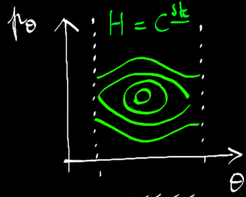
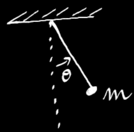
$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = +\dfrac{\partial H}{\partial p} \end{cases}$$

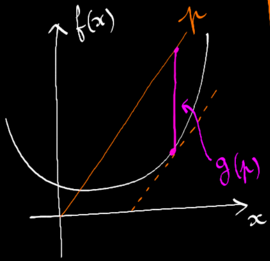
1740 - 1780

1820 - 1840





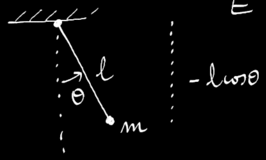
Energie et formalisme Hamiltonien



$$dS = p\,dq + Q\,dP - (H - K)\,dt$$

Merci à {Calicadoka, Arno, Feutel, Yann B,
Mathieu et Samdam !

Exemple : pendule



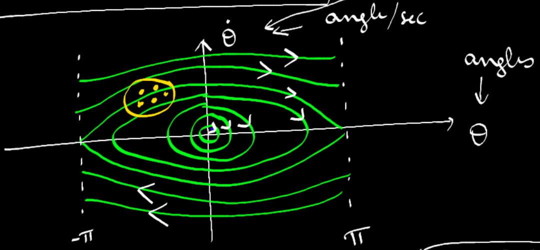
$$E = E_c + E_p = \frac{1}{2}mv^2 + mg(-\ell\cos\theta)$$

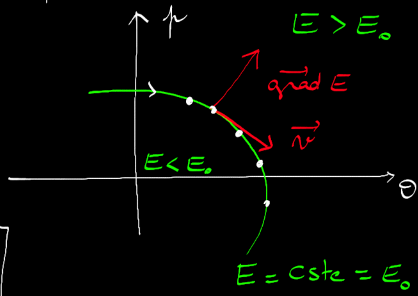
$$\boxed{\boxed{E = \frac{1}{2}m\ell^2\dot\theta^2 - mg\ell\cos\theta}}$$

$$\frac{dE}{dt} = m\ell^2\dot\theta\ddot\theta - mg\ell\,\dot\theta(-\sin\theta) = m\ell\dot\theta\left(\ell\ddot\theta + g\sin\theta\right) = 0$$

$$\boxed{\ddot\theta + \frac{g}{\ell}\sin\theta = 0}$$

$$p = m\ell^2\dot\theta$$





$$\boxed{\begin{aligned}\dot\theta &= \frac{\partial E}{\partial p}\\ \dot p &= -\frac{\partial E}{\partial\theta}\end{aligned}}$$

$$\overrightarrow{\text{grad}}\,E = \begin{pmatrix}\partial E/\partial\theta\\ \partial E/\partial p\end{pmatrix}$$

$$E = \frac{p^2}{2m\ell^2} - mg\ell\cos\theta$$

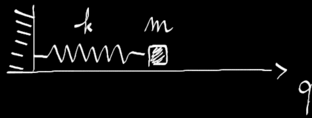
$$\vec v = \begin{pmatrix}\partial E/\partial p\\ -\partial E/\partial\theta\end{pmatrix} = \begin{pmatrix}\dot\theta\\ \dot p\end{pmatrix}$$

$$\begin{cases}\dot\theta = \dfrac{p}{m\ell^2}\\[1ex] \dot p = -mg\ell\sin\theta\end{cases}$$

$$\ddot\theta = \frac{\dot p}{m\ell^2} = -\frac{mg\ell\sin\theta}{m\ell^2} = -\frac{g}{\ell}\sin\theta$$

Exemple 2.

