Rigorous bounds for the thermodynamic potential of a maxwellian viscoelastic material

1 Position of the problem

The incremental potential:

$$\min_{\overline{\boldsymbol{\alpha}}} \min_{\boldsymbol{\alpha}_{\mathbf{s}} \in \mathcal{S}(\mathbf{C}_0)} \left\{ \min_{\mathbf{e} \in \mathcal{K}(\overline{\mathbf{e}})} w(\mathbf{e}, \boldsymbol{\alpha}_{\mathbf{s}}) + \Delta t \min_{\dot{\boldsymbol{\alpha}}_{\mathbf{c}} \in \mathcal{K}(\dot{\overline{\boldsymbol{\alpha}}})} \varphi(\dot{\boldsymbol{\alpha}}_{\mathbf{c}} + \dot{\boldsymbol{\alpha}}_{\mathbf{s}}) \right\} \tag{1}$$

The minimum of the Helmoltz free energy is equivalent to the thermo-elastic problem [...], and hence to the Lippmann-Schwinger equation:

$$\mathbf{e} + \boldsymbol{\Gamma}_{C_0}(\delta\mathbf{C} : \mathbf{e}) = \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}(\delta\mathbf{C} : \boldsymbol{\alpha}_{\mathbf{s}}) \tag{2}$$

where $\boldsymbol{\Gamma}_{C_0}$ is relative to $\mathbf{C}_0$.

The minimum of the dissipation potential is equivalent to the thermo-elastic problem [...], and hence to the Lippmann-Schwinger equation:

$$\dot{\boldsymbol{\alpha}}_{\mathbf{c}} + \boldsymbol{\Gamma}_{M_0}(\delta\mathbf{M} : \dot{\boldsymbol{\alpha}}_{\mathbf{c}}) = \dot{\overline{\boldsymbol{\alpha}}} - \boldsymbol{\Gamma}_{M_0}(\mathbf{M} : \dot{\boldsymbol{\alpha}}_{\mathbf{s}}) \tag{3}$$

where $\boldsymbol{\Gamma}_{M_0}$ is relative to $\mathbf{M}_0$.

The fact that $\boldsymbol{\alpha}_{\mathbf{s}}$ belongs to $\mathcal{S}(\mathbf{C}_0)$ is equivalent to:

$$\begin{array}{l} \text{There exists a } \mathbf{a} \text{ such that} \\ \boldsymbol{\alpha}_{\mathbf{s}} = \mathbf{a} - \overline{\mathbf{a}} - \boldsymbol{\Gamma}_{C_0}(\mathbf{C}_0 : \mathbf{a}) \end{array} \tag{4}$$

2 The polarization trial fields

We now use the incremental potential to derive a rigorous bound on it. If we take $\mathbf{e}^*$ in $\mathcal{K}(\overline{\mathbf{e}})$, $\dot{\boldsymbol{\alpha}}_{\mathbf{c}}{}^*$ in $\mathcal{K}(\dot{\overline{\boldsymbol{\alpha}}})$, and $\boldsymbol{\alpha}_{\mathbf{s}}{}^*$ in $\mathcal{S}(\mathbf{C}_0)$, we can state that the incremental potential computed with those fields will be greater or equal to the minimum. In the following we drop the *. We hence take:

$$\begin{array}{l} \mathbf{e} = \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}(\delta\mathbf{C} : \boldsymbol{\alpha}_{\mathbf{s}}) - \boldsymbol{\Gamma}_{C_0}(\chi\boldsymbol{\tau}_e) \\ \text{with} \quad \boldsymbol{\tau}_e \quad \text{uniform} \end{array} \tag{5}$$

which belongs to $\mathcal{K}(\overline{\mathbf{e}})$ for all $\boldsymbol{\alpha}_{\mathbf{s}}$, and

$$\begin{array}{l} \dot{\boldsymbol{\alpha}}_{\mathbf{c}} = \dot{\overline{\boldsymbol{\alpha}}} - \boldsymbol{\Gamma}_{M_0}(\mathbf{M} : \dot{\boldsymbol{\alpha}}_{\mathbf{s}}) - \boldsymbol{\Gamma}_{M_0}(\chi\boldsymbol{\tau}_c) \\ \text{with} \quad \boldsymbol{\tau}_c \quad \text{uniform} \end{array} \tag{6}$$

which belongs to $\mathcal{K}(\dot{\overline{\boldsymbol{\alpha}}})$ for all $\dot{\boldsymbol{\alpha}}_{\mathbf{s}}$, and

$$\begin{array}{l} \boldsymbol{\alpha}_{\mathbf{s}} = (\chi - f)\mathbf{a}_s - \boldsymbol{\Gamma}_{C_0}(\chi\mathbf{C}_0 : \mathbf{a}_s) \\ \text{with} \quad \mathbf{a}_s \quad \text{uniform} \end{array} \tag{7}$$

which belongs to $\mathcal{S}(\mathbf{C}_0)$.

