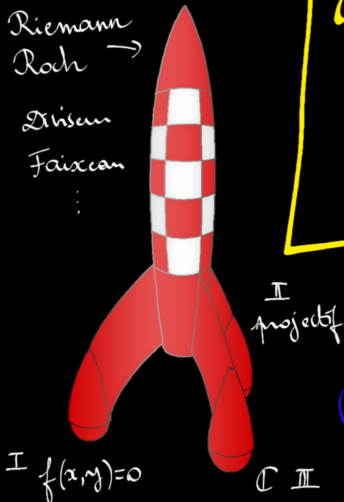
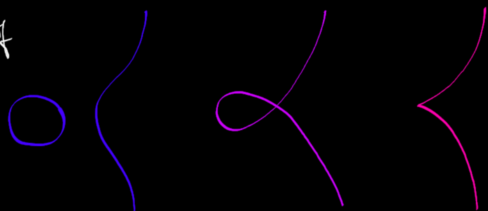
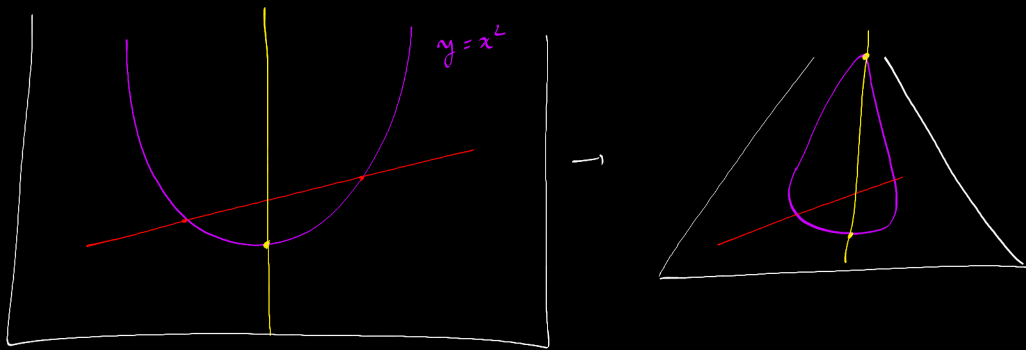
Vers la Géométrie Algébrique





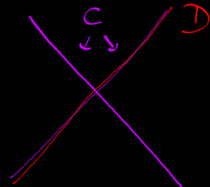


Question : Soit C une courbe algébrique plane $\{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ avec f polynôme de degré d. Soit D une droite.

Combien y a-t-il de points dans $C \cap D$?

Ex : $C : x^2 - y^2 = 0$

$D : x = y$



$C \cap D = D$

infinité de points.

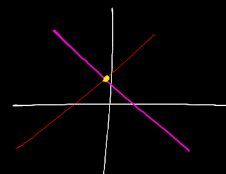
"cas stupide" car $D \subset C$.

↳ exclure ces cas dans la suite

Exemple : Droites

- Courbe de degré 1 : $C = \{ax + by + c = 0\}$

  $D = \{a'x + b'y + c' = 0\}$

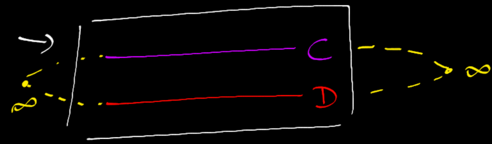


$$\#(C \cap D) = 1$$

- Courbe de degré 2 : ... (voir plus tard).

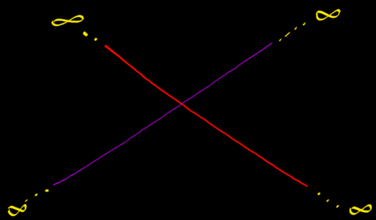
- Droites parallèles : $y = 0 \leftarrow C$

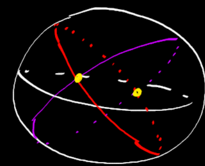
  $y = 1 \leftarrow D$



$\#(C \cap D) = 0$

$\# = 2$??



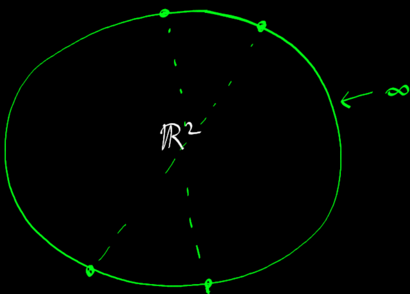


Ajouter 1 point à l'∞ n'est pas bon.

