Special Kähler Structures for $\mathcal{N} = 4$ SYM

Antoine Bourget
IPhT, Saclay

April 1st, 2024

KAIST Workshop on Aspects of Quantum Field Theory

Based on upcoming work with
Philip Argyres, Julius F. Grimminger, Matteo Lotito and Mitch Weaver

Motivations

The **moduli space of supersymmetric vacua** is a fundamental geometrical structure attached to SQFTs.

- Contains data on the *local structure* of a QFT (OPE of local operators, e.g. chiral rings).
- What about the *global structure* (additional data needed to properly define the theory on topologically non-trivial manifolds) ?

Our goal:

Show that global structures are also encoded in MSV
(with the appropriate bells and whistles)

Today, demonstrate on simple example of $\mathcal{N} = 4$ SYM.

Outline

Outline

4d $\mathcal{N} = 2$ SCFT Moduli Space of Vacua

4d $\mathcal{N} = 2$ superconformal algebra:

$$\mathfrak{su}(2,2|2) \supset \mathfrak{so}(4,2) \oplus \mathfrak{su}(2)_H \oplus \mathfrak{u}(1)_C$$

Moduli space of $\mathcal{N} = 2$ preserving vacua:

- Higgs branch ($\mathfrak{u}(1)_C$ invariant scalar operators VEVs) is singular hyperKähler.
- Coulomb branch ($\mathfrak{su}(2)_H$ invariant scalar operators VEVs) is singular special Kähler (SK).
- Mixed branches.

4d $\mathcal{N} = 2$ SCFT Moduli Space of Vacua

Nature of the singularities:

- **Complex singularities** are singularities of the spaces viewed as algebraic varieties
- **Metric singularities** are loci where the metric curvature diverges.

General observation :

The metric aspects are more difficult to grasp than the algebraic ones.

Branch	Metric aspects	Complex / Algebraic aspects
Higgs	hyperKähler	Symplectic singularity
Coulomb	special Kähler	Algebraic singularity (often trivial)

$\mathcal{N} = 4$ SYM moduli space

Consider $\mathcal{N} = 4$ SYM with simple complex gauge algebra $\mathfrak{g}$.

$$r = \mathrm{rank}(\mathfrak{g}) \qquad W = \mathrm{Weyl}(\mathfrak{g})$$

<u>Fact 1</u> : the moduli space of vacua is the <u>flat orbifold</u>

$$\mathcal{M} = \mathbb{C}^{3r}/W$$

where W acts on each of the three $\mathbb{C}$ factors in its fundamental reflection representation.

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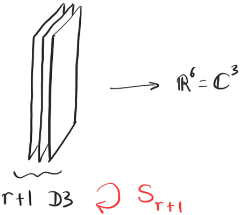

$\mathcal{N} = 4$ SYM moduli space

On a generic point on $\mathcal{M}$, gauge group broken to $\mathrm{U}(1)^r$.
Singularities (metric and algebraic) on loci fixed by some $w \in W$. The Hasse diagram [AB, Grimminger, 2022]:
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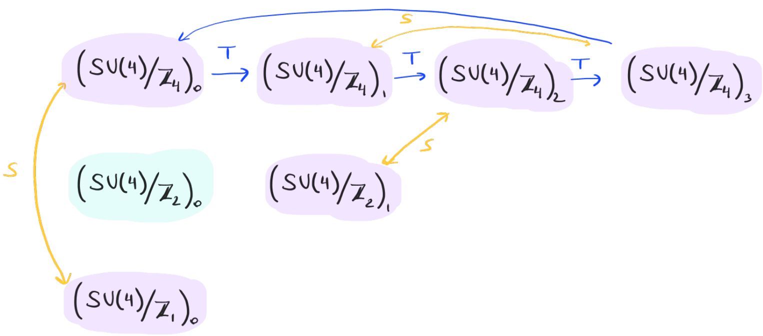
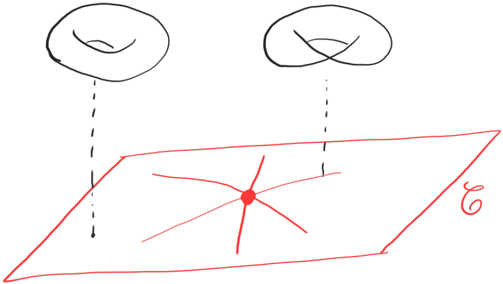


$\mathcal{N} = 4$ SYM moduli space

$\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4) \supset \mathfrak{su}(2,2|2)$
Choosing an $\mathcal{N} = 2$ subalgebra is equivalent to picking a "Higgs branch" within.
The "Coulomb branch" now is freely generated (but has metric singularities).