Note that (7) gives $\boldsymbol{\alpha}_\mathbf{s}$ which can be used in (5), and it gives $\dot{\boldsymbol{\alpha}}_\mathbf{s}$:

$$\dot{\boldsymbol{\alpha}}_\mathbf{s} = \frac{1}{\Delta t}\left[(\chi - f)\mathbf{a}_s - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \mathbf{a}_s) - \boldsymbol{\alpha}_{s,n}\right] \tag{8}$$

which can be used in (6). We can also assume that $\boldsymbol{\alpha}_{s,n}$ is of the same form (there exists $\mathbf{a}_{s,n}$), so that

$$\dot{\boldsymbol{\alpha}}_\mathbf{s} = (\chi - f)\dot{\mathbf{a}}_s - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \dot{\mathbf{a}}_s). \tag{9}$$

One advantage of the uniformity of the unknowns $\boldsymbol{\tau}_e, \boldsymbol{\tau}_s, \boldsymbol{\tau}_c, \mathbf{a}_s$ is that they can be taken out of the Green operators:

$$\begin{aligned}
\mathbf{e} &= \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}(\delta\mathbf{C} : \boldsymbol{\alpha}_\mathbf{s}) - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_e \\
&= \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}\left[\delta\mathbf{C} : ((\chi - f)\mathbf{a}_s - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \mathbf{a}_s))\right] - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_e \\
&= \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}\left[\chi(1 - f)\delta\mathbf{C} : \mathbf{a}_s\right] - \boldsymbol{\Gamma}_{C_0}\left[\chi\delta\mathbf{C} : \boldsymbol{\Gamma}_{C_0}\left(\chi \mathbf{C}_0 : \mathbf{a}_s\right)\right] - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_e \\
&= \overline{\mathbf{e}} + \boldsymbol{\Gamma}_{C_0}\left[\chi(1 - f)\delta\mathbf{C} : \mathbf{e}_i \otimes \mathbf{e}_j\right] : \mathbf{a}_s - \boldsymbol{\Gamma}_{C_0}\left[\chi\delta\mathbf{C} : \boldsymbol{\Gamma}_{C_0}\left(\chi \mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j\right)\right] : \mathbf{a}_s \\
&\quad - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_e
\end{aligned} \tag{10}$$

which still belongs to $\mathcal{K}(\overline{\mathbf{e}})$; and

$$\begin{aligned}
\dot{\boldsymbol{\alpha}}_\mathbf{c} &= \dot{\overline{\boldsymbol{\alpha}}} - \boldsymbol{\Gamma}_{M_0}\left[\mathbf{M} : ((\chi - f)\dot{\mathbf{a}}_s - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \dot{\mathbf{a}}_s))\right] - \boldsymbol{\Gamma}_{M_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_c \\
&= \dot{\overline{\boldsymbol{\alpha}}} - \boldsymbol{\Gamma}_{M_0}\left[(\chi - f)\mathbf{M} : \mathbf{e}_i \otimes \mathbf{e}_j\right] : \dot{\mathbf{a}}_s + \boldsymbol{\Gamma}_{M_0}\left[\mathbf{M} : \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j)\right] : \dot{\mathbf{a}}_s \\
&\quad - \boldsymbol{\Gamma}_{M_0}(\chi \mathbf{e}_i \otimes \mathbf{e}_j) : \boldsymbol{\tau}_c
\end{aligned} \tag{11}$$

which still belongs to $\mathcal{K}(\dot{\overline{\boldsymbol{\alpha}}})$; and

$$\begin{aligned}
&\boldsymbol{\alpha}_\mathbf{s} = (\chi - f)\dot{\mathbf{a}}_s - \boldsymbol{\Gamma}_{C_0}(\chi \mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j) : \dot{\mathbf{a}}_s \\
&\text{with} \quad \dot{\mathbf{a}}_s \quad \text{uniform}
\end{aligned} \tag{12}$$

which still belongs to $\mathcal{S}(\mathbf{C}_0)$.

Sum up

$$\begin{aligned}
&\mathbf{e}(\mathbf{x}) = \overline{\mathbf{e}} + \mathbf{P}_{CC}(\mathbf{x}) : \mathbf{a}_s - \mathbf{P}_{C_0}(\mathbf{x}) : \boldsymbol{\tau}_e \\
&\dot{\boldsymbol{\alpha}}_\mathbf{c}(\mathbf{x}) = \dot{\overline{\boldsymbol{\alpha}}} - \mathbf{P}_M(\mathbf{x}) : \dot{\mathbf{a}}_s + \mathbf{P}_{MC}(\mathbf{x}) : \dot{\mathbf{a}}_s - \mathbf{P}_{M_0}(\mathbf{x}) : \boldsymbol{\tau}_c \\
&\boldsymbol{\alpha}_\mathbf{s}(\mathbf{x}) = (\chi(\mathbf{x}) - f)\mathbf{a}_s - \mathbf{P}_C(\mathbf{x}) : \mathbf{a}_s \\
&\dot{\boldsymbol{\alpha}}_\mathbf{s}(\mathbf{x}) = (\chi(\mathbf{x}) - f)\dot{\mathbf{a}}_s - \mathbf{P}_C(\mathbf{x}) : \dot{\mathbf{a}}_s
\end{aligned} \tag{13}$$

