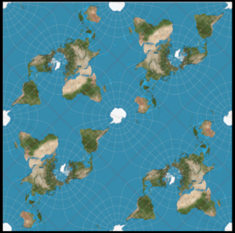
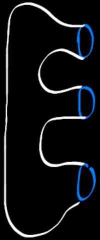


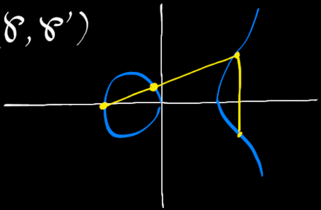
$$\int_0^a \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

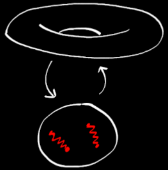
La Géométrie Révélée

Promenade en compagnie des fonctions elliptiques et des surfaces de Riemann



(γ, γ')



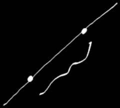


$G = 0,8346268\ldots$

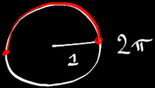
$$\boxed{\frac{r}{2} = d + g - 1}$$

Introduction.

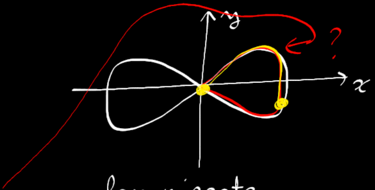
①



droite



cercle



lemniscate

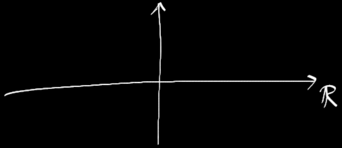
$$(x^2+y^2)^2 = x^2 - y^2$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

$$\int_{\cancel{-1}\;0}^{\cancel{1}\;a} \frac{dx}{\sqrt{1-x^4}}$$

$$= 2{,}62205755429\ldots$$

②



$$\cancel{\delta(x) = \begin{cases} 0 & \text{si } x \in \mathbb{Q} \\ 1 & \text{si } x \notin \mathbb{Q} \end{cases}}$$

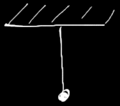
périodique de période $r \in \mathbb{Q}$

fonction lisse : $\longrightarrow$ périodique : $T > 0$

$\longrightarrow$ non périodique

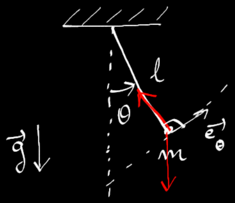
$\mathbb{C}$: 2 périodes ?

③



$$\ddot{\theta} + \omega^2 \underbrace{\sin\theta}_{\approx \theta} = 0 \longrightarrow \ddot{\theta} + \omega^2\theta = 0$$

I) Pendule



$$m\vec{a} = \sum \vec{F}$$

$$\omega^2 = g/\ell$$

$$\ell m \ddot{\theta} = -mg\sin\theta \longrightarrow \boxed{\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0}$$

Supposons les oscillations petites : $\boxed{\sin\theta \sim \theta}$

$$\boxed{\ddot{\theta} + \omega^2\theta = 0}$$

$$\underline{\theta(t) = A\sin(\omega t) + B\cos(\omega t)}$$

$\dot{\theta}$

$$\dot{\theta}\ddot{\theta} + \omega^2\theta\dot{\theta} = 0 = \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 + \frac{\omega^2}{2}\theta^2\right)$$

$$\boxed{\frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\theta^2 = E}$$

On prend $\underline{\omega = 1}$

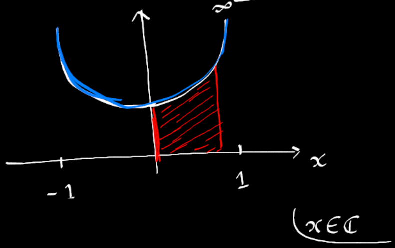
Posons : $x = \dfrac{\theta}{\sqrt{2E}}$

