Homework : a partial solution

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1 Geodesics in the hyperbolic plane

1. The Euler-Lagrange equations for the usual Lagrangian $\mathcal{L} = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2)$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{y^2}\right) = \frac{\ddot{x}}{y^2} - \frac{2\dot{x}\dot{y}}{y^3} = \frac{\partial \mathcal{L}}{\partial x} = 0 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{y^2}\right) = \frac{\ddot{y}}{y^2} - \frac{2\dot{y}\dot{y}}{y^3} = \frac{\partial \mathcal{L}}{\partial y} = -\frac{\dot{x}^2 + \dot{y}^2}{y^3} \tag{2}$$

from which we deduce

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0 \tag{3}$$

and

$$\ddot{y} + \frac{\dot{x}^2 - \dot{y}^2}{y} = 0\,. \tag{4}$$

Note that from these equation one can read the Christoffel coefficients :

$$-\Gamma^x_{xy} = -\Gamma^y_{yy} = \Gamma^y_{xx} = \frac{1}{y}\,. \tag{5}$$

The other method to find the geodesic equation is to compute these coefficients first from the general formula.

2. First we observe that the Lagrangian is independent of τ (the parameter along the geodesic). This implies that energy is conserved along the geodesic :

$$E = \dot{x}\frac{\partial \mathcal{L}}{\partial \dot{x}} + \dot{y}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \mathcal{L} = \frac{1}{2y^2}(\dot{x}^2 + \dot{y}^2) \tag{6}$$

is conserved. Furthermore, $\mathcal{L}$ doesn't depend on x, so we have a conserved conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{y^2}\,. \tag{7}$$

Another method to obtain conserved quantities is to find the Killing vectors. In the present case, the Killing equation $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ becomes

$$\begin{cases} y\partial_x \xi_x = \xi_y \\ y\partial_y \xi_y = -\xi_y \\ y\partial_x \xi_y + y\partial_y \xi_x = -2\xi_x \end{cases} \tag{8}$$

which we can solve. The second equation implies that

$$\xi_y = \frac{C(x)}{y} \tag{9}$$

Let D be a function such that $D'(x) = C(x)$. The first equation implies that

$$\xi_x = \frac{D(x)}{y^2} + A(y)\,. \tag{10}$$

Now the last equation gives $C'(x) = -2A(y) - yA'(y)$, and since this is valid for all x and y, it is a constant, that we call B. Therefore we find $C(x) = Bx + F$ and $A(y) = -\frac{B}{2} + \frac{G}{y^2}$ where F and G are constants. Finally, the solution of the Killing equation is

$$\begin{aligned} \xi_x &= \frac{Bx^2}{2y^2} - \frac{B}{2} + \frac{Fx}{y^2} + \frac{G}{y^2} && (11) \\ \xi_y &= \frac{Bx}{y} + \frac{F}{y} && (12) \end{aligned}$$

The space of solutions has dimension 3 and is generated by (we put the forms ξ_μ in the contravariant version ξ^μ):

$$\begin{aligned} \xi_1 &= (1,0) && (13) \\ \xi_2 &= (x,y) && (14) \\ \xi_3 &= \left(\frac{x^2}{2} - \frac{y^2}{2}, xy\right) && (15) \end{aligned}$$

One can check the commutators $[\xi_2, \xi_2] = -\xi_1$, $[\xi_2, \xi_3] = \xi_3$,$[\xi_1, \xi_3] = \xi_2$, they form the $su(2)$ algebra.

3. Let's review the basics of curvature in one dimension. The curvature is the rate of change of the tangential vector of unit length. The tangential vector has unit length, hence its derivative with respect to arc length is proportional to the normal vector. The constant of proportionality is the curvature. Therefore we see that the curvature is just the scalar product of the derivative of the tangent vector with the normal vector. This gives the formula $-y'x'' + x'y''$. Now if we take another parametrisation, we obtain the formula in the text.

4. Finally we use the equations of motion and the two conserved quantities to show that the curvature is constant :
$$\rho = \frac{-\dot{x}}{y(\dot{x}^2 + \dot{y}^2)^{1/2}} = \frac{-p}{\sqrt{2E}}\,. \tag{16}$$

5. So the geodesic is a circle (or a straight line if the curvature is 0). The tangent is vertical when y=0 because $\frac{\dot{x}}{y^2}$ is a constant. Therefore, the geodesics are arcs of circles whose center is on the real axis, or straight lines orthogonal to the real axis.

