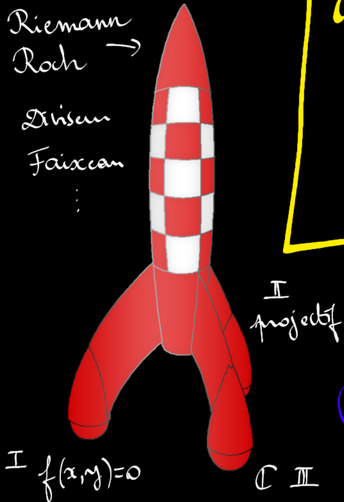
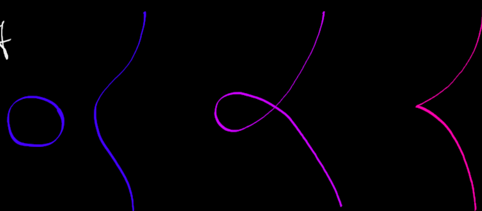
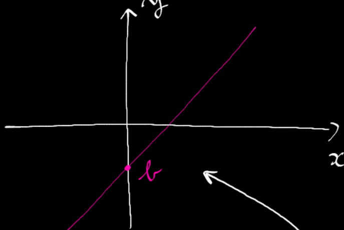
Vers la Géométrie Algébrique



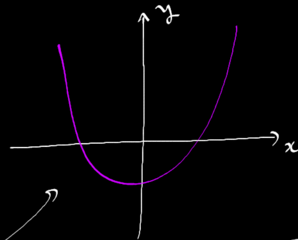


Géométrie algébrique :

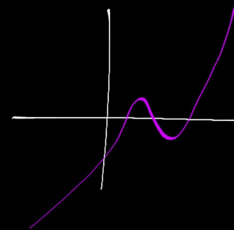
$$y = ax + b$$



$$y = ax^2 + bx + c$$



$$y = (x-1)(x-2)(x-3)$$

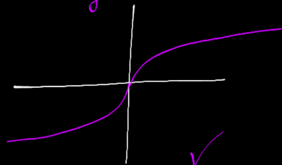


Courbes algébriques

$\cancel{y = e^{x}}$

$y = x^{1/3}$

$y^3 = x$



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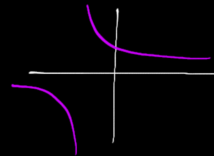
$y = \cancel{\sin}(x)$

$y = \dfrac{1}{1+x}$

$y(1+x) = 1$

$xy + y - 1 = 0$



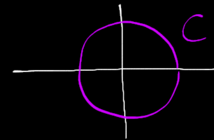
✓

"Définition". Une courbe algébrique C dans $\mathbb{R}^2$ est

$$C = \{ (x,y) \in \mathbb{R}^2 \,/\, f(x,y) = 0 \}$$

avec f polynôme à deux variables de degré $d \geqslant 1$.

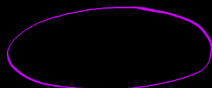
Exemples.



- $f(x,y) = x^2 + y^2 - 1$.

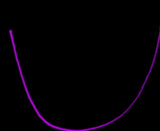
  $$C = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$$

- Coniques : f est de degré 2.



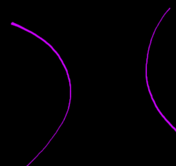
ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



parabole

$$y = a x^2$$

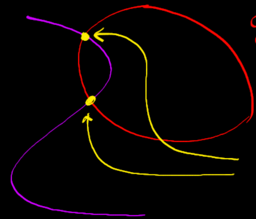


hyperbole

$$xy = a$$

- Intersection de deux courbes algébriques

$f(x,y)=0$ $\qquad$ $g(x,y)=0$

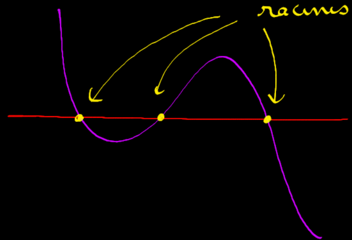


solutions de $\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases}$

- $y = P(x)$

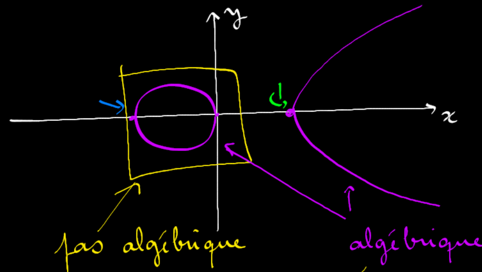
$y = 0$

racines du polynôme P.



La "complexité" d'une courbe algébrique dépend de son degré.

