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Vides et Modularité dans les théories de jauge supersymétriques $\mathcal{N} = 1^*$

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Résumé

Nous explorons la structure des vides dans une déformation massive de la théorie de Yang-Mills maximalement supersymétrique en quatre dimensions. Sur un espace-temps topologiquement trivial, la théorie des orbites nilpotentes dans les algèbres de Lie rend possible le calcul exact de l'indice de Witten. Nous en donnons les fonctions génératrices pour les algèbres classiques, et recourons à un calcul explicite pour les exceptionnelles. Après compactification sur un cercle, un lien entre les théories de jauge supersymétriques et les systèmes intégrables est exploitable pour réduire la chasse aux vides à une extrémisation du hamiltonien de Calogero-Moser elliptique twisté. Une analyse soigneuse des propriétés globales du groupe de jauge et des opérateurs de ligne est nécessaire pour obtenir un accord parfait. En combinant exploration numérique sur ordinateur et contrôle analytique grâce à la théorie des formes modulaires, nous exhibons la structure des vides massifs pour des algèbres de rang petit, et mettons en évidence de nouvelles propriétés modulaires. Nous montrons que des branches de vides de masse nulle existent, et nous en donnons la structure exacte pour les algèbres de rang deux.

Mots clés : Théories de jauge supersymétriques, Systèmes intégrables, Modularité

Abstract

We investigate the vacuum structure of a massive deformation of the maximally supersymmetric Yang-Mills gauge theory in four dimensions. When the topology of spacetime is trivial, the Witten index can be computed exactly for any gauge group using the theory of nilpotent orbits in Lie algebras. We provide generating functions for classical algebras and an explicit calculation for the exceptional ones. Upon compactification on a circle, one can use a bridge between supersymmetric gauge theories and complex integrable systems to reduce the analysis of vacua to the search of extrema of the twisted elliptic Calogero-Moser Hamiltonian. A careful inspection of global properties of the gauge group and line operators are needed to reach total agreement. Using a combination of numerical exploration on a computer and analytical control through the theory of modular forms, we determine the structure of massive vacua for low-rank gauge algebras and exhibit new modular properties. We also show that massless branches of vacua can exist, and provide an analytic description for rank two gauge algebras.

Keywords : Supersymmetric gauge theories, Integrable systems, Modularity

Preface

This work is based on three articles [1, 2, 3] by Jan Troost and myself that have been published in the Journal of High Energy Physics in 2015 and 2016. Large portions of these articles have been used with only formatting changes with the agreement of Jan Troost. The table 1 gives a list of sections of this thesis that are reproduced from the three papers. While I have chosen to focus this document on the content of the three aforementioned articles, I have also worked on other topics with Jan during the last three years [4, 5, 6].

Sections in this document	[1]	[2]	[3]
Sections 2.5 and 2.6		$\times$	
Section 3.3	$\times$		
Section 3.4			$\times$
Chapter 4	$\times$		
Section 5.3	$\times$		
Chapter 6 and 7			$\times$

Table 1: The sections listed here are in essence taken from the indicated papers.

The chapter associated to number 0 is roughly a translation in French of the presentation chapter 1, supplemented with some elements from other chapters and the conclusion.

Remerciements

Dans le travail qui est présenté ici, rien n'aurait vu le jour sans mon directeur de thèse, Jan Troost, qui a rempli son rôle avec une conscience et une présence sans égales. Il a su m'orienter dans l'univers fascinant des théories de jauge supersymétriques, sans toutefois me laisser accroire que l'on peut se passer de la théorie des cordes. Il m'a communiqué son enthousiasme ainsi que son audace inspirée face aux abstractions de la physique théorique. Et surtout, il m'a consacré un temps démesuré tout au long de cette thèse, de la rédaction des articles à la relecture de ce manuscrit, en passant par les longues discussions devant le tableau noir ou le mardi au ministère sur tous les sujets. Je conserverai un souvenir ému de cette collaboration.

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J'ai eu le réel privilège de côtoyer des doctorants et post-doctorants exceptionnels au cours des trois dernières années, et je me dois d'être reconnaissant au CPER[1] pour m'avoir fourni l'occasion d'avoir autant d'estimables collègues de bureau. L'attribution à chacun ce qu'il m'a apporté dépassant en difficulté la détermination de la structure des vides de E_8[2], je me contente de laisser cette tâche en exercice au lecteur, et de remercier par ordre d'apparition Jonathan, Sophie, Alice, Dario, Antoine, Thibaud, Bruno, Éric, Thimothée, et Mathieu pour, en vrac, les lundis-piscine, les demi-poulets, les places VIP au debating, les recommandations avisées de blogs, livres, et séries télé, l'eau chaude, Physique pour tous, le sympathique coin sieste du bureau

[1]La réelle signification de ce sigle peut et doit rester obscure, les intéressés comprendront.
[2]Non traité dans cette thèse.

obscur, ainsi que le château d'eau et la poudre du lumineux, les insectes séchés et les moutons, ShareLatex, les nombreux débats parfois enflammés – avec nos sincères excuses à Amir – ; merci de m'avoir éduqué sur les monuments du cyclisme ou sur comment on fait les bébés, en passant par les sous-préfectures ; merci pour l'assertivité savoureuse de certains, mais aussi pour votre tolérance affable envers mes maintes obsessions compulsives, et enfin merci pour l'atmosphère de travail enrichissante et infiniment stimulante que vous avez créée. Et par delà la rue Lhomond, ces trois années de pérégrinations n'auront pas été solitaires car d'autres ont gravi de concert les pentes du doctorat, en particulier Ricardo qu'anime la passion universelle, Jean-Baptiste qui restera toujours délicieux, Siavash qui m'a fait voir Paris autrement, Amaury qui m'a expliqué le monde, Grégoire pour nos inénarrables débats, Ruoting pour nos pots interdisciplinaires, Harold qui m'a prodigué des avis inestimables, mais aussi Shira, Lætitia, Corinne, Jonathan, Jie, Nicolas. Et bien sûr, je remercie la cohorte de mes amis de plus longue date qui m'ont pardonné mes errements théoriques, depuis nos années d'études à Paris, Palaiseau, ou même pour certains depuis l'école primaire. Je ne peux tous les nommer ici, mais s'ils lisent ces lignes, ils sont plus que probablement concernés.

Enfin, mes parents et ma famille m'ont apporté tout au long de cette thèse, tout comme au cours des 24 années précédentes, un soutien indéfectible et une affection profonde, et ils sont pour plus qu'il ne le pensent sans doute dans l'achèvement de cette thèse. Au delà de l'aide matérielle de tous les instants pourtant déjà si cruciale, merci surtout à eux d'avoir fait de Sainte-Terre, de Meylan et de la rue Saint-Ferdinand les lieux qui demeurent pour moi les plus chers.

Contents

Chapter 0

Présentation

0.1 Difficultés en Théorie Quantique des Champs

Dans cette thèse, nous nous intéressons à des théories quantiques des champs en $3+1$ dimensions d'espace-temps. Ce choix est principalement motivé par le fait que l'espace dans lequel nous vivons semble posséder trois dimensions (du moins, à notre échelle), auxquelles il faut ajouter le temps. Les interactions fondamentales de la nature telles que nous les comprenons depuis les années 1970 sont régies par le *Modèle Standard* de la physique des particules, selon lequel les différentes particules élémentaires sont décrites par des champs quantiques décrits par une théorie de jauge, dite de Yang-Mills. Les particules élémentaires sont celles qui n'admettent pas de description en termes de sous-composants dans le cadre de la théorie considérée. Dans le modèle standard, les particules élémentaires sont de deux types : d'une part, les bosons de jauge qui sont les vecteurs des interactions fondamentales (ce sont le photon, responsable de l'électromagnétisme, les bosons Z et $W^{\pm}$ qui véhiculent la force nucléaire faible, et les gluons pour l'interaction nucléaire forte), et d'autre part les champs de matière. Ces champs de matière sont en général des fermions de spin $\frac{1}{2}$ (électrons, neutrinos et quarks), mais il faut leur adjoindre le boson de Higgs, de spin 0, observé en 2012. Un ingrédient fondamental des théories de Yang-Mills est leur groupe de jauge qui gouverne leurs symétries – les bosons de jauge sont alors associés à chacun des générateurs de ce groupe, et les champs de matière peuvent être regroupés en paquets qui se transforment de façon précise sous son action – ils forment une *représentation* du groupe de jauge.

Dans le cas du Modèle Standard, le groupe de jauge est $SU(3) \times SU(2) \times U(1)$, où le premier facteur $SU(3)$ est le secteur *fort* et le reste $SU(2) \times U(1)$ est appelé secteur *électrofaible*. En dépit d'une formulation mathématique très similaire pour les deux secteurs, que nous allons expliciter sous peu, la physique y présente de forts contrastes. Plus exactement, il s'agit de la physique à nos échelles de taille et d'énergie qui est différente quand on passe du secteur électrofaible au secteur fort : l'un des effets marquants est l'impossibilité d'observer un quark (particule chargée de l'interaction forte) isolé, alors que les électrons isolés sont légion. En revanche, à haute énergie, ou de façon équivalente, à des échelles de distance très petites, la formulation au travers du *Lagrangien* est très semblable en ce qui concerne les bosons de jauge.

Des changements radicalement importants se produisent donc lorsque l'on modifie l'échelle de distance d'observation ; l'ensemble de ces changements est appelé *flot de renormalisation*. Dans le domaine électrofaible, la dynamique non-abélienne est brisée à basse énergie par le mécanisme de Brout-Englert-Higgs, qui donne une masse aux bosons Z et $W^{\pm}$, et par conséquent ces bosons peuvent être négligés la plupart du temps. Seul le photon, qui reste de masse nulle, est partie intégrante de la physique à basse énergie décrite par l'electromagnétisme, dont le groupe de jauge est $U(1)_{\mathrm{em}}$. Dans le secteur fort, il n'y a pas de boson de Higgs disponible pour briser la symétrie $SU(3)$ qui demeure présente à notre échelle et qui est responsable du confinement des quarks décrit précédemment. Le confinement peut être décrit de façon simplifiée en utilisant l'analogie des trois couleurs fondamentales : chaque quark est associé à une de ces trois couleurs, et les seules particules observables de façon isolée (on parle d'état *asymptotique*) doivent correspondre à la couleur blanche. A cause de ce phénomène, les particules que l'on observe sont des mésons (paires d'un quark et d'un antiquark) ou des baryons (composés de trois quarks), qui ne correspondent pas directement aux champs fondamentaux présents dans le Lagrangien. Plusieurs questions surgissent alors:

- N'est-il pas possible de réécrire le Lagrangien en utilisant de nouveaux degrés fondamentaux, de sorte que ceux-ci soient observables à basse énergie ?
- Quel est le mécanisme responsable du confinement ?
- A quel point le confinement dépend-il du groupe de jauge et du contenu en matière d'une théorie donnée ?

Répondre à ces questions revient à comprendre les différentes *phases* des théories de Yang-Mills en fonction du groupe de jauge et du contenu en matière. Dans la section 1.1, nous passons en revue certains concepts de théorie quantique des champs liés à ces questions.

En particulier, la notion sous-jacente dans la première question est la notion de dualité, qui permet de formuler une même théorie de deux façons apparemment différentes. Un exemple que nous allons développer plus loin, puis généraliser amplement, est la dualité entre aspects électriques et magnétiques. Il s'agit d'un exemple de dualité reliant faible couplage et fort couplage, à l'instar de la S-dualité dont nous reparlerons. Cette dualité est une composante de la modularité, qui regroupe de nombreuses autres dualités et symétries au sens propre dans un groupe infini, et rend nécessaire l'introduction d'objets inaccessibles à la théorie des perturbations, comme les monopoles et les instantons. Ces objets jouent par ailleurs un rôle crucial dans notre compréhension du phénomène du confinement, et sont donc la clé de toute réponse à la seconde question ci-dessus. Enfin, concernant la troisième question, la richesse de la réponse repose sur le fait qu'en plus de dépendre du groupe de jauge et du contenu en matière, la phase à basse énergie dépend du *vide* précis autour duquel on observe la théorie. Rappelons qu'un vide dans une théorie classique correspond à un minimum d'énergie potentielle. Il se peut qu'à cause d'une symétrie de la théorie, il existe plusieurs vides à la même énergie, on parle alors de dégénérescence. Dans ce cas, la physique peut être différente dans les différents vides.

Bien que les physiciens aient acquis au cours des dernières décennies une compréhension satisfaisante des phénomènes décrits dans le paragraphe précédent, il demeure difficile d'obtenir des résultats exacts sans hypothèse supplémentaire, cette hypothèse prenant souvent la forme de la supersymétrie, reliant bosons et fermions. Quelques aspects importants des théories de Yang-Mills avec supersymétrie sont décrits dans la section 1.2. En particulier, l'algèbre de symétrie impose dans ce cas que l'énergie dans un vide qui préserve la supersymétrie soit exactement nulle, et il est donc typique d'observer une forte dégénérescence, et possiblement différentes phases. Ainsi, comprendre la physique à basse énergie d'une théorie supersymétrique donnée se ramène à un processus en deux étapes :

1. Trouver tous les vides de la théorie;
2. Dans chaque vide, caractériser la phase.

Grâce aux progrès réalisés au cours des dernières décennies, la seconde tâche est relativement facile à accomplir, du moins dans les théories avec supersymétrie, comme nous allons le voir dans ce chapitre introductif. La première étape, en revanche, est plus ou moins ardue, en fonction de la théorie considérée. Cette thèse est consacrée à cette seconde étape, dans le cas particulier d'une théorie supersymétrique appelée $\mathcal{N} = 1^*$, que l'on peut présenter comme le cas non trivial le plus intéressant – cette affirmation péremptoire sera étayée par des arguments dans les sections à venir. Mais avant cela, regardons de plus près les théories de jauge en quatre dimensions.

0.2 Théories de Yang-Mills en quatre dimensions

0.2.1 Dualité électromagnétique classique

Nous commençons par une rapide description de la dualité électro-magnétique, qui est un bon point de départ en tant que prétexte pour introduire des objets non perturbatifs comme les monopoles magnétiques, et aussi parce que cette dualité est l'inspiration de la dualité de Montonen-Olive qui peut être vue comme sa généralisation non-abélienne. Considérons donc la théorie classique de l'électromagnétisme, c'est-à-dire une théorie de jauge avec groupe $U(1)$, sans matière. Son Lagrangien se réduit aux termes cinétiques pour le champ de jauge $A = A_\mu \mathrm{d}x^\mu$, qui fait intervenir le tenseur $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\mathcal{L}_{\text{pure } QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \,. \tag{1}$$

Nous utilisons ici et dans le reste de la thèse une normalisation légèrement inhabituelle, qui se révélera plus pratique par la suite. Ce Lagrangien conduit directement aux équations de Maxwell dans le vide, que nous écrivons ici sous leur forme non explicitement relativiste :

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{2}$$
$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{0} \tag{3}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4}$$
$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0} \,. \tag{5}$$

Les deux premières lignes (2) et (3)sont les équations du mouvement pour le champ de jauge, tandis que (4) et (5) sont des équations de contrainte qui sont enracinées dans la description que nous avons choisie, qui se base sur un champ de jauge défini de façon globale. Une observation remarquable à propos de ces équations est leur symétrie sous l'action de la *dualité* électromagnétique décrite par

$$\vec{E} \to \vec{B} \tag{6}$$
$$-\vec{B} \to \vec{E} \,. \tag{7}$$

Si nous avions écrit les équations sous la forme $\mathrm{d}\tilde{F} = \mathrm{d}F = 0$, la symétrie aurait été encore plus évidente :

$$F \leftrightarrow \tilde{F}\,. \tag{8}$$

Cette symétrie est cependant violée dès que l'on ajoute des sources dans les équations (c'est-à-dire une densité de charges électriques et une densité de courants électriques), et elle ne peut être restaurée qu'à condition d'ajouter également des sources magnétiques (ce que l'on appelle des monopoles de Dirac, et les courants magnétiques associés), avec le postulat supplémentaire que la dualité échange les deux types de sources. Il est clair que dans une configuration où des monopoles magnétiques sont présents, $\vec{\nabla} \cdot \vec{B} \neq 0$ et il est simplement impossible de définir le champ A_μ. Cependant, supposons un instant que l'on ajoute uniquementy des sources magnétiques, et pas de sources électriques. Dans ce cas, les deux équations (2) et (3) peuvent être résolue à l'aide d'un nouveau potentiel vecteur que nous appelons A_μ^D et qui achève la description duale :

$$\boxed{\begin{array}{c} A_\mu \\ \vec{B} \;=\; \vec{\nabla} \times \vec{A} \\ \vec{E} \;=\; -\partial_t \vec{A} - \vec{\nabla} A_0 \end{array}} \longleftrightarrow \boxed{\begin{array}{c} A_\mu^D \\ \vec{E} \;=\; -\vec{\nabla} \times \vec{A}^D \\ \vec{B} \;=\; -\partial_t \vec{A}^D - \vec{\nabla} A_0^D \end{array}} \tag{9}$$

Jusqu'à maintenant, nous nous sommes restreint à de la physique classique. Bien qu'aucun monopole magnétique n'ait encore été observé dans la nature, les équations ci-dessus sont mathématiquement cohérentes. Mais c'est une autre paire de manche que de donner un sens à la théorie quantique : dans celle-ci, le degré de liberté fondamental est le photon A_μ. Si la théorie contient des monopoles électriques *et* magnétiques, alors le champ de jauge ne peut être défini,[1] et des contraintes d'ordre topologique émergent. Dans le cadre de la théorie $U(1)$, la charge électrique[2] e et la charge magnétique $\tilde{e}$ sont reliées par la condition de quantification de Dirac

$$e\tilde{e} = 2\pi\,, \tag{10}$$

dans laquelle nous avons posé $\hbar = 1$. En d'autres termes, pour chaque paire composée d'un objet électrique et d'un objet magnétique, le produit de leurs charges est un multiple entier de 2π. Cela implique par exemple la quantification de la charge électrique dès qu'il existe au moins un monopole magnétique. Si l'on considère des *dyons*, c'est-à-dire des objets chargés électriquement et magnétiquement, avec des charges (n_e, n_m) et (n'_e, n'_m), alors la condition de Dirac devient $n_e n'_m - n_m n'_e \in 2\pi\mathbb{Z}$.

Une autre conséquence cruciale de cette équation est que si la charge électrique élémentaire e est grande, alors sa duale magnétique est petite. Ainsi la dualité electromagnétique appartient à la famille des dualités reliant fort couplage et faible couplage. Il faut néanmoins insister sur le fait que la dualité électromagnétique est une équivalence entre deux théories $U(1)$ libres, alors que la S-dualité que nous utilisons par la suite identifiera des théories *interagissantes* avec des constantes de couplages différentes.

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[1] Notons cependant qu'avec de la matière dynamique, on peut construire des configuration régulières qui présentent une charge magnétique, comme par exemple le monopole de 't Hooft-Polyakov mentionné un peu plus loin.

[2] Nous notons e la constante de couplage dans le cas abélien. Dans le reste de la thèse, nous nous focaliserons sur le cas non-abélien, où la constante de couplage sera notée g_{YM}. Il n'y a pas de différence fondamentale entre ces deux constantes, il s'agit simplement d'une affaire de conventions.

0.2.2 Physique non-perturbative

Le langage de base de la théorie quantique des champs est celui de la théorie des perturbations autour d'un vide donné, sur lequel agissent des opérateurs d'excitation que l'on peut voir comme des particules. Une observable physique, comme une fonction de corrélation, peut être calculée au moyen d'une série infinie dont les termes successifs sont représentés par des diagrammes de Feynman. Cela mène généralement à une série divergente, mais si la constante de couplage (qui est le paramètre du développement perturbatif) est petite, on obtient une approximation correcte en tronquant la série après quelques ordres. Le résultat est alors donné sous la forme d'une série en puissances positives de g_{YM}. Cependant, il existe des quantités qui ne peuvent être représentées sous cette forme, et en cela requièrent d'aller au-delà de la théorie des perturbations. Les monopoles magnétiques sont de tels objets, comme nous l'expliquons maintenant.

La construction des monopoles magnétiques peut être illustrée de façon simple dans le modèle de Georgi-Glashow avec groupe de jauge $SU(2)$, et un champ de Higgs dans la représentation adjointe, donc le Lagrangien est

$$\mathcal{L}_{GG} = -\frac{1}{4g_{YM}^2} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \lambda(\phi^a\phi^a - v^2)^2 . \tag{11}$$

Les termes qui contribuent à l'énergie d'une configuration sont $F^a_{\mu\nu} F^{a,\mu\nu}$, $D_\mu \phi^a D^\mu \phi^a$ et $\lambda(\phi^a\phi^a - v^2)^2$. Ainsi, une configuration d'énergie minimale est obtenue en prenant $\langle A^a_\mu \rangle = 0$ et $\langle \phi^a \rangle$ constant de sorte que

$$\langle \phi^a \phi^a \rangle = v^2 . \tag{12}$$

Dans ce cas la symétrie de jauge est brisée selon le schéma $SU(2) \to U(1)$. Nous appelons cette configuration le vide de *Higgs*. Dans ce vide, le champ de jauge abélien résiduel peut être utilisé pour définir une charge magnétique, qui est ici nulle.

Il existe d'autres configurations stables qui sont des minima locaux d'énergie. On peut comprendre cela à partir de l'équation (12) qui doit être satisfaite sur la sphere S^2_∞ à l'infini spatial, et qui est l'équation d'une seconde sphère S^2_{Higgs}. Nous en concluons que toute configuration d'énergie finie est associée à une application $S^2_\infty \to S^2_{\text{Higgs}}$. Pour le vide de Higgs, cette application envoie S^2_∞ sur un point $\langle \phi^a \rangle$. Mais d'autres applications sont possibles, et cela coûterait une quantité infinie d'énergie de changer de secteur topologique. Ainsi chaque classe d'applications topologiquement équivalente, appartenant au second groupe d'homotopie $\pi_2 \left(S^2_{\text{Higgs}} \right) = \mathbb{Z}$ est un candidat pour un vide stable, où l'entier correspond à la charge magnétique. De telles solutions partout régulières ont été construites explicitement [7, 8] et sont connues sous le nom de monopoles de 't Hooft-Polyakov. Plus généralement, pour une brisure de symétrie de jauge $G \to H$, le groupe qui les classifie est $\pi_2 (G/H)$.

Les monopoles magnétiques sont des objets non perturbatifs. On peut déduire cela de la formule donnant leur masse classique,

$$m_{\text{monopole}} = 4\pi |n_m| \frac{v}{g_{YM}} \tag{13}$$

où n_m est la charge magnétique. On s'aperçoit que cette masse diverge quand la constante de couplage g_{YM} est très petite, ce qui empêche de l'écrire comme une série en puissances positives de g_{YM}.

Un autre aspect non-perturbatif dans les théories de Yang-Mills est la possibilité d'inclure dans le Lagrangien un terme

$$\mathcal{L}_\theta = \frac{\theta_{YM}}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} = \frac{\theta_{YM}}{8\pi^2} \mathrm{Tr}\,(F \wedge F) \ . \tag{14}$$

Bien qu'il viole la symétrie CP, il préserve la renormalisabilité de la théorie, et est invisible en théorie des perturbations. Cependant, il a un effet sur la masse des monopoles magnétiques : en présence de (14), la formule pour la masse (13) reçoit une correction additive proportionnelle à θ^2_{YM}.

0.2.3 Phases des théories de Yang-Mills

Le Lagrangien d'une théorie de Yang-Mills pure est

$$\mathcal{L}_{YM} = -\frac{1}{4g^2_{YM}} F^a_{\mu\nu} F^{a,\mu\nu} = -\frac{1}{2g^2_{YM}} \mathrm{Tr}\left(F^2\right) \ . \tag{15}$$

Notre but est d'étudier une théorie basée sur ce terme, avec un certain contenu en matière. Nous voulons une théorie suffisamment riche afin de comprendre des phénomènes physiques intéressants, et suffisamment simple pour qu'elle soit manipulable. Nous allons donc considérer une simplification du modèle standard :

- On prend un groupe de jauge G, qui peut être abélien ou non, mais nous supposons qu'il est simple et connexe. Nous ajoutons les bosons de jauge correspondants dans la théorie. L'algèbre de Lie associée est notée $\mathfrak{g}$. Quand $\mathfrak{g} = \mathfrak{su}(N)$, on dit que N est le nombre de couleurs de la théorie.

- Puis on ajoute un certain nombre de champs de matière ψ_k, que nous nommons génériquement des *quarks*. Nous entendons par là des champs de spin $\frac{1}{2}$ qui se transforment dans la représentation fondamentale. Nous mettons N_f quarks dans la théorie, de sorte que l'indice k va de 1 à N_f si $N_f \neq 0$.

- Nous nous réservons aussi la possibilité d'ajouter des champs scalaires complexes ϕ_j dans une représentation $\mathbf{R}$ du groupe de jauge. Nous en prenons N_s, avec $j = 1, \ldots, N_s$.

Le Lagrangien s'écrit alors

$$\mathcal{L} = -\frac{1}{4g^2_{YM}} F^a_{\mu\nu} F^{a,\mu\nu} + \nabla_\mu \bar{\phi}_j \nabla^\mu \phi_j - V(\phi) + \left[i\bar{\psi}_k \not{\nabla} \psi_k - \sum_k m_k \bar{\psi}_k \psi_k \right] . \tag{16}$$

Nous avons supposé qu'il n'y avait pas d'interaction entre les quarks et les champs scalaires. Le modèle standard est beaucoup plus compliqué que ce modèle-ci, car le groupe de jauge n'y est pas simple, les fermions se transforment dans des représentations différentes les unes des autres, et il y a des interactions entre les différents types de fermions et le champ de Higgs. Cependant, ce modèle reproduit la plupart des phénomènes intéressants du modèle standard, et nous servira de champ d'expérimentation pour le reste de cette thèse.[3] Maintenant que la scène

[3] En fait, ce n'est pas entièrement vrai. Les théories $\mathcal{N} = 1^*$ que nous allons étudier – nous les introduisons un peu plus loin –, contiennent un champ de jauge et trois scalaires massifs en ce qui concerne les bosons, ainsi que les fermions qui leur sont reliés par supersymétrie. Ces fermions doivent avoir la même masse que leurs partenaires bosoniques, et ils interagissent avec les bosons à cause de la forme du superpotentiel.

microscopique est en place, nous pouvons poser la question centrale : quelle physique à grande distance ce Lagrangien décrit-il ? La réponse à cette question est fort riche, et dépend de façon complexe des (quelques) paramètres de la théorie : le nombre de couleurs, de fermions, et le potentiel scalaire $V(\phi)$. Nous allons à présent donner un bref aperçu des différentes réponses possibles. On rappelle à cette fin la fonction β à une boucle pour la constante de couplage,

$$\beta_{\text{1-loop}} = \left[\frac{\mathrm{d}g_{YM}}{\mathrm{d}\log(\mu/\Lambda)}\right]_{\text{1-loop}} = -\frac{g_{YM}^3}{(4\pi)^2}\left[\frac{11}{3}T(\text{adjoint}) - \frac{2}{3}T(\text{fermions}) - \frac{1}{3}T(\text{scalars})\right] . \tag{17}$$

Il est de notoriété publique que les théories de Yang-Mills sont plus subtiles que les théories abéliennes : la non-commutativité du groupe se traduit par des termes additionnels $[A, A]$ dans la définition du champ de courbure F, qui donnent lieu à des interactions en A^3 et A^4. Supposons donc dans un premier temps que le groupe de jauge est $G = U(1)$. La physique à basse énergie dépend du contenu en matière. S'il n'y a pas de matière du tout, alors la théorie est libre, donc peu intéressante. Tournons-nous donc vers les deux cas les plus simples qui soient non triviaux.

(`C`) Supposons qu'on ne mette qu'un fermion de Dirac de masse m, que l'on peut appeler un électron. Alors $\beta_{\text{1-loop}} = \frac{e^3}{12\pi^2} > 0$, où e est la charge électrique. Ainsi, quand l'échelle d'énergie μ décroît, la constante de couplage décroît également,

$$e^2 \sim -\frac{1}{\log(\mu/\Lambda)} . \tag{18}$$

Quand μ devient de l'ordre de la masse m de l'électron, ce dernier découple et l'évolution de la constante de couplage s'arrête à une certaine valeur $(e^2)_0$.

(`H`) Maintenant mettons uniquement un champ complexe scalaire massif ϕ avec un potentiel $V(\phi)$ qui est un polynôme en $|\phi|^2$, de sorte que l'invariance de jauge soit préservée, avec un coefficient dominant positif, afin que l'énergie soit minorée. Il est clair que le point $\phi = 0$ est un extremum du potentiel, mais il peut être soit un maximum, soit un minimum. La situation intéressante est celle où il y a un minimum en un point $|\phi_0|^2 \neq 0$. Alors la symétrie $U(1)$ qui agit par $\phi \to e^{i\theta}\phi$ est spontanément brisée par le choix de ϕ_0. C'est le mécanisme de Brout-Englert-Higgs : le champ de jauge devient massif et le champ de Higgs complexe est réduit à une composante réelle.

Des électrons statiques permettent de sonder la physique du premier cas (`C`). Ils ressentent un potentiel de Coulomb,

$$V_{\text{Coulomb}} \sim \frac{(e^2)_0}{r} . \tag{19}$$

Voilà pourquoi on parle de phase de *Coulomb*. Le potentiel à longue portée avec la dépendance caractéristique en r^{-1} est du à la présence de champs de jauge de masse nulle. D'autre part, la phase (`H`) est naturellement appelée la phase de *Higgs*. Ici, à grande distance le potentiel ressenti est

$$V_{\text{Higgs}} \sim \frac{\exp(-m_{\text{photon}}r)}{r} , \tag{20}$$

où m_{photon} est la masse donnée au boson de jauge par le mécanisme BEH.

Tournons-nous maintenant vers les théories de jauge non abéliennes. La fonction β peut recevoir une forte contribution négative de la part du boson de jauge, qui sera dominante si le nombre de champs de matière est suffisamment faible. Cela signifie que la constante de couplage est grande dans l'infrarouge, et que la théorie entre dans sa phase de confinement. Le spectre est très différent de ce à quoi on pourrait s'attendre naïvement en inspectant le Lagrangien. Ainsi, en plus des phases déjà présentes dans les théories abéliennes, nous devons prendre en compte la phase de confinement, qui est caractérisée par un potentiel

$$V_{\text{Confining}} \sim r \tag{21}$$

entre des charges test. On peut se représenter cette interaction par une corde reliant ces charges, qui est constituée par un tube de flux.

Un critère pratique peut être utilisé pour diagnostiquer le confinement, en utilisant des lignes de Wilson. On associe à toute boucle fermée $\mathscr{C}$ de l'espace-temps un opérateur

$$W(\mathscr{C}) = \mathrm{tr}\left[\mathcal{P}\exp\left(i\oint_{\mathscr{C}} A\right)\right] \tag{22}$$

où $\mathcal{P}\exp$ est l'exponentielle ordonnée. Il est très utile de calculer la valeur moyenne dans le vide d'un tel opérateur car cela permet de déterminer le comportement asymptotique du potentiel entre deux quarks sondes. En général, $\langle W(\mathscr{C})\rangle$ se comporte dans la limite d'un grand contour comme $\exp(-\kappa\mathscr{A})$ ou $\exp(-\kappa\mathscr{P})$, avec $\mathscr{A}$ l'aire entourée par la boucle et $\mathscr{P}$ son périmètre. En règle générale, et à certaines subtilités près [9], une loi en $\exp(-\kappa\mathscr{A})$ signale que l'on est dans la phase confinante.

0.2.4 Structure globale du groupe de jauge

Jusqu'à présent, nous avons essentiellement utilisé l'algèbre de Lie $\mathfrak{g}$, et pas le groupe de jauge G qui présente une structure plus riche. La raison en est que les quantités physiques semblent ne dépendre que de la structure algébrique de G, et pas de ses propriétés topologiques. Cela est correct pour une large classe d'observables, qui contient en particulier tous les corrélateurs d'opérateurs locaux d'une théorie formulée sur $\mathbb{R}^4$. Cependant, comme démontré récemment par Aharony, Seiberg et Tachikawa, [10], les aspects globaux ont au moins trois effets qui auront de l'importance dans cette thèse :

- La structure des phases de la théorie sur $\mathbb{R}^4$;
- La dynamique des opérateurs locaux quand la théorie est compactifiée sur $\mathbb{R}^3 \times S^1$;
- L'indice de Witten de la théorie compactifiée.

Les outils appropriés pour tenir compte des propriétés topologiques sont les corrélateurs d'opérateurs de ligne, généralisant les quantités $\langle W(\mathscr{C})\rangle$ qui sondent le confinement. Mais quels sont les opérateurs possibles dans une théorie donnée ?

En plus des lignes de Wilson introduites précédemment, il est naturel de définir des lignes de 't Hooft [11] où le champ de jauge est remplacé par son dual magnétique. Ainsi, les propriétés des lignes de Wilson vis-à-vis du groupe de jauge G ont un équivalent dans les lignes de 't Hooft vis-à-vis du groupe magnétique $G^\vee$ qui est le dual de Langlands de G. Ainsi, les lignes de Wilson sont paramétrées par le réseaux des poids P de G alors que celles de 't Hooft sont paramétrées par le réseau dual, celui des co-poids $P^\vee$. Plus généralement, on peut définir des lignes dyoniques, indexées par des paires dans l'espace

$$\frac{P \times P^\vee}{\text{Weyl Group}}, \tag{23}$$

comme expliqué par Kapustin dans [12]. En fait, les lignes existent toujours par familles complètes générées par le réseau $Q \times Q^\vee$, où Q est le réseau des racines et $Q^\vee$ son dual, et ces familles sont en réalité indexées par $Z(\tilde{G})^2$, où $Z(\tilde{G})$ est le centre du recouvrement universel de G. Les charges possibles sont contraintes par une généralisation de la condition de quantification de Dirac (10). Pour résumer, afin de déterminer complètement le secteur de jauge d'une théorie, il faut procéder en trois temps : choisir une algèbre de jauge $\mathfrak{g}$, puis un groupe de jauge $G = \tilde{G}/H$ où H est un certain sous-groupe du centre $Z(\tilde{G})$, et enfin un spectre de lignes qui satisfasse à la condition de Dirac généralisée, les différents choix possibles étant numérotés par un indice i. La théorie qui en résulte est alors notée $(\tilde{G}/H)_i$.

Pour illustrer cela, nous décrivons la structure de phase de la théorie pure $\mathfrak{su}(N)$. Dans ce cas, nous avons $\tilde{G} = SU(N)$ et $Z(\tilde{G}) = \mathbb{Z}_N$. Les sous-groupes de $\mathbb{Z}_N$ sont les groupes $\mathbb{Z}_d$ pour d diviseur de N. La condition de quantification de Dirac requiert que deux classes de lignes ayant pour charges (n, m) et $(n', m') \in \mathbb{Z}_N \times \mathbb{Z}_N$ doivent satisfaire $nm' - mn' = 0$ modulo N. On s'aperçoit alors que pour un groupe de jauge $SU(N)/\mathbb{Z}_d$ donné il y a exactement d choix possibles pour un ensemble de lignes. Ainsi, le nombre de théories de jauges distinctes est la somme des diviseurs de N. Il est souhaitable de connaître la physique à grande distance dans chacune de ces théories : c'est une question délicate, et nous allons la contourner en ajoutant plus de symétrie à notre théorie. Plus précisément, ajoutons un champ fermionique très massif dans la représentation adjointe du groupe de jauge – cela ne change pas la physique à basse énergie, de laquelle ces fermions sont absents – puis diminuons progressivement sa masse jusqu'à atteindre zéro. Il n'est pas exclus qu'une transition de phase se produise au cours de cette opération, et c'est l'une des raisons pour lesquelles il est si difficile d'analyser les théories non supersymétriques. Car c'est bien la supersymétrie que nous sommes en train d'ajouter : lorsque la masse du fermion adjoint est nulle, on peut le relier au boson de jauge par une supercharge, ce fermion devenant un gaugino. Une myriade de nouvelles techniques sont alors disponibles pour sonder la structure des vides, comme nous allons le voir dans la prochaine section.

0.3 Théories supersymétriques et modularité

Comme nous l'avons vu, il est important de déterminer la structure des vides des théories de jauges en quatre dimensions, et la supersymétrie rend cette tâche plus accessible – nous y reviendrons. D'autre part, il serait agréable de disposer d'un principe supplémentaire qui permettrait de mettre de l'ordre dans les vides. Nous allons voir qu'un tel principe existe, il s'agit de la modularité, qui généralise la dualité électro-magnétique mentionnée dans la section 0.2.1.

0.3.1 La théorie maximalement supersymétrique

La première apparition de la S-dualité en physique remonte aux travaux sur les théories de Yang-Mills dans les années 1970. Dans [13], Montonen et Olive ont proposé une symétrie échangeant fort et faible couplage, champs électrique et magnétique, et le groupe de jauge G avec son dual de Langlands $G^\vee$. Cette symétrie est par essence non-perturbative, et il est par conséquent difficile, si ce n'est impossible, de la vérifier par des moyens classiques. L'introduction de la supersymétrie fournit le contrôle non-perturbatif suffisant pour effectuer des tests.

En fait, la version la plus simple de la symétrie proposée par Montonen et Olive n'est vraie qu'en présence de beaucoup de supersymétrie. Le point de départ est le travail de Witten et Olive [14], où il est démontré que lorsque des monopoles ou d'autres solitons sont inclus dans des théories supersymétriques, des charges centrales reliées aux nombres topologiques des solitons sont générées. Dans un contexte quadri-dimensionel, les charges électrique et magnétique joueront le rôle de charges centrales de l'algèbre de supersymétrie, et la masse m de tout état est bornée inférieurement par la condition de Bogomol'nyi-Prasad-Sommerfield (BPS). Dans la version $\mathcal{N} = 2$ du modèle étudié par Montonen et Olive (le nombre $\mathcal{N}$ compte le nombre de supercharges, comme expliqué dans l'appendice A), cette borne est *saturée* par toutes les particules au niveau classique, y compris les monopoles et les dyons. Ainsi, puisque le nombre d'états dans les supermultiplets ne peut pas sauter à mesure que la constante de Planck est activée, Witten et Olive concluent que la borne BPS doit rester saturée dans la théorie quantique, ce qui donne accès au spectre non-perturbatif (au moins en partie). Osborn [15] poursuit ce travail dans la théorie $\mathcal{N} = 4$ super Yang-Mills, qui est maximalement supersymétrique. En imposant une brisure spontanée de symétrie, il génère un multiplet $\mathcal{N} = 2$ massif qui sature la borne BPS, et montre qu'un monopole magnétique correspond au même multiplet, et avec la même masse ! Ainsi, dans la théorie $\mathcal{N} = 4$, l'accord entre les contenus des multiplets électrique et magnétique est une forte indication que la dualité est réalisée.

Bien sûr, comme nous l'avons expliqué, l'algèbre de jauge ne suffit pas pour spécifier complètement la théorie, et il faut également en donner les opérateurs de ligne? Ainsi, même si $\mathfrak{g}^\vee = \mathfrak{g}$, les théories ne sont pas nécessairement invariantes par S-dualité (et la transformation T n'est pas non plus nécessairement une symétrie). Un exemple est représenté sur la figure 1 extraite de [10].

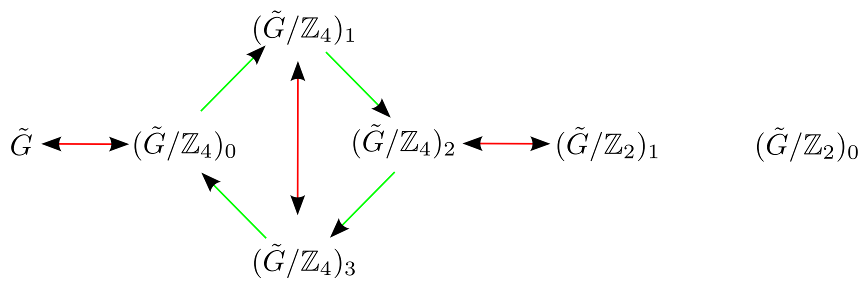


Figure 1: Ce diagramme représente comment les sept théories avec algèbre de jauge $\mathfrak{su}(4)$ sont échangées par les symétries modulaires. En rouge, nous montrons l'action de la S-dualité, et en vert, l'action de T. Quand il n'y a pas de flèche de la couleur correspondante, cela signifie que la théorie est invariante. Ce diagramme est à comparer avec la figure 4.1 dans le chapitre 4.

0.3.2 De la S-dualité à l'invariance modulaire

A l'origine observé dans des modèles sur réseaux [16, 17], l'effet de l'angle θ_{YM} a pu être intégré à notre histoire. D'après l'effet de Witten [18], l'inclusion d'un tel angle dans le Lagrangien donne aux dyons une charge électrique qui peut être un multiple non entier de la charge élémentaire. Cependant lorsque l'angle est varié continûment de zéro à 2π, le spectre est rebattu mais demeure globalement invariant. Dans le contexte des théories de jauge supersymétriques, avec un groupe de jauge simple, il est naturel d'assembler l'angle θ_{YM} et la constante de couplage g_{YM} dans une combinaison complexe

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta_{YM}}{2\pi}\,, \tag{24}$$

que l'on peut interpréter comme la composante scalaire d'un superchamp, de sorte que le Lagrangien de la théorie (A.14 puisse être considéré comme un terme de superpotentiel. La S-dualité et la symétrie $\theta_{YM} \to \theta_{YM} + 2\pi$ génèrent à eux deux un groupe modulaire $SL(2,\mathbb{Z})$ de dualités. Cependant, l'accord dans les multiplets trouvé par Osborn dans la théorie maximalement supersymétrique n'est plus réalisé dans les théories $\mathcal{N} = 2$, ce qui exclut une réalisation simple de la dualité électro-magnétique dans ces théories, et par la même occasion dans toutes les théories avec moins de supersymétrie. Cependant, une version différente des mêmes idées émerge du travail de Seiberg et Witten dans les années 1990.

Dans la théorie pure $SU(2)$ avec des monopoles [19], la partie scalaire du multiplet vecteur et de son dual (a_D, a) se transforme sous l'action du groupe modulaire $SL(2,\mathbb{Z})$. Le générateur

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{25}$$

est identifié avec la dualité électro-magnétique, et échange deux descriptions équivalentes de la même théorie. D'autre part, le générateur

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{26}$$

est la symétrie $\theta_{YM} \to \theta_{YM} + 2\pi$ décrite précédemment. Au niveau classique, $\tau_{\mathrm{cl}} = \frac{a_D}{a}$, et l'action du groupe modulaire est l'action habituelle sur τ_{cl}. Cela reste vrai dans la théorie quantique dans le cas de $\mathcal{N} = 4$, mais en général le flot de la constante de couplage sous l'action du groupe de renormalisation gâche la modularité.

Dans les théories avec encore moins de supersymétrie, la modularité est plus cachée. Ainsi, la chromodynamique quantique $\mathcal{N} = 1$ avec le nombre adéquat de quarks et de squarks est reliée par la dualité de Seiberg [20, 21] à une théorie différente, mais qui évolue dans l'infrarouge vers la même physique. Il s'agit ici encore d'une version de la dualité électro-magnétique, les quarks étant par exemple envoyés sur les quarks du groupe dual magnétique dans l'autre théorie, mais il est important de noter que les deux théories sont très différentes. Dans la version la plus simple, une théorie $SU(N)$ avec $\frac{3}{2}N < N_f < 3N$ quarks est reliée à une théorie $SU(N_f - N)$ avec N_f quarks ainsi que des mésons. De plus, la dualité n'est réalisée que dans la limite de basse énergie.

