Dimer models in Physics

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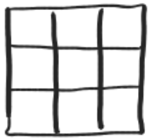
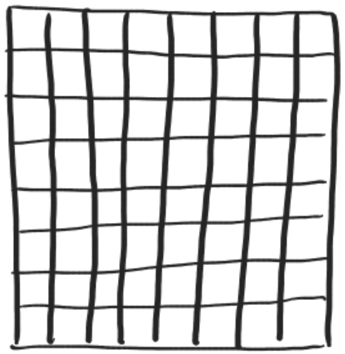
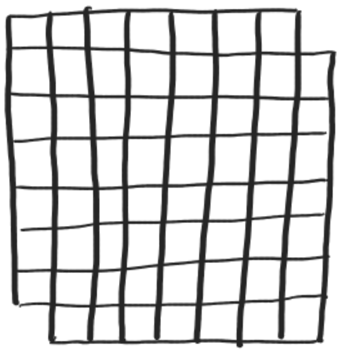
How many tilings for chessboard with dominos?







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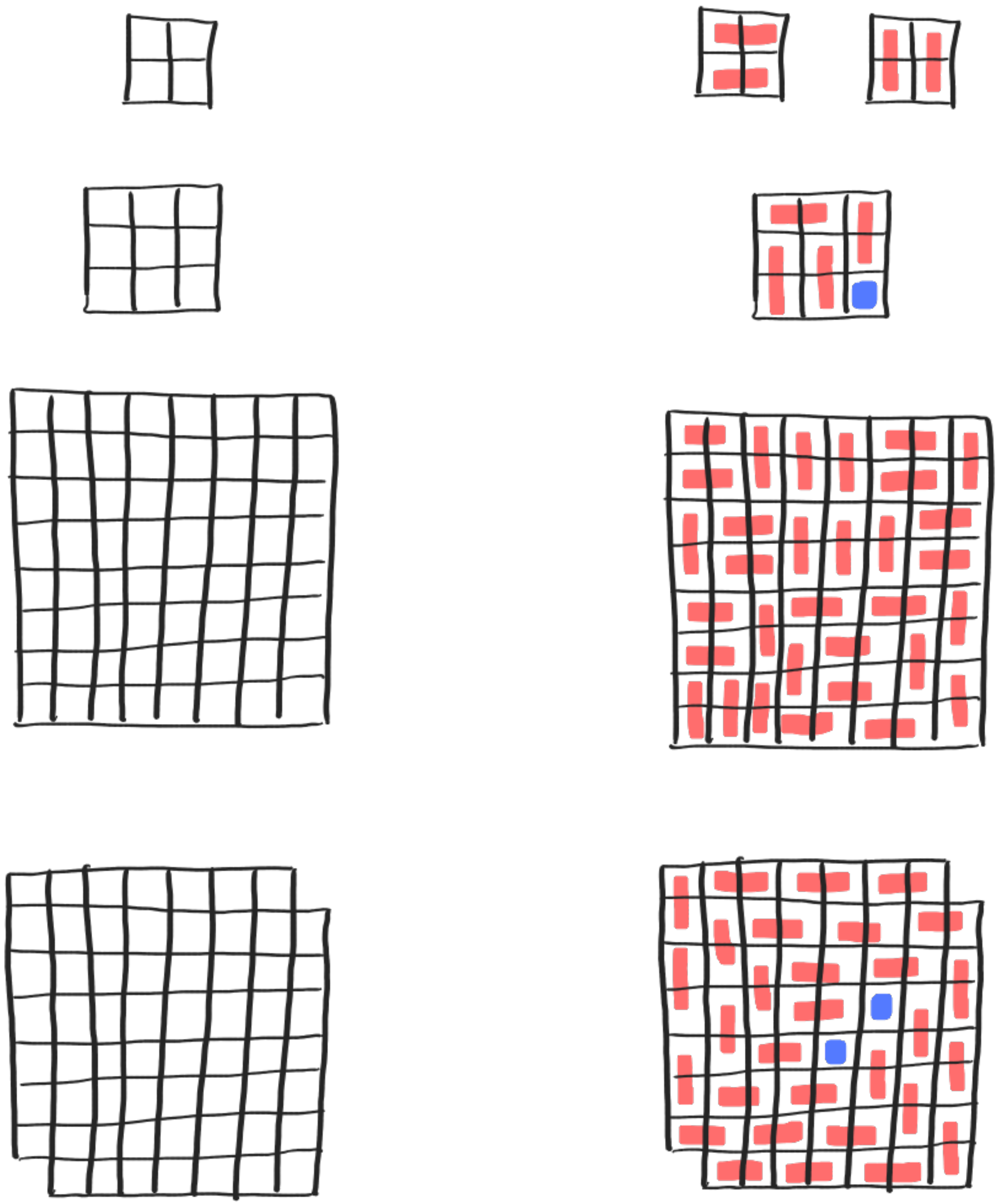
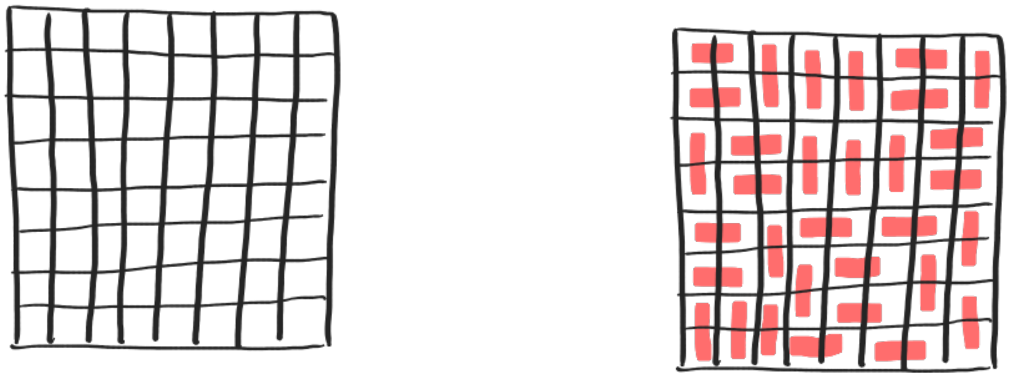


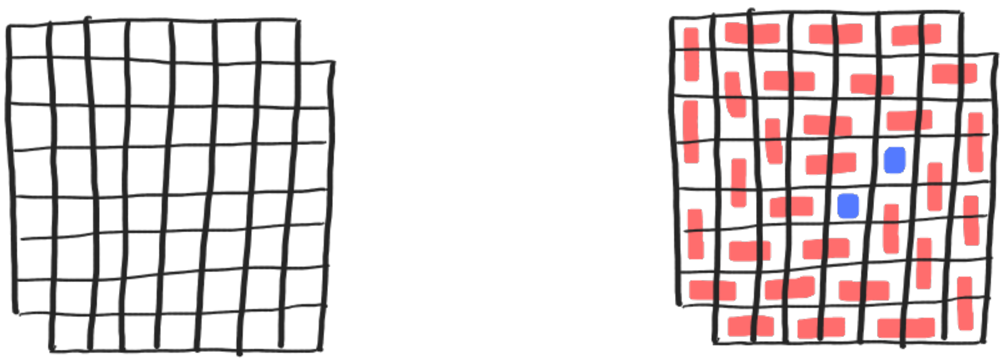
How many tilings for chessboard with dominos?

$Z = 2$

$Z = 0$



$Z = 12\,988\,816$



$Z = 0$

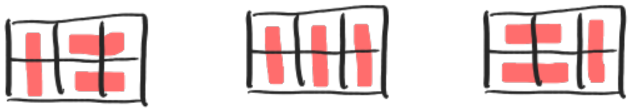
Plan :

- Introduction to dimers
- The Kasteleyn determinant
- Application in statistical physics
- Application to string theory and gauge theory.

Exercise : how many tilings for $2 \times n$ rectangle ?

 $F_1 = 1$

 $F_2 = 2$

 $F_3 = 3$

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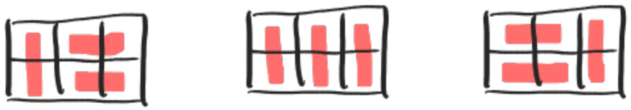

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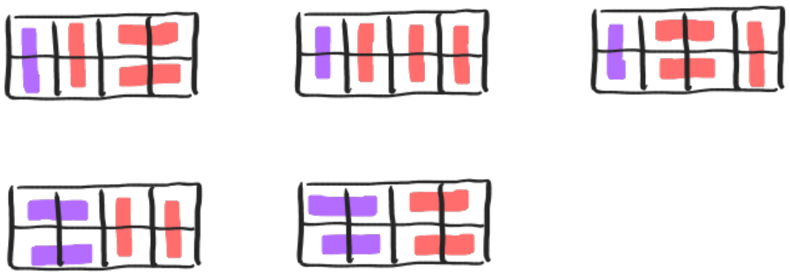

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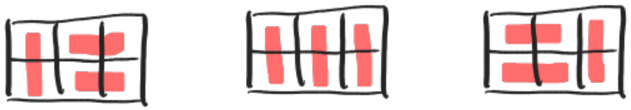

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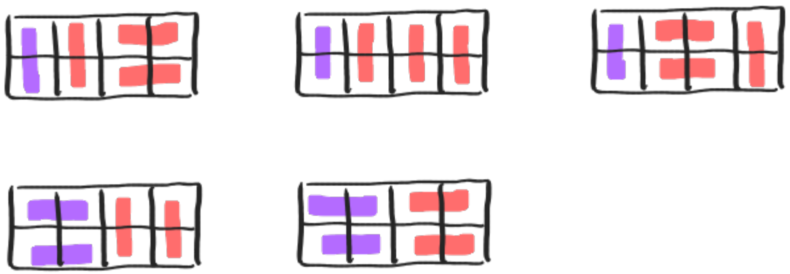


F_3

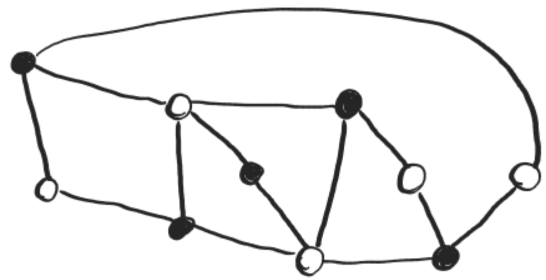
F_2

$F_3 + F_2 = F_4 = 5$

F_n = n^{th} Fibonacci number !

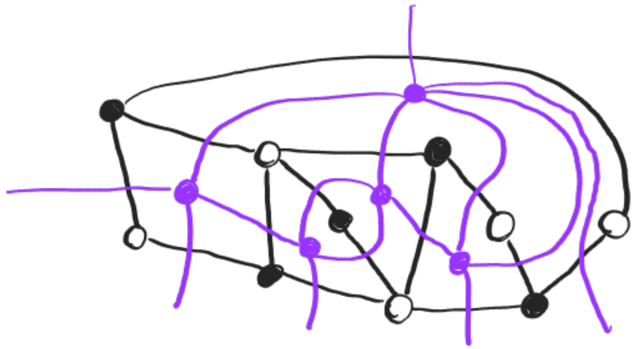
Let's formalize a bit:

- Let $G = (V, E)$ be a BIPARTITE graph: $V = V_b \sqcup V_w$ and $E \subset V_b \times V_w$.



- Assume PLANAR graph
  $\longrightarrow$ there is a DUAL graph

- A DIMER CONFIGURATION is a subset of E that covers all of V exactly once.



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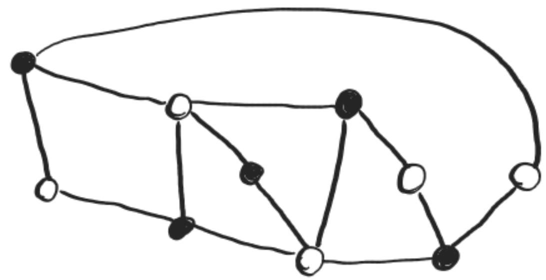

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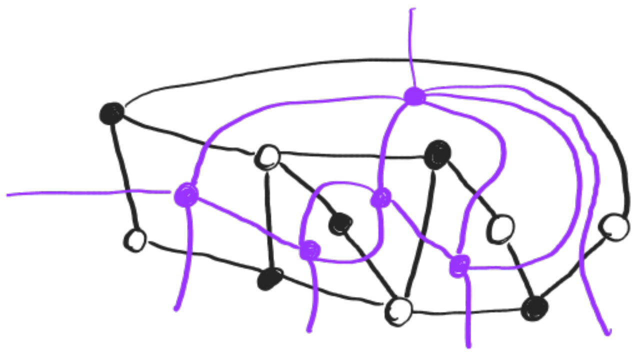

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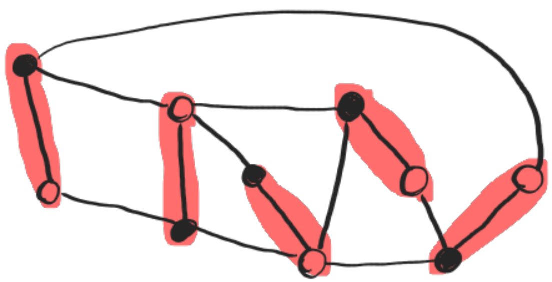

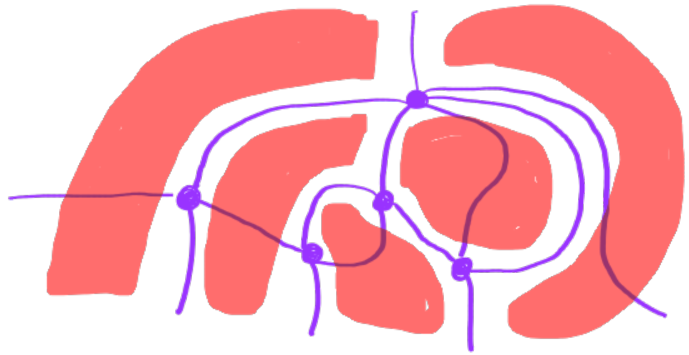

Dimers on G $\longleftrightarrow$ Domino tiling on $G^\vee$

Central question: Given G planar and bipartite, how many dimer configurations are there?