• $y^2 = (x-1)\,x\,(x+1)$

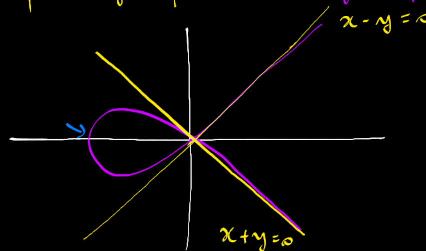


$y^2 = x^2(x+1)$

$\cancel{x^3} + x^2 - y^2 = 0$

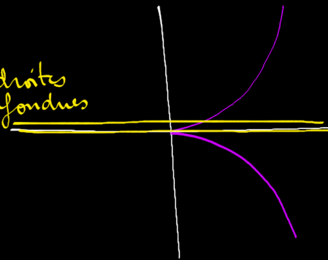
proche de l'origine

$(x+y)(x-y) = 0$

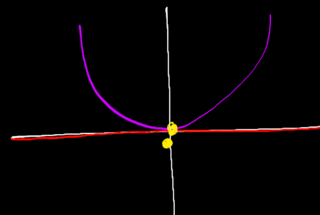


$$y^2 = x^3$$

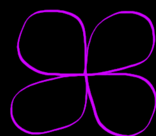
origine $\longrightarrow$ $y^2 = 0$ $\longrightarrow$ deux droites confondues



- $\begin{cases} y = x^2 \\ y = 0 \end{cases}$ $\qquad$ $\begin{cases} x^2 = 0 \\ y = 0 \end{cases}$



- $(x^2+y^2)^3 - 4x^2y^2 = 0$



Pourquoi s'intéresser aux courbes algébriques ?

* Simple à manipuler.
* Fonctionne avec d'autres corps que $\mathbb{R}$.

Remplaçons $\mathbb{R}$ par $\mathbb{Q}$: $\{(x,y) \in \mathbb{Q}^2 \mid f(x,y) = 0\}$

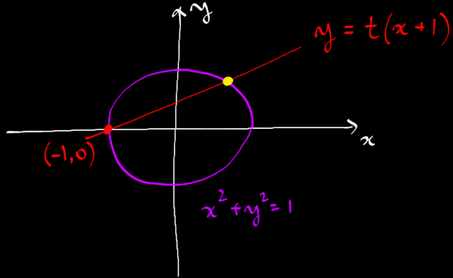
triplet pythagoricien

Exemple : $x^2 + y^2 = 1$ dans $\mathbb{Q}^2$.

$$x = \frac{p}{q} \qquad y = \frac{p'}{q'} \qquad \frac{p^2}{q^2} + \frac{(p')^2}{(q')^2} = 1 \qquad \boxed{(pq')^2 + (qp')^2 = (qq')^2}$$

équation diophantienne.

On va trouver explicitement toutes les solutions dans $\mathbb{Q}^2$.



$$t \rightsquigarrow (x,y) \in C$$

$$\|$$

$$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

$$\left(\frac{1-t^2}{1+t^2} \right)^2 + \left(\frac{2t}{1+t^2} \right)^2 = 1$$

$$\begin{cases} x^2+y^2=1 \\ y = t(x+1) \end{cases}$$

$$x^2 + t^2x^2 + 2t^2x + t^2 - 1 = 0$$

$$(1+t^2)x^2 + (2t^2)x + (t^2-1) = 0$$

$$\Delta = 4t^4 - 4(t^4-1) = 4$$

$$x = \frac{-2t^2 \pm 2}{2(1+t^2)} = \frac{\pm 1 - t^2}{1+t^2}$$

$$y = t \; \frac{1-t^2+1+t^2}{1+t^2} = \frac{2t}{1+t^2}.$$

$$\boxed{(1-t^2)^2 + (2t)^2 = (1+t^2)^2}$$

t	t^2-1	$2t$	t^2+1
0	-1	0	1
1	0	2	2
2	3	4	5
3	8	6	10
4	15	8	17
5	24	10	26
		$\vdots$	

$\boxed{12 \quad 5 \quad 13}$

Conjecture / Théorème. Si $n \geqslant 3$, alors $x^n + y^n = z^n$ n'a pas de solution entière non triviale.

↳ traduction : $C = \{(x,y) \in \mathbb{Q}^2 \mid x^n + y^n = 1\}$ n'a que des points triviaux.

$x^1 + y^1 = 1$ $\quad$ $x^2 + y^2 = 1$ $\quad$ $x^3 + y^3 = 1$ $\quad$ $x^4 + y^4 = 1$ $\quad$ $x^5 + y^5 = 1$

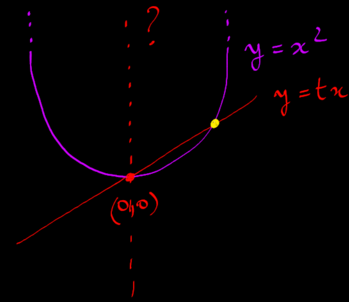
∞

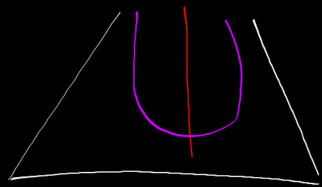
genre $g = 0$ $\qquad$ $g = 1$ $\qquad$ $g > 1$

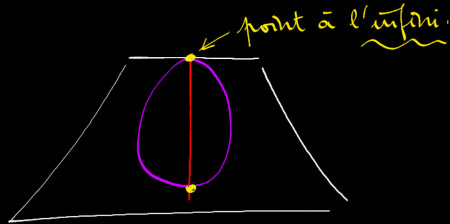
<u>Théorème</u>. Si $g > 1$, le nombre de points rationnels est fini.
(Faltings)

Question : • Qu'est-ce que le genre ?

• Comprendre les intersections...







Géométrie projective.