Special Relativity – Solutions

Special relativity

0.1 Accelerated observer in Minkowski space

1. $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ where τ is an invariant ($\mathrm{d}\tau^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$). The 4-velocity of O is $u^\mu = (1,0,0,0)$. For O' one applies a boost with $\beta = -v = -v/c$:
$$u^\mu = \Lambda^\mu{}_\nu u^\nu = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma \\ v\gamma \end{pmatrix} \tag{1}$$
with $\gamma = \frac{1}{\sqrt{1-\beta 2}}$.
2. $a^\mu = \mathrm{d}v^\mu/\mathrm{d}\tau = 0$ here.
3. ok
4. Equations $a \cdot u = 0$, $u^2 = 1$ and $a^2 = g^2$. Result $\mathrm{d}u^t/\mathrm{d}\tau = a^t = g\sqrt{(u^t)^2 - 1}$ and $\mathrm{d}u^x/\mathrm{d}\tau = a^x = g\sqrt{(u^x)^2 + 1}$.
5. $\int \frac{\mathrm{d}u^t}{\sqrt{(u^t)^2-1}} = \cosh^{-1 u^t}$, etc. Then $u^t = \cosh g\tau + C^t$ and $u^x = \sinh g\tau + C^x$. The constants are 0 due to synchronization. Then
$$t = \frac{1}{g}\sinh g\tau \qquad x = \frac{1}{g}(\cosh g\tau - 1) \tag{2}$$
6. A photon emitted after the asymptote can not be received. This asymptote is an event horizon.
7. In the frame of O' the trip takes $\Delta\tau = \frac{c}{g}\cosh^{-1}(1 + gx/c^2) = 3 \times 10^8$ seconds. This is 10 years. In the frame of O $\Delta t = \frac{c}{g}\sinh g\tau/c = 20000$ years.

0.2 Doppler effect

1. This quantity must be a Lorentz scalar. It is $k \cdot u$ (indeed this is the energy is the frame of the observer where tghe velocity is (1,0,0,0)).
2. The energy of the photons is $h\nu_E = k \cdot u$ where u is the velocity of Alice. In Bob's frame $k = h\nu_R(1, -\cos\theta, -\sin\theta, 0)$ and $u = (\gamma, \gamma\beta)$. Hence
$$\nu_E = \nu_R\gamma(1 + \beta\cos\theta). \tag{3}$$

0.3 Electromagnetism

1. We have

$$F_{\mu\nu} = -\begin{pmatrix} 0 & E^x & E^y & E^z \\ -E^x & 0 & -B^z & B^y \\ -E^y & B^z & 0 & -B^x \\ -E^z & -B^y & B^x & 0 \end{pmatrix} \tag{4}$$

2. Transforms as a tensor $F'_{\mu\nu} = (\Lambda^{-1})^{\rho}_{\mu}(\Lambda^{-1})^{\sigma}_{\mu}F_{\rho\sigma}$.

3.

$$F'_{\mu\nu} = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & -0 & 0 \\ 0 & B^z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} F_{\mu\nu} \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & -0 & 0 \\ 0 & B^z & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

$$-\begin{pmatrix} 0 & E^x & \cosh\phi E^y - \sinh\phi B^z & \cosh\phi E^z - \sinh\phi B^y \\ * & 0 & \sinh\phi E^y - \cosh\phi B^z & \sinh\phi E^y - \cosh\phi B^z \\ * & * & 0 & -B^x \\ * & * & * & 0 \end{pmatrix} \tag{6}$$

4. $S = \int \mathrm{d}\tau \mathcal{L}$ with $\mathcal{L} = m\sqrt{u^2} + eA_\mu u^\mu$.

5. Use $\partial\mathcal{L}/\partial x^\nu = e(\partial_\nu A_\mu)u^\mu$ and $\partial\mathcal{L}/\partial u^\nu = mu_\nu + eA_n u$.

6. $\vec{V}$ is the 3-velocity.

7. Use

$$m\frac{\mathrm{d}u_i}{\mathrm{d}\tau} = -m\gamma\frac{\mathrm{d}}{\mathrm{d}t}(\gamma V_i) \tag{7}$$

$$eu^\mu F_{i\mu} = \gamma e(-E_i - (\vec{V}\times\vec{B})_i) \tag{8}$$

   And find the Lorentz force

$$m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\vec{V}) = e(\vec{E} + \vec{V}\times\vec{B}) \tag{9}$$

   We see that the correct definition of $\vec{p}$ is $\gamma m\vec{V}$.

