1 Minimisation of potential energy

We consider a periodic cell Z of a random heterogeneous material. The conductivity at $\mathbf{x} \in Z$ is $\boldsymbol{\sigma}(\mathbf{x})$ (symmetric, positive definite, second-order tensor); $\mathbf{E}(\mathbf{x})$, $\phi(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ denote the electric field, the electric potential and the volumic current, respectively, at point $\mathbf{x}$. The apparent conductivity of the cell Z, $\boldsymbol{\sigma}^{\mathrm{app}}$, is found from the solution to the following problem

$$Z: \quad \operatorname{div}\mathbf{j} = 0, \tag{1}$$

$$Z: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{2}$$

$$Z: \quad \mathbf{E} = \overline{\mathbf{E}} + \mathbf{grad}\,\phi^{\mathrm{per}}, \tag{3}$$

where ϕ^{per} is a Z-periodic field, $\mathbf{j}$ is a Z-antiperiodic field, and $\overline{\mathbf{E}}$ is a prescribed constant vector. Eq. (3) ensures that the electric field is curl-free, with $\overline{\mathbf{E}} = \langle \mathbf{E} \rangle$, where angle brackets denote volume averages over the cell Z. We consider a biphasic cell with a matrix whose conductivity is $\boldsymbol{\sigma}_0$ and another phase whose conductivity is $\boldsymbol{\sigma}_1$.

We introduce the linear Green operator $\boldsymbol{\Gamma}_0^{\mathrm{per}}$, associated with the conductivity $\boldsymbol{\sigma}_0$.

The electrical field solution of problem (1)-(3) minimizes the following energy:

$$\mathcal{W}(\mathbf{E}) = \frac{1}{2} \int_Z \mathbf{E} \cdot \boldsymbol{\sigma} \cdot \mathbf{E}. \tag{4}$$

We then consider the following electrical field:

$$\mathbf{E} = \overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\chi\overline{\boldsymbol{\tau}}) \tag{5}$$

where $\overline{\boldsymbol{\tau}}$ is a constant vector and χ denotes characteristic function of medium whose conductivity is $\boldsymbol{\sigma}_1$. This field is curl-free, and its average is $\overline{\mathbf{E}}$. We then define

$$
\begin{aligned}
\mathcal{E}(\overline{\boldsymbol{\tau}}) &= \mathcal{W}\left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\chi\overline{\boldsymbol{\tau}})\right) \\
&= \frac{1}{2} \int_Z \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\chi\overline{\boldsymbol{\tau}})\right) \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\mathrm{per}}(\chi\overline{\boldsymbol{\tau}})\right) \\
&= \frac{1}{2} \int_Z \left(\overline{\mathbf{E}} - \mathbf{H} \cdot \overline{\boldsymbol{\tau}}\right) \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \mathbf{H} \cdot \overline{\boldsymbol{\tau}}\right)
\end{aligned} \tag{6}
$$

where

$$\mathbf{H}_{ij}(\mathbf{x}) = \boldsymbol{\Gamma}_i^{0,\mathrm{per}}(\chi\mathbf{e}_j)(\mathbf{x}), \qquad \text{(symmetric if isotropy of } \boldsymbol{\sigma}_0 \text{ ?)} \tag{7}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}$:

$$\frac{\partial \mathcal{E}}{\partial \overline{\boldsymbol{\tau}}} = -\int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \mathbf{H} \cdot \overline{\boldsymbol{\tau}}\right) \tag{8}$$

which is null if

$$\int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} = \int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \cdot \mathbf{H} \cdot \overline{\boldsymbol{\tau}}. \tag{9}$$

Because $\boldsymbol{\sigma}$ is positive definite we can (except if $\mathbf{H}$ is null everywhere) obtain a candidate $\overline{\boldsymbol{\tau}}$:

$$\overline{\boldsymbol{\tau}} = \mathbf{G}^{-1} \int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}}, \tag{10}$$

where

$$\mathbf{G} = \int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \cdot \mathbf{H} \tag{11}$$

defines a positive definite quadratic form of $\boldsymbol{\Gamma}_0^{\mathrm{per}}$. Coming back to the energy, we can state that

$$\frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\mathrm{app}} \cdot \overline{\mathbf{E}} \leq \mathcal{E}(\overline{\boldsymbol{\tau}})$$
$$\leq \frac{1}{2} \int_Z \overline{\mathbf{E}} \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} - \frac{1}{2}\overline{\boldsymbol{\tau}} \cdot \mathbf{G} \cdot \overline{\boldsymbol{\tau}} \tag{12}$$
$$\leq \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} - \frac{1}{2}\overline{\boldsymbol{\tau}} \cdot \mathbf{G} \cdot \overline{\boldsymbol{\tau}}$$