$$E(q,\dot q) = \frac{1}{2} m v^2 + \frac{1}{2} k q^2$$

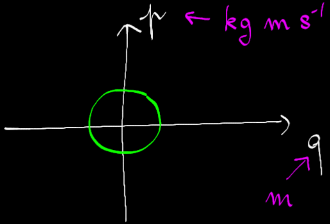


$$p = m\dot q$$

$$\boxed{E(p,q) = \frac{p^2}{2m} + \frac{1}{2} k q^2}$$

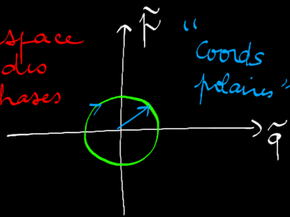
$$\ddot q = \frac{\dot p}{m} = -\frac{k}{m} q$$

$$\begin{cases} \dot p = -\dfrac{\partial E}{\partial q} = -kq \\[2mm] \dot q = +\dfrac{\partial E}{\partial p} = \dfrac{p}{m} \end{cases}$$



$$\boxed{\ddot q + \omega^2 q = 0 \quad \text{avec} \quad \omega^2 = \frac{k}{m}}$$

Espace des phases



"Coords polaires"

$$\begin{cases} \tilde p = \alpha p \\ \tilde q = \dfrac{1}{\alpha} q \end{cases} \qquad \alpha = \frac{1}{\sqrt{\omega m}}$$

$$\boxed{E(\tilde p,\tilde q) = \frac{\omega}{2}\left(\tilde p^2 + \tilde q^2\right)}$$

$$[\tilde p^2] = \text{kg m}^2\,\text{s}^{-1} = [\tilde q^2]$$

$$\begin{cases} P = \dfrac{1}{2}\left(\tilde p^2 + \tilde q^2\right) \\[2mm] Q = \arctan\left(\dfrac{\tilde q}{\tilde p}\right) \end{cases}$$

$$\boxed{E(P,Q) = \omega P}$$

$$\tan Q = \frac{\tilde q}{\tilde p}$$

$$\boxed{\begin{cases} \dot P = -\dfrac{\partial E}{\partial Q} = 0 \\[2mm] \dot Q = +\dfrac{\partial E}{\partial P} = \omega \end{cases}}$$

$$\dot Q = \frac{d}{dt}\left(\arctan \frac{\tilde q}{\tilde p}\right) = \frac{\dfrac{\dot{\tilde q}}{\tilde p} - \dfrac{\tilde q\,\dot{\tilde p}}{\tilde p^2}}{1 + (\tilde q/\tilde p)^2} = \frac{\omega + \omega\,\tilde q^2/\tilde p^2}{1 + (\tilde q/\tilde p)^2} = \omega$$

$$\dot{\tilde p} = -\omega \tilde q$$

$$\dot{\tilde q} = +\omega \tilde p$$

Formalisme Hamiltonien

- Lagrangien : $L(q, \dot q, \underline{\underline{t}}) \rightsquigarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = \frac{\partial L}{\partial q}}$

$p = \frac{\partial L}{\partial \dot q}$

- $E(q, \dot q, t) = p\dot q - L(q, \dot q, t)$ Energie.

$p = \frac{\partial L}{\partial \dot q}$

- $\boxed{H(q, p, t) \overset{?}{=} p\dot q - L(q, \dot q, t)}$ avec $\dot q(q, p)$ défini par

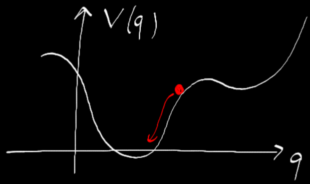
Hamiltonien

Transformation de Legendre

$$\boxed{\begin{cases} \dfrac{\partial H}{\partial p} = \dot q \\ \dfrac{\partial H}{\partial q} = -\dot p \end{cases}}$$

Exemple.