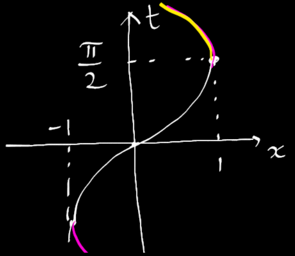
$$1 - x^2 = 1 - \frac{\theta^2}{2E} = \frac{1}{E}\left(E - \frac{\theta^2}{2}\right) = \frac{\dot\theta^2}{2E} = \dot x^2$$

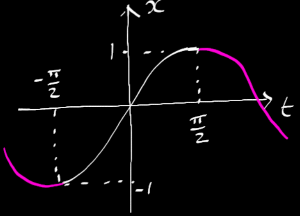
$$\left(\frac{dx}{dt}\right)^2 = 1 - x^2 \qquad \frac{dx}{\sqrt{1-x^2}} = dt$$

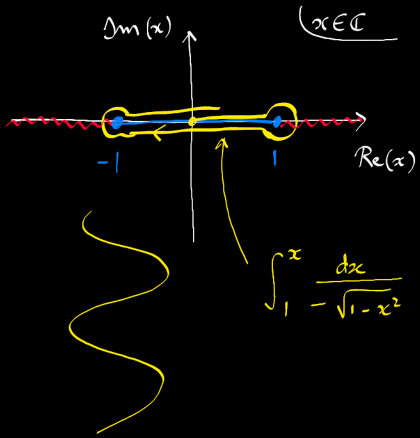
Conclusion : $\boxed{t = \int_0^{x_0} \frac{dx}{\sqrt{1-x^2}}}$ ← arcsin(x_0)

On définit $\boxed{\pi = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}}$









$$\int_1^x \frac{dx}{-\sqrt{1-x^2}}$$

$$\sqrt{z} \longrightarrow e^{i\pi} z = -z$$

$$z \to e^{2i\pi} z$$

Sans approximation $\sin\theta \sim \theta$

(Euler 1738)

$$\ddot{\theta} + \sin\theta = 0$$

$$\times\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{\theta}^2 - \cos\theta\right) = 0$$

$$E = \frac{1}{2}\dot{\theta}^2 + \underbrace{1 - \cos\theta}_{\approx \frac{1}{2}\theta^2}$$

Posons $x = \sqrt{\frac{2}{E}} \sin\frac{\theta}{2}$

$= \frac{1}{k} \sin\frac{\theta}{2}$

$$\boxed{k = \sqrt{\frac{E}{2}}}$$

$$1 - x^2 = 1 - \frac{2}{E}\sin^2\frac{\theta}{2} = \frac{1}{E}\left(\frac{1}{2}\dot{\theta}^2 + \underbrace{1 - \cos\theta - 2\sin^2\frac{\theta}{2}}_{0}\right) = \frac{\dot{\theta}^2}{2E}$$

$$1 - k^2x^2 = 1 - \sin^2\frac{\theta}{2} = \cos^2\frac{\theta}{2}$$

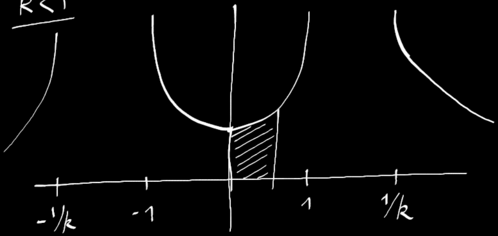
$$(1-x^2)(1-k^2x^2) = \frac{\dot{\theta}^2}{2E}\cos^2\frac{\theta}{2} = \frac{\dot{\theta}^2}{4k^2}\cos^2\frac{\theta}{2} = \dot{x}^2$$