2 Minimisation of the complementary energy

2.1 Current density version of Lippmann-Schwinger equation

It is known that another problem can be considered to compute another apparent conductivity:

$$Z: \quad \operatorname{div}\mathbf{j} = 0, \qquad \langle \mathbf{j} \rangle = \overline{\mathbf{J}}, \tag{13}$$
$$Z: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{14}$$
$$Z: \quad \mathbf{E} = \langle \mathbf{E} \rangle + \mathbf{grad}\,\phi^{\mathrm{per}}. \tag{15}$$

where $\overline{\mathbf{J}}$ is a prescribed constant vector, and $\mathbf{j}$ remains a Z-antiperiodic field.

The above problem can be rewritten as

$$Z: \quad \operatorname{div}\mathbf{j} = 0, \qquad \langle \mathbf{j} \rangle = \overline{\mathbf{J}}, \tag{16}$$
$$Z: \quad \mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T, \qquad \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}, \tag{17}$$
$$Z: \quad \mathbf{E} = \langle \mathbf{E} \rangle + \mathbf{grad}\,\phi^{\mathrm{per}} \tag{18}$$
$$\tag{19}$$

where $\boldsymbol{\rho} = \boldsymbol{\sigma}^{-1}$.

And the solution of this problem is also the solution of this equation:

$$\mathbf{j} = \overline{\mathbf{J}} + \boldsymbol{\sigma}_0 \langle \mathbf{E}^T \rangle + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \qquad \text{with} \quad \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}. \quad (20)$$

Indeed, (i) the average of this field is $\overline{\mathbf{J}}$, (ii) its divergence is null because the following equation is verified (by definition of $\boldsymbol{\Gamma}_0^{\mathrm{per}}$):

$$\mathrm{div} \left[\boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right] = 0 \quad (21)$$

and (iii) the corresponding field $\mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T$ can be written

$$\mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle + \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \mathbf{E}^T + \mathbf{E}^T = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle + \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right)$$
$$(22)$$

which is Eq.(18) with $\langle \mathbf{E} \rangle = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle$. Hence, rewriting Eq.(20),

$$\mathbf{j} - \overline{\mathbf{J}} = -\boldsymbol{\sigma}_0 \left(\mathbf{E}^T - \langle \mathbf{E}^T \rangle \right) + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) \qquad \text{with} \quad \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}. \quad (23)$$

Or

$$\mathbf{j}^* = \boldsymbol{\tau}^* - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\tau} \right), \qquad \text{where} \qquad \boldsymbol{\tau} = -\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T = -\boldsymbol{\sigma}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j} \quad (24)$$

and $\mathbf{a}^*$ is a notation for $\mathbf{a} - \langle \mathbf{a} \rangle$. This last equation is the current density version of Lippmann-Schwinger equation.

2.2 Minimisation of the complementary energy

It is well-known that solution of problem (13)-(15) is the solution of the minimisation problem:

$$\min_{\mathbf{j} \in \mathcal{S}(\overline{\mathbf{J}})} \mathcal{W}^*(\mathbf{j}) \quad (25)$$

where

$$\mathcal{W}^*(\mathbf{j}) = \frac{1}{2} \int_Z \mathbf{j} \cdot \boldsymbol{\rho} \cdot \mathbf{j}. \quad (26)$$

We then take the polarization $\boldsymbol{\tau} = \chi \overline{\boldsymbol{\tau}}$ and the correponding current density:

$$\mathbf{j} = \overline{\mathbf{J}} + \boldsymbol{\tau}^* - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\tau} \right) = \overline{\mathbf{J}} + \chi^* \overline{\boldsymbol{\tau}} - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\chi \overline{\boldsymbol{\tau}} \right). \quad (27)$$

This field is divergence-free, and its average is $\overline{\mathbf{J}}$.