Le travail récent de Gaiotto [22] éclaire d'un jour nouveau les dualités des théories $\mathcal{N} = 2$, en associant à ces dernières une courbe ultraviolette C avec insertions, de telle sorte que la courbe de Seiberg-Witten, qui contrôle la physique dans l'infrarouge, soit donnée par l'équation $\lambda^2 - \varphi(z) = 0$, où z est un paramètre de C, λ est la forme de Seiberg-Witten et les propriétés asymptotiques de $\varphi(z)$ sont déterminées par le superpotentiel. Un exemple important est celui de la théorie $\mathcal{N} = 2^*$, dans laquelle la matière est composé d'un hypermultiplet de masse m dans la représentation ajointe. Dans ce cas, la courbe C est un tore avec une insertion, et en utilisant les propriétés des fonctions elliptiques, on obtient [23]

$$\varphi(z) = (m^2 \wp(z;\tau) + u)\mathrm{d}z^2\,, \tag{27}$$

où $u = \operatorname{tr}\phi^2$, avec ϕ la composante scalaire du multiplet vecteur. Cela fournit une interprétation géométrique du groupe modulaire $SL(2,\mathbb{Z})$, qui apparaît ici comme le groupe des symétries discrètes de la structure complexe du tore.

La théorie $\mathcal{N}=4$ est un cas particulier de cette situation, et on peut comprendre le diagramme de la figure 1 de ce point de vue. En effet, il existe une interprétation des données discrètes nécessaires pour définir les théories dites de classe S à quatre dimensions, à savoir le spectre des opérateurs de ligne, en termes d'une géométrie à six dimensions [24], comme le sous-réseau maximal isotrope de $H^1(C, Z(\tilde{G}))$, où C est la courbe ultraviolette. Pour les théories $\mathcal{N}=4$, nous venons de voir que C est un tore, et donc $H^1(C, Z(\tilde{G}))$ est isomorphe à $Z(\tilde{G})^2$. Cela explique pourquoi le même diagramme de dualité apparaîtra plus tard dans notre analyse des extrema du potentiel de Calogero-Moser elliptique et des vides des théories $\mathcal{N}=1^*$. On pourra comparer avec les figures 4.1 et 5.1.

0.3.3 La structure des vides

L'espace des vides est un outil de base de l'analyse des théories de jauge supersymétriques, qui est basé sur l'holomorphie. La classification des vides massifs et de masse nulle, et l'analyse de leurs symétries et de leurs propriétés de dualité sont des traits fondamentaux d'une théorie. Rappelons quelques théorèmes centraux :

- Dans les théories $\mathcal{N}=1$, le superpotentiel n'est pas renormalisé en théorie des perturbations. Cependant, il peut être affecté par des effets instantoniques.

- Dans les théories $\mathcal{N}=2$, la fonction β à une boucle (A.25) est exacte.

- Les théories $\mathcal{N}=4$ sont finies.

Nous devons faire la distinction entre les espaces de modules classique et quantique. En général, une dégénérescence entre des vides d'une théorie classique qui n'est pas causée par une symétrie (on parle de dégénérescence accidentelle) est détruite par les effets quantiques. Cependant, en présence de supersymétrie, la situation générique correspond à une préservation quantique de la dégénérescence en théorie des perturbations, bien que des effets non-perturbatifs puissent générer un superpotentiel additionnel et détruire les vides classiques. Le superpotentiel effectif sur l'espace des modules contient toute l'information concernant la limite à basse énergie, avec son contenu en particules et sa structure de phase.

L'espace des modules classique est singulier aux points où il y a des champs additionnels de masse nulle par rapport à un point générique. Une question naturelle concerne alors le destin de ces singularités dans l'espace des modules quantique [25]. En fait, à peu près tout peut arriver : une singularité peut disparaître, rester mais changer de nature, ou ne pas être affectée. Nous verrons un exemple dans le cas de la théorie $SU(2)$ pure $\mathcal{N}=2$. Avant cela, penchons-nous sur la théorie maximalement supersymétrique.

$\mathcal{N}=4$

Dans le langage des théories $\mathcal{N}=1$, la théorie de Yang-Mills $\mathcal{N}=4$ possède six scalaires ϕ^i, $i=1,\ldots,6$ dans la représentation adjointe, en plus du multiplet vecteur, pour ce qui est du contenu bosonique. Le superpotentiel fournit alors un terme

$$\frac{1}{8g_{YM}^2}\sum_{i,j}[\phi^i,\phi^j]^2 \tag{28}$$

qui est nul si les ϕ^i commutent deux à deux. On peut ainsi les voir comme des éléments de la sous-algèbre de Cartan : ils brisent le groupe de jauge à $U(1)^r$, où r est le rang, et nous parlons donc de phase de Coulomb. Quand toutes les valeurs dans le vide sont nulles, on parle de phase superconforme – et on peut naturellement trouver toute une série de phases intermédiaires avec une symétrie résiduelle non abélienne.

Ainsi, nous avons trouvé l'espace des modules classique :

$$\mathcal{M}_{\mathcal{N}=4} = \frac{\mathbb{R}^{6r}}{\text{Weyl group}}, \tag{29}$$

qui est singulier aux points de $\mathbb{R}^{6r}$ qui sont fixés par le groupe de Weyl. Ces singularités signalent de nouveaux degrés de libertés de masse nulle, qui sont nécessaires pour construire les bosons de jauge non abéliens qui apparaissent à ces points. On peut montrer que cet espace classique n'est pas corrigé dans la théorie quantique [26], de sorte que (29) reste vrai dans la théorie complète.

$\mathcal{N}=2$

Maintenant que nous avons décrit la situation pour la théorie $\mathcal{N}=4$ très contrainte, nous nous tournons vers $\mathcal{N}=2$ où la physique est beaucoup plus riche. A titre d'exemple du mécanisme évoqué précédemment, dans la théorie $SU(2)$ pure l'espace des modules est $\mathbb{C}$ avec une singularité à l'origine où la symétrie non-abélienne est restaurée. L'espace des modules n'est pas détruit dans la théorie quantique, mais la singularité se scinde en deux singularités quantiques qui correespondent à des valeurs non nulles du paramètre de la branche de Coulomb.

$\mathcal{N}=1$

Les théories de jauge avec la quantité minimale de supersymétrie sont moins contraintes, et présentent des propriétés physiques très riches. L'exemple de base est la déformation massive de la théorie $\mathcal{N}=2$ du paragraphe précédent. Classiquement, ainsi qu'en théorie des perturbations, le superpotentiel se réduit au terme de masse, et il n'y a qu'un seul vide avec une valeur moyenne nulle pour le multiplet chiral. Cependant, en prenant en compte les effets non-perturbatifs, le superpotentiel Affleck-Dine-Seiberg est généré [27], induisant un effet répulsif dans la région fortement couplée, et scindant ainsi le vide classique en deux vides quantiques.

Dans la théorie de Yang-Mills pure, le groupe de R-symétrie $U(1)$ est brisé par des instantons à un sous-groupe discret, et le nombre de vides est le nombre de Coxeter dual $h^\vee$ de l'algèbre de jauge. L'existence de vides isolés implique la possibilité d'existence de murs de domaines entre les régions d'espace-temps avec différentes configurations de vides. Dans les théories $\mathcal{N}=1$, la densité d'énergie sur un tel mur peut être calculée de façon exacte dans le régime à fort couplage [28],

$$T_{\text{mur}} = |2\Delta\mathcal{W}|\,. \tag{30}$$

0.3.4 Compactification sur le cylindre

Après avoir décrit l'espace des vides des théories $\mathcal{N} = 2$ sur $\mathbb{R}^4$, Seiberg et Witten [29] ont poursuivi leur exploration sur un espace de topologie non triviale, $\mathbb{R}^3 \times S^1$. Un effet intéressant de cette compactification est alors que la courbe de Seiberg-Witten acquiert une signification physique. En effet, comme nous le verrons en détail dans le chapitre 5, deux scalaires additionnels apparaissent (la ligne de Wilson le long du cercle et le dual du photon dans les trois directions non compactes), que l'on peut combiner en un scalaire complexe Z, voir la figure 2. La théorie à basse énergie est alors un modèle sigma pour lequel l'espace cible est la courbe de Seiberg-Witten, et l'espace des modules est précisément cette courbe, avec ses modules. Un potentiel généré de façon non perturbative sur la branche de Coulomb peut être vu comme un potentiel additionnel (au sens de la mécanique classique) dans le modèle sigma, ce qui est l'essence du lien intime entre les théories de jauge supersymétriques et les systèmes mécaniques intégrables – c'est la supersymétrie $\mathcal{N} = 2$ qui fournit cette structure supplémentaire.

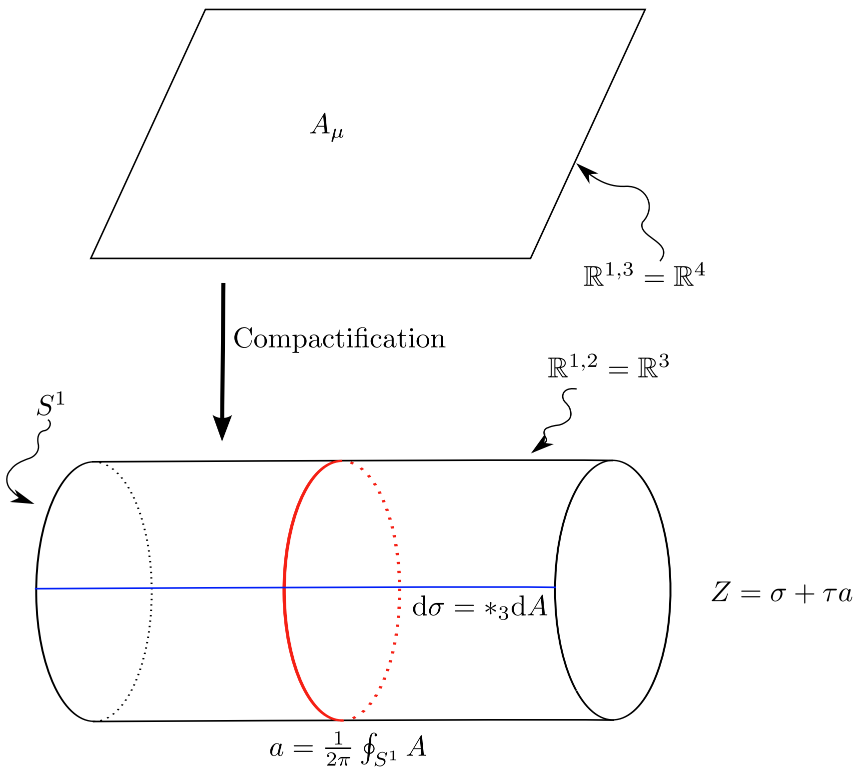


Figure 2: Compactification de la théorie sur le cylindre, et apparition des deux scalaires réels qui se combinent dans le scalaire complexe Z.

Ainsi, la compactification sur le cylindre est un outil extrêmement puissant pour sonder la dynamique de jauge, pour au moins trois raisons :

- Elle offre une réalité physique à la courbe de Seiberg-Witten ;
- Elle relie théories de jauges et systèmes intégrables ;
- Elle connecte la physique à quatre dimensions (quand le rayon $R \to \infty$) et la physique à trois dimensions (quand $R \to 0$).

De façon heuristique, si toute théorie $\mathcal{N} = 2$ donne naissance à un système intégrable complexifié, la compactification sur le cylindre le transforme en un espace hyperkähler manipulable [30]. Nous allons donc utiliser abondamment ces compactifications dans la suite. Il faut cependant noter que d'autres topologies permettent d'accéder à des résultats intéressants. Citons par exemple la compactification sur une sphère à quatre dimensions [31] ou un ellipsoïde [32] où la fonction de partition peut être calculée exactement en utilisant le principe de localisation supersymétrique. La correspondance d'Alday-Gaiotto-Tachikawa [33] fait alors un lien avec des théories conformes à deux dimensions dont les constantes de couplages sont contrôlées par cette géométrie.

0.4 Les vides des théories $\mathcal{N} = 1^*$

Faisons le point sur la situation, avant de présenter le plan de cette thèse. Comme nous l'avons expliqué, l'un des grands succès de la physique au vingtième siècle fut la découverte selon laquelle certaines forces de la nature sont décrites avec précision par des théories quantiques des champs avec invariance de jauge. Résoudre ces théories est difficile, et nous nous rabattons sur les théories avec supersymétrie, où le pouvoir de l'holomorphie est d'une aide précieuse. La dynamique à basse énergie de ces théories est un sujet riche et fructueux, et est reliée à des questions portant sur certains systèmes intégrables classiques. C'est ce pont que nous allons exploiter dans cette thèse pour explorer les vides d'une théorie $\mathcal{N} = 1$ particulière, et des avancées seront réalisées aux deux extrémités du pont, comme illustré par la figure 3 qui dépeint les relations de dépendance entre les chapitres.

A partir de maintenant, nous nous focalisons sur une déformation de la théorie de Yang-Mills $\mathcal{N} = 4$ obtenue en ajoutant à la main dans le Lagrangien une masse pour les trois multiplets chiraux, que nous appelons $\mathcal{N} = 1^*$, et qui est présentée en détail au début du chapitre 2. Il s'agit d'un terrain d'expérimentation très intéressant, car la théorie déformée allie certaines propriétés modulaires héritées de sa parente maximalement supersymétrique à la richesse de la structure des vides d'une théorie minimalement supersymétrique. D'une certaine façon, cette théorie est également parmi les plus naturelles, aux côtés de la théorie pure $\mathcal{N} = 1$ et de la chromodynamique quantique supersymétrique, de par son contenu en matière minimaliste, et surtout son caractère fini à haute énergie.

Dans le chapitre 2, nous calculons le nombre de vides massifs de la théorie $\mathcal{N} = 1^*$ sur $\mathbb{R}^4$ pour tous les groupes de jauges simples G. Nous utilisons des techniques semi-classiques et nous reproduisons avec succès les décomptes déjà connus pour les groupes de type A, B et C. Nous présentons des fonctions génératrices pour les groupes $O(2n)$ et $SO(2n)$, et nous calculons l'indice de Witten pour les groupes exceptionnels. Un rôle crucial est joué par la classification des orbites nilpotentes, ainsi que les propriétés globales de leurs centraliseurs dans les groupes de Lie. Des exemples illustratifs sont donnés pour illustrer les subtilités de l'analyse dans le cas des algèbres de type D. Les résultats sont résumés par les fonctions génératrices suivantes, pour l'index de Witten :

$$
\begin{aligned}
I_{SU(n)} &= \sum_{n=1}^{\infty} \sigma_2(n) q^n \\
I_{O(n)}(q) &= \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2(1-q^{4k})^2} \\
I_{SO(n)}(q) &= \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{4k})^2(1-q^{2k-1})^2} + \prod_{k=1}^{\infty} \frac{1+q^{8k-4}}{(1-q^{4k})^2}\,, \\
I_{Sp(2n)}(q) &= q^{-1} I_{SO(2n+1)}(q)\,.
\end{aligned}
\tag{31}
$$

Dans ces expressions, P_0 est un polynôme défini [34] par l'équation (2.61) dans le chapitre 2. Pour les groupes exceptionnels, le résultat est

$$
\begin{aligned}
I_{G_2} &= 10\,, \\
I_{F_4} &= 45\,, \\
I_{E_6} &= 44\,, \\
I_{E_7} &= 174\,, \\
I_{E_6} &= 301\,.
\end{aligned}
\tag{32}
$$

Dans le chapitre 3, nous laissons les théories de jauge de côté pour un moment, et nous nous tournons vers l'étude du système elliptique complexifié de Calogero-Moser. Nous commençons par introduire quelques concepts de base sur les systèmes intégrables, du point de vue classique comme du point de vue quantique, et nous présentons les systèmes de Calogero-Moser, dont la définition implique une algèbre de Lie simple. L'explication de la raison pour laquelle ces systèmes sont utiles pour comprendre la structure des vides des théories $\mathcal{N}=1^*$ est repoussée au chapitre 5. Un outil fort pratique qui est mis à profit pour appréhender les systèmes intégrables est la prise en compte de limites pour certains paramètres, après lesquelles le système reste intégrable, mais est plus simple. À partir du système elliptique, nous pouvons ainsi atteindre des systèmes de Calogero-Moser associés à des algèbres plus petites, ou encore des système de Toda. Nous introduisons pour classifier les limites des algèbres de Lie affines, ainsi que des sous-algèbres pseudo-Levi [35], qui permettent de généraliser les limites d'Inozemtsev [36]. Les systèmes elliptiques sont définis, comme leur nom l'indique, sur des courbes elliptiques, et possèdent d'agréables propriétés modulaires dont il sera fait un usage abondant dans le chapitre suivant.

Le chapitre 4 est, de fait, dédié à l'étude détaillée des extrema isolés dans plusieurs cas particuliers, du système elliptique Calogero-Moser. Cette analyse combine des explorations numériques utilisant Mathematica [37] alliées à la théorie des formes modulaires, et permet d'obtenir de nombreux résultats exacts. Nous déterminons la valeur du potentiel aux extrema, qui est une fonction du paramètre modulaire du tore τ, pour plusieurs algèbres de Lie de petit rang. Pour $\mathfrak{so}(5)$, nous obtenons des formes modulaires à valeurs vectorielles pour le sous-groupe de congruence $\Gamma_0(4)$. Pour $\mathfrak{so}(7)$ et $\mathfrak{so}(8)$, les extrema se séparent en deux sous-ensembles : le premier contient les extrema qui forment des formes modulaires vectorielles pour d'autres sous-groupes de congruence ($\Gamma_0(4)$, $\Gamma(2)$ et $\Gamma(3)$), alors que le second contient des extrema qui sont sujets à des monodromies autour de points particuliers à l'intérieur du domaine fondamental. Il s'agit là d'un phénomène assez surprenant ; par exemple, dans le cas de $\mathfrak{so}(8)$, le point critique τ_M

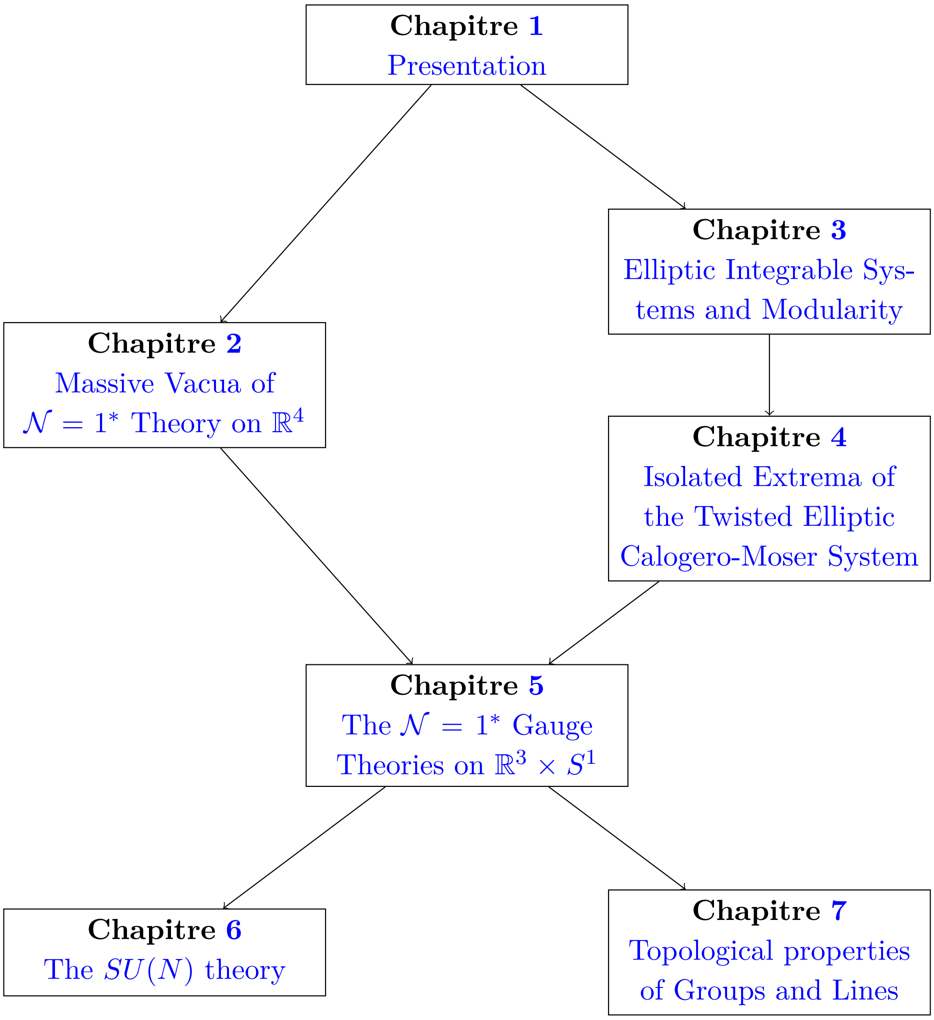


Figure 3: Interdépendances entre les chapitres de cette thèse

vaut approximativement $\tau_M \approx 2.41i$, et nous pouvons affirmer avec un haut degré de certitude qu'il vérifie l'équation

$$\frac{1}{1728} j(\tau_M) = \frac{7626496}{3375} \,. \tag{33}$$

Après cette excursion pendant deux chapitres dans le royaume des systèmes intégrables elliptiques, nous rejoignons notre sujet principal, les théories de jauge $\mathcal{N} = 1^*$. Grâce à la supersymétrie, il est possible de calculer le superpotentiel effectif $\mathcal{W}$ à basse énergie, de façon exacte. Pour une déformation massive de la théorie pure $\mathcal{N} = 2$, cela a été fait dès 1994 [19], et pour les théories $\mathcal{N} = 1^*$ avec groupe de jauge $SU(N)$, le superpotentiel exact a été proposé par Dorey [38], en utilisant les techniques de Seiberg et Witten [19, 29]. L'idée cruciale est de compactifier sur un cyclindre, comme nous l'expliquons dans les premières sections du chapitre 5. Pour les autres algèbres de Lie, le superpotentiel est alors exactement le potentiel du système elliptique complexifié de Calogero-Moser [39]. Par conséquent, nous devrions être capables de comparer l'analyse semi-classique déduite de notre analyse des vides sur $\mathbb{R}^4$ aux résultats numériques et exacts obtenus au cours du chapitre 4 : c'est l'objet de la section 5.3, dans laquelle un accord partiel est atteint. Cependant, nous trouvons que certains extrema du système intégrable n'ont pas de contrepartie dans la théorie de jauge semi-classique, ce qui est le signe que certaines subtilités n'ont pas été prises en compte. La résolution de cette énigme occupe les deux derniers chapitres, qui présentent une analyse plus profonde de la théorie quantique des champs étudiée.

Dans les deux derniers chapitres, nous utilisons donc toutes les techniques développées jusqu'à présent pour dévoiler la structure des vides de la théorie $\mathcal{N} = 1^*$ sur $\mathbb{R}^3 \times S^1$. Nous commençons avec les algèbres de type A dans le chapitre 6, car ce cas est le plus simple – le phénomène des lignes de Wilson discrètes n'apparaît pas ici. Néanmoins, nous obtenons des branches de masse nulle, qui sont également visibles du côté des systèmes intégrables. Sans le cas de l'algèbre $\mathfrak{su}(3)$, nous offrons une description analytique de ces branches. En termes des lignes de Wilson complexifiées, cette description fait usage de la technique d'Eichler-Zagier [40] pour inverser la fonction de Weierstrass elliptique. Dans ce cas particulier, après ajustement fin du couplage autour des points elliptiques d'ordre trois, nous identifions les singularités d'Argyres-Douglas [41].

Dans le chapitre 7, nous montrons qu'après compactification sur un cercle, l'analyse semi-classique des vides massifs et de masse nulle dépend de la classification des orbites nilpotentes, ainsi que de la classe de conjugaison du groupe des composantes de leurs centraliseurs. Les algèbres pseudo-Levi que nous avons utilisées pour définir les limites généralisées d'Inozemtsev permettent de classifier ces objets, et définissent un raffinement de la théorie des orbites nilpotentes présentée au chapitre 2, que nous expliquons en section 7.4, basée sur le travail de Sommers [35]. La topologie non triviale permet d'allumer des lignes de Wilson qui augmentent le nombre de vides massifs. Nous démontrons de façon semi-classique que des vides de masse nulle peuvent être transformés en vides massifs par ces lignes de Wilson dans des groupes de jauge discrets résiduels. Nous illustrons cette analyse dans les théories de jauge avec l'algèbre $\mathfrak{so}(5)$ qui nous a servi de guide jusqu'à présent, ainsi qu'avec l'algèbre exceptionnelle G_2.

Chapter 1

Presentation

In this thesis, our main concern will be quantum field theories in four spacetime dimensions. The fundamental interactions in Nature, as we understand them in the framework of the Standard Model of Particle Physics, are described by Yang-Mills gauge theories. According to this model, the fundamental fields can be divided into two categories, the gauge bosons with spin 1 that mediate the forces, and the matter fields that have spin $\frac{1}{2}$ for the fermions (quarks, electrons and neutrinos) or spin zero in the case of the Higgs boson. This means that the fundamental degrees of freedom are accurately described by a Lagrangian which contains gauge bosons associated with a gauge group, and matter fields that transform into certain given representations of this group. This description is correct at very short distance, or equivalently very high energy.

The gauge group of the standard model is $SU(3)\times SU(2)\times U(1)$, where the first factor $SU(3)$ is the *strong* sector and the remaining $SU(2) \times U(1)$ is the *electroweak* sector. The physics in these two sectors is very different at low energies, although the high-energy formulation is similar. In the electroweak sector, the non-abelian gauge dynamics is broken at low energies due to the Higgs mechanism, which gives a mass to the W and Z bosons, and the residual gauge theory is abelian, with a massless photon. On the other hand, the situation in QCD is more complicated because of the absence of an equivalent Brout-Englert-Higgs mechanism. Probably the strongest manifestation of the difficulty is that the fundamental degrees of freedom that are used to write down the Lagrangian (quarks and gluons) do not coincide with the asymptotic degrees of freedom that we can observe at our energy levels (mesons and baryons). The underlying reason for this discrepancy is *color confinement*, the fact that asymptotic physical states must be colorless, if we stick with the analogy between the three colors of the $SU(3)$ model and the usual fundamental colors. This immediately sparks several questions:

- Wouldn't it be possible to rewrite the Lagrangian in such a way that the fundamental degrees of freedom are the asymptotic states we observe?
- What is the mechanism that imposes color confinement?
- Is color confinement a fundamental law of nature? In other words, does it survive if we change the gauge group and the matter content? If confinement is fundamental, why doesn't it occur in the electroweak sector as well, which is described by a non-abelian gauge group $SU(2) \times U(1)$?

The answer to these questions boils down to understanding the various possible *phases* of Yang-Mills gauge theories, as a function of the gauge group and the matter content. In section 1.1, we review various concepts of gauge quantum field theories that are related to these questions. The word "phase" has to be understood from a statistical physicist point of view: the physics is supposed to be known at very small length scale where it is described by the microscopic Lagrangian, and the goal is to deduce from this the physics at large length scale, or low energy, where such phenomena as confinement occur. In some cases, it may be possible to write down an effective Lagrangian at low energies involving weakly coupled fields that may be different from the fields in the microscopic description.

The notion behind the first question above is the concept of duality, that will be a guiding principle in the exploration we are about to embark on. Modularity is an extension of this concept which intertwines in intricate ways dualities between theories and and symmetries of a given theory. Typically, it requires going beyond perturbation theory and involves characteristically non-perturbative objects such as monopoles and instantons, that also play a crucial role in our current understanding of confinement. As for the third set of questions, the richness of the answer lies in the fact that not only does the phase depend on the theory (its symmetries and matter content), but even a single theory can have different *vacua* which are in different phases. The vacuum in a classical theory is the state in the Hilbert space of minimal energy; because of a symmetry, it can have some degeneracy. There can also be local minima in the energy landscape, that can be interpreted as false-vacua, and that will eventually decay by tunnelling into a deeper minimum.

Although physicists have acquired during the last decades a satisfactory qualitative understanding of part of the answers – including resorting to lattice gauge theories –, it remains hard to obtain exact analytical results without the help of supersymmetry. Some salient successes of supersymmetric Yang-Mills theories are described in section 1.2. One important aspect of supersymmetry, which is related to the discussion in the previous paragraph, is that vacua are typically degenerate with zero energy, and therefore distinct phases can be observed in two true vacua of the same theory. As a consequence, understanding the (low-energy) physics of a given supersymmetric QFT can be decomposed in two stages:

1. Find all vacua of the theory;

2. In each vacuum, characterize the phase.

It appears that thanks to progress made in the last few decades, the second step is now a fairly easy one with the help of supersymmetry, as we will describe in this presentation chapter. The first step may be a tougher nut to crack, depending on which theory one chooses to study. This thesis is devoted to this step in the case of the $\mathcal{N} = 1^*$ theory with arbitrary gauge group. The last section of this introduction is an overview of this analysis.

1.1 Yang-Mills Gauge Theories in Four Dimensions

1.1.1 Classical Electric-Magnetic Duality

We begin with a quick description of electric-magnetic duality, which is a good starting point as it is the pretext for introducing non-perturbative objects as magnetic monopoles, and it is the inspiration of the Montonen-Olive duality which can be seen as its non-abelian version. Let us then take the classical theory of electromagnetism, which is the pure $U(1)$ theory. Its Lagrangian contains only the kinetic term for the gauge field $A = A_\mu \mathrm{d}x^\mu$, which involves the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\mathcal{L}_{\text{pure } QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \,. \tag{1.1}$$

We use here the somewhat unconventional normalization that is more convenient for generalizations later on. We now reproduce Maxwell's equations in the vacuum, which follow from this Lagrangian, and write them in the familiar non-relativistic fashion:

$$\vec{\nabla} \cdot \vec{E} = 0 \tag{1.2}$$
$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{0} \tag{1.3}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{1.4}$$
$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = \vec{0} \,. \tag{1.5}$$

The first two lines (1.2) and (1.3) are the equations of motion for the gauge field, while (1.4) and (1.5) are constraint equations that are rooted in the microscopic description chosen here, and based on a globally defined gauge field. A striking feature of these equations is that they are invariant under the *duality* transformation

$$\vec{E} \to \vec{B} \tag{1.6}$$
$$-\vec{B} \to \vec{E} \,. \tag{1.7}$$

Had we written the equations in the form $\mathrm{d}\tilde{F} = \mathrm{d}F = 0$, the symmetry would have been even more evident, and reads:

$$F \leftrightarrow \tilde{F} \,. \tag{1.8}$$

This symmetry is immediately spoiled if one adds electric sources (by which we mean and electric charge density and an electric current), and can be restored only upon addition of magnetic matter (Dirac magnetic monopoles and magnetic currents), with the postulate that the symmetry exchanges these two kinds of sources. Of course, in a magnetic monopole configuration, $\vec{\nabla} \cdot \vec{B} \neq 0$ and it is impossible to even define the field A_μ. However, suppose that we include only magnetic sources and no electric sources. Then the two equations (1.2) and (1.3) can be solved in terms of a new vector that we can call A_μ^D and that completes the duality description:

$$\boxed{\begin{array}{c} A_\mu \\ \vec{B} \;=\; \vec{\nabla} \times \vec{A} \\ \vec{E} \;=\; -\partial_t \vec{A} - \vec{\nabla} A_0 \end{array}} \longleftrightarrow \boxed{\begin{array}{c} A_\mu^D \\ \vec{E} \;=\; -\vec{\nabla} \times \vec{A}^D \\ \vec{B} \;=\; -\partial_t \vec{A}^D - \vec{\nabla} A_0^D \end{array}} \tag{1.9}$$

Up to now, we have discussed only classical physics. Although magnetic monopoles have not yet been observed in nature, the equations make perfect sense. But in the quantum theory, the fundamental degree of freedom of quantum electrodynamics is the vector boson A_μ. If there are both electric and magnetic monopoles in the theory, then a global gauge field can not be found,[1] and topological constraints emerge. In the simple playground of $U(1)$ theory, the electric charge[2] e and the magnetic charge $\tilde{e}$ are related by the Dirac quantization condition

$$e\tilde{e} = 2\pi \,, \tag{1.10}$$

[1]Note however that with matter in the theory, one can construct smooth gauge configurations that have magnetic charges, as for instance the 't Hooft-Polyakov monopole mentioned in the next section.

[2]We denote the abelian coupling constant e. In the following we will be mostly interested in non-abelian gauge theory, and the coupling will be denoted g_{YM}. There is no fundamental difference between e and g_{YM} but just a matter of convention.

where we have set $\hbar = 1$. In other words, for any pair of an electric object and a magnetic object, the product of the electric charge and the magnetic charge is an integer times 2π. This implies the quantization of (say) electric charges, provided at least one magnetic monopole exists. If instead of pure electric and magnetic objects we consider *dyons* with charges (n_e, n_m) and (n'_e, n'_m), then the Dirac quantization condition is that $n_e n'_m - n_m n'_e$ be an integer multiple of 2π.

Another crucial consequence of this equation is that if the electric coupling constant e is large, then its magnetic dual will be small. The electric-magnetic duality belongs to the family of strong-weak dualities. Nevertheless, it should be stressed that the electric-magnetic duality is an equivalence between free $U(1)$ theories (or more generally free $U(1)^r$ theories). This has to be contrasted with S-dualities (generalizing the Montonen-Olive discussed below) which identify *interacting* theories with a priori distinct couplings.

1.1.2 Non-perturbative Physics

The primitive language of quantum field theory is perturbation theory around a vacuum configuration, on which excitations are seen as interacting particles that can be computed using Feynman diagrams. A given correlation function can be decomposed as an infinite sum of such diagrams. This usually leads to a divergent series, but if the coupling constant g_{YM}, which is the perturbation parameter, is small, one can obtain a good approximation by truncating the summation after a few orders. The result is presented as a series expansion in increasing non-negative powers of g_{YM}. However, not all relevant quantities can be expressed as such series, and they indicate that going beyond perturbation theory is required to capture them. Magnetic monopoles belong to this category of objects, as we review now.

The construction of magnetic monopoles can be illustrated in a simple way in the Georgi-Glashow model with gauge group $SU(2)$, and a Higgs field in the adjoint representation, whose Lagrangian is

$$\mathcal{L}_{GG} = -\frac{1}{4g_{YM}^2} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - \lambda(\phi^a\phi^a - v^2)^2 \,. \tag{1.11}$$

See the appendix for the conventions used. The terms that contribute to the energy of a configuration are $F^a_{\mu\nu} F^{a,\mu\nu}$, $D_\mu \phi^a D^\mu \phi^a$ and $\lambda(\phi^a\phi^a - v^2)^2$. Therefore a minimal energy configuration is obtained with $\langle A^a_\mu \rangle = 0$ and $\langle \phi^a \rangle$ constant such that

$$\langle \phi^a \phi^a \rangle = v^2 \,. \tag{1.12}$$

In this case the gauge symmetry is broken $SU(2) \to U(1)$. This is called the *Higgs vacuum*. In this vacuum the abelian residual field can be used to define a magnetic charge, which is zero when $\langle A^a_\mu \rangle = 0$.

There are other stable configurations, that are local minima of the energy landscape. This can be understood from equation (1.12) which has to be satisfied on the sphere S^2_∞ at spatial infinity, and which is the equation of another sphere S^2_{Higgs}. We conclude that any finite-energy configuration is associated to a map $S^2_\infty \to S^2_{\text{Higgs}}$. For the Higgs vacuum considered above, this map sends S^2_∞ to a point $\langle \phi^a \rangle$. However other maps are possible, and it would cost an infinite amount of energy to evolve from one map to another: each class of topologically equivalent maps offers a candidate for a stable vacuum. These are classified by the second homotopy group $\pi_2\left(S^2_{\text{Higgs}}\right) = \mathbb{Z}$, and the corresponding integer is the magnetic charge. Such solutions can be found explicitly [7, 8] and are known as 't Hooft-Polyakov monopoles. They are smooth everywhere, and do not require the introduction of point-like magnetic charges as in the previous section. In the case of a more general gauge symmetry breaking $G \to H$, the classifying group is $\pi_2(G/H)$. For a thorough discussion the reader can consult [42, 43].

The magnetic monopoles are non-perturbative objects. This can be seen from their (classical) mass, which is given by

$$m_{\text{monopole}} = 4\pi |n_m| \frac{v}{g_{YM}} \tag{1.13}$$

where n_m is the magnetic charge. This mass diverges when the coupling constant g_{YM} becomes small, which prevents any form of series expansion using non-negative powers of g_{YM}. Note however that it is possible to use perturbation theory around the pseudo-vacuum corresponding to this topologically stable solution, for instance to find corrections to the mass formula (1.13).

Another aspect of non-perturbative Yang-Mills theories is the θ-term

$$\mathcal{L}_\theta = \frac{\theta_{YM}}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a,\mu\nu} = \frac{\theta_{YM}}{8\pi^2} \text{Tr}\,(F \wedge F) \tag{1.14}$$

that can be added to the Lagrangian. Although it violates the CP-symmetry (CP standing for Charge Parity), the renormalizability of the theory is preserved, and this term is invisible in perturbation theory. It has however a strong effect on the mass of a magnetic monopole: in the presence of (1.14), the mass formula (1.13) receives an additive correction proportional to θ^2_{YM}.

1.1.3 Phases of Yang-Mills Theories

The Lagrangian of a pure gauge theory is given by

$$\mathcal{L}_{YM} = -\frac{1}{4g^2_{YM}} F^a_{\mu\nu} F^{a,\mu\nu} = -\frac{1}{2g^2_{YM}} \text{Tr}\left(F^2\right) . \tag{1.15}$$

Let us consider a theory quite similar to the standard model, but simpler than that:

- We take a gauge group G, which may be abelian or not, but we assume that it is simple (and connected). We put the corresponding gauge boson in the theory. The corresponding Lie algebra is denoted $\mathfrak{g}$. When $\mathfrak{g} = \mathfrak{su}(N)$, we say that N is the number of *colors* of the theory.
- Then we add a certain number of standard matter fields ψ_k, that we call generically *quarks*. By this we mean spin $\frac{1}{2}$ fields that transform in the fundamental representation of the gauge group. We put N_f of them in our theory, so the index k runs from 1 to N_f if $N_f \neq 0$.
- We might also want to add (complex) scalar fields ϕ_j in some representation $\mathbf{R}$ of the gauge group. We put N_s such scalars, so we let the index $j = 1, \ldots, N_s$.

The Lagrangian, obtained by combining the various parts (A.1), (A.9) and (A.10) has the form

$$\mathcal{L} = -\frac{1}{4g^2_{YM}} F^a_{\mu\nu} F^{a,\mu\nu} + \nabla_\mu \bar{\phi}_j \nabla^\mu \phi_j - V(\phi) + \left[i\bar{\psi}_k \not{\nabla} \psi_k - \sum_k m_k \bar{\psi}_k \psi_k \right] . \tag{1.16}$$

We have assumed that there are no interactions between the quarks and the scalar fields. The Standard Model of particle physics is far more intricate that this simple model, as the gauge group is not simple, the fermions transform in various representations, there are many interactions between the different types of fermions and the Higgs field (the Yukawa couplings), etc. However, this simple model reproduces most of the interesting physical effects of the Standard Model, and will play the role of a toy model for the rest of this thesis.[3] Now that the microscopic scene is set, we can ask the main question: what does the physics look like at large distances? The answer to this question is quite rich, and it depends in an intricate way on the (few) parameters of the theory : the number of colors N, the number of fermions N_f, and the scalar potential $V(\phi)$. We will now give a brief glimpse of the different possible answers. We recall that the running of the coupling constant is known in general at one loop,

$$\beta_{\text{1-loop}} = \left[\frac{\mathrm{d}g_{YM}}{\mathrm{d}\log(\mu/\Lambda)}\right]_{\text{1-loop}} = -\frac{g_{YM}^3}{(4\pi)^2}\left[\frac{11}{3}T(\text{adjoint}) - \frac{2}{3}T(\text{fermions}) - \frac{1}{3}T(\text{scalars})\right] . \quad (1.17)$$

This is equation (A.11) in the appendix where the various notations are defined.

It is well known that Yang-Mills gauge theories are more involved than abelian ones: the non-abelianity of the group translates into the additional term $[A, A]$ in the definition of the field strength F (see equation (A.3)), which yields complicated A^3 and A^4 interactions. Let us then first assume that the gauge group is $G = U(1)$. The low-energy physics then depends on the matter content. If there is no matter at all, the theory is free because there is no self-interaction in the gauge field. Let us then consider two of the simplest non trivial cases in turn.

(C) Suppose we put only one *massive* Dirac fermion of mass m, that we can call the electron. This is precisely the theory of Quantum ElectroDynamics (QED). The β-function at one loop is obtained from (1.17) by setting $T(\text{fermions}) = 2$ for the two Weyl fermions that make up one Dirac fermion. The result is $\beta_{\text{1-loop}} = \frac{e^3}{12\pi^2} > 0$, where e is the electric coupling constant. As a consequence, when the energy scale μ decreases, the coupling constant decreases as well,

$$e^2 \sim -\frac{1}{\log(\mu/\Lambda)} . \quad (1.18)$$

When μ becomes of the order of the mass m of the electron, the latter decouples and the running of the coupling constant stops at some infrared value $(e^2)_0$.

(H) Now let's consider putting only a massive complex scalar field ϕ with a potential $V(\phi)$ which is a polynomial in $|\phi|^2$, so that gauge invariance is preserved, with positive leading coefficient, so that energy is bounded from below. It is clear that the point $\phi = 0$ is an extremum of the potential, but it may be a local maximum or a local minimum depending on the coefficients of the polynomial $V(\phi)$. The interesting situation is when there is a minimum at a nonzero point $|\phi_0|^2 \neq 0$. Then the $U(1)$ gauge symmetry that acts by $\phi \to e^{i\theta}\phi$ is spontaneously broken once such a ϕ_0 is chosen. This is the Brout-Englert-Higgs mechanism: the gauge field becomes massive and the complex Higgs field is reduced to a massive real scalar. Of course a gauge symmetry is not really a symmetry, it is rather

[3] This is not completely true. The $\mathcal{N} = 1^*$ theories that we will study, to be introduced later on, involve a gauge field and three complex massive scalars as described here. The fermions, which are related to the bosons by supersymmetry, will then also have mass terms with the same mass. Moreover, because the theory inherits interactions from its $\mathcal{N} = 4$ parent, there are interactions of two different types in the Lagrangian, namely terms of the type ϕ^4 that are described here by the potential $V(\phi)$, but also interactions terms of the type $\bar{\phi}\psi^2$ that are absent from the present description.

a redundancy in the description, which is why the would-be Goldstone boson associated to the spontaneous breaking of the $U(1)$ symmetry is not present there.