Flat orbifolds are easy! Let's study the SK structure on the Coulomb branch.



Outline

SK geometry

There is some amount of freedom in what is meant by Special Kähler geometry... Here we adopt the following definition:

[Donagi, Witten 1995] [Freed 1997]
[Argyres, Martone, Ray 2022]

A SK structure is a quadruple $(E \to \mathcal{C}^*, J, \Lambda, s)$ where

- $E \to \mathcal{C}^*$ is a rank r complex vector bundle over $\mathcal{C}^*$ with structure group $\mathrm{Sp}_J(2r, \mathbb{Z})$
- Λ is a rank $2r$ symplectic lattice in the fibers such that J induces an integral symplectic form on Λ.
- s is a holomorphic section of the dual bundle E^* such that $J(ds \overset{\wedge}{,} ds) = 0$ where d is the exterior derivative on $\mathcal{C}^*$.

The (positive) metric on $\mathcal{C}^*$ is given by $ds^2 = iP(ds, d\bar{s})$.
XXX draw XXX

SK geometry

Two simplifying assumptions will be used:

<u>Fact 2</u> : the CB of $\mathcal{N} \geq 3$ QFTs has isotrivial special Kähler geometry, i.e. the fibers are all isomorphic to a given fixed Abelian variety A.

[Cecotti, Del Zotto, Martone, Moscrop 2021]

<u>Fact 3</u> : Theories arising as limits of consistent quantum theories containing gravity have principally polarized SK structures (Banks-Seiberg, Caorsi-Cecotti)

[Banks, Seiberg, 2011]
[Caorsi, Cecotti 2018]

NB : the status of non principally polarized theories is still slightly unclear to me (is it equivalent to being a relative theory?).

SK geometry

Putting together Facts 1, 2 and 3 :

Fact 4 : An $\mathcal{N}=4$ SK structure for algebra $\mathfrak{g}$ is an $\mathrm{Sp}(2r,\mathbb{Z})$-orbit of pairs (S,τ) with

- S **a symplectic integral representations** $S : W \to \mathrm{Sp}(2r,\mathbb{Z})$ **which is** $\mathbb{Q}$**-equivalent to two copies of the fundamental reflection representation of** W**.**
- $\boldsymbol{\tau}$ **is an** $r \times r$ **matrix in the Siegel half space such that** $\boldsymbol{\tau} \in \mathrm{Fix}(\mathrm{Im}\, S)$**, i.e. for all** $w \in W$**,** $S(w) \circ \boldsymbol{\tau} = \boldsymbol{\tau}$**.**

under the action given by

$$M \cdot (S,\boldsymbol{\tau}) = (MSM^{-1}, M \circ \boldsymbol{\tau})$$

for $M \in \mathrm{Sp}(2r,\mathbb{Z})$**.**

Let's classify those!

Outline

$\mathrm{GL}(r, \mathbb{Z})$ representations of W

Step 1 : classify $\mathbb{Z}$-equivalence classes of representations of W that are $\mathbb{Q}$-equivalent to the reflection representation.

Equivalently, we need to find lattice representations.

Introduce

$$Z = \Gamma_{\mathrm{weight}}/\Gamma_{\mathrm{root}}$$

the center of $\mathfrak{g}$. For any subgroup $H \subseteq Z$, there is a lattice Γ_H such that $H = \Gamma_{\mathrm{weight}}/\Gamma_H$.

Theorem (Feit 1998): The Γ_H for $H \subseteq Z$ provide an exhaustive list.

We call R_H the corresponding representations.

$\mathrm{GL}(r, \mathbb{Z})$ representations of W

W	Number of lattices	Lattices
A_1	1	$\Gamma_{\mathrm{root}} \simeq \Gamma_{\mathrm{weight}} \simeq \mathbb{Z}$
A_ℓ $(\ell \geq 2)$	$\sigma_0(\ell+1)$	Γ_d (for $d \vert \ell+1$)
$B_2 = C_2$	2	$\Gamma_{\mathrm{root}}(C_2) \simeq \Gamma_{\mathrm{weight}}(B_2)$, $\Gamma_{\mathrm{root}}(B_2) \simeq \Gamma_{\mathrm{weight}}(C_2)$
B_ℓ, C_ℓ $(\ell \geq 3)$	3	$\Gamma_{\mathrm{root}}(C_\ell)$, $\Gamma_{\mathrm{root}}(B_\ell) \simeq \Gamma_{\mathrm{weight}}(C_\ell)$, $\Gamma_{\mathrm{weight}}(B_\ell)$
$D_{2\ell}$ $(\ell \geq 2)$	5	Γ_{root}, Γ_V, Γ_S, Γ_C, Γ_{weight}
$D_{2\ell+1}$ $(\ell \geq 1)$	3	$\Gamma_{\mathrm{root}} = \Gamma_4$, Γ_2, $\Gamma_{\mathrm{weight}} = \Gamma_1$
E_6, E_7	2	Γ_{root}, Γ_{weight}
E_8	1	Γ_{root}
F_4, G_2	2	Γ_{root}, Γ_{coroot}

$\mathrm{Sp}(2r, \mathbb{Z})$ representations of W

Step 2 : Find canonical forms for $\mathrm{Sp}(2r, \mathbb{Z})$ representations of W.