We also define

$$\begin{aligned}
\mathcal{E}^* \left(\overline{\boldsymbol{\tau}} \right) &= \mathcal{W}^* \left(\overline{\mathbf{J}} + \chi^* \overline{\boldsymbol{\tau}} - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\chi \overline{\boldsymbol{\tau}} \right) \right) \\
&= \mathcal{W}^* \left(\overline{\mathbf{J}} - \mathbf{F} \cdot \overline{\boldsymbol{\tau}} \right) \\
&= \frac{1}{2} \int_Z \left(\overline{\mathbf{J}} - \mathbf{F} \cdot \overline{\boldsymbol{\tau}} \right) \cdot \boldsymbol{\rho} \cdot \left(\overline{\mathbf{J}} - \mathbf{F} \cdot \overline{\boldsymbol{\tau}} \right)
\end{aligned} \quad (28)$$

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where

$$\mathbf{F}_{ij}(\mathbf{x}) = (\boldsymbol{\sigma}_0 \cdot \mathbf{\Gamma}_0^{\text{per}})_i (\chi \mathbf{e}_j)(\mathbf{x}) - \delta_{ij} \chi^*(\mathbf{x}), \qquad \text{(which can be computed from } \mathbf{H}_{ij} \text{ !)}. \tag{29}$$

Minimising this function with respect to $\overline{\boldsymbol{\tau}}$:

$$\frac{\partial \mathcal{E}^*}{\partial \overline{\boldsymbol{\tau}}} = - \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \cdot \left(\overline{\mathbf{J}} - \mathbf{F} \cdot \overline{\boldsymbol{\tau}} \right) \tag{30}$$

which is null if

$$\int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \cdot \overline{\mathbf{J}} = \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \cdot \mathbf{F} \cdot \overline{\boldsymbol{\tau}}. \tag{31}$$

Because $\boldsymbol{\rho}$ is positive definite we can (except if $\mathbf{F}$ is null everywhere) obtain a candidate $\overline{\boldsymbol{\tau}}$:

$$\overline{\boldsymbol{\tau}} = \mathbf{K}^{-1} \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \cdot \overline{\mathbf{J}}, \tag{32}$$

where

$$\mathbf{K} = \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \cdot \mathbf{F}. \tag{33}$$

Coming back to the energy, we can state that

$$\begin{aligned} \frac{1}{2}\overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{\text{app}} \cdot \overline{\mathbf{J}} &\leq \mathcal{E}^*(\overline{\boldsymbol{\tau}}) \\ &\leq \frac{1}{2} \int_Z \overline{\mathbf{J}} \cdot \boldsymbol{\rho} \cdot \overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}} \cdot \mathbf{K} \cdot \overline{\boldsymbol{\tau}} \\ &\leq \frac{1}{2}\overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{Reuss} \cdot \overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}} \cdot \mathbf{K} \cdot \overline{\boldsymbol{\tau}} \end{aligned} \tag{34}$$

which also gives

$$\begin{aligned} \frac{1}{2}\overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{\text{app}} \cdot \overline{\mathbf{J}} &\leq \frac{1}{2}\overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{Reuss} \cdot \overline{\mathbf{J}} - \frac{1}{2}\overline{\mathbf{J}} \cdot \mathbf{M} \cdot \overline{\mathbf{J}} \\ \boldsymbol{\sigma}^{\text{app}} &\geq \left[\boldsymbol{\rho}^{Reuss} - \mathbf{M} \right]^{-1} \end{aligned} \tag{35}$$

where

$$\mathbf{M} = \int_Z {}^t\boldsymbol{\rho} \cdot \mathbf{F} \cdot {}^t\mathbf{K}^{-1} \cdot \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \tag{36}$$

3 Convergence of the upper bound

We then consider the following electrical field:

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \mathbf{\Gamma}_0^{\text{per}} \left[\chi\overline{\boldsymbol{\tau}}_1 - \chi\boldsymbol{\delta\sigma} \cdot \mathbf{\Gamma}_0^{\text{per}} \left(\chi\overline{\boldsymbol{\tau}}_2 - \chi\boldsymbol{\delta\sigma} \cdot \mathbf{\Gamma}_0^{\text{per}} (...) \right) \right] = \overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{\Gamma}_0^{\text{per}} \left[\left(-\chi\boldsymbol{\delta\sigma} \cdot \mathbf{\Gamma}_0^{\text{per}} \right)^{k-1} \left(\chi\overline{\boldsymbol{\tau}}_k \right) \right]$$

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$$\tag{37}$$

where $\overline{\boldsymbol{\tau}}_k$ are constant vectors, and $\boldsymbol{\delta\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_0$. We can write this field as a linear combination of the $\overline{\boldsymbol{\tau}}_k$:

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \tag{38}$$

where

$$\begin{aligned}
\mathbf{H}_{ij}^k(\mathbf{x}) &= \boldsymbol{\Gamma}_i^{0,\mathrm{per}} \left[(-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}})^{k-1} \left(\chi \mathbf{e}_j \right) \right] (\mathbf{x}) \\
&= \boldsymbol{\Gamma}_i^{0,\mathrm{per}} \left[-\chi \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1} \right] (\mathbf{x})
\end{aligned} \tag{39}$$

We then define

$$\begin{aligned}
\mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) &= \mathcal{W} \left(\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) \right) \\
&= \frac{1}{2} \int_Z \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right)
\end{aligned} \tag{40}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}_i$:

$$\frac{\partial \mathcal{E}}{\partial \overline{\boldsymbol{\tau}}_i} = - \int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \tag{41}$$

which is null if

$$\int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} = \sum_{k=1}^{M} \int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k. \tag{42}$$

This is a linear matrix equation and we can write

$$\mathbf{G}_{ik} = \int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{H}^k \qquad \text{and} \quad \mathbf{U}_i = \int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} \tag{43}$$

In fact these expressions can be reduced to integrals over the inclusions only. We hence have

$$\begin{aligned}
\int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma}_0 \cdot \mathbf{H}^k &= \int_Z {}^t \mathbf{H}^i \cdot \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0 \left(-\chi \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1} \right) \\
&= \int_Z {}^t \mathbf{H}^i \cdot \left[\boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0 \left(-\chi \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1} \right) + \chi \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1} - \chi \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1} \right] \\
&= - \int_Z \chi {}^t \mathbf{H}^i \cdot \boldsymbol{\delta\sigma} \cdot \mathbf{H}^{k-1}
\end{aligned}$$

$$\tag{44}$$

where we have used the fact that $\boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0 \left(-\chi \delta \boldsymbol{\sigma} \cdot \mathbf{H}^{k-1} \right) + \chi \delta \boldsymbol{\sigma} \cdot \mathbf{H}^{k-1}$ is a statically admissible field. We obtain

$$\mathbf{G}_{ik} = \int_Z \chi\, {}^t\mathbf{H}^i \cdot \delta\boldsymbol{\sigma} \cdot \left[\mathbf{H}^k - \mathbf{H}^{k-1} \right] \tag{45}$$

and

$$\mathbf{G}_{i1} = \int_Z \chi\, {}^t\mathbf{H}^i \cdot \left(\delta\boldsymbol{\sigma} \cdot \mathbf{H}^k + \mathbf{1} \right) \tag{46}$$

And we also have

$$\mathbf{U}_i = \int_Z \chi\, {}^t\mathbf{H}^i \cdot \delta\boldsymbol{\sigma} \cdot \overline{\mathbf{E}} \tag{47}$$

Coming back to the energy, we can state that

$$\frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\mathrm{app}} \cdot \overline{\mathbf{E}} \leq \mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M)$$

$$\leq \frac{1}{2}\int_Z \overline{\mathbf{E}} \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} - \frac{1}{2}\overline{\boldsymbol{\tau}}_i \cdot \mathbf{G}_{ik} \cdot \overline{\boldsymbol{\tau}}_k$$

$$\leq \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} - \frac{1}{2}\overline{\boldsymbol{\tau}}_i \cdot \mathbf{G}_{ik} \cdot \overline{\boldsymbol{\tau}}_k = \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} - \frac{1}{2}\overline{\boldsymbol{\tau}}_i \cdot \mathbf{U}_i \tag{48}$$

(Einstein's convention on summation of indices) which gives

$$\boldsymbol{\sigma}^{\mathrm{app}} \leq \boldsymbol{\sigma}^{Voigt} - \mathbf{N} \tag{49}$$

with

$$\mathbf{N} = \int_Z {}^t\boldsymbol{\sigma} \cdot \mathbf{H} \cdot {}^t\mathbf{G}^{-1} \cdot \int_Z {}^t\mathbf{H} \cdot \boldsymbol{\sigma} \tag{50}$$

Remark 1. *Is there a convergence of the right-hand side member to the apparent conductivity when $M \to \infty$? I guess yes, because we know that the field obtained with the "basic scheme" is a special case of the field (37), with all $\overline{\boldsymbol{\tau}}_k$ equal to the initial polarization of the basic scheme.*

