Abstract

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1. The linear comparison composite

We consider a periodic cell Z of a random heterogeneous material. The conductivity at $\mathbf{x} \in Z$ is $\boldsymbol{\sigma}(\mathbf{x})$ (symmetric, positive definite, second-order tensor); $\mathbf{E}(\mathbf{x})$, $\phi(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ denote the electric field, the electric potential and the volumic current, respectively, at point $\mathbf{x}$. The apparent conductivity of the cell Z, $\boldsymbol{\sigma}^{\text{app}}$, is found from the solution to the following problem

$$Z: \quad \operatorname{div} \mathbf{j} = 0, \tag{1}$$

$$Z_r: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E} + \mathbf{P}, \tag{2}$$

$$Z: \quad \mathbf{E} = \overline{\mathbf{E}} + \mathbf{grad}\, \phi^{\text{per}}, \tag{3}$$

where ϕ^{per} is a Z-periodic field, $\mathbf{j}$ is a Z-antiperiodic field, and $\overline{\mathbf{E}}$ is a prescribed constant vector. Eq. (3) ensures that the electric field is curl-free, with $\overline{\mathbf{E}} = \langle \mathbf{E} \rangle$, where angle brackets denote volume averages over the cell Z. We consider a biphasic cell with a matrix whose conductivity is $\boldsymbol{\sigma}_0$ and another phase whose conductivity is $\boldsymbol{\sigma}_1$. The polarization $\mathbf{P}$ is a Z-antiperiodic field.

We introduce the linear Green operator $\boldsymbol{\Gamma}_0^{\text{per}}$, associated with *a given arbitrary* conductivity $\boldsymbol{\sigma}_0$.

The electrical field solution of problem (1)-(3) is also the solution of the minimization problem, which also defines the effective energy:

$$\mathcal{W}^{\text{eff}}\left(\overline{\mathbf{E}}\right) = \min_{\mathbf{E} \in C(\overline{\mathbf{E}})} \mathcal{W}(\mathbf{E}) \tag{4}$$

where

$$\mathcal{W}(\mathbf{E}) = \int_Z \frac{1}{2} \mathbf{E} \cdot \boldsymbol{\sigma} \cdot \mathbf{E} + \mathbf{P} \cdot \mathbf{E}. \tag{5}$$

The Lippmann-Schwinger equation is

$$\mathbf{E} = \overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\boldsymbol{\tau}), \qquad \text{where} \qquad \boldsymbol{\tau} = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_0) \cdot \mathbf{E} + \mathbf{P} \tag{6}$$

We then consider the following electrical field:

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \tag{7}$$

where $\overline{\boldsymbol{\tau}}_k$ are constant vectors and χ denotes characteristic function of medium whose conductivity is $\boldsymbol{\sigma}_1$. This field is curl-free, and its average is $\overline{\mathbf{E}}$. We then define

$$\begin{aligned} \mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) &= \mathcal{W}\left(\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M)\right) \\ &= \int_Z \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) \cdot \left\{\frac{\boldsymbol{\sigma}}{2} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) + \mathbf{P}\right\} \end{aligned} \tag{8}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}_i$:

$$\frac{\partial \mathcal{E}}{\partial \overline{\boldsymbol{\tau}}_i} = -\int_Z {}^t\mathbf{H}^i \cdot \left\{\boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) + \mathbf{P}\right\} \tag{9}$$

which is null if

$$\int_Z {}^t\mathbf{H}^i \cdot \left\{\boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) + \mathbf{P}\right\} = \sum_{k=1}^{M} \int_Z {}^t\mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k. \tag{10}$$

This is a linear matrix equation and we can define

$$\mathbf{G}_{ik} = \int_Z {}^t\mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{H}^k \qquad \text{and} \quad \mathbf{U}_i = \int_Z {}^t\mathbf{H}^i \cdot \left\{\boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) + \mathbf{P}\right\} = \mathbf{a}_i \cdot \overline{\mathbf{E}} + \mathbf{b}_i \tag{11}$$

where ...

