Symmetries in quantum theory : Old and New

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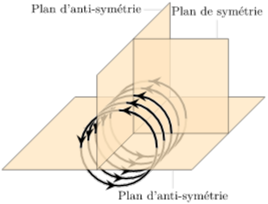
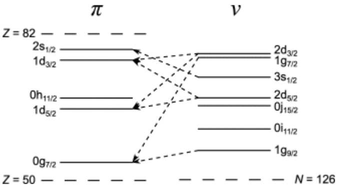
Why Symmetries?

Uses in classical and quantum physics:

- In classical electromagnetism:

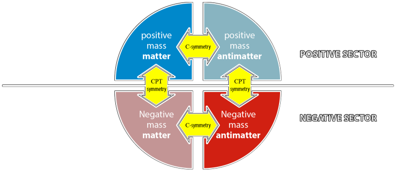
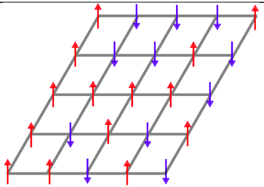
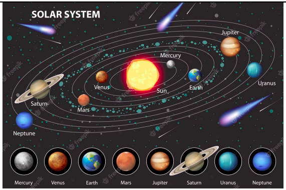
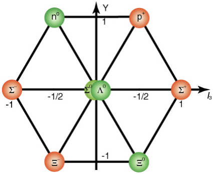


- In Particle physics:



Why Symmetries ?

Physics is hard. Need to simplify it to make computations.

	Space-time	Internal
Discrete		
Continuous		

Plan

Today:

1. Symmetries in Classical and Quantum physics
2. Generalized Symmetries
3. Applications and further extensions

References:

- Gaiotto-Kapustin-Seiberg-Willet, Generalized Global Symmetries, 2014.
- Aharony-Seiberg-Tachikawa, Reading between the lines of four-dimensional gauge theories, 2013.
- See the Snowmass White Paper 2205.09545 for a complete list of references.

Conservation laws

In electromagnetism, electric charge contained in a volume V:

$$Q = \iiint_V \rho(x) \, \mathrm{d}^3 x$$

with ρ the charge density.

Conservation equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho + \mathrm{div}\,\vec{j} = 0$$

with $\vec{j}$ the electric current. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = -\iiint_V \mathrm{div}\,\vec{j}\,\mathrm{d}^3 V = -\iint_{\partial V} \vec{j} \cdot \mathrm{d}^2 \vec{S}$$

Noether's Theorem in Classical Mechanics

This is a general motto:

Noether's Theorem :
Symmetries imply conservation laws.

Today, add another motto:

Symmetries are topological operators.

Noether's Theorem in Classical Mechanics

Let (M, L) be a Lagrangian system, i.e.

- M is a smooth finite-dimensional manifold, the configuration space.
- $L : TM \to \mathbb{R}$ the Lagrangian function.

Among all paths $\gamma : [t_0, t_1] \to M$ with $\gamma(t_i) = q_i$, $i = 0, 1$, those which describe physical motions are the critical points of the action

$$S(\gamma) = \int_{t_0}^{t_1} L(\gamma'(t))\,\mathrm{d}t$$

In standard coordinates on TM, Euler-Lagrange:

$$\frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))\right)$$

System of second order ODE, very difficult in general.

Noether's Theorem in Classical Mechanics

A symmetry of the system is a diffeomorphism $g : M \to M$ such that $L(g_\star v) = L(v)$ for all $v \in TM$.
If there is a 1-parameter group $\{g_s\}_{s\in\mathbb{R}}$ of symmetries, then there is an integral of motion

$$Q(q,\dot{q}) = \sum_\mu \frac{\partial L}{\partial \dot{q}^\mu}(q,\dot{q}) \left(\frac{\mathrm{d}g_s^\mu(q)}{\mathrm{d}s}\Big|_{s=0} \right)$$

i.e. for all extremals γ of the action functional,

$$\frac{\mathrm{d}}{\mathrm{d}t} Q(\gamma'(t)) = 0$$

Examples:

- Invariance under time translations $\Rightarrow$ Conservation of energy
- Invariance under translations $\Rightarrow$ Conservation of momentum
- Invariance under rotations $\Rightarrow$ Conservation of angular momentum

Noether's Theorem in Classical Field Theory

Classical Mechanics:

$$\frac{\mathrm{d}}{\mathrm{d}t} Q = 0 \qquad \text{with} \qquad Q = \frac{\partial L}{\partial \dot{\mathsf{q}}} \delta_s \mathsf{q}$$

Classical Field Theory:

$$\partial_\mu J^\mu = 0 \qquad \text{with} \qquad J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_s \phi$$

Conserved current implies existence of a conserved charge

$$Q = \int_{\text{space}} J^0 \, \mathrm{d}^{d-1} x = \int_{\text{space}} \star J$$

provided the fields vanish at infinity fast enough.