$$\boxed{L(q,\dot q) = \tfrac{1}{2} m \dot q^2 - V(q)}$$



$$H(p,q) = p\dot q - \tfrac{1}{2} m \dot q^2 + V(q)$$

$$\left(p = \frac{\partial L}{\partial \dot q} = m \dot q \right) \qquad \boxed{H(p,q) = \frac{p^2}{2m} + V(q)}$$

$$\begin{cases} \dot p = -\dfrac{\partial H}{\partial q} = -\dfrac{\partial V}{\partial q} = \text{Force} \\[2ex] \dot q = \dfrac{\partial H}{\partial p} = \dfrac{p}{m} \end{cases}$$

Remarque. Coordonnées $(q^1, q^2, \dots, q^n)$ de position → (vecteurs)

$(p_1, p_2, \dots, p_n)$ de $q^{\text{té}}$ de mouvement. → (1-formes)

$$L(q,\dot q) = \tfrac{1}{2} m \sum (\dot q^i)^2 - V(q^i)$$

$$H(p,q) = \sum_i p_i \dot q^i - L(\dots)$$

Plan :

- Legendre ??
- Trajectoires dans l'espace des phases
- Principe de moindre action.
- Structure symplectique (T^*M)
- Puissance du formalisme

Transformée de Legendre

f fonction $\mathcal{C}^\infty$, $f'' > 0$

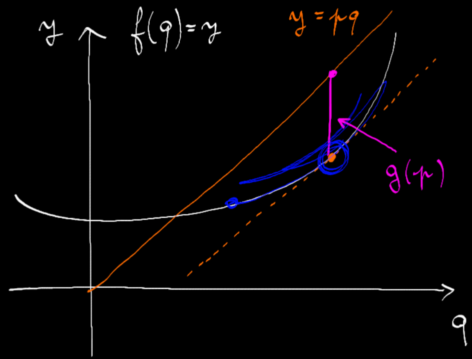
$p \in \mathbb{R}$

$$\boxed{g(p) = pq - f(q) \quad \text{avec } f'(q) = p.}$$

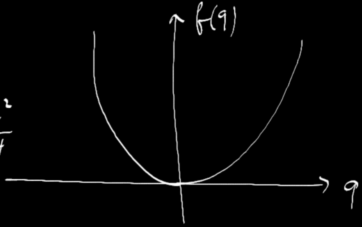
$F(p,q) = pq - f(q)$

définit $g(p) = F(p, q(p))$

avec $\dfrac{\partial F}{\partial q}(p, q(p)) = 0$



Exemple . (1) $f(q) = q^2$



$$g(p) = pq - f(q) = \frac{p^2}{2} - \frac{p^2}{4} = \frac{p^2}{4}$$

avec $f'(q) = p = 2q$

$$q^2 \rightleftarrows \frac{p^2}{4}$$

(2) $f(q) = e^q$ $\qquad e^q \rightleftarrows p(\ln p - 1)$

(3) $\frac{q^\alpha}{\alpha} \rightleftarrows \frac{p^\beta}{\beta} \Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = 1$

$f(q) \rightleftarrows g(p)$. $\qquad df = \frac{\partial f}{\partial q} dq \qquad \frac{\partial f}{\partial q} = p$

$$dg = d(pq - f) = p\,dq + q\,dp - \frac{\partial f}{\partial q} dq = q\,dp.$$

$p = \frac{\partial f}{\partial q}$ $\qquad \begin{matrix} f(q) \\ df = p\,dq \end{matrix} \rightleftarrows \begin{matrix} g(p) \\ dg = q\,dp \end{matrix} \qquad q = \frac{\partial g}{\partial p}$