Coming back to the energy, we write

$$\begin{aligned}
\mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) &= \int_Z \left[\left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) \cdot \left\{\frac{\boldsymbol{\sigma}}{2} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) + \frac{\mathbf{P}}{2} + \frac{\mathbf{P}}{2}\right\}\right] \\
&= \int_Z \left[\left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) \cdot \left\{\frac{\boldsymbol{\sigma}}{2} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right) + \frac{\mathbf{P}}{2}\right\} - \sum_{k=1}^{M} \frac{\mathbf{P}}{2} \cdot \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right] \\
&= \int_Z \left[\left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) + \frac{\mathbf{P}}{2}\right) - \sum_{k=1}^{M} \left(\left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) \cdot \frac{\boldsymbol{\sigma}}{2} + \frac{\mathbf{P}}{2}\right) \cdot \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k\right] \\
&= \left\{\int_Z \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) \cdot \left(\frac{\boldsymbol{\sigma}}{2} \cdot \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) + \frac{\mathbf{P}}{2}\right)\right\} - \frac{1}{2} \sum_{k,i=1}^{M} \overline{\boldsymbol{\tau}}_i \cdot \mathbf{U}_i
\end{aligned} \tag{12}$$

where we used (9) from line 1 to line 2, and also definition of $\mathbf{U}_i$ from line 3 to line 4. Hence, we have

$$\mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} + \left\{\int_Z -\boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} + \frac{1}{2}\boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) + \left(\overline{\mathbf{E}} - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) \cdot \frac{\mathbf{P}}{2}\right\} - \frac{1}{2} \sum_{k,i=1}^{M} \overline{\boldsymbol{\tau}}_i \cdot \mathbf{U}_i \tag{13}$$

and we have also

$$\sum_{k,i=1}^{M} \overline{\boldsymbol{\tau}}_i \cdot \mathbf{U}_i = \left(\overline{\mathbf{E}} \cdot {}^t\mathbf{a} + {}^t\mathbf{b}\right) \cdot \mathbf{G}^{-1} \cdot \left(\mathbf{a} \cdot \overline{\mathbf{E}} + \mathbf{b}\right) = \overline{\mathbf{E}} \cdot {}^t\mathbf{a} \cdot \mathbf{G}^{-1} \cdot \mathbf{a} \cdot \overline{\mathbf{E}} + \left({}^t\mathbf{b} \cdot \mathbf{G}^{-1} \cdot \mathbf{a} + {}^t\mathbf{a} \cdot \mathbf{G}^{-1} \cdot \mathbf{b}\right) \cdot \overline{\mathbf{E}} + {}^t\mathbf{b} \cdot \mathbf{G}^{-1} \cdot \mathbf{b} \tag{14}$$

And then

$$\mathcal{W}^{\text{eff}}\left(\overline{\mathbf{E}}\right) \leq \frac{1}{2}\overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\text{eff}} \cdot \overline{\mathbf{E}} + \mathbf{P}^{\text{eff}} \cdot \overline{\mathbf{E}} + w^0 \tag{15}$$

where

$$\begin{aligned}
\boldsymbol{\sigma}^{\text{eff}} &= \boldsymbol{\sigma}^{Voigt} - {}^t\mathbf{a} \cdot \mathbf{G}^{-1} \cdot \mathbf{a} \qquad \mathbf{P}^{\text{eff}} = \int_Z \left(\frac{\mathbf{P}}{2} - \boldsymbol{\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P})\right) - \frac{1}{2}\left({}^t\mathbf{b} \cdot \mathbf{G}^{-1} \cdot \mathbf{a} + {}^t\mathbf{a} \cdot \mathbf{G}^{-1} \cdot \mathbf{b}\right) \\
w^0 &= \frac{1}{2}\int_Z \left[\boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) \cdot \boldsymbol{\sigma} \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) - \boldsymbol{\Gamma}_0^{\text{per}}(\mathbf{P}) \cdot \mathbf{P}\right] - \frac{1}{2}{}^t\mathbf{b} \cdot \mathbf{G}^{-1} \cdot \mathbf{b}
\end{aligned} \tag{16}$$

In case of convergence, the inequality (15) becomes an equality.

2. If P is a constant-piecewise field

3. Minimisation of the complementary energy

3.1. Current density version of Lippmann-Schwinger equation

It is known that another problem can be considered to compute another apparent conductivity:

$$Z: \quad \operatorname{div} \mathbf{j} = 0, \qquad \langle \mathbf{j} \rangle = \overline{\mathbf{J}}, \tag{17}$$
$$Z: \quad \mathbf{j} = \boldsymbol{\sigma} \cdot \mathbf{E}, \tag{18}$$
$$Z: \quad \mathbf{E} = \langle \mathbf{E} \rangle + \mathbf{grad}\, \phi^{\mathrm{per}}. \tag{19}$$

where $\overline{\mathbf{J}}$ is a prescribed constant vector, and $\mathbf{j}$ remains a Z-antiperiodic field.