3.1 Contrast expansions

We assume in the following that the two phases are isotropic, so that we can define the contrast $c = \sigma_1/\sigma_0$. In a first time, we want to find an expansion that permits to compute the bounds easily when changing the contrast. To that extent, let us rewrite $\mathbf{H}_{ij}^k(\mathbf{x})$ as

$$\mathbf{H}_{ij}^k(\mathbf{x}) = (c-1)^{k-1}\boldsymbol{\Gamma}_i^{0,\mathrm{per}} \left[\left(-\chi\sigma_0\boldsymbol{\Gamma}_0^{\mathrm{per}} \right)^{k-1} (\chi\mathbf{e}_j) \right](\mathbf{x}) \tag{51}$$

and hence rewrite Eq. (38) as

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{A}^k \cdot \overline{\boldsymbol{\pi}}_k \tag{52}$$

where

$$\overline{\boldsymbol{\pi}}_k = (c-1)^{k-1}\overline{\boldsymbol{\tau}}_k \tag{53}$$

and

$$\mathbf{A}_{ij}^k(\mathbf{x}) = \boldsymbol{\Gamma}_i^{0,\mathrm{per}}\left[\left(-\chi\sigma_0\boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^{k-1}(\chi\mathbf{e}_j)\right](\mathbf{x}) = \boldsymbol{\Gamma}_i^{0,\mathrm{per}}\left[-\chi\sigma_0\mathbf{A}_{\bullet j}^{k-1}\right](\mathbf{x}) \qquad (k \geq 2) \tag{54}$$

and

$$\mathbf{A}_{ij}^1(\mathbf{x}) = \boldsymbol{\Gamma}_i^{0,\mathrm{per}}(\chi\mathbf{e}_j)(\mathbf{x}) \tag{55}$$

which is independent of the contrast.

Minimising the energy w. r. t. $\overline{\boldsymbol{\pi}}_k$, Eq. (42) must be replaced by

$$\int_Z {}^t\mathbf{A}^i \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} = \sum_{k=1}^{M} \int_Z {}^t\mathbf{A}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{A}^k \cdot \overline{\boldsymbol{\pi}}_k \tag{56}$$

but this equation depends on the contrast. Hence, we rewrite it

$$\left[\sigma_0 \int_Z (1-\chi)\,{}^t\mathbf{A}^i + \sigma_1 \int_Z \chi\,{}^t\mathbf{A}^i\right] \cdot \overline{\mathbf{E}} = \left[\sigma_0 \sum_{k=1}^{M} \int_Z (1-\chi)\,{}^t\mathbf{A}^i \cdot \mathbf{A}^k + \sigma_1 \sum_{k=1}^{M} \int_Z \chi\,{}^t\mathbf{A}^i \cdot \mathbf{A}^k\right] \cdot \overline{\boldsymbol{\pi}}_k \tag{57}$$

which can be re-written:

$$\left[(\sigma_1 - \sigma_0) \int_Z \chi\,{}^t\mathbf{A}^i\right] \cdot \overline{\mathbf{E}} = \left[\sum_{k=1}^{M} \int_Z \chi\,{}^t\mathbf{A}^i \cdot \left((\sigma_1 - \sigma_0)\mathbf{A}^k - \sigma_0\mathbf{A}^{k-1}\right)\right] \cdot \overline{\boldsymbol{\pi}}_k \tag{58}$$

or also

$$(c-1)\mathbf{P}_i = \sum_{k=1}^{M} ((c-1)\mathbf{R}_{ik} - \mathbf{S}_{ik}) \cdot \overline{\boldsymbol{\pi}}_k \tag{59}$$

where

$$\mathbf{P}_i = \int_Z \chi\,{}^t\mathbf{A}^i \cdot \overline{\mathbf{E}}$$

$$\mathbf{R}_{ik} = \int_Z \chi\,{}^t\mathbf{A}^i \cdot \mathbf{A}^k \qquad \mathbf{S}_{ik} = \mathbf{R}_{i,(k-1)}. \tag{60}$$

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The upper bound can also be rewritten

$$\frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\mathrm{app}} \cdot \overline{\mathbf{E}} \le \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} - (c-1)\,\sigma_0 \overline{\boldsymbol{\pi}}_i \cdot \mathbf{P}_i. \tag{61}$$

We could rewrite this inequality as

$$\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\mathrm{app}} \cdot \overline{\mathbf{E}} \le (1-f)\,\sigma_0\,\overline{\mathbf{E}} \cdot \overline{\mathbf{E}} + c\,f\,\sigma_0\,\overline{\mathbf{E}} \cdot \overline{\mathbf{E}} - \sigma_0(c-1)^2\,\mathbf{P} \cdot ((c-1)\mathbf{R} - \mathbf{S})^{-1} \cdot \mathbf{P} \tag{62}$$

where $\mathbf{P}, \mathbf{R}, \mathbf{S}$ do not depend on contrast, and whose expressions are given by the matrix sub-blocks and sub-vectors given by (62).