Let $H \subseteq Z$ be a subgroup and $D \in \mathrm{Mat}(r, \mathbb{Q})$ be a *symmetric* matrix such that for all $w \in W$, the combination

$$L_{(H,D)}(w) := R_H(w)D - DR_H^{-t}(w)$$

has *integer coefficients*. We then call $S_{(H,D)}$ the following symplectic integral representation of W:

$$\begin{aligned} S_{(H,D)} \quad : \quad & W \to \mathrm{Sp}(2r, \mathbb{Z}) \\ & w \mapsto \begin{pmatrix} R_H(w) & L_{(H,D)}(w) \\ 0 & R_H^{-t}(w) \end{pmatrix} . \end{aligned}$$

Theorem: Any representation $W \to \mathrm{Sp}(2r, \mathbb{Q})$ that is $\mathbb{Q}$-equivalent to the direct sum of two copies of the fundamental reflection representation is $\mathbb{Z}$-equivalent to some $S_{(H,D)}$.

$\mathrm{Sp}(2r, \mathbb{Z})$ representations of W

Step 3 : Now that we know that the $S_{(H,D)}$ are sufficient, count how many $\mathbb{Z}$-equivalences there are.

$$\begin{pmatrix} \mathbf{1} & D \\ 0 & \mathbf{1} \end{pmatrix}^{-1} S_{(H,0)} \begin{pmatrix} \mathbf{1} & D \\ 0 & \mathbf{1} \end{pmatrix} = S_{(H,D)}$$

Group of *bindings*:

$$\mathcal{B} = \{L : W \to \mathrm{Mat}(r, \mathbb{Z}) | \exists D \in \mathrm{Sym}(r, \mathbb{Q}) \,, \forall w \in W \,, L(w) = R(w)D - DR(w)^{-t}\}$$

Subgroup of *inner bindings*:

$$\mathcal{B}_0 = \{L : W \to \mathrm{Mat}(r, \mathbb{Z}) | \exists D \in \mathrm{Sym}(r, \mathbb{Z}) \,, \forall w \in W \,, L(w) = R(w)D - DR(w)^{-t}\}$$

The $\mathrm{Sp}(2r, \mathbb{Z})$-equivalence classes of representations of the form $S_{(H,D)}$ for H fixed are labelled by $\mathcal{B}/\mathcal{B}_0$.

$\mathrm{Sp}(2r, \mathbb{Z})$ representations of W

Step 4 : Now that we know that the $S_{(H,D)}$ are sufficient and how many they are, find when two of them are equivalent.

Introduce the intertwiners (normalized so that $I_{H,H} = 1$ and $K_{H,H}$ is the Killing form):

$$I_{H,H'} R_{H'}(w) = R_H(w) I_{H,H'}$$

$$K_{H,H'} R_{H'}(w) = R_H^{-t}(w) K_{H,H'}$$

Then any matrix $M \in \mathrm{Sp}(2r, \mathbb{Z})$ that satisfies $MS_{(H_1,D_1)}(w) = S_{(H_2,D_2)}(w)M$ for all $w \in W$ is of the form

$$M = \begin{pmatrix} aI_{21} - cD_2K_{21} & bK_{12}^{-1} + aI_{21}D_1 - dD_2I_{12}^t - cD_2K_{21}D_1 \\ cK_{21} & dI_{12}^t + cK_{21}D_1 \end{pmatrix}$$

for $a, b, c, d \in \mathbb{Q}$ such that $ad - bc = 1$.

This allows to fully classify the SK structures

Outline

A_3 example

Step 1 : There are three $\mathbb{Z}$-inequivalent representations R_H, for $H = 1, 2, 4$.