The above problem can be rewritten as

$$Z: \quad \operatorname{div} \mathbf{j} = 0, \qquad \langle \mathbf{j} \rangle = \overline{\mathbf{J}}, \tag{20}$$
$$Z: \quad \mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T, \qquad \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}, \tag{21}$$
$$Z: \quad \mathbf{E} = \langle \mathbf{E} \rangle + \mathbf{grad}\, \phi^{\mathrm{per}} \tag{22}$$
$$\tag{23}$$

where $\boldsymbol{\rho} = \boldsymbol{\sigma}^{-1}$.

And the solution of this problem is also the solution of this equation:

$$\mathbf{j} = \overline{\mathbf{J}} + \boldsymbol{\sigma}_0 \langle \mathbf{E}^T \rangle + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \qquad \text{with } \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}. \tag{24}$$

Indeed, (i) the average of this field is $\overline{\mathbf{J}}$, (ii) its divergence is null because the following equation is verified (by definition of $\boldsymbol{\Gamma}_0^{\mathrm{per}}$):

$$\operatorname{div} \left[\boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right] = 0 \tag{25}$$

and (iii) the corresponding field $\mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T$ can be written

$$\mathbf{E} = \boldsymbol{\rho}_0 \cdot \mathbf{j} + \mathbf{E}^T = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle + \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) - \mathbf{E}^T + \mathbf{E}^T = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle + \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) \tag{26}$$

which is Eq.(22) with $\langle \mathbf{E} \rangle = \boldsymbol{\rho}_0 \cdot \overline{\mathbf{J}} + \langle \mathbf{E}^T \rangle$. Hence, rewriting Eq.(24),

$$\mathbf{j} - \overline{\mathbf{J}} = -\boldsymbol{\sigma}_0 \left(\mathbf{E}^T - \langle \mathbf{E}^T \rangle \right) + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T \right) \qquad \text{with } \mathbf{E}^T = (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j}. \tag{27}$$

Or

$$\mathbf{j}^* = \boldsymbol{\tau}^* - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} (\boldsymbol{\tau}), \qquad \text{where} \qquad \boldsymbol{\tau} = -\boldsymbol{\sigma}_0 \cdot \mathbf{E}^T = -\boldsymbol{\sigma}_0 \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}_0) \cdot \mathbf{j} \tag{28}$$

and $\mathbf{a}^*$ is a notation for $\mathbf{a} - \langle \mathbf{a} \rangle$. This last equation is the current density version of Lippmann-Schwinger equation.

3.2. Minimisation of the complementary energy

It is well-known that solution of problem (17)-(19) is the solution of the minimisation problem:

$$\min_{\mathbf{j} \in \mathcal{S}(\overline{\mathbf{J}})} \mathcal{W}^*(\mathbf{j}) \tag{29}$$

where

$$\mathcal{W}^*(\mathbf{j}) = \frac{1}{2} \int_Z \mathbf{j} \cdot \boldsymbol{\rho} \cdot \mathbf{j}. \tag{30}$$

We then take the polarization $\boldsymbol{\tau} = \chi\overline{\boldsymbol{\tau}}$ and the correponding current density:

$$\mathbf{j} = \overline{\mathbf{J}} + \boldsymbol{\tau}^* - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\boldsymbol{\tau}) = \overline{\mathbf{J}} + \chi^*\overline{\boldsymbol{\tau}} - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\chi\overline{\boldsymbol{\tau}}). \tag{31}$$

This field is divergence-free, and its average is $\overline{\mathbf{J}}$.

We also define

$$\begin{aligned}\mathcal{E}^*(\overline{\boldsymbol{\tau}}) &= \mathcal{W}^*\left(\overline{\mathbf{J}} + \chi^*\overline{\boldsymbol{\tau}} - \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\text{per}}(\chi\overline{\boldsymbol{\tau}})\right)\\ &= \mathcal{W}^*\left(\overline{\mathbf{J}} - \mathbf{F}\cdot\overline{\boldsymbol{\tau}}\right)\\ &= \frac{1}{2}\int_Z \left(\overline{\mathbf{J}} - \mathbf{F}\cdot\overline{\boldsymbol{\tau}}\right)\cdot\boldsymbol{\rho}\cdot\left(\overline{\mathbf{J}} - \mathbf{F}\cdot\overline{\boldsymbol{\tau}}\right)\end{aligned} \tag{32}$$

where

$$\mathbf{F}_{ij}(\mathbf{x}) = \left(\boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\text{per}}\right)_i(\chi\mathbf{e}_j)(\mathbf{x}) - \delta_{ij}\chi^*(\mathbf{x}), \qquad \text{(which can be computed from } \mathbf{H}_{ij}\text{ !).} \tag{33}$$