Symmetries in Quantum Mechanics

In Quantum Mechanics, a symmetry is implemented by unitary and linear (or antiunitary and antilinear) operators $U : \mathcal{H} \to \mathcal{H}$.

For a 1-parameter group,

$$U_\alpha = \exp\left(i\alpha Q\right)$$

where

$$[Q, H] = 0.$$

In QFT, for a 1-parameter group, current J satisfying $\mathrm{d} \star J = 0$,

$$U_\alpha(M^{d-1}) = \exp(i\alpha Q(M^{d-1}))$$

with

$$Q(M^{d-1}) = \int_{M^{d-1}} \star J.$$

Unifying formalism for discrete and continuous

Consider a Euclidean QFT in spacetime dimension d.

Definition. The theory has a global symmetry group G iff there is a topological defect $U_g(M^{d-1})$ for each $g \in G$ and each $d-1$ dimensional submanifold, satisfying the group law:

$$U_g(M^{d-1})U_{g'}(M^{d-1}) = U_{gg'}(M^{d-1})$$

The charged objects are local operators.

Unifying formalism for discrete and continuous

Important properties:

- The symmetry is *invertible* (consequence of the group law).

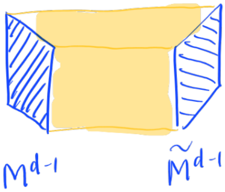
Unifying formalism for discrete and continuous

Important properties:

- The symmetry is *invertible* (consequence of the group law).
- $U_g(M^{d-1})$ is topological. For continuous symmetries, this is Noether's theorem:

$$Q(M^{d-1}) - Q(\tilde{M}^{d-1}) = \int_{\partial X^d} \star J = \int_{X^d} \mathrm{d} \star J = 0$$

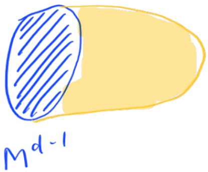
with $M^{d-1} - \tilde{M}^{d-1} = \partial X^d$.



Unifying formalism for discrete and continuous

- In particular if $M^{d-1} = \partial X^d$ then

$$Q(M^{d-1}) = 0.$$



Unifying formalism for discrete and continuous

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Unifying formalism for discrete and continuous

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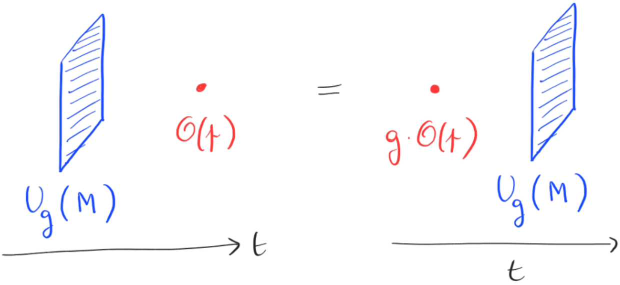
- Action of $U_g(M^{d-1})$ on local operators:

Unifying formalism for discrete and continuous

- Equal time commutation relation:

$$U_g(M^{d-1})\mathcal{O}_i(p) = R_i^j(g)\mathcal{O}_j(p)U_g(M^{d-1}) \qquad \text{at equal time.}$$



Higher form symmetries

Problem to interpret electromagnetism in this relativistic framework:

$$Q_{\text{elec}} = \int_{M^3} \mathrm{d} \star_3 j_{\text{elec}} = \int_{(\partial M)^2} \star_3 j_{\text{elec}}$$

Mismatch by 1 dimension !

Need to extend the formalism...

Higher form symmetries

Consider a Euclidean QFT in spacetime dimension d.

Definition. The theory has a q-form global symmetry group $G^{(q)}$ iff there is a topological defect $U_g(M^{d-q-1})$ for each $g \in G^{(q)}$ and each $d-q-1$ dimensional submanifold, satisfying the group law:

$$U_g(M^{d-q-1})U_{g'}(M^{d-q-1}) = U_{gg'}(M^{d-q-1})$$

The charged objects are q-dimensional operators.

Higher form symmetries

Important properties:

- For continuous symmetries, we have a $q+1$-form Noether current:

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Higher form symmetries

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- For continuous symmetries, we have a $q+1$-form Noether current:

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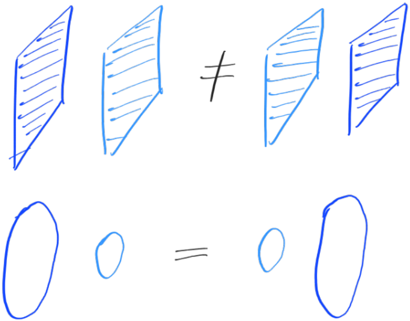
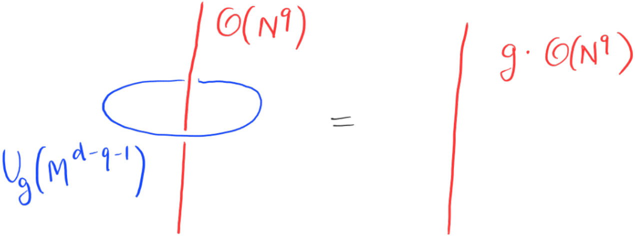
- The topological property implies that for $q > 0$, $G^{(q)}$ is Abelian.