Static test electrons in the first case (C) experience a Coulomb potential as a function of the distance r between them,

$$V_{\text{Coulomb}} \sim \frac{(e^2)_0}{r} \,. \tag{1.19}$$

This is why this is called the *Coulomb phase*. The long range potential with the characteristic dependence r^{-1} can be traced back to the presence of massless gauge fields. On the other hand, the phase (H) where the Higgs mechanism occurs is not surprisingly called the *Higgs phase*. In this phase, at large distances the potential felt by test charges is

$$V_{\text{Higgs}} \sim \frac{\exp(-m_{\text{photon}} r)}{r} \,, \tag{1.20}$$

where m_{photon} is the mass given to the gauge boson by the Higgs mechanism. We should also mention the existence of a third phase corresponding to a matter content which reduces to massless electrons. In this case, nothing can stop the running of the coupling constant (1.18) in the IR, and the potential (1.19) becomes

$$V_{\text{Free-Electric}} \sim \frac{1}{r \log(\Lambda r)} \,. \tag{1.21}$$

Now let us turn to non-abelian gauge theories. The β-function (A.11) can receive a strong negative contribution from the gauge boson, which is dominant if the number of matter fields is low enough. This means that the coupling constant grows in the IR, and the theory enters the *confining* phase. The spectrum is very different from what one could naively expect from the Lagrangian. So in addition to the phases already present in abelian gauge theories, we have to add the confining phase, which is characterized by a potential between test charges

$$V_{\text{Confining}} \sim r \,. \tag{1.22}$$

This can be pictured by a string between the two probe quarks, where the string consists of the electric flux-tube.

1.1.4 Line Operators

A convenient criterion that can be used to know whether a given theory is confining uses *Wilson loops*. To any closed path $\mathscr{C}$ in spacetime we can associate the operator[4]

$$W(\mathscr{C}) = \operatorname{tr}\left[\mathcal{P}\exp\left(i\oint_{\mathscr{C}} A\right)\right] \tag{1.23}$$

where $\mathcal{P}\exp$ denotes the path ordered exponential. This path ordered exponential is not gauge invariant in itself (see equation (A.7)), but the operator $W(\mathscr{C})$ is thanks to the trace. Moreover, it is a very useful operator to compute the vacuum expectation value of, because it can diagnose the forces between probe quarks. In general, the behaviour of $\langle W(\mathscr{C})\rangle$ when the contour $\mathscr{C}$ becomes large is either $\exp(-\kappa\mathscr{A})$ or $\exp(-\kappa\mathscr{P})$, where $\mathscr{A}$ and $\mathscr{P}$ are the area enclosed by and

[4] Note that there is no coupling constant in this Wilson line: this is a consequence of our choice of normalization for the gauge field.

the perimeter of $\mathscr{C}$. Accordingly, we say that the Wilson line obeys the area or perimeter law. As a general rule of thumb, an area law signals confinement.

However, we hasten to add that this criterion should not be taken too seriously. Even in one of the simplest models which exhibits both the Higgs mechanism and confinement, namely the abelian lattice Higgs model where the Higgs[5] has $U(1)$-charge 1, there is no clear distinction between the two phases. In this model [9] there is a *total screening* phase where the Wilson line always scale with perimeter law even in the would-be confining regime. This phenomenon, dubbed *complementarity*, is generic in any theory, where there are scalars in the fundamental representation of the gauge group. Such a theory can be called a *screening* theory, because the colored scalars can be used to screen any given color charge, with the effect of diminishing its strength. This manifests itself in the behaviour of the Wilson line, which always obeys a perimeter law in such a theory, even if there is confinement in the sense that asymptotic states are always color singlets. A way out would be to consider probe charges that can not be screened in the Wilson loop: it is possible to define such a loop for any representation $\mathbf{R}$ of the gauge group. Considering whether these generalized Wilson loops follow an area or a perimeter law is then a good order parameter for determining the phase of the theory. We will come back to this issue later on, when dealing with supersymmetric theories in section 1.2.3.

1.1.5 The Global Structure of the Gauge Group

Until now, we have not taken too much care about the group G, and we have focused on its Lie algebra $\mathfrak{g}$. The possible choices of gauge groups with a given Lie algebra $\mathfrak{g}$ are given by subgroups of the center $Z(\tilde{G})$, where $\tilde{G}$ is the unique simply connected group with Lie algebra $\mathfrak{g}$. The possible groups are then $\tilde{G}/H$ where H is a subgroup of $Z(\tilde{G})$. It seems that physical quantities depend only on the algebraic structure and not on topological aspects of the gauge group itself. This is true indeed for a large class of observable that includes all correlators of *local* operators when the theory is formulated on $\mathbb{R}^4$. However, as has been studied in detail recently by Aharony, Seiberg and Tachikawa [10], these global properties have an impact on at least three aspects of four-dimensional gauge theories that will be of importance in this work:

- The phase structure of the theory on $\mathbb{R}^4$
- The dynamics of local operators when the theory is compactified on $\mathbb{R}^3 \times S^1$.
- The Witten index of the compactified theory.

The appropriate tool that can be used to probe and characterize the physical aspects related to the global structure of the gauge group are correlators of lines, which generalizes the quantity $\langle W(\mathscr{C})\rangle$ that probes confinement. But what are the line operators in a given theory ?

Beyond the Wilson line defined above, it is natural to introduce the *'t Hooft line* [11] where the gauge field A_μ is replaced by its dual A_μ^D. We could as well say that the 't Hooft line is a magnetic Wilson line, and it is natural to expect that all properties of the Wilson line with respect to the gauge group G translate into the same properties of the 't Hooft line with respect to the Langlands dual $G^\vee$. For instance, Wilson lines are labeled by the weight lattice P while 't Hooft lines are labeled by the co-weight lattice $P^\vee$ – the symbol $\vee$ on top of a lattice name is

[5] This statement about the charge is equivalent to saying that the Higgs transforms in the fundamental representation of $U(1)$.

used to denote the dual of this lattice. Our notations and further information about Lie algebra related lattices can be found in section B.2 in the appendix and in particular in equation (B.20). More generally, there are dyonic lines that are labeled by pairs in

$$\frac{P \times P^{\vee}}{\text{Weyl Group}}, \tag{1.24}$$

as explained in [12]. In fact, lines come into full families generated by the lattice $Q \times Q^{\vee}$, where Q is the root lattice, which can therefore be labeled by pairs in $Z(\tilde{G})^2$. The possible charges carried by these lines are constrained by the generalization of the Dirac quantization condition (1.10) whose expression is given in section 6.3 of [10]. To sum up, we see that determining the gauge sector of the theory, including line operators[6], is a three step program: first, choose a gauge algebra $\mathfrak{g}$, then choose a subgroup H of the center of the covering group, and finally choose a full spectrum of lines satisfying the Dirac quantization condition, the different possible choices being labeled by an index i. The resulting theory is denoted $(\tilde{G}/H)_i$.

From the fact that 't Hooft lines are the magnetic duals of Wilson lines, we also deduce that an area law for the 't Hooft line signals that the theory is in the Higgs phase, which is the magnetic dual of the confining phase.

As an example, we describe the phase structure of pure $\mathfrak{su}(N)$ gauge theories along the lines of section 1.1.4. In this case we have $\tilde{G} = SU(N)$ and $Z(\tilde{G}) = \mathbb{Z}_N$. The subgroups of $\mathbb{Z}_N$ are the groups $\mathbb{Z}_d$ for d a divisor of N. The Dirac quantization condition states that any two classes of lines labeled by (n, m) and $(n', m') \in \mathbb{Z}_N \times \mathbb{Z}_N$ must satisfy $nm' - mn' = 0$ modulo N. Then it is not hard to see that for a given gauge group $SU(N)/\mathbb{Z}_d$ there are precisely d possible choices of set of lines. As a consequence, the number of physically distinct $\mathfrak{su}(N)$ gauge theories is the sum of divisors of N. One would now like to infer the large distance physics in each of these vacua using the lines. This is a delicate issue, and we will bypass it by adding a new symmetry in the theory. Namely, we add a very massive fermionic field in the adjoint representation of the gauge group – this can not change the low-energy structure of the theory – and then decrease the mass until it reaches zero. It can not be excluded that there be a phase transition when this mass is decreased, and this is part of the reasons why it is so difficult to analyze non-supersymmetric theories. However, when the mass of the fermion field vanished, the theory becomes supersymmetric – the fermion is the gaugino, and a wealth of new techniques become available to probe the vacuum structure, as we explain in the next section.

1.2 Supersymmetric Gauge Theories and Modularity

As we have seen, it's an important task to determine the vacuum structure of four-dimensional gauge theories. It would be helpful to have an organizing principle to put some structure in the set of vacua, be it discrete or continuous. In the theory we will finally be interested in, such a principle indeed exists. Its inspiration can be traced to the electric-magnetic duality of classical electromagnetism that we mentioned in section 1.1.1.

[6] We don't consider higher-dimensional operators here, such as surface operators. For a discussion of the surface defects group in six-dimensional theories, see for instance [44].

1.2.1 The maximally supersymmetric theory

The first occurrence of S-duality in physics can be traced to the works about non-supersymmetric Yang-Mills theories in the 70's. In [13], Montonen and Olive proposed a symmetry exchanging strong and weak coupling, electric and magnetic fields, and the gauge group G with its Langlands dual $G^\vee$. The argument is based on the Georgi-Glashow Lagrangian (A.12) with gauge group $SO(3)$. In a ground state, this gauge group is broken to a $U(1)$, and the particle content is made of a massless photon, a massive Higgs scalar field and two massive charged gauge bosons. Besides this perturbative spectrum, there are also the magnetic monopoles. This symmetry is entirely non-perturbative, and is therefore difficult, if not impossible, to test using classical methods. A possible semiclassical approach, taken by Montonen and Olive, is to use the Bogomol'nyi-Prasad-Sommerfield (BPS) limit (which can be thought of as the limit $\lambda \to 0$ in the Lagrangian), in which the BPS bound on the mass of a dyon can be saturated. The introduction of supersymmetry provides the necessary non-perturbative control on the quantum field theory that allows very non-trivial tests of the duality.

In fact, the simplest version of the symmetry proposed by Montonen and Olive is true only when (a lot of) supersymmetry is present. The starting point was the work of Witten and Olive [14], where it was shown that when monopoles or other solitons are included in supersymmetric theories, central charges related to the topological numbers of the solitons are generated. In the four-dimensional framework, the electric and magnetic charges will appear as central charges, and the mass m of any state satisfies the BPS lower bound coming from the electric and magnetic charges. In the $\mathcal{N}=2$ supersymmetric version of the Georgi-Glashow model studied by Montonen and Olive, a remarkable feature is that the bound is saturated at the classical level for all particles, including the monopoles and the dyons. Then because the number of states in supermultiplets can not jump as Planck's constant is turned on, Witten and Olive conclude that the bound must remain saturated in the full quantum theory, thus resulting in (at least part of) the quantum spectrum. Osborn [15] then pushed on this work to $\mathcal{N}=4$ Yang-Mills theory. Imposing a spontaneous symmetry breaking, he generates the massive $\mathcal{N}=2$ supermultiplet[7] that saturates a BPS bound, and shows that a magnetic monopole corresponds to the same multiplet with the same mass. In the $\mathcal{N}=4$ then, this matching of multiplet content is a strong indication that the duality between the usual electric formulation and the dual magnetic one holds.

More precisely, the conjecture which is strongly believed to be true [45, 46] is that the $\mathcal{N}=4$ super Yang-Mills theory with a simple gauge Lie algebra $\mathfrak{g}$ and complexified coupling constant τ is isomorphic to a similar theory with gauge algebra $\mathfrak{g}^\vee$ and coupling

$$\tau^\vee = -\frac{1}{\nu\tau}\,, \tag{1.25}$$

and the self-duality group of the original theory is[8] $\Gamma_0(\nu)$ where ν is the ratio of the squared-lengths of long and short roots in $\mathfrak{g}$. This time, the duality provides an equivalence of full *quantum* theories, and an intuitive observation indicates that this can hold only in theories where

[7] It is made of five scalars, four spin $\frac{1}{2}$ and one spin 1, for a total of sixteen states.

[8] There is a small subtlety here because of the fact that modular groups in general can be seen to act either on the gauge coupling τ or on electric and magnetic objects as in (1.6). In the first case, S^2 is trivial while in the second case it changes the sign of electric and magnetic charges. If this effect is to be included, the duality group is really $(\Gamma_0(\nu) \rtimes \mathbb{Z}_4)/\mathbb{Z}_2$, see [46].

the coupling constant g_{YM} does not run since otherwise the relation 1.25 doesn't make sense. A geometric interpretation of the duality can be found [47], in which the four-dimensional $\mathcal{N}=4$ theories are realized as low-energy limits of type II string theories, and where the Montonen-Olive duality is seen as T-duality on a 2-torus.

Of course, as explained in section 1.1.4, the gauge algebra is not enough to specify completely the theory, and one should also provide an appropriate set of lines. Then even when $\mathfrak{g}^\vee = \mathfrak{g}$, various theories are not invariant under S-duality (nor are they invariant under the T transformation). As an example, we show how the $\mathfrak{su}(4)$ theories transform in figure 1.1 taken from [10].

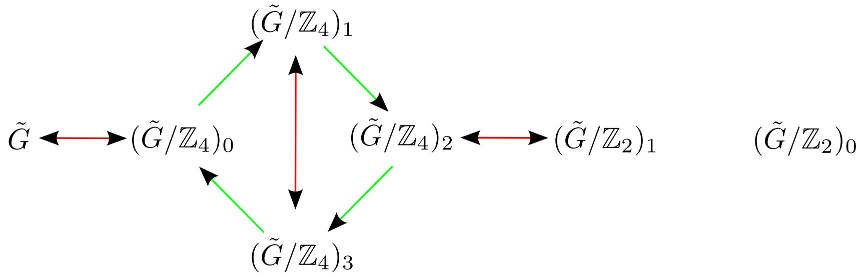


Figure 1.1: This diagram represents how the seven theories with gauge algebra $\mathfrak{su}(4)$ are exchanged by modular symmetries. In red we exhibit the action of S-duality, in green, T-duality. When no arrow is present, the theory is invariant. The notation used to define the theories is as in [10], and $\tilde{G} = SU(4)$. This figure should be compared with figure 4.1 in chapter 4.

1.2.2 From S-duality to modular invariance

Starting with lattice gauge models with discrete abelian gauge groups [16, 17], the effect of the θ_{YM} angle was incorporated in the story. According to the Witten effect [18], the inclusion of such an angle in the Lagrangian gives dyons an electric charge that can be different from an integer multiple of the elementary charge. When θ_{YM} is varied continuously from 0 to 2π, the spectrum of dyons is reshuffled but remains invariant. In the context of supersymmetric gauge theories, it is natural to assemble the angle θ_{YM} and the gauge coupling g_{YM} is the complex combination

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta_{YM}}{2\pi}\,, \tag{1.26}$$

that can be interpreted as the scalar component of a superfield, so that the super Yang-Mills Lagrangian (A.14) can be seen as a superpotential term. There is one such coupling constant for each simple factor in the gauge group. When the gauge group itself is simple, the electric-magnetic duality and the shift $\theta_{YM} \to \theta_{YM} + 2\pi$ generate a $SL(2,\mathbb{Z})$ modular group of dualities. If there are k simple factors, the modular group becomes $Sp(2k,\mathbb{Z})$, as first seen in abelian lattice theories [48].

However, the matching in multiplet structure found by Osborn ceases to exist in $\mathcal{N}=2$ theories, which rules out a simple realization of the simple electric-magnetic duality in those theories, and by the same token, in all theories with less supersymmetry. Nevertheless, a different version of these ideas can be brought to life in $\mathcal{N}=2$ theories, as first demonstrated by Seiberg and Witten.

Let us focus on the case of gauge group $SU(2)$, which is broken to $U(1)$ on a generic point of the Coulomb branch, so that the expected modular group is $SL(2,\mathbb{Z})$. In the pure $SU(2)$ theory [19] with monopoles, the scalar parts of the vector multiplet and its dual (a_D, a) can be acted on by this modular group. The action of the generator

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.27}$$

can be identified with the electric-magnetic duality, and exchanges two equivalent descriptions of the same theory. On the other hand, the generator

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{1.28}$$

is the symmetry $\theta_{YM} \to \theta_{YM} + 2\pi$ that we have described previously. At the classical level, we have $\tau_{\mathrm{cl}} = \frac{a_D}{a}$ and therefore the modular group acts in the usual way on τ_{cl}. This remains true at the quantum level in the $\mathcal{N} = 4$ theory, but in a generic $\mathcal{N} = 2$ theory the running of τ at one loop spoils modularity.[9]

In theories with even less supersymmetry, duality is more remote. The $\mathcal{N} = 1$ quantum chromodynamics, with the appropriate number of quarks, is related by Seiberg duality [20, 21] to a different theory which flows to the same IR physics. The original version relates a $SU(N)$ theory with $\frac{3}{2}N < N_f < 3N$ quarks in the fundamental representation (and the same number of anti-quarks) to the $SU(N_f - N)$ theory with N_f quarks and an additional gauge singlet in the bifundamental representation of the flavor group. This is another version of an electric-magnetic, strong-weak duality: the electric particles (quarks and gluons) are mapped to their magnetic duals (quarks of the dual magnetic gauge group and non-abelian 't Hooft-Polyakov monopoles). However, it relates very different theories, with different gauge groups and matter contents, and the duality holds only in the low-energy limit. We will come back to these ideas in the sequel.

The recent work of Gaiotto has shed new light on $\mathcal{N} = 2$ dualities [22]. As an example, consider the $SU(2)$ theory with four massive quarks. The four quarks can be repacked into two trifundamental fields of $SU(2)^3$, and the form of the superpotential indicates that they should be connected by one leg corresponding to the gauge group, giving finally a four-punctured sphere as *ultraviolet* curve C. Then the Seiberg-Witten curve is given by the equation $\lambda^2 - \varphi(z) = 0$ where the asymptotic properties of $\varphi(z)$ are determined by the superpotential. The $Spin(8) \ltimes SL(2,\mathbb{Z})$ modular invariance now has a geometrical interpretation as the different ways a sphere with four punctures can degenerate into two spheres with three punctures each (there are $\frac{1}{2}\binom{4}{2} = 3$ ways to do that, that correspond to the vector and the two spinor representations of $SO(8)$).

Another example which is more relevant for the present work is the $\mathcal{N} = 2^*$ theory, still with gauge group $SU(2)$. The adjoint hypermultiplet can be seen as part of a trifundamental where now two indices have to be contracted. This construction also spits a singlet, but it decouples and can be ignored. Geometrically, the three-punctured sphere becomes a one-punctured torus in the process. Again, the Seiberg-Witten curve is given by $\lambda^2 - \varphi(z) = 0$, where now $\varphi(z)$ is easily determined, since it only has one singularity controlled by the mass m of the adjoint hyper. We obtain readily [23]

$$\varphi(z) = (m^2 \wp(z;\tau) + u)\mathrm{d}z^2\,, \tag{1.29}$$

[9] The theory with four massive hypermultiplets, which is conformal invariant if there are no bare masses, also possesses a manifest modular invariance [49], under the guise of an extended $Spin(8) \ltimes SL(2,\mathbb{Z})$.

where $u = \operatorname{tr} \phi^2$ as usual. This provides a geometrical interpretation of the modular group $SL(2, \mathbb{Z})$ as the mapping class group of the ultraviolet torus.

The $\mathcal{N} = 4$ theory is a particular case of this situation, and one can understand the diagram of figure 1.1 from this point of view. Indeed, there is a nice interpretation of the discrete data needed to define properly the so-called class S four-dimensional theories, namely the spectrum of line operators, in terms of six-dimensional geometry [24] as a maximal isotropic sublattice of $H^1(C, Z(\tilde{G}))$, where C is the ultraviolet curve. For $\mathcal{N} = 4$ theories, we just argued that C is a torus, and $H^1(C, Z(\tilde{G}))$ is isomorphic to $Z(\tilde{G})^2$. This explains why the same duality diagram will appear later in our analysis of the extrema of the Calogero-Moser potential and the vacua of the $\mathcal{N} = 1^*$ theories – compare with figures 4.1 and 5.1.

1.2.3 The vacuum structure

The space of vacua is a basic tool in the analysis of supersymmetric gauge theories based on holomorphy. The classification of massless and massive vacua, and the analysis of their symmetry and duality properties are fundamental features of a theory. Let us recall that using arguments essentially based on holomorphy, one can prove three important non-renormalization theorems :

- In $\mathcal{N} = 1$ theories, the superpotential is not renormalized in perturbation theory. However it can be affected by instantonic effects.
- In $\mathcal{N} = 2$ theories, the β-function at one loop (A.25) is exact.[10]
- The $\mathcal{N} = 4$ theories are finite.

We have to distinguish between the classical moduli space and the quantum one. Generically, a degeneracy between vacua in a classical theory that is not due to a symmetry (it is an accidental degeneracy) is lifted by quantum effects. However, with supersymmetry the generic situation is that the degeneracy of classical vacuum states is not lifted in perturbation theory, although non-perturbative effects can generate an additional superpotential and destroy classical vacua. The effective superpotential on the moduli space contains all the information about the low energy limit, with its particle content and its phase structure. Note also that in general a moduli space (at least classically) is parametrized by the gauge invariant monomials in the fields [52].

The classical moduli space is singular at points where there are additional massless fields in comparison with a generic point. A natural question is the fate of these singularities in the quantum moduli space [25], and the answer is that pretty much anything can happen: a singularity may disappear, or undergo no qualitative modification, or remain but change in nature. We will see an example in the $\mathcal{N} = 2$ pure $SU(2)$ theory. Before that, we examine the maximally supersymmetric theory.

$\mathcal{N} = 4$

In $\mathcal{N} = 1$ language, the $\mathcal{N} = 4$ super Yang-Mills theory has six real scalars ϕ^i, $i = 1, \ldots, 6$ in the adjoint representation as bosonic matter content, in addition to the vector multiplet. The

[10]The analog result for $\mathcal{N} = 1$ theories is the Novikov-Shifman-Vainshtein-Zakharov formula [50, 51], which involves the anomalous dimensions of the matter fields.

superpotential gives a term

$$\frac{1}{8g_{YM}^2}\sum_{i,j}[\phi^i,\phi^j]^2 \tag{1.30}$$

which vanishes if the ϕ^i mutually commute. They can be thought of as taking value in a Cartan subalgebra of the gauge algebra, they generically break the gauge group to the abelian $U(1)^r$, where r is the rank, and hence we call this the Coulomb phase. When all the expectation values are zero, we are in the superconformal phase. Intermediate phases with non-abelian residual gauge symmetry can be found.

Thus we have found the classical moduli space of vacua. It can be conveniently written

$$\mathcal{M}_{\mathcal{N}=4}=\frac{\mathbb{R}^{6r}}{\text{Weyl group}} \tag{1.31}$$

and it is singular at points of $\mathbb{R}^{6r}$ fixed by the Weyl group. These singularity signal that new degrees of freedom become massless, which is necessary to build the non-abelian gauge bosons that appear there. One can show that this classical moduli space is not quantum corrected [26], so that (1.31) remains true in the full quantum theory.

$\mathcal{N}=2$

Now that we have described the situation for the very constrained $\mathcal{N}=4$ theory, we move on to $\mathcal{N}=2$ where the physics is far richer. As an example of the mechanism evoked above, in the $\mathcal{N}=2$ pure $SU(2)$ theory the classical moduli space is $\mathbb{C}$ with a singularity at the origin where the non-abelian gauge symmetry is restored. This moduli space is *not* lifted in the quantum theory (this can be seen as a consequence of holomorphy), and in fact is a (rigid) special Kähler manifold because of the very structure of $\mathcal{N}=2$ supersymmetry.[11] However the unique singularity of the classical moduli space splits into two singularities corresponding to finite values of the parameter of the Coulomb branch.

The quantum moduli space of vacua for pure $SU(N)$ theories with $\mathcal{N}=2$ is obtained as a family of hyper-elliptic curves with $N-1$ parameters [56, 57, 58]. When $N_f \leq 2N$ fundamental quarks are added to these pure theories, the quantum moduli space in the Coulomb phase has been found in [59, 60], and this analysis has been extended to the other classical gauge groups [61, 62]. As for the Higgs branch, it can not receive quantum corrections and is its metric is determined by the classical equations of motion alone [63]. In all cases, the solution is found as the moduli space of families of hyper-elliptic curves, with meromorphic one-forms whose periods generate the spectrum of low-energy excitations. At a generic point on the Coulomb branch, the gauge group is broken to $U(1)^r$, and the abelian effective theory in invariant under a low-energy modular group, which is the generalization of the electric-magnetic duality discussed in section 1.1.1. Along singular submanifolds of the moduli space, more complicated phenomena occur, and circling around these singularities produces monodromies in this modular group. To be more precise, the scalar components a of the photon and a_D of its dual can be assembled in a

[11] In the $\mathcal{N}=2$ superspace formalism, the Lagrangian density is an analytic function of the superfield called the *prepotential*, from which we can deduce the low energy Kähler potential and complexified gauge couplings. The classical prepotential can be read directly from the tree-level Lagrangian of the theory, but it receives corrections at one loop [53] and from instantons [54, 55].

vector (a_D, a) which is a section of a $Sp(2r, \mathbb{Z})$ bundle over the quantum moduli space when the masses of the quarks vanish. When they don't, this structure is supplemented by constant shifts that are taken into account with a low-energy modular group equal to $Sp(2r, \mathbb{Z}) \ltimes \mathbb{Z}^{N_f}$.

$\mathcal{N} = 1$

The gauge theories with the minimal amount of supersymmetry, $\mathcal{N} = 1$, are less constrained, and present very rich physical properties. The constraints are essentially limited to the holomorphy principle, with no possible Lorentz-invariant central charge in the supersymmetry algebra being able to play its role in a BPS-like bound. Note however that it is possible to define *generalized central charges*, which are not proper central charges because they don't commute with the full super-Poincaré algebra (they are not Lorentz-invariant), but which allow for BPS objects, as we will see.

The basic example is the massive deformation of the $\mathcal{N} = 2$ theory considered in the previous paragraph. Classically, and in perturbation theory as well, the superpotential reduces to the mass term, and there is only one vacuum with a vanishing expectation value for the chiral multiplet. Taking into account non-perturbative effects, the Affleck-Dine-Seiberg superpotential is generated [27], inducing a repulsive effect in the strongly coupled region and splitting the single classical vacuum into two quantum vacua.

In the pure super Yang-Mills theory, the classical R-symmetry group $U(1)$ is broken to a discrete subgroup by instantons. This will be reviewed in section 2.2.3, as this plays a fundamental role in our calculations later on. The number of vacua is equal to the dual Coxeter number $h^\vee$ of the gauge algebra, in each vacuum the discrete $U(1)$ R-symmetry is further broken to $\mathbb{Z}_2$ by gaugino condensation $\langle\lambda\lambda\rangle$, and they all contribute $(-1)^F = +1$ to the Witten index, giving a total of $h^\vee$ for the Witten index. It is not possible to find a semi-classical configuration corresponding to these vacua, since the gaugino condensate would have a *fractional* instanton number. One strategy to obtain these results is to solve a theory with massive matter which is simpler to deal with, and then send the mass of the matter to large values, thereby reducing to the pure $\mathcal{N} = 1$ theory [64]. The normalization of $\langle\lambda\lambda\rangle$ was a long-standing problem, with contradicting computations, until the technology of multi-instanton calculus was firmly established [65].

The existence of isolated vacua implies the possibility of having domain walls between regions of spacetime in different vacuum configurations. In $\mathcal{N} = 1$ super Yang-Mills theories in 4 dimensions, the exact energy density of a domain wall can be computed in the strong coupling regime [28]. The reason why this computation is possible is because the wall is half-BPS, with a tensorial central charge [66] defined by the anticommutator $\{Q_\alpha, Q_\beta\} = -4\Sigma_{\alpha\beta}\mathcal{Z}$, where $\Sigma_{\alpha\beta}$ is the wall area tensor. In this case, assuming a few hypothesis, the wall tension T_{wall} is proportional to the modulus of the difference between the values of the superpotential in the two vacua the wall interpolates between,

$$T_{\text{wall}} = |\mathcal{Z}| \tag{1.32}$$

where

$$\mathcal{Z} = 2\Delta\mathcal{W}\,. \tag{1.33}$$

We now add matter to the pure $\mathcal{N} = 1$ considered up to now in this section. The one-loop

β-function (A.23), which reads

$$\left[\frac{\mathrm{d}g_{YM}}{\mathrm{d}\log\mu}\right]_{1\text{ loop}} = -\frac{g_{YM}^3}{(4\pi)^2}\left[3h^\vee - T(\text{chiral mult.})\right] , \tag{1.34}$$

plays again a crucial role in determining the low-energy and high-energy physics. Asymptotic freedom is preserved as long as there is not too much matter, namely $T(\text{chiral mult.}) \leq 3h^\vee$. Beyond this bound, it is not really interesting to say more about the low-energy limit of the theory, as it is weakly coupled, and we will assume that the bound is satisfied, $\beta \leq 0$. With matter now included, there may be a continuous classical moduli space of vacua, which as we mentioned before can receive only non-perturbative corrections in the quantum theory. Whether this correction is generated or not depends on the matter content. If there is not too much matter then the Affleck-Dine-Seiberg [27] superpotential is generated by instanton effects. The precise statement is that $T(\text{chiral mult.}) < h^\vee$; for super-QCD with N_f quarks and antiquarks and gauge group $SU(N)$, this condition is just $N_f < N$. In this case the superpotential behaves as

$$\mathcal{W}_{ADS} \sim \left(\frac{1}{\det(\bar{Q}Q)}\right)^{(N-N_f)^{-1}} \tag{1.35}$$

where Q denotes the scalar component of the quarks chiral multiplets, and $\bar{Q}Q$ is the meson constructing by contracting the color indices. The important feature is that $\mathcal{W}_{ADS}$ is large at small values of $\det(\bar{Q}Q)$ and if no other superpotential is added at tree level, there is a runaway vacuum – that is, no vacuum at all. As already alluded to at the beginning of this section, a classical superpotential (for instance, a mass term in the example discussed there) can open the possibility of a stable vacuum.

In super-QCD theories where the one-loop β-function vanishes, the solution exhibit a version of modular invariance, the duality acting as a subgroup of $SL(2,\mathbb{Z})$ on the couplings and on the masses by outer automorphisms of the flavor symmetry. For gauge algebras of type B and C, this subgroup is the congruence subgroup $\Gamma^0(2)$ generated by $T^2 : \tau \mapsto \tau + 2$ and $S : \tau \mapsto -1/\tau$.

Between these two regimes, the physics is also very complex, both from the point of view of dualities and physical phases. We already mentioned Seiberg duality, which relates a strong coupled theory to a weak coupled one with a different matter content. Depending on the matter content, one can for instance observe confinement with or withour chiral symmetry breaking, a free magnetic phase (dual to the free electric phase evoked in section 1.1.3, etc.

Finally, let us come back to a general issue of Yang-Mills theories, the existence of the total screening phase (see section 1.1.4). This can occur only if there are matter fields in faithful representations of the center of the gauge group.[12] When no such matter fields are present, for instance if the matter fields belong to the adjoint representation, then distinct branches of vacua can exist, with radically different low-energy behavior (Higgs, confinement, oblique confinement). As an example, the $U(N)$ theory with one adjoint chiral multiplet and polynomial superpotential has been analyzed in detail in [67], where the phase structure is indicated by an order parameter constructed out of powers $W^r H^s$ of the Wilson and 't Hooft lines.

[12] A faithful representation is one where different elements of the group are represented by different operators on the representation space. Obviously the adjoint representation is *not* faithful, unless the center of the gauge group is trivial. On the other hand the defining (fundamental) representation of $SU(N)$ is faithful.

1.2.4 Argyres-Douglas points

Let us come back to the $\mathcal{N} = 2$ theory with $SU(2)$ gauge group and one fundamental quark examined by Seiberg and Witten [49]. In their notation, the curve is

$$y^2 = x^2(x - u) + \frac{1}{4} m\Lambda_1^3 x - \frac{1}{64}\Lambda_1^6 , \tag{1.36}$$

where m is the mass of the quark, $u = \operatorname{tr} \phi^2$ is the gauge invariant parameter constructed from the scalar component of the vector multiplet and Λ_1 is the renormalization scale parameter. There are three singular points, corresponding to a monopole point and two dyon points. If we adjust the mass and u so that the monopole and the dyon points coincide, by putting $m = \frac{3}{4}\Lambda_1$ and $u = \frac{3}{4}\Lambda_1^2$, the curve becomes

$$y^2 = \left(x - \frac{1}{4}\Lambda_1^2\right)^3 . \tag{1.37}$$

In this case, all the roots coincide, and mutually non-local particles become massless [68]. Similar points can be found for two or three quarks as well. This situation is closely related to one observed earlier in the $SU(3)$ theory by Argyres and Douglas [41], and is dubbed an Argyres-Douglas point. Around these points, one can show that the dual photon a_D receives no logarithmic corrections, which indicates that the theory is conformal.

1.2.5 Compactification on a cylinder

After having described the moduli space of vacua of $\mathcal{N} = 2$ theories on $\mathbb{R}^4$, Seiberg and Witten [29] pursued their exploration on a topologically non-trivial spacetime $\mathbb{R}^3 \times S^1$. One very interesting effect of this compactification is that the Seiberg-Witten curve acquires a physical significance. Indeed, as we will see in more detail in chapter 5, two additional scalars appear upon compactification (the Wilson line along the compact direction and the dual of the photon in the non-compact directions) that can be combined into a complex scalar that we call Z, as depicted on figure 1.2. The low-energy action is then a sigma model in which the target space is the Seiberg-Witten curve, and the moduli space *is* this curve, along with its moduli. A non-perturbatively generated superpotential on the Coulomb branch can then be seen as an additional potential (in the classical mechanics sense of this term) in the sigma model, and this is the essence of the intimate link between supersymmetric gauge theories and mechanical (and even integrable, as we will see) systems.

In this setup, the area of the curve is proportional to R^{-1}, if R is the radius of the compact circle S^1. One of the advantages of compactification is that its analysis should provide information about three-dimensional as well as four-dimensional physics, which appear as limits, respectively, of small and large R. In a way, the theory on the cylinder encapsulates all this information, and much more. When $R \to \infty$, the size of the curve goes to zero, and the new degree of freedom Z on the moduli space is lost. On the other hand when $R \to 0$, the elliptic curve decompactifies, and the moduli space of a three-dimensional theory is recovered.

We have seen that compactification on a cylinder is a tremendously powerful probe for gauge dynamics, for several motives:

- It gives a physical reality to the Seiberg-Witten curve;

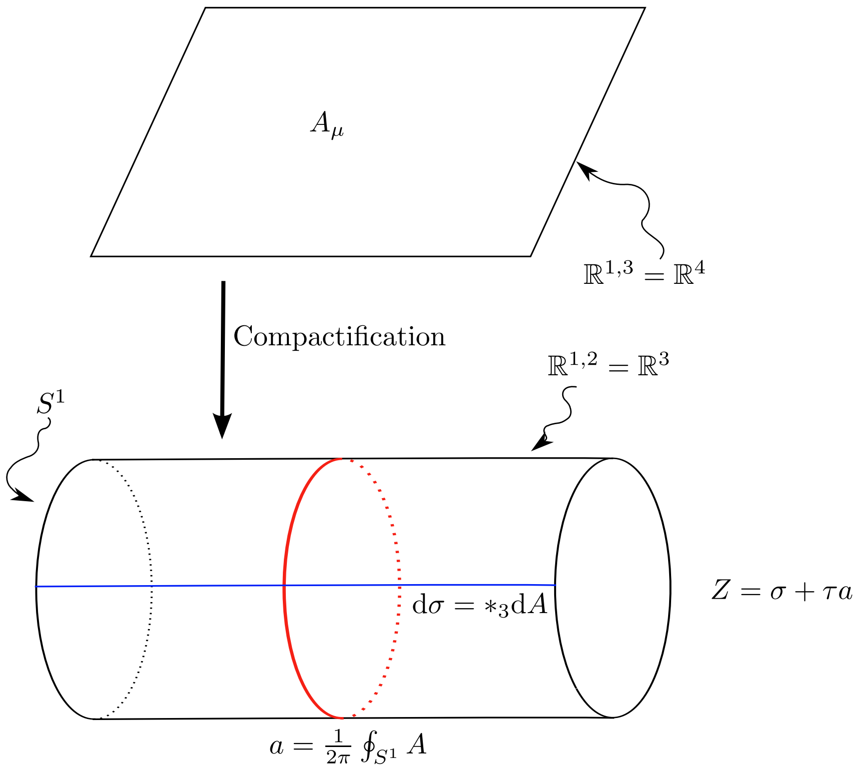


Figure 1.2: Compactification of the theory on the cylinder and appearance of two additional real scalars fields that combine in the complex field Z.

- It connects supersymmetric gauge dynamics and integrable systems;
- It connects four-dimensional physics to three-dimensional physics.

Heuristically, if any $\mathcal{N} = 2$ theory gives rise to a complex integrable system, compactification on the cylinder turns the integrable system into an honest hyperkähler space [30]. For these reasons, and in particular the bridge to integrable systems, we will focus on this kind of compactification. It should be noted however that other topologies are equally interesting to understand better the physics of supersymmetric gauge theories. A striking example is compactification of $\mathcal{N} = 2$ theories on a four-sphere [31] or four-ellipsoids [32] where the partition function can be computed exactly using the supersymmetric localization principle. The Alday-Gaiotto-Tachikawa relation [33] connects the partition function of certain such theories to two-dimensional Liouville or Toda conformal field theories with coupling controlled by the geometry of the ellipsoid.

1.3 Vacua of the $\mathcal{N} = 1^*$ Theory: a Summary

Let us summarize the last few sections before presenting a detailed plan of the rest of this thesis. As we have explained, one of the great successes of physics in the twentieth century has been to discover that the forces of Nature are accurately described by four-dimensional gauge quantum field theories. Since solving them is hard, we may revert to studying supersymmetric gauge theories, in which the power of holomorphy lends a helping hand. The infrared dynamics of such theories is a rich and fruitful subject, as the Seiberg-Witten solution for the low-energy effective action on the Coulomb branch of $\mathcal{N} = 2$ gauge theories in four dimensions demonstrates

[19, 49]. The techniques used were soon recognised to lie close to those studied in integrable systems [69, 70]. For pure $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions, which is massive in the infrared, we understand the supersymmetric index [71, 72, 73, 74, 75], as well as the transformation properties of the vacua under the broken non-anomalous R-symmetry. It is natural to extend the study of the vacua to other $\mathcal{N} = 1$ supersymmetric gauge theories.

It is the bridge between integrable models and supersymmetric gauge theories that we will further explore in this thesis to explore the vacua of a particular $\mathcal{N} = 1$ gauge theory. We will attempt to reinforce both sides separately, and present results in a manner such that the contributions to these two domains may be read independently, as illustrated by figure 1.3 where we depicted the main inter-relations of the chapters.

From now on we will focus on the deformation of $\mathcal{N} = 4$ supersymmetric Yang-Mills obtain by adding by hand masses in the Lagrangian for the three adjoint chiral multiplets, called $\mathcal{N} = 1^*$, and introduced in detail in section 2.2. This is an interesting playground due to duality symmetries inherited from the celebrated duality properties of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory reviewed in the introduction. In a way, this theory is also arguably a very natural $\mathcal{N} = 1$ supersymmetric model along with pure Yang-Mills and super-QCD, and has the additional property of having a vanishing one-loop β-function. This is of course intimately related with the fact that it is a mere deformation of $\mathcal{N} = 4$, but in this respect it makes a link with other theories like $\mathcal{N} = 1$ with gauge group $SU(N)$ and $2N$ fundamental quarks, to which part of the ideas presented here should apply as well.

In chapter 2, we compute the number of massive vacua of $\mathcal{N} = 1^*$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$ for any gauge group G. We use semi-classical techniques and efficiently reproduce the known counting for A, B and C type gauge groups, present the generating function for both $O(2n)$ and $SO(2n)$, and compute the supersymmetric index for gauge groups of exceptional type. A crucial role is played by the classification of nilpotent orbits, as well as global properties of their centralizers. We give illustrative examples of new features of our analysis for the D-type algebras.

In chapter 3, we leave gauge theories aside for a while and switch to the study of complexified elliptic Calogero-Moser systems. We first introduce the necessary background on integrable systems, both quantum and classical, and present Calogero-Moser systems, whose definition involves a simple Lie algebra. The explanation of the reason why such systems are useful in order to understand the structure of vacua of the $\mathcal{N} = 1^*$ gauge theory is postponed until chapter 5, where we will uncover a beautiful bridge between the two subject matters. The further detailed comparison of the global features of the theory on $\mathbb{R}^{2,1} \times S^1$ will add ornaments to this bridge in chapter 7. A useful tool that can be used to get a grasp on integrable systems is to take limits in some parameters, after which the system is still integrable, but simpler. From the elliptic system, we can reach in this way simpler Calogero-Moser systems as well as Toda systems. We will introduce affine Lie algebras and pseudo-Levi subalgebras [35] that are key in defining generalized Inozemtsev limits of (twisted) elliptic integrable systems. The elliptic Calogero-Moser potential, being defined on an elliptic curve, transforms in a modular-covariant way that we make explicit and use abundantly in the following chapter.

Chapter 4 is henceforth dedicated to a detailed analysis of isolated extrema in particular cases of the elliptic Calogero-Moser integrable system. This analysis combines numerical explorations using Mathematica [37] and the theory of modular forms, and provides compelling evidence for many exact results. We determine the value of the potential at such extrema, as a function

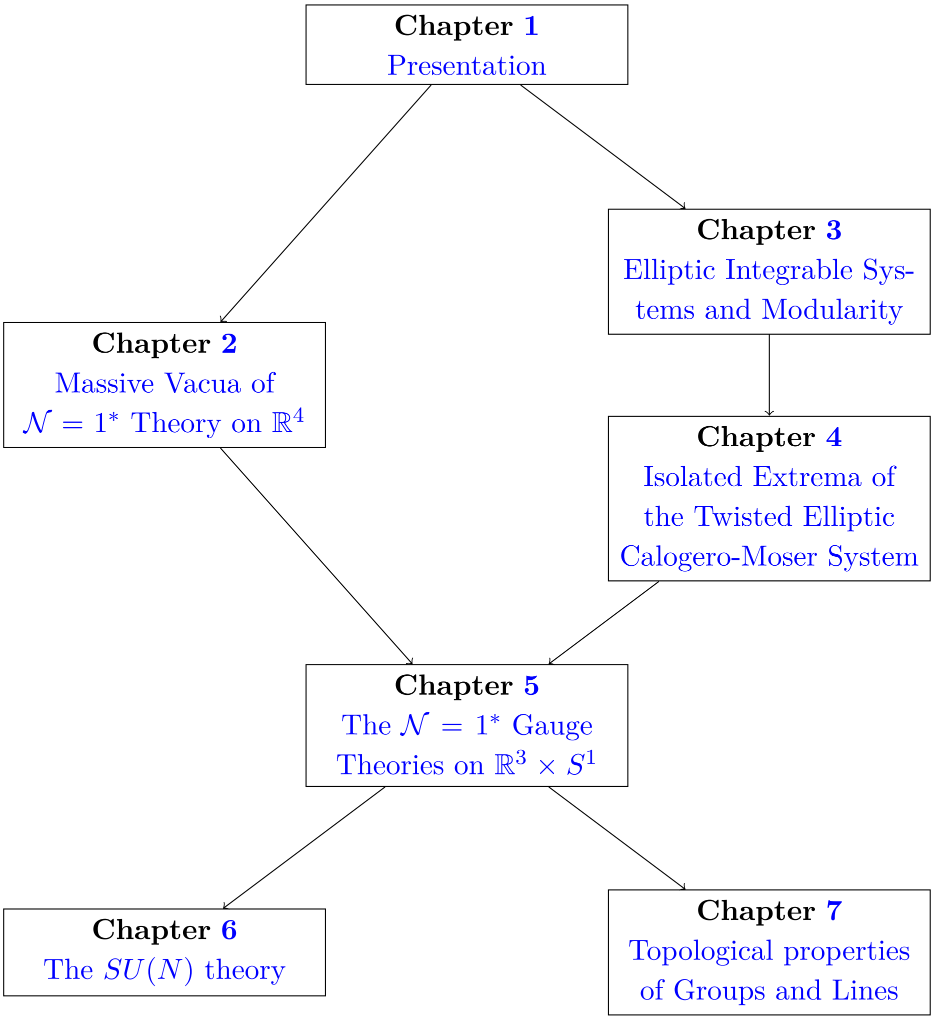


Figure 1.3: Interdependence of chapters in this thesis.

of the modular parameter of the torus on which the integrable system lives. We calculate the extrema for low rank B, C, D root systems using a mix of analytical and numerical tools. For $\mathfrak{so}(5)$ we find convincing evidence that the extrema constitute a vector valued modular form for the $\Gamma_0(4)$ congruence subgroup of the modular group. For $\mathfrak{so}(7)$ and $\mathfrak{so}(8)$, the extrema split into two sets. One set contains extrema that make up vector valued modular forms for congruence subgroups (namely $\Gamma_0(4)$, $\Gamma(2)$ and $\Gamma(3)$), and a second set contains extrema that exhibit monodromies around points in the interior of the fundamental domain. The former set can be described analytically, while for the latter, we provide an analytic value for the point of monodromy for $\mathfrak{so}(8)$, as well as extensive numerical predictions for the Fourier coefficients of the extrema.