In fact the $\mathbf{P}_i, \mathbf{R}_{ik}, \mathbf{S}_{ik}$ can be expressed with the Hill tensors. Indeed, assuming that each $\mathbf{A}^k$ is uniform on the inclusion (which is obviously a big approximation),

$$\mathbf{P}_i = f\,(-\sigma_0)^i\,({}^t\mathbf{P}^0)^i\,\frac{1}{-\sigma_0} \cdot \overline{\mathbf{E}} = -\frac{f}{\sigma_0}\left(\frac{f-1}{3}\right)^i \overline{\mathbf{E}}$$

$$\mathbf{R}_{ik} = f\,(-\sigma_0)^{i+k}\,({}^t\mathbf{P}^0)^i \cdot (\mathbf{P}^0)^k\,\frac{1}{\sigma_0^2} = \frac{f}{\sigma_0^2}\left(\frac{f-1}{3}\right)^{i+k}\mathbf{1} \tag{63}$$

$$\mathbf{S}_{ik} = \frac{f}{\sigma_0^2}\left(\frac{f-1}{3}\right)^{i+k-1}\mathbf{1}$$

so that

$$-\frac{f}{\sigma_0}(c-1)\left(\frac{f-1}{3}\right)^i \overline{\mathbf{E}} = f\frac{1}{\sigma_0^2}\left(\frac{f-1}{3}\right)^{i+k}\frac{c+2-f(c-1)}{1-f}\overline{\boldsymbol{\pi}}_k \tag{64}$$

which is

$$-\delta\sigma\,\overline{\mathbf{E}} = \sum_k \left(\frac{f-1}{3}\right)^k \frac{c+2-f(c-1)}{1-f}\overline{\boldsymbol{\pi}}_k \tag{65}$$

There is no possibility to invert the system. If we had assumed all $\overline{\boldsymbol{\pi}}_k$ equals, we would have

$$\overline{\boldsymbol{\pi}} = \delta\sigma\frac{4-f}{c+2-f(c-1)}\overline{\mathbf{E}} \tag{66}$$

and the apparent conductivity is bounded by the equation

$$\frac{\sigma^{\mathrm{app}}}{\sigma_1} \le \frac{1-f}{c} + f - f\,(1-f)\frac{(c-1)^2}{c}\frac{1}{c+2-f(c-1)} \tag{67}$$

We can also rewrite the inverse matrix in the right hand-side (assuming its existence):

$$\left(\frac{1}{c}\mathbf{R} + \mathbf{S}\right)^{-1} = \mathbf{S}^{-1} \cdot (\frac{1}{c}\mathbf{R} \cdot \mathbf{S}^{-1} + \mathbf{Id})^{-1} = \mathbf{S}^{-1} \cdot \sum_{p=0}^{\infty}\frac{(-1)^p}{c^p}\left(\mathbf{R} \cdot \mathbf{S}^{-1}\right)^p \tag{68}$$

Assuming the convergence of our bounds to the apparent conductivity, this last expression would provide us a series expansion (on the contrast) that would converge to the effective conductivity (when $M \to \infty$, and hence, increasing the size of the matrices and vectors $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}$).