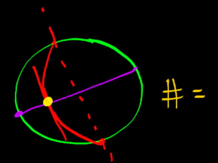
(correspond à $\mathbb{R}^2 \rightsquigarrow S^2$)

Solution : On ajoute un point à l'infini par direction, en identifiant les directions opposées.

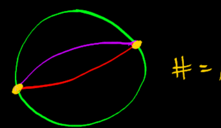
↪ PLAN PROJECTIF REEL $\mathbb{RP}^2$



Droites non parallèles

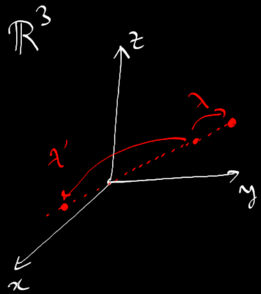


Droites parallèles

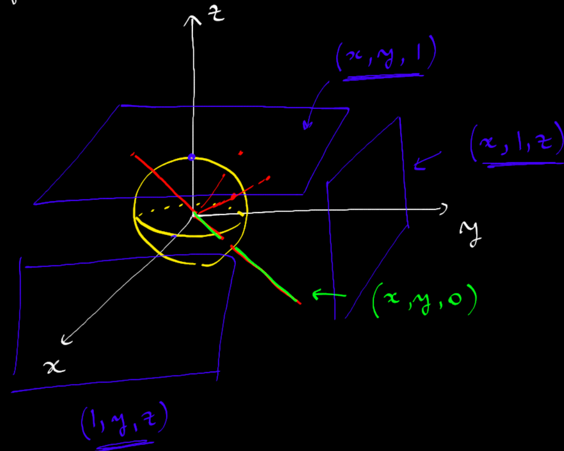


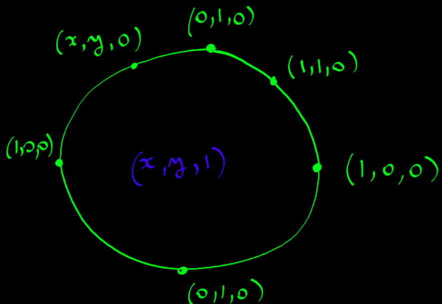
Plan projetif : $\mathbb{RP}^2 = \{(x,y,z)\in\mathbb{R}^3 - \{(0,0,0)\}\} \Big/ \left\{\begin{array}{l}(x,y,z)\sim(\lambda x,\lambda y,\lambda z)\\ \forall\, \lambda\in\mathbb{R}^*\end{array}\right\}$

$= \{\text{droites vectorielles dans } \mathbb{R}^3\}$



On veut faire de la géométrie algébrique dans $\mathbb{RP}^2$.



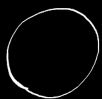


Courbe algébrique : $f(x,y)=0$.

Dans $\mathbb{RP}^2$, problème !

$x^2+y^2=1$ $\rightsquigarrow$ dans $\mathbb{RP}^2$, soit (x,y,z) tel que $\underline{x^2+y^2=1}$.

C



alors $(\lambda x, \lambda y, \lambda z)$ appartient à C

pour tout $\underline{\lambda \in \mathbb{R}^*}$, $\lambda^2 x^2 + \lambda^2 y^2 = \underline{\lambda^2 = 1}$

Cette équation n'a pas de sens dans $\mathbb{RP}^2$.

$$\boxed{\text{Pour faire de la géométrie projective, on ne regarde que les polynômes } \underline{\text{homogènes}}.}$$

$f(x,y)$ polynome $\rightsquigarrow$ $F(x,y,z)$ homogène.

$$x^2+y^2-1 \rightsquigarrow x^2+y^2-z^2$$

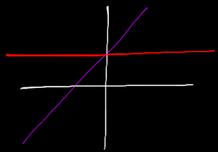
$$y^2-x^3+1 \rightsquigarrow y^2z-x^3+z^3$$

$z=1$

Exemple.

Droite $D = \{ax+by+cz=0\}$

- Droites non parallèles : $\begin{cases} C=\{y=x+z\} \\ D=\{y=z\} \end{cases}$



Points d'intersection :

- $(x,y,1)$: $\begin{cases} y = x+1 \\ y = 1 \end{cases} \iff (x,y,z) = (0,1,1)$

$$\boxed{\text{Un point } (0,1,1)}$$

- $(x,1,z)$: $\begin{cases} 1 = x+z \\ 1 = z \end{cases} \iff (x,y,z) = (0,1,1)$

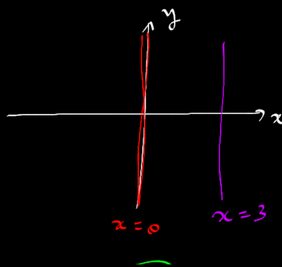
- $(1,y,z)$: $\begin{cases} y = 1+z \\ y = z \end{cases}$ $\times$

- <u>Droites parallèles</u>.