0.4 Photoproducing a pion

1. We use the conservation of momentum and obtain $2p_\gamma \cdot p_p - m_p^2 = -(m_n + m_\pi)^2$.

2. In the frame where the proton is at rest $-2E_\gamma m_p - m_p^2 = -(m_n + m_\pi)^2$, hence

$$E_\gamma = m_\pi\left(1 + \frac{m_\pi}{2m_p}\right) = 150\,\mathrm{MeV} \tag{10}$$

3. No.

Differential geometry – Solutions

0.5 The sphere as a manifold – Solution

1. $x = \frac{x_1}{1-x_3}$, $y = \frac{x_2}{1-x_3}$, $\bar{x} = \frac{x_1}{1+x_3}$, $\bar{y} = \frac{x_2}{1+x_3}$
2. $\bar{x} = \frac{x}{x^2+y^2}$, $\bar{y} = \frac{y}{x^2+y^2}$ using $\frac{1-x_3}{1+x_3} = \frac{1}{x^2+y^2}$.
3. ok
4. ok
5. $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$; $\Gamma^\phi_{\theta\phi} = \cot\theta$. This can also be derived from the lagrangian $\mathcal{L} = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$.
6. (a) $\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0$ and $\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$ (be careful, there is a factor 2).
   (b) Use $\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \frac{\mathrm{d}^2\theta}{\mathrm{d}\phi^2}\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 + \frac{\mathrm{d}\theta}{\mathrm{d}\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}$.
   (c) We find $\frac{\mathrm{d}^2 f}{\mathrm{d}\phi^2} + f = 0$. Solve, and find $A\sin\theta\cos\phi + B\sin\theta\sin\phi - \cos\theta = 0$. This is the equation of the great circle in the plane whose normal vector is $(A, B, -1)$.

0.6 Playing with a metric – Solution

$t = t' + x'$ and the other coordinates are left unchanged.

0.7 Embedding diagram of a Wormhole – Solution

1. Minkowski.
2. The coefficients don't depend on t, and a slice at constant r is just the metric of a sphere.
3. $\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + (b^2 + r^2)\mathrm{d}\phi^2)$.
4. We have axisymmetry. The metric of ambient space is $\mathrm{d}S^2 = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2 + \mathrm{d}z^2$.
5. Use axisymmetry.
6. The identification with
$$\mathrm{d}\Sigma^2 = \left[\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2\right]\mathrm{d}r^2 + \rho^2\mathrm{d}\phi^2 \tag{11}$$
gives a differential and an algebraic equation.
7. Use $\rho^2 = r^2 + b^2$. We find a differential equation for $z(r)$, and this gives $z(r) = b\sinh^{-1}(r/b)$. Then $\rho(z) = b\cosh(z/b)$.

Curvature – Solutions

0.8 Playing with a metric – Solution

1. Set $\xi = R\sqrt{A}$. Then $s\int_0^R (1 - Ar^2)\mathrm{d}r = R(1 - \xi^2/3)$.
2. The metric on the sphere is the usual one, so the area is $4\pi R^2$.
3. $V = \int (1 - Ar^2) r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{4}{3}\pi R^3 (1 - 3\xi^2/5)$.
4. $\int (1 - Ar^2)^2 r^2 \sin\theta \mathrm{d}t\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{4}{3}\pi R^3 T (1 - 6\xi^2/5 + 3\xi^4/7)$.
5. The Christoffel are

$$\begin{array}{cc} \Gamma[1,\,1,\,1] & \frac{2Ar}{Ar^2-1} \\ \Gamma[1,\,2,\,2] & -\frac{r}{(Ar^2-1)^2} \\ \Gamma[1,\,3,\,3] & -\frac{r\sin^2(\theta)}{(Ar^2-1)^2} \\ \Gamma[1,\,4,\,4] & \frac{2Ar}{Ar^2-1} \\ \Gamma[2,\,2,\,1] & \frac{1}{r} \\ \Gamma[2,\,3,\,3] & \sin(\theta)(-\cos(\theta)) \\ \Gamma[3,\,3,\,1] & \frac{1}{r} \\ \Gamma[3,\,3,\,2] & \cot(\theta) \\ \Gamma[4,\,4,\,1] & \frac{2Ar}{Ar^2-1} \end{array} \tag{12}$$

with time as the 4th coordinate, and the first index is the upper one. The Ricci tensor is

$$\begin{array}{cc} \mathrm{R}[1,\,1] & \frac{2A\left(3Ar^2-1\right)}{(Ar^2-1)^2} \\ \mathrm{R}[2,\,2] & \frac{Ar^2\left(Ar^2-2\right)}{(Ar^2-1)^2} \\ \mathrm{R}[3,\,3] & \left(1-\frac{1}{(Ar^2-1)^2}\right)\sin^2(\theta) \\ \mathrm{R}[4,\,4] & \frac{2A\left(Ar^2-3\right)}{(Ar^2-1)^2} \end{array} \tag{13}$$