$$Z_G = \sum_{\mathscr{C}} 1$$

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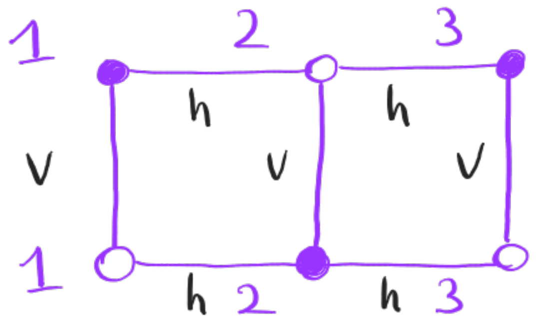
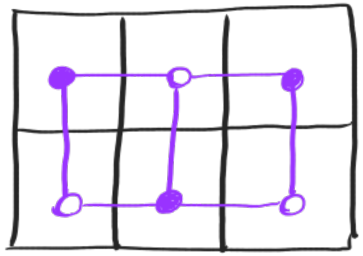
$$Z_G = \sum_{\mathcal{C}} 1$$

Refine using $\pi : E \longrightarrow R$ (R commutative ring e.g. $R = \mathbb{R}[x,y]$)

$$Z_G(\pi) = \sum_{\mathcal{C}} \prod_{e \in \mathcal{C}} \pi(e)$$

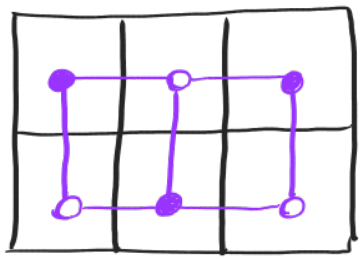
Fundamental example



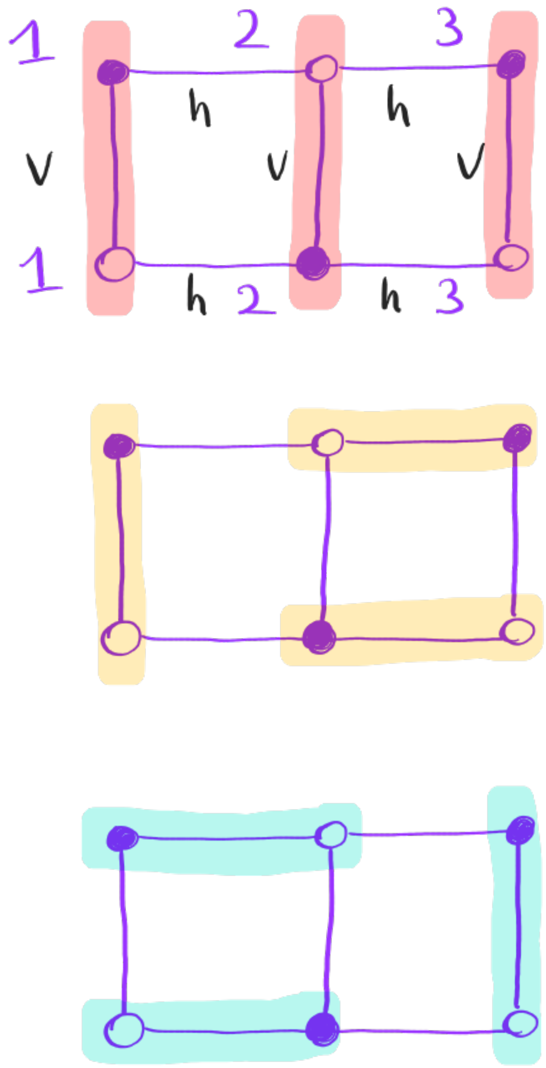


$$
M = \begin{array}{c c}
\begin{array}{ccc} 1 & 2 & 3 \\ \circ & \circ & \circ \end{array} & \\
\begin{pmatrix} v & h & 0 \\ h & v & h \\ 0 & h & v \end{pmatrix} & \begin{array}{c} \bullet\ 1 \\ \bullet\ 2 \\ \bullet\ 3 \end{array}
\end{array}
$$

Fundamental example



$$Z_G = 3 = v^3 + 2h^2 v$$



$$M = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

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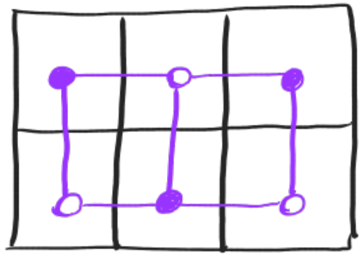

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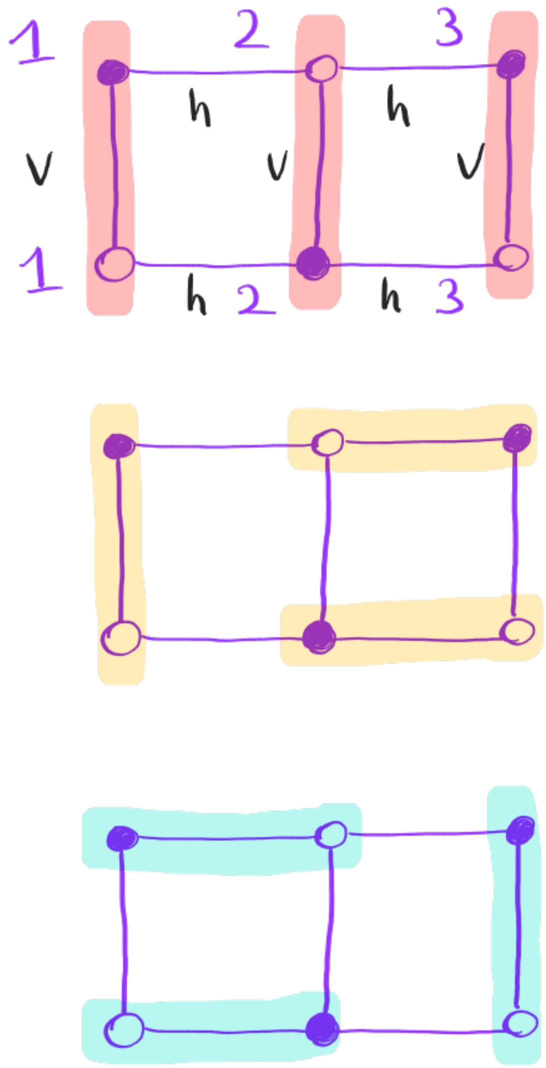
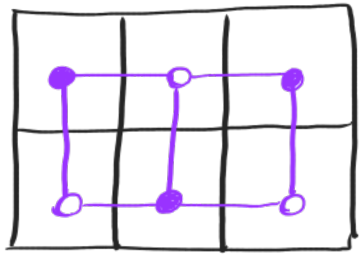


$$M = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \\ \circ & \circ & \circ \end{matrix} \\ \hline \begin{matrix} \bullet\ 1 \\ \bullet\ 2 \\ \bullet\ 3 \end{matrix} & \begin{pmatrix} v & h & 0 \\ h & v & h \\ 0 & h & v \end{pmatrix} \end{array}$$

$$Z_G = v \cdot v \cdot v + v \cdot h \cdot h + h \cdot h \cdot v$$

Fundamental example



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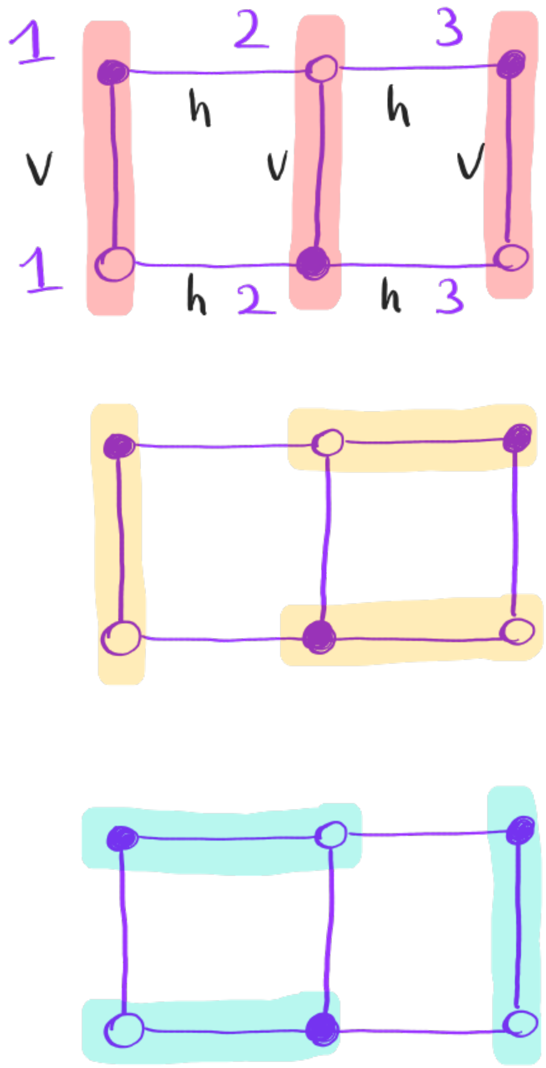


$$M = \begin{array}{cccc} 1 & 2 & 3 & \\ \left(\begin{array}{c} v \\ h \\ 0 \end{array}\right. & \begin{array}{c} h \\ v \\ h \end{array} & \left.\begin{array}{c} 0 \\ h \\ v \end{array}\right) & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$Z_G = v \cdot v \cdot v + v \cdot h \cdot h + h \cdot h \cdot v$$

$$= \mathrm{Perm}(M)$$

$$\mathrm{Perm}(M) = \sum_{\sigma \in S_N} \prod_{i=1}^{N} M_{i,\sigma(i)}$$

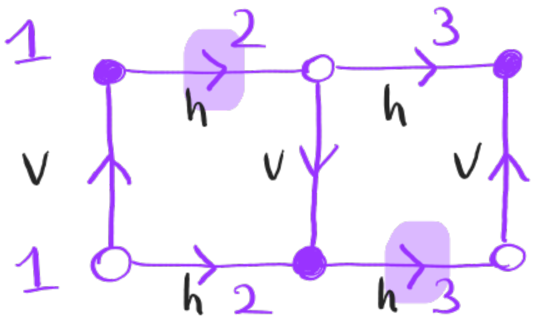
$$\mathrm{Det}(M) = \sum_{\sigma \in S_N} \prod_{i=1}^{N} (-1)^{\sigma} M_{i,\sigma(i)}$$

Permanent versus Determinant

$$\mathrm{Perm}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc \qquad \mathrm{Det}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- Multilinear forms, symmetric / antisymmetric
- $\mathrm{Perm}\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} = \mathrm{Det}\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} = \lambda_1 \cdots \lambda_N$
- But $\mathrm{Perm}(AB) \neq \mathrm{Perm}(A)\,\mathrm{Perm}(B) \Longrightarrow$ No Gauss Pivot!
- Super hard to compute: If $\exists$ polynomial time algorithm to compute $\mathrm{Perm}(M)$ then $P = NP$!

Kasteleyn Matrix



$+$: 

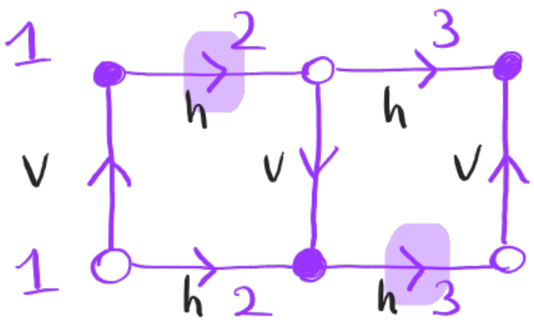
$-$: 

$$K = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \\ \circ & \circ & \circ \end{array} \\ \begin{pmatrix} v & -h & 0 \\ h & v & -h \\ 0 & h & v \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \bullet\ 1 \\ \bullet\ 2 \\ \bullet\ 3 \end{array}$$

Kasteleyn Matrix

+ : 

− : 



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A KASTELEYN ORIENTATION is an orientation of E such that for any face,

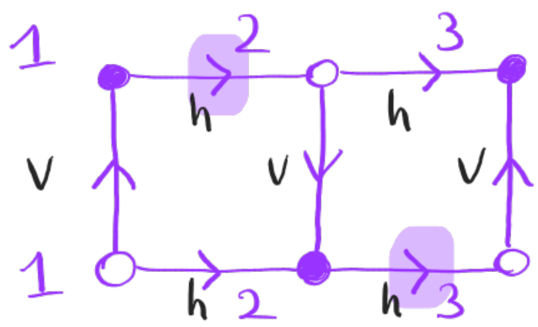
* If 2 mod 4 sides, EVEN number of 
* If 0 mod 4 sides, ODD number of 

$$\boxed{\underline{\text{Theorem}}:\quad Z = |\det K|}$$

Kasteleyn Matrix

$+ : \bullet \leftarrow \circ$

$- : \bullet \rightarrow \circ$



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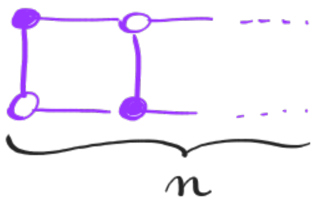
$$\boxed{\textbf{Theorem :} \quad Z = |\det K|}$$

Exercise :

$F_n = Z\Bigg($  $\Bigg) = \mathrm{Det}\begin{pmatrix} 1 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$

The chessboard problem

Take a chessboard of size $n \times m$ with $\underline{n \text{ even}}$. Then

$$\boxed{Z_G = \left|\operatorname{Det} K_G\right| = \prod_{x=1}^{n} \prod_{y=1}^{m} \left(4h^2\cos^2\left(\frac{x\pi}{n+1}\right) + 4v^2\cos^2\left(\frac{y\pi}{m+1}\right)\right)^{1/4}}$$

$Z_G(2,2) = 2$

$Z_G(4,4) = 36$

$Z_G(6,6) = 6728$

$Z_G(8,8) = 12988816$

$\vdots$

$Z_G(20,20) = 1269\,984\,011\,256\,235\,834\,242\,602\,753\,102\,293\,934\,298\,576\,249\,856$

Statistical Physics

Idea : $\begin{cases} \to \text{Give weight to what costs energy} \\ \to \text{Take large number limit.} \end{cases}$ $\longrightarrow$ Probability of $\mathcal{C}$:

$$P(\mathcal{C}) = \frac{1}{Z} e^{-\beta E(\mathcal{C})}$$

$$\Downarrow$$

$$Z = \sum_{\mathcal{C}} e^{-\beta E(\mathcal{C})}$$

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- Give weight to what costs energy $\longrightarrow$ Probability of $\mathcal{C}$:
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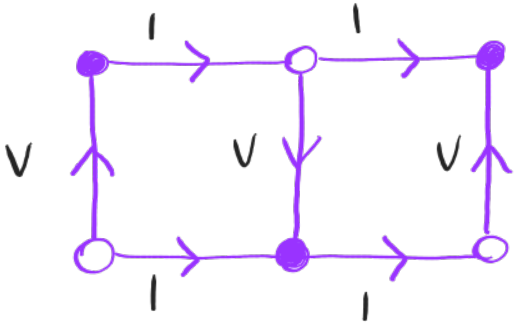
Example :

$E(◫) = 0$

$E(⊟) = \varepsilon$



$$V = e^{-\beta\varepsilon}$$

$$Z = V^3 + 2V$$

What is the average energy ?