Hors-Sujet

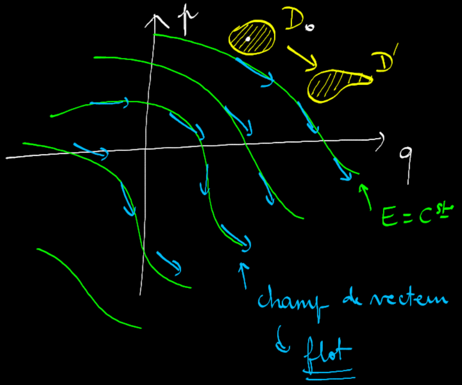
$$
\begin{tikzcd}
 & \boxed{\begin{array}{c} U(S,V) \\ dU = \underline{\underline{T}}dS - PdV \end{array}} \arrow[dl, "F = U - TS"'] \arrow[dr, "H = U + PV"] & \\
\boxed{\begin{array}{c} F(T,V) \\ dF = -SdT - PdV \end{array}} \arrow[dr, "G = F + PV"'] & & \boxed{\begin{array}{c} H(S,P) \\ dH = TdS + VdP \end{array}} \arrow[dl, "G = H - TS"] \\
 & \boxed{\begin{array}{c} G(T,P) \\ dG = -SdT + VdP \end{array}} &
\end{tikzcd}
$$

$T = \dfrac{\partial U}{\partial S}$

fonction à n variables

↳ 2^n transformées de Legendre.

Etude du flot hamiltonien



Espace des configurations : dim n

Espace des phases : dim $2n$

Théorème (Liouville) : le flot hamiltonien préserve les volumes dans l'espace des phases

$\uparrow$ $dp\,dq$

$\begin{cases} p \to \alpha p \\ q \to \frac{1}{\alpha} q \end{cases}$

$(p,q)_0 \rightsquigarrow (p,q)_t$ (on suppose que $\{E = c\}$ compacts)

Evolution D_0 domaine $\rightsquigarrow D_t$

$$\mathrm{vol}(D_t) = \int_{D_t} dp_t\,dq_t = \int_D \left| \frac{\partial(p,q)_t}{\partial(p,q)} \right| dp\,dq$$

$$\begin{vmatrix} 1 - t\dfrac{\partial^2 H}{\partial p \partial q} & -t\dfrac{\partial^2 H}{\partial q^2} \\ t\dfrac{\partial^2 H}{\partial p^2} & 1 + t\dfrac{\partial^2 H}{\partial q \partial p} \end{vmatrix} + O(t^2)$$

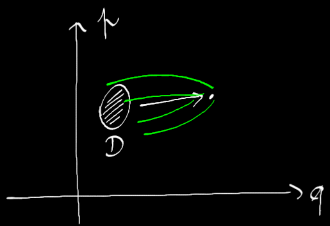
$$= 1 + O(t^2)$$

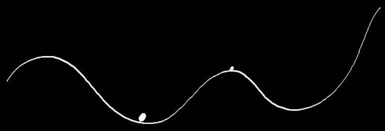
$$\frac{d}{dt}\mathrm{vol}\,D_t = \int_D \left| \frac{\partial\left(-\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)}{\partial(p,q)} \right| dp\,dq = 0$$

$$\begin{cases} p(t) = p(0) - t\dfrac{\partial H}{\partial q} + O(t^2) \\ q(t) = q(0) + t\dfrac{\partial H}{\partial p} + O(t^2) \end{cases}$$

Conséquences.

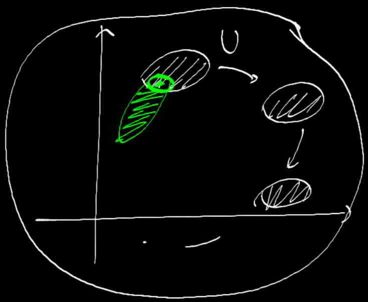
(1) Il est impossible d'avoir une position d'équilibre stable, même asymptotiquement.



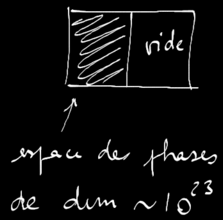


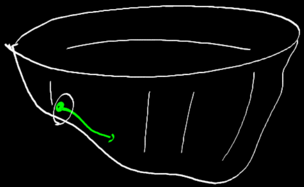
(2) De même pour un cycle stable

(3) Théorème de Poincaré : Pour un flot hamiltonien à orbites bornées, alors $\forall U$ ouvert de l'espace des phases, toute orbite passant par U y passe une infinité de fois



On regarde U toutes les secondes





Principe de "moindre" action : Soient t_1, t_2, q_1, q_2.