Minimising this function with respect to $\overline{\boldsymbol{\tau}}$:

$$\frac{\partial\mathcal{E}^*}{\partial\overline{\boldsymbol{\tau}}} = -\int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho}\cdot\left(\overline{\mathbf{J}} - \mathbf{F}\cdot\overline{\boldsymbol{\tau}}\right) \tag{34}$$

which is null if

$$\int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho}\cdot\overline{\mathbf{J}} = \int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho}\cdot\mathbf{F}\cdot\overline{\boldsymbol{\tau}}. \tag{35}$$

Because $\boldsymbol{\rho}$ is positive definite we can (except if $\mathbf{F}$ is null everywhere) obtain a candidate $\overline{\boldsymbol{\tau}}$:

$$\overline{\boldsymbol{\tau}} = \mathbf{K}^{-1}\int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho}\cdot\overline{\mathbf{J}}, \tag{36}$$

where

$$\mathbf{K} = \int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho}\cdot\mathbf{F}. \tag{37}$$

Coming back to the energy, we can state that

$$\begin{aligned}\frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{\text{app}}\cdot\overline{\mathbf{J}} &\le \mathcal{E}^*(\overline{\boldsymbol{\tau}})\\ &\le \frac{1}{2}\int_Z \overline{\mathbf{J}}\cdot\boldsymbol{\rho}\cdot\overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}}\cdot\mathbf{K}\cdot\overline{\boldsymbol{\tau}}\\ &\le \frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{Reuss}\cdot\overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}}\cdot\mathbf{K}\cdot\overline{\boldsymbol{\tau}}\end{aligned} \tag{38}$$

which also gives

$$\begin{aligned}\frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{\text{app}}\cdot\overline{\mathbf{J}} &\le \frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{Reuss}\cdot\overline{\mathbf{J}} - \frac{1}{2}\overline{\mathbf{J}}\cdot\mathbf{M}\cdot\overline{\mathbf{J}}\\ \boldsymbol{\sigma}^{\text{app}} &\ge \left[\boldsymbol{\rho}^{Reuss} - \mathbf{M}\right]^{-1}\end{aligned} \tag{39}$$

where

$$\mathbf{M} = \int_Z {}^t\boldsymbol{\rho}\cdot\mathbf{F}\cdot{}^t\mathbf{K}^{-1}\cdot\int_Z {}^t\mathbf{F}\cdot\boldsymbol{\rho} \tag{40}$$

4. Convergence of the upper bound

We then consider the following electrical field:

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \sum_{k=1}^{M} \boldsymbol{\Gamma}_0^{\mathrm{per}} \left[\left(-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \right)^{k-1} (\chi \overline{\boldsymbol{\tau}}_k) \right] \tag{41}$$

where $\overline{\boldsymbol{\tau}}_k$ are constant vectors, and $\boldsymbol{\delta\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_0$. We can write this field as a linear combination of the $\overline{\boldsymbol{\tau}}_k$:

$$\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \tag{42}$$

where

$$\mathbf{H}^k_{ij}(\mathbf{x}) = \boldsymbol{\Gamma}_i^{0,\mathrm{per}} \left[\left(-\chi \boldsymbol{\delta\sigma} \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \right)^{k-1} (\chi \mathbf{e}_j) \right] (\mathbf{x}), \qquad \text{(symmetric if isotropy of } \boldsymbol{\sigma}_0 \text{ ?).} \tag{43}$$

We then define

$$\begin{aligned} \mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) &= \mathcal{W}\left(\mathbf{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) \right) \\ &= \frac{1}{2} \int_Z \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \end{aligned} \tag{44}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}_i$:

$$\frac{\partial \mathcal{E}}{\partial \overline{\boldsymbol{\tau}}_i} = - \int_Z {}^t\mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \left(\overline{\mathbf{E}} - \sum_{k=1}^{M} \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k \right) \tag{45}$$

which is null if

$$\int_Z {}^t\mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} = \sum_{k=1}^{M} \int_Z {}^t\mathbf{H}^i \cdot \boldsymbol{\sigma} \cdot \mathbf{H}^k \cdot \overline{\boldsymbol{\tau}}_k. \tag{46}$$

This is a linear matrix equation.

Coming back to the energy, we can state that

$$\begin{aligned} \frac{1}{2} \overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{\mathrm{app}} \cdot \overline{\mathbf{E}} &\leq \mathcal{E}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) \\ &\leq \frac{1}{2} \int_Z \overline{\mathbf{E}} \cdot \boldsymbol{\sigma} \cdot \overline{\mathbf{E}} - \frac{1}{2} \overline{\boldsymbol{\tau}} \cdot \mathbf{G} \cdot \overline{\boldsymbol{\tau}} \\ &\leq \frac{1}{2} \overline{\mathbf{E}} \cdot \boldsymbol{\sigma}^{Voigt} \cdot \overline{\mathbf{E}} - \frac{1}{2} \overline{\boldsymbol{\tau}} \cdot \mathbf{G} \cdot \overline{\boldsymbol{\tau}} \end{aligned} \tag{47}$$

4.1. n^{th} order bound on the contrast

We can show that previous expansion provides the n^{th} order bound on the contrast.

