Higher Structures in Homotopy Type Theory

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PhD defence

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Plan

1. An informal introduction to algebra in HoTT
2. A universe of polynomial monads
3. Opetopic methods in type theory

Sets

Mathematics are classically based on sets: collections of discrete elements.

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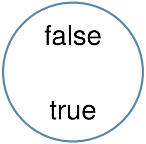

Figure: the set of booleans

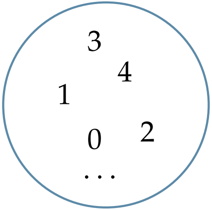

Figure: the set of natural numbers

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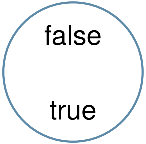

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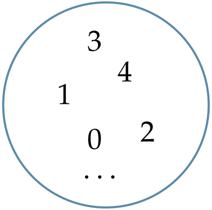

Figure: the set of natural numbers

Two elements of a set are *equal* if they have the same *definition*.

Principle of equivalence

The principle of equivalence states that mathematical reasoning should be invariant under the proper notion of equivalence.

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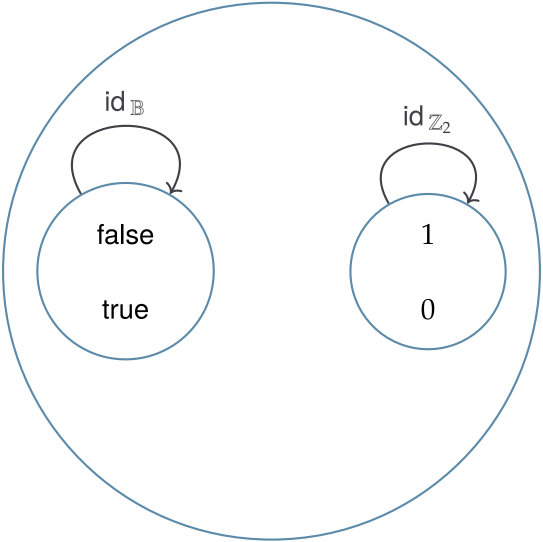
Set theory does not respect this principle (e.g., two bijective sets are not necessarily equal).

Principle of equivalence

Equality in sets does not account for all equivalences.



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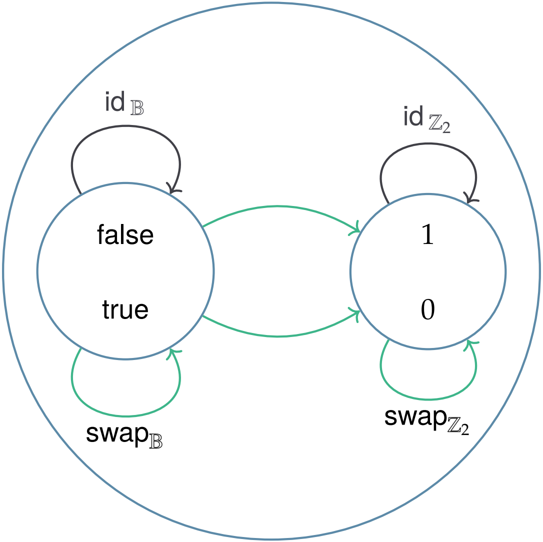

Homotopy type theory

Homotopy type theory is a foundation for constructive mathematics in which this principle holds.

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Type theory is a language rich enough to unify mathematical constructions and logical propositions.

The correspondence goes as follows.

Logic	Type theory
$\bot$	$\mathbf{0}$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$
$A \implies B$	$A \to B$
$\exists(x \in A).B(x)$	$(x : A) \times B(x)$
$\forall(x \in A).B(x)$	$(x : A) \to B(x)$

Univalence

In HoTT, the equality between two types X and Y is equivalent to the type of equivalences between these two types.

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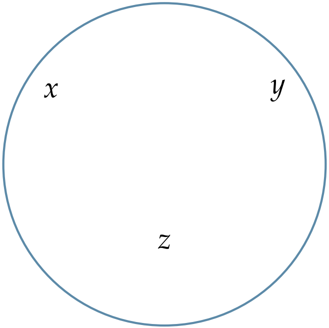
Not only are equivalent types identified, but the different ways they are identified is recorded.

Types can be regarded as *spaces* and equalities as *paths* in a space.

Spaces

Spaces contain elements—like sets—that we call points.



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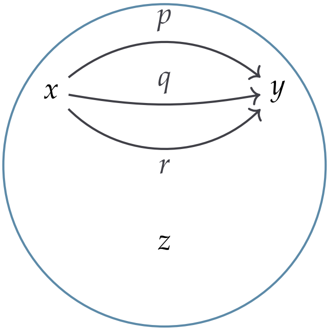
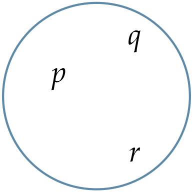
Spaces contain elements—like sets—that we call points.