T^*M Hamiltonien

dim $2n$

PMA ds l'espace des phases

$$S(\sigma) = \int_{t_1}^{t_2} (p\dot{q} - H)\, dt$$

PMA : Chemin qui extrémise S parmi tous les chemins de l'espace des phases $\boxed{\sigma(t) = (q(t), p(t))}$ tels que $q(t_1) = q_1$ et $q(t_2) = q_2$.

$$S(\sigma) = \int_\sigma p\, dq - H\, dt$$

M Lagrangien

dim n

Principe de Hamilton

$$S(\gamma) = \int_{t_1}^{t_2} L(\gamma(t), \dot{\gamma}(t), t)\, dt$$

PMA : Chemin qui extrémise S sur tous les chemins tels que $\begin{cases}\gamma(t_1) = q_1 \\ \gamma(t_2) = q_2\end{cases}$

$\boxed{\gamma(t) = q(t)}$

$(\dot{q} = \gamma'(t))$

Chemin $\sigma(t) = (q(t), p(t))$, variation $(\delta q(t), \delta p(t))$ tq $\begin{cases}\delta q(t_1) = 0 \\ \delta q(t_2) = 0\end{cases}$

$$\delta S = \int_{t_1}^{t_2} \left(\delta p\, \dot{q} + p\, \dot{(\delta q)} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q\right) dt$$

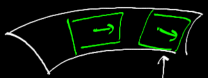
$$= \int_{t_1}^{t_2} \left[\underline{\delta p}\left(\dot{q} - \frac{\partial H}{\partial p}\right) + \underline{\delta q}\left(-\dot{p} - \frac{\partial H}{\partial q}\right)\right] dt + \cancel{\left[p\,\delta q\right]_{t_1}^{t_2}}$$

$\delta S = 0$ $(\forall\, \delta p, \delta q)$ $\iff$ $\boxed{\dot{q} = \frac{\partial H}{\partial p} \text{ et } \dot{p} = -\frac{\partial H}{\partial q}}$

Structure mathématique

Lagrangien. M = espace des configurations (coordonnées $q^1, q^2, \dots, q^n$)

TM = fibré tangent (coord standard $q^1, \dots, q^n, \dot q^1, \dots, \dot q^n$)



base de T_qM est $\left(\frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n}\right)$ et les composantes du vecteur v sont (par définition) les $(\dot q^1, \dots, \dot q^n)$

$$\boxed{v = \dot q^1 \frac{\partial}{\partial q^1} + \dots + \dot q^n \frac{\partial}{\partial q^n}}$$

$q \longrightarrow \alpha q$

$\dot q \longrightarrow \alpha \dot q$

Système lagrangien : (M, L) avec

$$L : TM \longrightarrow \mathbb{R}$$
$$(q, \dot q) \longmapsto L(q, \dot q)$$

• <u>T^*M</u> .

base de T_q^*M est $(dq^1, \dots, dq^n)$.

Toute forme linéaire α se décompose

en $\alpha = p_1\, dq^1 + \dots + p_n\, dq^n$

Coordonnées standard pour T^*M sont

$(q^1, \dots, q^n, p_1, \dots, p_n)$

$$\boxed{T^*M \text{ est “mieux” que } TM}$$

Sur T^*M il existe une 1-forme canonique :

$$\boxed{\theta = p\, dq}$$ LIOUVILLE

$$\boxed{\omega = dp \wedge dq}$$

$$\boxed{\underbrace{d\theta}_{\text{1-forme}} = \underbrace{\omega}_{\text{2-forme}}}$$

2-FORME SYMPLECTIQUE CANONIQUE

fermée $d\omega = 0$

<u>forme</u> : $\omega(\text{vect}, \text{vect}) = \text{nombre}$

ω permet de créer des liens :

