

A solution for homogenization of isotropic Maxwellian viscoelastic media

1 Position of the problem

We consider a periodic cell Z of a random heterogeneous material composed of two phases 1 and 2. The behaviour is given by

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^v \quad (1)$$

with

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}^e &= \frac{1}{3\kappa^e} \dot{\boldsymbol{\sigma}}^m \mathbf{1} + \frac{1}{2\mu^e} \dot{\boldsymbol{\sigma}}^d \\ \dot{\boldsymbol{\varepsilon}}^v &= \frac{1}{3\kappa^v} \boldsymbol{\sigma}^m \mathbf{1} + \frac{1}{2\mu^v} \boldsymbol{\sigma}^d \end{aligned} \quad (2)$$

where $\boldsymbol{\sigma}^m = \frac{1}{3} \text{tr} \boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \boldsymbol{\sigma}^m \mathbf{1}$.

For a biphasic medium, we note κ_1^e and κ_2^e , μ_1^e and μ_2^e , etc.
Hence,

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}_i^m &= \frac{1}{3\kappa_i^e} \dot{\boldsymbol{\sigma}}_i^m + \frac{1}{3\kappa_i^v} \dot{\boldsymbol{\sigma}}_i^m \\ \dot{\boldsymbol{\varepsilon}}_i^d &= \frac{1}{2\mu_i^e} \dot{\boldsymbol{\sigma}}_i^d + \frac{1}{2\mu_i^v} \dot{\boldsymbol{\sigma}}_i^d \end{aligned} \quad (3)$$

where $i = 1, 2$. Let us introduce $\tau_i^m = \frac{\kappa_i^v}{\kappa_i^e}$ and $\tau_i^d = \frac{\mu_i^v}{\mu_i^e}$.

We consider the following time-dependent mechanical problem:

$$\begin{aligned} t = 0 : \quad &\boldsymbol{\sigma}(\mathbf{x}, t) = \boldsymbol{\sigma}_0(\mathbf{x}), \quad \boldsymbol{\varepsilon}(\mathbf{x}, t) = \boldsymbol{\varepsilon}_0(\mathbf{x}) \\ t \geq 0 : \quad &\boldsymbol{\varepsilon}(\mathbf{x}, t) = \bar{\mathbf{E}}(t) + \mathbf{grad}^s \boldsymbol{\xi}^{\text{per}}(\mathbf{x}, t) \\ &\dot{\boldsymbol{\varepsilon}}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) : \dot{\boldsymbol{\sigma}}(\mathbf{x}, t) + \mathbf{M}(\mathbf{x}) : \boldsymbol{\sigma}(\mathbf{x}, t) \\ &\text{div } \boldsymbol{\sigma}(\mathbf{x}, t) = 0 \end{aligned} \quad (4)$$

where $\bar{\mathbf{E}}(t)$ is given. We hence have

$$\dot{\boldsymbol{\varepsilon}}^m(\mathbf{x}, t) = \bar{\mathbf{E}}^m(t) + \dot{\boldsymbol{\varepsilon}}^m(\mathbf{x}, t) \quad \dot{\boldsymbol{\varepsilon}}^d(\mathbf{x}, t) = \bar{\mathbf{E}}^d(t) + \dot{\boldsymbol{\varepsilon}}^d(\mathbf{x}, t) \quad (5)$$

where $\dot{\boldsymbol{\varepsilon}}^m(\mathbf{x}, t) + \dot{\boldsymbol{\varepsilon}}^d(\mathbf{x}, t) = \mathbf{grad}^s \dot{\boldsymbol{\xi}}^{\text{per}}(\mathbf{x}, t)$. We also assume that $\bar{\mathbf{E}}(t)$ can be decomposed as follows:

$$\begin{aligned} \bar{\mathbf{E}}^m(t) &= \sum_{k=0}^M \dot{E}_k^m e^{-t/\tau_k} \\ \bar{\mathbf{E}}^d(t) &= \sum_{k=0}^M \dot{E}_k^d e^{-t/\tau_k} \end{aligned} \quad (6)$$

where $\tau_1 > \dots > \tau_M$ are characteristic times. We let the possibility that one of the \dot{E}_k^m or $\dot{\mathbf{E}}_k^d$ is null but not at the same time $\dot{E}_k^m = 0$ and $\dot{\mathbf{E}}_k^d = \mathbf{0}$.

Remark 1. It is equivalent to the hypothesis that $\bar{\mathbf{E}}(t)$ can be decomposed as a finite Prony serie.

2 Decomposition proposed for the strain field

We consider the following decomposition for the strain field:

$$\begin{aligned}\dot{\varepsilon}^m(\mathbf{x}, t) &= \sum_{k=0}^M s_k(\mathbf{x}) e^{-t/\tau_k} \\ \dot{\varepsilon}^d(\mathbf{x}, t) &= \sum_{k=0}^M \mathbf{d}_k(\mathbf{x}) e^{-t/\tau_k}\end{aligned}\tag{7}$$

where

$$\begin{aligned}s_k(\mathbf{x}) &= \dot{E}_k^m + \dot{e}_k^m(\mathbf{x}) \\ \mathbf{d}_k(\mathbf{x}) &= \dot{\mathbf{E}}_k^d + \dot{\mathbf{e}}_k^d(\mathbf{x})\end{aligned}\tag{8}$$

and $\dot{e}_k^m(\mathbf{x}) + \dot{\mathbf{e}}_k^d(\mathbf{x}) = \text{grad}^s \dot{\xi}_k^{\text{per}}(\mathbf{x})$. Putting (7) into (3), we obtain a first-order linear time-differential equation with source terms. Hence, by linearity, the solution $\boldsymbol{\sigma}$ can be decomposed as

