e}$, and also $\mathbf{grad}\, \phi_i = \mathbf{e}_i$ (for all i). Hence, we have the following problems:

$$\begin{aligned}
&\operatorname{div}\left(\boldsymbol{\epsilon} \cdot (\overline{\mathbf{E}}_1 + \mathbf{e}_1)\right) = 0\\
&\operatorname{div}\left(\boldsymbol{\sigma} \cdot (\overline{\mathbf{E}}_1 + \mathbf{e}_1) + \boldsymbol{\epsilon} \cdot (\overline{\mathbf{E}}_2 + \mathbf{e}_2)\right) = 0\\
&\operatorname{div}\left(\boldsymbol{\sigma} \cdot (\overline{\mathbf{E}}_2 + \mathbf{e}_2) + \boldsymbol{\epsilon} \cdot (\overline{\mathbf{E}}_3 + \mathbf{e}_3)\right) = 0\\
&\operatorname{div}\left(\boldsymbol{\sigma} \cdot (\overline{\mathbf{E}}_3 + \mathbf{e}_3) + \boldsymbol{\epsilon} \cdot \mathbf{e}_4\right) = 0\\
&\operatorname{div}\left(\boldsymbol{\sigma} \cdot \mathbf{e}_4 + \boldsymbol{\epsilon} \cdot \mathbf{e}_5\right) = 0\\
&\operatorname{div}\left(\boldsymbol{\sigma} \cdot \mathbf{e}_5 + \boldsymbol{\epsilon} \cdot \mathbf{e}_6\right) = 0\\
&\ldots
\end{aligned} \tag{7}$$

These problems can be resolved in an iterative way.

3 Resolution of the problems

3.1 first problem

We consider a biphasic cell with a matrix whose conductivity is isotropic σ_0 and another phase whose conductivity is isotropic σ_1. The permittivity is also isotropic, $\epsilon_0 = \tau_0\sigma_0$ for medium 0 and $\epsilon_1 = \tau_1\sigma_1$ for medium 1. Hence, the first problem to solve is

$$\text{div}\left(\epsilon(\overline{\mathbf{E}}_1 + \mathbf{e}_1)\right) = 0 \tag{8}$$

and its solution can be expressed as

$$\overline{\mathbf{E}}_1 + \mathbf{e}_1 = \overline{\mathbf{E}}_1 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^1 \tag{9}$$

where the subblocks of $\mathbf{G}$ are:

$$\mathbf{G}_{ik} = \int_Z \epsilon\, {}^t\mathbf{A}^i \cdot \mathbf{A}^k \tag{10}$$

and

$$\mathbf{U}_i^1 = \int_Z \epsilon\, {}^t\mathbf{A}^i \cdot \overline{\mathbf{E}}_1 \qquad \text{and} \quad \mathbf{A}_{ij}^k(\mathbf{x}) = \mathbf{\Gamma}_i^{0,\text{per}}\left[\left(-\chi\epsilon_0\mathbf{\Gamma}_0^{\text{per}}\right)^{k-1}(\chi\mathbf{e}_j)\right](\mathbf{x}) \tag{11}$$

3.2 second problem

The second problem to solve is

$$\text{div}\left(\epsilon(\overline{\mathbf{E}}_2 + \mathbf{e}_2) + \mathbf{P}_1\right) = 0 \tag{12}$$

with $\mathbf{P}_1 = \sigma(\overline{\mathbf{E}}_1 + \mathbf{e}_1) = \sigma\left(\overline{\mathbf{E}}_1 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^1\right)$. Its solution can be expressed as

$$\overline{\mathbf{E}}_2 + \mathbf{e}_2 = \overline{\mathbf{E}}_2 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^2 - \mathbf{\Gamma}_0^{\text{per}}(\mathbf{P}_1) \tag{13}$$

where

$$\mathbf{U}_i^2 = \int_Z {}^t\mathbf{A}^i \cdot \left[\epsilon\left(\overline{\mathbf{E}}_2 - \mathbf{\Gamma}_0^{\text{per}}(\mathbf{P}_1)\right) + \mathbf{P}_1\right] \tag{14}$$

3.3 third problem

The third problem to solve is

$$\text{div}\left(\epsilon(\overline{\mathbf{E}}_3 + \mathbf{e}_3) + \mathbf{P}_2\right) = 0 \tag{15}$$

with $\mathbf{P}_2 = \sigma(\overline{\mathbf{E}}_2 + \mathbf{e}_2) = \sigma\left(\overline{\mathbf{E}}_2 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^2 - \mathbf{\Gamma}_0^{\text{per}}(\mathbf{P}_1)\right)$. Its solution can be expressed as

$$\overline{\mathbf{E}}_3 + \mathbf{e}_3 = \overline{\mathbf{E}}_3 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^3 - \mathbf{\Gamma}_0^{\text{per}}(\mathbf{P}_2) \tag{16}$$

where

$$\mathbf{U}_i^3 = \int_Z {}^t\mathbf{A}^i \cdot \left[\epsilon\left(\overline{\mathbf{E}}_3 - \mathbf{\Gamma}_0^{\text{per}}(\mathbf{P}_2)\right) + \mathbf{P}_2\right] \tag{17}$$

3.4 fourth problem

The fourth problem to solve is

$$\operatorname{div}\left(\epsilon\, \mathbf{e}_4 + \mathbf{P}_3\right) = 0 \tag{18}$$

with $\mathbf{P}_3 = \sigma(\overline{\mathbf{E}}_3 + \mathbf{e}_3) = \sigma\left(\overline{\mathbf{E}}_3 - \mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^3 - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_2)\right)$. Its solution can be expressed as

$$\mathbf{e}_4 = -\mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^4 - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_3) \tag{19}$$

where

$$\mathbf{U}_i^4 = \int_Z {}^t\mathbf{A}^i \cdot \left[-\epsilon\, \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_3) + \mathbf{P}_3\right] \tag{20}$$

3.5 k^{th} problem with $k \geq 5$

The k^{th} problem to solve is

$$\operatorname{div}\left(\epsilon\, \mathbf{e}_k + \mathbf{P}_{k-1}\right) = 0 \tag{21}$$

with $\mathbf{P}_{k-1} = \sigma \mathbf{e}_{k-1} = \sigma\left(-\mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^{k-1} - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_{k-2})\right)$. Its solution can be expressed as

$$\mathbf{e}_k = -\mathbf{A}(\mathbf{x}) \cdot \mathbf{G}^{-1} \cdot \mathbf{U}^k - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_{k-1}) \tag{22}$$

where

$$\mathbf{U}_i^k = \int_Z {}^t\mathbf{A}^i \cdot \left[-\epsilon\, \boldsymbol{\Gamma}_0^{\mathrm{per}}(\mathbf{P}_{k-1}) + \mathbf{P}_{k-1}\right] \tag{23}$$

4 Numerical application

We choose $\sigma_0 = 1$, $\tau_0 = 1e-6$, so that $\epsilon_0 = 1e-6$; and $\sigma_1 = 1e6$, $\tau_1 = 1e-6$, so that $\epsilon_1 = 1$. For the remote field, we choose $\overline{\mathbf{E}}_1 = 4e5$, $\overline{\mathbf{E}}_2 = -4e5$, $\overline{\mathbf{E}}_3 = 0$ and hence the time of interest is between $1e-6$ and $1e-5$.