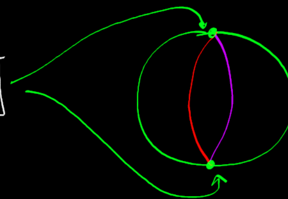
$C : \{x = 0\}$

$D : \{x = 3z\}$

- $(x,y,1)$: $\begin{cases} x = 0 \\ x = 3 \end{cases}$ $\times$



- $(x,y,1)$: $\begin{cases} x=0 \\ x=3 \end{cases}$ $\times$
- $(x,1,z)$: $\begin{cases} x=0 \\ x=3z \end{cases}$ $\boxed{(x,y,z)=(0,1,0)}$

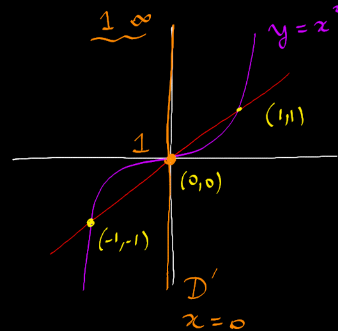


- $(1,y,z)$: $\begin{cases} 1=0 \\ 1=3z \end{cases}$ $\times$

Exemple. C de degré 3. $y = x^3$

D droite $y = x$

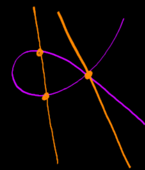
D' droite $x = 0$



Calculons $C \cap D'$ dans $\mathbb{RP}^2$.

$C : \{ yz^2 = x^3 \}$

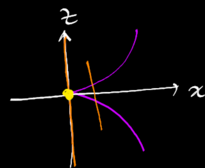
$D' : \{ x = 0 \}$.



$\#(C \cap D') = 3$

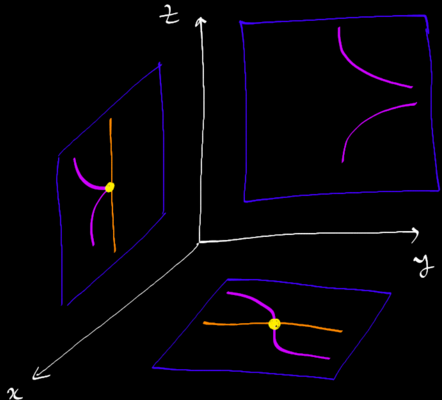
$\ast \ (x,y,1) : \begin{cases} y = x^3 \\ x = 0 \end{cases} \qquad (0,0,1) \longrightarrow 1$

$\ast \ (x,1,z) : \begin{cases} z^2 = x^3 \\ x = 0 \end{cases} \iff \begin{cases} z^2 = 0 \\ x = 0 \end{cases}$



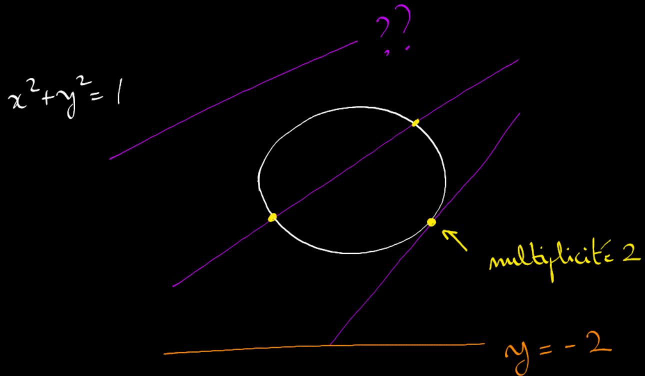
$(0,1,0)$ <u>double</u>. $\longrightarrow 2$

$\ast \ (1,y,z) : \begin{cases} yz^2 = 1 \\ 1 = 0 \end{cases} \quad \times$



$yz^2 = x^3$

$x = 0$

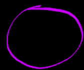


$$\begin{cases} x^2+y^2=1 \\ y=-2 \end{cases}$$

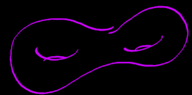
$$x^2 = 1-4 = -3$$

$$x = \pm i\sqrt{3}$$

Il faut aller dans $\mathbb{C}$.



$\mathbb{R}^2 \rightsquigarrow \mathbb{C}^2$



$f(x,y)=0$

$\mathbb{C} \quad \mathbb{C}$

$\mathbb{Q}$

arithmétique

diophante ...

Fermat