After this two-chapter long incursion in the realm of elliptic integrable systems, we catch up with the main topic, the $\mathcal{N} = 1^*$ gauge theory. With the help of $\mathcal{N} = 1$ supersymmetry, it is sometimes possible to calculate the effective superpotential $\mathcal{W}$ at low energies exactly. For an adjoint mass deformation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ this was done in the original work [19] in certain cases. For $\mathcal{N} = 1^*$ and gauge group $G = SU(N)$, the exact superpotential was proposed in [38] following the techniques of [19, 29]. The key idea is to compactify the theory on a cylinder, as we will explain in the first sections of chapter 5. The superpotential is the potential of the complexified elliptic Calogero-Moser integrable system associated to the root lattice of type A_{N-1}. In [39] the exact superpotential for $\mathcal{N} = 1^*$ with more general gauge algebra was argued to be the potential of the twisted elliptic Calogero-Moser system[13] with root lattice associated to the Lie algebra of the gauge group G. As a consequence, we should be able to compare the semiclassical analysis that follows from the vacuum structure on $\mathbb{R}^4$ to the exact and numerical results obtained from the integrable system in chapter 4. This is the object of section 5.3, in which partial agreement is reached. We map out the transformation properties under the infrared electric-magnetic duality group as well as under triality for $\mathcal{N} = 1^*$ with gauge algebra $\mathfrak{so}(8)$. However we find that some of the extrema of the elliptic Calogero-Moser system have no counterpart in the semiclassical gauge theory analysis. Solving this conundrum will require a deeper analysis of the quantum field theory, presented in the two subsequent chapters.

In the last two chapters, we use all the techniques developed so far to unveil the vacuum structure of the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^3 \times S^1$ for various gauge groups. We begin with gauge algebras of type A in chapter 6, because this case is simpler – the phenomenon of discrete Wilson lines doesn't appear here. Nevertheless it displays massless branches of vacua. We provide an analytic description of the branches of massless vacua in the case of the $\mathfrak{su}(3)$. The description of the branch in terms of the complexified Wilson lines on the circle invokes the Eichler-Zagier technique for inverting the elliptic Weierstrass function. In this particular case, after fine-tuning the coupling to elliptic points of order three, we identify the Argyres-Douglas singularities of the $\mathfrak{su}(3)$, $\mathcal{N} = 1^*$ theory.

In chapter 7, we show that upon compactification on a circle, the semi-classical analysis of the massless and massive vacua depends on the classification of nilpotent orbits, as well as on the conjugacy classes of the component group of their centralizer. The pseudo-Levi algebras that we used in defining the generalized Inozemtsev limits of the Calogero-Moser system play a classifying role in this refined nilpotent orbit theory, as we review in section 7.4. The non-trivial topology allows for turning on Wilson lines that can increase the number of massive vacua. We demonstrate that semi-classically massless vacua can be lifted by these Wilson lines in unbroken discrete gauge groups. We illustrate our analysis in the $\mathcal{N} = 1^*$ theories with gauge algebras $\mathfrak{so}(5)$ and for the exceptional gauge algebra G_2. We map out modular duality diagrams of the massive and massless vacua, and give analytical equations for the $\mathfrak{so}(5)$ massless branch.

[13]We note that further generalizations to $\mathcal{N} = 1^*$ theories with twisted boundary conditions on $\mathbb{R}^3 \times S^1$ are possible [76], although we leave the exploration of these twisted theories for future work.

Chapter 2

Massive Vacua of $\mathcal{N} = 1^*$ Theory on $\mathbb{R}^4$

2.1 Introduction

In this chapter, we introduce the $\mathcal{N} = 1^*$ theory with arbitrary gauge group on the four-dimensional spacetime $\mathbb{R}^4$ with trivial topology.

We wish to compute the number of quantum vacua of the $\mathcal{N} = 1^*$ theory semi-classically. The infrared dynamics of the gauge theory will be governed by the infrared dynamics of the gauge group left unbroken by the vacuum expectation values of the chiral multiplets. A novel feature compared to pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory is that there are both massive and massless phases in the infrared. This renders the calculation of the number of vacua through compactification of space-time even more subtle since in $\mathcal{N} = 1^*$ theory with generic gauge group G, Wilson lines on circles can for instance lift Coulomb to massive vacua, as we will see in chapter 7. In this chapter, we work directly in $\mathbb{R}^4$, albeit semi-classically. By carefully classifying vacuum expectation values of the three adjoint chiral multiplets, as well as the corresponding unbroken gauge groups and their global properties, we obtain a prediction for the number of massive vacua of $\mathcal{N} = 1^*$ on $\mathbb{R}^4$.

For A-type gauge groups, the number of massive vacua was counted in [77, 38], while for B, C and D type gauge groups, there were some remarks in [78], while almost complete results were presented in [79]. In this chapter, we will perform the semi-classical calculation of the supersymmetric index using a different and more efficient technique. It will allow us to complete the count in the case of the D-type gauge groups, and permit us to recuperate beautiful mathematical classification results which enable us to the predict the supersymmetric index for all exceptional gauge groups as well. These results complete the count of massive vacua of $\mathcal{N} = 1^*$ gauge theories in four dimensions. The main mathematical tool we use are nilpotent orbits.[1]

We start in section 2.2 with an introduction to the $\mathcal{N} = 1^*$ theory on the topologically trivial spacetime $\mathbb{R}^4$. Then in section 2.4, we explain the three-step program for computing the supersymmetric index, and the role played by nilpotent orbits of the gauge algebra, and their centralizers. In section 2.5 we recompute the supersymmetric index for A, B and C type gauge groups, and present the calculation for gauge groups of type D. The exceptional gauge algebras

[1]Nilpotent orbits also play a role in classifying surface operators in supersymmetric gauge theories [80] and co-dimension two defects in $(2,0)$ theories [81].

E, F and G are treated in section 2.6.

2.2 The $\mathcal{N} = 1^*$ Theory on $\mathbb{R}^4$

The $\mathcal{N} = 1^*$ theories first appeared under this name in [82]. They are obtained from four-dimensional $\mathcal{N} = 4$ Yang-Mills theory by adding finite masses to the three chiral multiplets, reducing the supersymmetry to $\mathcal{N} = 1$. There is a similar construction where less supersymmetry is broken: using an $\mathcal{N} = 2$ theory language, one gives a mass to the hypermultiplet in the adjoint, thereby obtaining the so-called $\mathcal{N} = 2^*$ theory. Many features of $\mathcal{N} = 1^*$ theories were studied before they acquired their name. In [83] the classical vacua of the theory are formally found, and they are used to write down a partition function on hyper-Kähler manifolds and perform a test of S-duality.

In this section, we first introduce the celebrated parent $\mathcal{N} = 4$ super Yang-Mills theory in $1 + 3$ dimensions, and then perform the mass deformation. Various aspects of the theory obtained thereof are analyzed in the following subsections.

2.2.1 The $\mathcal{N} = 4$ Gauge Theory

Let us remind the reader of a few important features of $\mathcal{N} = 4$ super Yang-Mills theory first. This four-dimensional theory can be obtained by dimensional reduction from the minimal super Yang-Mills theory in 10 dimensions; upon dimensional reduction, six components of the 10-dimensional gauge field become scalar fields ϕ_i, $i = 1, \ldots, 6$, that can be combined into the three complex scalars. Taking into account the 10-dimensional gauginos, the field content of the four-dimensional theory thus obtained can be summarized in terms of $\mathcal{N} = 1$ supermultiplets as

- One vector multiplet, containing the gauge field and the gauginos
- Three chiral multiplets containing the three complex scalars Φ_1, Φ_2 and Φ_3 in the adjoint representation of the gauge group.

The Lagrangian is entirely fixed by supersymmetry (see appendix A), and we stress that the way the chiral multiplets interact, i.e. the superpotential, is given by (A.26) that we reproduce here for convenience:

$$\mathcal{W}_{\mathcal{N}=4} = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr} \left(\Phi_1 [\Phi_2, \Phi_3] \right) . \tag{2.1}$$

The theory is invariant under electric-magnetic duality. In the conformal phase this translates into modular invariance of the complexified coupling constant, as we saw in chapter 1.

2.2.2 The Mass Deformation

We now add bare masses m_i to the chiral multiplets Φ_i. This of course doesn't change the kinetic terms and boils down to adding a contribution $\mathrm{Tr}(m_1 \Phi_1^2 + m_2 \Phi_2^2 + m_3 \Phi_3^2)$ to the superpotential. We obtain

$$\mathcal{W}_{\mathcal{N}=1^*} = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr} \left(\Phi_1 [\Phi_2, \Phi_3] + m_1 \Phi_1^2 + m_2 \Phi_2^2 + m_3 \Phi_3^2 \right) . \tag{2.2}$$

In order to remove annoying factors in the final expression, we rescale in the appropriate way the superfields. This rescaling of the fields is only possible if the three masses m_1, m_2 and m_3

are nonzero. Let us define an "average" bare mass $m = (m_1 m_2 m_3)^{1/3}$ and rescale the fields as [84]

$$\hat{\Phi}_i = \frac{1}{(N g_{YM}^2)^{1/3}} \sqrt{\frac{m_i}{m}} \Phi_i \,. \tag{2.3}$$

We also introduce the rescaled mass

$$\tilde{m} = \frac{m}{(N g_{YM}^2)^{1/3}} \tag{2.4}$$

The superpotential is now

$$\mathcal{W}_{\mathcal{N}=1^*} = 2\sqrt{2} N \mathrm{Tr} \left(\hat{\Phi}_1 [\hat{\Phi}_2, \hat{\Phi}_3] + \tilde{m} \left(\hat{\Phi}_1^2 + \hat{\Phi}_2^2 + \hat{\Phi}_3^2 \right) \right) \,. \tag{2.5}$$

We have introduced an integer N in these expressions. *A priori* it can be chosen at will, although it is very natural to relate it to the rank of the gauge group, and in particular if the gauge group is of type A_r, we will take $N = r + 1$. The normalization (2.5) is adapted to taking the large N limit. Finally we again renormalize the chiral multiplets according to

$$\tilde{\Phi}_i = -\frac{1}{2\tilde{m}} \hat{\Phi}_i \,, \tag{2.6}$$

so that the superpotential (A.26) supplemented with the massive contribution reduces to

$$\mathcal{W}_{\mathcal{N}=1^*} = -16\sqrt{2} N \tilde{m}^3 \,\mathrm{Tr} \left(\tilde{\Phi}_1 [\tilde{\Phi}_2, \tilde{\Phi}_3] - \frac{1}{2} (\tilde{\Phi}_1^2 + \tilde{\Phi}_2^2 + \tilde{\Phi}_3^2) \right) \,. \tag{2.7}$$

Note that this expression is invariant under cyclic permutation of the three superfields $\tilde{\Phi}_i$.

2.2.3 Field Theoretic Properties of $\mathcal{N} = 1^*$

The importance of an understanding of the $\mathcal{N} = 1^*$ theories lies in the fact that they have a very rich phase structure and exhibit confinement, a feature that the conformal $\mathcal{N} = 4$ theory don't possess. Actually, the $\mathcal{N} = 1^*$ theories realizes all the possible massive phases of the 't Hooft classification [85]. We want to give here an overview of these profound field theoretic features, starting with the analysis of pure $\mathcal{N} = 1$ theories, which is both important in itself and will be crucial in our analysis throughout.

Pure $\mathcal{N} = 1$ gauge theory

Let us first consider the pure $\mathcal{N} = 1$ theory, whose Lagrangian is given by (A.14), as a *classical* field theory with a simple gauge group G. The fact that it is supersymmetric is reflected into the fact that there is a conserved spin $\frac{3}{2}$ current. Moreover, the conserved energy-momentum tensor has a vanishing trace, which means that the theory is classically superconformal invariant. Finally, we have the R-symmetry $U(1)_R$ which rotates[2] the gluinos λ that appear in the expanded Lagrangian (A.18), and leaves the gauge field invariant. There is an associated R-current with spin $\frac{1}{2}$, which is conserved. These currents can all be packed into the so-called *hypercurrent*.

Now we turn to the quantum theory. The $U(1)_R$ symmetry is broken by an anomaly[3] to a discrete subgroup $\mathbb{Z}_{2h^\vee} \subset U(1)_R$, where $h^\vee$ is the dual Coxeter number of the Lie algebra of G,

[2] More precisely, we will take λ to have R-charge $+1$ while $\bar{\lambda}$ has R-charge -1.

[3] This is the mixed triangle anomaly between one $U(1)_R$ current and two gluons.

see (B.6). To be more precise, in a theory with Weyl fermions ψ coupled to a gauge field, the classically conserved current $\bar{\psi}\bar{\sigma}^\mu\psi$ has an anomaly which can be computed at one loop,

$$\partial_\mu\left(\bar{\psi}\bar{\sigma}^\mu\psi\right) = \frac{T(\text{Representation})}{16\pi^2} F^a_{\mu\nu}\tilde{F}^{a,\mu\nu} \,. \tag{2.8}$$

Here the fermions ψ are the gauginos λ in the adjoint representation, and the half Dynkin index T is just the dual Coxeter number $h^\vee$, see (B.9). This means that an axial rotation of the gauginos $\lambda \to e^{i\alpha}\lambda$ is equivalent to a shift of the θ-angle $\theta_{YM} \to \theta_{YM} + 2\alpha h^\vee$ because of the coupling (A.13) in the Lagrangian. We find that such a rotation is no longer a symmetry, unless α is an integer multiple of $\pi/h^\vee$, which leaves us with the announced discrete subgroup. This subgroup transforms the gluinos in the natural way

$$\lambda \to \exp\left(\pi i \frac{k}{h^\vee}\right)\lambda \,, \tag{2.9}$$

where $k = 0, \ldots, 2h^\vee - 1$. The gluino condensate $\langle \operatorname{Tr} \lambda\lambda\rangle$ play the role of an order parameter, and in a given vacuum, the $\mathbb{Z}_{2h^\vee}$ anomaly-free symmetry is further broken to $\mathbb{Z}_2$, with

$$\langle \operatorname{Tr} \lambda\lambda\rangle \propto \exp\left[\frac{2\pi i}{h^\vee}\left(k + \frac{\theta_{YM}}{2\pi}\right)\right] \,. \tag{2.10}$$

This means that there are $h^\vee$ physically distinct vacua in the theory. They are permuted by $\theta_{YM} \to \theta_{YM} + 2\pi$. It is believed that these are the *only* supersymmetric vacua

2.3 The Vacuum Structure

2.3.1 Classical Vacua

Let us find the vacuum structure of the classical theory with the superpotential (2.7). As we already mentioned, this was already described by Vafa and Witten [83]. In order for the kinetic energy term to be minimal, we impose that the fields $\tilde{\Phi}_i$ are *constant*[4] as functions of the spacetime position $x \in \mathbb{R}^4$. We compute the partial derivatives using the identity

$$\frac{\partial}{\partial X}\operatorname{Tr}(XY) = Y^T \,, \tag{2.11}$$

where Y^T is the transpose of Y:

$$\frac{\partial \mathcal{W}_{\mathcal{N}=1^*}}{\partial \tilde{\Phi}_1} = [\tilde{\Phi}_2, \tilde{\Phi}_3]^T - \tilde{\Phi}_1^T \,, \tag{2.12}$$

and circular permutations of the indices $1, 2, 3$. Therefore the equations for a critical point of $\mathcal{W}_{\mathcal{N}=1^*}$ are

$$\begin{aligned}
\left[\tilde{\Phi}_2, \tilde{\Phi}_3\right] &= \tilde{\Phi}_1 \\
\left[\tilde{\Phi}_3, \tilde{\Phi}_1\right] &= \tilde{\Phi}_2 \\
\left[\tilde{\Phi}_1, \tilde{\Phi}_2\right] &= \tilde{\Phi}_3 \,.
\end{aligned} \tag{2.13}$$

[4] This benign assumption will be re-evaluated in section 7.4 when we compactify the theory.

These are the F-term equations. Of course, if a gauge transformation is performed on a given configuration satisfying (2.13), then the resulting configuration is physically indistinguishable from the first, and still solves (2.13). We must therefore divide the space of solutions by the gauge group in order to find the physical vacua. However there is a further constraint that must be satisfied for a configuration to be a supersymmetric vacuum, namely the vanishing of the D-terms. It is a general fact that ([86], section 27.4) if there are no Fayet-Iliopoulos terms and if there is a solution to the F-terms equations then there exists a solution of both the F- and D-terms equations, which is related to the initial solution by a complexified gauge transformation. Therefore the classical vacua of our theory are the solutions of (2.13) divided by the complexified gauge group $G_{\mathbb{C}}$.

One easy although crucial point about the three equations (2.13) is that they are just the equations satisfied by the $\mathfrak{su}(2)$ generators. As noticed in [83], this means that the *classical vacua* of the theory are in one to one correspondence with complex conjugacy classes of homomorphisms of $\mathfrak{su}(2)$ to the complexified Lie algebra $\mathfrak{g}$ of G. We will come back to this problem in great detail in section 2.4.

2.3.2 The Witten Index

A basic property of four-dimensional gauge theories that are massive in the infrared is their number of quantum vacua. This number can for instance be determined for pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in four dimensions. Arguments from chiral symmetry breaking correctly predict the count of massive vacua. The tally is confirmed by calculating the supersymmetric index [71, 75]. The supersymmetric index, also known as the *Witten index*, is defined by

$$I_W = \mathrm{Tr}(-1)^F = \sum_{\psi} \langle \psi | (-1)^F | \psi \rangle \,. \tag{2.14}$$

where F is the fermion number operator, and the sum runs over the physical states of the theory. Let us pause a moment to focus on the definition of operator F. From a Fock space perspective, this number is well defined in a state built from the vacuum acted on by creation operators, and since these states form a basis of the total Hilbert space, this defines F as an operator. Since any supercharge Q changes the fermion number by one unit, either decreasing it by transforming a fermion into a boson or increasing it by the opposite process, we can write $(-1)^F Q = -Q(-1)^F$, or

$$\left\{ (-1)^F, Q \right\} = 0 \,. \tag{2.15}$$

As we see, $(-1)^F$ behaves nicely, by which we mean that it has good algebraic properties. If we write no such formula for F itself, there is a good reason for that: it may be the case that there be no conserved fermionic current associated to this charge. Let us for instance focus on pure $\mathcal{N} = 1$ super-Yang-Mills theory, with $U(1)_R$ symmetry broken to $\mathbb{Z}_{2h^\vee}$ and further dynamically broken to $\mathbb{Z}_2$ by gaugino condensation. The fermion number is not well defined in such a theory, but thanks to the remaining $\mathbb{Z}_2$ symmetry, $(-1)^F$ is.

In a supersymmetric theory with discrete spectrum, the numbers of bosonic and fermionic states at a given energy $E > 0$ are equal, and therefore, if one defines n_B and n_F to be the number of supersymmetric states of zero energy that are bosonic or fermionic respectively, we can write

$$I_W = \mathrm{Tr}(-1)^F = n_B - n_F \,, \tag{2.16}$$

and the trace can be thought of as being taken only on the states with vanishing energy. However, it is useful to keep in mind that the trace actually spans the whole of the Hilbert space, since it can include a deformation parameter β, to be interpreted as an inverse temperature, and become

$$I_W = \mathrm{Tr}(-1)^F e^{-\beta E}\,. \tag{2.17}$$

The advantage of this formulation is that this can be computed as a path integral with periodic time and appropriate boundary conditions.

The computation of the index has been performed in [75], and the solution of the famous discrepancy between the originally computed Witten index (which was equal to one plus the rank of the gauge group) and the number of supersymmetric vacua (the dual Coxeter number $h^\vee$, which is the correct Witten index) was already pointed out in the appendix of [72].

Note that even for pure $\mathcal{N}=1$ supersymmetric Yang-Mills theory, some subtleties remain. In [75], the supersymmetric index was computed by compactification on $T^3 \times S^1$, and an analysis of commuting triples in the gauge group. On the other hand, it was argued in [10] that the number of quantum vacua after compactification on S^1 depends on the global choice of gauge group and the spectrum of line operators. The analysis of the Witten index has been revisited [87] in light of this new perspective.

2.3.3 Quantum Vacua

Now we would like to turn to the *quantum vacua* of our $\mathcal{N}=1^*$ theory on $\mathbb{R}^4$. Take for instance the easiest solution of our equations (2.13), namely

$$\tilde{\Phi}_1 = \tilde{\Phi}_2 = \tilde{\Phi}_3 = 0\,. \tag{2.18}$$

The gauge group G is then unbroken, and since the three fields $\tilde{\Phi}_i$ are massive, they decouple at low energies. Therefore we recover the pure $\mathcal{N}=1$ theory, which we have studied in section 2.2.3. It has a rich vacuum structure: if $G = SU(N)$ for instance, then there are N massive vacua. On the other hand, if we consider a solution that completely breaks the gauge symmetry, all the potentially strong quantum effects disappear and there is only one quantum vacuum. We will call this situation the *Higgs phase*. There is of course a whole family of intermediate cases. We can summarize the strategy for determining the vacuum structure as a three steps program:

1. First one solves the equations of motion for constant scalar field configurations which are equivalent to the statement that the three complex scalars satisfy a $\mathfrak{su}(2)$ algebra. The enumeration of inequivalent embeddings of $\mathfrak{su}(2)$ in the gauge algebra then provides the set of classical solutions.

2. Then one determines the unbroken gauge group for each classical vacuum

3. Finally one counts the number of vacua that the corresponding pure $\mathcal{N}=1$ quantum theory gives rise to in the infrared.

2.3.4 Modularity of the Theory

The parent $\mathcal{N}=4$ is believed to possess the full modular invariance, where the modular group acts on the complexified coupling constant τ. It is a very natural and interesting question to examine to which extent the S-duality symmetry of the parent $\mathcal{N}=4$ theory is preserved in the deformed $\mathcal{N}=1^*$ theory. One of the main feature of this thesis is that this symmetry is realized in the global vacuum structure of the $\mathcal{N}=1^*$ theory, as we will see. This symmetry is spontaneously broken in a given vacuum.[5]

To each quantity $\mathcal{O}$ is assigned a weight $w(\mathcal{O})=(w,\bar{w})$ such that under the usual modular transformation,

$$\mathcal{O} \to (c\tau+d)^{w}(c\bar{\tau}+d)^{\bar{w}}\mathcal{O}\,. \tag{2.19}$$

This weight is additive, and knowing it for each operator allows to deduce the modular transformation of any correlation function. One of the first calculations that anyone interested in modular forms performs,

$$\Im\left(\frac{a\tau+b}{c\tau+d}\right)=\frac{\Im(\tau)}{|c\tau+d|^2}\,, \tag{2.20}$$

shows that

$$w\left(\frac{1}{g_{YM}^2}\right)=(-1,-1)\,. \tag{2.21}$$

It is then argued in [88] that the supercharge Q transforms with weight $(+\frac{1}{4},-\frac{1}{4})$, while $\bar{Q}$ transforms with opposite weight.

This is part of the answer to the profound question of the action of modular transformation on the supersymmetry algebra. The answer can not be found in the Lagrangian, which is perturbative by essence. The idea is to go on the Coulomb branch where one of the three adjoint scalars Φ is non-vanishing, where the algebra acquires a central extension made of an electric and a magnetic charge,

$$\mathcal{Z}=\sqrt{\frac{2}{\Im(\tau)}}(n_e+\tau n_m)\cdot\Phi\,. \tag{2.22}$$

In this expression n_e and n_m are the electric and magnetic charges, they belong respectively to the weight lattice and the coweight lattice of the gauge algebra, while Φ lives in the Cartan subalgebra. The action of $SL(2,\mathbb{Z})$ on the charges is

$$\Phi\to\Phi \qquad \text{and} \qquad \begin{pmatrix} n_e \\ n_m \end{pmatrix} \to \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}\begin{pmatrix} n_e \\ n_m \end{pmatrix}. \tag{2.23}$$

This implies that the central charge (2.22) transforms as

$$\mathcal{Z}\to(c\tau+d)^{-1/2}(c\bar{\tau}+d)^{+1/2}\mathcal{Z}\,, \tag{2.24}$$

and has weight $(-\frac{1}{2},\frac{1}{2})$.

Let us examine in more detail the modular properties of the various ingredients. The main character of our story so far is the superpotential $\mathcal{W}$, which appears in the Lagrangian in the form (A.20). From the point of view of modular transformations, the Grassmann integration $\int \mathrm{d}^2\theta$

[5]Note however that, in a given vacuum, a new modular symmetry can be found [84] in the chiral sector of the theory, which acts on a different coupling $\tilde{\tau}$. This new modular symmetry allows to pin down preferred chiral operators in the mass deformed theory, despite the mixing that this deformation causes, allowing to compare the results to the supergravity calculation in the dual theory.

can be seen as two supercharges Q^2, and transforms with weight $(\frac{1}{2}, -\frac{1}{2})$. From the invariance of the Lagrangian, we deduce that $\mathcal{W}$ transforms with weight

$$w(\mathcal{W}) = \left(-\frac{1}{2}, \frac{1}{2}\right) . \tag{2.25}$$

This can also be seen [89] from the transformation properties of the central charge (2.24) and the fact that this central charge is equal to the difference of superpotentials values (1.33). Recalling the way the gauge coupling constant g_{YM} transforms (2.21) and using [88] the fact that $\mathrm{Tr}\Phi_i^2$ transforms with weight $(1,1)$ we deduce that $w(m) = w(m_i) = w(\mathcal{W})$. Finally, after renormalization (2.4), this gives [84]

$$w(\tilde{m}) = \left(-\frac{5}{6}, \frac{1}{6}\right) . \tag{2.26}$$

As a consequence, we have

$$w\left(\frac{\mathcal{W}}{\tilde{m}^3}\right) = (2,0) , \tag{2.27}$$

a fact that we will use extensively later on.

2.4 The Semi-Classical Configurations and the Classification Problem

In this section, we discuss how the classification of vacuum expectation values for the adjoint scalars in the chiral multiplets of $\mathcal{N} = 1^*$ reduces to the problem of the classification of nilpotent orbits of the complexified Lie algebra $\mathfrak{g}$ of the gauge group. The idea of using nilpotent orbits in the context of classifying vacua in $\mathcal{N} = 1^*$ theory was mentioned in [90]. See also [91] for an application of nilpotent orbit theory to a supersymmetric index calculation.

2.4.1 Semi-classical Configurations and $\mathfrak{sl}(2)$ algebras

Our starting point is the $\mathcal{N} = 1^*$ super Yang-Mills theory with compact gauge group G on $\mathbb{R}^4$. As we explained in section 2.3, the vacua are classified by solving the F-term equations for constant fields, and dividing the solution space by the complexified gauge group $G_\mathbb{C}$. The equations dictate that the (rescaled) adjoint scalar fields $\tilde{\Phi}_i$ (where $i \in \{1,2,3\}$) form an $\mathfrak{sl}(2)$ algebra, see equation (2.13). Thus, the scalars provide us with a map from an $\mathfrak{sl}(2)$ algebra into the complexified Lie algebra $\mathfrak{g}$ of the gauge group. To find the supersymmetric vacua, we are to classify all $\mathfrak{sl}(2)$ triples inside the Lie algebra $\mathfrak{g}$, up to gauge equivalence. Configurations are gauge equivalent if they are mapped to each other by the adjoint action of the complexified gauge group $G_\mathbb{C}$ on the Lie algebra $\mathfrak{g}$. Thus, our first step is to review what is known about the classification of inequivalent $\mathfrak{sl}(2)$ triples embedded in the adjoint representation.

2.4.2 The Gauge Group, Triples and Nilpotent Orbits

From now on, we will denote the complexified gauge group as $G_\mathbb{C} \equiv G$. We need to make a distinction between various groups that have the same Lie algebra. One canonical group associated to the Lie algebra $\mathfrak{g}$ is the adjoint group $G_{ad} = \mathrm{Aut}(\mathfrak{g})^o$, namely the identity component of the group of automorphisms of the Lie algebra $\mathfrak{g}$. The adjoint group G_{ad} is alternatively characterized by the fact that it is the group with algebra $\mathfrak{g}$ and trivial center.

We can now lay the groundwork for the first classification problem. Note that amongst our complex adjoint fields $\tilde{\Phi}_i$, we can identify a linear combination $\tilde{\Phi}_+$ which is nilpotent by the equations of motion. Indeed, we can consider the complex combinations $\tilde{\Phi}_\pm = \pm\tilde{\Phi}_1 + i\tilde{\Phi}_2$ and $\tilde{\Phi}_0 = 2i\tilde{\Phi}_3$. Then the non-vanishing commutation relations amongst these fields are

$$\begin{aligned} \left[\tilde{\Phi}_0, \tilde{\Phi}_+\right] &= 2\tilde{\Phi}_+ \\ \left[\tilde{\Phi}_0, \tilde{\Phi}_-\right] &= -2\tilde{\Phi}_- \\ \left[\tilde{\Phi}_+, \tilde{\Phi}_-\right] &= \tilde{\Phi}_0. \end{aligned} \tag{2.28}$$

Of course, these remain standard commutation relations of the algebra $\mathfrak{sl}(2)$. We can describe them by stating that $(\tilde{\Phi}_0, \tilde{\Phi}_+, \tilde{\Phi}_-)$ form an $\mathfrak{sl}(2)$ triple. The vacuum expectation value $\tilde{\Phi}_+$ is now a nilpotent element of the Lie algebra $\mathfrak{g}$ of the gauge group. Reciprocally, given a nilpotent element in a complex semisimple Lie algebra $\mathfrak{g}$, the Jacobson-Morozov theorem states that we can always extend it to an $\mathfrak{sl}(2)$ triple. The relation between nilpotent elements and $\mathfrak{sl}(2)$ triples is a bijection in the following sense : there is a one-to-one correspondence between G-conjugacy classes of $\mathfrak{sl}(2)$ triples in $\mathfrak{g}$ and non-zero nilpotent G-orbits in $\mathfrak{g}$. This follows for instance from Theorem 3.2.10 in [92] when $G = G_{ad}$, and it remains true for connected gauge groups of any isogeny type (i.e. with non-zero center) because the adjoint action of the center is trivial. Moreover, if G and G' are two connected groups with the same Lie algebra, then G-conjugacy classes and G'-conjugacy classes of $\mathfrak{sl}(2)$ triples in this Lie algebra coincide, as do their nilpotent orbits. For instance, the $SO(2n)$ and $PSO(2n)$ classes of triples and orbits are the same. The assumption that the groups are connected is essential, as we will see in detail in the case of $O(2n)$ and $SO(2n)$.

The bottom line is that it will be sufficient for us to study the nilpotent orbits of $\mathfrak{g}$ in order to enumerate gauge inequivalent vacuum configurations for the triplet of adjoint scalars in the chiral multiplets. These nilpotent orbits are finite in number and they have been classified, as we review in the next section.

2.4.3 Nilpotent Orbit Theory

In this section we review the theory of nilpotent orbits that allow us to complete the first step of our classification. Our presentation is mainly based on [92, 93, 94, 95].

Definitions and notations

For any element $Z \in \mathfrak{g}$, we denote by $\mathcal{O}_Z$ its G_{ad}-orbit. A standard triple is a set $\{H, X, Y\}$ of elements of $\mathfrak{g}$ that satisfied the relations

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H\,. \end{aligned} \tag{2.29}$$

The elements X and Y are nilpotent, while H is semisimple. We will denote $\mathfrak{trip}$ the algebra generated by the triple inside $\mathfrak{g}$. For any standard triple, its G_{ad}-conjugacy class contains an element $\{H, X, Y\}$ such that H lies in the fundamental dominant Weyl chamber. One can then prove that for any simple root α, we have $\alpha(H) \in \{0, 1, 2\}$. Therefore we can build from H a *weighted Dynkin diagram*, where the node α is weighted by $\alpha(H)$. We insist that this is our definition of a weighted Dynkin diagram : not all diagrams with weights belonging to $\{0, 1, 2\}$ are such diagrams, only those that can be constructed from a standard triple as just described are. Given a standard triple (2.29), we also define

$$\mathfrak{g}_\lambda = \{Z \in \mathfrak{g} \,|\, [H, Z] = \lambda Z\} \; . \tag{2.30}$$

One can then prove that

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{Z}} \mathfrak{g}_\lambda \, . \tag{2.31}$$

Let us then define the following subalgebras:

$$\mathfrak{q} = \bigoplus_{\lambda \in \mathbb{N}} \mathfrak{g}_\lambda \, , \qquad \mathfrak{u} = \bigoplus_{\lambda \in \mathbb{N}^*} \mathfrak{g}_\lambda \, , \qquad \mathfrak{l} = \mathfrak{g}_0 \, . \tag{2.32}$$

We say that $\mathfrak{q}$ is the Jacobson-Morozov parabolic subalgebra associated to the standard triple, and $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is its Levi decomposition. The subalgebra $\mathfrak{l}$ is called the Levi subalgebra associated to the standard triple.

Finally, let $J \subset \Delta^s$ be a set of simple roots of $\mathfrak{g}$. To this set we can associate a standard Levi subalgebra in the following way:

$$\mathfrak{l}_J = \mathfrak{h} \oplus \sum_{\alpha \in \langle J \rangle} \mathfrak{g}_\alpha \, , \tag{2.33}$$

where $\langle J \rangle$ is the subroot system generated by J.

Fundamental theorem for nilpotent orbits

We now state the fundamental theorem[6] of nilpotent orbit theory. Let $\mathfrak{g}$ be a semisimple Lie algebra. We use the notations and definitions previously introduced. Then the five following finite sets are in **one-to-one correspondence**:

(1) The set of G_{ad}-conjugacy classes of standard triples in $\mathfrak{g}$.

(2) The set of nonzero nilpotent G_{ad}-orbits in $\mathfrak{g}$.

(3) The set of G_{ad}-orbits of semisimple elements that appear in some standard triple in $\mathfrak{g}$.[7]

(4) The set of weighted Dynkin diagrams of $\mathfrak{g}$.

(5) The set of G_{ad}-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$.

[6] We have collected various difficult theorems here, that have their independent life under various names. The fact that there is a surjective map (1) → (2) is called the Jacobson-Morozov theorem. The fact that it is also injective is due to Kostant. The injectivity of (1) → (3) is known as Mal'cev's theorem (and its surjectivity is trivial). The bijection (2) → (4) is also a theorem of Kostant. Finally the set (5) was constructed by Bala and Carter, and the bijection (2) → (5) is known as Bala-Carter theorem. The fact that all these sets are finite follows from the obvious finiteness of set (4).

[7] These are sometimes called *distinguished semisimple* orbits, not to be confused with the distinguished nilpotent orbits that we will use extensively in the following.

For a full proof of this theorem, the reader is referred to [92, 93]. In this paragraph we modestly provide a few ideas that are part of the proof and that we will use soon, along with the definitions needed to understand the statement of the theorem. So we want to write down an exhaustive list of the nilpotent orbits of $\mathfrak{g}$ in terms only of its Dynkin diagram. Let $n \in \mathfrak{g}$ be a nilpotent element. One can prove that any two *minimal Levi* subalgebras containing n are conjugate. Therefore we have a well defined map

$$n \to [\mathfrak{l}] \tag{2.34}$$

that sends a nilpotent element to a conjugacy class of Levi subalgebras. The "hardest" nilpotent elements are those for which this class is $\{\mathfrak{g}\}$: when this is the case, we say that n is *distinguished nilpotent* in $\mathfrak{g}$. Obviously, with the notation of the map (2.34), n is always distinguished in $\mathfrak{l}$, and one can further prove that it is distinguished in the semisimple Lie algebra $[\mathfrak{l}, \mathfrak{l}]$. We conclude from this discussion that we can restrict our attention to distinguished nilpotent orbits. Note that distinguished nilpotent orbits are those for which the nilpotent element does not commute with a non-central semi-simple element. Using the notations from (2.32) and the fundamental theorem, we then state the following crucial lemma, which gives a characterization of distinguished nilpotent orbits:

$$\mathcal{O}_X \text{ is distinguished} \quad \Leftrightarrow \quad \dim \mathfrak{l} = \dim \left(\frac{\mathfrak{u}}{[\mathfrak{u}, \mathfrak{u}]} \right) . \tag{2.35}$$

As we have seen, it is of tremendous importance to know the distinguished nilpotent orbits of a given semisimple Lie algebra. This can be deduced from the list of distinguished nilpotent orbits of simple Lie algebras. The so-called *principal* orbit, which can be characterized equivalently by the fact that it is the orbit with higher dimension or by the fact that its weighted Dynkin diagram has all its indices equal to 2, is always distinguished. For the classical algebras, there is also a characterization in terms of partitions which is given in table 2.2. In algebras of type A, it is the only one. However, in the other simple Lie algebras, there are more distinguished nilpotent orbits, as shown in table 2.3. Constructing these tables is easy using the characterization 2.35, and an example of how this is done is presented later on in the case of G_2. For the exceptional algebras, we also indicate the Bala-Carter name of the distinguished orbit. This name has the generic form Algebra(x_i) where Algebra is the type of the algebra in which the orbit is distinguished, i is the number of zeros in the weighted Dynkin diagram, and x is a unless there are several orbits with the same number of zeros, in which case the letter b is also used.

In order to illustrate this notation, and to make more concrete the various concepts used here, let us consider the algebra $D_4 = \mathfrak{so}(8)$. According to table 2.1, the nilpotent orbits are in bijection with partitions of 8 where even numbers appear an even number of times, and where the very even partitions $2+2+2+2$ and $4+4$ where only even numbers appear, are counted twice. This gives the list of 12 orbits presented in table 2.5. Among these orbits, two possess no repeated entry, namely $7+1$ and $5+3$, and these are the two distinguished orbits according to the characterization of table 2.3. The partition $7+1$ is the principal orbit and will be denoted D_4, while $5+3$ has a weighted diagram with one zero, as shown in table 2.5, and we call it $D_4(a_1)$.

Finally, we note that when an algebra is a sum of several simple algebras, the number of distinguished nilpotent orbits is just the product of the number of distinguished nilpotent orbits in each simple algebra. As an example, there are two distinguished nilpotent orbits in $D_4 + A_1$, that we call $D_4 + A_1$ and $D_4(a_1) + A_1$. All these notations will be used in section 2.6.

Lie Algebra	Labelling of Nilpotent Orbits
$\mathfrak{sl}(n)$	Partitions of n
$\mathfrak{so}(2n+1)$	Partitions of $2n+1$ in which even parts occur with even multiplicity
$\mathfrak{sp}(2n)$	Partitions of $2n$ in which odd parts occur with even multiplicity
$\mathfrak{so}(2n)$	Partitions of $2n$ in which even parts occur with even multiplicity with *very even* partitions corresponding to two orbits

Table 2.1: The nilpotent orbits in the Lie algebra $\mathfrak{g}$ in the left column are in one-to-one correspondence with the set of partitions described in the right column. The orbits are assumed to be generated by a connected group with algebra $\mathfrak{g}$.

Algebra	Principal nilpotent orbit
A_{n-1}	Partition $n = n$
B_n	Partition $2n+1 = (2n+1)$
C_n	Partition $2n = (2n)$
D_n	Partition $2n = (2n-1) + (1)$

Table 2.2: This table gives the partition that corresponds to the principal orbit for each type of classical algebras. The parenthesis indicate that the integer inside should be understood as one chunk in the partition.

Algebra	Distinguished nilpotent orbits
A_{n-1}	Principal
B_n, C_n, D_n	Partitions with no repeated part
E_6	E_6 principal
	$E_6(a_1)$: $\begin{array}{ccccccccc} &&&&2&&&& \\ &&&&\vert&&&& \\ 2&-&2&-&0&-&2&-&2\end{array}$
	$E_6(a_3)$: $\begin{array}{ccccccccc} &&&&0&&&& \\ &&&&\vert&&&& \\ 2&-&0&-&2&-&0&-&2\end{array}$
E_7	E_7 principal
	$E_7(a_1)$: $\begin{array}{ccccccccccc} &&&&&&2&&&& \\ &&&&&&\vert&&&& \\ 2&-&2&-&2&-&0&-&2&-&2\end{array}$
	$E_7(a_2)$: $\begin{array}{ccccccccccc} &&&&&&2&&&& \\ &&&&&&\vert&&&& \\ 2&-&0&-&2&-&0&-&2&-&2\end{array}$
	$E_7(a_3)$: $\begin{array}{ccccccccccc} &&&&&&0&&&& \\ &&&&&&\vert&&&& \\ 2&-&2&-&0&-&2&-&0&-&2\end{array}$
	$E_7(a_4)$: $\begin{array}{ccccccccccc} &&&&&&0&&&& \\ &&&&&&\vert&&&& \\ 2&-&0&-&0&-&2&-&0&-&2\end{array}$

	$E_7(a_5)$: $2 - 0 - 0 - \overset{0}{2} - 0 - 0$
E_8	E_8 principal
	$E_8(a_1)$: $2 - 2 - 2 - 2 - \overset{2}{0} - 2 - 2$
	$E_8(a_2)$: $2 - 2 - 0 - 2 - \overset{2}{0} - 2 - 2$
	$E_8(a_3)$: $2 - 2 - 2 - 0 - \overset{0}{2} - 0 - 2$
	$E_8(a_4)$: $2 - 0 - 2 - 0 - \overset{0}{2} - 0 - 2$
	$E_8(b_4)$: $2 - 2 - 0 - 0 - \overset{0}{2} - 0 - 2$
	$E_8(a_5)$: $0 - 2 - 0 - 0 - \overset{0}{2} - 0 - 2$
	$E_8(b_5)$: $2 - 2 - 0 - 0 - \overset{0}{2} - 0 - 0$
	$E_8(a_6)$: $0 - 2 - 0 - 0 - \overset{0}{2} - 0 - 0$
	$E_8(b_6)$: $2 - 0 - 0 - 0 - \overset{0}{2} - 0 - 0$
	$E_8(a_7)$: $0 - 0 - 0 - 2 - \overset{0}{0} - 0 - 0$
F_4	$2 - 2 \Rightarrow 2 - 2$
	$2 - 2 \Rightarrow 0 - 2$
	$0 - 2 \Rightarrow 0 - 2$
	$0 - 2 \Rightarrow 0 - 0$
G_2	$2 \Rrightarrow 0$
	$2 \Rrightarrow 2$

Table 2.3: For each type of simple Lie algebra, we give the list of distinguished nilpotent orbits, as presented in [93] for instance. The orbits are presented in terms of partitions for the classical algebras since there is a simple characterization. We recall that the principal orbit corresponds to a weighted diagram with all indices equal to 2.