0.9 Symmetries, Lie derivatives and Killing vectors

0.9.1 Killing vectors

1. $\delta \mathrm{d}x^\mu = \epsilon \frac{\partial \xi^\mu}{\partial x^\nu} \mathrm{d}x^\nu$ and $\delta g_{\mu\nu} = \epsilon \frac{\partial g_{\mu\nu}}{\partial x^\nu} \xi^\nu$. Then compute $\delta \mathrm{d}s^2$ and obtain :

$$\xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu \xi^\rho + g_{\rho\mu}\partial_\nu \xi^\rho = 0 \tag{14}$$

2. The change of the metric at first order dut to the change of coordinate is $g'_{\mu\nu} = g_{\mu\nu} - \epsilon g_{\rho\nu}\partial_\mu \xi^\rho - \epsilon g_{\rho\mu}\partial_\nu \xi^\rho$ and the change due to the change of point is the same as in the previous question.
3. Use the fact that the metric is covariantly constant. Then $D_\mu \xi_\nu + D_\nu \xi_\mu = 0$.

4. The Lie derivative is $\mathcal{L}_V\phi(x) = V^\mu\partial_\mu\phi$ for a scalar. It is extended to tensors by $\mathcal{L}_V U^\nu = V^\mu\partial_\mu U^\nu - (\partial_\mu V^\nu)U^\nu$. Note that for vectors $\mathcal{L}_V U = [V, U]$, and that if the connexion is symmetric we can replace ∂ by ∇ everywhere. For the metric $\mathcal{L}_V g_{\mu\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu$. (See Erdmenger).

5. Use Riemann - Killing - Riemann - Killing - Riemann - Killing to obtain $2\nabla_\mu\nabla_\nu\xi_\rho = (-R^\sigma{}_{\rho\mu\nu} + R^\sigma{}_{\mu\nu\rho} - R^\sigma{}_{\nu\rho\mu})$ and then use Bianchi identity.

6. Trivial : $\xi^\mu = (0, ..., 1, ..., 0)$.

7. Trivial.

8. Expand the derivatives using Leibniz rule. USe Killing equation $\nabla_\mu\xi^\rho\nabla_\rho\omega_\nu = \nabla_\rho\xi_\mu\nabla^\nu\omega^\rho$ to show that the terms with first order derivatives vanish. Then express the other terms using Riemann tensor, and show that they vanish using the symmetries.

0.9.2 The sphere

1. Trivial.

2. $\partial_\theta\xi_\theta = 0$, $\partial_\phi\xi_\phi + \sin\theta\cos\theta\xi_\theta = 0$ and $\partial_\theta\xi_\phi + \partial_\phi\xi_\theta - 2\cot\theta\xi_\phi = 0$.

3. $\xi_\theta = f(\phi)$, $\xi_\phi = -\sin\theta\cos\theta F(\phi) + g(\theta)$ using the first two equations. Then the third gives $g' - 2\cot\theta g = -f' - F = C$. The first part can be written $(\sin^{-2}\theta g(\theta))' = C\sin^{-2}\theta$, hence $g(\theta) = (C_1 - C\cot\theta)\sin^2\theta$. The other part gives an harmonic equation, hence $F(\phi) = -A\cos\phi + \sin\phi - C$. Then we substitute and find the answer.

4. We find the $so(3)$ algebra. Tis reflects $S^2 = SO(3)/SO(2)$.

5. $\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta$, $\Gamma^\phi_{\theta\phi} = \cot\theta$. $R^\theta_{\phi\phi\theta} = -\sin^2\theta$ and $R^\phi_{\theta\phi\theta} = 1$. Ricci tensor $R_{\theta\theta} = 1$, $R_{\phi\phi} = \sin^2\theta$, $R_{\theta\phi} = 0$. Scalar $R = 2$. If we set back the r, $R = 2/r^2$ and the other tensors don't change.

0.9.3 Conserved quantities

1. Construct $I^\mu = \xi_\nu T^{\nu\mu}$. $\nabla_\mu I^\mu = 0$ because $\nabla_\mu\xi_\nu$ is an antisymmetric tensor.