They also contain *paths* between points.



Spaces

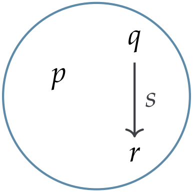
Paths between x and y assemble into a space.



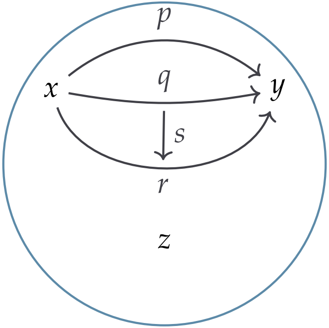
Spaces

Paths between x and y assemble into a space.

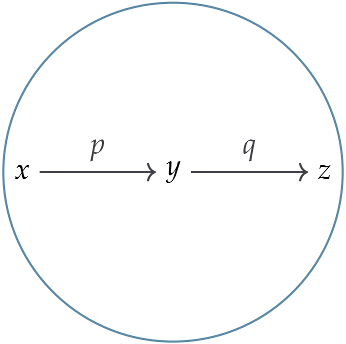
Suppose there is a path s between q and r



These paths between paths can be displayed on the original figure. In general, there can be paths of arbitrary dimensions.



Paths can be composed and their properties mirror those of equalities.



Algebra of paths

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$$
\begin{tikzcd}
x \arrow[r, "p"] \arrow[rr, bend right, "p \cdot q"'] & y \arrow[r, "q"] & z
\end{tikzcd}
$$

Algebra of paths

Path composition is associative up to a *higher path*.

$$a \xrightarrow{p} b \xrightarrow{q} c \xrightarrow{r} d$$

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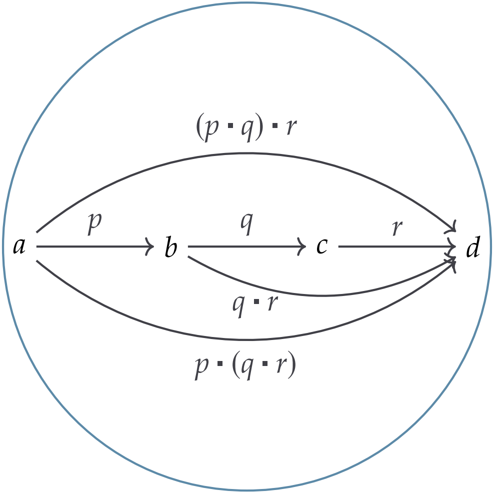

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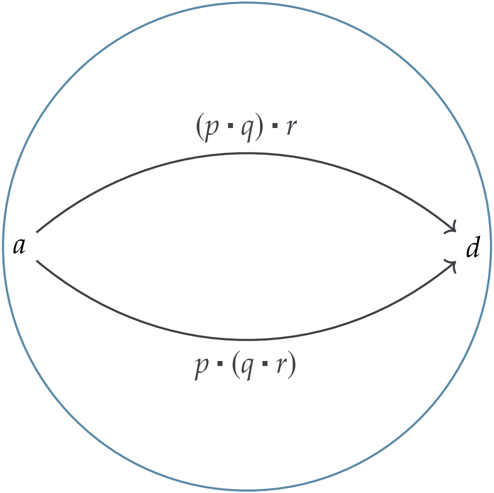

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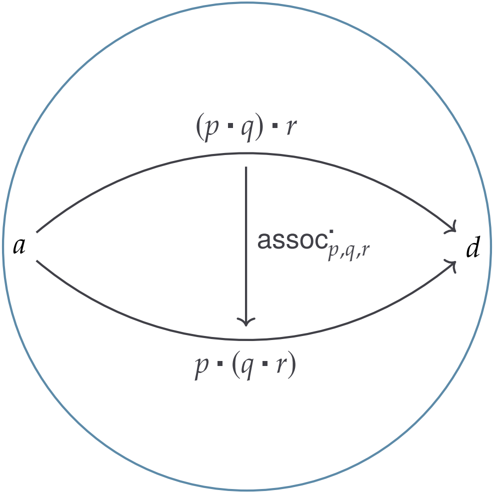
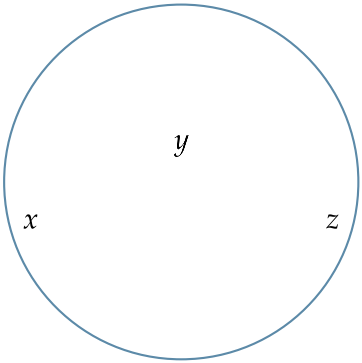


Any point comes with a distinguished loop called *identity*—think physical paths of length 0.