- Action on q-dimensional operators:

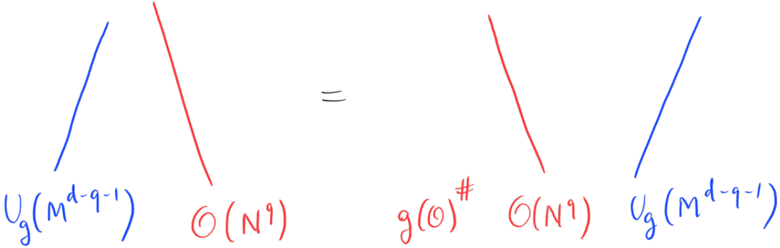


Higher form symmetries

- Equal time commutation relation:

$$U_g(M^{d-q-1})\mathcal{O}(N^q) = g(\mathcal{O})^{\#}\mathcal{O}(N^q)U_g(M^{d-q-1}) \qquad \text{at equal time}$$

with $\#$ the linking number between M^{d-q-1} and N^q.



Example: Maxwell theory

In pure Maxwell theory, two 2-form conserved currents:

- $j_{\mathrm{magn}} = \star F$. Conservation $\mathrm{d}F = 0$ is the absence of magnetic charge (spatial form $\mathrm{div}\vec{B} = 0$). Symmetry operator is

$$U_\alpha(M^2) = \exp\left(i\alpha \int_{M^2} F\right).$$

  On a 2-sphere, $\int_{S^2} F$ is the magnetic flux $\iint_S^2 \vec{B} \cdot \mathrm{d}\vec{S}$! The charged objects are the 't Hooft lines (world-lines of magnetic monopoles).

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- $j_{\mathrm{elec}} = F$. The conservation $\mathrm{d} \star F = 0$ is the absence of electric charge. It is broken in QED. Symmetry operator is

$$U_\alpha(M^2) = \exp\left(i\alpha \int_{M^2} \star F\right) .$$

  On a 2-sphere, $\int_{S^2} F$ is the magnetic flux $\iint_S^2 \vec{E} \cdot \mathrm{d}\vec{S}$! The charged objects are the Wilson lines (also world-lines of electric charges).

Conclusion : pure Maxwell has $U(1)^{(1)}_{\mathrm{elec}} \times U(1)^{(1)}_{\mathrm{magn}}$ symmetry.

Reading between the lines of gauge theories

Standard Model based on Yang-Mills theory. What is the gauge group?

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- We consider only topologically trivial spacetime geometries. But then we can't compute indices, sphere partition functions, finite temperature, etc.

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Need to be more precise...

Reading between the lines of gauge theories

Consider 4d gauge theory with gauge algebra $\mathfrak{g}$. Gauge group :

$$G = \tilde{G}/H$$

with $\tilde{G}$ simply connected, H subgroup of the center Z.
Line operators:

- Wilson lines are labeled by representations of G.

$$\text{Wilson Lines} \quad \leftrightarrow \quad \Lambda_{\text{weight}}(G)/W$$

- 't Hooft lines are labeled by representations of $G^\vee$.

$$\text{'t Hooft Lines} \quad \leftrightarrow \quad \Lambda_{\text{weight}}(G^\vee)/W$$

Also dyonic lines are present:

$$\text{Dyonic Lines} \quad \leftrightarrow \quad (\Lambda_{\text{weight}}(G) \times \Lambda_{\text{weight}}(G^\vee))/W$$

Reading between the lines of gauge theories

To study the spectrum of lines, it is enough to restrict to

$$\Lambda_{\mathrm{weight}}(\mathfrak{g})/\Lambda_{\mathrm{root}}(\mathfrak{g}) = Z$$

For $\mathfrak{g} = \mathfrak{su}(N)$, $Z = \mathbb{Z}_N$. Mutual locality imposes that for any pair of lines of charges $(m, e), (m', e') \in Z \times Z$,

$$\begin{vmatrix} m & m' \\ e & e' \end{vmatrix} \in N\,\mathbb{Z}.$$

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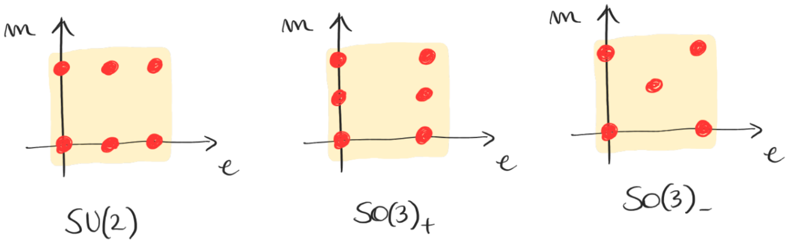
Examples:



Application: Confinement

Conclusion: pure $SU(N)$ Yang-Mills has a $\mathbb{Z}_N^{(1)}$ symmetry.