2 Static and weak field metric – Solution

2.1 Gravitational field of a star

1. At first order we have $g^{tt} = -(1 + \frac{2M}{r})$ and $g^{ij} = \delta^{ij}(1 - \frac{2M}{r})$. This gives (XXX include Christoffel).

2. The geodesic equation, using the fact that $p^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ is

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} + \frac{1}{m}\Gamma^i_{\mu\nu}p^\mu p^\nu = 0\,. \tag{17}$$

Using the Christoffel coefficients computed before, we find

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} + XXX = 0 \tag{18}$$

3. The fact that the particle is non-relativistic can be written $|p^i| \ll p^0$. We can also remark that the normalization of the velocity $u^2 = -1$ gives $p^2 = -m^2$ and here $(p^0)^2 = m^2 + O(M/r)$.

4. We keep only terms of order 1 in M/r in the geodesic equation for p^i :

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} + \Gamma^i_{tt}m^2 = 0 \tag{19}$$

This can be written

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = -m\vec{\nabla}\Phi = -\frac{mM}{r^2}\vec{e}_r \tag{20}$$

which is Newton's law for gravitation. In units where $G = 1$ and $c = 1$ the constant M is the mass of the object which creates the gravitational potential.

5. If we go back to standard units, we see that

$$\Phi = -\frac{GM}{c^2 r} \tag{21}$$

When M is the mass of the Sun and r is the radius of the orbit of the Earth around the Sun, this is about 10^{-8}. The approximation is very good, and we don't need the exact Schwarzschild solution to study the movement of the Earth.

2.2 Bending of light

1. The geodesic equation for p^y is

$$\frac{\mathrm{d}p^y}{\mathrm{d}s} + \Gamma^y_{tt}(p^0)^2 + \sum_j \Gamma^y_{jj}(p^j)^2 + 2\sum_{i\neq y}\Gamma^y_{yi}p^y p^i = 0 \tag{22}$$

2. The last term can be neglected because $p^y \ll p^x \sim p^0$. In the third term only $j = x$ contributes, and we find

$$\frac{\mathrm{d}p^y}{\mathrm{d}s} + \Gamma^y_{tt}(p^x)^2 = 0 \tag{23}$$

which is the equation we wanted to get.

3. Integrating this equation gives

$$\int_0^{p^y} \mathrm{d}p' = \int_{-\infty}^{x} \mathrm{d}x' \frac{-2Mbp^x}{(x'^2 + b^2)^{3/2}} \tag{24}$$

and then

$$p^y(x) = -2Mp^x\left(\frac{x}{b\sqrt{b^2+x^2}} + \frac{1}{b}\right) \tag{25}$$

which gives

$$p^y(x = \infty) = \frac{-4Mp^x}{b}\,. \tag{26}$$

4. The deflection angle is $\Delta\varphi = |p^y/p^x| = 4M/R \times R/b$, with
$$\frac{4M}{R} \sim 1.75'' \tag{27}$$
Therefore the prediction is
$$\Delta\varphi = 1.75''\frac{R}{b}\,. \tag{28}$$

2.3 A derivation

2.3.1 Linearised gravity

1. The Christoffel coefficients are given by
$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \tag{29}$$
(the terms in the bracket are first order in h, so we can take the term in front to zeroth order). We see that the Christoffel are first order, and therefore to compute the Riemann tensor we take only the terms that are first order in the Christoffel symbols. This gives
$$R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma} = \partial_\mu\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\mu} \tag{30}$$
We are in the vacuum, so the Einstein equations are just $R_{\nu\sigma} = 0$. We obtain the equation in the text. In this equation the D'Alembertian is $\eta^{\mu\nu}\partial_\mu\partial_\nu$ and the vector V_μ is
$$V_\mu = \eta^{\nu\rho}\partial_\nu h_{\mu\rho} - \frac{1}{2}\eta^{\nu\rho}\partial_\mu h_{\nu\rho}\,. \tag{31}$$

2. We have $h_{tt} = -2\Phi$ and $h_{ij} = -2\Phi\delta_{ij}$. It is easy to check that this implies $V_\mu = 0$, so the equation reduces to $\Delta\Phi(\vec{x}) = 0$. This is the vacuum equation of Newtonian gravity !