with

$$
\begin{aligned}
&\mathbf{P}_{CC}(\mathbf{x}) = \mathbf{\Gamma}_{C_0}\left[\chi(1-f)\delta\mathbf{C} : \mathbf{e}_i \otimes \mathbf{e}_j\right] - \mathbf{\Gamma}_{C_0}\left[\chi\delta\mathbf{C} : \mathbf{\Gamma}_{C_0}\left(\chi\mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j\right)\right]\\
&\mathbf{P}_{C_0}(\mathbf{x}) = \mathbf{\Gamma}_{C_0}(\chi\mathbf{e}_i \otimes \mathbf{e}_j)(\mathbf{x})\\
&\mathbf{P}_{M}(\mathbf{x}) = \mathbf{\Gamma}_{M_0}\left[(\chi - f)\mathbf{M} : \mathbf{e}_i \otimes \mathbf{e}_j\right]\\
&\mathbf{P}_{MC}(\mathbf{x}) = \mathbf{\Gamma}_{M_0}\left[\mathbf{M} : \mathbf{\Gamma}_{C_0}(\chi\mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j)\right]\\
&\mathbf{P}_{M_0}(\mathbf{x}) = \mathbf{\Gamma}_{M_0}(\chi\mathbf{e}_i \otimes \mathbf{e}_j)\\
&\mathbf{P}_{C}(\mathbf{x}) = \mathbf{\Gamma}_{C_0}(\chi\mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j)
\end{aligned}
\tag{14}
$$

3 The resulting potentials

3.1 Helmoltz free energy

We have to minimize the Helmoltz energy associated to the trial fields $\mathbf{e}$ and $\boldsymbol{\alpha}_\mathbf{s}$

$$w(\mathbf{e}, \boldsymbol{\alpha}_\mathbf{s}) = \int \frac{1}{2}(\mathbf{e} - \boldsymbol{\alpha}_s) : \mathbf{C} : (\mathbf{e} - \boldsymbol{\alpha}_s) \tag{15}$$

where

$$\mathbf{e}(\mathbf{x}) - \boldsymbol{\alpha}_\mathbf{s}(\mathbf{x}) = \overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s - \mathbf{P}_{C_0}(\mathbf{x}) : \boldsymbol{\tau}_e \tag{16}$$

with

$$\mathbf{P}(\mathbf{x}) = \mathbf{P}_{CC}(\mathbf{x}) - (\chi(\mathbf{x}) - f)\mathbf{I} + \mathbf{P}_C(\mathbf{x}) \tag{17}$$

The minimization is on $\boldsymbol{\tau}_e$ only. This minimization (derivation w. r. t. $\boldsymbol{\tau}_e$) gives:

$$\int {}^t\mathbf{P}_{C_0} : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P} : \mathbf{a}_s) = \int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P}_{C_0} : \boldsymbol{\tau}_e \tag{18}$$

and the resulting free energy is

$$
\begin{aligned}
w(\overline{\mathbf{e}}, \mathbf{a}_s) &= \int \frac{1}{2}(\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s - \mathbf{P}_{C_0}(\mathbf{x}) : \boldsymbol{\tau}_e)\\
&= \int \frac{1}{2}(\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) - \int \frac{1}{2}(\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) : \mathbf{C} : \mathbf{P}_{C_0}(\mathbf{x}) : \boldsymbol{\tau}_e\\
&= \int \frac{1}{2}(\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) - \boldsymbol{\tau}_e : \int \frac{1}{2} {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P}_{C_0} : \boldsymbol{\tau}_e
\end{aligned}
\tag{19}
$$

3.2 Dissipation potential

We have to minimize the dissipation potential associated to the trial fields $\dot{\boldsymbol{\alpha}}_\mathbf{c}$ and $\dot{\boldsymbol{\alpha}}_\mathbf{s}$

$$\varphi(\dot{\boldsymbol{\alpha}}_\mathbf{c} + \dot{\boldsymbol{\alpha}}_\mathbf{s}) = \int \frac{1}{2}(\dot{\boldsymbol{\alpha}}_\mathbf{c} + \dot{\boldsymbol{\alpha}}_\mathbf{s}) : \mathbf{M} : (\dot{\boldsymbol{\alpha}}_\mathbf{c} + \dot{\boldsymbol{\alpha}}_\mathbf{s}) \tag{20}$$

