1 Minimisation of Hashin-Shtrikman functional

We consider a periodic cell Z of a random heterogeneous material. The conductivity at $\mathbf{x} \in Z$ is $\boldsymbol{\sigma}(\mathbf{x})$ (symmetric, positive definite, second-order tensor); $\mathbf{E}(\mathbf{x})$, $\phi(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ denote the electric field, the electric potential and the volumic current, respectively, at point $\mathbf{x}$. The apparent conductivity of the cell Z, $\boldsymbol{\sigma}^{\mathrm{app}}$, is found from the solution to the following problem

$$Z: \quad \operatorname{div} \mathbf{j} = 0, \tag{1}$$

$$Z: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{2}$$

$$Z: \quad \mathbf{E} = \overline{\mathbf{E}} + \mathbf{grad}\, \phi^{\mathrm{per}}, \tag{3}$$

where ϕ^{per} is a Z-periodic field, $\mathbf{j}$ is a Z-antiperiodic field, and $\overline{\mathbf{E}}$ is a prescribed constant vector. Eq. (3) ensures that the electric field is curl-free, with $\overline{\mathbf{E}} = \langle \mathbf{E} \rangle$, where angle brackets denote volume averages over the cell Z. We consider a biphasic cell with a matrix whose conductivity is $\boldsymbol{\sigma}_0$ and another phase whose conductivity is $\boldsymbol{\sigma}_1$.

We introduce the linear Green operator $\boldsymbol{\Gamma}_0^{\mathrm{per}}$, associated with the conductivity $\boldsymbol{\sigma}_0$.

The electrical field solution of problem (1)-(3) minimizes the Hashin-Shtrikman functional

$$\mathcal{H}(\boldsymbol{\tau}) = \frac{1}{2} \overline{\mathbf{E}} \cdot \boldsymbol{\sigma}_0 \cdot \overline{\mathbf{E}} - \frac{1}{2} \langle \boldsymbol{\tau} \cdot \boldsymbol{\rho} \cdot \boldsymbol{\tau} \rangle - \frac{1}{2} \langle \boldsymbol{\tau} \cdot \boldsymbol{\Gamma}(\boldsymbol{\tau}) \rangle + \overline{\mathbf{E}} \cdot \langle \boldsymbol{\tau} \rangle \tag{4}$$

with $\boldsymbol{\rho} = \boldsymbol{\delta\sigma}^{-1}$.

We then consider the following polarization field:

$$\boldsymbol{\tau} = \sum_{k=0}^{N} \left(-\chi \boldsymbol{\delta\sigma} \boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^k (\chi \overline{\boldsymbol{\tau}}_k) \equiv \sum_{k=0}^{N} \boldsymbol{\tau}_k(\mathbf{x}) \tag{5}$$

For $N = 0$, we obtain the following polarization field:

$$\boldsymbol{\tau}_0 = \chi \overline{\boldsymbol{\tau}}_0 \tag{6}$$

and for $N = 1$, the following one:

$$\boldsymbol{\tau}_1 = \chi \overline{\boldsymbol{\tau}}_0 - \chi \boldsymbol{\delta\sigma} \boldsymbol{\Gamma}_0^{\mathrm{per}} (\chi \overline{\boldsymbol{\tau}}_1) \tag{7}$$

The different terms of the Hashin-Shtrikman functional hence are:

$$-\frac{1}{2} \langle \boldsymbol{\tau} \cdot \boldsymbol{\rho} \cdot \boldsymbol{\tau} \rangle = -\frac{1}{2} \sum_k \sum_l \overline{\boldsymbol{\tau}}_k \cdot \langle {}^t\mathbf{A}^k : \boldsymbol{\rho} : \mathbf{A}^l \rangle \cdot \overline{\boldsymbol{\tau}}_l \tag{8}$$

with

$$\left[\mathbf{A}^k\right]_{ij} (\mathbf{x}) = \left[\left(-\chi \boldsymbol{\delta\sigma} \boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^k\right]_i (\chi \mathbf{e}_j)(\mathbf{x}) \tag{9}$$