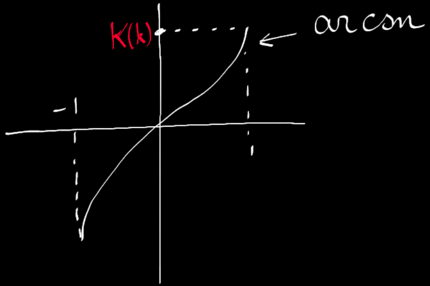
$$\dot{x} = \frac{1}{k}\,\frac{\dot{\theta}}{2}\cos\frac{\theta}{2} \qquad \dot{x}^2 = \frac{\dot{\theta}^2}{4k^2}\cos^2\frac{\theta}{2}$$

$$\dot{x} = \frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)}$$

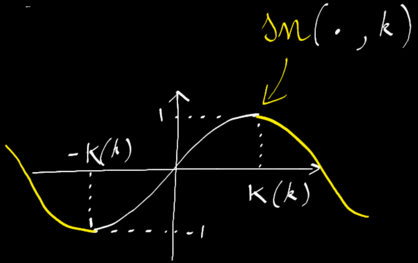
$$\boxed{t = \int_0^{x_0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}}$$

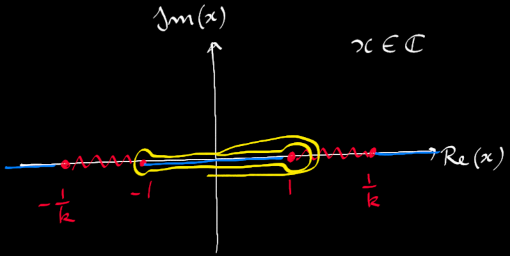
$k<1$





On définit $K(k) = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$





$x \in \mathbb{C}$

$t = \operatorname{arcsn}(x)$

$x = \operatorname{sn}(t)$

$$\sin\frac{\theta}{2} = k\ \mathrm{sn}(t,k)$$

$$\boxed{\theta(t) = 2\arcsin(k\ \mathrm{sn}(t,k))}$$

Petites oscillations : $\theta \simeq 2k\sin(t)$

période $4K(k) \approx 2\pi$

Résumé :

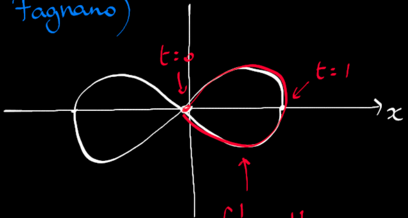
Équation	$\ddot{\theta} + \theta = 0$	$\longleftrightarrow$	$\ddot{\theta} + \sin\theta = 0$		
Fonction multivaluée	$\int \frac{dx}{\sqrt{1-x^2}}$ "arcsin"	$\longleftrightarrow$	$\int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ "arcsn"	$\longleftrightarrow$	$\int \frac{dx}{1-x^2}$ "arctanh"
$\updownarrow$ réciproque	$\updownarrow$		$\updownarrow$		$\updownarrow$
Fonction "périodique"	$\sin(\cdot)$	$\longleftrightarrow$	$\mathrm{sn}(\cdot, k)$ Jacobi : sinus amplitudinus (1830)	$\longleftrightarrow$	$\tanh(\cdot)$
Période	2π	$\longleftrightarrow$	$4K(k)$	$\longleftrightarrow$	∞
k	$k=0$	$\longleftrightarrow$	$0<k<1$	$\longleftrightarrow$	$k=1$

Lemniscate

$$(x^2+y^2)^2 = x^2 - y^2$$

(1718 Fagnano)

On veut calculer la longueur du lemniscate.



$$x = \sqrt{\frac{t^2}{2}(1+t^2)}$$

$$y = \sqrt{\frac{t^2}{2}(1-t^2)}$$

$$\int_0^{t_0} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)}\, dt$$

$$\int_{-1}^{1} \frac{dt}{\sqrt{1-t^4}} = 2K(i)$$

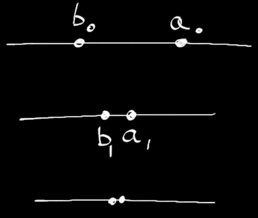
$$\parallel$$

$$2{,}622057554\ldots$$

Ellipses : $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\qquad k^2 = 1 - \dfrac{b^2}{a^2}$ $\qquad (b < a)$

Périmètre de l'ellipse : $4 \displaystyle\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx$ (avec $a=1$ pour simplifier)