A detailed example of how this theory can be used in practice to obtain the list of distinguished nilpotent orbits and then the list of nilpotent orbits will be presented in section (7.4.2) in the case of G_2. We postpone this discussion until this point, because we will then have discussed a generalization of the Bala-Carter theory, and we want to present this example only once. For the other exceptional groups, we will not make the computation in detail, but it can be done along the same lines. The resulting list of nilpotent orbits will be used in section 2.6.

2.4.4 Number of Partitions

Before closing this review of nilpotent orbits theory, we present now a computation of the generating functions that give the number of such orbits in all classical algebras as a function of their rank. This can be seen as a warm-up before tackling the problem of finding the generating functions for the index in section 2.5.3.

Generating functions

Let c_n be the number of partitions of type C of $2n$, namely partitions where odd numbers occur an even number of times. There are precisely c_n nilpotent orbits in $\mathfrak{sp}_{2n}$. Then we have:

$$\sum_{n=0}^{+\infty} c_n q^n = \frac{1}{\theta_4(q^2)} . \tag{2.36}$$

Our conventions for θ-functions are given in appendix C. Now let b_n be the number of partitions of type BD of n, namely partitions where odd numbers occur an even number of times (for instance $b_4 = 3$ and $b_5 = 4$). There are precisely b_{2n+1} nilpotent orbits in $\mathfrak{so}_{2n+1}$. Then we have

$$\sum_{n=0}^{+\infty} b_n q^n = \frac{2(-q)^{1/8}}{\theta_2(-q)} . \tag{2.37}$$

Now let us call b'_n the number of partitions of n of type BD, counting twice the *very even* partitions, namely those partitions that contain no odd integer. There are precisely b'_{2n} nilpotent orbits in $\mathfrak{so}_{2n}$. Then

$$\sum_{n=0}^{+\infty} b'_n q^n = \frac{2(-q)^{1/8}}{\theta_2(-q)} + \frac{q^{1/6}}{\eta(q^4)} . \tag{2.38}$$

In the statements above, the orbits are to be understood as adjoint orbits, i.e. orbits under the action of the adjoint group corresponding to the Lie algebra. For $\mathfrak{so}_{2n}$ one might want to consider $O(2n)$-orbits: there are precisely b_{2n} such nilpotent orbits (there is only one $O(2n)$-orbit attached to a very even partition, and it is the reunion of the two $PSO(2n)$-orbits attached to this partition).

Proof of the formulas

Recall that the usual way to write the generating function of the number of partitions of n is to start from $(1+q+q^2+...)(1+q^2+q^4+...)(1+q^3+q^6+...)...$ and simplify the expression. Here we want to add the condition that the odd integers occur an even number of times. There is a one-to-one correspondence between partitions of type C of $2n$ in $\mathbb{N}$ and partitions of n in $\mathbb{N} \cup \{\bar{1}, \bar{3}, \bar{5}, \bar{7}, ...\}$, using the rules $2k \leftrightarrow k$ and $(2k-1)+(2k-1) \leftrightarrow \overline{2k-1}$. For instance the C-partition $26 = 1+1+2+3+3+3+3+4+6$ corresponds to $13 = \bar{1}+1+2+\bar{3}+\bar{3}$. Using this correspondence, the trick is now to take the square of the terms corresponding to odd factors, since there are now two "types" of odd factors:

$$\sum_{n=0}^{+\infty} c_n q^n = (1+q+q^2+...)^2(1+q^2+q^4+...)(1+q^3+q^6+...)^2...$$

$$
\begin{aligned}
&= \prod_{m=1}^{+\infty}(1-q^{2m})^{-1}(1-q^{2m-1})^{-2} \\
&= \theta_4^{-1}(q^2)\,.
\end{aligned}
$$

For the BD-partitions we use a similar method. There is a bijection between BD-partitions of n in $\mathbb{N}$ and partitions of n in $(2\mathbb{N}-1)\cup(4\mathbb{N})$, given by $2k-1 \leftrightarrow 2k-1$ and $2k+2k \leftrightarrow 4k$. From this we deduce

$$
\begin{aligned}
\sum_{n=0}^{+\infty} b_n q^n &= \prod_{m=1}^{+\infty}(1-q^{2m-1})^{-1}(1-q^{4m})^{-1} \\
&= \prod_{m=1}^{+\infty}\left[(1-q^{2m-1})^{-1}(1+q^{2m})^{-1}\right](1-q^{2m})^{-1} \\
&= \prod_{m=1}^{+\infty}(1+(-1)^m q^m)^{-1}(1-(-q)^m)^{-1}(1+(-q)^m)^{-1} \\
&= \prod_{m=1}^{+\infty}(1+(-q)^m)^{-2}(1-(-q)^m)^{-1} \\
&= \frac{2(-q)^{1/8}}{\theta_2(-q)}
\end{aligned}
$$

Finally, taking into account the very even partitions is easy, since there is an obvious bijection between very even partitions of $4n$ and partitions of n. The generating function of these very even partitions is therefore $q^{1/6}(\eta(q^4))^{-1}$.

Numerical values

Since we will often need to enumerate partitions of integers with the aforementioned constraints in the following sections and chapters, we provide here the numbers of such partitions as a function of n:

n	b_n	b'_n	c_n
0	1	2	1
1	1	1	2
2	1	1	4
3	2	2	8
4	3	4	14
5	4	4	24
6	5	5	40
7	7	7	64
8	10	12	100
9	13	13	154
10	16	16	232
11	21	21	344
12	28	31	504

(2.39)

2.4.5 The Centralizer and the Index

The second step in our program is to determine the unbroken gauge group in a given semi-classical configuration for the adjoint scalar fields. Thus, for each nilpotent orbit with its associated $\mathfrak{sl}(2)$ triple, we need to determine the centralizer of the triple, i.e. the unbroken gauge group. This determination has been performed as well [92, 93, 94, 95]. The results are lined with intricacies that we will discuss in due course. Before we do so, we introduce some notation for the relevant mathematical objects.

We denote by $C_G(n)$ the centralizer in G of a nilpotent element $n \in \mathfrak{g}$. We call $\mathfrak{trip}$ the image in $\mathfrak{g}$ of a $\mathfrak{sl}(2)$ triple associated to the G-orbit of the nilpotent element n. We can then define $C_G(\mathfrak{trip})$, the centralizer of $\mathfrak{trip}$ in the group G. We are primarily interested in the structure of $C_G(\mathfrak{trip})$, the unbroken gauge group in a given semi-classical configuration for the adjoint scalar fields. It will be important to us that in general this group is not connected, and that therefore its component group $Comp(\mathfrak{trip}) = C_G(\mathfrak{trip})/C_G(\mathfrak{trip})^o$ is non-trivial. It is crucial for the computation of the supersymmetric index to understand both the structure of the part of the centralizer connected to the identity *and* the action of the component group on the connected components.

The third step will be to apply our knowledge of the infrared dynamics of pure $\mathcal{N} = 1$ supersymmetric gauge theories to the theory with gauge group $C_G(\mathfrak{trip})$. If the unbroken gauge group $C_G(\mathfrak{trip})$ is non-abelian, the number of massive vacua on $\mathbb{R}^4$ will be given by the product of the dual Coxeter numbers of the simple factors.[8] The set of vacua possibly still needs to be divided out by the action of the component group $Comp(\mathfrak{trip})$.

2.5 The Counting for the Classical Groups

In this section, we apply the three-step program set out in section 2.4 to the case of all classical gauge groups of types A, B, C and D.

2.5.1 The Nilpotent Orbits

In a first step, we describe the nilpotent orbits of the classical Lie algebras $A_{n-1} = \mathfrak{sl}(n)$, $B_n = \mathfrak{so}(2n+1)$, $C_n = \mathfrak{sp}(2n)$ and $D_n = \mathfrak{so}(2n)$.

The nilpotent orbits can be represented in a variety of ways. For the classical algebras, the nilpotent orbits in connected groups are in one-to-one correspondence with sets of partitions given explicitly in table 2.1. Since partitions will play an important role, we introduce a few useful notations. For any partition of an integer $N = d_1 + d_2 + ... + d_N$, we define $n_d = |\{k|d_k = d\}|$, namely the number of times the integer $d > 0$ appears in the partition, as well as the sets

$$D_o = \{d|d \text{ odd and } n_d > 0\} \tag{2.40}$$

$$D_e = \{d|d \text{ even and } n_d > 0\}\,. \tag{2.41}$$

The sets D_o (respectively D_e) are the sets of odd (respectively even) integers that appear in the partition of N.

As can be seen in table 2.1, there is a subtlety for $\mathfrak{so}(2n)$. We say that a given partition of an integer is *very even* if it has only even parts, each occurring with even multiplicity. A very even partition corresponds to precisely two inequivalent orbits of $\mathfrak{so}(2n)$. These orbits are interchanged under the outer $\mathbb{Z}_2$ automorphism that acts by exchanging the two extremal nodes

[8] The number of vacua for pure $\mathcal{N} = 1$ super Yang-Mills theory on $\mathbb{R}^4$ is independent of the center of the gauge group.

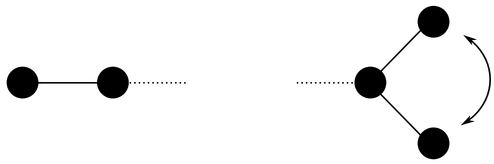


Figure 2.1: The action of the $\mathbb{Z}_2$ outer automorphism of the D_n algebra (for $n > 4$) on the Dynkin diagram.

on the fork of the D-type Dynkin diagram, as illustrated in figure 2.1. This subtlety disappears if instead of $SO(2n)$ orbits, one considers $O(2n)$ orbits : in this case, *every* partition of $2n$ with even parts occurring with even multiplicity correspond to a single orbit.[9] It will be useful to include the gauge group $O(n)$ into our discussion in the following, since it serves as a stepping stone to obtain results for $SO(n)$ gauge groups.

2.5.2 The Centralizers

For each partition, the centralizer $C_G(\mathfrak{trip})$ of the corresponding triple is explicitly known [92, 93, 94, 95] – the result is sometimes called the Springer-Steinberg [96]. We will make particular choices for the center of the gauge group, which influence the centralizers, but the final result will be independent of this choice. Since the massive vacua of the A-type $\mathcal{N} = 1^*$ theory are well-understood [77, 38], we concentrate on type B, C and D gauge algebras. To describe the centralizers of the triple for the gauge groups $(S)O(n)$ and $Sp(2n)$ we introduce more notation. Firstly, if H is a matrix group, we use the notation H^p_Δ to denote the diagonal copy of the group H inside the direct product group H^p. Secondly, we denote by

$$S\left(\prod_{d=1}^{m} H_d\right) \tag{2.42}$$

the subgroup of the product of matrix groups

$$\prod_{d=1}^{m} H_d \tag{2.43}$$

consisting of m-tuples of matrices whose determinants have product one.

The Centralizer in $O(n)$ and $SO(n)$

The basic result for the centralizer of a triple associated to a nilpotent orbit (labelled by a partition satisfying $\sum_d n_d\, d = n$) in $O(n)$ can then be written as

$$C_{O(n)}(\mathfrak{trip}) = \prod_{d\in D_e} [Sp(n_d)]^d_\Delta \times \prod_{d\in D_o} [O(n_d)]^d_\Delta\,, \tag{2.44}$$

which is isomorphic to the group

$$C_{O(n)}(\mathfrak{trip}) \cong \prod_{d\in D_e} Sp(n_d) \times \prod_{d\in D_o} O(n_d). \tag{2.45}$$

[9] We note that the classification of orbits in [79] for D-type gauge groups therefore corresponds to the choice of gauge group $G = O(2n)$.

The second expression (2.45) is simpler, but the first one realises the group $C_{O(n)}(\mathfrak{trip})$ as an explicit subgroup of $O(n)$ in the fundamental representation. From the first expression then, it is straightforward to derive the centralizer in $SO(n)$. Indeed we merely have to enforce the constraint that the determinant be 1, which gives:

$$C_{SO(n)}(\mathfrak{trip}) = S\left(\prod_{d\in D_e}[Sp(n_d)]^d_\Delta \times \prod_{d\in D_o}[O(n_d)]^d_\Delta\right). \tag{2.46}$$

We note that in a generic situation, the centralizer group is not connected, even if we start out with a connected gauge group. For further clarity, we construct an explicit matrix representation of the centralizer subgroup. For each dimension d such that $n_d > 0$, take a matrix M_{n_d} in $Sp(n_d)$ if the dimension d is even and in $O(n_d)$ if d is odd. Then build the matrix $M = Diag(M^d_{n_d})$ (i.e. where the matrix M_{n_d} is present d times along the diagonal). The centralizer is then the group of such matrices M with determinant 1. It then is manifest that

$$C_{SO(n)}(\mathfrak{trip}) \cong \prod_{d\in D_e} Sp(n_d) \times S\left(\prod_{d\in D_o} O(n_d)\right). \tag{2.47}$$

Let us show more explicitly that the centralizer is not necessarily connected. For any matrix M constructed as above,

$$\det M = \prod_{d\in D_e\cup D_o}(\det M_{n_d})^d = \prod_{d\in D_o}\det M_{n_d} \tag{2.48}$$

If the set of odd dimensions appearing in the partitions is not empty, $D_o \neq \emptyset$, the constraint on the determinant removes half of the connected components, leaving $2^{|D_o|-1}$ connected components. When $D_o = \emptyset$, the constraint is automatically enforced. Hence we have

$$Comp_{SO(n)}(\mathfrak{trip}) = \mathbb{Z}_2^{\max(0,|D_o|-1)}. \tag{2.49}$$

The centralizer in $Sp(2n)$

Similarly, the centralizer of a triple inside $Sp(2n)$ is given explicitly by

$$C_{Sp(2n)}(\mathfrak{trip}) = \prod_{d\in D_o}[Sp(n_d)]^d_\Delta \times \prod_{d\in D_e}[O(n_d)]^d_\Delta \tag{2.50}$$

and is isomorphic to

$$C_{Sp(2n)}(\mathfrak{trip}) \cong \prod_{d\in D_o} Sp(n_d) \times \prod_{d\in D_e} O(n_d). \tag{2.51}$$

The component group is equal to:

$$Comp_{Sp(2n)}(\mathfrak{trip}) = \mathbb{Z}_2^{|D_e|}. \tag{2.52}$$

2.5.3 The Supersymmetric Index for the Classical Groups

We have determined the set of inequivalent semi-classical configurations for the adjoint scalar fields $\tilde{\Phi}_i$, as well as the subgroup of the gauge group that is left unbroken by the vacuum expectation values. To compute the semi-classical number of massive vacua, we compute the Witten indices of the pure $\mathcal{N} = 1$ theories that arise upon fixing a given semi-classical configuration for the fields $\tilde{\Phi}_i$, and add them up for all possible inequivalent semi-classical configurations. The global properties of the unbroken gauge group come into play at this stage – we have already stressed that generically, the unbroken gauge group will not be connected (even if we started out with a connected gauge group). We start with a demonstration of how to take into account this complication in the elementary case of the groups $O(4)$ and $SO(4)$. With this example in mind, we can generalize this third step and compute the supersymmetric index for all $\mathcal{N} = 1^*$ theories with classical gauge groups.

$O(4)$ and $SO(4)$

We firstly recall that we have the equivalence of groups $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$ where the element we divide out by is the diagonal center $(-1, -1)$ in the product of the $SU(2)$ groups. Secondly, the $O(4)$ group contains two components, and in particular, it contains a parity operation that exchanges the two $su(2)$ algebras that make up the $\mathfrak{so}(4)$ algebra. This can be viewed as a special case of the $\mathbb{Z}_2$ outer automorphism operation on the $\mathfrak{so}(2n)$ Dynkin diagram of figure 2.1.

The extra gauge symmetry present in the $O(4)$ theory has direct consequences for the counting of inequivalent vacua. For a pure $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(2)$, the number of ground states is two, and they are labelled by a gaugino bilinear $\langle \lambda\lambda \rangle = \pm 1$. When the gauge group is $SU(2) \times SU(2)$, we therefore have $2 \times 2 = 4$ ground states. In pure $\mathcal{N} = 1$ with gauge group $SO(4)$, the number of ground states remains 4, because the action of the diagonal center $\mathbb{Z}_2$ is trivial on each fermion bilinear. The theory with gauge group $O(4)$, on the other hand has the extra element in the gauge group that exchanges the two $SU(2)$ gauge group factors. We must restrict to those vacua that are invariant under this element of the gauge group as well, and these are schematically represented as $(+, +)$, $(-, -)$ and $\frac{1}{\sqrt{2}}((+, +) + (-, -))$. Thus, the Witten index is three for the $O(4)$ theory. The component group matters.

The Consequences of the Component Group

In fact, the example of $O(4)$ is central for the following reason. We know that for a pure $\mathcal{N} = 1$ theory with connected simple Lie group as a gauge group, the Witten index is given by the dual Coxeter number of the underlying Lie algebra. (See appendix B for the relevant table.) If the gauge group is not connected, but has a gauge algebra with a connected Dynkin diagram, then the Witten index remains unchanged. (The extra elements of the gauge group leave the fermion bilinear invariant.) Thus, the only non-trivial case that we need to keep in mind when studying the centralizers (2.47) and (2.50) is precisely the case of $O(4)$. All other cases correspond to either trivial groups, a Coulomb vacuum (if an $SO(2)$ factor is present), or the product of simple factors for whom the component group will not influence the Witten index. Thus, we must keep an eye out only for the difference between the disconnected Dynkin diagram of $\mathfrak{so}(4) = su(2) \oplus su(2)$ which can be subject or not to an exchange identification, depending on the component group of the centralizer.

The Contribution of Each Centralizer

We apply the results derived for the simple groups to the centralizers (2.47) and (2.50). The general strategy is first to compute the number of vacua in pure $\mathcal{N}=1$ with gauge group equal to the connected component of the centralizer, and then modify this counting if necessary by taking into account the action of the component groups (2.49) and (2.52).

As a crucial warm-up exercise, we consider the case of the gauge group $S(O(4)^m)$ for $m \geq 1$. We wish to compute the number of vacua for a pure $\mathcal{N}=1$ theory with this gauge group. The gauge group has 2^{m-1} connected components, all isomorphic (as manifolds) to $SO(4)^m$. We can write

$$S\left(O(4)^m\right) = SO(4)^m \rtimes \mathbb{Z}_2^{m-1} = \left(\frac{SU(2)\times SU(2)}{\mathbb{Z}_2}\right)^m \rtimes \mathbb{Z}_2^{m-1} . \tag{2.53}$$

The Witten index for gauge group $SO(4)^m$ is 4^m. Now we have to take into account the $\mathbb{Z}_2^{m-1}$ factor. This group acts by exchanging an even number of $SU(2)$ factors in the product above. Then the gauge invariant vacua, described in terms of $SU(2)$ factor vacua, are enumerated as follows :

- 2 configurations in which all summands consist of factors equal to $(+-)$ or $(-+)$ (namely, $(+-)^m$ and $(-+)(+-)^{m-1}$, symmetrized appropriately)
- 3^m-1 configurations containing at least one $(++)$ or one $(--)$ (namely, all words with m letters chosen from the alphabet $\{(++),(+-),(--)\}$, except the word $(+-)^m$).

In total, this gives 3^m+1 gauge invariant vacua for gauge group $S(O(4)^m)$.

We have now gathered all elementary ingredients to proceed with the computation of the contribution to the Witten index of each given nilpotent orbit with associated centralizer. Let's start with the centralizer (2.50) for gauge group $Sp(2n)$. The identity component is isomorphic to

$$\prod_{d\in D_o} Sp(n_d) \times \prod_{d\in D_e} SO(n_d) \tag{2.54}$$

and the component group (2.52) ignores the $Sp(n_d)$ factors and acts on all the $SO(n_d)$ factors. In particular, the $SU(2)$ components inside the $SO(4)$ factors are exchanged independently, and the corresponding index is always 3. Taking into account all the factors, we conclude that the Witten index is

$$\prod_{d\in D_o}\left(\frac{n_d}{2}+1\right)\prod_{d\in D_e} I(n_d) \tag{2.55}$$

where the function I counts the number of massive vacua for a pure $\mathcal{N}=1$ theory with gauge group $O(n)$.

Next, we turn to the centralizer (2.47) for the $SO(n)$ nilpotent orbits. The identity component is

$$\prod_{d\in D_e} Sp(n_d) \prod_{d\in D_o} SO(n_d) \tag{2.56}$$

and the component group (2.49) acts on the O factors in the centralizer (2.47) as follows : we can change the sign of the determinant of an even number of matrices in the $O(n_d)$ factors. If there are no $O(4)$ factors, the index follows immediately. The opposite extreme, in which all factors of type $O(n_d)$ are $O(4)$ factors (and there is at least one such $O(4)$ factor), we have

Type	Unbroken Gauge Group	Case	Index
B, D	$S\Big(\prod_{d\in D_e}[Sp(n_d)]^d_\Delta \times \prod_{d\in D_o}[O(n_d)]^d_\Delta\Big)$	No $O(4)$ factor or at least one $O(n)$ factor with $n \geq 1$ and $n \neq 4$.	$\prod_{d\in D_e}(\frac{n_d}{2}+1)\prod_{d\in D_o}I(n_d)$
D	$S(\prod_{d\in D_e}[Sp(n_d)]^d_\Delta \times \prod_{d\in D_o}[O(4)]^d_\Delta)$	Only $O(4)$ factors.	$\Big(3^{\lvert D_o\rvert}+1\Big)\prod_{d\in D_e}(\frac{n_d}{2}+1)$
C	$\prod_{d\in D_o}[Sp(n_d)]^d_\Delta \times \prod_{d\in D_e}[O(n_d)]^d_\Delta$	No restriction	$\prod_{d\in D_o}(\frac{n_d}{2}+1)\prod_{d\in D_e}I(n_d)$

Table 2.4: The supersymmetric index contributions for unbroken subgroups in the connected gauge groups $SO(2n+1)$, $Sp(2n)$ and $SO(2n)$. The function I counts the number of massive vacua for a pure $\mathcal{N}=1$ theory with gauge group $O(n)$, and is given explicitly by $I(1)=1$, $I(3)=2$, $I(4)=3$ and $I(n)=n-2$ for $n \neq 1,3,4$.

treated in our warm-up exercise, and we just take the contribution of the $Sp(n_d)$ factors into account to obtain in that case

$$\left(3^{\lvert D_o\rvert}+1\right)\prod_{d\in D_e}\left(\frac{n_d}{2}+1\right). \tag{2.57}$$

Note that in this case,

$$N=\sum_{d\in D_e} n_d\, d + 4\sum_{d\in D_o} d \tag{2.58}$$

is even by construction. Thus, this particular case arises only for algebras of type D i.e. $SO(2n)$ gauge groups.

Finally, if there exists at least one $n_d \in D_o$ such that $n_d \neq 4$, then we can change the sign of the determinant of any number (odd or even) of matrices in the $O(4)$ factors. Hence the index is again given by the naive formula

$$\prod_{d\in D_e}\left(\frac{n_d}{2}+1\right)\prod_{d\in D_o} I(n_d)\,. \tag{2.59}$$

The supersymmetric indices for the centralizers for all classical groups are summarised in table 2.4.

2.5.4 The Generating Functions

In this subsection, we write down the generating function for the supersymmetric indices that we have computed in subsection 2.5.3. For A_{n-1}-type Lie algebras, no subtleties arise and the supersymmetric index is the sum of the divisors of n [77, 38]. A generating function is therefore $I_{SU(n)}=\sum_{n=1}^{\infty}\sigma_1(n)q^n$, where $\sigma_{k-1}(n)$ is defined in C.15.

Next, we write down the $SO(n)$ generating function. Firstly, we refer to a detailed discussion of the generating function for partitions satisfying the constraint that even parts occur with even multiplicity in [34], and we have derived that it codes the Witten index contribution for the centralizer in $O(n)$. We therefore state that the generating function of [34] captures the number of vacua in the $\mathcal{N}=1^*$ theory with $O(n)$ gauge group. The supersymmetric index of the $O(n)$ theory is then the coefficient of q^n in the series expansion of

$$I_{O(n)}(q) = \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2(1-q^{4k})^2} \tag{2.60}$$

where we introduce the polynomial

$$P_0(x) = 1 - x - x^2 + 3x^3 - x^4 - x^5 + x^6 \,. \tag{2.61}$$

To write down the generating function for gauge group $SO(n)$, we will need to take into account the doubling of the very even partitions and the fact that the centralizers in $SO(n)$ satisfy the constraint that their overall determinant is equal to one. The very even partitions are made out of elementary blocks of the form $2k + 2k + ... + 2k$ with $2m$ terms, and they contribute a factor $Sp(2m)$ in the residual symmetry, giving rise to $m+1$ quantum vacua. The corresponding contribution in the partition of n is $4mk$, so the number of vacua corresponding to the very even partitions is the coefficient of q^n in the generating function

$$\prod_{k=1}^{\infty} \sum_{m=0}^{\infty} (m+1) q^{4km} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{4k})^2} \,. \tag{2.62}$$

A partition is either very even, or not. Moreover, the very even partitions are already counted once in the generating function (2.60) – it is the origin of the second factor in the denominator [34]. Thus, to count them with the required double multiplicity, we add the generating function (2.62) to the generating function (2.60), obtaining

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^{4k})^2} \left(1 + \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2} \right) \,. \tag{2.63}$$

However, we still have to take special care of the partitions that fall under the restrictions of the second line in table 2.4. In the expression (2.63), all such partitions received an index $3^{|D_o|} \prod_{d \in D_e} (\frac{n_d}{2} + 1)$ (corresponding to imposing all $O(4)$ gauge invariances), and we must add those vacua that are gauge invariant under the smaller unbroken gauge group arising from the unit determinant requirement. To add this contribution, we first write the generating function of the number of partitions that give rise to an unbroken $S\left(O(4)^m\right)$ gauge group factor, possibly supplemented with $Sp(2n)$ gauge group factors. These are partitions of $N = 2n$ into odd integers, each of which appear exactly four times. Equivalently, they are partitions of $N/4$ into distinct odd integers. The coefficient of x^n in

$$-1 + \prod_{k=1}^{\infty} (1 + x^{2k-1}) \tag{2.64}$$

is the number of partitions of n into distinct odd integers, so that setting $x = q^4$ gives us the generating function for this partition problem. Including the supplementary $\mathfrak{sp}(2n)$ factors boils down to multiplying by the generating function (2.62), and we thus get the extra gauge invariant massive vacua counting function:

$$\prod_{k=1}^{\infty} \frac{1}{(1-q^{4k})^2} \left(-1 + \prod_{k=1}^{\infty} (1 + q^{8k-4}) \right) \,. \tag{2.65}$$

Finally, the generating function for $SO(n)$ follows from adding the $O(n)$ generating function, supplemented with the second copy of the very even partitions coded in (2.63) as well as the extra gauge invariant vacua of equation (2.65). We obtain that the number of massive vacua in the $SO(n)$ theory is the coefficient of q^n in

$$I_{SO(n)}(q) = \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{4k})^2(1-q^{2k-1})^2} + \prod_{k=1}^{\infty} \frac{1+q^{8k-4}}{(1-q^{4k})^2} \,. \tag{2.66}$$

The generating function for $SO(2n+1)$ gauge group is obtained by taking the odd part

$$I_{SO(2n+1)}(q) = I_{O(2n+1)}(q) = \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2(1-q^{4k})^2} - \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(-q^{2k-1})}{(1+q^{2k-1})^2(1-q^{4k})^2} \,. \tag{2.67}$$

This result was obtained in [34]. In the case of B-type gauge group, the counting function is identical for $O(2n+1)$ and $SO(2n+1)$. For the D-type gauge groups, we find

$$I_{O(2n)}(q) = \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2(1-q^{4k})^2} + \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(-q^{2k-1})}{(1+q^{2k-1})^2(1-q^{4k})^2} \,, \tag{2.68}$$

and

$$I_{SO(2n)}(q) = \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{4k})^2(1-q^{2k-1})^2} + \frac{1}{2}\prod_{k=1}^{\infty} \frac{P_0(-q^{2k-1})}{(1-q^{4k})^2(1+q^{2k-1})^2} + \prod_{k=1}^{\infty} \frac{1+q^{8k-4}}{(1-q^{4k})^2} \,. \tag{2.69}$$

The last term in equation (2.66) contributes to the $SO(2n)$ generating function only. In the next section, we will discuss a few detailed examples that illustrate the special features of the D_n supersymmetric index.

Finally, for reference we also give the generating function for $Sp(2n)$:

$$I_{Sp(2n)}(q) = \prod_{k=1}^{\infty} \frac{P_0(q^{2k})}{(1-q^{2k})^2(1-q^{4k-2})^2} = q^{-1} I_{SO(2n+1)}(q) \,. \tag{2.70}$$

The last equality is highly non-trivial and was proven in [34]. It is a consequence of the S-duality between B and C-type $\mathcal{N}=1^*$ gauge theories. The q^{-1} factor is here to take into account the shift in powers of q when going from $SO(2n+1)$ to $Sp(2n)$. The identity of the left and right hand side was already noted by Ramanujan [97].

2.5.5 Illustrative Examples

Before moving on to the exceptional groups, we illustrate some of the salient features of the analysis by two telling cases in which the special features of $(S)O(4)$ (sub)groups come into play. We will consider in turn gauge algebras $\mathfrak{so}(4)$ and $\mathfrak{so}(8)$. We first give the first few terms of the expansion of the generating functions given in the previous section, to facilitate numerical checks:

$$I_{O(n)}(q) = 1 + q + 3q^3 + 6q^4 + 6q^5 + 7q^6 + 15q^7 + 26q^8 + 31q^9 + 36q^{10} + \ldots \tag{2.71}$$
$$I_{SO(n)}(q) = 2 + q + 3q^3 + 9q^4 + 6q^5 + 7q^6 + 15q^7 + 33q^8 + 31q^9 + 36q^{10} + \ldots \tag{2.72}$$
$$I_{Sp(2n)}(q) = 1 + 3q^2 + 6q^4 + 15q^6 + 31q^8 + 59q^{10} + 115q^{12} + 208q^{14} + \ldots \tag{2.73}$$

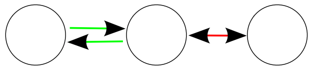


Figure 2.2: The diagram of dualities for $SU(2)$. Each circle represents a massive vacuum. In red, we show the action of S-duality on the massive vacua, in green T-duality (when non-trivial).

The gauge algebra $\mathfrak{so}(4)$

In a first example, we discuss the $\mathcal{N}=1^*$ theories with $\mathfrak{so}(4)$ gauge algebra. The $\mathfrak{so}(4)$ algebra is the direct sum of two $su(2)$ algebras. Thus, the theory with $SO(4)$ gauge group has the same vacuum structure as the theory with product gauge group $SU(2)\times SU(2)$, and therefore has 3×3 massive vacua. Let us check this elementary statement using our semi-classical method, based on the classification of nilpotent orbits:

Orbit	Residual symmetry	Number of vacua
$\{2,2\}_1$	$Sp(2)$	2
$\{2,2\}_2$	$Sp(2)$	2
$\{3,1\}$	$S(O(1)\times O(1))$	1
$\{1,1,1,1\}$	$SO(4)$	2×2

(2.74)

Yet another way to establish this result is by using the low energy effective superpotential on $\mathbb{R}^3\times S^1$ (under the condition that no extra vacua arise upon compactification, which is satisfied in this case). This analysis has been done in [38]. One can map out how the massive vacua behave under the infrared duality group [77], and one finds the tensor product of two duality diagrams of $SU(2)$ – the latter is depicted in figure 2.2. The result is drawn in figure 2.3.

When we consider the $\mathcal{N}=1^*$ theory with $O(4)$ gauge group, on the other hand, we have the table of nilpotent orbits and centralizers

Orbit	Residual symmetry	Number of vacua
$\{2,2\}$	$Sp(2)$	2
$\{3,1\}$	$O(1)\times O(1)$	1
$\{1,1,1,1\}$	$O(4)$	$2\times 2-1$

(2.75)

for a total of 6 vacua, in agreement with (2.71). The diagram of dualities is now the direct sum of two $SU(2)$ duality diagrams (i.e. twice figure 2.2). We see the crucial role played by the $\mathbb{Z}_2$ identification of the two $su(2)$ summands in the Lie algebra when the gauge group is $O(4)$, as well as the fact that very even partitions correspond to a single orbit for $O(2n)$ groups.

The gauge algebra $\mathfrak{so}(8)$

Another interesting case that we wish to put forward is the case of the Lie algebra $\mathfrak{so}(8)$. We note first of all that the group of outer automorphisms is the triality group S_3 that acts on the $\mathfrak{so}(8)$ Dynkin diagram (in figure 2.4) by permuting the three external nodes. In the following table, we list the partitions that correspond to the $SO(8)$ nilpotent orbits. We also provide the Dynkin labels of the orbit (see [93] for the necessary background). The latter labelling is useful here, since it provides a direct handle on the behaviour of the orbits under the triality group.

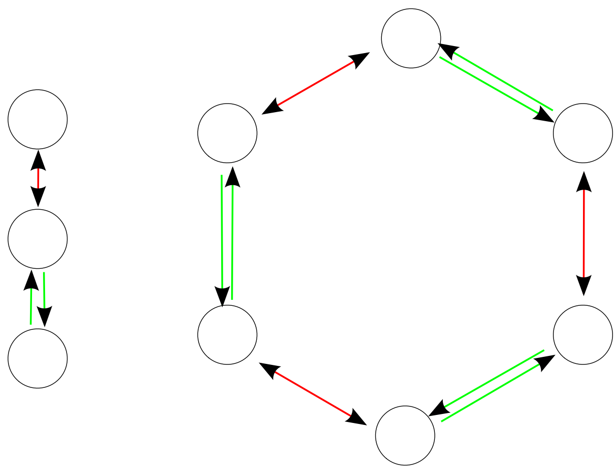


Figure 2.3: The diagram of dualities for $SO(4)$. In red, we show the action of S-duality on the massive vacua, in green T-duality (when non-trivial).

Orbit	Dynkin diagram	Residual symmetry	Massive vacua	Triality Rep
$\{4,4\}_1$	$(0,2,0,2)$	$Sp(2)$	2	3_1
$\{4,4\}_2$	$(0,2,2,0)$	$Sp(2)$	2	3_1
$\{5,3\}$	$(2,0,2,2)$	$S(O(1)^2)$	1	
$\{7,1\}$	$(2,2,2,2)$	$S(O(1)^2)$	1	
$\{2,2,2,2\}_1$	$(0,0,0,2)$	$Sp(4)$	3	3_2
$\{2,2,2,2\}_2$	$(0,0,2,0)$	$Sp(4)$	3	3_2
$\{3,2,2,1\}$	$(1,0,1,1)$	$Sp(2)\times S(O(1)^2)$	2	
$\{3,3,1,1\}$	$(0,2,0,0)$	$S(O(2)^2)$	0	
$\{5,1,1,1\}$	$(2,2,0,0)$	$S(O(3)\times O(1))$	2	3_1
$\{2,2,1,1,1,1\}$	$(0,1,0,0)$	$Sp(2)\times SO(4)$	8	
$\{3,1,1,1,1,1\}$	$(2,0,0,0)$	$S(O(5)\times O(1))$	3	3_2
$\{1,1,1,1,1,1,1,1\}$	$(0,0,0,0)$	$SO(8)$	6	

Table 2.5: Summary of the $\mathfrak{so}(8)$ analysis.

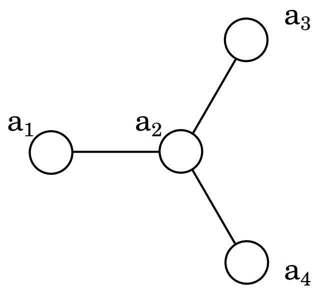


Figure 2.4: This weighted Dynkin diagram for $\mathfrak{so}(8)$ is denoted (a_1, a_2, a_3, a_4) in the text.

The three orbits labelled 3_1 in the last column form a triplet under triality, as do the three orbits labelled 3_2. There are a total of 33 massive vacua for the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^4$ with gauge group $SO(8)$. When we mod out by a $\mathbb{Z}_2$ outer automorphism, which is equivalent to studying the gauge group $O(8)$, we find 26 massive vacua. To find this final tally, we need to realize that the $(0,1,0,0)$ orbit contributes $2 \times (2 \times 2 - 1) = 6$ vacua when the gauge group is $O(8)$.

2.6 The Counting for the Exceptional Groups

In this section, we count the number of massive vacua for mass-deformed $\mathcal{N} = 4$ super Yang-Mills theory with exceptional gauge group. For determining the centralizer subgroups, we assume that our gauge group is the adjoint group $G = G_{ad}$. The supersymmetric indices for other choices of centers are identical.

2.6.1 The Orbits and the Centralizers

The first two steps in our program consist of listing the nilpotent orbits, in bijection with the $\mathfrak{sl}(2)$ triples, and the centralizers, the subgroup of the gauge group left unbroken by the adjoint scalar field vacuum expectation values. While there is a handy list of nilpotent orbits (see e.g. [93]) of exceptional Lie algebras available, we need to delve slightly deeper into the mathematics to understand the centralizers of the associated triples.

Let n be a nilpotent element of a Lie algebra $\mathfrak{g}$ and $\mathfrak{trip}$ the span of an $\mathfrak{sl}(2)$ triple corresponding to n. The centralizer of the triple is reductive (i.e. semi-simple plus abelian factors) and is a factor in the centralizer of the nilpotent element $C_G(n) = C_G(\mathfrak{trip}) \ltimes U$ where U is the unipotent radical of $C_G(n)$. From chapter 13 of [93] we can read off both the type and the component group of $C_G(\mathfrak{trip})$ (called C there), which is almost all we need to compute the contribution to the Witten index for a given orbit. However, we still have to know precisely how the component group of $C_G(\mathfrak{trip})$ acts on its connected components. These are final gauge equivalence identifications that we will need to perform on the vacua of effective pure $\mathcal{N} = 1$ supersymmetric Yang-Mills theories. These actions can be deduced from the detailed reference [95].

In the following subsection, we explicitly compute the supersymmetric index by going through the lists of nilpotent orbits, following the order of the lists provided in [93],[10] and for each of these orbits, we compute the contribution to the index as follows.

[10]The numbering used in [95] is one lower than our numbering.

(a) If the unbroken gauge group $C_G(\mathfrak{trip})$ contains an abelian factor, we have a massless, Coulomb vacuum, and the contribution is zero.

(b) If the unbroken gauge group $C_G(\mathfrak{trip})$ is simple (respectively trivial), the contribution is the dual Coxeter number of the corresponding Lie algebra (respectively one). (The action of a possible component group on the gaugino bilinear is trivial.)

(c) If the unbroken gauge group $C_G(\mathfrak{trip})$ contains several simple factors, and the component group is trivial, then the contribution is the product of the dual Coxeter numbers of the simple factors.

(d) If the unbroken gauge group $C_G(\mathfrak{trip})$ contains several simple factors, and the component group is non-trivial, then we consider the tensor product of the quantum vacua of each simple factor, and count those vacua which are also gauge invariant with respect to the action of the component group.

2.6.2 The Supersymmetric Index for the Exceptional Groups

We proceed on a case-by-case basis.

The algebra G_2

The number of orbits is 5. For each of these orbits, we read off the Lie algebra of the unbroken gauge group in [93] (see our table 2.6) and observe that they are all simple. Hence the supersymmetric index is:

$$I_{G_2} = 4 + 2 + 2 + 1 + 1 = 10. \tag{2.76}$$

Number	Unbroken Gauge Algebra	Component group	Number of Massive Vacua
1	G_2	1	4
2	A_1	1	2
3	A_1	1	2
4	1	S_3	1
5	1	1	1

Table 2.6: The number of the orbit in the table in [93], the unbroken gauge algebra, the non-identity component of the unbroken gauge group and the number of massive vacua the orbit gives rise to.

The algebra F_4

There are 16 orbits for the algebra F_4, enumerated in table 2.7. All centralizers are either simple or trivial (case (b) above), except for the orbits number 4 and 8. Note that for orbits number 3 and 5 (as a few examples amongst many), the component group acts non-trivially on the gauge algebra. The outer automorphism action leaves the fermion bilinear of pure $\mathcal{N} = 1$ super Yang-Mills theory invariant.

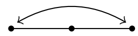
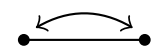
Number	Gauge Algebra	Component group	Action	Massive Vacua
1	F_4	1		9
2	C_3	1		4
3	A_3	S_2		4
4	$A_1 + A_1$	1		2×2
5	A_2	S_2		3
6	G_2	1		4
7	A_1	1		2
8	$A_1 + A_1$	S_2		$2 \times 2 - 1$
9	A_1	1		2
10	A_1	S_2		2
11	1	S_4		1
12	A_1	1		2
13	A_1	1		2
14	1	S_2		1
15	1	S_2		1
16	1	1		1

Table 2.7: The number of the orbit in the [93] table, the unbroken gauge algebra, the non-identity component of the gauge group, as well as its action on the gauge algebra, in a selection of cases. The last column indicates the contribution to the supersymmetric index.

Orbit number 4 has trivial component group, and therefore falls into case (c) where we count all tensor product vacua. More interestingly, in the case of orbit 8, the Lie algebra of the unbroken gauge group is $A_1 \oplus A_1$, and the disconnected component of the centralizer acts to exchange the two $su(2)$ algebras. This is a familiar phenomenon, and we realize that the number of gauge invariant vacua is 3. The total number of massive vacua is therefore:

$$I_{F_4} = 9 + 4 + 4 + (2 \times 2) + 3 + 4 + 2 + (3) + 2 + 2 + 1 + 2 + 2 + 1 + 1 + 1 = 45. \qquad (2.77)$$

We have put the contribution of orbit number 4 and orbit number 8 between parentheses in equation (2.77) to clearly exhibit the different number of vacua which they contribute to the total supersymmetric index, despite the fact that the algebra of the centralizer is identical for both orbits.

The algebra E_6

For the E_6 algebra, there are 21 orbits, listed in table 2.8.

Number	Gauge Algebra	Component group	Action	Massive Vacua
1	E_6	1		12
2	A_5	1		6

Number	Gauge Algebra	Component group	Action	Massive Vacua
3	$B_3 + T_1$	1		0
4	$A_2 + A_1$	1		2×3
5	$A_2 + A_2$	S_2	$\updownarrow$	$3 \times 3 - 3$
6	$A_2 + T_1$	1		0
7	G_2	1		4
8	$A_1 + T_1$	1		0
9	$B_2 + T_1$	1		0
10	A_1	1		2
11	$A_1 + T_1$	1		0
12	T_2	S_3		0
13	$A_1 + T_1$	1		0
14	A_2	1		3
15	T_1	1		0
16	A_1	1		2
17	T_1	1		0
18	1	S_2		1
19	T_1	1		0
20	1	1		1
21	1	1		1

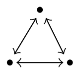
Table 2.8: The number of the orbit, the unbroken gauge algebra, the non-identity component of the gauge group, as well as its action on the gauge algebra (when relevant), and the resulting number of massive vacua.

All orbits have either a simple centralizer, or a trivial component group (and therefore fall into classes (b) or (c)), except orbit number 5. The component group of orbit number 5 exchanges the two $su(3)$ algebras in the unbroken gauge group. We therefore need to count the $SU(3) \times SU(3)$ pure $\mathcal{N} = 1$ super Yang-Mills vacua which are invariant under this interchange. The 3×3 representation of the $\mathbb{Z}_2$ exchange splits into 6 invariants and 3 non-trivial representations. We therefore find the index

$$\begin{aligned} I_{E_6} &= 12 + 6 + 0 + 2 \times 3 + (3 \times 3 - 3) + 0 + 4 + 0 + 0 + 2 \\ &\quad +0 + 0 + 0 + 3 + 0 + 2 + 0 + 1 + 0 + 1 + 1 \\ &= 44. \end{aligned} \tag{2.78}$$

The algebra E_7

There are 45 orbits, among which only orbits number 16 and 19 need special care.