Application:

- $\mathbb{Z}_N^{(1)}$ preserved $\Leftrightarrow$ Confinement
- $\mathbb{Z}_N^{(1)}$ spontaneously broken $\Leftrightarrow$ Deconfined phase

Application : Chern Simons Theories

In 3d, symmetry operators and charged objects are lines. In $U(1)_k$ Chern-Simons theory

$$S = \frac{k}{4\pi} \int A \wedge \mathrm{d}A$$

define

$$W_q(\gamma) = \exp\left(iq \int_\gamma A \right).$$

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define

$$W_q(\gamma) = \exp\left(iq \int_\gamma A \right) .$$

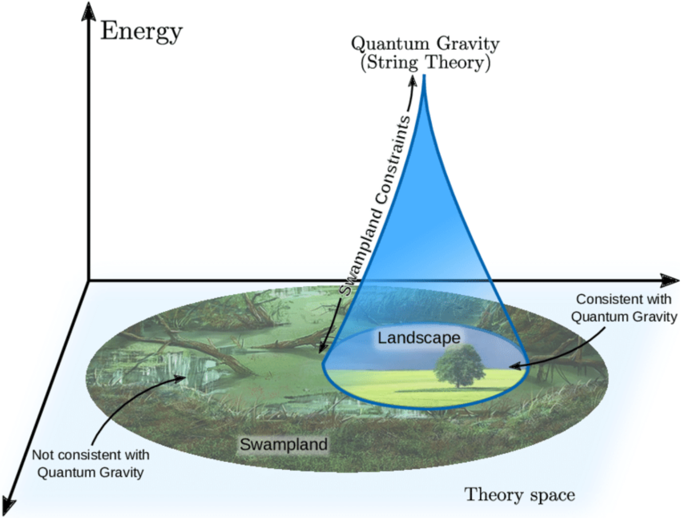
Then we have the braiding relation

$$W_q(\gamma) W_{q'}(\gamma') = e^{2\pi i \frac{qq'}{k}} W_{q'}(\gamma') W_q(\gamma)$$

This is the commutation relations for the $\mathbb{Z}_k^{(1)}$ symmetry.

This symmetry is *anomalous*, and the story continues...

Application : Swampland Program



Picture from 2102.01111

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No Global symmetry conjecture. There are no global symmetries in quantum gravity (i.e. any symmetry is either broken or gauged).

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Cobordism conjecture. For QG theory in d dimensions compactified on a n-dimensional internal manifold,

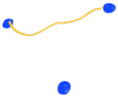
$$\Omega_n^{QG} = 0\,,$$

otherwise there would be a $(d-n-1)$-form global symmetry with charges in Ω_n^{QG}.

Leads to the prediction of new objects (see 2302.00007 and references therein).

Application : Swampland Program

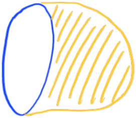
$(2n)\,pt \equiv 0$



$(2n+1)\,pt \not\equiv 0$

$\Omega_0 = \mathbb{Z}_2$

Application : Swampland Program



$\Omega_1 = 0$

$$\Omega_2 = \mathbb{Z}_2$$

Beyond Higher Forms

Summary:

- New language to deal with symmetries, a fundamental concept in physics.
- Gives new ways to understand QFT, makes predictions in QG, etc.

Only the beginning of the story:

- Higher group symmetries: when there are both q-form and p-form symmetries with $q < p$, the subset of q-form symmetries might not close.
- Non invertible symmetries
- Categorical symmetries
- ...

Thank you for your attention!

Application : Neutrino Masses

In the standard model, it is possible to gauge the symmetry $U(1)_{\tau-\mu}$.

Then $U(1)_{e-\mu}$ becomes anomalous : the fusion of two codimension 1 symmetry operators leave behind codimension 2 operators

$$\mathcal{D}_{kN}[\Sigma_3] \times \overline{\mathcal{D}}_{kN}[\Sigma_3] \sim \sum_{\text{two-cycles } S_2} U^M_{2\pi/N}[S_2]$$

This non-invertible symmetry protects the neutrino masses. A breaking by non-perturbative effects in the UV leads to exponentially small masses.