Jeu : moyenne arithmétique - géométrique



$a, b \in \mathbb{R}$. Moyenne arithmétique $\frac{a+b}{2}$

$0 < b < a < 1$. Moyenne géométrique $\sqrt{ab}$

$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab} = 2\Big(\underbrace{\frac{a+b}{2} - \sqrt{ab}}_{\geq 0}\Big)$$

$0 < b_0 < a_0 < 1$

$$\begin{cases} b_1 = \sqrt{a_0 b_0} \\ a_1 = \frac{a_0 + b_0}{2} \end{cases} \quad \ldots \quad \begin{cases} b_{n+1} = \sqrt{a_n b_n} \\ a_{n+1} = \frac{a_n + b_n}{2} \end{cases}$$

Question : $\lim_{n \to \infty} a_n = ?$

Gauss (179.)

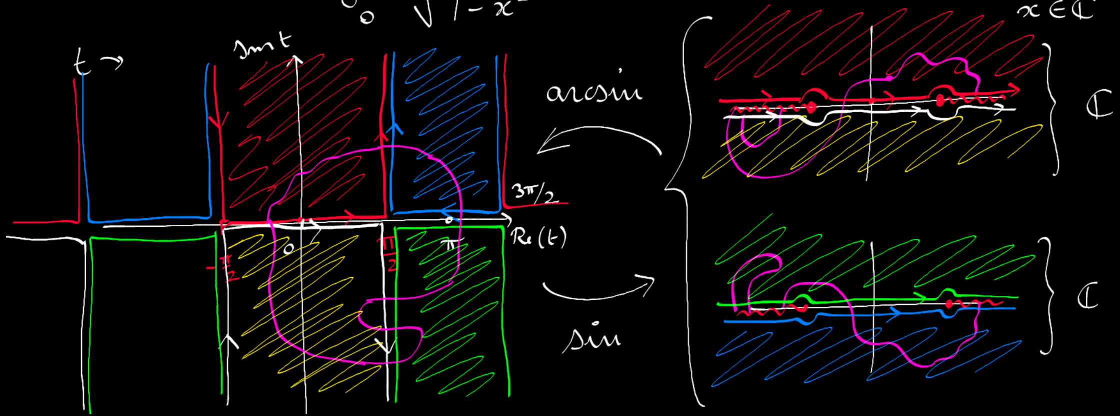
$$\boxed{\begin{array}{l} \lim a_n = \lim b_n = \dfrac{\pi a}{2K(k)} \\ \text{avec } k = 1 - \dfrac{b^2}{a^2} \end{array}}$$

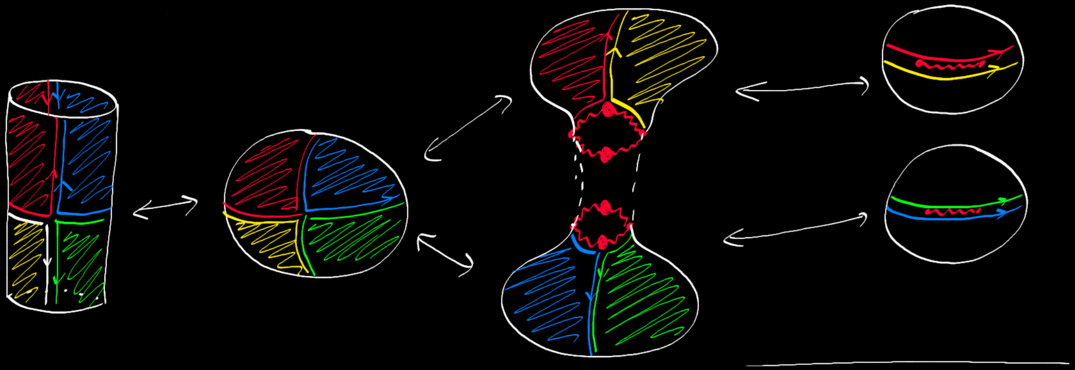
Géométrie cachée

$\sin$?

$$\int_1^{x_0} \frac{dx}{\sqrt{1-x^2}} = i\int_1^{x_0} \frac{dx}{\sqrt{x^2-1}}$$

$$t = \arcsin(x_0) = \int_0^{x_0} \frac{dx}{\sqrt{1-x^2}} \longleftrightarrow \sin(t) = x_0$$

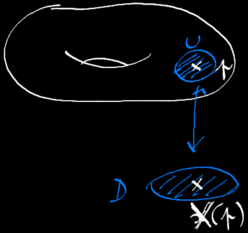




Interlude : Surfaces de Riemann.