3. The transformation law has been derived several times in the lectures and the TD. We find $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$.

4. We have 4 degrees of freedom that we can use so choose a coordinate system such that $V_\mu = 0$.

5. The basic quantities in electromagnetism are the vector and scalar potentials, they correspond to $h_{\mu\nu}$ here. The electromagnetic fields correspond to the curvature ; a "gauge" transformation can be applied on the basic quantities without changing the fields. A gauge condition can be imposed in both cases, which simplifies the field equations (Maxwell equations and $\Box h_{\mu\nu} = 0$).

2.3.2 Deriving the static weak field metric

1. Assume the general metric perturbations are independent of t. Then h_{ti} has to vanish if the metric is to be unchanged under $t \to -t$.

2. We can use the residual symmetry given by $\xi^\mu = (0, \vec{\xi})$. There are 3 independent functions there, which preserve the components h_{it}. They can be used to bring the three non-diagonal components h_{ij} to zero.

3. The Lorentz gauge condition $V_x = 0$ becomes $\partial_x(-h^x_x + h^y_y + h^z_z + h^t_t) = 0$. Therefore the quantity inside the bracket is constant, and it must vanish at infinity, so it is 0. We have similar conditions coming from $V_y = V_z = 0$. These three equations can be satisfied only if $h^x_x = h^y_y = h^z_z = -h^t_t$.

3 Are wormholes physical? – Solution

1. The Christoffel coefficients are (Γ[i,j,k] stands for Γ^i_{jk}, with $1 = r$, $2 = \theta$, $3 = \phi$ and $4 = t$) :

$$
\begin{array}{cc}
\Gamma[1,\,2,\,2] & -r \\
\Gamma[1,\,3,\,3] & -r\sin^2(\theta) \\
\Gamma[2,\,2,\,1] & \frac{r}{b^2+r^2} \\
\Gamma[2,\,3,\,3] & -\sin(\theta)\cos(\theta) \\
\Gamma[3,\,3,\,1] & \frac{r}{b^2+r^2} \\
\Gamma[3,\,3,\,2] & \cot(\theta)
\end{array}
\tag{32}
$$

The Riemann tensor is (R[i,j,k,l] stands for $R^i{}_{jkl}$) :

$$
\begin{array}{cc}
\mathrm{R}[1,\,2,\,2,\,1] & \frac{b^2}{b^2+r^2} \\
\mathrm{R}[1,\,3,\,3,\,1] & \frac{b^2\sin^2(\theta)}{b^2+r^2} \\
\mathrm{R}[2,\,1,\,2,\,1] & -\frac{b^2}{(b^2+r^2)^2} \\
\mathrm{R}[2,\,3,\,3,\,2] & -\frac{b^2\sin^2(\theta)}{b^2+r^2} \\
\mathrm{R}[3,\,1,\,3,\,1] & -\frac{b^2}{(b^2+r^2)^2} \\
\mathrm{R}[3,\,2,\,3,\,2] & \frac{b^2}{b^2+r^2}
\end{array}
\tag{33}
$$

We find that the only non-vanishing component of the Ricci tensor is $R_{rr} = -\frac{2b^2}{(b^2+r^2)^2}$, and this is also the scalar curvature. The Einstein tensor is given by

$$
\begin{array}{cc}
\mathrm{G}[1,\,1] & -\frac{b^2}{(b^2+r^2)^2} \\
\mathrm{G}[2,\,2] & \frac{b^2}{b^2+r^2} \\
\mathrm{G}[3,\,3] & \frac{b^2\sin^2(\theta)}{b^2+r^2} \\
\mathrm{G}[4,\,4] & -\frac{b^2}{(b^2+r^2)^2}
\end{array}
\tag{34}
$$

2. We have found

$$T^{tt} = \frac{1}{8\pi G}\frac{b^2}{(b^2+r^2)^2} \tag{35}$$

which means that the energy density as measured by a stationary observer is negative. SInce all realistic matter described classically has positive energy density, this means it is impossible to construct a wormhole with the metric we consider with classical means.