- Naïve (wrong) answer : $\dfrac{3\varepsilon + \varepsilon + \varepsilon}{3} = \dfrac{5}{3}\varepsilon$

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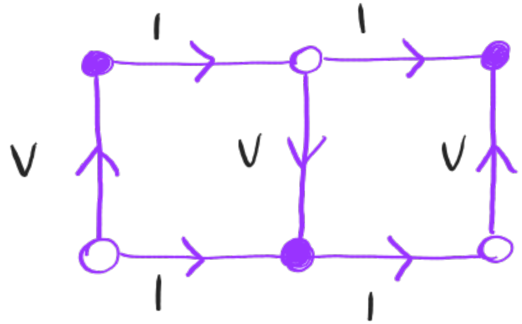

$$E(\square\!\square) = 0$$

$$E(\boxminus) = \varepsilon$$

$$V = e^{-\beta\varepsilon}$$

$$Z = V^3 + 2V$$

What is the average energy ?

- Naïve (wrong) answer : $\dfrac{3\varepsilon + \varepsilon + \varepsilon}{3} = \dfrac{5}{3}\varepsilon$

- Correct answer : $\langle E \rangle = \frac{1}{Z}\left(3\varepsilon\, e^{-3\varepsilon\beta} + 2\varepsilon\, e^{-\varepsilon\beta}\right)$

$$\langle E \rangle = \frac{1}{Z}\sum_{\mathcal{C}} E(\mathcal{C})\, P(\mathcal{C}) = \frac{1}{Z}\sum_{\mathcal{C}}\left(-\frac{\partial}{\partial\beta} e^{-\beta E(\mathcal{C})}\right) = -\frac{\partial \log Z}{\partial \beta}$$

- $\langle E \rangle = - \dfrac{\partial \log Z}{\partial \beta}$

- Entropy $S = - \sum_{\sigma} P(\sigma) \log P(\sigma) = \log Z + \beta \langle E \rangle$

- Free energy $F = \langle E \rangle - \frac{1}{\beta} S = - \frac{1}{\beta} \log Z$

$\boxed{\text{Take } \beta = 1}$

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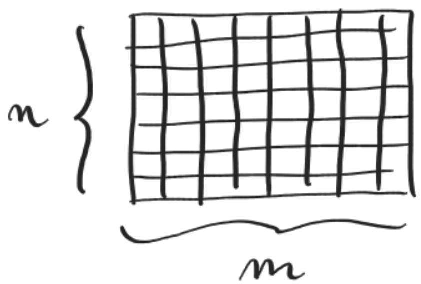
<u>Example</u> : MOLECULAR FREEDOM

Intuitively : on average when adding a domino, what is the number of possibilities that I have ?

Take



$$\varphi = \exp\left[\lim_{n,m \to \infty} \frac{\log Z(m,n)}{\left(\frac{mn}{2}\right)} \right]$$

* If there were no constraints at all, ⊟ or ◫ randomly $\longrightarrow \varphi = 2$

(indeed $Z = 2^{nm/2}$)

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Real case: $$\frac{1}{nm} \log Z \longrightarrow \frac{1}{\pi^2} \int_0^{\pi/2} d\omega \int_0^{\pi/2} d\omega' \log\left(4\cos^2\omega + 4\cos^2\omega'\right)$$

And $$\varphi = \exp\left[\frac{2G}{\pi}\right] \approx 1,7916\ldots$$

with $$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \approx 0.916\ldots$$ Catalan's constant

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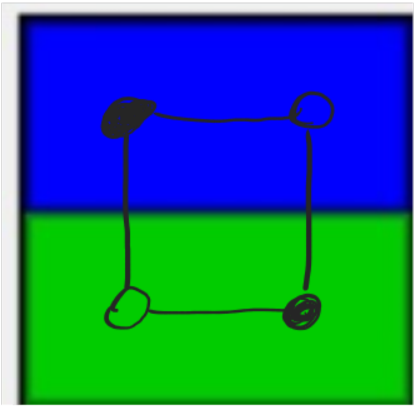
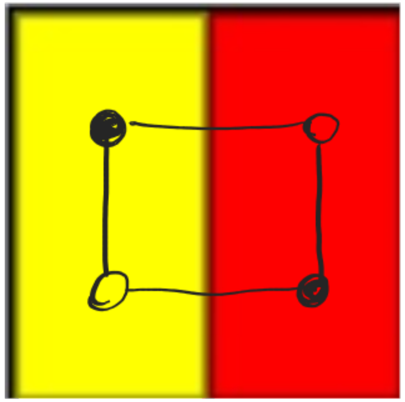
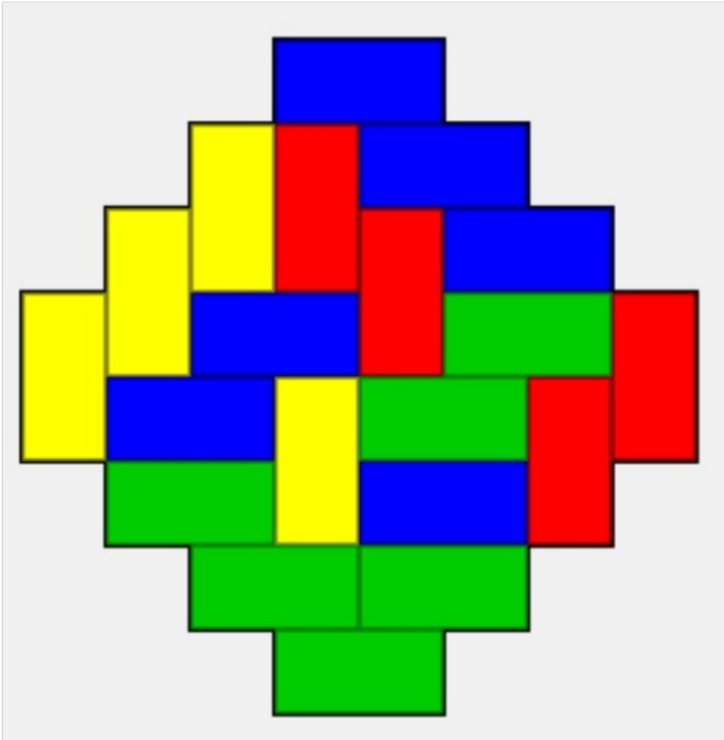
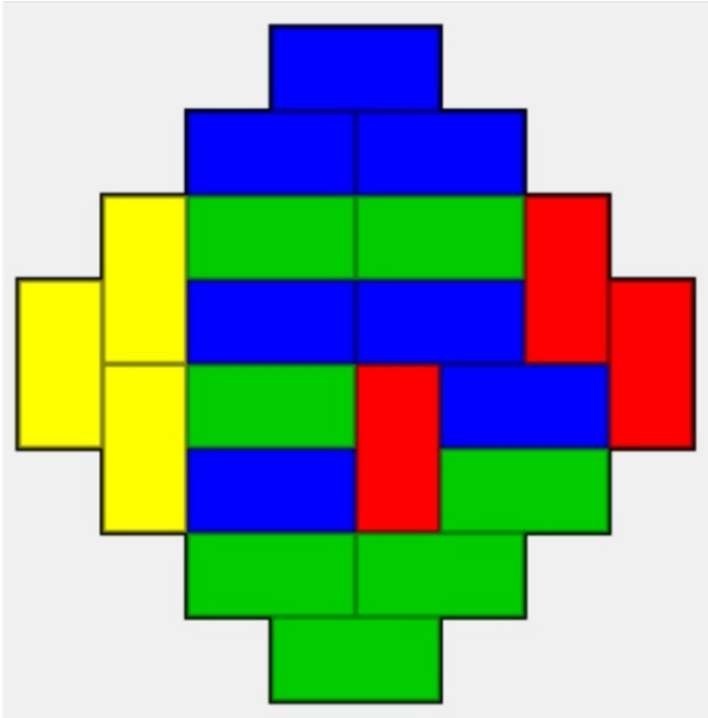
Further topics: $\begin{cases} \text{correlation functions, field theory, etc}\ldots \\ \text{Phase transitions}, \ldots \end{cases}$