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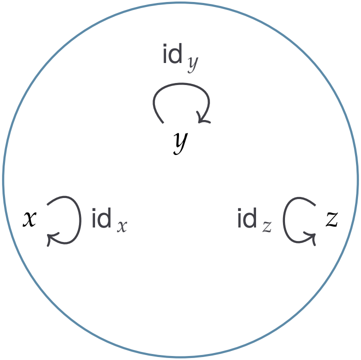

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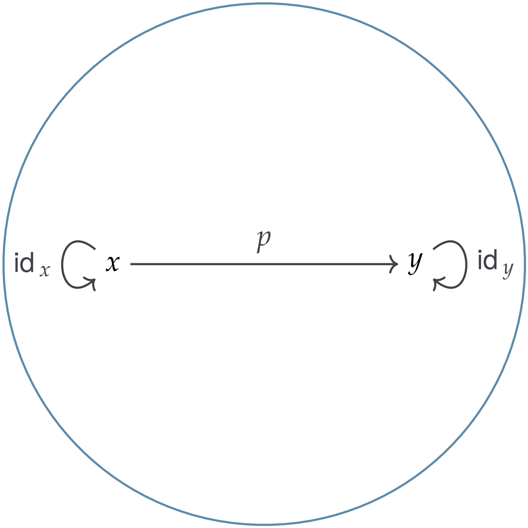

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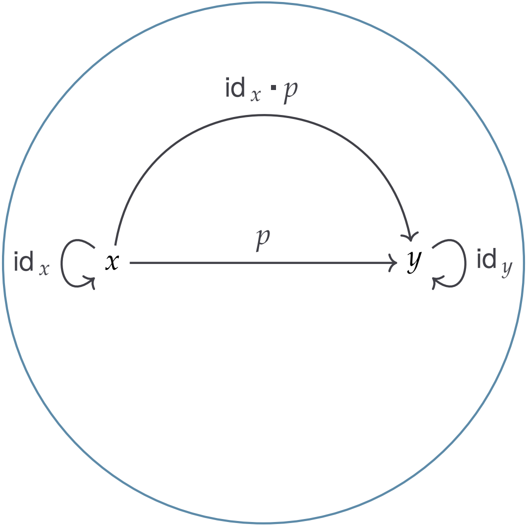

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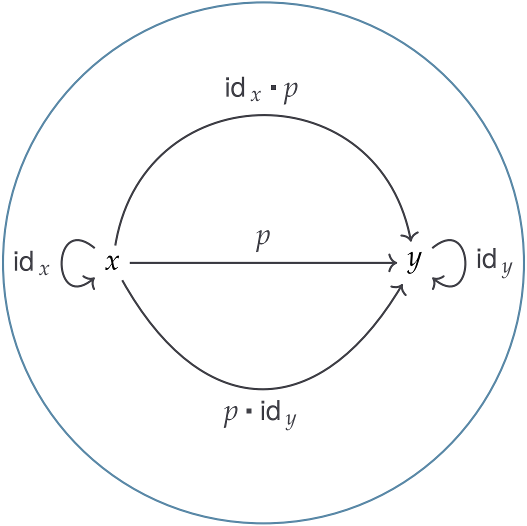

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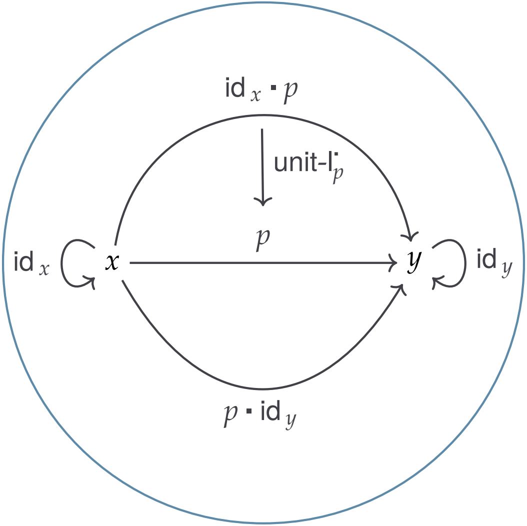

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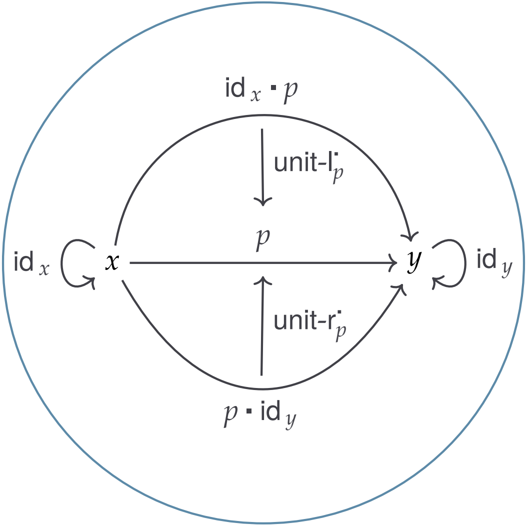