where

$$\begin{aligned}\dot{\boldsymbol{\alpha}}_{\mathbf{c}}(\mathbf{x})+\dot{\boldsymbol{\alpha}}_{\mathbf{s}}(\mathbf{x}) &= \dot{\overline{\boldsymbol{\alpha}}}-\mathbf{P}_M(\mathbf{x}):\dot{\mathbf{a}}_s+\mathbf{P}_{MC}(\mathbf{x}):\dot{\mathbf{a}}_s-\mathbf{P}_{M_0}(\mathbf{x}):\boldsymbol{\tau}_c\\ &\quad +(\chi(\mathbf{x})-f)\dot{\mathbf{a}}_s-\mathbf{P}_C(\mathbf{x}):\dot{\mathbf{a}}_s\\ &= \dot{\overline{\boldsymbol{\alpha}}}+\mathbf{Q}(\mathbf{x}):\dot{\mathbf{a}}_s-\mathbf{P}_{M_0}(\mathbf{x}):\boldsymbol{\tau}_c\end{aligned} \tag{21}$$

with

$$\mathbf{Q}(\mathbf{x})=\mathbf{P}_{MC}(\mathbf{x})-\mathbf{P}_M(\mathbf{x})+(\chi(\mathbf{x})-f)\mathbf{I}-\mathbf{P}_C(\mathbf{x}). \tag{22}$$

The minimization is on $\boldsymbol{\tau}_c$ only. It gives:

$$\int {}^t\mathbf{P}_{M_0}:\mathbf{M}:(\dot{\overline{\boldsymbol{\alpha}}}+\mathbf{Q}(\mathbf{x}):\dot{\mathbf{a}}_s)=\int {}^t\mathbf{P}_{M_0}:\mathbf{M}:\mathbf{P}_{M_0}:\boldsymbol{\tau}_c \tag{23}$$

and the resulting dissipation potential is

$$\begin{aligned}\varphi(\dot{\overline{\boldsymbol{\alpha}}},\boldsymbol{\alpha}_{s,n},\mathbf{a}_s) &= \int\frac{1}{2}(\dot{\overline{\boldsymbol{\alpha}}}+\mathbf{Q}(\mathbf{x}):\dot{\mathbf{a}}_s):\mathbf{M}:(\dot{\overline{\boldsymbol{\alpha}}}+\mathbf{Q}(\mathbf{x}):\dot{\mathbf{a}}_s)\\ &\quad -\boldsymbol{\tau}_c:\int\frac{1}{2}{}^t\mathbf{P}_{M_0}:\mathbf{M}:\mathbf{P}_{M_0}:\boldsymbol{\tau}_c\end{aligned} \tag{24}$$

3.3 Incremental potential

We then minimize the incremental potential w. r. t. $\mathbf{a}_s$, and also w. r. t. $\overline{\boldsymbol{\alpha}}$. First, (18) gives

$$\begin{aligned}&\int {}^t\mathbf{P}_{C_0}:\mathbf{C}:\mathbf{P}=\int {}^t\mathbf{P}_{C_0}:\mathbf{C}:\mathbf{P}_{C_0}:\frac{\partial\boldsymbol{\tau}_e}{\partial\mathbf{a}_s}\\ &-\int {}^t\mathbf{P}_{C_0}:\mathbf{C}=\int {}^t\mathbf{P}_{C_0}:\mathbf{C}:\mathbf{P}_{C_0}:\frac{\partial\boldsymbol{\tau}_e}{\partial\overline{\boldsymbol{\alpha}}}\end{aligned} \tag{25}$$

and (23) gives

$$\begin{aligned}&\frac{1}{\Delta t}\int {}^t\mathbf{P}_{M_0}:\mathbf{M}:\mathbf{Q}=\int {}^t\mathbf{P}_{M_0}:\mathbf{M}:\mathbf{P}_{M_0}:\frac{\partial\boldsymbol{\tau}_c}{\partial\mathbf{a}_s}\\ &\frac{1}{\Delta t}\int {}^t\mathbf{P}_{M_0}:\mathbf{M}=\int {}^t\mathbf{P}_{M_0}:\mathbf{M}:\mathbf{P}_{M_0}:\frac{\partial\boldsymbol{\tau}_c}{\partial\overline{\boldsymbol{\alpha}}}\end{aligned} \tag{26}$$