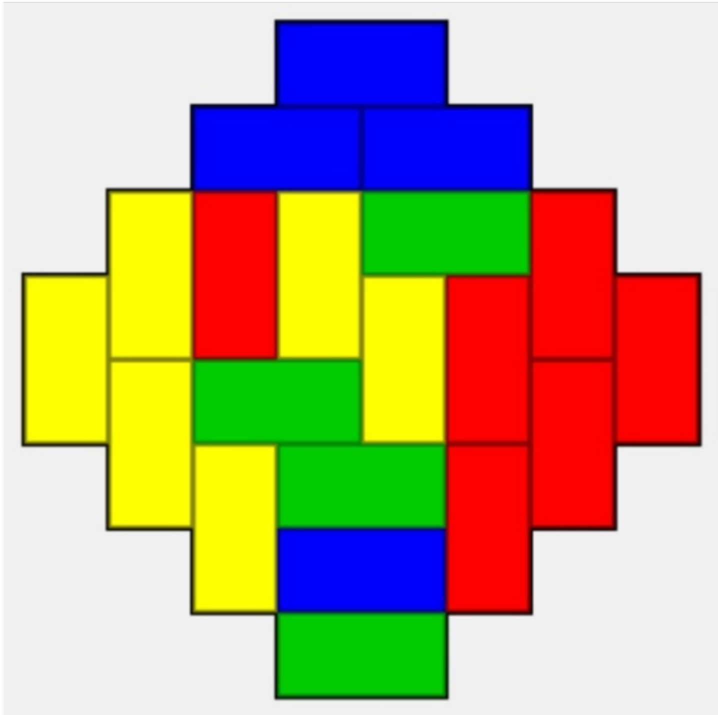
The Aztec Diamond

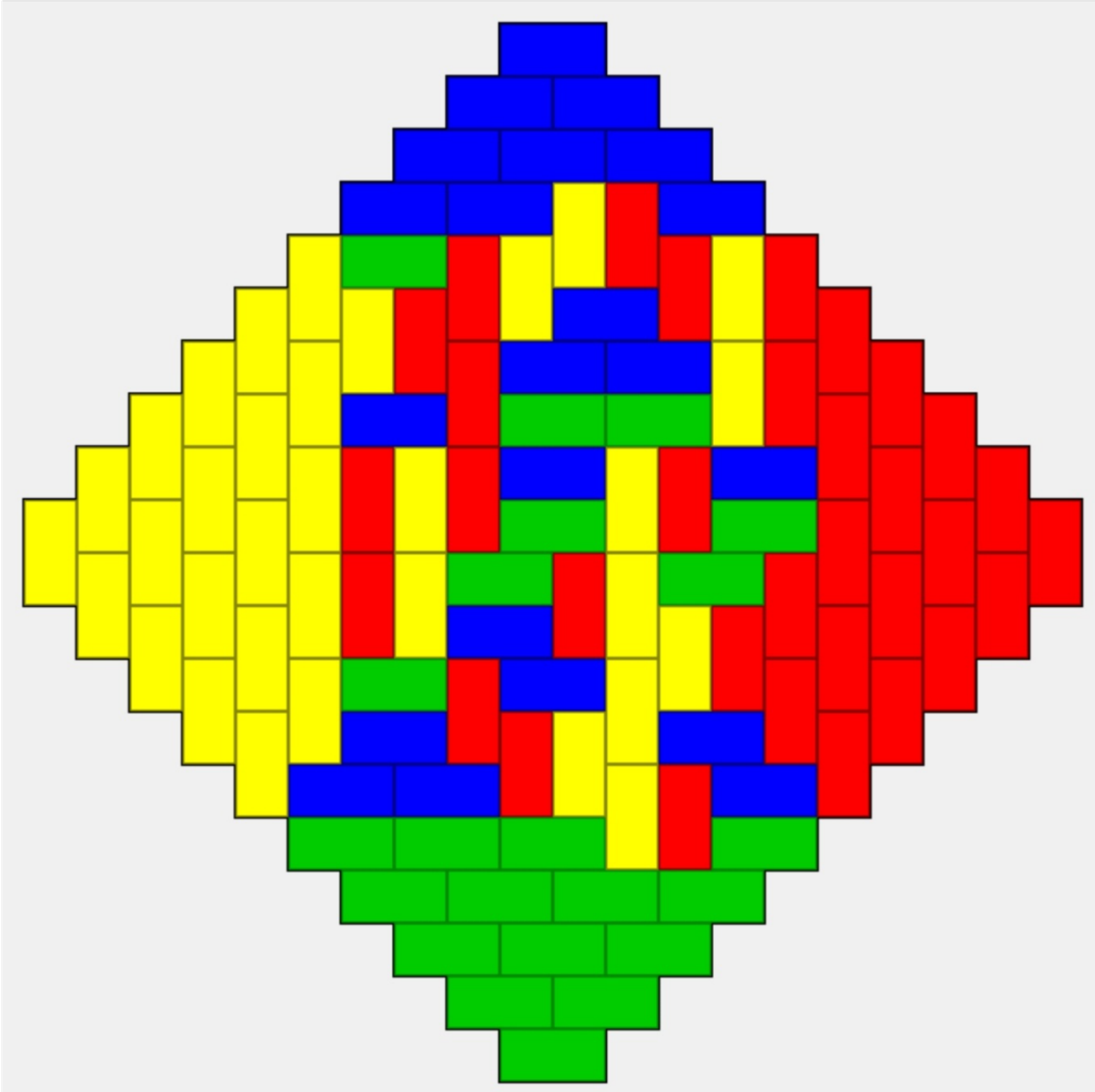


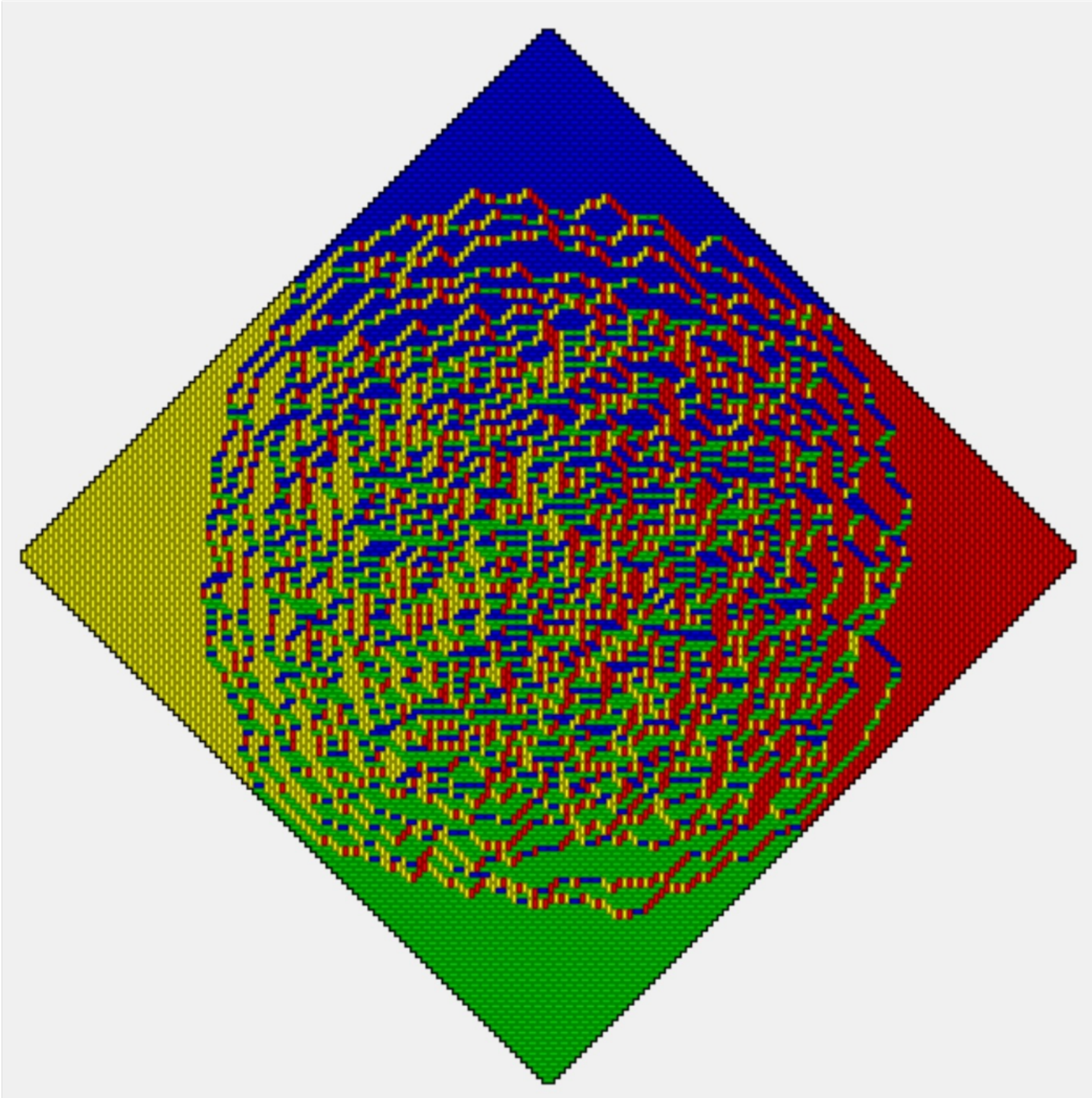












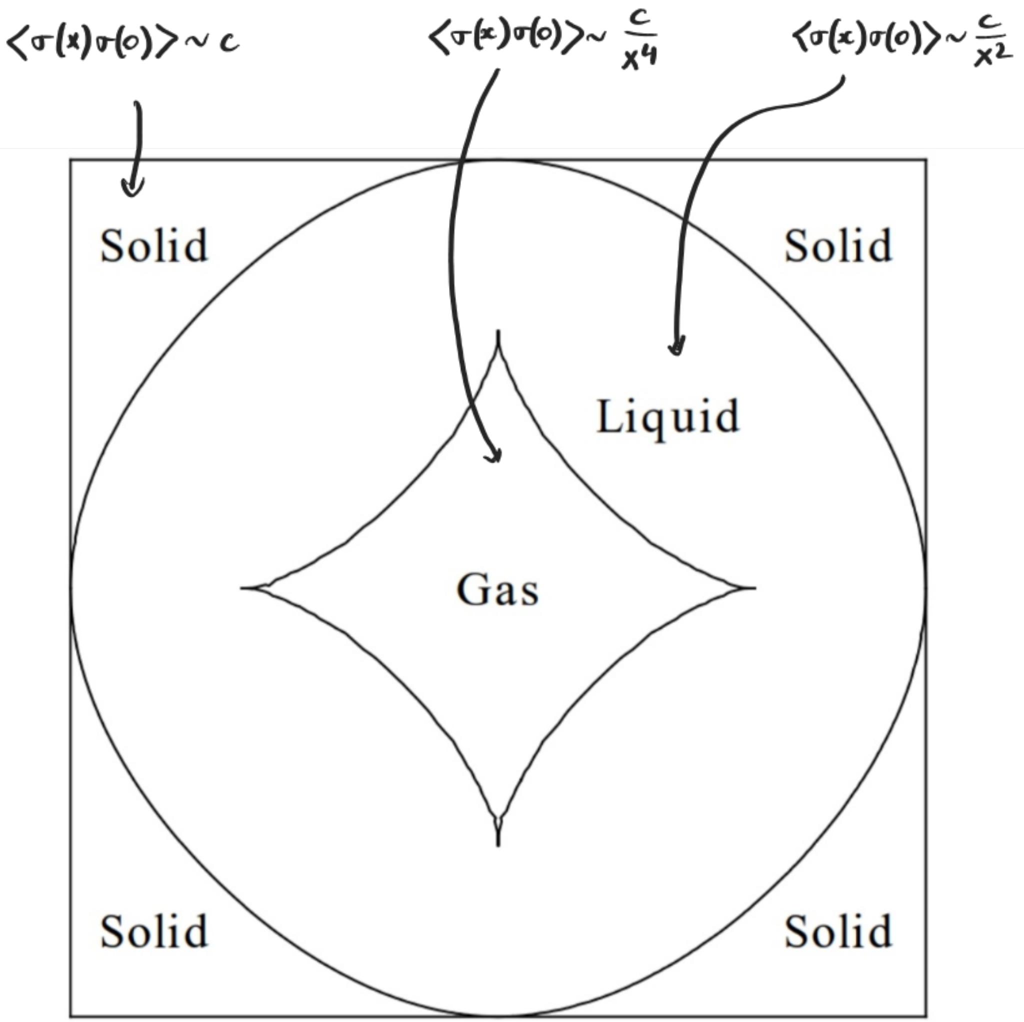


Figure: Phases of two-period weighted Aztec diamond from Chhita and Johansson (2016)

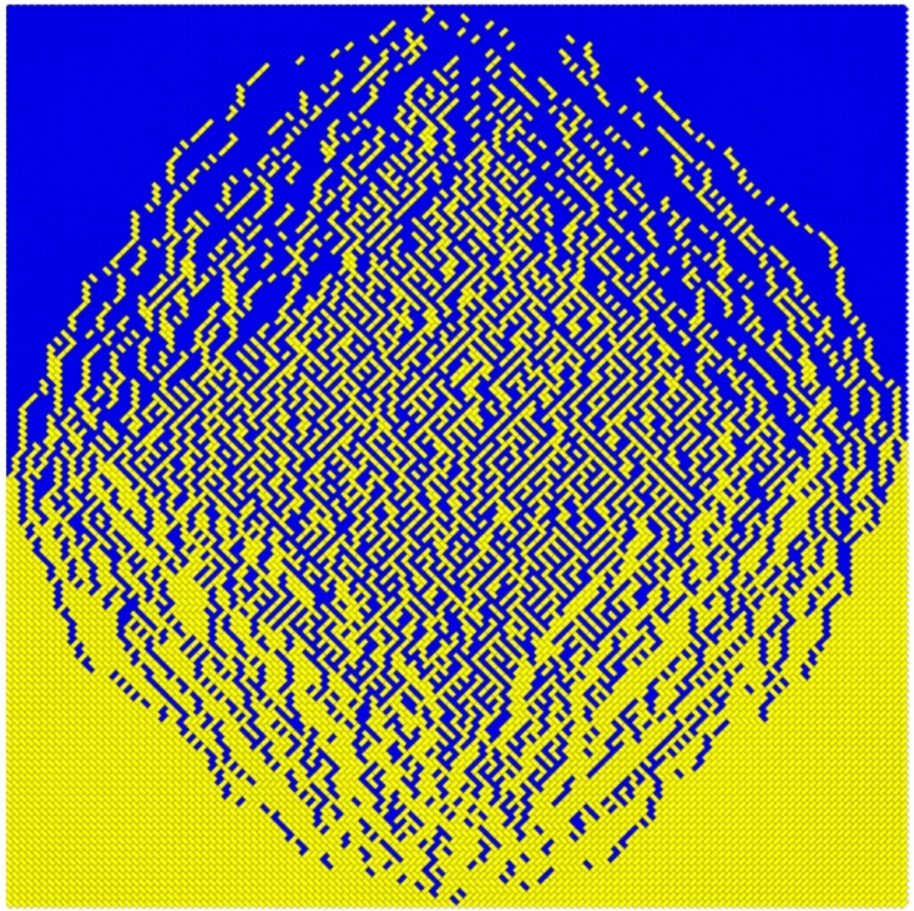


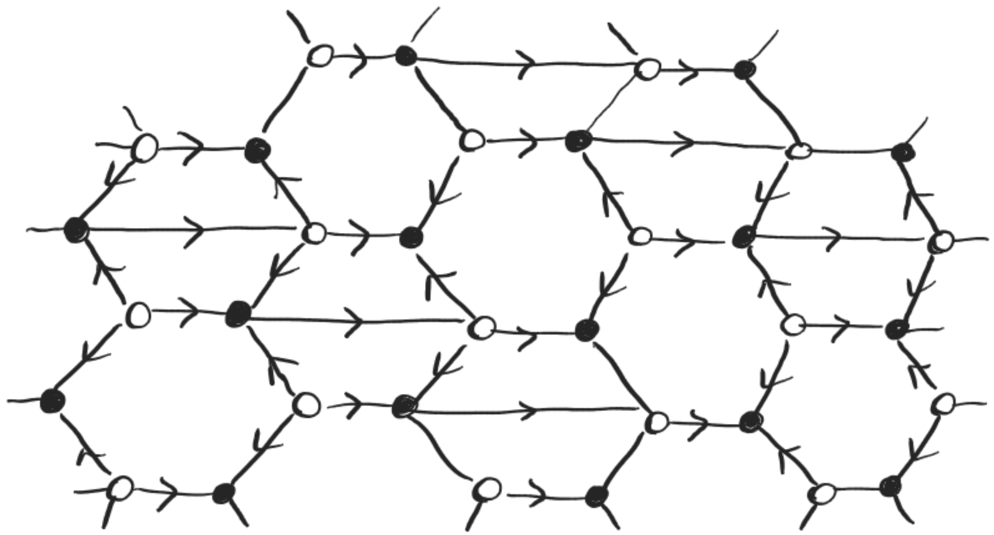
Figure: Aztec diamond of size 128 with weights 1 and 2

[E. Bain, Dimer models and the 2-periodic weighted Aztec diamond, 2022]

Periodic dimers, string theory and quantum field theory

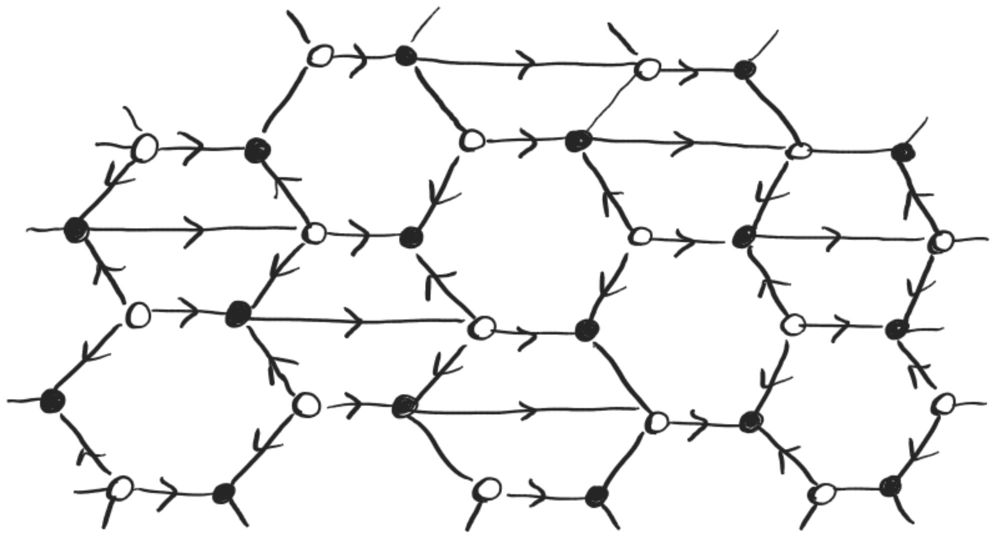
Toric dimers and supersymmetric gauge theories

Start from periodic graph:

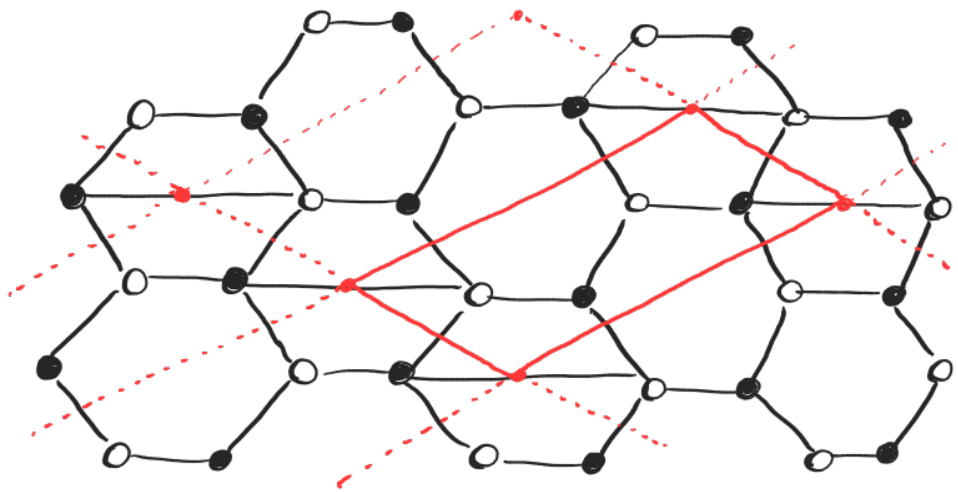


Toric dimers and supersymmetric gauge theories

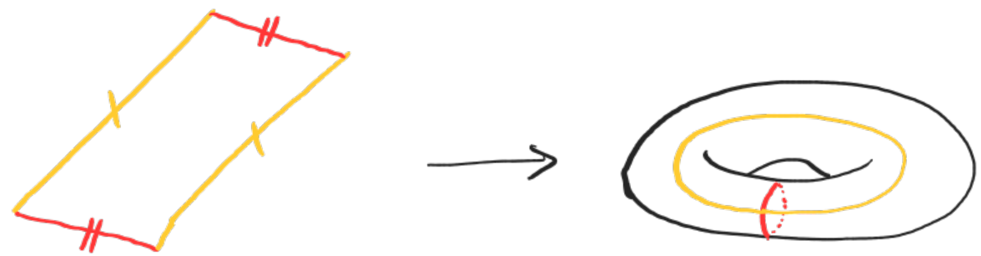
Start from periodic graph:



Fundamental cell:

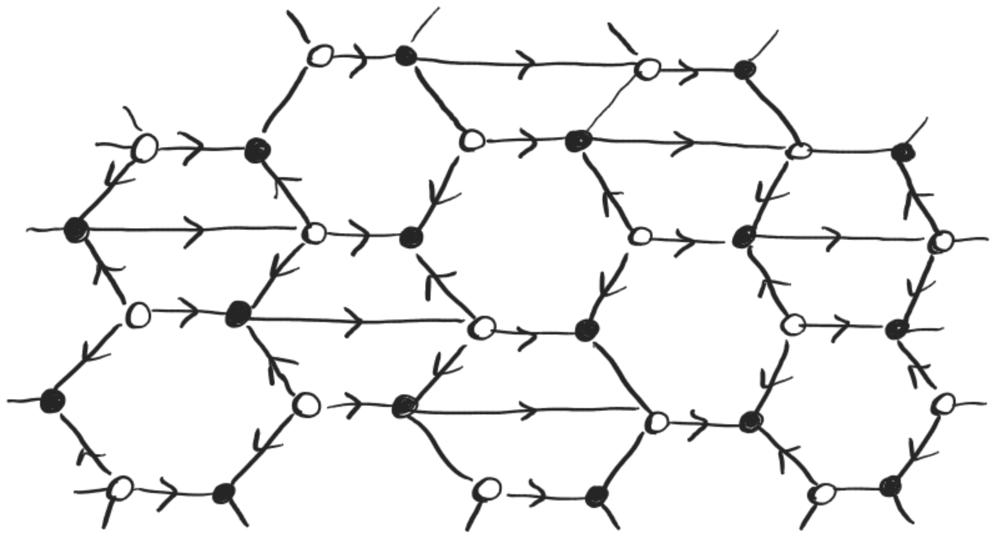


Recall that $\mathbb{R}^2/\mathbb{Z}^2 \simeq \mathbb{T}^2$

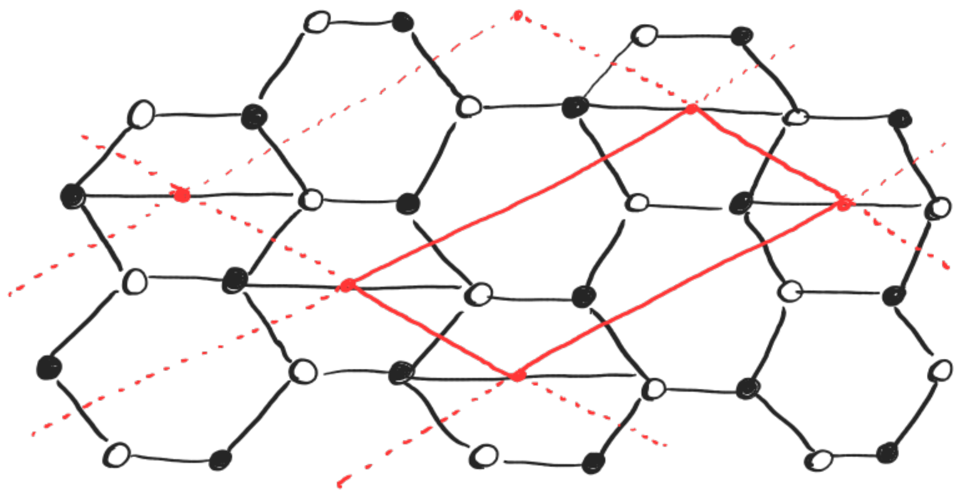


Toric dimers and supersymmetric gauge theories

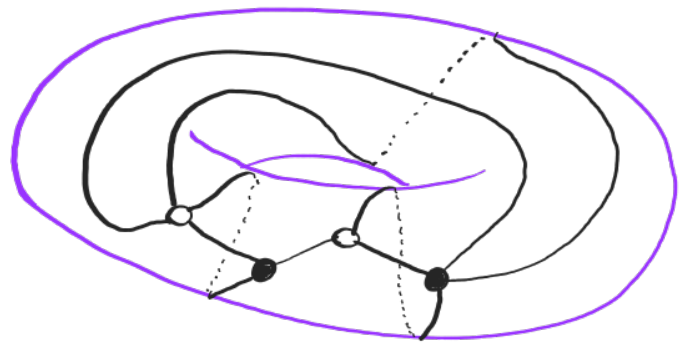
Start from periodic graph:



Fundamental cell:



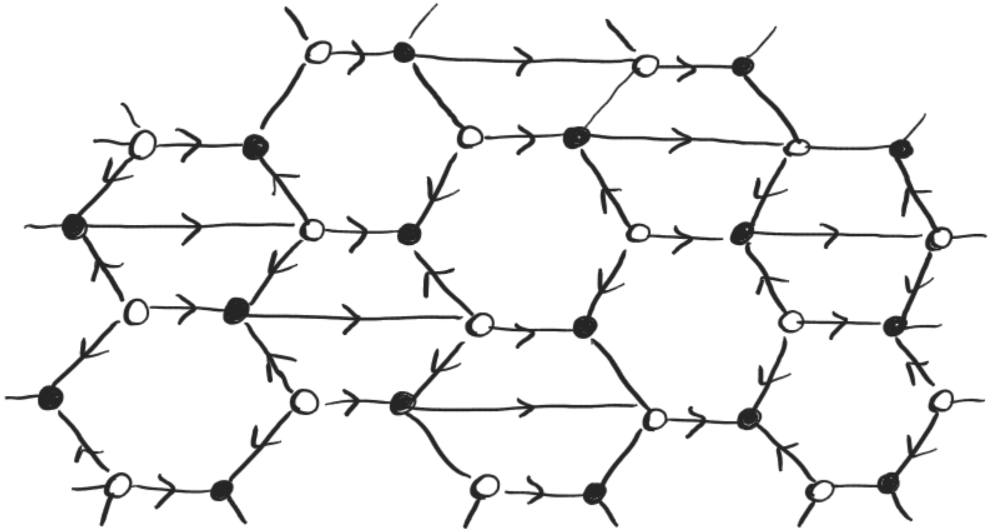
$\Longrightarrow$



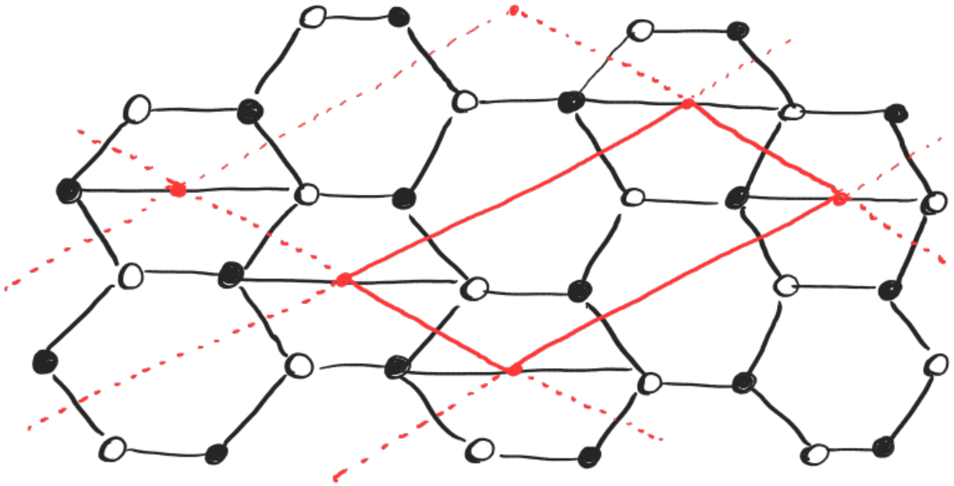
Toric dimer problem

Toric dimers and supersymmetric gauge theories

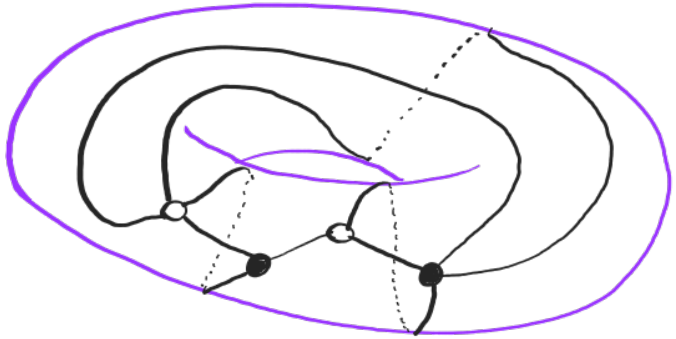
Start from periodic graph:



Fundamental cell:

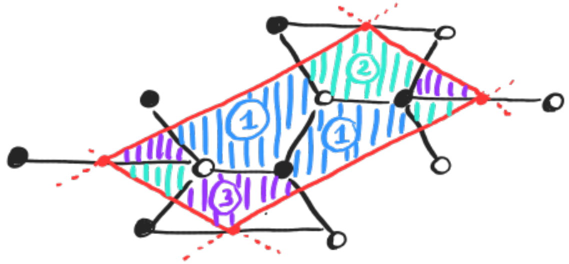


$\Rightarrow$



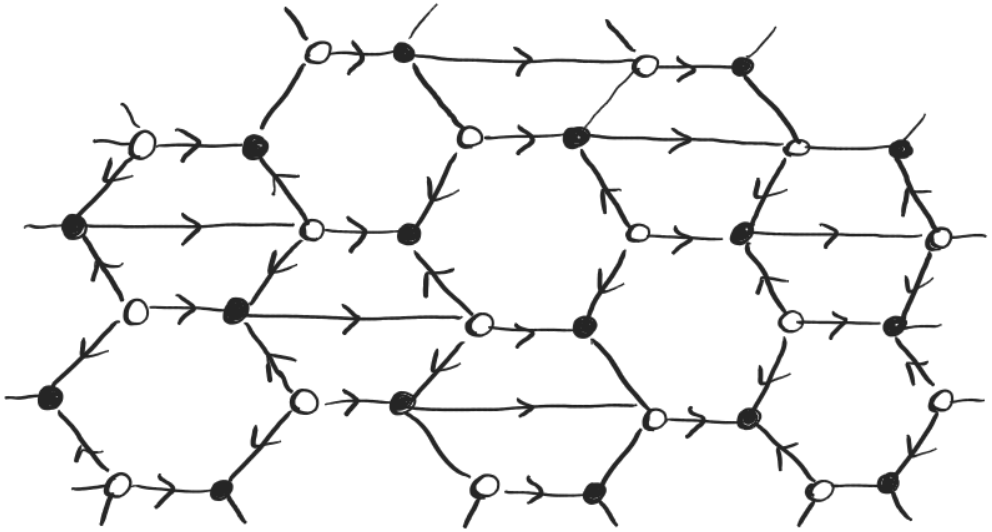
Toric dimer problem

Dual: periodic quiver
gauge theory:

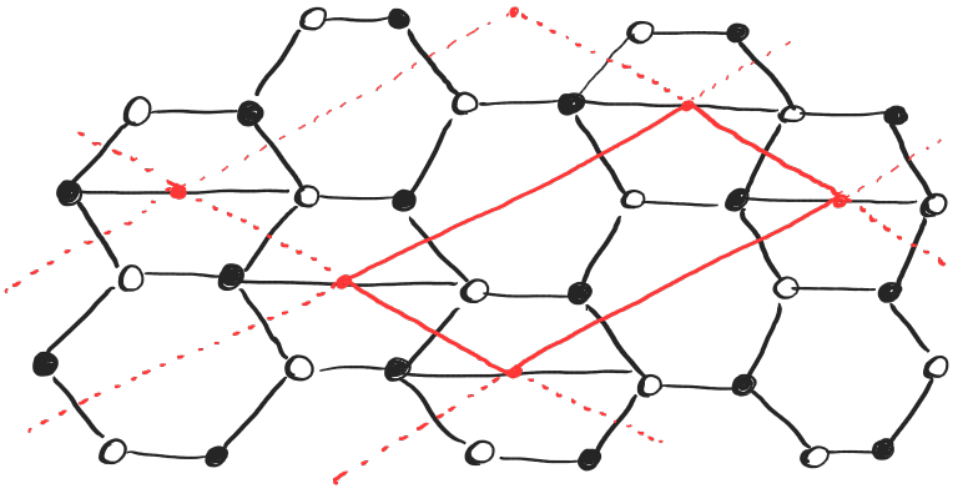


Toric dimers and supersymmetric gauge theories

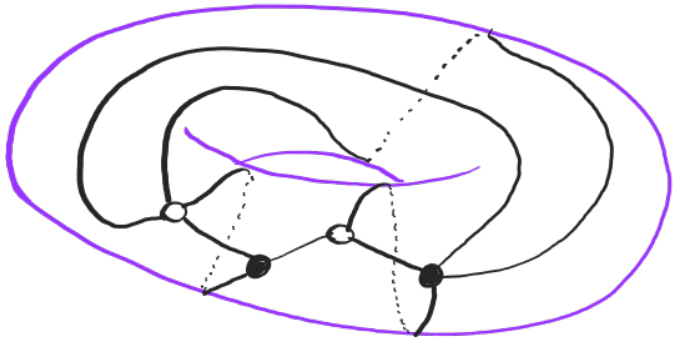
Start from periodic graph:



Fundamental cell:



$\Longrightarrow$



Toric dimer problem

Dual: periodic quiver

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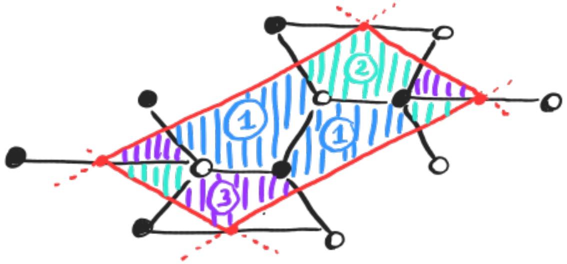
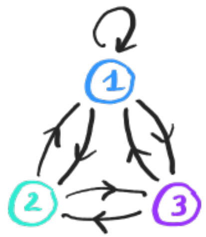
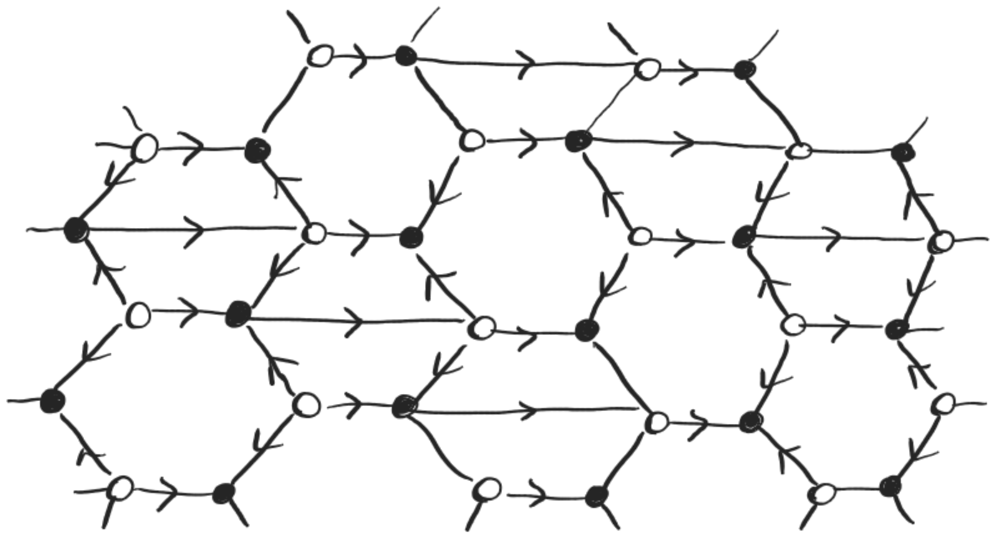


$\Longleftrightarrow$

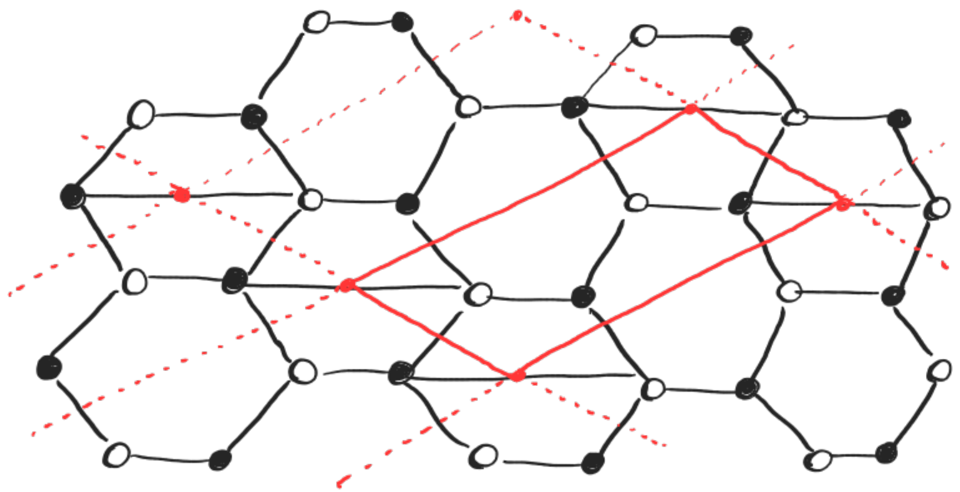


Toric dimers and supersymmetric gauge theories

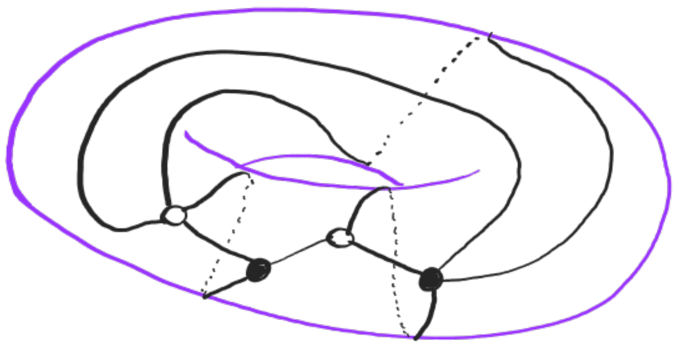
Start from periodic graph:



Fundamental cell:

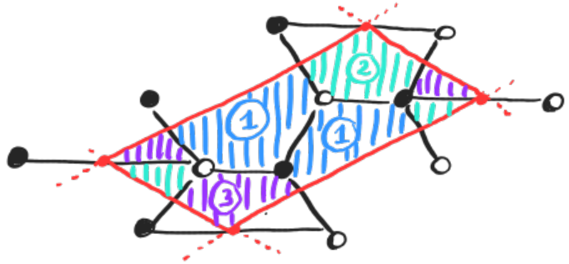


$\Longrightarrow$

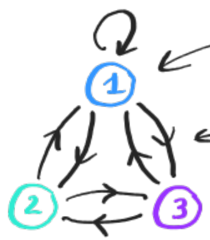


Toric dimer problem

Dual: periodic quiver gauge theory:



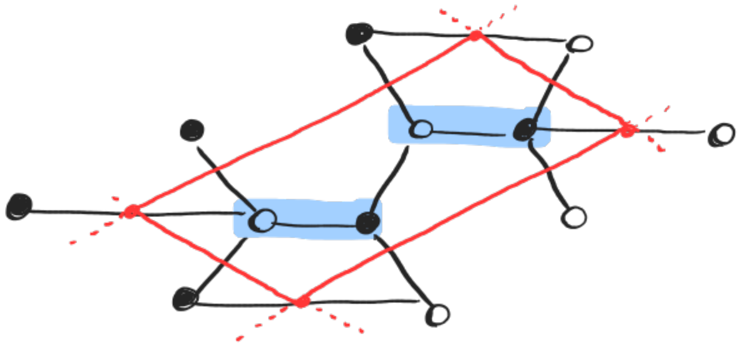
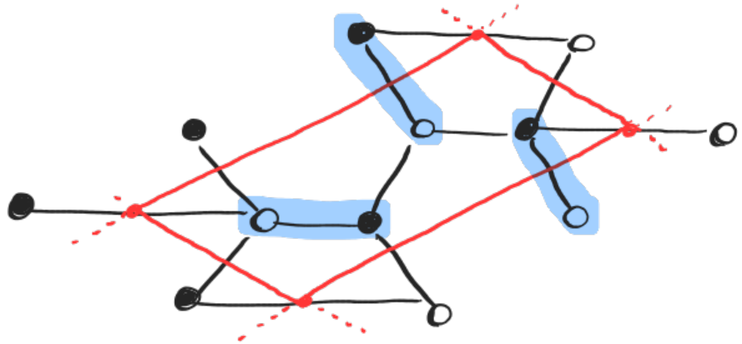
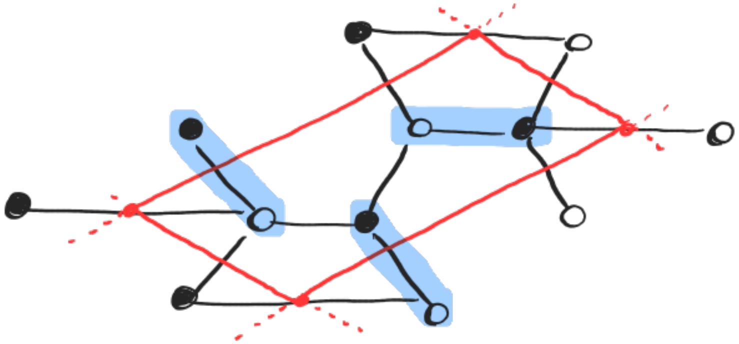
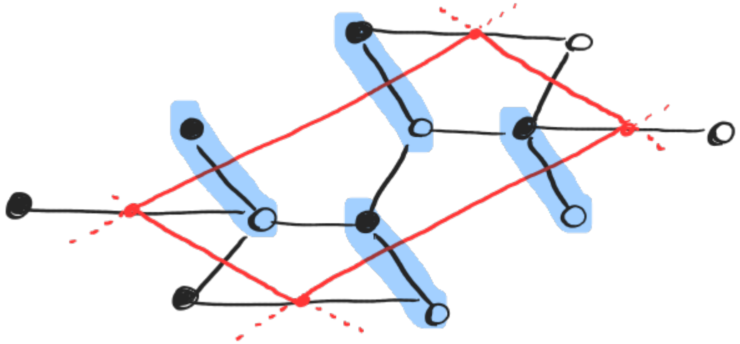
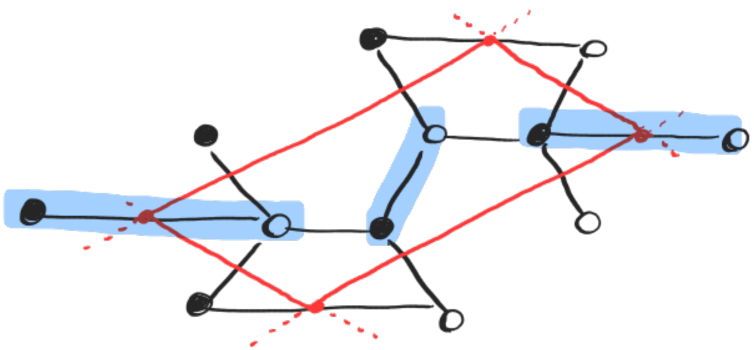
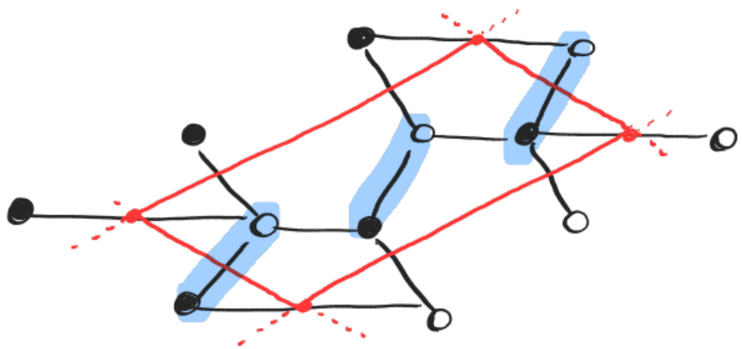
$\Longleftrightarrow$