- Une surface (2d) est un espace topologique qui peut être couvert par des ouverts U difféomorphes au disque $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2<1\}$,

$$\hookrightarrow X : U \longrightarrow D, \quad p \longmapsto X(p)$$



tels que si $U_1 \cap U_2 \neq \emptyset$ alors si

$$X_1 : U_1 \longrightarrow D$$
$$X_2 : U_2 \longrightarrow D$$

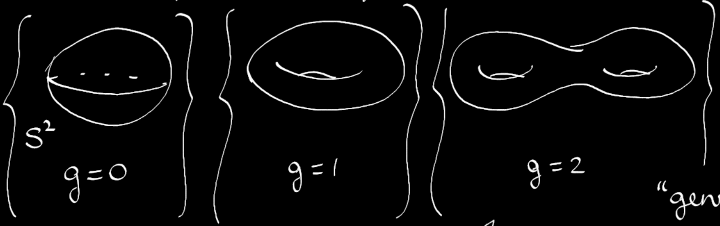
alors $\varphi_{12} : X_1(U_1 \cap U_2) \longrightarrow X_2(U_1 \cap U_2)$

soit C^∞ et bijective.

- Une <u>surface de Riemann</u> est une surface où les applications induites φ_{12} sont <u>holomorphes</u>

(avec $z = x + iy$)

- La <u>liste complète</u> des surfaces de Riemann <u>compactes</u> (orientables) est :



...

"genre" $\boxed{g \in \mathbb{N}}$

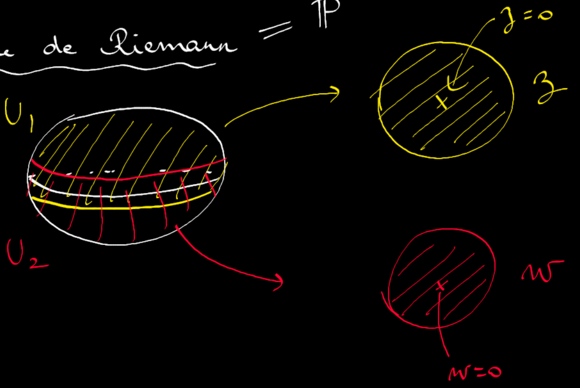
1 élément

∞ éléments

• La raison d'être d'une surface de Riemann est de traiter les fonctions multivaluées.

Exemples.

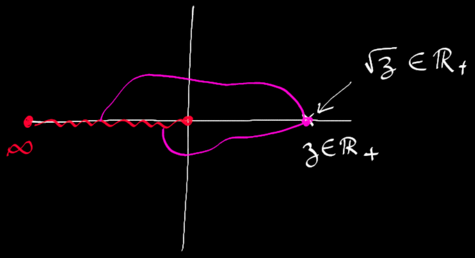
• Sphère de Riemann $= \mathbb{P}^1$



Sur $U_1 \cap U_2 \neq \emptyset$, $z \neq 0$ et $w \neq 0$.