Hence,

$$\begin{aligned}\frac{\partial w}{\partial\mathbf{a}_s} &= \int {}^t\mathbf{P}:\mathbf{C}:(\overline{\mathbf{e}}+\mathbf{P}(\mathbf{x}):\mathbf{a}_s)-\boldsymbol{\tau}_e:\int {}^t\mathbf{P}_{C_0}:\mathbf{C}:\mathbf{P}_{C_0}:\frac{\partial\boldsymbol{\tau}_e}{\partial\mathbf{a}_s}\\ &= \int {}^t\mathbf{P}:\mathbf{C}:(\overline{\mathbf{e}}+\mathbf{P}(\mathbf{x}):\mathbf{a}_s)-\boldsymbol{\tau}_e:\int {}^t\mathbf{P}_{C_0}:\mathbf{C}:\mathbf{P}\\ &= \int\left[{}^t\mathbf{P}(\mathbf{x})-{}^t\boldsymbol{\rho}_0:\boldsymbol{\Lambda}^{-1}:{}^t\mathbf{P}_{C_0}(\mathbf{x})\right]:\mathbf{C}:(\overline{\mathbf{e}}+\mathbf{P}(\mathbf{x}):\mathbf{a}_s)\end{aligned} \tag{27}$$

and

$$\frac{\partial \varphi}{\partial \mathbf{a}_s} = \frac{1}{\Delta t} \int \left[{}^t\mathbf{Q}(\mathbf{x}) - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) \tag{28}$$

with

$$\begin{aligned}
\boldsymbol{\Lambda} &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P}_{C_0} \\
\boldsymbol{\Theta} &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} : \mathbf{P}_{M_0} \\
\boldsymbol{\rho}_0 &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P} \\
\boldsymbol{\pi}_0 &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} : \mathbf{Q}
\end{aligned} \tag{29}$$

The first evolution equation is then given by

$$\begin{aligned}
&\int \left[{}^t\mathbf{P}(\mathbf{x}) - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) \right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\
&\quad + \int \left[{}^t\mathbf{Q}(\mathbf{x}) - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0
\end{aligned} \tag{30}$$

which is also

$$\begin{aligned}
&\int \left[{}^t\mathbf{P}(\mathbf{x}) - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) \right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\
&\quad + \int \left[{}^t\mathbf{Q}(\mathbf{x}) - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0
\end{aligned} \tag{31}$$

with

$$\begin{aligned}
\boldsymbol{\rho} &= \int {}^t\mathbf{P}(\mathbf{x}) : \mathbf{C} : \mathbf{P}(\mathbf{x}) \\
\boldsymbol{\pi} &= \int {}^t\mathbf{Q}(\mathbf{x}) : \mathbf{M} : \mathbf{Q}(\mathbf{x})
\end{aligned} \tag{32}$$

As for the derivation w. r. t. $\overline{\boldsymbol{\alpha}}$,

$$\begin{aligned}
\frac{\partial w}{\partial \overline{\boldsymbol{\alpha}}} &= -\int \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) + \boldsymbol{\tau}_e : \int {}^t\mathbf{P}_{C_0} : \mathbf{C} \\
&= \int \left[{}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) - \mathbf{I} \right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s)
\end{aligned} \tag{33}$$

and

$$\frac{\partial \varphi}{\partial \overline{\boldsymbol{\alpha}}} = \int \left[\frac{1}{\Delta t}\mathbf{I} - \frac{1}{\Delta t} {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) \tag{34}$$

with

$$
\begin{aligned}
\boldsymbol{\eta}_0 &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} \\
\boldsymbol{\zeta}_0 &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} \\
\boldsymbol{\eta} &= \int {}^t\mathbf{P} : \mathbf{C} \\
\boldsymbol{\zeta} &= \int {}^t\mathbf{Q} : \mathbf{M}
\end{aligned} \tag{35}
$$

The second evolution equation is then

$$
\begin{aligned}
&\int \left[{}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) - \mathbf{I}\right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\
&\quad + \int \left[\mathbf{I} - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x})\right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0
\end{aligned} \tag{36}
$$

3.4 Summary

We then have two differential equations:

$$
\begin{aligned}
\boldsymbol{\gamma}_1 : (\overline{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\alpha}}) + \boldsymbol{\beta}_1 : \mathbf{a}_s + \boldsymbol{\lambda}_1 : \dot{\overline{\boldsymbol{\alpha}}} + \boldsymbol{\mu}_1 : \dot{\mathbf{a}}_s = 0 \\
\boldsymbol{\gamma}_2 : (\overline{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\alpha}}) + \boldsymbol{\beta}_2 : \mathbf{a}_s + \boldsymbol{\lambda}_2 : \dot{\overline{\boldsymbol{\alpha}}} + \boldsymbol{\mu}_2 : \dot{\mathbf{a}}_s = 0
\end{aligned} \tag{37}
$$

with

$$
\begin{aligned}
\boldsymbol{\gamma}_1 &= \boldsymbol{\eta} - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\eta}_0 \\
\boldsymbol{\beta}_1 &= \boldsymbol{\rho} - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\rho}_0 \\
\boldsymbol{\lambda}_1 &= \boldsymbol{\zeta} - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\zeta}_0 \\
\boldsymbol{\mu}_1 &= \boldsymbol{\pi} - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\pi}_0 \\
\boldsymbol{\gamma}_2 &= -\mathbf{C}^V + {}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\eta}_0 \\
\boldsymbol{\beta}_2 &= -{}^t\boldsymbol{\eta} + {}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\rho}_0 \\
\boldsymbol{\lambda}_2 &= \mathbf{M}^V - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\zeta}_0 \\
\boldsymbol{\mu}_2 &= {}^t\boldsymbol{\zeta} - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\pi}_0
\end{aligned} \tag{38}
$$