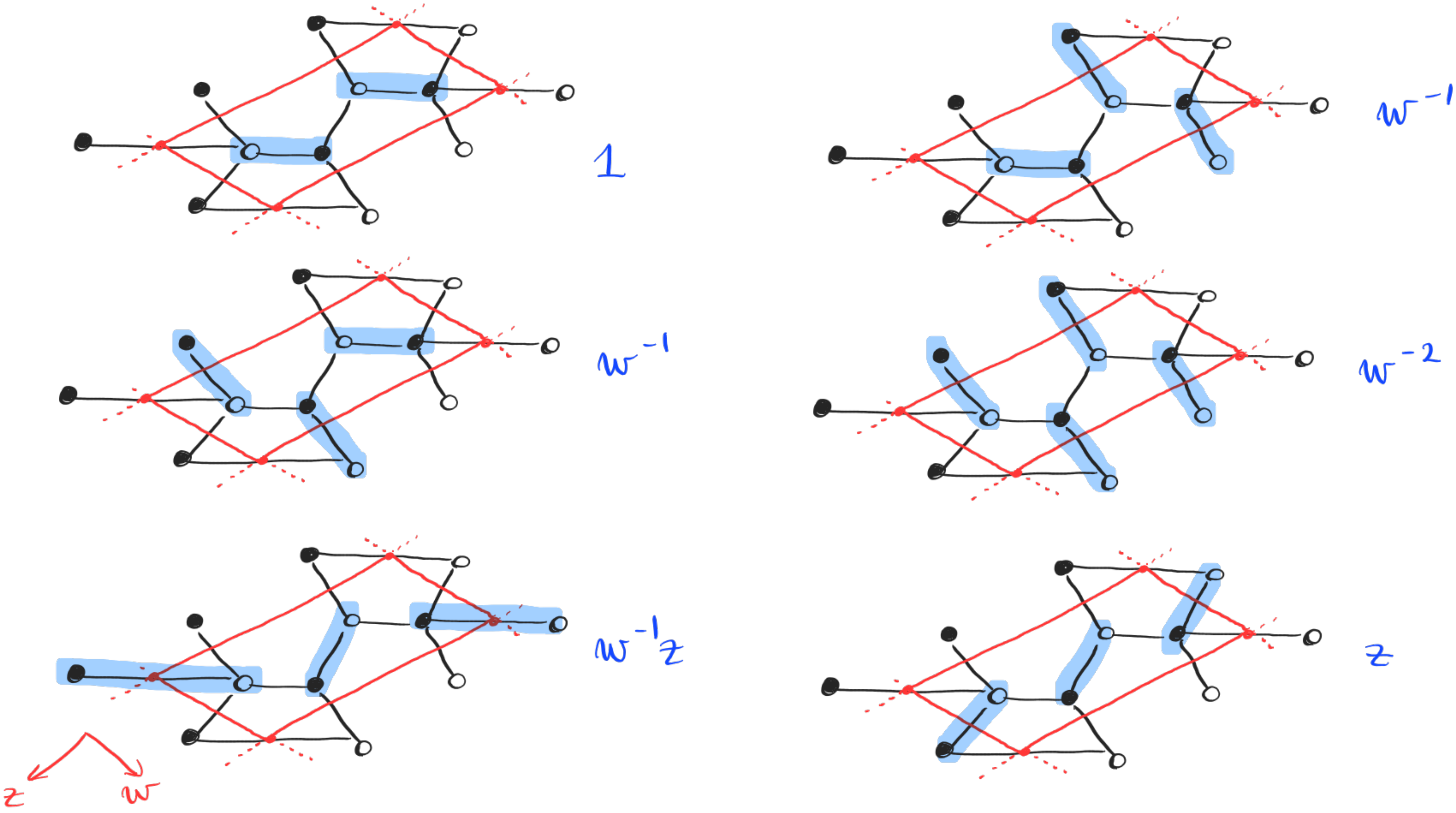


gauge group

bifundamental matter

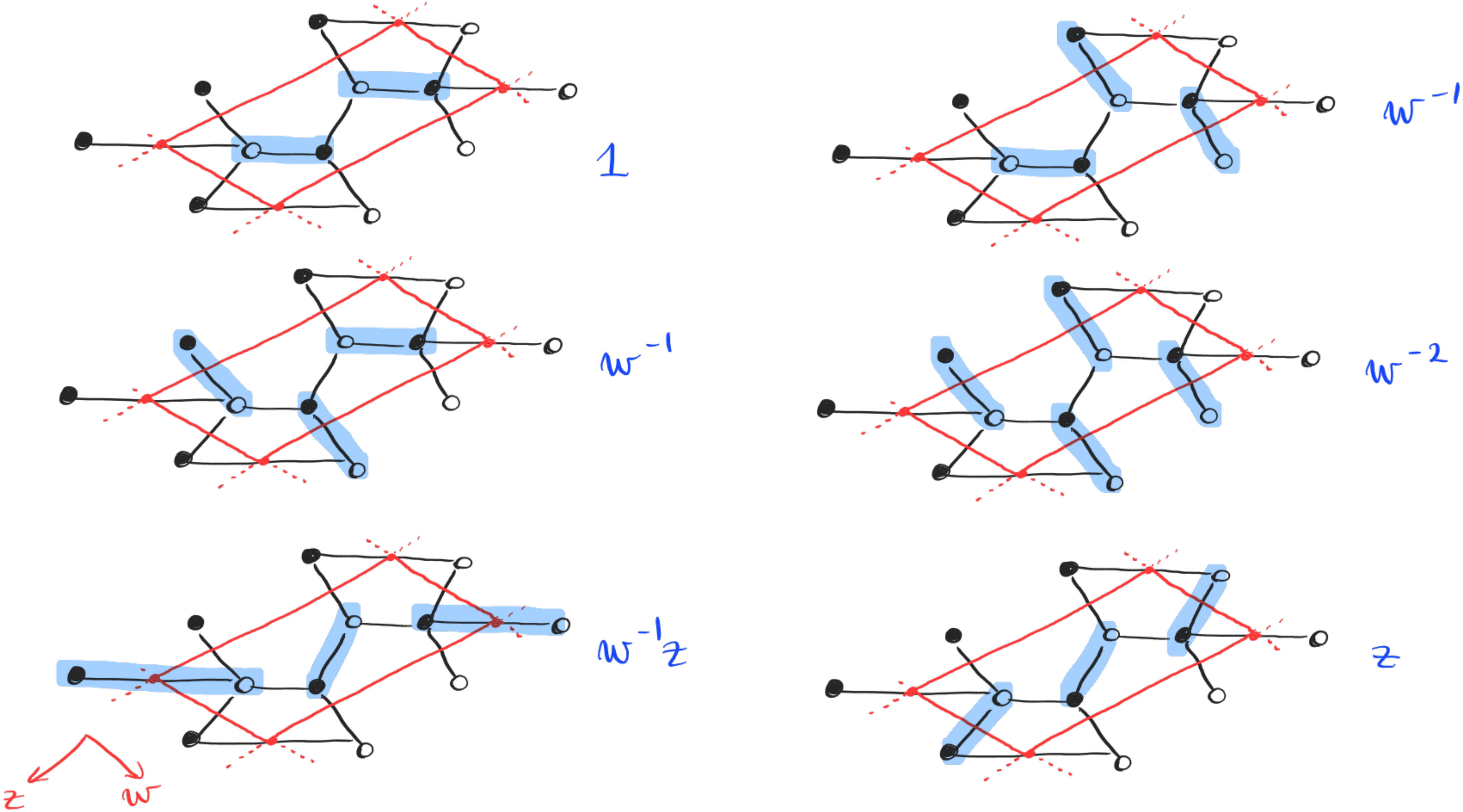
(+ superpotential)

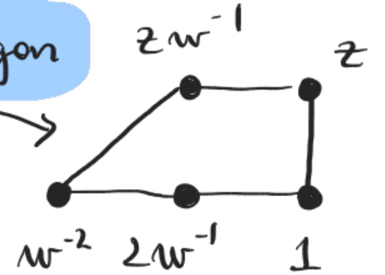


$$K = \begin{array}{cc} \begin{matrix} \circ & \circ \end{matrix} & \\ \begin{pmatrix} 1 + w^{-1} & 1 \\ z + zw^{-1} & 1 + w^{-1} \end{pmatrix} & \begin{matrix} \bullet \\ \bullet \end{matrix} \end{array}$$

$$\det K = 1 + w^{-2} + 2w^{-1} - z - zw^{-1}$$

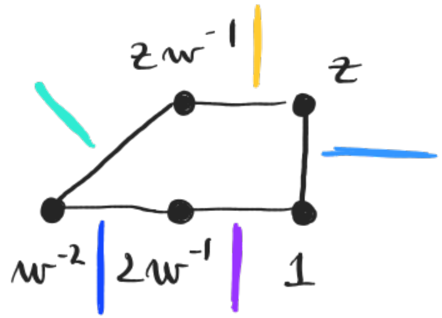


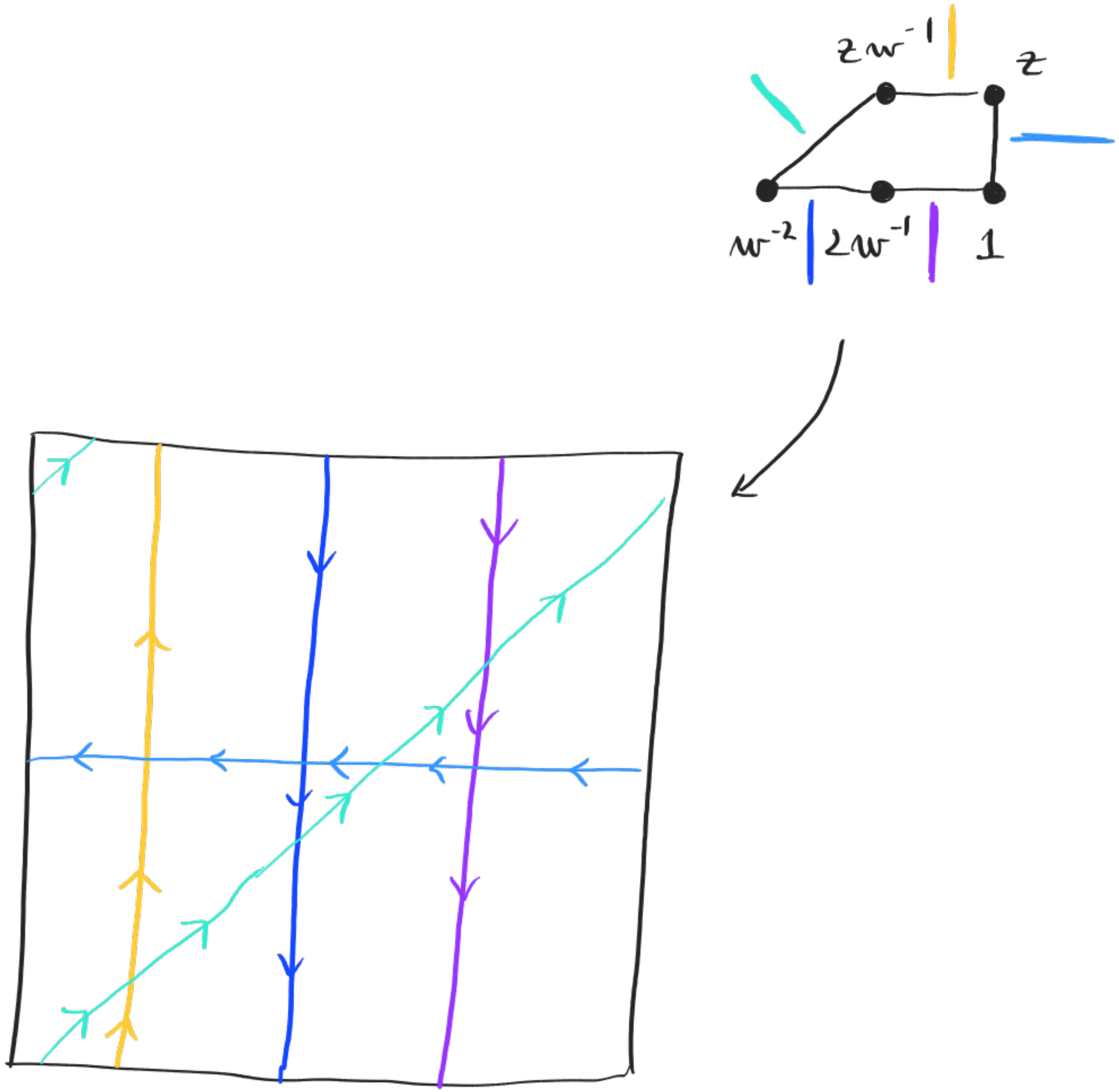
Newton polygon

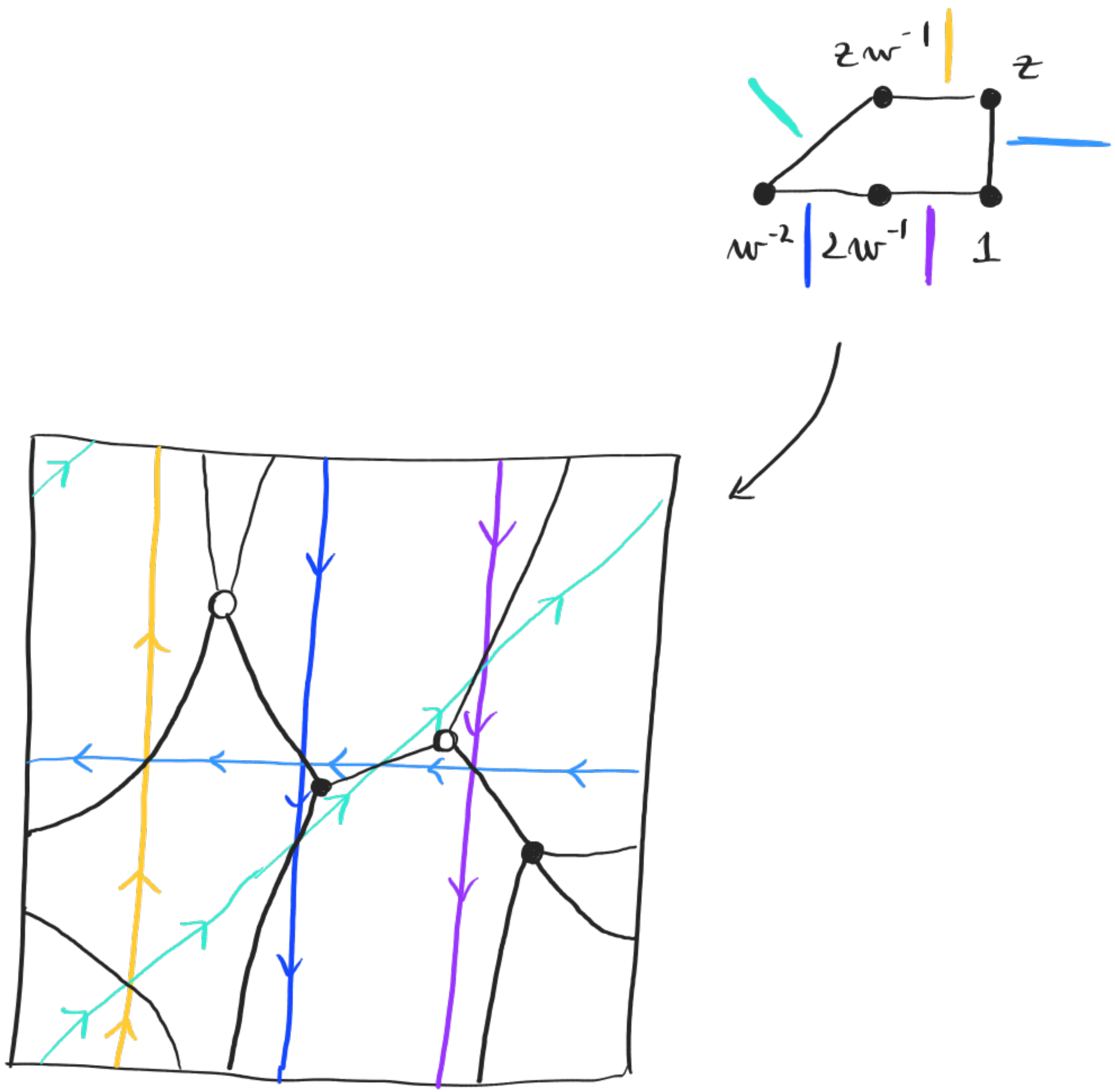


$$K = \begin{array}{cc} \circ & \circ \\ \left(\begin{array}{cc} 1 + w^{-1} & 1 \\ z + zw^{-1} & 1 + w^{-1} \end{array}\right) & \begin{array}{c} \bullet \\ \bullet \end{array} \end{array}$$