5. Convergence of the lower bound

We then consider the following electrical current density field:

$$\mathbf{j}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{J}} - \sum_{k=1}^{M} \left[\langle \boldsymbol{\tau}^k \rangle - \boldsymbol{\tau}^k + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\Gamma}_0^{\mathrm{per}} \left(\boldsymbol{\tau}^k \right) \right] \tag{48}$$

where

$$\boldsymbol{\tau}^k(\mathbf{x}) = \left(-\chi\boldsymbol{\delta\sigma}\cdot\boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^{k-1}(\chi\overline{\boldsymbol{\tau}}_k)(\mathbf{x}) \tag{49}$$

and $\overline{\boldsymbol{\tau}}_k$ are constant vectors, and $\boldsymbol{\delta\sigma} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_0$. We can rewrite the current density as

$$\mathbf{j}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) = \overline{\mathbf{J}} - \sum_{k=1}^{M} \mathbf{F}^k \cdot \overline{\boldsymbol{\tau}}_k \tag{50}$$

where

$$\mathbf{F}^k_{ij}(\mathbf{x}) = \langle\left[\left(-\chi\boldsymbol{\delta\sigma}\cdot\boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^{k-1}\right]_i(\chi\mathbf{e}_j)(\mathbf{x})\rangle - \left[\left(-\chi\boldsymbol{\delta\sigma}\cdot\boldsymbol{\Gamma}_0^{\mathrm{per}}\right)^{k-1}\right]_i(\chi\mathbf{e}_j)(\mathbf{x}) + \boldsymbol{\sigma}_0\cdot\mathbf{H}^k_{ij}(\mathbf{x}). \tag{51}$$

We then define

$$\begin{aligned}\mathcal{E}^*(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M) &= \mathcal{W}^*\left(\mathbf{j}(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M)\right)\\ &= \frac{1}{2}\int_Z\left(\overline{\mathbf{J}} - \sum_{k=1}^{M}\mathbf{F}^k\cdot\overline{\boldsymbol{\tau}}_k\right)\cdot\boldsymbol{\rho}\cdot\left(\overline{\mathbf{J}} - \sum_{k=1}^{M}\mathbf{F}^k\cdot\overline{\boldsymbol{\tau}}_k\right)\end{aligned} \tag{52}$$

and we derive this function with respect to $\overline{\boldsymbol{\tau}}_i$:

$$\frac{\partial\mathcal{E}^*}{\partial\overline{\boldsymbol{\tau}}_i} = -\int_Z {}^t\mathbf{F}^i\cdot\boldsymbol{\rho}\cdot\left(\overline{\mathbf{J}} - \sum_{k=1}^{M}\mathbf{F}^k\cdot\overline{\boldsymbol{\tau}}_k\right) \tag{53}$$

which is null if

$$\int_Z {}^t\mathbf{F}^i\cdot\boldsymbol{\rho}\cdot\overline{\mathbf{J}} = \sum_{k=1}^{M}\int_Z {}^t\mathbf{F}^i\cdot\boldsymbol{\rho}\cdot\mathbf{F}^k\cdot\overline{\boldsymbol{\tau}}_k. \tag{54}$$

This is a linear matrix equation.

Coming back to the energy, we can state that

$$\begin{aligned}\frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{\mathrm{app}}\cdot\overline{\mathbf{J}} &\leq \mathcal{E}^*(\overline{\boldsymbol{\tau}}_1, ..., \overline{\boldsymbol{\tau}}_M)\\ &\leq \frac{1}{2}\int_Z\overline{\mathbf{J}}\cdot\boldsymbol{\rho}\cdot\overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}}\cdot\mathbf{K}\cdot\overline{\boldsymbol{\tau}}\\ &\leq \frac{1}{2}\overline{\mathbf{J}}\cdot\boldsymbol{\rho}^{Reuss}\cdot\overline{\mathbf{J}} - \frac{1}{2}\overline{\boldsymbol{\tau}}\cdot\mathbf{K}\cdot\overline{\boldsymbol{\tau}}\end{aligned} \tag{55}$$

6. Hashin-Shtrikman principle

7. Choice of the reference medium