4 Macroscopic stress and tangent operator

The macroscopic stress is given by

$$
\begin{aligned}
\overline{\boldsymbol{\sigma}} &= \frac{\partial \overline{w}}{\partial \overline{\boldsymbol{\varepsilon}}} \\
&= \mathbf{C}^V : \overline{\mathbf{e}} + {}^t\boldsymbol{\eta} : \mathbf{a}_s - \boldsymbol{\tau}_e : \boldsymbol{\eta}_0 \\
&= \mathbf{C}^V : \overline{\mathbf{e}} + {}^t\boldsymbol{\eta} : \mathbf{a}_s - {}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : (\boldsymbol{\eta}_0 : \overline{\mathbf{e}} + \boldsymbol{\rho}_0 : \mathbf{a}_s) \\
&= -\boldsymbol{\gamma}_2 : \overline{\mathbf{e}} - \boldsymbol{\beta}_2 : \mathbf{a}_s
\end{aligned} \tag{39}
$$

And the macroscopic tangent operator is given by

$$\frac{\partial \overline{\boldsymbol{\sigma}}}{\partial \Delta \overline{\boldsymbol{\varepsilon}}} = -\boldsymbol{\gamma}_2 : \left(\mathbf{I} - \frac{\partial \overline{\boldsymbol{\alpha}}}{\partial \overline{\boldsymbol{\varepsilon}}} \right) - \boldsymbol{\beta}_2 : \frac{\partial \mathbf{a}_s}{\partial \overline{\boldsymbol{\varepsilon}}} \tag{40}$$

and the derivatives are obtained by deriving the two differential equations, which gives:

$$\begin{pmatrix} \dfrac{\boldsymbol{\lambda}_1}{\Delta t} - \boldsymbol{\gamma}_1 & \dfrac{\boldsymbol{\mu}_1}{\Delta t} + \boldsymbol{\beta}_1 \\ \\ \dfrac{\boldsymbol{\lambda}_2}{\Delta t} - \boldsymbol{\gamma}_2 & \dfrac{\boldsymbol{\mu}_2}{\Delta t} + \boldsymbol{\beta}_2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \overline{\boldsymbol{\alpha}}}{\partial \overline{\boldsymbol{\varepsilon}}} \\ \\ \dfrac{\partial \mathbf{a}_s}{\partial \overline{\boldsymbol{\varepsilon}}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\gamma}_1 \\ \\ \\ -\boldsymbol{\gamma}_2 \end{pmatrix} \tag{41}$$

A Evolution equations

A.1 First evolution equation

$$\begin{aligned} &\int \left[{}^t\mathbf{P}(\mathbf{x}) - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) \right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\ &\quad + \int \left[{}^t\mathbf{Q}(\mathbf{x}) - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0 \end{aligned} \tag{42}$$

We can note that

$$\frac{\mathbf{H}_n(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) : \mathbf{a}_s}{\Delta t} = \mathbf{u} - \boldsymbol{\Gamma}_{M_0}(\mathbf{M} : \mathbf{u}) \qquad (= \mathbf{M} : \dot{\boldsymbol{\alpha}}_{\mathbf{s}}) \tag{43}$$

with [...] Hence assuming that

$$\boldsymbol{\alpha}_{s,n}(\mathbf{x}) = (\chi(\mathbf{x}) - f)\mathbf{a}_{s,n} - \mathbf{P}_C(\mathbf{x}) : \mathbf{a}_{s,n} \tag{44}$$

with $\mathbf{a}_{s,n}$ uniform, we have

$$\mathbf{u} = (\chi(\mathbf{x}) - f)\dot{\mathbf{a}}_s - \mathbf{P}_C(\mathbf{x}) : \dot{\mathbf{a}}_s \tag{45}$$

and

$$\begin{aligned} \frac{\mathbf{H}_n(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) : \mathbf{a}_s}{\Delta t} &= [(\chi(\mathbf{x}) - f)\mathbf{I} - \mathbf{P}_C(\mathbf{x})] : \dot{\mathbf{a}}_s - \boldsymbol{\Gamma}_{M_0} \left[\mathbf{M} : ((\chi(\mathbf{x}) - f) - \mathbf{P}_C(\mathbf{x})) : \mathbf{e}_i \otimes \mathbf{e}_j \right] : \dot{\mathbf{a}}_s \\ &= [(\chi(\mathbf{x}) - f)\mathbf{I} - \mathbf{P}_C(\mathbf{x}) - \mathbf{P}_M(\mathbf{x}) + \mathbf{P}_{MC}(\mathbf{x})] : \dot{\mathbf{a}}_s \\ &(= \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) \end{aligned} \tag{46}$$