Representation	$H = 1$	$H = 2$	$H = 4$
Reflection 1	$\begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$
Reflection 2	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Reflection 3	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

A_3 example

Step 2 and 3 : Representations

$$S_{(H,D)} \quad : \quad W \to \mathrm{Sp}(2r, \mathbb{Z})$$
$$w \mapsto \begin{pmatrix} R_H(w) & L_{(H,D)}(w) \\ 0 & R_H^{-t}(w) \end{pmatrix}.$$

where we have to compute $\mathcal{B}/\mathcal{B}_0$ for each case. For $H = 1$, generic form of D such that $L_{(H,D)}(w)$ has integer coefficients for all $w \in \mathfrak{S}_4$ is

$$\begin{pmatrix} \beta & \frac{3\beta}{2} - \frac{3n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} & 2\beta - 2n_1 + n_3 \\ \frac{3\beta}{2} - \frac{3n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} & 3\beta - 3n_1 + 2n_3 & 4\beta - 4n_1 + 2n_3 \\ 2\beta - 2n_1 + n_3 & 4\beta - 4n_1 + 2n_3 & 6\beta - 6n_1 + 3n_3 \end{pmatrix}$$

for $\beta \in \mathbb{Q}$ and $n_1, n_2, n_3 \in \mathbb{Z}$. Then $\mathcal{B}/\mathcal{B}_0 = \{0\}$, as for any $n_1, n_2, n_3 \in \mathbb{Z}$, there is a $\beta \in \mathbb{Q}$ such that the above has integer coefficients.

A_3 example

Step 2 and 3 : Representations

$$
\begin{aligned}
S_{(H,D)} \quad &: \quad W \to \mathrm{Sp}(2r, \mathbb{Z}) \\
& \quad w \mapsto \begin{pmatrix} R_H(w) & L_{(H,D)}(w) \\ 0 & R_H^{-t}(w) \end{pmatrix}.
\end{aligned}
$$

where we have to compute $\mathcal{B}/\mathcal{B}_0$ for each case. For $H = 2$, generic form of D such that $L_{(H,D)}(w)$ has integer coefficients for all $w \in \mathfrak{S}_4$ is

$$
\begin{pmatrix}
\beta & \frac{3\beta}{2} - \frac{3n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} & \beta - n_1 \\
\frac{3\beta}{2} - \frac{3n_1}{2} + \frac{n_2}{2} + \frac{n_3}{2} & 3\beta - 3n_1 + n_3 & 2\beta - 2n_1 + n_3 \\
\beta - n_1 & 2\beta - 2n_1 + n_3 & \frac{3\beta}{2} - \frac{3n_1}{2} + \frac{n_3}{2}
\end{pmatrix}
$$

for $\beta \in \mathbb{Q}$ and $n_1, n_2, n_3 \in \mathbb{Z}$. Then $\mathcal{B}/\mathcal{B}_0 = \mathbb{Z}_2$, as for any $(n_1, n_2, n_3) = (0, 1, 0)$, one would need β integer and both odd and even so that the above has integer coefficients.

A_3 example

Step 2 and 3 : In total, there are four $\mathbb{Z}$-equivalence classes (H, D):

$$(1,0) \qquad (2,0) \qquad (2,1) \qquad (4,0)\,.$$

Step 4 : Consider for instance $(H, D) = (1, 0)$. Then the Killing form and its inverse are

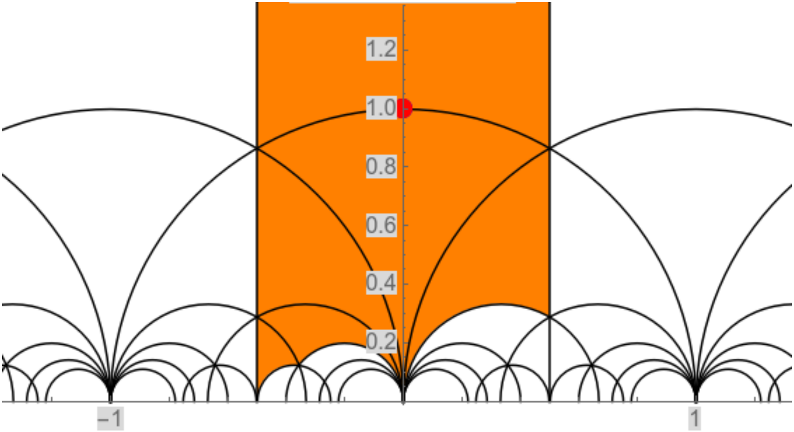
$$K = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \frac{3}{4} \end{pmatrix} \qquad K^{-1} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 6 & 8 \\ 4 & 8 & 12 \end{pmatrix}.$$

So the intertwiner reads

$$M = \begin{pmatrix} a\mathbf{1} & bK^{-1} \\ cK & d\mathbf{1} \end{pmatrix}$$

and for it to have integer entries, one needs that $c \equiv 0$ mod 4. In other words, the orbits of the form $(S_{(1,0)}, \gamma \circ \tau)$ for $\gamma \in \mathrm{SL}(2, \mathbb{Z})/\Gamma_0(4)$ are all disjoint. One can show something similar for $(S, D) = (2, 1), (4, 0)$.
For $(S, D) = (2, 0)$, one finds instead no condition on a, b, c, d so the orbit of $(S_{(2,0)}, \tau)$ is isolated.