Number	Gauge Algebra	Component group	Action	Massive Vacua
1	E_7	1		18
2	D_6	1		10
3	$B_4 + A_1$	1		14
4	F_4	1		9
5	$C_3 + A_1$	1		8
6	A_5	S_2		6
7	C_3	1		4
8	$A_3 + T_1$	S_2		0
9	$3A_1$	1		8
10	$B_3 + A_1$	1		10
11	$G_2 + A_1$	1		8
12	G_2	1		4
13	B_3	1		5
14	$2A_1$	1		4
15	$3A_1$	1		8
16	$3A_1$	S_3		4
17	$2A_1$	1		4
18	C_3	1		4
19	$2A_1$	S_2	$\bullet \longleftrightarrow \bullet$	3
20	$A_1 + T_1$	S_2		0
21	$A_2 + T_1$	S_2		0
22	A_1	1		2
23	G_2	1		4
24	B_2	1		3
25	T_2	S_2		0
26	$A_1 + T_1$	S_2		0
27	A_1	1		2
28	$2A_1$	1		4
29	A_1	1		2
30	A_1	1		2
31	A_1	1		2
32	A_1	S_2		2
33	$2A_1$	1		4
34	1	S_3		1
35	A_1	1		2

Number	Gauge Algebra	Component group	Action	Massive Vacua
36	A_1	1		2
37	A_1	1		2
38	1	S_2		1
39	A_1	1		2
40	T_1	S_2		0
41	A_1	1		2
42	1	S_2		1
43	1	1		1
44	1	1		1
45	1	1		1

Table 2.9: The number of the orbit in the [93] table, the unbroken gauge algebra, the non-identity component of the gauge group, as well as the action on the gauge algebra, when relevant. The final column is the tally of massive vacua.

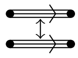
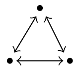
On the $A_1 \oplus A_1 \oplus A_1$ gauge algebra left unbroken by the triple associated to orbit number 16, the S_3 component group acts as permutations of the summands. We therefore compute the number of the 2^3 tensor product vacua which are invariant under S_3. The number is equal to 4, which is then the contribution to the supersymmetric index associated to orbit number 16. In the case of orbit number 19, the unbroken $\mathbb{Z}_2$ factor acts by exchanging the two gauge groups of type A_1, and the number of vacua is 3. The final tally is

$$\begin{aligned} I_{E_7} &= 18+10+2\times 7+9+2\times 4+6+4+0+2^3+2\times 5+2\times 4+4+5+2^2+2^3 \\ &\quad +(4)+2^2+4+(3)+0+0+2+4+3+0+0+2+2^2+2+2 \\ &\quad +2+2+2^2+1+2+2+2+1+2+0+2+1+1+1+1 \\ &= 174. \end{aligned} \tag{2.79}$$

The algebra E_8

The algebra E_8 exhibits 70 nilpotent orbits, catalogued in table 2.10.

Number	Gauge Algebra	Component group	Action	Massive Vacua
1	E_8	1		30
2	E_7	1		18
3	B_6	1		11
4	$F_4 + A_1$	1		18
5	E_6	S_2		12
6	C_4	1		5
7	A_5	S_2		6
8	$B_3 + A_1$	1		10

Number	Gauge Algebra	Component group	Action	Massive Vacua
9	B_5	1		9
10	$G_2 + A_1$	1		8
11	$2G_2$	S_2		10
12	$G_2 + A_1$	1		8
13	$B_3 + A_1$	1		10
14	D_4	S_3		6
15	F_4	1		9
16	B_2	1		3
17	$B_2 + A_1$	1		6
18	$3A_1$	S_3		4
19	$B_2 + T_1$	S_2		0
20	A_4	S_2		5
21	$2A_1$	1		4
22	C_3	1		4
23	A_2	S_2		3
24	$A_2 + T_1$	S_2		0
25	B_2	1		3
26	A_3	S_2		4
27	$A_1 + T_1$	S_2		0
28	$2A_1$	1		4
29	$G_2 + A_1$	1		8
30	$2A_1$	1		4
31	A_1	1		2
32	A_2	S_2		3
33	G_2	S_2		4
34	B_3	1		5
35	A_1	1		2
36	$2A_1$	1		4
37	A_1	1		2
38	$2A_1$	S_2	$\bullet\!\longleftrightarrow\!\bullet$	3
39	A_1	S_2		2
40	A_1	S_3		2
41	$2A_1$	1		4
42	1	S_5		1
43	$2A_1$	1		4

Number	Gauge Algebra	Component group	Action	Massive Vacua
44	$2A_1$	S_2	$\bullet\!\longleftrightarrow\!\bullet$	3
45	A_1	1		2
46	A_1	S_2		2
47	A_2	S_2		3
48	T_1	S_2		0
49	B_2	1		3
50	G_2	1		4
51	T_1	S_2		0
52	A_1	1		2
53	T_1	S_2		0
54	A_1	S_2		2
55	1	S_3		1
56	T_1	S_2		0
57	A_1	1		2
58	A_1	1		2
59	1	S_3		1
60	A_1	1		2
61	1	S_3		1
62	A_1	1		2
63	1	S_2		1
64	1	S_2		1
65	A_1	1		2
66	1	S_2		1
67	1	S_2		1
68	1	1		1
69	1	1		1
70	1	1		1

Table 2.10: The orbit in the table of [93] , the type of centralizer, the component group, as well as its action on the gauge algebra, when relevant. The number of massive vacua results.

There are 70 orbits, among which orbits number 11, 18, 38 and 44 deserve special care. In orbit 11, the algebra of the unbroken gauge group is $G_2 \oplus G_2$, and the two summands are exchanged by the S_2 component group. Of the 4×4 vacua of pure $\mathcal{N} = 1$ with gauge group $G_2 \times G_2$, 10 are invariant under the component group. In orbit 18, the three $su(2)$ algebras are exchanged by the S_3 group as in orbit 16 of E_7, leading to 4 vacua. In the case of orbits 38 and 44, two $su(2)$ algebras are exchanged by the S_2 component group, so both contribute 3 vacua. The census yields

$$I_{E_8} \quad = \quad 301\,. \tag{2.80}$$

We have thus computed all Witten indices of $\mathcal{N} = 1^*$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$.

Chapter 3

Elliptic Integrable Systems and Modularity

3.1 Introduction

In this chapter, we leave aside supersymmetric gauge theories and begin an exploration of a certain class of integrable systems. The reason why this particular class is of interest to us will become clear in chapter 5, and no mention of gauge theories will appear in the present chapter and the next one, which can be read independently. As we will explain shortly, the Calogero-Moser systems to be considered here are fundamental examples of integrable models, on the one hand because they appear in a natural way in classical mechanics and are easily defined, and on the other hand possess deep modular and Lie theoretic properties that we will review in due course. In addition, studying these models is very rewarding because other systems can be reached from them by various limiting processes.

In the first section, we define the Calogero-Moser systems and lay bare some of its elementary properties. We then show explicitly how the associated potential transforms under the modular group action. Finally we will present in section 3.4 an analysis that extends the limits performed by Inozemtsev [36] and further investigated by Khastgir, Sasaki and Takasaki [98]. The limits that can be taken are classified by data from the affine Lie algebra associated with the potential. We will see in the last chapter of this thesis that this structure also plays a fundamental role on the gauge theory side.

3.2 The Calogero-Moser system

3.2.1 General Definitions

It is interesting to identify and study dynamical systems that are integrable. It is common to discover in a given theory of physical interest that a subsector has an integrable structure. In this case, one can hope for exact solutions, and we will see that this is the case for the Calogero-Moser systems that we want to study: in the most elementary case (rational potential), the Newtonian equations of motion can be solved exactly usign purely algebraic operations. Before diving into the details of Caloger–Moser systems, we remind the reader of various useful definitions.

In classical mechanics, any function $F(p,q)$ on phase space evolves according to $\dot{F} = \{H, F\}_P$

where H is the Hamiltonian. We say that a dynamical system on a phase space of dimension $2n$ is *Liouville integrable* if one knows n independent conserved quantities F_i which Poisson-commute (and then the Hamiltonian is a function of the F_i).

The notion of Lax pairs is fundamental (for a pedagogical introduction see [99]). It consists in presenting the equations of motion in the form

$$\dot{L}(z) = [L(z), M(z)]\,, \tag{3.1}$$

where the matrices $L(z)$ and $M(z)$ depend on the dynamical variables and on the *spectral parameter* z. Using the well-known formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\det L = (\det L)\mathrm{tr}\left(L^{-1}\dot{L}\right)\,, \tag{3.2}$$

we find that the equation $\det(L(z)-k)=0$, for $k\in\mathbb{C}$ a constant, is time-independent.[1] This means that the eigenvalues of $L(z)$ are time-independent, and so are the symmetric functions built from them, and the curve Γ it defines,

$$\Gamma = \{(k,z)\in\mathbb{C}^2 |\, \det(L(z)-k)=0\}\,. \tag{3.3}$$

This curve is called the *spectral curve*, it can be seen as a Riemann surface, and its moduli contain the conserved quantities.

An equivalent but more abstract way of presenting a classical integrable system, which emphasizes the essential data in the presentation above, is the following. One has to exhibit a $2n$-dimensional manifold with a symplectic form ω and an $\mathbb{R}^n$-valued function $(F_1,\ldots,F_n)$ whose components Poisson-commute, defined on this manifold. The preimage of a given point in $\mathbb{R}^n$ is the *level* manifold, and is isomorphic to a n-dimensional torus. One can then define action-angle variables $(I_j,\varphi_j)_{j=1,\ldots,n}$ that correspond to a given basis $(A_j)_{j=1,\ldots,n}$ of cycles of the torus via

$$I_j = \frac{1}{2\pi}\oint_{A_j}\mathrm{d}^{-1}\omega\,. \tag{3.4}$$

We come back to this formalism in section 3.2.5.

3.2.2 Why Calogero-Moser Systems ?

Let us first consider the very practical problem of the dynamics of a given number of classical non-relativistic particles in our 3-dimensional space with a given interaction potential (for instance, a Coulomb or gravitational potential). In general if the number of particles is not very small ($n\le 2$), it is impossible to obtain an exact solution of this problem. There is a situation in which we know such a solution for any number n of particles, namely when the particles interact pairwise with a quadratic potential. But the reason why this situation is tractable is that it reduces to $n-1$ *independent* particles, each moving in a harmonic potential, and therefore it is really a one-particle problem.

The situation is different in one-dimensional spaces. We will present systems on n-particles evolving in one-dimensional spaces with non-trivial pairwise interactions that are integrable. They can be generalized and embedded into a family of systems indexed by simple Lie algebras, and their integrability results from the presence of higher hidden symmetries [100, 101, 102].

[1] Note that the equation (3.1) can be solved by $L(z,t)=U(t)L(z,0)U^{-1}(t)$ and $M(z,t)=-\dot{U}(t)U^{-1}(t)$.

The coordinates of the particles that we consider are denoted $(Z_1, \dots, Z_n)$. We take the usual non-relativistic[2] kinetic term

$$\frac{1}{2}\dot{Z}^2 = \frac{1}{2}\sum_{i=1}^{N} P_i^2\,, \tag{3.5}$$

where the P_i are the canonical momenta associated to the positions Z_i, and a potential

$$V(Z) = g\sum_{i<j} f\left(Z_i - Z_j\right) \tag{3.6}$$

where f is an even function. In this thesis, this function f will mostly be the Weierstrass function $\wp$ associated to two half-periods ω_1 and ω_2, in which case we call the physical system the *Elliptic Calogero-Moser* system. The definition and useful properties of the Weierstrass function and other elliptic functions can be found in section C.2. Two other choices are frequent, and we summarize the terminology[3] as follows, and in figure 3.1 :

$$f(z) = \begin{cases} \frac{1}{z^2} & \text{Rational Calogero-Moser System} \\ \frac{1}{\sin^2 z} & \text{Trigonometric Calogero-Moser System} \\ \wp(z;\omega_1,\omega_2) & \text{Elliptic Calogero-Moser System} \end{cases} \tag{3.7}$$

The rational system accurately models particles on a line with a strong repulsive interaction. It is possible to add a confining harmonic potential on top, so that the particles don't go to infinity, and one can show that this doesn't spoil the integrability properties stated below. The trigonometric system can be thought of as the rational system on a circle, keeping in mind the relation

$$\sum_{k\in\mathbb{Z}} \frac{1}{(z+k\pi)^2} = \frac{1}{\sin^2 z}\,. \tag{3.8}$$

The elliptic potential is harder to interpret with a real z, but if we allow z to take complex values, it obviously lives on a torus.

The form of the interactions in (3.6) seems to be the most natural one from the physical point of view, but it sheds light on its symmetries to write it in an apparently more contrived form. The root system of the Lie algebra A_{N-1} can be described as follows: we take N vectors $\epsilon_1, \dots, \epsilon_N \in \mathfrak{h}^*$, where $\mathfrak{h}$ is the Cartan subalgebra, that satisfy

$$\sum_{i=1}^{N} \epsilon_i = 0 \tag{3.9}$$

in a Euclidean space, with the scalar products

$$(\epsilon_i, \epsilon_j) = \delta_{ij} - \frac{1}{N}\,. \tag{3.10}$$

Then the set of positive roots of A_{N-1} is

$$\Delta^+_{A_{N-1}} = \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq N\}\,. \tag{3.11}$$

[2]There exists a relativistic version of the Calogero-Moser system, which is often called the Ruijsenaars-Schneider system.

[3]The Trigonometric Calogero-Moser System is also known as the Sutherland System.

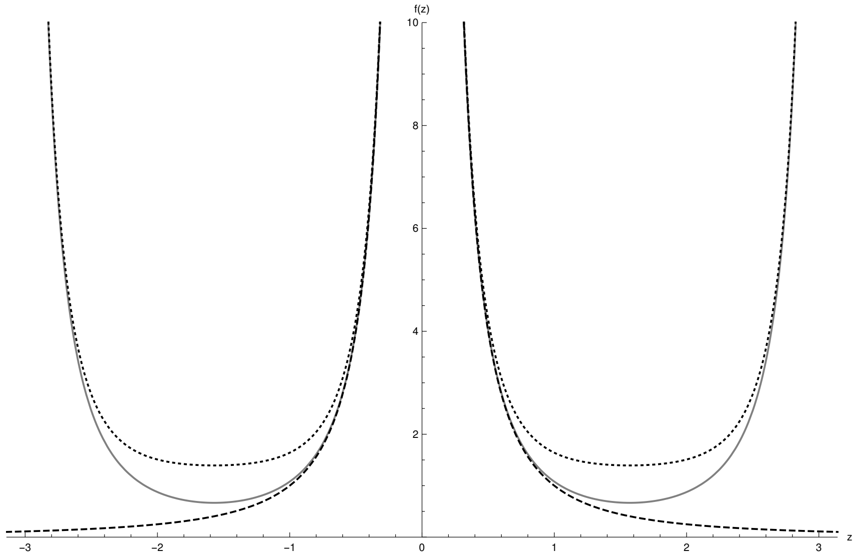


Figure 3.1: The dashed line is the graph of $z \mapsto 1/z^2$, the gray line is the graph of $z \mapsto 1/\sin^2 z$ and the dotted line is the graph of $z \mapsto \wp\left(z; \frac{\pi}{2}, i\frac{\pi}{4}\right)$, corresponding to $\tau = \frac{i}{2}$.

Now we can also embed the Z_i in $\mathfrak{h}^*$ via

$$Z \equiv \sum_{i=1}^{N} Z_i \epsilon_i \, . \tag{3.12}$$

Then if $\alpha = \epsilon_i - \epsilon_j \in \Delta^+_{A_{N-1}}$, one can compute $(\alpha, Z) = \alpha(Z) = Z_i - Z_j$. Hence we rewrite (3.6) as

$$V_{A_{N-1}}(Z) = g \sum_{\alpha \in \Delta^+_{A_{N-1}}} f\left(\alpha(Z)\right) \, . \tag{3.13}$$

We have added a subscript on V to indicate the root system that is used. A generalization is now possible to any root system. We denote by r the dimension of the root space, which is also the rank of the associated Lie algebra. The vector Z lives in the space dual to the root lattice. The coupling constant g in front of the potential played a spectator role until now, since it is just a multiplicative constant, but it becomes important when all the positive roots in the chosen root system are not equivalent (we say that two roots are equivalent if they are connected by the action of the Weyl group). In this case there are as many coupling constants as there are equivalence classes. For the simply-laced root systems there is only one such class, and for the non simply-laced root systems associated to a simple Lie algebra (namely B_r, C_r, F_4 and G_2) there are two classes, the long and the short roots. In this last case we will therefore have two coupling constants g_l and g_s. The most general potential is therefore

$$V_{\mathfrak{g}}(Z) = V_{\Delta_{\mathfrak{g}}}(Z) = \sum_{\alpha \in \Delta^+_{\mathfrak{g}}} g_{\nu(\alpha)} f\left(\alpha(Z)\right) \, , \tag{3.14}$$

where the index $\nu(\alpha)$ is defined by

$$\nu(\alpha) = \frac{|\alpha_{long}|^2}{|\alpha|^2} = \frac{2}{|\alpha|^2} \tag{3.15}$$

and the coupling constant $g_{\nu(\alpha)}$ then depends only on the length of the root α. With our conventions (B.12), this means that there are three possible coupling constants g_1, g_2 and g_4, and at most two are involved in any integrable system based on a root system.[4] We use interchangeably the notations $\alpha \cdot Z$ and $\alpha(Z)$ throughout thesis.

The Weierstrass function becomes a $\frac{1}{\sin^2}$ function when the period $\omega_2 \to i\infty$, as shown by the limit (C.27) in the appendix. Moreover, if one sends the two periods to infinity, $\omega_1 \to \infty$ and $\omega_2 \to i\infty$, then we have $\wp(z) \sim z^{-2}$. Therefore the rational and trigonometric potentials are limits of the elliptic potential.

Now we know how to generalize (3.6) properly, but we haven't seen yet why this potential and its generalization (3.14) are so interesting. The answer is that *these systems are integrable for all Lie algebra root systems.*

Let us focus on the simplest form based on the A_{N-1} Lie algebra with potential (3.6). Calogero and Moser proposed [103, 102, 104] an ansatz which depends on three unknown functions. Requiring that the Lax equation (3.1) be equivalent to Hamilton's equations yields functional differential equations for the unknown functions. These can be solved exactly in some cases, and the solution determines the function f, basically resulting in (3.7). For the Elliptic Calogero-Moser system, the explicit solution obtained from the ansatz is

$$L_{ij}(z) = P_i\delta_{ij} - \sqrt{2g}(1-\delta_{ij})\Phi(Z_i - Z_j; z)\,, \tag{3.16}$$

$$M_{ij}(z) = \sqrt{2g}\sum_{k\neq i}\wp(Z_i - Z_k)\delta_{ij} + \sqrt{2g}(1-\delta_{ij})\Phi'(Z_i - Z_j; z)\,, \tag{3.17}$$

where Φ is the Lamé function given by

$$\Phi(x; z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)}e^{x\zeta(z)} \tag{3.18}$$

and the Weierstrass functions σ and ζ are related to $\wp$ by $\wp = -\zeta'$ and $\zeta = \sigma'/\sigma$. We then have very explicit formulas for a Lax pair, but this is not enough to assert that the system is integrable: we need to check that the time-independent quantities $I_k \propto \mathrm{tr}L^k$, for $k = 2, \ldots, N$ are functionally independent and satisfy $\{I_k, I_l\} = 0$. This is a difficult task which we are not going to present here, but the interested reader can find the proof in [104].

It should be noted that for other Lie algebras the explicit construction of the Lax matrices is known but there are additional subtleties.[5] One of them is that $L(z)$ doesn't belong to the Lie algebra $\mathfrak{g}$ for generic value of z. In general the parameters g_ν for $\nu = 1, 2, 4$ are free; however for the case C_r, the construction of the Lax pair given in [105] requires[6] that

$$g_4^{C_r} = 2g_2^{C_r}\,. \tag{3.19}$$

3.2.3 Classical and Quantum Integrability

Until now, when speaking about integrability, we always meant *classical* integrability. The notion of integrability extends to the quantum case in a very natural way. As an illustration, let us solve the basic quantum problem: one particle in a potential $V(Z) = g\wp(Z)$. This corresponds

[4]This last statement does not hold for the so-called BC_r system, which is not a usual root system but is a *crystallographic* root system, that can be used to study all classical algebras.

[5]As far as I know, there is no explicit construction for G_2 algebras.

[6]See equation (6.16c) there.

to the Elliptic Calogero-Moser system with the simplest root system A_1. We look for a solution of the time-independent Schrödinger equation

$$-\frac{1}{2}\psi''(Z) + g\wp(Z)\psi(Z) = E\psi(Z)\,. \tag{3.20}$$

It is convenient to write $g = \frac{1}{2}l(l+1)$. Then one can show that for en arbitrary integer l and arbitrary energy E there are two solutions

$$\psi_\epsilon = e^{\epsilon k Z}\frac{\sigma(Z-\epsilon b_1)\ldots\sigma(Z-\epsilon b_l)}{\sigma(Z)^l} \tag{3.21}$$

for $\epsilon = \pm 1$. The constants k and b_j are fixed by equation (3.20). Then one should require that a physical solution be regular at $Z = 0$, which leaves us with one linear combination of the two solutions above, and at $Z = 2\omega_1$, which gives a quantification condition.

The quantum integrability is very similar to the classical discussion of section 3.2.1. Using the canonical quantization procedure we can construct operators $\hat{I}_k$ from the classical quantities I_k. These operators are well defined, since they are sums of products of commuting operators, but the commutator $[\hat{I}_k, \hat{I}_l]$ is not well defined and we need to introduce a regularization procedure, for instance normal order. This means that $\{I_k, I_l\} = 0$ does not imply $[\hat{I}_k, \hat{I}_l] = 0$, and a separate proof is needed. This proof is presented in [106].

Not only are the energy levels of the Calogero-Moser systems quantified, they also have integrality properties. For instance the energy levels E of the rational system with an additional confining potential $\frac{1}{2}\omega^2 Z^2$ satisfy ([107])

$$\frac{\Delta E}{\omega} \in \mathbb{Z}\,, \tag{3.22}$$

and this integer can be determined in terms of the degrees of fundamental invariants of the root system. The analogous quantization for the trigonometric Calogero-Moser system states that the energy levels are proportional to a quantity that depends on a dominant weight λ. For simply laced root systems, we have for instance

$$E \propto (\lambda+\rho)^2\,, \tag{3.23}$$

where ρ is the Weyl vector. For non simply-laced root systems, we have to replace in this formula ρ by a deformed version of the Weyl vector [108]. In addition we can also deduce that the frequencies of small oscillations around equilibrium points have integrality properties.

3.2.4 Twisted and Untwisted Elliptic Calogero-Moser Models

Let us first recall that the untwisted Elliptic Calogero-Moser system is just (3.14) with the third option in (3.7):

$$V_{\Delta,un}(Z) = \sum_{\alpha\in\Delta^+} g_{\nu(\alpha)}\wp(\alpha(Z);\omega_1,\omega_2)\,. \tag{3.24}$$

The subscript *un* stands for "untwisted".

The twisted elliptic Calogero-Moser model is defined in terms of twisted Weierstrass functions (C.29) which are summed over shifts by fractions of periods (thus in effect modifying that period). We have a twisted elliptic Calogero-Moser model for all non-simply laced root systems and the value of n is then given by the ratio of the length squared of the long versus the short roots. We will be interested in the twisted elliptic Calogero-Moser model with potential:

$$V_{\Delta,tw}(Z) = \sum_{\alpha\in\Delta^+} g_{\nu(\alpha)} \wp_{\nu(\alpha)}(\alpha(Z);\omega_1,\omega_2) \tag{3.25}$$

When Δ is a Lie algebra root system (as opposed to other crystallographic or non-crystallographic root systems), there are at most two possible lengths for the roots and we can rewrite this potential as

$$V_{\Delta,tw}(Z) = g_l \sum_{\alpha_l\in\Delta_l^+} \wp(\alpha_l(Z);\omega_1,\omega_2) + g_s \sum_{\alpha_s\in\Delta_s^+} \wp_n(\alpha_s(Z);\omega_1,\omega_2)\,, \tag{3.26}$$

where α_l denote the long and α_s the short roots in the root system $\Delta = \Delta_l \cup \Delta_s$, and g_l and g_s are two coupling constants.

Universal Lax pairs have been constructed for this twisted model [109].

3.2.5 Complexified Models

It is crucial for the later application to supersymmetric gauge theories that we allow for complex values for the components of the vector Z. We will see in section 5.2.2 that Z will encapsulate two real scalars of the compactified gauge theory.

Although it might seem that complexifying the variable Z does not make a big difference with the real case, it is worth pointing that the analysis is harder in this case. For instance, it is difficult to find analytically the extrema of the complexified Calogero-Moser potential even in the simplest cases of algebras of type A, and we will see that otherwise absent non-isolated extrema show up in the complexification process.

From a more formal point of view, we can reuse the abstract formalism alluded to at the end of section 3.2.1, see for instance [110]. The manifold we begin with is now a complex manifold of dimension $2n$, ω is a non-degenerate closed $(2,0)$-form and the function $(F_1,\ldots,F_n)$ is now $\mathbb{C}^n$-valued. As analogues of (3.4), we now have $2n$ complex action variables $(a^j)_{j=1,\ldots,n}$ corresponding to cycles A_j and $(a_{D,j})_{j=1,\ldots,n}$ corresponding to cycles B_j. Because there can not be more than n independent Poisson-commuting functions, there should be a relation between the a^j and the $a_{D,j}$. As the notation suggests, such a relation exists through a prepotential $\mathscr{F}$:

$$a_{D,j} = \frac{\partial\mathscr{F}}{\partial a^j}\,. \tag{3.27}$$

We see that a^j and $a_{D,j}$ are the special Kähler coordinates required by $\mathcal{N}=2$ supersymmetry, and this justifies our interest on complexified integrable systems.

3.2.6 The Symmetries of the Potential

Let us discuss in detail the symmetries of the twisted elliptic Calogero model that act on the set of variables Z. We first observe that the Weyl group action leaves invariant the scalar product $\alpha(Z) = (\alpha, Z)$ and that the root system is Weyl invariant. See also appendix B for the definitions of the different lattices discussed hereafter. This implies that the Weyl group action on Z leaves the potential invariant. Secondly, we note that the outer automorphisms of the Lie algebra, which correspond to symmetries of the Dynkin diagram, also leave the set of roots and the scalar product invariant. Therefore, outer automorphisms as well form a symmetry of the model.

Moreover, the periodicities of the model in the two directions of the torus are as follows. By the definition of the dual weight, or co-weight lattice, we have that $\alpha(\lambda^\vee) \in \mathbb{Z}$ for all roots α. This implies that shifts of Z by $2\omega_2\, P^\vee$, namely shifts by periods times co-weights, leave the potential invariant.

To discuss the periodicity in the ω_1 direction, we concentrate for simplicity on the algebras A, B, C and D, and normalize their long roots to have length squared two. We then have that for a long root α_l and a weight λ, the equation $(\alpha_l, \lambda) \in \mathbb{Z}$ holds while for a short root α_s of the B or C algebras we have $(\alpha_s, \lambda) \in \frac{1}{2}\mathbb{Z}$, for all weights λ. As a consequence, the periodicity in the (twisted) ω_1 direction is the lattice $2\omega_1\, P$ where P is the weight lattice. The group of all symmetries is a semi-direct product of the lattice shifts, the Weyl group as well as the outer automorphism group.

3.3 Elliptic Integrable Systems and Modularity

In this section, we study properties of (twisted) elliptic Calogero-Moser systems. We analyze the complexified model, defined on a torus with modular parameter τ. In particular, we examine the extrema of the complexified potential, and exhibit their curious characteristics.

3.3.1 Langlands Duality

Beyond the many features of these integrable systems already discussed in the literature, the first supplementary property that will be pertinent to our study of isolated extrema, is their behaviour under an inversion of the modular parameter τ. We therefore briefly digress in this subsection to discuss a few of the details of the duality. Models associated to simply laced Lie algebras map to themselves under the modular S-transformation $S : \tau \to -1/\tau$. This is easily confirmed using the transformation rule (C.23) of the Weierstrass $\wp$ function under modular transformations. We do have a non-trivial Langlands or short-long root duality between the twisted elliptic Calogero-Moser model of B-type and the twisted model of C-type. For the non-simply laced cases, we will always work with the twisted model (3.26), and *we will drop the corresponding subscript on the potential from now on.* In order to exhibit the duality, we make the potential for the $B_r = \mathfrak{so}(2r+1)$ theory more explicit:

$$\begin{aligned} V_B(x_i; \omega_1, \omega_2) \quad &= \quad b_l \left[\sum_{i<j} \wp(x_i - x_j; \omega_1, \omega_2) + \wp(x_i + x_j; \omega_1, \omega_2) \right] \\ &\quad + b_s \left[\sum_{i=1}^{r} \wp(x_i; \omega_1, \omega_2) + \wp(x_i + \omega_1; \omega_1, \omega_2) \right] , \end{aligned}$$

and for the $C_r = \mathfrak{sp}(2r)$ theory as well:

$$\begin{aligned} V_C(y_i; \omega_1', \omega_2') \quad &= \quad c_s \left[\sum_{i<j} \wp(y_i - y_j; \omega_1', \omega_2') + \wp(y_i + y_j; \omega_1', \omega_2') \right. \\ &\quad \left. + \wp(y_i - y_j + \omega_1'; \omega_1', \omega_2') + \wp(y_i + y_j + \omega_1'; \omega_1', \omega_2') \right] \end{aligned}$$

$$+c_l \sum_{i=1}^{r} \wp(2y_i;\omega_1',\omega_2')\,. \tag{3.28}$$

We have chosen the parametrization of the vector Z as well as the root systems given by (B.12) and (B.13) in appendix B, with z_i relabelled x_i and y_i for clarity. We have assigned half-periods ω_i to the B-system and ω_i' to the C-system. We have also made explicit the twisted Weierstrass functions $\wp_2$ with twisting index 2, which is the ratio of lengths squared of the long and short roots. Finally, we have chosen natural names for the coupling constants g_l and g_s in (3.26).To demonstrate the duality between these models, we use the elliptic function identities (C.31), valid for $Im(\omega_2/\omega_1) > 0$, to manipulate the B_r potential such that it becomes of the form of the C_r potential:

$$\begin{aligned}
V_B \;=\;& b_l \left[\sum_{i<j} \wp(x_i - x_j; 2\omega_2, -\omega_1) + \wp(x_i - x_j + 2\omega_2; 2\omega_2, -\omega_1) + \wp(x_i + x_j; 2\omega_2, -\omega_1) + \right.\\
& \left. \wp(x_i + x_j + 2\omega_2; 2\omega_2, -\omega_1) \right] + b_s \sum_{i=1}^{r} \wp(x_i; \omega_2, -\omega_1/2)\\
& - \frac{\pi^2 r(r-1)}{24\omega_2^2} b_l \left[2E_2\left(-\frac{\omega_1}{\omega_2}\right) - E_2\left(-\frac{\omega_1}{2\omega_2}\right)\right] + \frac{\pi^2 r}{6\omega_1^2} b_s \left[2E_2\left(2\frac{\omega_2}{\omega_1}\right) - E_2\left(\frac{\omega_2}{\omega_1}\right)\right]\\
=\;& b_l \left[\sum_{i<j} \wp(x_i - x_j; 2\omega_2, -\omega_1) + \wp(x_i - x_j + 2\omega_2; 2\omega_2, -\omega_1) + \wp(x_i + x_j; 2\omega_2, -\omega_1) + \right.\\
& \left. \wp(x_i + x_j + 2\omega_2; 2\omega_2, -\omega_1) \right] + b_s \sum_{i=1}^{r} \wp(x_i; \omega_2, -\omega_1/2)\\
& + \frac{\pi^2}{12\omega_1^2} (2rb_s + r(r-1)b_l) \left[2E_2\left(2\frac{\omega_2}{\omega_1}\right) - E_2\left(\frac{\omega_2}{\omega_1}\right)\right]\,.
\end{aligned} \tag{3.29}$$

In the last equality, we used the modular transformation rule (C.38) for a combination of second Eisenstein series. We observe that the end result (3.29) can be identified with the C_r potential (3.28), provided we match parameters as follows:

$$\begin{gathered}
\omega_1' = 2\omega_2 \qquad \omega_2' = -\omega_1 \qquad y_i = x_i\\
c_s = b_l \qquad c_l = 4b_s\,,
\end{gathered} \tag{3.30}$$

and we allow for a τ-dependent shift of the potential that invokes the second Eisenstein series E_2, defined in appendix C. These identifications imply a duality (which we will denote S_2) between the modular parameters of the B and C-type integrable systems:

$$\tau^B \equiv -\frac{1}{2\tau^C}\,. \tag{3.31}$$

In the following, we will be interested in B and C models in which the ratio of the long to short root coupling constants is equal to two, i.e. we put $b_l = b = 2b_s$ and $c_l = c = 2c_s$. Various particular choices of parameters and observables that we make in this section are motivated by the gauge theory applications that we will discuss in section 5.[7] It is important that this relation

[7]It is also of interest to study the integrable systems more generally.

be compatible with the duality map (3.30). We rewrite the identity of the potentials for this specific ratio of parameters:

$$\sum_{i<j} \wp(x_i - x_j; \omega_1, \omega_2) + \wp(x_i + x_j; \omega_1, \omega_2) + \frac{1}{2}\left(\sum_i \wp(x_i; \omega_1, \omega_2) + \wp(x_i + \omega_1; \omega_1, \omega_2)\right) =$$
$$\sum_{i<j} \wp(x_i - x_j; 2\omega_2, -\omega_1) + \wp(x_i - x_j + 2\omega_2; 2\omega_2, -\omega_1) + \wp(x_i + x_j; 2\omega_2, -\omega_1) +$$
$$\wp(x_i + x_j + 2\omega_2; 2\omega_2, -\omega_1) + 2\sum_i \wp(2x_i; 2\omega_2, -\omega_1) + \frac{\pi^2 r^2}{12\omega_1^2}\left[2E_2\left(2\frac{\omega_2}{\omega_1}\right) - E_2\left(\frac{\omega_2}{\omega_1}\right)\right],$$

and the integrable system duality can be summarised as:

$$V_B(x_i, \tau) = \frac{1}{2\tau^2} V_C\left(\frac{x_i}{2\tau}, -\frac{1}{2\tau}\right) + \frac{\pi^2 r^2}{3}\left[2E_2(2\tau) - E_2(\tau)\right], \tag{3.32}$$

when we use the rescaling (C.21). The duality may be viewed as a standard Langlands duality. We went through its detailed derivation since the τ-dependent shift in the duality transformation (3.32) is important for later purposes. This shift is a general feature of the duality, and appears similarly in equation (7.32) where the G_2 twisted Calogero-Moser Hamiltonian is used, and where the duality

$$S_3 : \tau \mapsto -\frac{1}{3\tau} \tag{3.33}$$

is involved.

3.3.2 Langlands Duality at Rank Two

There is a further special case of low rank which is of particular interest to us in the following. The B and C type Lie algebras of rank two are identical: $\mathfrak{so}(5) \equiv \mathfrak{sp}(4)$. If we apply the duality of B and C type potentials to this special case, we derive that the following transformations leave the potential invariant:

$$\begin{aligned} \omega_1' &= 2\omega_2 & \omega_2' &= -\omega_1 & c' &= 2b \\ x_2' - x_1' &= 2x_1 & x_1' + x_2' &= 2x_2\,. \end{aligned} \tag{3.34}$$

If we parameterise the potential in terms of the modular parameter $\tau = \omega_2/\omega_1$, the duality transformation for $\mathfrak{so}(5)$ reads:

$$V_{\mathfrak{so}(5)}(x_1, x_2, \tau) = \frac{1}{2\tau^2} V_{\mathfrak{so}(5)}\left(\frac{x_1 + x_2}{2\tau}, \frac{x_1 - x_2}{2\tau}, -\frac{1}{2\tau}\right) + \frac{4\pi^2}{3}\left[2E_2(2\tau) - E_2(\tau)\right]. \tag{3.35}$$

In summary, we derived a Langlands duality between B and C type (twisted) elliptic Calogero-Moser models. The resulting identities captured in equations (3.32) and (3.35) and the shifts appearing in these duality transformations will be useful. We return to the more general discussion of the integrable systems, and in particular their extrema.

3.4 Semi-Classical Limits of Elliptic Integrable Systems

In this section, we firstly propose new limits of elliptic integrable systems that generalize the Inozemtsev limits performed in [36, 111]. We are motivated by the fact that these limits describe semi-classical physics of supersymmetric gauge theories in four dimensions, and we will make use of them in chapters 6 and 7. The existence of these limiting behaviors may also be of interest in the theory of integrable systems [36, 111, 98]. Each limit is associated to a choice of subset of the set of simple roots of (the dual of) the affine root system that enters the definition of the twisted elliptic integrable system.

3.4.1 Calogero-Moser systems and Toda systems

Let $\mathfrak{g}$ be a simple complex Lie algebra. Under certain scaling behaviors of τ and $m \to \infty$, D'Hoker and Phong show in [111] that the elliptic Calogero-Moser system becomes a new integrable system which can be of Toda or trigonometric Calogero-Moser type. The trigonometric Calogero-Moser potential having been defined in (3.14) and (3.7), we give now the expression of the Toda potential associated to any finite-dimensional or affine Lie algebra. Let Δ^s be the set of simple roots of this algebra. Then we define

$$V_{\Delta}^{\text{Toda}}(Z^*) = \frac{1}{2} \sum_{\alpha \in \Delta^s} g^*_{\nu(\alpha)} \exp\left(-\alpha(Z^*)\right) , \tag{3.36}$$

where Z^* is again an element of the Cartan algebra and g^*_ν are coupling constants. We stress that the main difference between Toda systems and the other integrable systems that we have mentioned until now is that in the latter, a sum over all the positive roots is involved, while the Toda Hamiltonian is a sum over simple roots only.

The precise statement of [111] is now as follows, where we define, as usual, $q = e^{2\pi i \tau}$ and $\tau = \frac{\omega_2}{\omega_1}$.

- In the limit $\tau \to i\infty$ with $g^*_\nu = g_\nu q^\delta$ and $Z^* = Z - 2\omega_2 \delta \rho^\vee$ kept constant (where $\rho^\vee$ is the dual Weyl vector (B.8)), the untwisted Calogero-Moser system (3.24) tends towards the following system, according to the value of δ:
  - the affine Toda systems associated with untwisted affine Lie algebra $\mathfrak{g}^{(1)}$, when $\delta = 1/h$;
  - the ordinary Toda system associated with Lie algebra $\mathfrak{g}$ when $0 < \delta < 1/h$;
  - the trigonometric Calogero-Moser system associated with Lie algebra $\mathfrak{g}$ when $\delta = 0$.
- In the limit $\tau \to i\infty$ with $g^*_\nu = g_\nu q^\delta$ and $Z^* = Z - 2\omega_2 \delta \rho$ kept constant (where ρ is the Weyl vector (B.7)), the twisted Calogero-Moser system (3.25) tends towards the following system, according to the value of δ:
  - the affine Toda systems associated with dual affine Lie algebra $\left(\mathfrak{g}^{(1)}\right)^\vee$ when $\delta = 1/h^\vee$;
  - the ordinary Toda system associated with dual Lie algebra $\mathfrak{g}^\vee$ when $0 < \delta < 1/h^\vee$;
  - the trigonometric Calogero-Moser system associated with dual Lie algebra $\mathfrak{g}^\vee$ when $\delta = 0$.

In the following paragraphs, we will see that more general limits can be taken.

3.4.2 The Dual Affine Algebra and Non-Perturbative Contributions

This subsection is concerned with generalizing this analysis to include combinations of trigonometric and Toda integrable systems. These limits code possible symmetry breaking patterns of the gauge theory, as will be made explicit in chapters 6 and 7. The limit we take can be described as a limit towards large imaginary modular parameter τ, or as the semi-classical limit from the perspective of the $\mathcal{N}=1^*$ gauge theory where this parameter is identified with the complex coupling constant (A.17). The procedure gives an analytical handle on the extrema of the superpotential in the semi-classical regime.

The large imaginary τ expansion of the (twisted) elliptic integrable potential is known to be governed by affine algebras [112, 113, 39, 76]. Thus, it will be useful to introduce some affine algebra notation.[8] The (untwisted) affine algebra $\hat{\mathfrak{g}} = \mathfrak{g}^{(1)}$ is built from the loop algebra of $\mathfrak{g}$, the central extension $\hat{k}$ and the derivation d. We build a Cartan subalgebra of $\hat{\mathfrak{g}}$ from a Cartan subalgebra of $\mathfrak{g}$ by adding the generators $\hat{k}$ and d. Elements of the dual of the Cartan are denoted (λ, k, n) with the Lorentzian scalar product

$$(\lambda, k, n) \cdot (\lambda', k', n') = \lambda \cdot \lambda' + kn' + k'n\,. \tag{3.37}$$

A root is a weight of the adjoint representation, so it must have $k=0$. If we define the imaginary root δ to be equal to

$$\delta = (0, 0, 1)\,, \tag{3.38}$$

the set of affine roots is

$$\hat{\Delta} = \{\alpha + m\delta | m \in \mathbb{Z} \text{ and } \alpha \in \Delta\} \cup \{m\delta | m \in \mathbb{Z} \text{ and } m \neq 0\}\,, \tag{3.39}$$

and the set of positive affine roots is

$$\hat{\Delta}^+ = \Delta^+ \cup \{\alpha + m\delta | m \in \mathbb{N}^* \text{ and } \alpha \in \Delta\} \cup \{m\delta | m \in \mathbb{N}^*\}\,. \tag{3.40}$$

A set of positive simple roots is given by adjoining the affine root $\alpha_0 = \delta - \vartheta$, where ϑ is the highest root (B.4) of $\mathfrak{g}$, to a simple root system of $\mathfrak{g}$. The theory of twisted affine algebras, their classification, their (simple, positive) roots is also pertinent here, and can be looked up in [114].