4   Convergence of the lower bound

We have the same result for the lower bound. We consider the following electrical current density field:

$$\mathbf{j}(\overline{\boldsymbol{\tau}}_1,...,\overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{J}} - \sum_{k=1}^{M} \left[\langle \boldsymbol{\tau}^k \rangle - \boldsymbol{\tau}^k + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\text{per}} \left(\boldsymbol{\tau}^k \right) \right] \tag{69}$$

where

$$\boldsymbol{\tau}^k(\mathbf{x}) = (-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}})^{k-1} \left(\chi \overline{\boldsymbol{\tau}}_k \right) (\mathbf{x}) \tag{70}$$

and $\overline{\boldsymbol{\tau}}_k$ are constant vectors, and $\boldsymbol{\delta\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_0$. We can rewrite the current density as

$$\mathbf{j}(\overline{\boldsymbol{\tau}}_1,...,\overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{J}} - \sum_{k=1}^{M} \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k \tag{71}$$

where

$$\mathbf{F}_{ij}^k(\mathbf{x}) = \langle \left[(-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}})^{k-1} \right]_i (\chi \mathbf{e}_j)(\mathbf{x}) \rangle - \left[(-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}})^{k-1} \right]_i (\chi \mathbf{e}_j)(\mathbf{x}) + \boldsymbol{\sigma}_0 \cdot \mathbf{H}_{ij}^k(\mathbf{x}). \tag{72}$$

We then define

$$\mathcal{E}^*(\overline{\boldsymbol{\tau}}_1,...,\overline{\boldsymbol{\tau}}_M) = \mathcal{W}^* \left(\mathbf{j}(\overline{\boldsymbol{\tau}}_1,...,\overline{\boldsymbol{\tau}}_M) \right)$$

$$= \frac{1}{2} \int_Z \left(\overline{\mathbf{J}} - \sum_{k=1}^{M} \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \cdot \boldsymbol{\rho} \cdot \left(\overline{\mathbf{J}} - \sum_{k=1}^{M} \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \tag{73}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}_i$:

$$\frac{\partial \mathcal{E}^*}{\partial \overline{\boldsymbol{\tau}}_i} = - \int_Z {}^t\mathbf{F}^i \cdot \boldsymbol{\rho} \cdot \left(\overline{\mathbf{J}} - \sum_{k=1}^{M} \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \tag{74}$$

which is null if

$$\int_Z {}^t\mathbf{F}^i \cdot \boldsymbol{\rho} \cdot \overline{\mathbf{J}} = \sum_{k=1}^{M} \int_Z {}^t\mathbf{F}^i \cdot \boldsymbol{\rho} \cdot \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k. \tag{75}$$

This is a linear matrix equation.

Coming back to the energy, we can state that

$$\frac{1}{2} \overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{\text{app}} \cdot \overline{\mathbf{J}} \leq \mathcal{E}^*(\overline{\boldsymbol{\tau}}_1,...,\overline{\boldsymbol{\tau}}_M)$$

$$\leq \frac{1}{2} \int_Z \overline{\mathbf{J}} \cdot \boldsymbol{\rho} \cdot \overline{\mathbf{J}} - \frac{1}{2} \overline{\boldsymbol{\tau}}_i \cdot \mathbf{K}_{ik} \cdot \overline{\boldsymbol{\tau}}_k \tag{76}$$

$$\leq \frac{1}{2} \overline{\mathbf{J}} \cdot \boldsymbol{\rho}^{Reuss} \cdot \overline{\mathbf{J}} - \frac{1}{2} \overline{\boldsymbol{\tau}}_i \cdot \mathbf{K}_{ik} \cdot \overline{\boldsymbol{\tau}}_k$$

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which gives

$$\boldsymbol{\sigma}^{\text{app}} \geq \left[\boldsymbol{\rho}^{Reuss} - \mathbf{M}\right]^{-1} \tag{77}$$

with

$$\mathbf{M} = \int_Z {}^t\boldsymbol{\rho} \cdot \mathbf{F} \cdot {}^t\mathbf{K}^{-1} \cdot \int_Z {}^t\mathbf{F} \cdot \boldsymbol{\rho} \tag{78}$$

5 Equality of the bounds

$$\begin{aligned}
\boldsymbol{\sigma}^{Voigt} - \mathbf{N} &= \left[\boldsymbol{\rho}^{Reuss} - \mathbf{M}\right]^{-1} \\
\boldsymbol{\sigma}^{Voigt} \cdot \boldsymbol{\rho}^{Reuss} - \mathbf{N} \cdot \boldsymbol{\rho}^{Reuss} - \boldsymbol{\sigma}^{Voigt} \cdot \mathbf{M} + \mathbf{N} \cdot \mathbf{M} &= \mathbf{I} \\
(fc + 1 - f)\,(\mathbf{N} + c\mathbf{M}) - c\mathbf{N} \cdot \mathbf{M} &= (1 - c)\,(1 - f)(fc + 1 - f)\mathbf{I}
\end{aligned} \tag{79}$$