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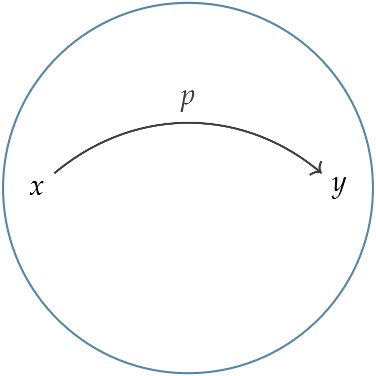

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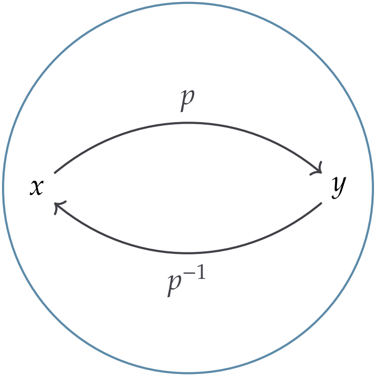

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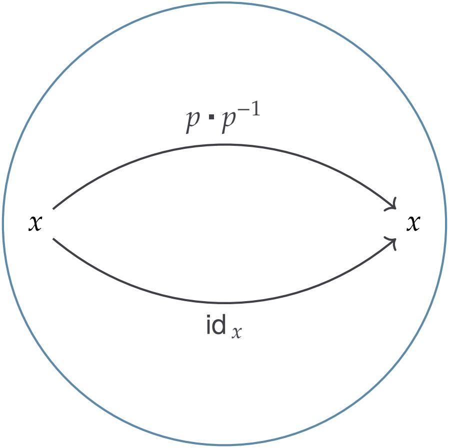

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Paths can be inverted.

$$\begin{tikzcd} x \arrow[r, bend left, "p \cdot p^{-1}"] \arrow[r, bend right, "\mathsf{id}_x"'] \arrow[r, phantom, "\Downarrow \mathsf{inv\text{-}r}^{\cdot}_{p}"] & x \end{tikzcd}$$

Algebra of paths

Similarly, there is a path

$$\begin{tikzcd} y \arrow[r, bend left, "p^{-1} \cdot p"{name=U}] \arrow[r, bend right, "\mathsf{id}_y"'{name=D}] & y \arrow[from=U, to=D, Rightarrow, "\mathsf{inv}\text{-}\mathsf{r}^{\cdot}_p"] \end{tikzcd}$$

In addition, the paths witnessing the laws satisfy *coherence* conditions. More on that later.

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This structure we just described is the one of ∞-groupoid.

Algebra on spaces

We want to generalise usual algebraic structures (groups, monoids, rings, ...) to spaces.

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An algebraic structure on a space is an operation acting on the points of this space satisfying *coherent* laws expressed in terms of paths.

An associative magma on a space X is the data of a binary operation

$$\_ \otimes \_ : X \times X \to X$$

along with a path

$$(a \otimes b) \otimes c \xrightarrow{\mathsf{assoc}^{\otimes}_{a,b,c}} a \otimes (b \otimes c)$$

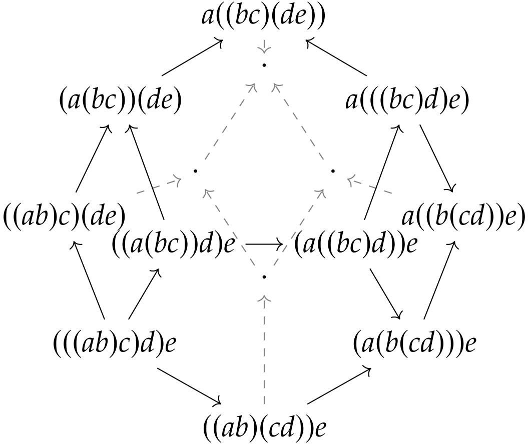
for any $a, b, c : X$.