Armed with this knowledge, let's analyze how the potential behaves in the large imaginary τ limit, and how the low-energy effective superpotential codes non-perturbative corrections to gauge theory on $\mathbb{R}^{2,1} \times S^1$. The low-energy effective superpotential for the $\mathcal{N}=1^*$ gauge theory with gauge algebra $\mathfrak{g}$ is given by

$$V_{tw}(Z) = \sum_{\alpha \in \Delta^+} g_{\nu(\alpha)} \wp_{\nu(\alpha)}(\alpha \cdot Z; \tau) \tag{3.41}$$

where the index $\nu(\alpha)$ is defined by (3.15) and the short and long root coupling constants are expressed in terms of a single constant g by

$$g_\nu = \frac{g}{\nu}\,. \tag{3.42}$$

We have dropped the index Δ, and we have switched to the τ-notation for the Weierstrass function defined by (C.21) in the appendix on elliptic functions, which will be more practical for

[8] See e.g. [114] for the theory of affine Kac-Moody algebras.

taking the limit and is closer to the gauge theory analysis. We normalize the long roots to have length squared two. To perform the semi-classical large imaginary τ expansion, we can exploit the result

$$\wp(2\omega_1 x;\omega_1,\omega_2) = -\frac{\pi^2}{12\omega_1^2}E_2(q) + \frac{\pi^2}{4\omega_1^2}\csc^2(\pi x) - \frac{2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos 2\pi n x\,, \tag{3.43}$$

where $q = e^{2\pi i\tau}$ as usual. This expansion is valid whenever the series is convergent, which requires $|q|<1$ or equivalently that $\tau\in\mathcal{H}$, and also $|\Im(x)|<\Im(\tau)$. The space $\mathcal{H}$ is the upper-half plane of complex numbers with positive imaginary part. For $x\in\mathcal{H}$ or $x\in\mathbb{R}\setminus\mathbb{Z}$ we can use the further expansion:

$$-4\sum_{k=1}^{\infty}ke^{2\pi ikx} = \csc^2\pi x\,, \tag{3.44}$$

to find

$$\wp(2\omega_1 x;\omega_1,\omega_2) = -\frac{\pi^2}{12\omega_1^2}E_2(q) - \frac{\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}n\left[e^{2\pi inx} + \sum_{m=1}^{\infty}q^{nm}\left(e^{-2\pi inx}+e^{2\pi inx}\right)\right]. \tag{3.45}$$

For the twisted Weierstrass function $\wp_\nu$ defined for $\nu\in\mathbb{N}^*$ by (C.29) we have the counterpart

$$\begin{aligned}\wp_\nu(2\omega_1 x;\omega_1,\omega_2) &= -\frac{\nu\pi^2}{12\omega_1^2}E_2(q) + \frac{\nu^2\pi^2}{4\omega_1^2}\csc^2(\pi\nu x) - \frac{2\nu^2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{nq^{n\nu}}{1-q^{n\nu}}\cos 2\pi n\nu x \\ &= -\frac{\nu\pi^2}{12\omega_1^2}E_2(q) - \frac{\nu^2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}n\left[e^{2\pi in\nu x} + \sum_{m=1}^{\infty}q^{nm\nu}\left(e^{-2\pi in\nu x}+e^{2\pi in\nu x}\right)\right].\end{aligned} \tag{3.46}$$

Again, this expansion is valid for $x\in\mathbb{C}\setminus\mathbb{Z}$ such that $0\le\Im(x)<\Im(\tau)$. It should be clear that the part of the argument of the Weierstrass function proportional to τ plays a crucial role in the Taylor series in the large τ limit. This is illustrated by the fact that for any $0<a<1$ and any $b\in\mathbb{R}$,

$$\lim_{\tau\to i\infty}\wp(a\tau+b;\tau) = -\frac{\pi^2}{3}\,. \tag{3.47}$$

This can be deduced from the limit (C.27). It is therefore useful to separate the argument into a part proportional to τ and a part that will not grow with τ, by setting

$$Z = X + \tau Y\,, \tag{3.48}$$

where X and Y are complex variables. At this stage, this decomposition is arbitrary. We have doubled the number of degrees of freedom, and we will use this redundancy in subsection 3.4.3 to impose the value of Y. Plugging this parametrization into the (twisted) Weierstrass function yields

$$\begin{aligned}\wp_\nu(2\omega_1\alpha\cdot Z;\omega_1,\omega_2) = -\frac{\nu\pi^2}{12\omega_1^2}E_2(q) - \frac{\nu^2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}n\Bigg[&q^{n\nu\alpha\cdot Y}e^{2in\pi\nu\alpha\cdot X} \\ &+\sum_{m=1}^{\infty}q^{nm\nu}\left(q^{-n\nu\alpha\cdot Y}e^{-2in\pi\nu\alpha\cdot X}+q^{n\nu\alpha\cdot Y}e^{2in\pi\nu\alpha\cdot X}\right)\Bigg].\end{aligned} \tag{3.49}$$

Using these expansion formulas for the potential, we arrive at a sum of exponential terms, each associated to a positive affine root. From now on, we set $2\omega_1 = 1$ and $2\omega_2 = \tau$ to avoid cluttered expressions. The potential now reads

$$V_{tw}(\hat{Z}) = -4\pi^2 g \left(\frac{|\Delta^+|}{12} E_2(q) + \sum_{n=1}^{\infty} n \left[\sum_{\hat{\alpha}\in\hat{\Delta}^+} \nu(\hat{\alpha}) q^{n\nu(\hat{\alpha})\hat{\alpha}\cdot\hat{Y}} e^{2\pi i n\nu(\hat{\alpha})\hat{\alpha}\cdot X} \right] \right) . \tag{3.50}$$

We have used the notations $\hat{X} = (X, 0, 0) = X$, $\hat{Y} = (Y, 1, 0)$ and $\hat{Z} = \hat{X} + \tau\hat{Y}$ so that for any affine root $\hat{\alpha} = \alpha + m\delta \in \hat{\Delta}$, we have the equalities $\hat{\alpha}\cdot\hat{Y} = \alpha\cdot Y + m$ and $\hat{\alpha}\cdot\hat{Z} = \alpha\cdot Z + m\tau$. We also define ν on affine roots with non-zero real part by $\nu(\hat{\alpha}) = \nu(\alpha + m\delta) = \nu(\alpha)$, so that

$$\hat{\alpha}^\vee = \frac{2}{|\hat{\alpha}|^2}\hat{\alpha} = \frac{2}{|\alpha|^2}\hat{\alpha} = \nu(\hat{\alpha})\hat{\alpha} \, . \tag{3.51}$$

We have arbitrarily declared $\nu(m\delta) = 0$.[9] The form of the exponents in equation (3.50) suggests switching from the affine root system to its dual

$$V_{tw}(\hat{Z}) = -4\pi^2 g \left(\frac{|\Delta^+|}{12} E_2(q) + \sum_{n=1}^{\infty} n \left[\sum_{\hat{\alpha}^\vee\in(\hat{\Delta}^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} q^{n\hat{\alpha}^\vee\cdot\hat{Y}} e^{2\pi i n\hat{\alpha}^\vee\cdot X} \right] \right) . \tag{3.52}$$

In the sum, we again disregard the terms associated to purely imaginary roots. This expression can be written in a more compact way, to which we will give a semiclassical interpretation as a sum over three-dimensional monopole-instanton contributions [39] in section 5.2.3:

$$V_{tw}(\hat{Z}) = -4\pi^2 g \left(\frac{|\Delta^+|}{12} E_2(q) + \sum_{n=1}^{\infty} n \left[\sum_{\hat{\alpha}^\vee\in(\hat{\Delta}^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n\hat{\alpha}^\vee\cdot\hat{Z}} \right] \right) . \tag{3.53}$$

We have two forms for the final expression. One expression (namely (3.50)) is in terms of the root system we started out with, the other (namely (3.52)) in terms of co-roots. Both forms are equally canonical, due to the fact that both the electric Wilson line variable and the dual photon variable are present in the potential and are interchanged under Langlands duality. This is a manifestation of the S-duality of the parent $\mathcal{N} = 4$ theory. In a given semi-classical expansion (i.e. $\tau \to i\infty$), we may more easily read expression (3.52), which has an interpretation as a sum over magnetic monopole instantons in this limit.

3.4.3 Semi-Classical Limits

Concretely, we take semi-classical limits as follows. We consider a particular isolated extremum whose positions Z depend only on τ (up to discrete equivalences that depend on the gauge group). We assume that at weak coupling, the limit

$$Y \equiv \lim_{\tau\to i\infty} \frac{1}{\tau} Z(\tau) \tag{3.54}$$

exists and we define $X(\tau) = Z(\tau) - \tau Y$. Note that for any $\tau \in \mathcal{H}$ we have $Z(\tau) = X(\tau) + \tau Y$ as before, and the parametrization Y is a vector that is independent of τ and which characterizes the extremum (or several extrema) under consideration. It is a non-trivial task to enumerate the set of vectors Y that give rise to isolated extrema. We will also deal with continuous

[9] We note that low-energy effective superpotential is ambiguous up to a purely q-dependent term.

branches of extrema, for which the definition (3.54) has no intrinsic meaning. In this case we can nevertheless choose an arbitrary set of coordinates of the branch, and take the limit while keeping these coordinates fixed. Depending on the choice of parametrization, this may lead to a continuous set of values for the vector Y. From now on, when studying a given extremum, we trade the variable $Z(\tau)$ for the variable $X(\tau)$ which is finite in the limit we want to perform, and use the expansion (3.52).

Before doing so, let's choose a basis of simple roots $(\alpha_1, \ldots, \alpha_r)$ in the root system Δ. Then $(\alpha_0, \alpha_1, \ldots, \alpha_r)$ are the simple roots of the affine root system $\hat{\Delta}$. The dual root system has a set of simple roots $((\alpha_0)^\vee, \alpha_1^\vee, \ldots, \alpha_r^\vee)$. To be more explicit about the semi-classical limit, we must distinguish between variables that sit on the boundary of the fundamental alcove, and those that reside inside. We therefore choose a vector Y in the fundamental affine Weyl chamber (or fundamental alcove), which implies that $(\alpha_i)^\vee \cdot \hat{Y} \geq 0$ for $i = 0, 1, ..., r$. We decompose the positive roots in terms of simple roots of the dual of the affine algebra, and the vector Y in the weight space in terms of affine fundamental weights $\hat{\pi}_i$:

$$\hat{\alpha}^\vee = \sum_{i=0}^{r} n_i \alpha_i^\vee \qquad \hat{Y} = \sum_{i=0}^{r} Y_i \hat{\pi}_i \tag{3.55}$$

where the n_i are non-negative integers,

$$\hat{\pi}_i = (\pi_i; a_i^\vee; 0) \qquad \hat{\pi}_0 = (0; 1; 0)\,, \tag{3.56}$$

and $a_i^\vee$ denote the co-marks of the Lie algebra. The fundamental weights satisfy the orthonormality conditions $(\hat{\pi}_i, \alpha_j^\vee) = \delta_{ij}$, so that $Y_i = \alpha_i^\vee \cdot \hat{Y} \geq 0$ and

$$\hat{\alpha}^\vee \cdot \hat{Y} = \sum_{i=0}^{r} n_i Y_i\,. \tag{3.57}$$

Note that the definition of $\hat{Y} = (Y, 1, 0)$ gives a linear relation between the $r+1$ coordinates Y_i,

$$1 = Y_0 + \sum_{i=1}^{r} Y_i a_i^\vee = \sum_{i=0}^{r} Y_i a_i^\vee\,. \tag{3.58}$$

Similarly we define $X_i = \alpha_i^\vee \cdot X$, and have the constraint $\sum_{i=0}^{r} X_i a_i^\vee = 0$. The distinction we now make is between those variables Y_i that lie on the boundary of the fundamental alcove, and those that lie inside. This will fix the leading behavior of the extrema that we focus on. For $Y_i = 0$, we note that there is an infinite set of non-perturbative contributions that needs to be taken into account in the semi-classical limit, and in particular, we need to resum them to the trigonometric term (as in equation (3.44)). The set of roots $\alpha^\vee$ for which this phenomenon occurs will again form a root system. Thus, to leading order in the modular parameter $q = e^{2\pi i \tau}$, we will have a trigonometric integrable system corresponding to a choice of subset of simple roots inside the affine simple root system. In a second step, by assumption, we have the remaining coordinates Y_j that do not vanish to leading order in τ. As a consequence (of formula (3.52)), these directions Y_j lead to subleading exponential terms.

More in detail, let's group positive roots by their inner products with $\hat{Y}$ and form the sets:

$$(\hat{\Delta}_t^+(Y))^\vee \quad = \quad \left\{ \hat{\alpha}^\vee \in (\hat{\Delta}^+)^\vee | \hat{\alpha}^\vee \cdot \hat{Y} = 2\omega_1 t \right\}\,, \tag{3.59}$$

and also the spectrum S of such inner products

$$S(Y) \quad = \quad \left\{ t \in \mathbb{R} | (\hat{\Delta}_t^+(Y))^\vee \neq \emptyset \right\} . \tag{3.60}$$

The spectrum of inner products without zero will be denoted $S(Y)^* = S(Y) - \{0\}$. The set of roots with zero inner product is finite while the full spectrum $S(Y)$ is generically infinite, due to the infinite nature of the affine root system. The superpotential

$$V_{tw}(\hat{Z}) = -4\pi^2 g \left(C(\tau) + \sum_{n=1}^{\infty} n \left[\sum_{t \in S(Y)} q^{nt} \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_t^+)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X} \right] \right) , \tag{3.61}$$

will split into two sets of terms. Note that the exponents of q are non-negative, so that the expression remains finite when we take the limit $q \to 0$. As mentioned previously, the first split happens between terms with zero inner product and non-zero inner product:

$$V_{tw}(\hat{Z}) \quad = \quad -4\pi^2 g \left(C(\tau) - \frac{1}{4} \sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_0^+(Y))^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} \csc^2 \left(\pi \hat{\alpha}^\vee \cdot X \right) \right. \tag{3.62}$$

$$\left. + \sum_{t \in S(Y)^*} \sum_{n=1}^{\infty} n q^{nt} \left[\sum_{\hat{\alpha}^\vee \in (\hat{\Delta}_t^+(Y))^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X} \right] \right) . \tag{3.63}$$

We obtain a sum of a trigonometric and an exponential system

$$V_{tw}(\hat{Z}) = -4\pi^2 g \left(C(\tau) - \frac{1}{4} V_{trig}^Y(X) + \sum_{t \in S(Y)^*} \sum_{n=1}^{\infty} n q^{nt} V_{exp}^{(n,t,Y)}(X) \right) , \tag{3.64}$$

where

$$V_{trig}^Y(X) \quad = \quad \sum_{\hat{\alpha}^\vee \in \left(\hat{\Delta}_0^+(Y)\right)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} \csc^2 \left(\pi \hat{\alpha}^\vee \cdot X \right) \tag{3.65}$$

$$V_{exp}^{(n,t,Y)}(X) \quad = \quad \sum_{\hat{\alpha}^\vee \in \left(\hat{\Delta}_t^+(Y)\right)^\vee} \frac{1}{\nu(\hat{\alpha}^\vee)} e^{2\pi i n \hat{\alpha}^\vee \cdot X} . \tag{3.66}$$

The behavior of the subdominant system is intricate. A first stab at the subdominant system consists in realizing that the remaining variables (indexed by the set $\bar{J}_0 = \{0, 1, \ldots, r\} \setminus J_0$ where J_0 is the set of coordinates with zero inner product) will all have a leading exponential term. These exponentials, combined with the constraint equation (3.58), may give rise to exponential interactions, stabilized by an exponential interaction of opposite sign. The affine Toda potential is an example of this type of subdominant potential. Roughly speaking, this reasoning goes through, but the devil is in the details. The first complicating factor is the influence of the dominant terms on the subdominant terms when searching for an equilibrium position. In particular, corrections to equilibrium positions for leading coordinates may strongly influence subdominant contributions. Particular equilibrium configurations for the leading trigonometric system can also give rise to subtle and persistent cancellations in the coefficients of subdominant exponential terms. There may also be a staircase of subdominant terms, each with its own limiting behavior. Even a continuous set of limiting behaviors can occur. Moreover, the solutions to the trigonometric system are only known as zeroes of orthogonal polynomials, making this process hard to carry through analytically in full generality.

Therefore, we develop only a partial picture of the integrable systems that result in the limit. Still, we provide a generalization of the limit discussed in [111] in the following subsection, and useful heuristics based on the examples in sections 7.2, 6 and 7.4.

3.4.4 The Trigonometric, Affine Toda and Intermediate Limits

Here we will treat the special case in which no cancellation of (sub)leading exponentials occurs, and in which the subleading exponential integrable system stabilizes all the remaining coordinates and leads to an isolated extremum. We can then analytically solve for the remaining variables. Due to the constraint equation we have that the set $J_0 \subsetneq \{0, 1, ..., r\}$ is a true subset of the set of simple roots (we identify a simple root α_i with its index i). We obtain a trigonometric integrable system for the root system corresponding to the simple roots in J_0. This system gives solutions for $|J_0|$ of the $r+1$ variables X_i. Let t_1 be the smallest non-zero element of the spectrum $S(Y)$. At the next level in the q-expansion, we find contributions corresponding to the set $(\hat{\Delta}^+_{t_1}(Y))^\vee$, which is equal to a set of positive roots.

The final Toda integrable system is a sum over $|\bar{J}_0|$ vectors where $\bar{J}_0$ is the complement of the set of affine simple roots that enter the trigonometric system, by assumption. We then obtain

$$V^{(n=1,t_1,Y)}_{Toda}(X) = C \sum_{i \in \bar{J}_0} \frac{1}{\tilde{\nu}(\alpha_i^\vee)} e^{2\pi i \alpha_i^\vee \cdot X} \,. \tag{3.67}$$

In the last equality we have indicated the fact that for each individual index i, there may be a renormalization of the constant $\tilde{\nu}$ in front of the exponential term, due to various roots contributing to the same exponential behavior. The constraint equation then gives

$$1 = \sum_{i \in \bar{J}_0} Y_i a_i^\vee = t_1 \sum_{i \in \bar{J}_0} a_i^\vee \tag{3.68}$$

from which we extract t_1 and finally

$$\hat{Y} = \frac{\sum\limits_{i \in \bar{J}_0} \hat{\omega}_i}{\sum\limits_{i \in \bar{J}_0} a_i^\vee} \,. \tag{3.69}$$

After projection on the finite part, we find:

$$Y = \frac{\sum\limits_{i \in \bar{J}_0} \omega_i}{\sum\limits_{i \in \bar{J}_0} a_i^\vee} \,. \tag{3.70}$$

where we define $\omega_0 = 0$. Here the dependence on t_1 has disappeared. We can simply use $Y_{\bar{J}_0}$ as an ansatz, for every non-empty set $\bar{J}_0 \subset \{0, 1, ..., r\}$, where we have defined

$$Y_J = \frac{\sum\limits_{i \in J} \omega_i}{\sum\limits_{i \in J} a_i^\vee} \,. \tag{3.71}$$

On the condition that the subleading exponentials have non-vanishing coefficients, this gives the semi-classical (linear order in τ) values for the Y coordinates of the integrable system. Namely, a first set sits at an extremum of the trigonometric integrable system, and a second set at the extrema of the affine Toda system. Known applications of this ansatz are the following. A first extreme case is $\bar{J}_0 = \emptyset$ and $Y = 0$. Then $\Delta_0^+(Y) = \Delta^+$ and we recover the trigonometric potential only. The other extreme case is $\bar{J}_0 = \{0, 1, ..., r\}$ and $Y = \rho/h^\vee$ (where ρ is the Weyl vector and $h^\vee$ the dual Coxeter number of the gauge algebra). We then obtain the affine Toda potential for the algebra $(\mathfrak{g}^{(1)})^\vee$, as described in [111]. There are many intermediate cases that follow the above pattern, or an even more intricate one.[10] Examples are provided in sections 6, 7.2 and 7.4. It would be desirable to have a full classification of semi-classical limits. The $\mathcal{N} = 1^*$ gauge theory provides intuition in the case of the (twisted) elliptic Calogero-Moser system with particular coupling constants – the question in the integrable system context is even more general.

[10] In the gauge theory, these cases correspond respectively to a fully Higgsed vacuum, confining pure $\mathcal{N} = 1$ dynamics, and partial Higgsing.

Chapter 4

Isolated Extrema of the Twisted Elliptic Calogero-Moser System

This chapter is dedicated to an extensive study of the extrema of the complexified and twisted elliptic Calogero-Moser potential for various low-rank Lie algebras. The many studies of classical integrable models at equilibrium present in the literature have uncovered remarkable properties, like the integrality of the minimum of the potential and of the frequencies of small oscillations around the minimum for the rational and trigonometric Calogero-Moser systems, as reviewed in section 3.2.3. We will analyze the potential of the elliptic integrable system evaluated at generalised equilibrium positions. We show that they give rise to interesting vector valued modular forms as well as more general non-analytic modular vectors. Modularity provides a more conceptual way of understanding the integrality properties of the integrable system. Indeed, this rationale then continues to hold for the integrable systems that can be obtained from the elliptic Calogero-Moser systems by limiting procedures presented in section 3.4. Thus, studying *elliptic* integrable systems, depending on a modular parameter, is found to have an additional pay-off. We will use a combination of analytical work, principally based on modular properties of the potential, and extensive numerics.

It is known that A-type integrable systems often have simpler properties than do the integrable systems associated with other root systems. As a relevant example, let us quote the fact that the real trigonometric Calogero-Moser system with potential of A-type has equally spaced equilibrium positions along the real axis, while the B, C, D-type potentials have minima associated to zeroes of Jacobi polynomials [108], which satisfy known relations [115], but are not known explicitly in general. The elliptic Calogero-Moser systems that we examine show a similar dichotomy. Extrema of the (complex) elliptic A-model are equally spaced. This fact leads to relatively easily constructable values for the potential at extrema, for any rank [77, 38, 84]. For the B, C, D-type models that we study in this chapter, much less is known, and we need to combine numerical searches with analytic approaches to determine the extremal values of the potential, for low rank cases.

To be more precise, we will be interested in extrema of the complexified potential, satisfying

$$\partial_{Z_i} V(Z_j) = 0 \qquad \forall i \,. \tag{4.1}$$

This will correspond, in section 5, to a supersymmetric vacuum in the $\mathcal{N} = 1^*$ gauge theory, where the effective superpotential $\mathcal{W}$ is identified with the potential V of the integrable system.

We moreover demand that at the extremum (4.1) the function

$$\sum_{i=1}^{r}\left|\frac{\partial V(Z^j)}{\partial Z^i}\right|^2 \tag{4.2}$$

not possess any flat directions. We say that such an extremum is *isolated*. In the gauge theory language, this condition implies that the vacuum is massive.

Recall that the group of symmetries acting on the variables Z were a lattice group of translations, the Weyl group as well as the outer automorphisms of the Lie algebra. Using these symmetries, we will introduce a notion of equivalence on the variables Z. We will consider the vector Z to be identified by the periodicities of the model. The periodicity in the ω_1 direction is given by the weight lattice P, while in the ω_2 direction it is the co-weight lattice $P^\vee$. Furthermore, we will consider extrema that are related by the action of the Weyl group of the Lie algebra to be equivalent. By contrast, outer automorphisms are taken to be global symmetries of the problem. When the global symmetry group is broken by a given extremum, the global symmetries will generate a set of degenerate extrema.

4.1 The Case $A_{N-1} = \mathfrak{su}(N)$

The extrema of the elliptic Calogero-Moser model of type A_r have been studied in great detail, mostly in the context of supersymmetric gauge theory dynamics (see e.g. [77, 38, 84]). Firstly, we remark that in this case, the equivalence relations that follow from the periodicity of the potential as well as the Weyl symmetry group of the Lie algebra are straightforwardly implemented. We use the parametrization of simple roots given in (B.12). We parametrize the coordinates of our integrable system via equation (3.12). As we have already seen, the Calogero-Moser potential then reads

$$V_{A_{N-1}}(Z;\omega_1,\omega_2) = g_2 \sum_{1\le i<j\le N} \wp\left(Z_i - Z_j;\omega_1,\omega_2\right) . \tag{4.3}$$

The Weyl group S_N acts by permuting the components Z_i. We can shift one of the components Z_i to zero by convention. The equivalence under shifts by fundamental weights is identical to the toroidal periodicity relations for the individual coordinates Z_i. The inequivalent extrema of the potential (satisfying the additional condition (4.2) of non-flatness) are then argued to correspond one-to-one to sublattices of order N of the torus with modular parameter τ [77, 38, 116].

These extrema are classified by pairs of integers (p,k) satisfying that p is a divisor of N and $k \in \{0,1,\ldots,p-1\}$. For a given pair (p,k), the associated extremum is given explicitly by

$$\{Z_i|i=1\ldots,N\} = \left\{\frac{2\omega_1}{(N/p)}\left(r+s\frac{k}{p}\right)+\frac{2\omega_2}{p}s\middle| r=0,\ldots,\left(\frac{N}{p}-1\right) \text{ and } s=0,\ldots,p-1\right\} . \tag{4.4}$$

The number of extrema is therefore equal to the sum of the divisors of N. The $\mathbb{Z}_2$ outer automorphism of $\mathfrak{su}(N)$ for $N\ge 3$ acts trivially on the minima, since it acts by permutation, combined with a sign flip for all Z_i, which leaves a sublattice ankered at the origin invariant. The value of the potential at one of these extrema is

$$V_{A_{N-1}}(\tau) = -g_2 N^2 \frac{\pi^2}{6}\left(E_2(\tau) - \frac{N}{p^2}E_2\left(\frac{N}{p^2}\tau+\frac{k}{p}\right)\right) . \tag{4.5}$$

Under the $SL(2,\mathbb{Z})$ action on the torus modular parameter τ, the sublattices of order N of the torus are permuted into each other (in a way that depends intricately on the integer N). The permutation of the sublattices also entails the permutation of the values (4.5) at these extrema under $SL(2,\mathbb{Z})$. For instance, if we take $N = 4$, there are $1+2+4 = 7$ extrema, and the duality diagram is represented in figure 4.1. The list of extremal values of the elliptic Calogero-Moser model therefore form a vector valued modular form (see e.g. [117, 118, 119]) of weight two under the group $SL(2,\mathbb{Z})$. The associated representation of the modular group is a representation in terms of permutations specified by the $SL(2,\mathbb{Z})$ action on sublattices of order N. One can identify a subgroup of the modular group under which a given component of the vector-valued modular form is invariant, and then use minimal data to fix it [89].

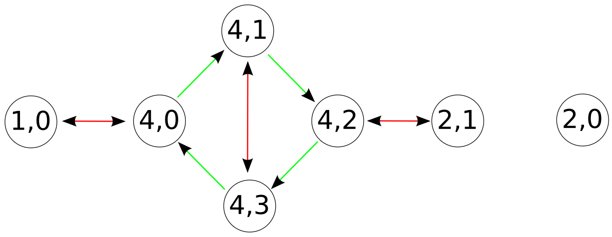


Figure 4.1: The diagram of the action of dualities on the $A_3 = \mathfrak{su}(4)$ extrema. Each extremum is labeled by a pair of integers (p,k) as defined in the text. In red we exhibit the action of S-duality, in green, T-duality.

In summary, the extrema of the Calogero-Moser model of type $A_{N-1} = \mathfrak{su}(N)$ are under analytic control. The positioning of the extrema can be expressed linearly in terms of the periods of the model, and the vector valued modular form of extremal values for the potential has an automorphy factor that can be characterised by sublattice permutation properties. The extremal values are generalised Eisenstein series of weight two under congruence subgroups of the modular group.

4.2 The B, C, D Models

For other algebras, we are at the moment only able to study low rank cases. From the analysis, it is clear that crucial simplifying properties of the A_r case are absent. Nevertheless, generic features of the A_r case persist in a subclass of extrema, in that we find vector-valued modular forms as extremal values for the potential. We also find a class of extremal values that exhibit new features.

To describe in detail which extrema are considered to be equivalent, we must discuss the equivalence relations that we mod out by for the B, C and D root systems individually.

$D_r = \mathfrak{so}(2r)$

For the D_r case, we can parameterise the roots as $\alpha_i = e_i - e_{i+1}$ (for $i \in \{1, 2, \ldots, r-1\}$) and $\alpha_r = e_{r-1} + e_r$. We put $X = X_j e^j$ and imply that the relation $e_i(e^j) = \delta_i{}^j$ holds. The equivalence of the vector X under shifts proportional to the weight lattice implies that each variable X_j lives on a torus with modular parameter τ. It moreover identifies the vector X with the vector X shifted by a half-period in each variable simultaneously. The Weyl group is

$W(\mathfrak{so}(2r)) = S_r \ltimes \mathbb{Z}_2^{r-1}$, and acts by permutation of the components X_j, as well as the sign change of an even number of them. The outer automorphism group (for $r \neq 4$) is equal to $\mathbb{Z}_2$ and acts as $X_r \to -X_r$. For $r = 4$, the global symmetry group is S_3 triality.

$B_r = \mathfrak{so}(2r+1)$

For B_r, the roots are $\alpha_i = e_i - e_{i+1}$ (for $i \in \{1, 2, \ldots, r-1\}$) and $\alpha_r = e_r$. We recall that the periodicity is the weight lattice in the ω_1 direction (due to the twist), and the co-weight lattice in the ω_2 direction. Thus, we can shift components of the vector $X = X_j e^j$ by periods, or all components simultaneously by a half period in the ω_1 direction. In the ω_2 direction, we allow shifts of the individual components by periods. The Weyl group acts by combinations of permutations and any sign flip of the coordinates.

$C_r = \mathfrak{sp}(2r)$

The roots are $\alpha_i = (e_i - e_{i+1})/\sqrt{2}$ (for $i \in \{1, 2, \ldots, r-1\}$) and $\alpha_r = \sqrt{2} e_r$.[1] We can shift components X_j of $X = \sqrt{2} X_j e^j$ by half-periods in the ω_1 direction, while in the ω_2 direction, we can allow shifts by any period, as well as a half-period shift of all X_j simultaneously. The Weyl group allows any permutation and sign flip of the coordinates. The equivalence relations and symmetries in the B, C and D cases, beyond permutation symmetries and toroidal periodicity, are summarized in the table 4.1.

B_r	Individual $X_i \to -X_i$ Collective $X_i \to X_i + \omega_1$
C_r	Individual $X_i \to -X_i$ and $X_i \to X_i + \omega_1$ Collective $X_i \to X_i + \omega_2$
D_r	Even number of sign flips $X_i \to -X_i$ Collective $X_i \to X_i + \omega_1$ and $X_i \to X_i + \omega_2$ Global symmetries : $\mathbb{Z}_2$ generically and S_3 for D_4.

Table 4.1: Symmetries of the potentials based on various root systems.

Armed with this detailed knowledge about the equivalence of configurations, we programmed a numerical search for isolated extrema. In the following subsections, we list the results we found by root system. For simply laced root systems we studied the elliptic Caloger-Moser model, while results for non-simply laced root systems correspond to the twisted elliptic Calogero-Moser model with a coefficient for the short root term which is equal to one half the coefficient in front of the long root terms (as described below equation (3.31)).

4.3 The Case $C_2 = \mathfrak{sp}(4) = \mathfrak{so}(5)$ and Vector Valued Modular Forms

Since the root system C_2 is the first example of our series, we provide a detailed discussion. We discuss the positions of the isolated extrema, the series expansions relevant to the potential at

[1] By our conventions, we normalise the long roots such that they have length squared two.

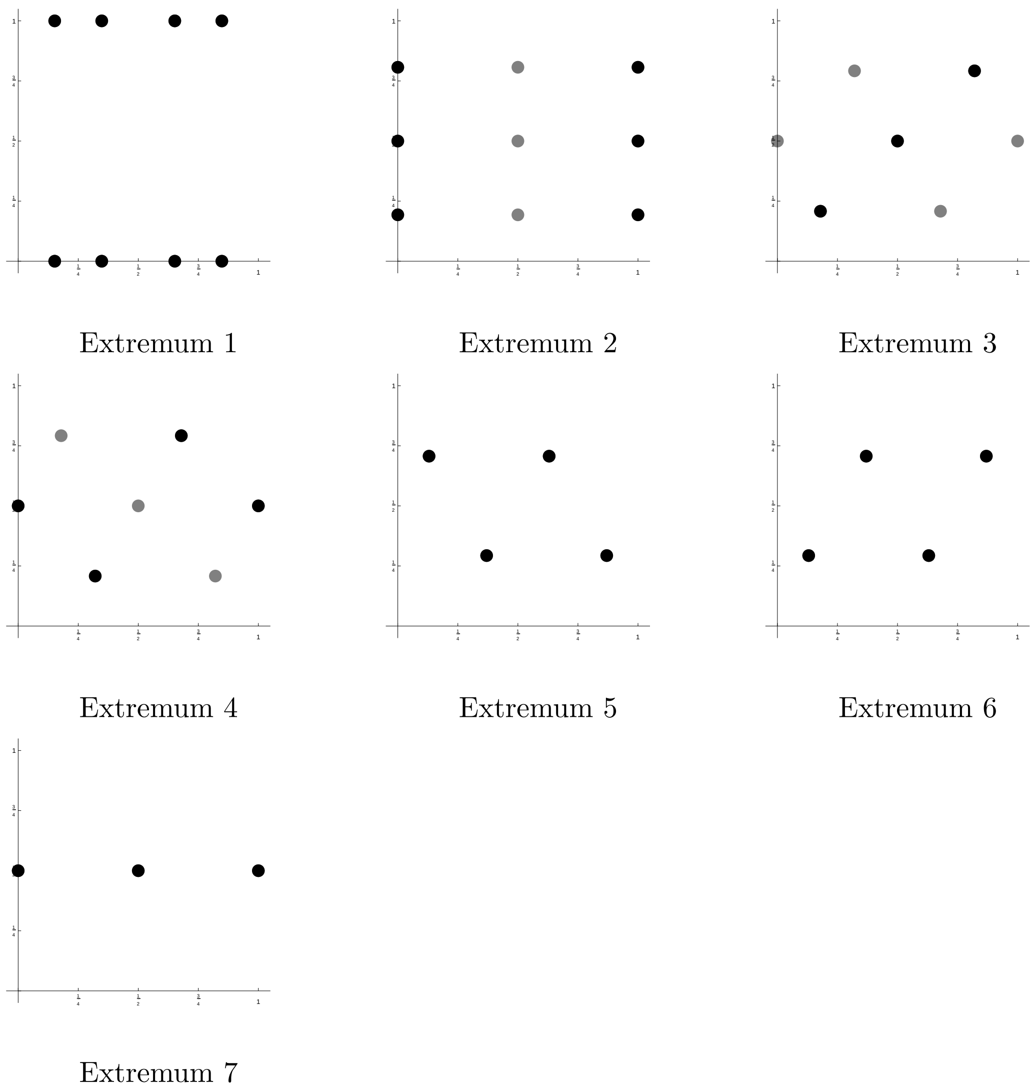


Figure 4.2: Extrema for the superpotential at coupling $\tau = i$ for the Lie algebra $so(5)$ are drawn in dark. Configurations obtained by translation by ω_1 are drawn in light gray.

these extrema, the action of the duality group, as well as the identification of the relevant vector valued modular forms.

4.3.1 The Positions of the Extrema

For the Lie algebra $\mathfrak{so}(5) = \mathfrak{sp}(4)$ we found 7 isolated extrema of the potential. We provide their positioning at $\tau = i$ in figure 4.2. We have drawn in bold the positions of the extrema as well as their opposites, in a fundamental cell of the torus.[2]

These numerical results were found using a Mathematica program, which was written around the built-in function `FindMinimum`. Careful programming augments the precision of the algorithm to at least two hundred digits. The most costly part of the algorithm is the random search for extrema. Indeed, the intricate landscape drawn by the potential can hide extrema. We gave

[2]We have indicated reflections over other half-periods in grey, to illustrate that the minima are close to forming sublattice structures.

a drawing of the position of the numbered extrema on the torus with modular parameter $\tau = i$. The positions of the extrema for other values of the modular parameter can be reached by interpolation. We have analytic control over a few extra properties of the extrema. E.g. if we follow extremum 1 to $\tau = i\infty$, we find that the equilibrium positions are given by $\frac{1}{2\pi}\arccos(\pm 1/\sqrt{3})$ where $\pm 1/\sqrt{3}$ are the zeroes of the Jacobi polynomial $P_2^{(0,0)}$. The first extremum, which we label 1, lies on the real axis and is the equilibrium position of the real integrable system. The extremum 2 lies on the imaginary axis, while extrema 3 and 4 are then approximately obtained by applying the transformation $\tau \to \tau + 1$. The extrema 5 and 6 are S_2 Langlands duals of extrema 3 and 4. It is easy to deduce from the potential that the positions of the extrema generically behave non-linearly as a function of τ.

4.3.2 Series Expansions of the Extrema

By numerically evaluating the extrema of the potential for a range of values of the modular parameter τ, we are able to write the extrema as an expansion in terms of a power of the modular parameter $q = e^{2\pi i\tau}$. The extremal values can be written in terms of the series:

$$\begin{aligned}
A_0(q) &= \frac{1}{24} + q + q^2 + 4q^3 + q^4 + 6q^5 + 4q^6 + 8q^7 + q^8 + 13q^9 + 6q^{10} + 12q^{11} \\
&\quad +4q^{12} + 14q^{13} + 8q^{14} + \dots \qquad (4.6)\\
A_1(q) &= 1 + 48q + 828q^2 + 8064q^3 + 109890q^4 + 1451520q^5 + 11198088q^6 + 141212160q^7 \\
&\quad +1666682811q^8 + 9413050176q^9 + 145022264892q^{10} + 1838450006784q^{11} \\
&\quad +11103941590326q^{12} + 138638111404032q^{13} + \dots \\
A_2(q) &= 2 + 48q + 576q^2 + 9792q^3 + 99576q^4 + 743904q^5 + 13146624q^6 + 115737984q^7 \\
&\quad +1015727364q^8 + 14338442448q^9 + 102050482176q^{10} + 935515738944q^{11} \\
&\quad +12532363069968q^{12} + 122390111091744q^{13} + \dots \\
A_3(q) &= \frac{13}{216} + 7q + 541q^2 + 24508q^3 + 939669q^4 + 19944842q^5 + 764752180q^6 \\
&\quad +21016537080q^7 + 905672825157q^8 + 38827071780859q^9 + 827503353279726q^{10} + \dots \\
A_4(q) &= 1 + 148q + 7446q^2 + 154344q^3 + 5100349q^4 + 352720380q^5 + 10627587582q^6 \\
&\quad +166124184888q^7 + 5419843397586q^8 + 294399334337124q^9 + \dots \\
A_5(q) &= -\frac{1}{216} + 29q + 431q^2 + 80468q^3 - 231081q^4 + 948846414q^5 + 1301490428q^6 \\
&\quad +90560563752q^7 - 529100109849q^8 + 93349951292249q^9 + \dots . \qquad (4.7)
\end{aligned}$$

The integer coefficients have been determined up to an accuracy of at least 10^{-6}. For the first order terms, the accuracy can be up to 10^{-200}. In terms of these series, the potential in extremum number 1, on the real axis is

$$V_1 = 144\pi^2 A_3\left(\frac{q}{27}\right) . \qquad (4.8)$$

The potential in the other extrema are:

$$\begin{aligned}
V_2 &= -12\pi^2\left(\frac{8}{3}A_0(q) + (2q)^{1/3}A_1(q/9) + (2q)^{2/3}A_2(q/9)\right) \\
V_3 &= -12\pi^2\left(\frac{8}{3}A_0(q) + (2q)^{1/3}e^{2\pi i/3}A_1(q/9) + (2q)^{2/3}e^{4\pi i/3}A_2(q/9)\right)
\end{aligned}$$

$$V_4 = -12\pi^2 \left(\frac{8}{3} A_0(q) + (2q)^{1/3} e^{4\pi i/3} A_1(q/9) + (2q)^{2/3} e^{2\pi i/3} A_2(q/9) \right) , \tag{4.9}$$

and

$$\begin{aligned} V_{5,6} &= 72\pi^2 \left(A_5 \left(\frac{q}{27} \right) \pm i \sqrt{\frac{q}{27}} A_4 \left(-\frac{q}{27} \right) \right) \\ V_7 &= \frac{48}{3} \pi^2 A_0(q) . \end{aligned} \tag{4.10}$$

The growth properties of these series, as well as the fact that we are dealing with a physical system living on a torus suggests turning these numerical data into an analytic understanding, based on the theory of modular forms. In the following, we show that this is possible for the rank 2 root system B_2.

4.3.3 Modular Forms of the Hecke Group and the $\Gamma_0(4)$ Subgroup

We need to introduce a few groups related to the modular group. Let $\tilde{\tau}$ be a variable in the upper half-plane $\mathcal{H}$, and let $\lambda \geq 0$. We define the group $G(\lambda)$ of transformations of $\mathcal{H}$ as the group generated by

$$\mathcal{T}_\lambda \quad : \quad \tilde{\tau} \to \tilde{\tau} + \lambda \tag{4.11}$$

$$\mathcal{S} \quad : \quad \tilde{\tau} \to -\frac{1}{\tilde{\tau}} . \tag{4.12}$$

The abstract Hecke group $G(\lambda)$ is of course independent of the different ways it can act on $\mathcal{H}$, but its matrix representations are not. Let us illustrate this by changing our variable $\tilde{\tau}$ to $\tau = \kappa\tilde{\tau}$. Then $\mathcal{T}_\lambda$ and $\mathcal{S}$ act on τ as follows:

$$\mathcal{T}_\lambda \quad : \quad \tau \to \tau + \kappa\lambda \tag{4.13}$$

$$\mathcal{S} \quad : \quad \tau \to -\frac{\kappa^2}{\tau} . \tag{4.14}$$

If $0 \leq \lambda < 2$ the Hecke group $G(\lambda)$ is also denoted H_q with $\lambda = 2\cos\frac{\pi}{q}$. If $q \in \mathbb{N}$ then the Hecke group H_q is isomorphic to the free product of the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_q$. For more information on Hecke groups and associated modular forms see e.g. the lectures [120].

Let us consider the group $G(\sqrt{2}) = H_4$ with $\lambda = \frac{1}{\kappa} = \sqrt{2}$. Then

$$\mathcal{T}_{\sqrt{2}} = T \quad : \quad \tau \to \tau + 1 \tag{4.15}$$

$$\mathcal{S} = S_2 \quad : \quad \tau \to -\frac{1}{2\tau} . \tag{4.16}$$

We already noted the duality transform for the B, C-type twisted Calogero-Moser system under the map S_2, see equation (3.31). For the $\mathfrak{so}(5)$ Lie algebra, which is identical to the $\mathfrak{sp}(4)$ Lie algebra, this transformation maps the integrable system to itself (up to a τ dependent shift of the potential and an overall factor – see equation (3.35)). The map T also maps the integrable system to itself. Together, these transformations generate the action of $G(\sqrt{2})$ on the modular parameter τ. Let us define the element $U \in G(\sqrt{2})$ by

$$U = S_2 T^{-2} S_2 : \tau \to \frac{\tau}{4\tau + 1} . \tag{4.17}$$

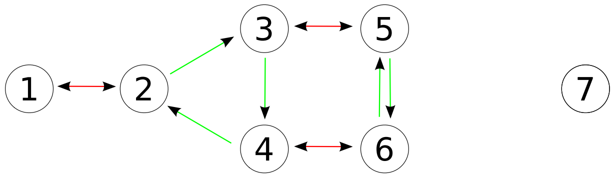


Figure 4.3: The diagram of dualities on the extrema of the integrable system for $so(5)$. In red, we draw the action of Langlands S_2-duality, and in green, T-duality, when the action on a given extremum is non-trivial.

The subgroup of $G(\sqrt{2})$ generated by T and U is isomorphic to the congruence subgroup $\Gamma_0(4)$ of the modular group $SL(2,\mathbb{Z})$ via the identifications

$$\begin{aligned} T &: \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ U &: \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} . \end{aligned} \tag{4.18}$$

The group $\Gamma_0(N)$ is defined in (C.1).