A_3 example



* Add n_H number.
* Ref to AST

* Add table mat four SU(4).

A_3 example

Summary:
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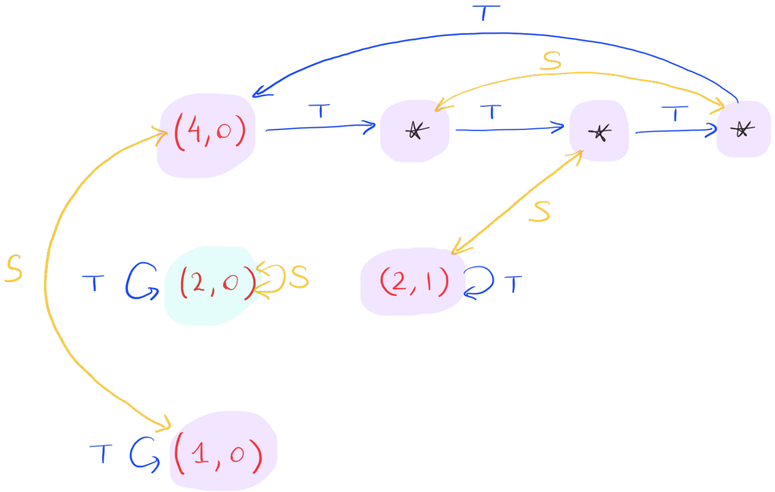

D_4 example

In a similar way one finds :

Lattice	$\mathcal{B}/\mathcal{B}_0$
Γ_{root}	$\mathbb{Z}_4$
Γ_V	$\mathbb{Z}_2$
Γ_S	$\mathbb{Z}_2$
Γ_C	$\mathbb{Z}_2$
Γ_{weight}	$\mathbb{Z}_1$

D_4 example

The congruence subgroups and the orbits can be read from

	$\{R,1\}$	$\{R,2\}$	$\{R,3\}$	$\{R,4\}$	$\{V,1\}$	$\{V,2\}$	$\{S,1\}$	$\{S,2\}$	$\{C,1\}$	$\{C,2\}$	$\{W,1\}$
$\{R,1\}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$
$\{R,2\}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_4 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_3 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_3 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 8n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 8n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$
$\{R,3\}$	$\begin{pmatrix} 4n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 16n_3 - 8n_4 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 - 2n_3 - 3n_4 & n_2 \\ 16n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 - 2n_3 - 2n_4 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$
$\{R,4\}$	$\begin{pmatrix} 4n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & n_2 \\ 16n_3 - 8n_4 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 - 2n_3 - 2n_4 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 - 2n_3 - 3n_4 & n_2 \\ 16n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 8n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 16n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 8n_1 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$
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$\{V,2\}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_4 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 2n_3 & n_4 \end{pmatrix}$
$\{S,1\}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ n_3 & n_4 \end{pmatrix}$
$\{S,2\}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_4 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 2n_3 & n_4 \end{pmatrix}$
$\{C,1\}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & 2n_2 \\ n_3 & n_4 \end{pmatrix}$
$\{C,2\}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 8n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 4n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 & n_2 \\ 2n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 2n_1 - n_4 & n_2 \\ 4n_3 & n_4 \end{pmatrix}$	$\begin{pmatrix} 4n_1 & 2n_2 \\ 2n_3 & n_4 \end{pmatrix}$
$\{W,1\}$	$\begin{pmatrix} n_1 & n_2 \\ n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & n_2 \\ 4n_3 & 4n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ 2n_3 & 2n_4 \end{pmatrix}$	$\begin{pmatrix} n_1 & 2n_2 \\ n_3 & n_4 \end{pmatrix}$

Outline

Conclusion

- One reproduces the results of [Aharony, Seiberg, Tachikawa 2013] from SK geometry and group theory.
- The method can be readily applied to more general orbifolds, yielding e.g. the global forms of $\mathcal{N} = 3$ SCFTs.
- We can also treat non-principal polarizations in the same way.
- Can similar ideas apply to more general SK geometries (non orbifold, non isotrivial, etc)?
- The same information about the global structure of SCFTs should be hidden in the metric aspects of Higgs branch geometry (?)