In addition, we ask for a path making the following diagram commute up to this higher path:

$$\begin{tikzcd}
 & (a \otimes (b \otimes c)) \otimes d \arrow[rr, "\mathsf{assoc}^{\otimes}_{a,b\otimes c,d}"] & & a \otimes ((b \otimes c) \otimes d) \arrow[dr, "\mathsf{id}_a \otimes \mathsf{assoc}^{\otimes}_{b,c,d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\mathsf{assoc}^{\otimes}_{a,b,c} \otimes \mathsf{id}_d"] \arrow[drr, "\mathsf{assoc}^{\otimes}_{a\otimes b,c,d}"'] & & & & a \otimes (b \otimes (c \otimes d)) \\
 & & (a \otimes b) \otimes (c \otimes d) \arrow[urr, "\mathsf{assoc}^{\otimes}_{a,b,c\otimes d}"'] & &
\end{tikzcd}$$

Coherence

In turn, this new data has to satisfy its own coherence conditions leading to an infinite tower of data described by Stasheff's associahedra K_n.



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Sets are degenerate spaces whose only paths are identities. Laws of algebraic structures on sets are therefore trivially coherent.

Algebra in type theory

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Spaces are primitive in type theory, any algebraic structure must be stated coherently in the first place.

A theory of structures

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Type theory seems to be missing a theory of structures.

Proposal

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This approach is compatible with univalence.

Applications

In this type theory, the following results have been established:

- Fibrant opetopic types are equivalent to Baez-Dolan coherent algebras whose morphisms are invertible.
- The internal ∞-groupoid associated to a type.
- The $(\infty, 1)$-category of types.
- Adjunctions between $(\infty, 1)$-categories.
- Fibrant opetopic types are closed under dependent sums.

Setting

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When restricted to opetopic sets, we are able to prove the postulates.

Polynomial monads

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- $\mathsf{Typ}_M : \{i : \mathsf{Idx}_M\}\ \{c : \mathsf{Cns}_M(i)\} \to \mathsf{Pos}_M(c) \to \mathsf{Idx}_M$

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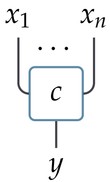
Elements depicted as corollas:



Polynomial monads

The structure of cartesian polynomial monad is defined by a unit operation η and a multiplication operation μ:

$$\eta_M : (i : \mathsf{Idx}_M) \to \mathsf{Cns}_M(i)$$

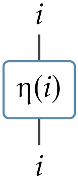
$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i)) \to \overrightarrow{\mathsf{Cns}_M}(c) \to \mathsf{Cns}_M(i)$$

Polynomial monads

The unit

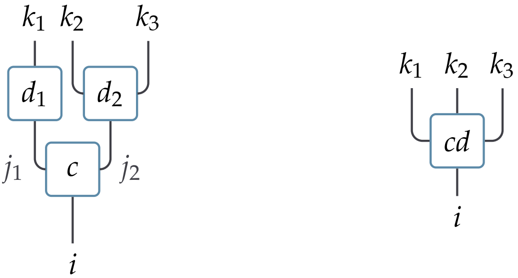
$$\eta_M : (i : \mathsf{Idx}_M) \longrightarrow \mathsf{Cns}_M(i)$$

Units $\eta(i)$ are *unary* constructors whose source and target have the same sort:



$$\mu_M : \{i : \mathsf{Idx}_M\}\ (c : \mathsf{Cns}_M(i))\ (d : \overrightarrow{\mathsf{Cns}_M}(c)) \to \mathsf{Cns}_M(i)$$

The multiplication "contracts" a tree of constructors while preserving the type of positions and their typing.



Polynomial monads

Laws

The operation μ_M is associative and unital with units η_M:

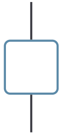
$$\mu_M(c, \lambda\, p \to \eta_M(\mathsf{Typ}_M(c,p))) \equiv c$$

$$\mu_M(\eta_M(i), d) \equiv d(\eta\text{-}\mathsf{pos}\,(i))$$

$$\mu_M(\mu_M(c,d), e) \equiv \mu_M(c, (\lambda\, p \to \mu_M(d(p), (\lambda\, q \to e(\mathsf{pair}^{\mu}(p,q))))))$$

Identity monad

The identity monad $\boxed{\mathsf{Id} : \mathcal{M}}$ has a single unary constructor.



Its monad structure is trivial.

Baez-Dolan slice construction

The universe $\mathcal{M}$ is closed under the Baez-Dolan slice construction. For any monad $M : \mathcal{M}$ and family $X : \mathsf{Fam}_M$ with

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Iterating this construction allows to capture the combinatorics of

- the composition of X-cells
- its laws
- …

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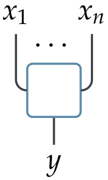
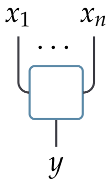
The indices of M/X are *frames*: constructors of M decorated with elements in X:



Baez-Dolan slice construction

The indices of M/X are *frames*: constructors of M decorated with elements in X:

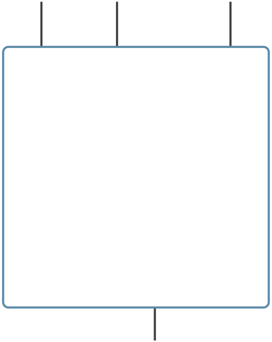


Defined as quadruplets $(i, y) \triangleleft (c, x)$ of type

$$\mathsf{Idx}_{M/X} :\equiv (i : \mathsf{Idx}_M) \times (y : X(i)) \times (c : \mathsf{Cns}_M(i)) \times (x : \overrightarrow{X}(c))$$

Baez-Dolan slice construction

Constructors of M/X are well-founded trees of frames which multiply to their indexing frame.