- Définit isomorphisme $J : T^*(T^*M) \longrightarrow T(T^*M)$ par :

$$\begin{array}{ccc} \text{1-forme sur } T^*M & \longmapsto & \text{Champ de vecteurs sur } T^*M \\ \alpha & & J(\alpha) \end{array}$$

$$\boxed{\omega(X, J(\alpha)) = \alpha(X)}$$ pour tous $\begin{cases} X \text{ champ de vecteur sur } T^*M \\ \alpha \text{ 1-forme} \end{cases}$

- Définit un crochet sur les fonctions $\mathcal{C}^\infty(T^*M)$:

Si $f, g : T^*M \longrightarrow \mathbb{R}$, on pose :

$$\boxed{\{f, g\} = \omega(J(df), J(dg))}$$

<u>Exemple</u>. $\omega\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial q}\right) = (dp \wedge dq)\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial q}\right)$

$$= \underbrace{dp\left(\frac{\partial}{\partial q}\right)}_{0} \underbrace{dq\left(\frac{\partial}{\partial q}\right)} - dq\left(\frac{\partial}{\partial q}\right) \underbrace{dp\left(\frac{\partial}{\partial q}\right)}_{0} = 0$$

$$\begin{cases} \omega\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial q}\right) = 1 \\ \omega\left(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\right) = -1 \\ \omega\left(\frac{\partial}{\partial p}, \frac{\partial}{\partial p}\right) = 0. \end{cases}$$

$J(dp) = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p}$

$\omega\left(\frac{\partial}{\partial p}, J(dp)\right) = 1 = a$

$\omega\left(\frac{\partial}{\partial q}, J(dp)\right) = 0 = -b$

$$\boxed{J(dp) = \frac{\partial}{\partial q}}$$

$$\omega = \begin{array}{cc} \frac{\partial}{\partial q} & \frac{\partial}{\partial p} \\ \end{array}$$

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{array}{c} \frac{\partial}{\partial q} \\ \frac{\partial}{\partial p} \end{array}$$

$$\boxed{J(dp) = \frac{\partial}{\partial q} \qquad J(dq) = -\frac{\partial}{\partial p}}$$

$$\{f,g\} = \omega\big(J(df), J(dg)\big)$$

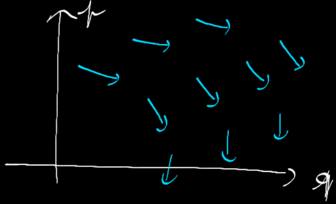
$$= \omega\left(J\left(\frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial q}dq\right), J\left(\frac{\partial g}{\partial p}dp + \frac{\partial g}{\partial q}dq\right)\right)$$

$$= \omega\left(\frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}, \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}\right)$$

$$\boxed{\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}}$$ Crochet de Poisson.

Définition. Un système hamiltonien (M, H) est un couple
M espace des configurations (variété $\mathcal{C}^\infty$ de dimension n)
et $H : T^*M \longrightarrow \mathbb{R}$.

Une quantité naturelle est $J(dH) = X_H$ champ de vecteurs hamiltonien



$$X_H = J\left(\frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial q}dq\right)$$

$$\boxed{X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}}$$

flot : $\frac{d}{dt}(p,q) = X_H(p,q)$

$$\boxed{\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\[2ex] \dot{q} = +\dfrac{\partial H}{\partial p} \end{cases}}$$

Comment construit-on un hamiltonien à partir d'un Lagrangien ?