$$\begin{aligned}\boldsymbol{\sigma}^m(\mathbf{x}, t) &= \sum_{k=0}^M \boldsymbol{\sigma}_k^m(\mathbf{x}, t) \\ \boldsymbol{\sigma}^d(\mathbf{x}, t) &= \sum_{k=0}^M \boldsymbol{\sigma}_k^d(\mathbf{x}, t)\end{aligned}\tag{9}$$

where each term (indexed by k) of the sums is solution of one of the following differential equations:

$$\begin{aligned}s_k(\mathbf{x}) e^{-t/\tau_k} &= \frac{1}{3\kappa^e(\mathbf{x})} \dot{\sigma}_k^m(\mathbf{x}, t) + \frac{1}{3\kappa^v(\mathbf{x})} \sigma_k^m(\mathbf{x}, t) \\ \mathbf{d}_k(\mathbf{x}) e^{-t/\tau_k} &= \frac{1}{2\mu^e(\mathbf{x})} \dot{\boldsymbol{\sigma}}_k^d(\mathbf{x}, t) + \frac{1}{2\mu^v(\mathbf{x})} \boldsymbol{\sigma}_k^d(\mathbf{x}, t)\end{aligned}\tag{10}$$

for $k = 0, \dots, M$. Let us fix k in $[[0, M]]$. Hence, we can state that the solutions σ_k^m and $\boldsymbol{\sigma}_k^d$ are given by (classical resolution of this type of equation):

$$\begin{aligned}\sigma_k^m(\mathbf{x}, t) &= \zeta_k(\mathbf{x}) e^{-t/\tau^m(\mathbf{x})} + \frac{3\kappa^v(\mathbf{x}) s_k(\mathbf{x})}{1/\tau^m(\mathbf{x}) - 1/\tau_k} e^{-t/\tau_k} \\ \boldsymbol{\sigma}_k^d(\mathbf{x}, t) &= \boldsymbol{\delta}_k(\mathbf{x}) e^{-t/\tau^d(\mathbf{x})} + \frac{2\mu^v(\mathbf{x}) \mathbf{d}_k(\mathbf{x})}{1/\tau^d(\mathbf{x}) - 1/\tau_k} e^{-t/\tau_k}\end{aligned}\tag{11}$$

We can then introduce

$$\tilde{\kappa}_k(\mathbf{x}) = \frac{\kappa^v(\mathbf{x})}{1/\tau^m(\mathbf{x}) - 1/\tau_k} \quad \tilde{\mu}_k(\mathbf{x}) = \frac{\mu^v(\mathbf{x})}{1/\tau^d(\mathbf{x}) - 1/\tau_k} \quad (12)$$

and hence

$$\begin{aligned} \boldsymbol{\sigma}^m(\mathbf{x}, t) &= \zeta(\mathbf{x})e^{-t/\tau^m(\mathbf{x})} + \sum_{k=0}^M 3\tilde{\kappa}_k(\mathbf{x})s_k(\mathbf{x})e^{-t/\tau_k} \\ \boldsymbol{\sigma}^d(\mathbf{x}, t) &= \boldsymbol{\delta}(\mathbf{x})e^{-t/\tau^d(\mathbf{x})} + \sum_{k=0}^M 2\tilde{\mu}_k(\mathbf{x})\mathbf{d}_k(\mathbf{x})e^{-t/\tau_k} \end{aligned} \quad (13)$$

where $\zeta(\mathbf{x}) = \sum_k \zeta_k(\mathbf{x})$ and $\boldsymbol{\delta}(\mathbf{x}) = \sum_k \boldsymbol{\delta}_k(\mathbf{x})$.

3 Case of a finite number of phases

3.1 Case of a biphasic

We can show that the equilibrium equation leads to

$$\operatorname{div}(3\tilde{\kappa}_k(\mathbf{x})s_k(\mathbf{x})\mathbf{1} + 2\tilde{\mu}_k(\mathbf{x})\mathbf{d}_k(\mathbf{x})) = 0 \quad \forall k = 0, \dots, M \quad (14)$$

where

$$\begin{aligned} \tilde{\kappa}_k^1 &= \frac{\kappa_1^v}{1/\tau_1^m - 1/\tau_k} & \tilde{\mu}_k^2 &= \frac{\mu_2^v}{1/\tau_2^d - 1/\tau_k} \\ \tilde{\kappa}_k^2 &= \frac{\kappa_2^v}{1/\tau_2^m - 1/\tau_k} & \tilde{\mu}_k^2 &= \frac{\mu_2^v}{1/\tau_2^d - 1/\tau_k} \end{aligned} \quad (15)$$

This is a linear elastic homogenization problem of a biphasic medium.

Moreover, by the mean of equations (13), $\boldsymbol{\delta}(\mathbf{x})$ and $\zeta(\mathbf{x})$ are obtained with the initial condition on the stress field. They must also check the following equations which are a consequence of the fact that they are statically admissible, but also that their contribution to the stress field is vanishing with different characteristic times.

$$\begin{aligned} \operatorname{grad}(\chi_i(\mathbf{x})\zeta(\mathbf{x})) &= 0 \quad (i = 1, 2) \\ \operatorname{div}(\chi_i(\mathbf{x})\boldsymbol{\delta}(\mathbf{x})) &= 0 \quad (i = 1, 2) \end{aligned} \quad (16)$$

4 Numerical application