We then have

$$\begin{aligned} &\int \left[{}^t\mathbf{P}(\mathbf{x}) - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) \right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\ &\quad + \int \left[{}^t\mathbf{Q}(\mathbf{x}) - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x}) \right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0 \end{aligned} \tag{47}$$

$$\begin{aligned} &\boldsymbol{\rho} : \mathbf{a}_s + \boldsymbol{\eta} : \overline{\mathbf{e}} - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\rho}_0 : \mathbf{a}_s - {}^t\boldsymbol{\rho}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\eta}_0 : \overline{\mathbf{e}} \\ &\quad + \boldsymbol{\pi} : \dot{\mathbf{a}}_s + \boldsymbol{\zeta} : \dot{\overline{\boldsymbol{\alpha}}} - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\pi}_0 : \dot{\mathbf{a}}_s - {}^t\boldsymbol{\pi}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\zeta}_0 : \dot{\overline{\boldsymbol{\alpha}}} = 0 \end{aligned} \tag{48}$$

A.2 Second evolution equation

$$\begin{aligned}&\int \left[{}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) - \mathbf{I}\right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\ &\quad + \int \left[\mathbf{I} - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x})\right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0\end{aligned} \tag{49}$$

is also

$$\begin{aligned}&\int \left[{}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : {}^t\mathbf{P}_{C_0}(\mathbf{x}) - \mathbf{I}\right] : \mathbf{C} : (\overline{\mathbf{e}} + \mathbf{P}(\mathbf{x}) : \mathbf{a}_s) \\ &\quad + \int \left[\mathbf{I} - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : {}^t\mathbf{P}_{M_0}(\mathbf{x})\right] : \mathbf{M} : (\dot{\overline{\boldsymbol{\alpha}}} + \mathbf{Q}(\mathbf{x}) : \dot{\mathbf{a}}_s) = 0\end{aligned} \tag{50}$$

$$\begin{aligned}&-{}^t\boldsymbol{\eta} : \mathbf{a}_s - \int {}^t\mathbf{C}(\mathbf{x}) : \overline{\mathbf{e}} + {}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\eta}_0 : \overline{\mathbf{e}} + {}^t\boldsymbol{\eta}_0 : \boldsymbol{\Lambda}^{-1} : \boldsymbol{\rho}_0 : \mathbf{a}_s \\ &\quad + \int \mathbf{M}(\mathbf{x}) : \dot{\overline{\boldsymbol{\alpha}}} + {}^t\boldsymbol{\zeta} : \dot{\mathbf{a}}_s - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\zeta}_0 : \dot{\overline{\boldsymbol{\alpha}}} - {}^t\boldsymbol{\zeta}_0 : \boldsymbol{\Theta}^{-1} : \boldsymbol{\pi}_0 : \dot{\mathbf{a}}_s = 0\end{aligned} \tag{51}$$

B Computation of some tensors

B.1

$$\mathbf{A}(\mathbf{x}) = (\chi(\mathbf{x}) - f)\mathbf{I} - \boldsymbol{\Gamma}_{C_0}(\chi\mathbf{C}_0 : \mathbf{e}_i \otimes \mathbf{e}_j) \tag{52}$$

$$\mathbf{P}(\mathbf{x}) = \boldsymbol{\Gamma}_{C_0}\left[\chi\delta\mathbf{C} : \mathbf{A}\right] - \mathbf{A}(\mathbf{x}) \tag{53}$$

$$\mathbf{Q}(\mathbf{x}) = -\boldsymbol{\Gamma}_{M_0}\left[\mathbf{M} : \mathbf{A}\right] + \mathbf{A}(\mathbf{x}) \tag{54}$$

$$\begin{aligned}\boldsymbol{\Lambda} &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P}_{C_0} = {}^t\mathbf{P}_0^C : \left(\delta\mathbf{C} : \mathbf{P}_0^C - \mathbf{I}\right) \\ \boldsymbol{\Theta} &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} : \mathbf{P}_{M_0} = {}^t\mathbf{P}_0^M : \left(\delta\mathbf{M} : \mathbf{P}_0^M - \mathbf{I}\right) \\ \boldsymbol{\rho}_0 &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P} \\ \boldsymbol{\pi}_0 &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} : \mathbf{Q}\end{aligned} \tag{55}$$

$$\begin{aligned}\boldsymbol{\rho} &= \int {}^t\mathbf{P}(\mathbf{x}) : \mathbf{C} : \mathbf{P}(\mathbf{x}) \\ \boldsymbol{\pi} &= \int {}^t\mathbf{Q}(\mathbf{x}) : \mathbf{M} : \mathbf{Q}(\mathbf{x})\end{aligned} \tag{56}$$

$$\begin{aligned}\boldsymbol{\eta}_0 &= \int {}^t\mathbf{P}_{C_0} : \mathbf{C} \\ \boldsymbol{\zeta}_0 &= \int {}^t\mathbf{P}_{M_0} : \mathbf{M} \\ \boldsymbol{\eta} &= \int {}^t\mathbf{P} : \mathbf{C} \\ \boldsymbol{\zeta} &= \int {}^t\mathbf{Q} : \mathbf{M}\end{aligned} \tag{57}$$