(version math de la transformée de Legendre)

$$(M,L) \rightsquigarrow \theta_L = \frac{\partial L}{\partial \dot q^i}\, dq^i \rightsquigarrow \omega_L = d\theta_L \quad \text{2-forme sur } TM.$$

Une transformation de Legendre est une application $\tau_L : TM \to T^*M$, $(\underline{q},\dot q) \mapsto (\underline{q},p)$, $p = \frac{\partial L}{\partial \dot q}$

préservant la base M telle que $\tau_L^*(\theta) = \theta_L$

Si L est non dégénéré, ie $\left(\frac{\partial L}{\partial \dot q^i \partial \dot q^j}\right)_{i,j}$ inversible,

alors on peut définir $H : T^*M \to \mathbb{R}$

par $\boxed{H \circ \tau_L = \text{Energie} = \dot q \frac{\partial L}{\partial \dot q} - L}$

Reformulation : Une fonction $f : T^*M \to \underline{\mathbb{R}}$, $\mathcal{C}^\infty$, est appelée une observable (classique)

$$\{fg,h\} = \{f,h\}g + f\{g,h\}$$

Hamilton $\Longleftrightarrow$ $\boxed{\frac{df}{dt} = \{H,f\}}$

$$\Big(\underbrace{\underbrace{\mathcal{C}^\infty(T^*M)\,,\ \times}_{\text{algèbre (ass).}}\,,\ \{\,,\,\}}_{\text{algèbre de Lie}}\Big)$$

algèbre de Poisson

<u>Action revisitée</u> ("Action en fonction des coordonnées")

$$S(\underline{\gamma}) = \int_{t_1}^{t_2} L(\gamma'(t))\,dt$$

$$\gamma : [t_1, t_2] \longrightarrow M$$

<u>fixons t_1 et q_1.</u>

$$S(\underline{t_2, q_2}) = \int_{t_1}^{t_2} L(\gamma'(t))\,dt \longrightarrow \boxed{dS = p\,dq - H\,dt}$$

(en (t_2, q_2))

avec γ satisfaisant Euler Lagrange

$$\boxed{\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0}$$

<u>Hamilton-Jacobi</u>

Transformations canoniques.

Def. Une transformation canonique est un difféomorphisme $T^*M \to T^*M$ préservant ω

Ex : $(p, q) \to (\alpha p, \frac{1}{\alpha} q)$ $\qquad (p,q) \to (P, Q)$

Comment les trouver ? $\qquad P = \frac{1}{2}(\tilde{q}^2 + \tilde{p}^2)$

$$\boxed{\delta \int p\,dq - H\,dt = 0} \iff \boxed{\delta \int P\,dQ - K\,dt = 0}$$

Il suffit qu'il existe une fonction $F(q, Q, t)$ ← bizarre.

$$dF = p\,dq - H\,dt - P\,dQ + K\,dt$$

$$dF = p\,dq - P\,dQ - (H - K)\,dt$$

Leg.

En fait on voudrait une dépendance en (q, P).

$$\Phi(q, P, t) = QP + F(q, Q, t)$$

$$\boxed{\frac{\partial \Phi}{\partial q} = p \qquad \frac{\partial \Phi}{\partial P} = Q}$$

$$\boxed{d\Phi = p\,dq + Q\,dP - (H - K)\,dt}$$

$\Phi = qP$: $\begin{cases} p = P \\ Q = q \end{cases}$

(fonction génératrice)

$$\frac{\partial \Phi}{\partial t} = -(H-K) = -H$$

$$\boxed{\Phi(q, P, t) \quad \boxed{\frac{\partial \Phi}{\partial t} + H\left(\frac{\partial \Phi}{\partial q}, q\right) = 0}}$$

Un système hamiltonien trivial est $H=0$

On va essayer de trouver

$$(p, q) \to (P, Q) \text{ tel que } \underline{K=0}$$

$$\dot{P} = 0$$
$$\dot{Q} = 0$$

<u>Changement de point de vue</u> :

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0 \quad \text{avec} \quad S(q, t)$$

Trouver des solutions dépendant de n constantes :

$$S(q^1, \dots, q^n, \underbrace{\alpha_1, \dots, \alpha_n}_{\text{interprétés comme les } P_i}, t)$$