2. Use the relation from the introduction and the fact that $\nabla_\mu I^\mu = 0$.

3. $\partial_0 Q = -\int\partial_i\hat{I}^i\mathrm{d}^3x = \int_\infty\hat{I}^i\mathrm{d}\Sigma_i = 0$ if it decreases sufficiently rapidly.

0.10 Mathematics of curvature

1. Use several times the Bianchi identity...

2. Λ^2E^* has dimension 1, so the dimension is at most 1. $R_{1212} = 1$ can satisfy all the properties, so the dimension is exactly 1.

3. (a) The dimension is at most 6 due to the inclusion. Now to any symmetric tensor h_{mn} we can associate a tensor $R_{ijkl} = h_{mn}$ where ijm and kln are even permutations of 123. So the dimension is 6.

   (b) It is just a symmetric tensor.

   (c) $R_{\mu\nu\rho\sigma} = -\frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) + (g_{\mu\rho}R_{\nu\sigma} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} + g_{\nu\sigma}R_{\mu\rho})$

4. easy
5. The image is $\Lambda^4 E^*$. The decomposition is $S^2(\Lambda^2 E^*) = \mathcal{R}E \oplus \Lambda^4 E^*$. Then the dimension of $\mathcal{R}E$ is
$$\frac{1}{2}(\frac{n(n-1)}{2})(\frac{n(n-1)}{2}-1) - \frac{n(n-1)(n-2)(n-3)}{2.3.4} = \frac{n^2(n^2-1)}{12} \tag{15}$$

Einstein equations and the Energy-momentum tensor – Solutions

0.11 Energy-momentum-stress tensor – Solutions

1. Densities are a local concept. We can use the free fall frame.
2. (a) $n = \mathcal{N}/\mathcal{V}_*$ and the volume of the box in the frame where it is moving at speed V is $\mathcal{V} = \mathcal{V}_*\sqrt{1-V^2}$. The total number $\mathcal{N}$ doexn't change, so $N^0 = \mathcal{N}/\mathcal{V} = \frac{n}{\sqrt{1-V^2}}$
   (b) We have $N^0 = nu^0$ and nu^μ is a 4-vector since n is a scalar quantity. We have
   $$\vec{N} = \frac{n\vec{V}}{\sqrt{1-V^2}} \tag{16}$$
   (c) Now $N^\mu(x) = n(x)u^\mu(x)$ is considered to be a field. The conservation equation is
   $$\frac{\partial N^0}{\partial t} + \vec{\nabla}\cdot\vec{N} = 0 \leftrightarrow \partial_\mu N^\mu = 0 \tag{17}$$
   (d) This is the only way to build a scalar...
   (e) Densities are fluxes in timelike directions through spacelike 3-surfaces? Currents are fluxes in spacelike directions through timelike 3-surfaces.
   (f) The charge is $J^\mu n_\mu \delta\mathcal{V}$.
3. (a) Trivial.
   (b) The 3-volume is spacelike. Then $\delta p^\mu = T^{\mu 0}\delta\mathcal{V}$. Then the energy density is $\delta p^0/\delta\mathcal{V} = T^{00}$ and the momentum density is T^{0i}.
   (c) Consider a timelike 3-surface spanned by δy, δz and δt. Then $\delta p^\mu = T^{x\mu}\delta A\delta t$ and $T^{x0} = \frac{\delta p^0}{\delta A\delta t}$ is the flux of energy in the x direction. The spacial part is
   $$T^{ix} = \frac{\delta p^i/\delta t}{\delta A} \tag{18}$$
   where the numerator is a force. So this is a force per unit area, ie the *stress*.
   (d) easy.

0.12 Electromagnetic field – Solution

1. Easy.
2. $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ because the Christoffel are symmetric. Then use the general expression of $\nabla_\mu F_{\nu\rho}$, and antisymmetrize to find $\nabla_{[\mu}F_{\nu\rho]} = \partial_{[\mu}F_{\nu\rho]} = 2\partial_{[\mu}\partial_\nu A\rho] = 0$.

3. bla...

4. $\nabla_\mu F^{\nu\mu} = \mu_0 J^\nu$ gives $\nabla^\mu T_{\mu\nu} = -F_{\mu\nu} J^\nu$. Therefore it is not conserved in presence of charged matter. This is because there is an exchange of energy and momentum between matter and the electromagnetic field. However, the total energy-momentum tensor must be conserved.

5. Take the trace of Einstein equations. This gives that the scalar curvature is proportional to the trace of the energy-momentum tensor. Here this is 0 !