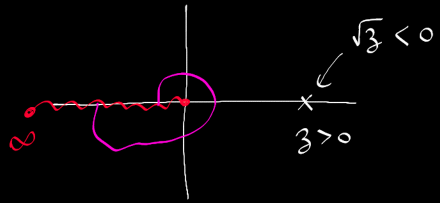
On définit $\varphi_{12}(z) = \boxed{\frac{1}{z} = w}$

- Surface de $z \longmapsto \sqrt{z}$



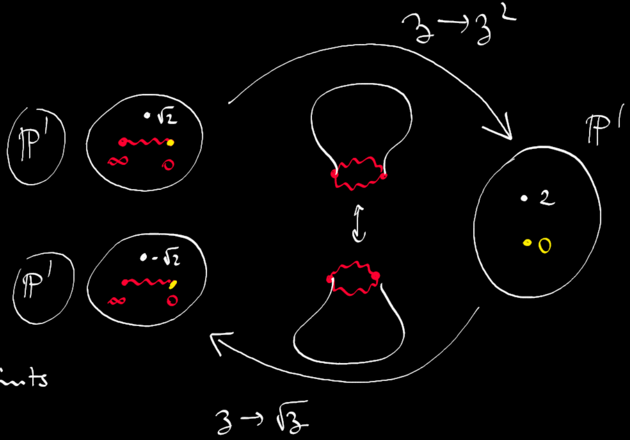
$$\begin{cases} z \longrightarrow e^{2\pi i} z \\ \sqrt{z} \longrightarrow -\sqrt{z} \end{cases}$$

$$\begin{cases} z \longrightarrow e^{4\pi i} z \\ \sqrt{z} \longrightarrow \sqrt{z} \end{cases}$$



$\underbrace{\text{Revêtement}}_{}$ à 2 feuillets

Revêtement ramifié en 2 points



Généralisation (Riemann-Hurwitz)

point où n feuillets se croisent, l'indice est $n-1$

Si on a un revêtement Σ à d feuillets, d'indice de ramification r, de la sphère de Riemann $\mathbb{P}^1$, alors Σ est de genre $\boxed{g = 1 + \frac{r}{2} - d}$

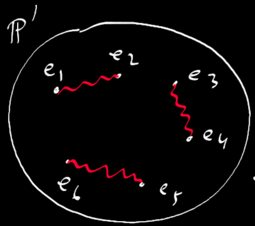
Exemple $\sqrt{z}$.

$d = 2$

$r = 2 = 1 + 1$

$$g = 1 + \frac{2}{2} - 2 = 0$$

Exemple : $f(z) = \sqrt{(z-e_1)(z-e_2)\cdots(z-e_n)}$ $\qquad e_1 \neq e_2 \neq e_3 \cdots$

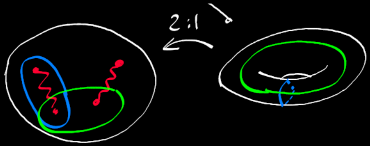


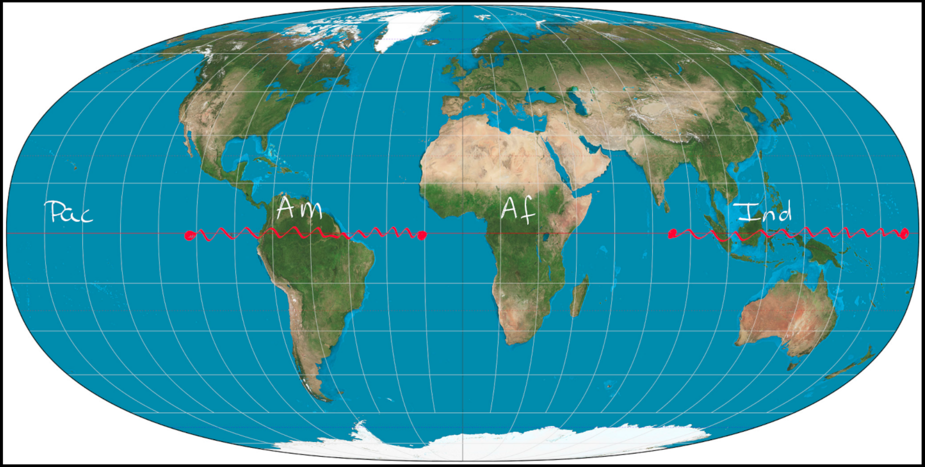
$g=2$

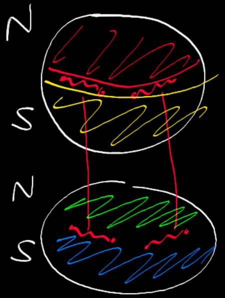
Si n pair : $g = 1 + \frac{n}{2} - 2 = \frac{n}{2} - 1$