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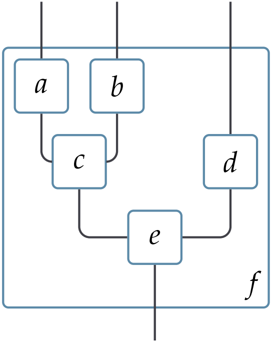

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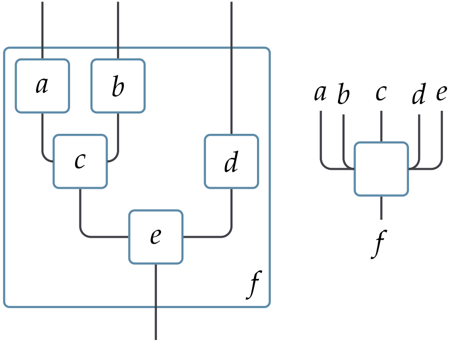

Baez-Dolan slice construction

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Provide a notion of pasting diagram.

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A lot of proofs in this framework go by induction on pasting diagrams.

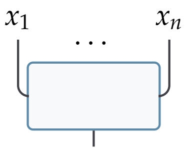
Particularly suited to type theory.

0-ALGEBRAS

A 0-algebra for a monad M is

- a family $X_0 : \mathsf{Fam}_M$,
- a family $X_1 : \mathsf{Fam}_{M/X_0}$.

such that for any constructor $c : \mathsf{Cns}_M(i)$ and values $x : \overrightarrow{X_0}(c)$, there exists is a unique pair composed of



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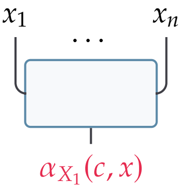

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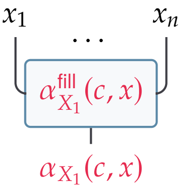

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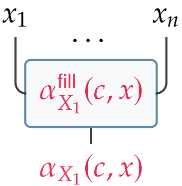

X_1 is an *entire* and *functional* relation.

Fundamental thm. of identity types

Theorem (Fundamental thm. of identity types)

Let $A : \mathcal{U}$ and $B : A \to \mathcal{U}$ such that $(x : A) \times B(x)$ is contractible with centre of contraction (x, p), then for any $y : A$,

$$B(y) \simeq (x = y)$$

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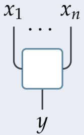
Corollary

Let (X_0, X_1) be a M-0-algebra, for any constructor $c : \mathsf{Cns}_M(i)$, values $x : \overrightarrow{X_0}(c)$, and value $y : X_0(i)$,

$X_1($ 

$) \simeq (\alpha_{X_1}(c, x) = y)$

Opetopic types

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$\mathcal{O}_M$ denotes the type of M-opetopic types.

Opetopic types

Some definitions of higher algebraic structures:

- $\infty\text{-}\mathsf{Grp} = (X : \mathcal{O}_{\mathsf{Id}}) \times \mathsf{is\text{-}fibrant}(X)$
- $(\infty, 1)\text{-}\mathsf{Cat} = (X : \mathcal{O}_{\mathsf{Id}}) \times \mathsf{is\text{-}fibrant}(X_{>0})$

Let X be a fibrant Id-opetopic type (i.e., an ∞-groupoid).

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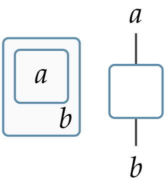
X_0 is the type of objects.

∞-GROUPOIDS

1-CELLS

The family of 1-cells $\boxed{X_1 : \mathsf{Fam}_{\mathsf{Id}/X_0}}$ is a binary relation on X_0.