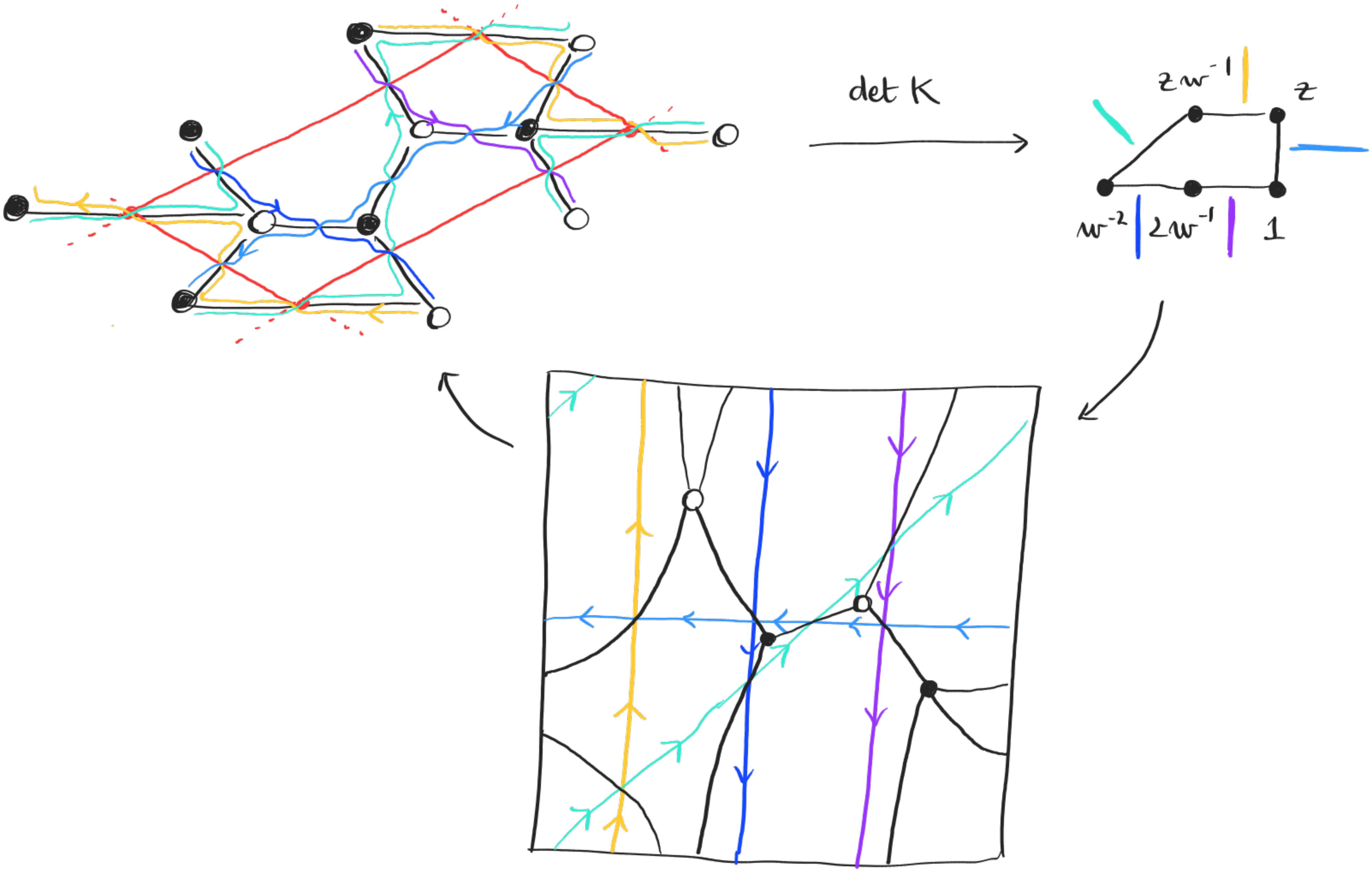
$$\det K = 1 + w^{-2} + 2w^{-1} - z - zw^{-1}$$


zw^{-1}
z
w^{-2}
$2w^{-1}$
1


zw⁻¹
z
w⁻²
2w⁻¹
1



zw^{-1}
z
w^{-2}
$2w^{-1}$
1

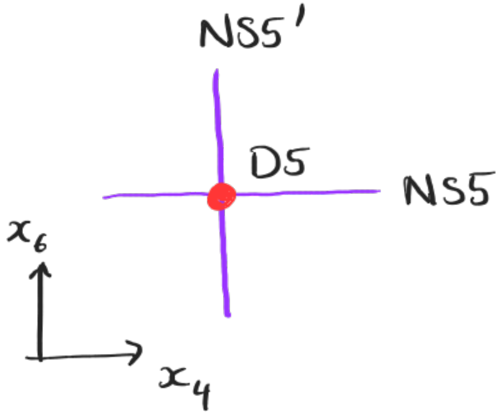

det K
zw^{-1}
z
w^{-2}
$2w^{-1}$
1

Brane systems

Basic setup:

	0	1	2	3	4	5	6	7	8	9
D5	–	–	–	–		–		–		
NS5	–	–	–	–	–	–				
NS5'	–	–	–	–			–	–		

$\Longrightarrow$ on $T^2_{(x_4,x_6)}$



Brane systems

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	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	—	—				
NS5'	—	—	—	—			—	—		

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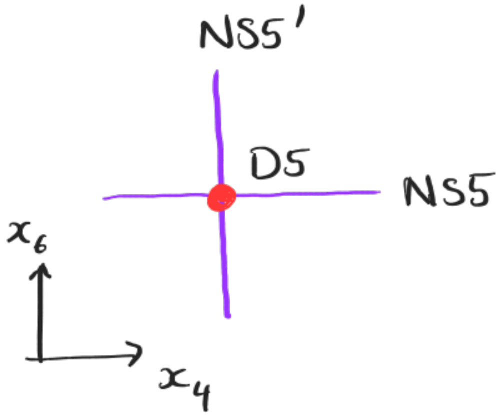


Actually:

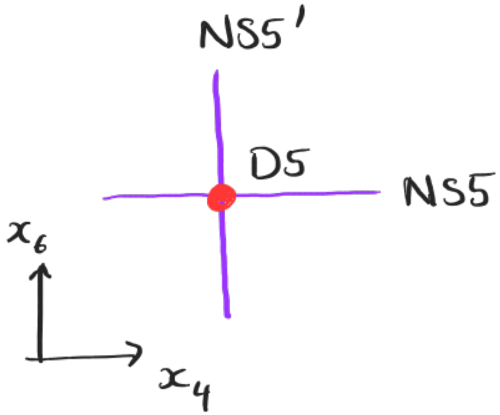
	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	{surface Σ}					

Brane systems

Basic setup:

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	—	—				
NS5'	—	—	—	—			—	—		

$\Longrightarrow$ on $T^2_{(x_4, x_6)}$

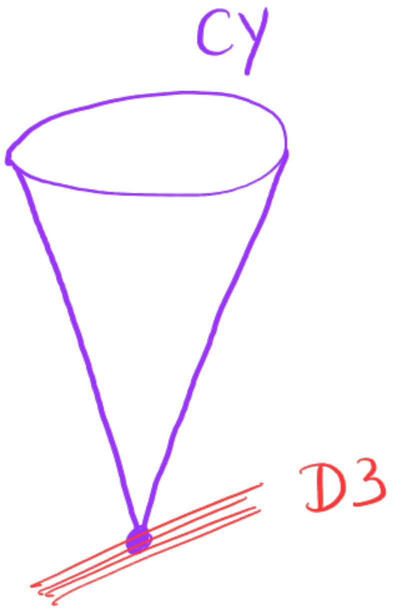


Actually:

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	{surface Σ}					

$\updownarrow$ T-duality along (x_4, x_6)

	0	1	2	3	4	5	6	7	8	9
D3	—	—	—	—						
CY					—	—	—	—	—	—

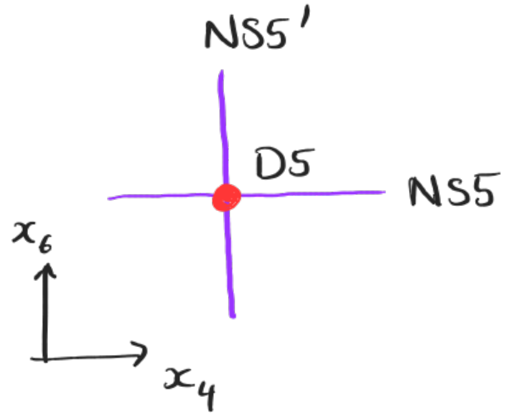


Brane systems

Basic setup:

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	—	—				
NS5'	—	—	—	—			—	—		

$\Longrightarrow$ on $T^2_{(x_4, x_6)}$



Actually:

	0	1	2	3	4	5	6	7	8	9
D5	—	—	—	—		—		—		
NS5	—	—	—	—	{surface Σ}					

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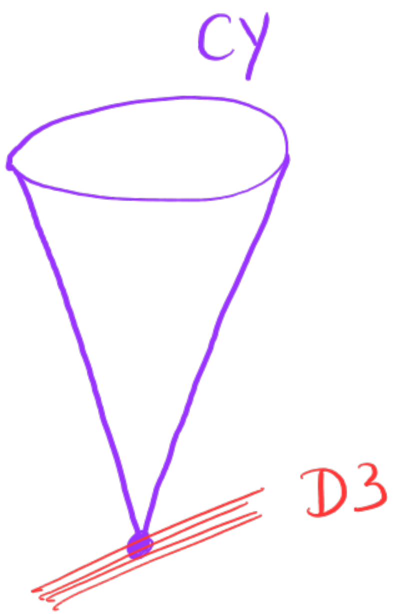
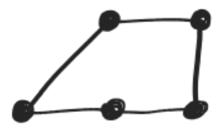


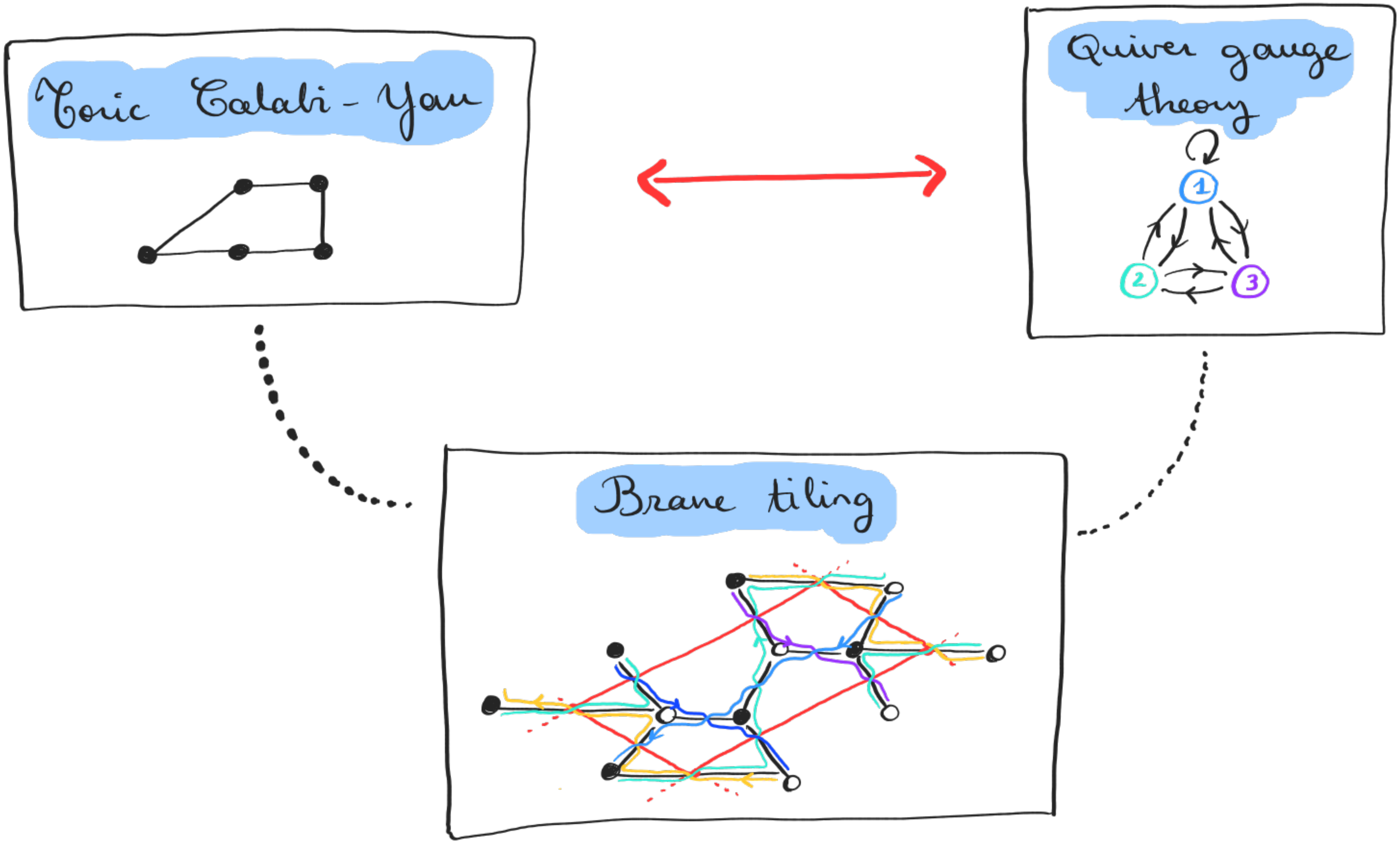
	0	1	2	3	4	5	6	7	8	9
D3	—	—	—	—						
CY					—	—	—	—	—	—

toric Calabi-Yau

$\uparrow$

determined by polygon in $\mathbb{Z}^2$.




Toric Calabi-Yau
Quiver gauge theory
1
2
3
Brane tiling

Thank you for your attention!