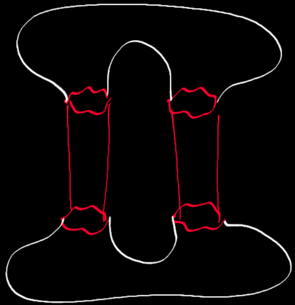
Si n impair : $g = 1 + \frac{n+1}{2} - 2 = \frac{n}{2} - \frac{1}{2}$.

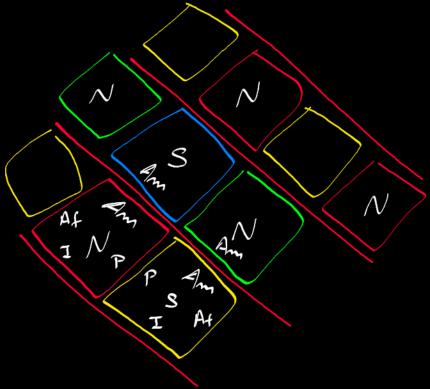
n	1	2	3	4	5	6	7	...
g	0	0	1	1	2	2	3	...

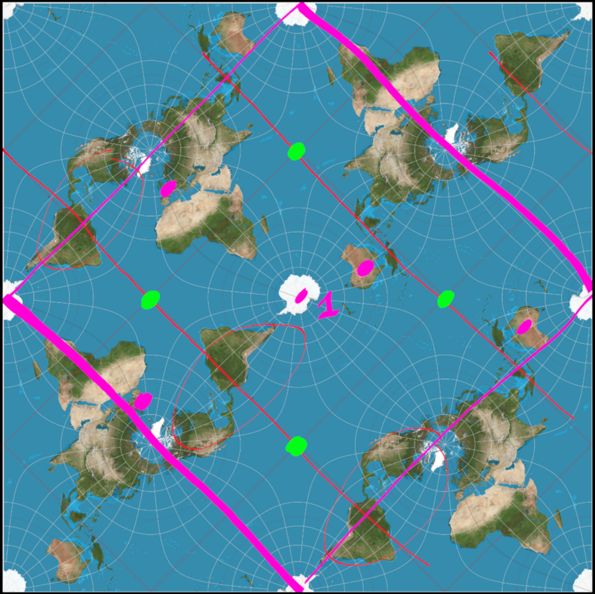
Interlude : Double recouvrement de 




N
S
N
S




N
N
S
Am
N
Af
Am
I
N
P
N
Am
P
Am
S
I
Af

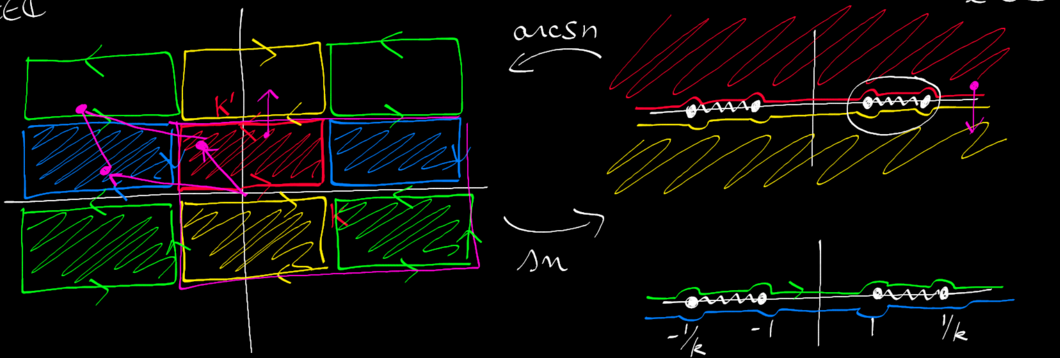

1

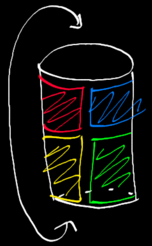
Retour au sn

$$\operatorname{arcsn}(x_0) = \int_0^{x_0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$t \in \mathbb{C}$