Les nouveaux Q sont $\boxed{\frac{\partial S}{\partial \alpha_i} = \beta^i}$

Exemple. Oscillateur harmonique $\quad H = \frac{\omega}{2}(p^2+q^2)$

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0$$

$$\frac{\partial S}{\partial t} + \frac{\omega}{2}\left(\left(\frac{\partial S}{\partial q}\right)^2 + q^2\right) = 0$$

$$S(q,t) = S_t(t) + S_q(q)$$

$$\left[\frac{\partial S_t}{\partial t}\right] + \left[\frac{\omega}{2}\left(\left(\frac{\partial S_q}{\partial q}\right)^2 + q^2\right)\right] = 0$$

$$\frac{\partial S_t}{\partial t} = -E \qquad \left(\frac{\partial S_q}{\partial q}\right)^2 + q^2 = \frac{2E}{\omega}$$

$$\frac{\partial S_q}{\partial q} = \sqrt{\frac{2E}{\omega} - q^2}$$

$$\boxed{S = -Et + \int \sqrt{\frac{2E}{\omega} - q^2}\, dq}$$

$$S(q, t, E)$$

$$(p, q) \to (P, Q)$$

amplitude phase

$$\frac{\partial S}{\partial E} = Q = \underline{cste} \qquad (K=0)$$

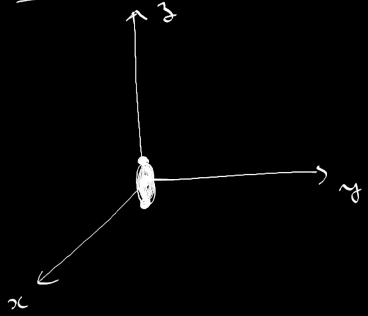
$$\frac{\partial S}{\partial E} = -t + \int \frac{\frac{2}{\omega}\, dq}{2\sqrt{\frac{2E}{\omega} - q^2}}$$

$$t_0 = -t + \frac{1}{\omega}\arcsin\left(\sqrt{\frac{\omega}{2E}}\, q\right)$$

$$\boxed{\sqrt{\frac{2E}{\omega}}\sin(\omega(t_0 + t)) = q}$$

Exemple non trivial.

Problème de Kepler asymétrique.



$$\boxed{V(r,\theta,\varphi) = a(r) + \frac{b(\theta)}{r^2}}$$

$$\boxed{\begin{aligned} &H(r,\theta,\varphi,p_r,p_\theta,p_\varphi) = \\ &\frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) + V(r,\theta)\end{aligned}}$$

$$\left[\frac{\partial S}{\partial t}\right] + \frac{1}{2m}\left(\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\underbrace{\frac{\partial S}{\partial \varphi}}_{p_\varphi}\right)^2\right) + a(r) + \frac{b(\theta)}{r^2} = 0$$

$\alpha_2 \to p_\varphi$

$$\frac{\partial S}{\partial t} = -E \to \alpha_1$$

$$S(t,r,\theta,\varphi) = -Et + p_\varphi\varphi + S'(r,\theta)$$

$$r^2E = \frac{1}{2m}\left[r^2\left(\frac{\partial S'}{\partial r}\right)^2 + \left(\frac{\partial S'}{\partial \theta}\right)^2 + \frac{p_\varphi^2}{\sin^2\theta}\right] + r^2 a(r) + b(\theta)$$

$\to \alpha_3$

$$S(t, \underbrace{r, \theta, \varphi}_{q}, \underbrace{\alpha_1, \alpha_2, \alpha_3}_{P}) = -\alpha_1 t + \alpha_2 \varphi + \int \sqrt{\alpha_3 - 2mb - \frac{\alpha_2^2}{\sin^2\theta}}\, d\theta$$

$$+ \int \sqrt{2m\left[\alpha_1 - a(r)\right] - \frac{\alpha_3}{r^2}}\, dr$$

Dans (P, Q) le hamiltonien $K = 0$

donc $\begin{cases} \dot{P} = 0 \\ \dot{Q} = 0 \end{cases}$ et $Q = \dfrac{\partial S}{\partial P}$

Si on calcule les intégrales, on trouve la solution.