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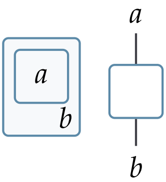
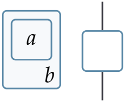


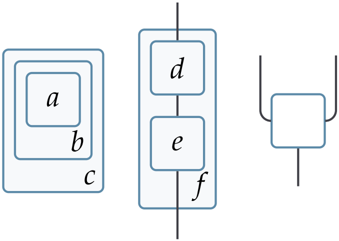
$$a \dashrightarrow b$$

X being fibrant,

$X_1($ 

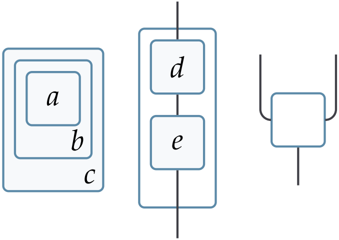
$) \simeq (a = b)$

The family of 2-cells $\boxed{X_2 : \mathsf{Fam}_{\mathsf{Id}/X_0/X_1}}$ relates a source pasting diagram of 1-cells with a target *parallel* 1-cell.



$$\begin{tikzcd} & b \arrow[dr, "e"] \arrow[d, phantom, "\Downarrow"] & \\ a \arrow[ur, "d"] \arrow[rr, "f"'] & {} & c \end{tikzcd}$$

X being fibrant, pasting diagrams of 1-cells can be composed.



$$\begin{tikzcd} & b \arrow[dr, "e"] \arrow[d, phantom, "\Downarrow"] & \\ a \arrow[ur, "d"] \arrow[rr, dashed] & {} & c \end{tikzcd}$$

∞-GROUPOIDS

2-CELLS

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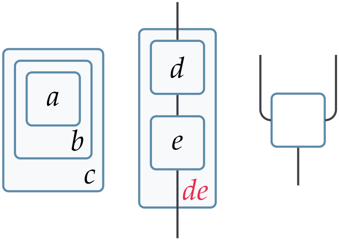

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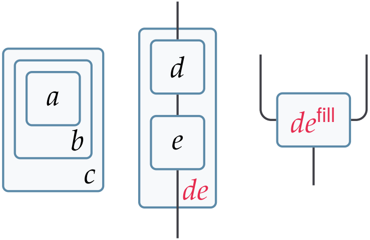


$$\begin{tikzcd} & b \arrow[dr, "e"] \arrow[d, Rightarrow, "de^{\mathsf{fill}}"] & \\ a \arrow[ur, "d"] \arrow[rr, "de"'] & {} & c \end{tikzcd}$$

The family of 3-cells $\boxed{X_3 : \mathsf{Fam}_{\mathsf{Id}/X_0/X_1/X_2}}$ relates a source pasting diagram of 2-cells to a target 2-cell.

Fibrancy makes the of composition of 1-cells associative and unital.

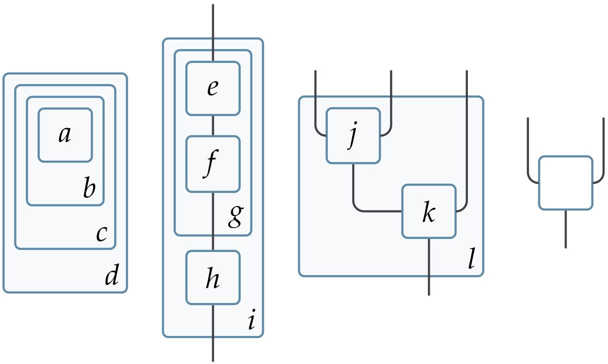
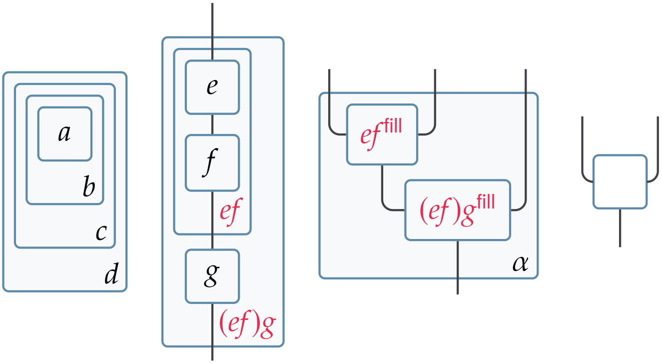
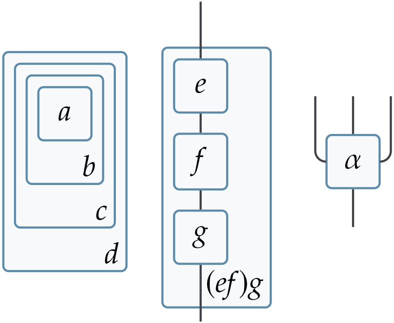


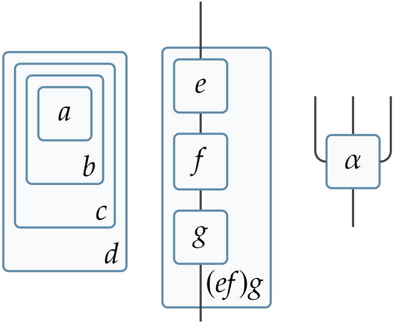
Figure: A 3-dimensional frame

3-CELLS



3-CELLS





X is fibrant therefore

$$(ef)g = efg$$

Applications

In this type theory, the following results have been established:

- Fibrant opetopic types are equivalent to Baez-Dolan coherent algebras whose morphisms are invertible.
- The internal ∞-groupoid associated to a type.
- The $(\infty, 1)$-category of types.
- Adjunctions between $(\infty, 1)$-categories defined as bifibrations over the interval.
- Fibrant opetopic types are closed under dependent sums.