$x \in \mathbb{C}$





$$\begin{cases} K(k) = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\[2ex] K'(k) = \displaystyle\int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}} \end{cases}$$

La fonction $sn(\cdot\,,k)$ est <u>doublement</u> périodique

$$\begin{cases} \omega_1 = 4K(k) \\ \omega_2 = 2i\,K'(k) \end{cases}$$

La structure complexe du tore est définie par le rapport $\boxed{\dfrac{\omega_2}{\omega_1} = \dfrac{i\,K'(k)}{2\,K(k)} = \tau}$

<u>Définition</u> : Une fonction méromorphe doublement périodique est appelée <u>fonction elliptique</u>.

La surface de Riemann associée  est appelée <u>courbe elliptique</u>.

<u>Fait</u>. « L'ensemble des fonctions elliptiques de période (ω_1, ω_2) est un corps généré par 1 fonction »

$$= \mathbb{C}(\wp)[\wp']$$

Conclusion.

1736 : Euler et le pendule

1797 : Gauss trouve la double périodicité

$$\int \frac{dx}{\sqrt{1-x^4}}$$

Analyse complexe ⋮

182... : Abel / Jacobi inversent les intégrales

185... : Riemann → géométrie cachée

⋮

2001 : $\boxed{\text{Théorème de Modularité}}$ ⟹ Théorème de Fermat

Trichotomie :

$g = 0$ courbure > 0

GROUPE

$g = 1$ courbure $= 0$ Surface de Riemann $=$ Variété Abélienne

$g \geq 2$ courbure < 0 Variété Abélienne dim > 1

hyperbolique

← géométrie hyperbolique

6 angles droits

Variétés Abéliennes

$$\int \frac{dx}{\sqrt{P(x)}}$$

$$y^2 - P(x) = 0$$

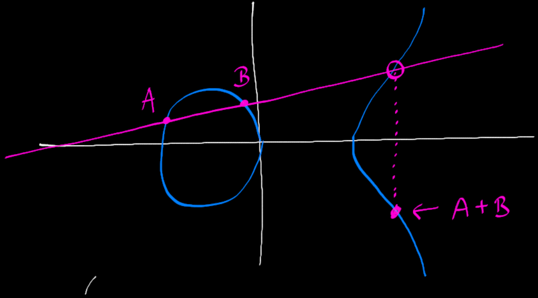
$$\int \frac{dx}{y}$$

$$P(x, y) = 0$$

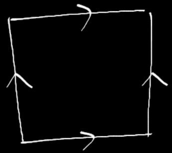
$$\int \frac{dx}{y}$$

Structure de Groupe.

$$\boxed{y^2 = x^3 - x}$$



$\Longleftrightarrow$

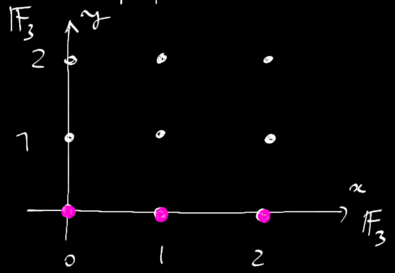


$$\int_0^1 \frac{du}{\sqrt{u-u^3}} = 2{,}62\ldots$$

↳ Cryptographie avec courbes elliptiques

$\mathbb{F}_3 = \{0,1,2\}.$

$y^2 = x^3 - x$



Autre trichotomie.

0	périodes	$1/z^2$	Rationnel
	$\uparrow$	$\uparrow$	$\uparrow$
1	période	$\dfrac{1}{\sin^2(z)}$	Trigonométrique
	$\uparrow$	$\uparrow$	$\uparrow$
2	périodes	$\wp(z)$	Elliptique