B.2

We have

$$\int {}^t\mathbf{P}_{C_0} : \mathbf{C} : \mathbf{P}_{C_0} = \int {}^t\mathbf{P}_{C_0} : \mathbf{C}_0 : \mathbf{P}_{C_0} + \int {}^t\mathbf{P}_{C_0} : \delta\mathbf{C} : \mathbf{P}_{C_0} \tag{58}$$

The first integral can be written as

$$\begin{aligned}\int {}^t\mathbf{P}_{C_0} : \mathbf{C}_0 : \mathbf{P}_{C_0} &= \int {}^t\mathbf{P}_{C_0} : [\mathbf{C}_0 : \mathbf{\Gamma}_{C_0}(\chi\mathbf{e}_i \otimes \mathbf{e}_j) - \chi\mathbf{e}_i \otimes \mathbf{e}_j] - \int {}^t\mathbf{P}_{C_0} : \chi\mathbf{e}_i \otimes \mathbf{e}_j \\ &= 0 - \int {}^t\mathbf{P}_{C_0} : \chi\mathbf{e}_i \otimes \mathbf{e}_j \\ &= -f\,{}^t\mathbf{P}_0^C\end{aligned} \tag{59}$$

and the second is

$$\int {}^t\mathbf{P}_{C_0} : \delta\mathbf{C} : \mathbf{P}_{C_0} = {}^t\mathbf{P}_0^C : (\mathbf{C}_1 - \mathbf{C}_0) : \mathbf{P}_0^C \tag{60}$$

B.3

Moreover,

$$\int {}^t\mathbf{P}_{M_0} : \mathbf{M} : \mathbf{P}_{M_0} = \int {}^t\mathbf{P}_{M_0} : \mathbf{M}_0 : \mathbf{P}_{M_0} + \int {}^t\mathbf{P}_{M_0} : \delta\mathbf{M} : \mathbf{P}_{M_0} \tag{61}$$

The first integral can be written as

$$\begin{aligned}\int {}^t\mathbf{P}_{M_0} : \mathbf{M}_0 : \mathbf{P}_{M_0} &= \int {}^t\mathbf{P}_{M_0} : [\mathbf{M}_0 : \mathbf{\Gamma}_{M_0}(\chi\mathbf{e}_i \otimes \mathbf{e}_j) - \chi\mathbf{e}_i \otimes \mathbf{e}_j] - \int {}^t\mathbf{P}_{M_0} : \chi\mathbf{e}_i \otimes \mathbf{e}_j \\ &= 0 - \int {}^t\mathbf{P}_{M_0} : \chi\mathbf{e}_i \otimes \mathbf{e}_j \\ &= -f\,{}^t\mathbf{P}_0^M\end{aligned} \tag{62}$$

and the second is

$$\int {}^t\mathbf{P}_{M_0} : \delta\mathbf{M} : \mathbf{P}_{M_0} = {}^t\mathbf{P}_0^M : (\mathbf{M}_1 - \mathbf{M}_0) : \mathbf{P}_0^M \tag{63}$$

B.4

$$\int \mathbf{C}(\mathbf{x}) : \mathbf{P}(\mathbf{x}) = \int \mathbf{C}_0 : \mathbf{P}(\mathbf{x}) + \int \delta\mathbf{C} : \mathbf{P}(\mathbf{x}) \tag{64}$$

The first integral is, because of the fact that $\int \mathbf{P}_{CC} = 0$ and $\int \mathbf{P}_C = 0$,

$$\int \mathbf{C}_0 : \mathbf{P}(\mathbf{x}) = -\int (\chi - f)\mathbf{C}_0 = 0 \tag{65}$$

The second one is

$$\begin{aligned}\int \delta\mathbf{C}(\mathbf{x}) : \mathbf{P}(\mathbf{x}) &= \delta\mathbf{C} : \int \chi(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ &= -f(1-f)\delta\mathbf{C} + \delta\mathbf{C}\end{aligned} \tag{66}$$

Hence,

B.5

In the same way,

$$\int {}^t\mathbf{P}(\mathbf{x}) : \mathbf{C}(\mathbf{x}) : \mathbf{P}(\mathbf{x}) = \int {}^t\mathbf{P}(\mathbf{x}) : \mathbf{C}_0 : \mathbf{P}(\mathbf{x}) + \int {}^t\mathbf{P}(\mathbf{x}) : \delta\mathbf{C} : \mathbf{P}(\mathbf{x}) \tag{67}$$

The first integral is
and the second
Hence