Conclusion

Fibrant opetopic types are internal presentations of types which enables the definition of higher algebraic structures on arbitrary types.

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Paves the way for the development of higher category theory in univalent opetopic foundations.

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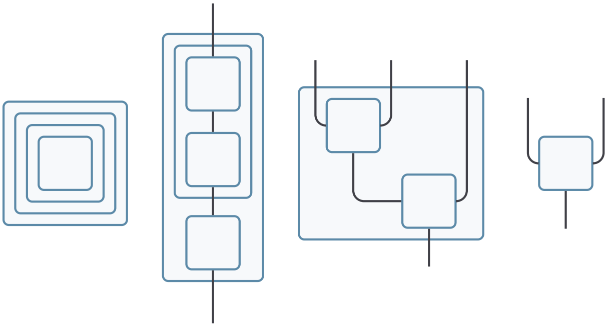
The geometry of opetopes is particularly suited to a type-theoretical approach.

Paves the way for the development of higher category theory in univalent opetopic foundations.

Future work

- Have a more synthetic notion of opetopic type.
- Develop a dedicated type checker or an agda mode.
- Develop higher category theory in univalent opetopic foundations.
- Investigate the semantics of this opetopic type theory.

Thank you for your attention.



Slice monad constructors

The type of constructors of the slice monad is an inductive type with two constructors:

$$\mathsf{lf} : (x : \mathsf{Idx}_M) \to \mathsf{Cns}_{M/}\ (x \triangleleft \eta_M\ x)$$

$$\begin{aligned}
&\mathsf{nd} : (x : \mathsf{Idx}_M)\ (y : \mathsf{Cns}_M\ x)\ \{z : \overrightarrow{\mathsf{Cns}_M}\ y\} \\
&\quad \to (t : \overrightarrow{\mathsf{Cns}_{M/}}\ (y \blacktriangleleft z)) \\
&\quad \to \mathsf{Cns}_{M/}\ (x \triangleleft \mu_M\ y\ z)
\end{aligned}$$

Types and their fibrant relations assemble into the $(\infty, 1)$-category

$$\mathcal{U}^o : \mathcal{O}_{\mathsf{Id}}$$

The universe

0-cells

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$$\mathcal{U}^o : \mathcal{O}_{\mathsf{Id}}$$

Its family of objects $\mathcal{U}^o_0$ is the universe of types $\mathcal{U}$:

$$\mathcal{U}^o_0(*) \equiv \mathcal{U}$$

The family of 1-cells

$$\mathcal{U}_1^o : \mathsf{Idx}_{\,\mathsf{Id}/\mathcal{U}_0^o} \to \mathcal{U}$$

is a binary relation on $\mathcal{U}$.

The universe

1-cells

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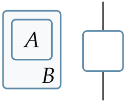
For example,

$\mathcal{U}_1^o($ 

$) \simeq (R : (a : A)\ (b : B) \to \mathcal{U}) \times \text{is-fibrant}(R)$

The universe

2-cells

The family of 2-cells

$$\mathcal{U}_2^o : \mathsf{Idx}_{\mathsf{Id}/\mathcal{U}_0^o/\mathcal{U}_1^o} \to \mathcal{U}$$

relates a *source* pasting diagram of 1-cells to a *target* 1-cell.

The universe

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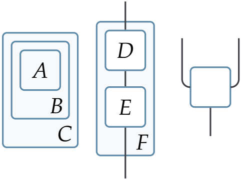

$$) \simeq (R : (a : A)\ (b : B)\ (c : C) \\ \to (d : D(a, b))\ (e : E(b, c))\ (f : F(a, c)) \to \mathcal{U}) \\ \times \text{is-fibrant}(R)$$

The universe

Fibrant relations

Formally, the domain of our relations are frames of the universal fibration $\mathcal{U}^o_\bullet \to \mathcal{U}^o$.

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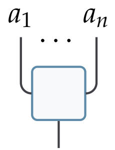

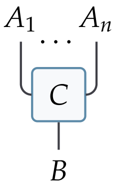

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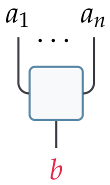

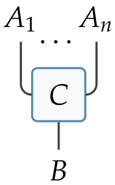

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