The extremal values of the potential may therefore form a vector valued modular form with respect to the Hecke group $G(\sqrt{2})$, and as a consequence also with respect to the congruence subgroup $\Gamma_0(4)$, since we expect extrema to be at most permuted and/or rescaled under the group. Here, we assume analyticity in the interior of the fundamental domain. We will mostly exploit the group $\Gamma_0(4)$ in the following, since the literature on the subject of modular forms with respect to congruence subgroups is abundant. For starters, we determine the action of the operations T and S_2 on the vector V_i of extremal values of the twisted Calogero-Moser potential:

$$\begin{aligned} T &: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \\ S_2 &: \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \end{aligned} \tag{4.19}$$

See figure 4.3 for a summary of the action of the duality group. To this information, we add the last column in the matrix S_2, which originates in the shift of the potential under Langlands duality. From these data, we easily calculate the action of the generator $U = S_2 T^{-2} S_2$ on the vector valued modular form:

$$U \quad : \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.20}$$

We thus find the action of $\Gamma_0(4)$ on the vector valued modular form, and we observe the following pattern: there is one entry (the seventh) which is an ordinary modular form of weight 2 under $\Gamma_0(4)$, and there are two sets of three components (namely $\{2, 3, 4\}$ and $\{1, 5, 6\}$) that mix under $\Gamma_0(4)$. Thus, our vector valued modular form of dimension seven splits into a singlet and a sextuplet. Concentrating on the ordinary modular form of weight 2, we have that it is a linear combination of Eisenstein series $E_{2,N}$ defined by:

$$E_{2,N}(\tau) \quad = \quad E_2(\tau) - N E_2(N\tau) \, . \tag{4.21}$$

Indeed, the dimension of the space $\mathcal{M}_2(\Gamma_0(4))$ of modular forms of $\Gamma_0(4)$ is two, and it is spanned by $E_{2,2}$ and $E_{2,4}$. We thus only need two Fourier coefficients to fix the entire modular form, and we find that:

$$A_0(q) \quad = \quad -\frac{1}{24} E_{2,2}(\tau) = \frac{1}{48}(\theta_3^4 + \theta_4^4)(\tau) \tag{4.22}$$

$$V_7 \quad = \quad \frac{\pi^2}{3}(\theta_3^4 + \theta_4^4)(\tau) \, . \tag{4.23}$$

We then have a slew of consistency checks on all the other integers that we determined numerically (see (4.6)). These thirteen checks work out. We do therefore claim that the result (4.23) is exact. This is a simple example illustrating our methodology.

Next, we consider the triplet consisting of the components $\{2, 3, 4\}$. We find three eigenvectors of T, with eigenvalues corresponding to the cubic roots of unity. The eigenvector with eigenvalue 1 is also mapped to itself under the U transformation, and forms again a modular form of weight 2 under $\Gamma_0(4)$. It is indeed proportional to $E_{2,2}$:

$$V_2 + V_3 + V_4 = -2\pi^2(\theta_3^4 + \theta_4^4)(\tau) \, . \tag{4.24}$$

The other two eigenvectors, we raise to the power three, such that they become invariant under the T-transformation. These forms belong to the space $\mathcal{M}_6(\Gamma_0(4))$ of weight six modular forms. The dimension of this vector space is 4 (see theorem 3.5.1 in [121] with $g = \varepsilon_2 = \varepsilon_3 = 0$ and $\varepsilon_\infty = 3$), and it consists of three Eisenstein series, and one cusp form. A basis for these vector spaces is given by:

$$E_6^1 \quad = \quad -\frac{1}{252} E_6(\tau) \tag{4.25}$$

$$E_6^2 \quad = \quad -\frac{1}{252} E_6(2\tau) \tag{4.26}$$

$$E_6^4 \quad = \quad -\frac{1}{252} E_6(4\tau) \tag{4.27}$$

$$S_6 = \eta(q^2)^{12}, \tag{4.28}$$

where E_6 is the Eisenstein series of weight six, and η is the η-function, also recorded in appendix C. We need four coefficients to fix the eigenvectors in terms of this basis and we find (using the notation $\omega_3 = \exp(2\pi i/3)$) :

$$(V_2 + \omega_3 V_3 + \omega_3^2 V_4)^3 = -23328\pi^6 (E_6^1 - E_6^2 - 2S_6) \tag{4.29}$$
$$(V_2 + \omega_3^2 V_3 + \omega_3 V_4)^3 = -23328\pi^6 (E_6^1 - E_6^2 + 2S_6)\,. \tag{4.30}$$

The consistency checks using the numerics work out.

For the second triplet, we diagonalise U first, and proceed very analogously as above, except that we have to take a higher power for the second combination to find a modular form of weight 12 with respect to $\Gamma_0(4)$. We find the relations:

$$\begin{aligned} V_1 + V_5 + V_6 &= 4\pi^2 \left(\theta_3^4 + \theta_4^4\right) \\ (V_1 + \omega_3 V_5 + \omega_3^2 V_6)^3 + (V_1 + \omega_3^2 V_5 + \omega_3 V_6)^3 &= 5832\pi^6 (E_6^1(q) - 64 E_6^2) \\ ((V_1 + \omega_3 V_5 + \omega_3^2 V_6)^3 - (V_1 + \omega_3^2 V_5 + \omega_3 V_6)^3)^2 &= 136048896\pi^{12} \eta(q)^{24}\,. \end{aligned} \tag{4.31}$$

Note that the sum of all potentials is necessarily a modular form with weight 2 of $\Gamma_0(4)$. Indeed, this sum is equal to $112\pi^2 A_0(q)$ (as follows from the identity $A_5(q/27) + A_3(q/27) = \frac{4}{3} A_0(q)$).

4.3.4 A Remark on a Manifold of Extrema

There are also branches of extrema, namely, non-isolated extrema. These too are expected to behave well under a modular subgroup. Although this was not the focus of our investigation, we did find numerical evidence for a manifold of extrema at which the potential takes the $\Gamma_0(4)$ covariant value $-\frac{2\pi^2}{3} E_{2,2}$. We will come back to this observation in section 7.2 and provide an analytical proof of these statements in section 7.5.

Summary

In summary, we have full analytic control over the value of the potential for all isolated extrema of the $\mathfrak{so}(5)$ twisted Calogero-Moser integrable system. We have found a vector valued modular form of weight two of $\Gamma_0(4)$, and we were able to explicitly express its seven components in terms of ordinary modular forms of $\Gamma_0(4)$. The vector valued septuplet splits into a singlet modular form and a sextuplet vector valued modular form. The plot will thicken at higher rank.

4.4 The Case $D_4 = \mathfrak{so}(8)$ and the Point of Monodromy

At this stage, we choose to present our results on the rank four $D_4 = \mathfrak{so}(8)$ model first, since they are simpler than those on the non-trivial rank three cases to be presented in subsection 4.5. The $\mathfrak{so}(8)$ model is simply laced and we therefore expect the ordinary modular group $SL(2,\mathbb{Z})$ to play the leading role. The integrable system exhibits a global symmetry group S_3 that permutes the three satellite simple roots of the Dynkin diagram of $\mathfrak{so}(8)$. We will refer to the S_3 permutation group as triality. We turn to the enumeration and classification of the extrema of the potential. We found 34 extrema. These are listed and labelled in section 4.4.5. If we mod out by the global symmetry group, we are left with 20 extrema. The latter fall into multiplets of the duality group of size 1, 3, 4 and 12. We discuss these multiplets in the following paragraphs.

4.4.1 The Singlet

There is a singlet under S and T duality as well as triality. It has zero potential: $V_1 = 0$.

4.4.2 The Triplet

There is also a triplet under the duality group, labelled $\{2, 3, 4\}$, and the dualities act as:

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The relations $S^2 = 1$ and $(ST)^3 = 1$ are satisfied. We note that in these extrema, the positions belong to the lattice generated by $\omega_1/2$ and $\omega_2/2$. For this multiplet,T-duality acts geometrically.

We would like to deduce again from the S and T matrices and from the known first coefficients of the series expansions (see section 4.4.5) the exact expressions of the potentials in these extrema. The functions are expected to transform well under some congruence subgroup of the modular group. Note that the sum of the three functions must be a full-fledged modular form – indeed, the sum $V_2(q) + V_3(q) + V_4(q)$ vanishes. A brute force strategy leading to the identification of the appropriate congruence subgroup is the following. We decompose the generators of congruence subgroups [3] in terms of a product of S and T operations. We evaluate the product using the representation at hand (here 3×3 matrices) and check whether it is trivial for every generator.

It turns out that the subgroup $\Gamma(2)$ acts trivially on the extremal potentials. Hence all the potentials V_2, V_3 and V_4 belong to $\mathcal{M}_2(\Gamma(2))$. This space has dimension 2, and it is the set of linear combinations of the three Eisenstein functions associated to the three vectors of order 2 in $(\mathbb{Z}_2)^2$ which have the property that the sum of the three coefficients vanishes. (See appendix C for details and conventions). Matching a few coefficients, we find that

$$\begin{aligned} V_2 &= 12\left(2G_{2,2}\begin{bmatrix}0\\1\end{bmatrix} - G_{2,2}\begin{bmatrix}1\\1\end{bmatrix} - G_{2,2}\begin{bmatrix}1\\0\end{bmatrix}\right) \\ V_3 &= 12\left(-G_{2,2}\begin{bmatrix}0\\1\end{bmatrix} - G_{2,2}\begin{bmatrix}1\\1\end{bmatrix} + 2G_{2,2}\begin{bmatrix}1\\0\end{bmatrix}\right) \\ V_4 &= 12\left(-G_{2,2}\begin{bmatrix}0\\1\end{bmatrix} + 2G_{2,2}\begin{bmatrix}1\\1\end{bmatrix} - G_{2,2}\begin{bmatrix}1\\0\end{bmatrix}\right). \end{aligned}$$

This can also be written in terms of the Weierstrass $\wp$ function :

$$\begin{aligned} V_2(\tau) &= 3\left(2\wp\left(\frac{1}{2};\tau\right) - \wp\left(\frac{\tau+1}{2};\tau\right) - \wp\left(\frac{\tau}{2};\tau\right)\right) \\ V_3(\tau) &= 3\left(-\wp\left(\frac{1}{2};\tau\right) - \wp\left(\frac{\tau+1}{2};\tau\right) + 2\wp\left(\frac{\tau}{2};\tau\right)\right) \\ V_4(\tau) &= 3\left(-\wp\left(\frac{1}{2};\tau\right) + 2\wp\left(\frac{\tau+1}{2};\tau\right) - \wp\left(\frac{\tau}{2};\tau\right)\right). \end{aligned}$$

These two ways of writing the potentials make the action of dualities manifest. For instance, the transformation properties (C.23) show that under S-duality, $\wp(\frac{1}{2},\tau)$ becomes $\wp(\frac{1}{2},\frac{-1}{\tau}) =$

[3] There exist algorithms to find the generators. These are for instance implemented in Sage.

$\tau^2\wp(\frac{\tau}{2},\tau)$ while $\wp(\frac{\tau+1}{2},\tau)$ becomes $\tau^2\wp(\frac{\tau+1}{2},\tau)$, so that V_2 and V_3 are S-dual, et cetera. The result can also be written using perhaps more familiar modular forms

$$\begin{aligned}
V_2(q) &= -6\pi^2 E_{2,2}(q) \\
V_3(q) &= \frac{3}{2}\pi^2\left(2E_{2,2}(q) - 3\theta_2^4(q)\right) \\
V_4(q) &= \frac{3}{2}\pi^2\left(2E_{2,2}(q) + 3\theta_2^4(q)\right) .
\end{aligned}$$

The action of T-duality is again clear from these expressions. For S-duality it is slightly more intricate. Given that $E_{2,2}(q) = -\theta_2^4(q^2) - \theta_3^4(q^2)$, it relies on the identities

$$\begin{aligned}
2\theta_3^4(2\tau) + 2\theta_2^4(2\tau) + 3\theta_2^4(\tau) &= -\theta_2^4(\tau/2) + 2\theta_3^4(\tau/2) \\
\theta_3^4(\tau/2) + \theta_4^4(\tau/2) - 6\theta_4^4(\tau) &= -4\theta_3^4(2\tau) - 4\theta_2^4(2\tau) + 6\theta_2^4(\tau) ,
\end{aligned}$$

for S-duality between extrema 2 and 3, and self-S-duality for extremum 4, respectively.

4.4.3 The Quadruplet

We move on to discuss the extremal values of the potential in the quadruplet. We can arrive at the following closed form for the potential in extremum 6:

$$V_6(q) = -24\pi^2(-\frac{1}{24}E_{2,3}(q) + (\eta(q)^3 + 9\eta(q^9)^3)\eta(q^3)^2/\eta(q) + 3(\eta(q^3)^3/\eta(q))^2) .$$

Note that this can alternatively be written as

$$V_6(q) = -24\pi^2(g_0(q) + q^{1/3}g_1(q) + 3q^{2/3}g_2(q)) ,$$

where the g_i are functions that can be expanded into series with only integer powers of q (and the three summands in this expression correspond to the same summands in the expression above). Thus we know how the operation $\tau \to \tau+1$ acts on the extremum, and it generates two other extrema, whose potential we also know exactly. These are extrema 7 and 8:

$$\begin{aligned}
V_7(q) &= -24\pi^2(g_0(q) + e^{2i\pi/3}q^{1/3}g_1(q) + 3e^{-2i\pi/3}q^{2/3}g_2(q)) \\
V_8(q) &= -24\pi^2(g_0(q) + e^{-2i\pi/3}q^{1/3}g_1(q) + 3e^{2i\pi/3}q^{2/3}g_2(q)) .
\end{aligned}$$

The potential for the extremum 5 is:

$$V_5(q) = -3\pi^2 E_{2,3}(q) .$$

In the basis $\{5, 6, 7, 8\}$ the matrices for S- and T-dualities are :

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

We can also apply the same method as above. The generators of $\Gamma(3)$ are all trivial in this basis. Thus the potentials are weight 2 modular forms of this congruence subgroup. The latter form a 3-dimensional space, generated by the zero-sum linear combinations of the 4 Eisenstein series associated to the order 3 vectors in $(\mathbb{Z}_3)^2$ (there are 8 such vectors, but the Eisenstein series are invariant under $v \to -v$, leaving only 4 distinct functions, see appendix). We find

$$\begin{aligned}
V_5 &= \frac{27}{2}\left(3G_{2,3}\begin{bmatrix}0\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\0\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\2\end{bmatrix}\right)\\
V_6 &= \frac{27}{2}\left(-G_{2,3}\begin{bmatrix}0\\1\end{bmatrix} + 3G_{2,3}\begin{bmatrix}1\\0\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\2\end{bmatrix}\right)\\
V_7 &= \frac{27}{2}\left(-G_{2,3}\begin{bmatrix}0\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\0\end{bmatrix} + 3G_{2,3}\begin{bmatrix}1\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\2\end{bmatrix}\right)\\
V_8 &= \frac{27}{2}\left(-G_{2,3}\begin{bmatrix}0\\1\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\0\end{bmatrix} - G_{2,3}\begin{bmatrix}1\\1\end{bmatrix} + 3G_{2,3}\begin{bmatrix}1\\2\end{bmatrix}\right),
\end{aligned}$$

or alternatively,

$$\begin{aligned}
V_5(\tau) &= \frac{3}{2}\left(3\wp\left(\frac{1}{3};\tau\right) - \wp\left(\frac{\tau}{3};\tau\right) - \wp\left(\frac{\tau+1}{3};\tau\right) - \wp\left(\frac{\tau+2}{3};\tau\right)\right)\\
V_6(\tau) &= \frac{3}{2}\left(-\wp\left(\frac{1}{3};\tau\right) + 3\wp\left(\frac{\tau}{3};\tau\right) - \wp\left(\frac{\tau+1}{3};\tau\right) - \wp\left(\frac{\tau+2}{3};\tau\right)\right)\\
V_7(\tau) &= \frac{3}{2}\left(-\wp\left(\frac{1}{3};\tau\right) - \wp\left(\frac{\tau}{3};\tau\right) + 3\wp\left(\frac{\tau+1}{3};\tau\right) - \wp\left(\frac{\tau+2}{3};\tau\right)\right)\\
V_8(\tau) &= \frac{3}{2}\left(-\wp\left(\frac{1}{3};\tau\right) - \wp\left(\frac{\tau}{3};\tau\right) - \wp\left(\frac{\tau+1}{3};\tau\right) + 3\wp\left(\frac{\tau+2}{3};\tau\right)\right).
\end{aligned}$$

The dualities act on the vectors characterising the modular forms as follows

$$\begin{aligned}
T &: \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}0\\1\end{bmatrix}\\
&\quad \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix} \to \begin{bmatrix}1\\2\end{bmatrix} \to \begin{bmatrix}1\\0\end{bmatrix},\\
S &: \begin{bmatrix}0\\1\end{bmatrix} \leftrightarrow \begin{bmatrix}1\\0\end{bmatrix}\\
&\quad \begin{bmatrix}1\\1\end{bmatrix} \leftrightarrow \begin{bmatrix}1\\2\end{bmatrix}.
\end{aligned}$$

This reproduces the action of the dualities on the associated extrema. Thus, while the pattern of the positions of the extrema is non-linear, the arguments of the values of the potential at certain extrema do provide a linear realisation of the duality group.

Finally, we note that triality generates three copies of the triplet as well as of the quadruplet. Indeed, each of these extrema is left invariant by a $\mathbb{Z}_2$ subgroup of S_3 (as described in section 4.4.5).

Up to now, we have discussed the singlet, triplet and quadruplet whose duality diagrams are summarised in figure 4.5.

4.4.4 The Duodecuplet and a Point of Monodromy

In the multiplet of size twelve, also depicted in figure 4.5, a new feature appears. We find that the extrema exhibit a monodromy around a point in the interior of the fundamental domain of the parameter τ. Thus, to be able to describe the multiplet structure in this case we must first discuss the monodromy.

The point of monodromy

We find a single point in the interior of the fundamental domain around which there is monodromy amongst extrema. It is possible to determine this point numerically[4] and its value is close to $\tau_M \sim 2.41558i$. In particular, the extrema 13 and 16 are exchanged when we follow a loop in the τ-plane that closely circles the value τ_M. Moreover, using the geometry of the positions of the extrema 13 and 16, one can show that τ_M is a solution of the system of equations

$$\begin{cases} \wp(z;\tau)^2 + \wp(z-\omega_3;\tau)^2 + \wp(2z-\omega_3;\tau)^2 = \frac{\pi^4}{3}E_4(\tau) \\ 2\wp'(z;\tau) + 2\wp'(z-\omega_3;\tau) + \wp'(2z-\omega_3;\tau) = 0\,, \end{cases} \tag{4.32}$$

where $\omega_3 = \omega_1 + \omega_2$, which gives the numerical result

$$\tau_M = 2.415576987549484510777262081474158860468152563579077460...i\,.$$

Using the large accuracy of the value of the point of monodromy τ_M, we find the corresponding rational Klein invariant (with the normalisation (C.35)):

$$j(\tau_M) = \frac{488095744}{125} = 1728 \times \frac{7626496}{3375}\,.$$

This can be considered as an exact statement – the uncertainty is as low as 10^{-200}. Elliptic curves with rational Klein invariant have interesting arithmetic properties (see e.g. [121]).

The extended duality group

We can add the monodromy group to the set of generators S and T that act on our vector of extrema. The resulting diagram of dualities then becomes the one in figure 4.5. The generators satisfy the relations:

- $S^2 = M^2 = 1$ and $T^6 = 1$, while $(TM)^8 = 1$
- $SM = MS$
- $(MST)^3 = 1$.

Once we are underneath the point of monodromy in the canonical fundamental domain, the matrix MT plays the role usually taken by the matrix T in $SL(2,\mathbb{Z})$. In particular, relations like $(ST)^3 = 1$ implied by the geometry of the fundamental domain of the modular group take on the form $(SMT)^3 = 1$, et cetera. Triality leaves each extremum invariant.

[4]The most immediate manifestation of the monodromy phenomenon can be seen as a symmetry breaking in the equilibrium positions for extrema 13 and 16 when moving on the imaginary axis across the point of monodromy τ_M (which is purely imaginary). Below this critical value, as can be seen in the diagrams drawn at $\tau = i$ (in section 4.4.5), the two extrema are exchanged by the $\mathbb{Z}_2$ action $X_i \leftrightarrow -\bar{X}_i$, while above the critical value, they are both invariant with respect to this action. This makes it possible to determine $2.41557 \leq \mathrm{Im}\tau_M \leq 2.41558$.

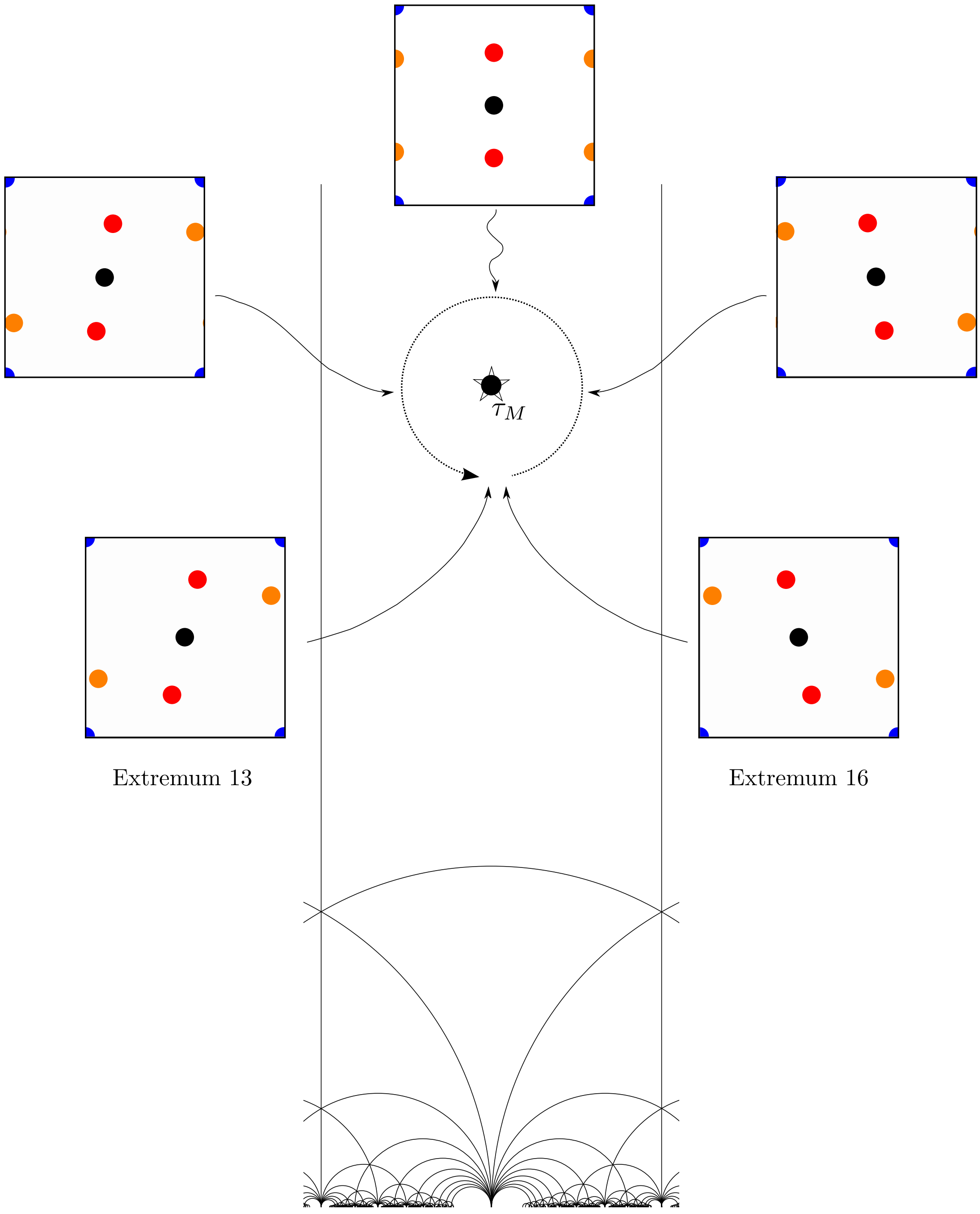


Figure 4.4: We illustrate the monodromy that exchanged extrema 13 and 16 in the case of algebra $\mathfrak{so}(8)$. The point of monodromy is figured by a star, and the circular arrow defines a path in the τ-plane. We start with extremum 16, and after a full circle, the configuration is that of extremum 13.

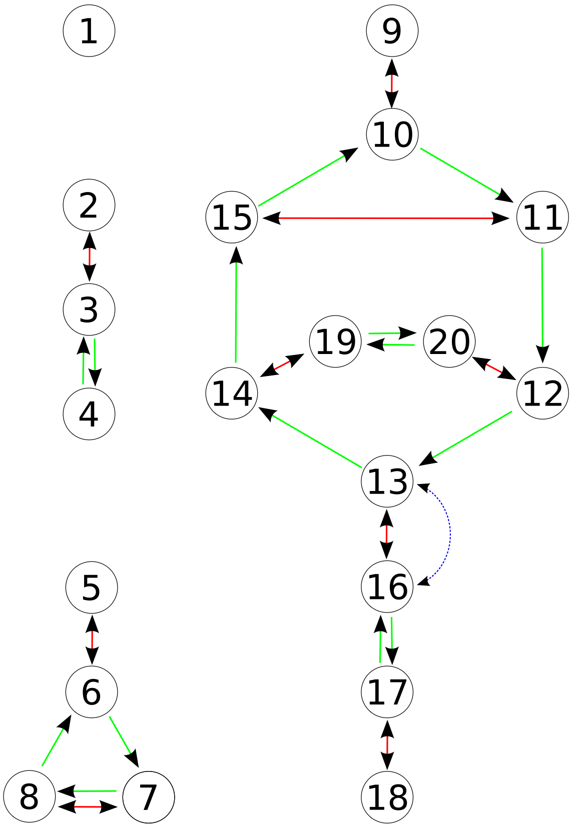


Figure 4.5: The diagram of the action of dualities on the $D_4 = \mathfrak{so}(8)$ extrema. In red we exhibit the action of S-duality, in green, T-duality, and in dotted blue, the monodromy.

4.4.5 The List of Extrema for $\mathfrak{so}(8)$

In this section, we give terms in the Fourier expansion of the extremal values of the potential in the duodecuplet. We note that a consistency and exhaustivity check on all multiplets is provided by the fact that the sum of all extrema in a given multiplet of $SL(2,\mathbb{Z})$ has to be a weight 2 modular form. The check works out: the sum equals zero in each multiplet separately, as it must. An analytic understanding of the duodecuplet extrema remains desirable.

The strategy we used to find extrema boils down to finding all the minima (which are also zeros) of the (auxiliary, gauge theory) potential (4.2) with non vanishing mass (5.22) using a simple gradient algorithm with random initial conditions. Then we identify those configurations that are related by one of the symmetries we quotient by. This procedure is executed at a given value of τ. Once the complete list of extrema is known, we can follow a given extremum along any curve in the upper half plane, by adiabatically varying τ. The T-dual extrema and the monodromies are obtained in this way, while the action of S-duality is known exactly. We thus unfold the whole web of dualities.

In order to determine the potential at the extrema, we first make use of our knowledge of T-duality, which dictates the Fourier expansion variable $q^{1/n}$, where n is the smallest positive integer such that T^n acts trivially on the extremum under consideration. Then we evaluate the extremal potential at many different values of τ and find recursively the rational Fourier coefficients.

The diagrams of the extrema

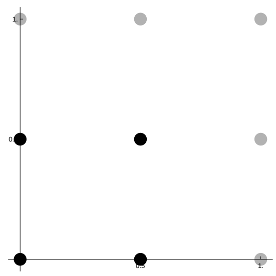
In the diagrams that follow, the black dots represent the values of the components X_i, $i = 1, 2, 3, 4$ at the extrema. The dark grey dots are images under the symmetries discussed in section 4.2.

In some of the diagrams, there are five black dots instead of four, reflecting the fact that they represent three extrema related by the global S_3 symmetry. For every such extremum, one subgroup $\mathbb{Z}_2 \subset S_3$ acts trivially. One of the three extrema is obtained by choosing one of the circled black dots and the three ordinary black dots, a second one is obtained by choosing the other circled black dot and the three black dots, while the third is determined by the four small black dots. (The pale grey dots show the possible translations of this extremum by half-periods.)

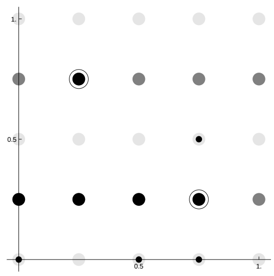
We note in passing that some exact information on the positioning of the extrema is available. For instance, for extremum number 9, some exact information on the positions is the following. At $\tau \to i\infty$, the system reduces to the Sutherland system (with trigonometric potential). According to [108], the positions at equilibrium are related to the roots of a Jacobi polynomial. Explicitly in the case of D_4, the polynomial is $P_2^{(1,1)}(y) = \frac{15}{4}(y-1)^2 + \frac{15}{2}(y-1) + 3$, from which we deduce the positions $X_1 = 0$, $X_{2,3} = \frac{1}{2\pi}\arccos(\pm 1/\sqrt{5})$ and $X_4 = \frac{1}{2}$. For $\tau \to 0$, the positions converge on $X_1 = 0$, $X_2 = 1/6$, $X_2 = 1/3$ and $X_4 = \frac{1}{2}$. This numerical convergence is slow.

S-duality guarantees that the situation is similar for the extremum on the imaginary axis, with the two limits exchanged. Moreover, T-duality then acts in the $\tau \to i\infty$ limit as $X_0 \to X_0$, $X_1 \to X_1 + 1/6$, $X_2 \to X_2 + 1/3$, $X_3 \to X_3 + 1/2$. (These transformations are exact within the precision of the numerics.) This generates the 6-cycle. Et cetera.

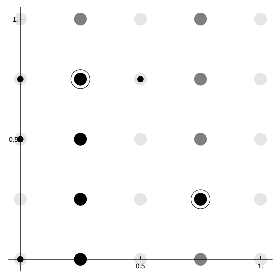
Extrema at $\tau = i$ for $\mathfrak{so}(8)$



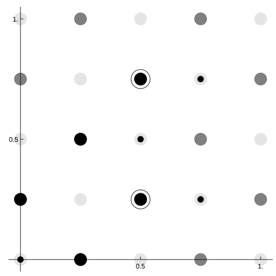
Extremum 1



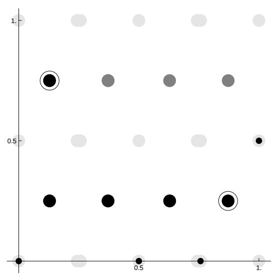
Extremum 2



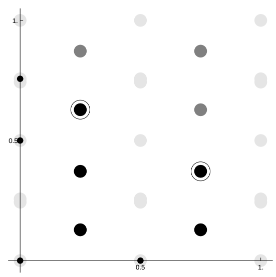
Extremum 3



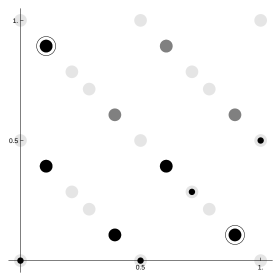
Extremum 4



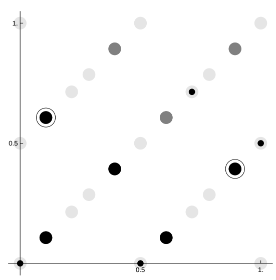
Extremum 5



Extremum 6

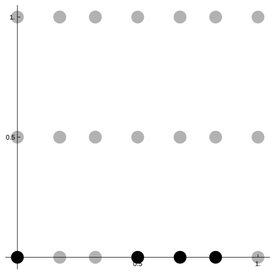


Extremum 7

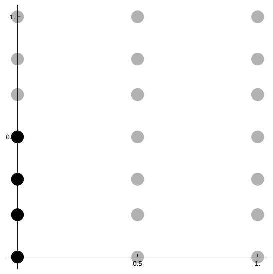


Extremum 8

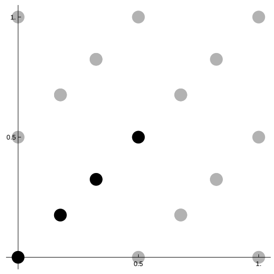
Extrema at $\tau = i$



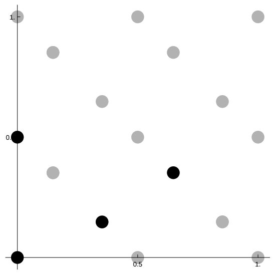
Extremum 9



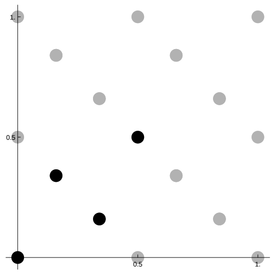
Extremum 10



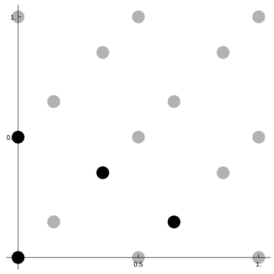
Extremum 11



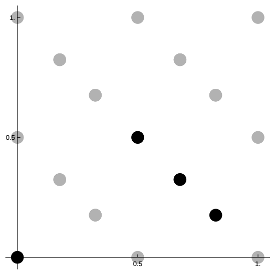
Extremum 12



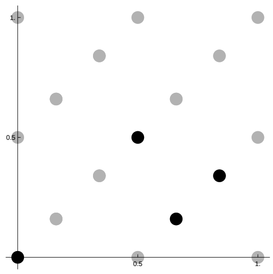
Extremum 13



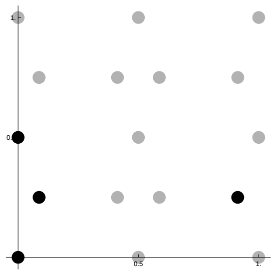
Extremum 14



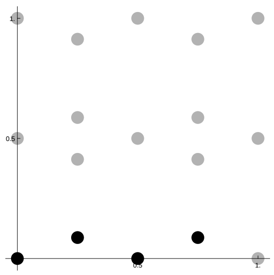
Extremum 15



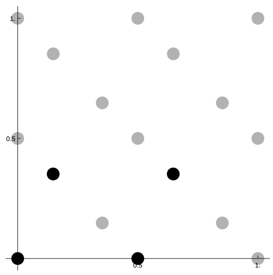
Extremum 16



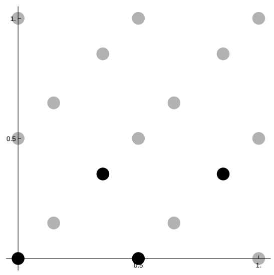
Extremum 17



Extremum 18



Extremum 19



Extremum 20

The series for the $\mathfrak{so}(8)$ extremal potentials

We have been able to determine the q-expansions of the potentials in each extremum with great accuracy, in terms of functions with integer coefficients. For extrema 1 to 8, we gave the exact expression in section 4.4. To list the series for the remaining extrema, we introduce 11 functions, for which we only reproduce the first few coefficients – more can be obtained – :

$$
\begin{aligned}
f_1(q) &= \tfrac{1}{1800} - 467q + 45379q^2 - 23993958092q^3 - 44044347374301q^4 \\
&\quad -711960536580667762q^5 + \ldots \\
f_2(q) &= 1 - 15172q + 51582918q^2 - 397077052296q^3 + 5101142359347277q^4 \\
&\quad +94300056917523369780q^5 + \ldots \\
f_3(q) &= \tfrac{1}{600} + q + 369q^2 + 68644q^3 + 11490041q^4 + 1579638246q^5 + \ldots \\
f_4(q) &= 1 + 3096q + 1818378264q^2 + 2446348866170976q^3 \\
&\quad +4535490919062930456600q^4 + \ldots \\
f_5(q) &= 1 - 142284q - 2825331513294q^2 - 110241726267588876840q^3 + \ldots \\
f_6(q) &= 2 + 780960q + 18367562372664q^2 + 762875530342634406144q^3 + \ldots \\
f_7(q) &= 1 - 4478868q - 121113750523626q^2 - 5314750232983801186536q^3 \\
f_8(q) &= \tfrac{1}{3}(14 + 79929712q + 2425403175787968q^2 \\
&\quad +111756708524847535116096q^3 + \ldots) \\
f_9(q) &= \tfrac{1}{3}(-37 - 489421748q - 16364614670173794q^2 \\
&\quad -787663906596039662206584q^3 + \ldots) \\
f_{10}(q) &= 1 - 12264q - 7273512936q^2 - 9785395464683424q^3 \\
&\quad -18141963676251721826280q^4 + \ldots \\
f_{11}(q) &= 1 + 110596q + 110757888006q^2 + 180011523750912008q^3 \\
&\quad +367762906594569664954381q^4 + \ldots
\end{aligned}
$$

The potentials then read

$$
\begin{aligned}
V_9 &= 14400\pi^2 f_3\left(\tfrac{q}{5^3}\right) \\
V_{10+k} &= -4\pi^2 \sum_{j=0}^{5} (16q)^{j/6} \exp\left(2\pi i \tfrac{kj}{6}\right) f_{4+j}\left(\tfrac{q}{3^3}\right) \\
V_{16} &= -3\pi^2(f_{10}(q) - 72\sqrt{q} f_{11}(q)) \\
V_{17} &= -3\pi^2(f_{10}(q) + 72\sqrt{q} f_{11}(q)) \\
V_{19} &= -24\pi^2(75 f_1(q/15^3) + i\sqrt{5q/3} f_2(q/15^3)) \\
V_{20} &= -24\pi^2(75 f_1(q/15^3) - i\sqrt{5q/3} f_2(q/15^3))\,,
\end{aligned}
$$

where $k = 0, \ldots, 5$. The last series V_{18} can then be deduced from the fact that the sum of all potentials in the duodecuplet vanishes. Note that the coefficients grow rapidly, preventing the functions above to be modular forms. The monodromy is responsible for this phenomenon, as can be confirmed by the estimation of the convergence radius given by the successive ratios of the coefficients (see figure 4.6).

4.5 The Dual Cases $B_3 = \mathfrak{so}(7)$ and $C_3 = \mathfrak{sp}(6)$

4.5.1 Exact Multiplets

For the twisted elliptic integrable models associated to the dual Lie algebra root systems $\mathfrak{so}(7)$ and $\mathfrak{sp}(6)$, we present our results succinctly. We have found 17 isolated extrema for each, and they are Langlands dual. We have therefore 34 extrema in total. We identified two quadruplets of the full duality group for which we found analytic expressions for the potential at the extrema. The list of the corresponding extrema is given in section 4.5.3. We find the following duality properties and analytic values for the extrema of the potential. The extrema labelled $\{1, 2\}$ have

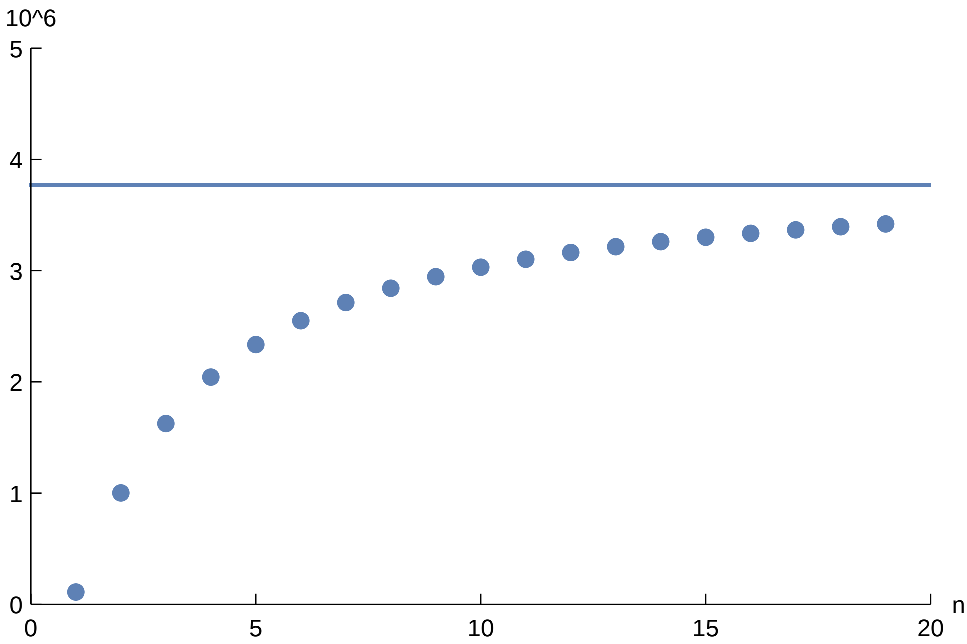


Figure 4.6: The dots show the successive ratios of the coefficients of f_{11}, and a line has been drawn, for comparison, at the value $1/q_M = e^{-2\pi i \tau_M}$.

extremal values for the $\mathfrak{so}(7)$ potential equal to $V_1(\tau)$ and $V_2(\tau)$. From the diagram of dualities (figure 4.7), we read off that these extremal values are modular forms of $\Gamma_0(4)$ with weight 2. Moreover, Langlands duality then implies that $V_{1^\vee}(2\tau)$ and $V_{2^\vee}(2\tau)$ are also of that ilk. The space $\mathcal{M}_2(\Gamma_0(4))$ of these weight 2 forms has the two generators

$$-E_{2,2}(\tau) = \theta_2^4(2\tau) + \theta_3^4(2\tau) = 1/2(\theta_3^4(\tau) + \theta_4^4(\tau))$$
$$-E_{2,4}(\tau) = 3\theta_3^4(2\tau) = 3/4(\theta_3^2(\tau) + \theta_4^2(\tau))^2 \,.$$

In terms of the generators, the extrema are:

$$\begin{aligned}
V_1(\tau) &= \pi^2 \left(-E_{2,2}(\tau) - 2E_{2,4}(\tau)\right) \\
V_2(\tau) &= \pi^2 \left(-7E_{2,2}(\tau) + 2E_{2,4}(\tau)\right) \\
V_{1^\vee}(2\tau) &= \pi^2 \left(+E_{2,2}(\tau) + 0E_{2,4}(\tau)\right) \\
V_{2^\vee}(2\tau) &= \pi^2 \left(-2E_{2,2}(\tau) + 1E_{2,4}(\tau)\right) \,.
\end{aligned}$$

For the other quadruplet under the full duality group, we have a similar story, with the happy ending:

$$\begin{aligned}
V_3(2\tau) &= \pi^2/6 \left(-15E_{2,2}(\tau) + 7E_{2,4}(\tau)\right) \\
V_4(2\tau) &= \pi^2/6 \left(+9E_{2,2}(\tau) - 1E_{2,4}(\tau)\right) \\
V_{3^\vee}(\tau) &= 8\pi^2/3 \left(-3E_{2,2}(\tau) + 1E_{2,4}(\tau)\right) \\
V_{4^\vee}(\tau) &= 8\pi^2/3 \left(0E_{2,2}(\tau) - 1E_{2,4}(\tau)\right) \,.
\end{aligned}$$

The action of Langlands S_2 duality as well as T-duality can be found explicitly using these exact expressions, for instance by exploiting properties of θ functions. As an example, we note that the action of T-duality is summarised in the equalities:

$$\begin{aligned} E_{2,2}\left(\tau+\frac{1}{2}\right) &= -2E_{2,2}(\tau)+E_{2,4}(\tau) \\ E_{2,4}\left(\tau+\frac{1}{2}\right) &= -3E_{2,2}(\tau)+2E_{2,4}(\tau)\,. \end{aligned}$$

Moreover, on the extrema, the Langlands duality S_2 acts as

$$\frac{1}{2\tau^2}V_1\left(-\frac{1}{2\tau}\right)=V_{1^\vee}(\tau)+3\pi^2E_{2,2}(\tau)\,,$$

and similar relations hold for the other S_2-dual couples, as predicted by the duality formula (3.32).

4.5.2 The Duodecuplet, the Quattuordecuplet and the Points of Monodromy

We further identified a duodecuplet and a quattuordecuplet under the duality group (for a total of $(4+4+12+14)/2 = 17$ extrema for $B_3 = \mathfrak{so}(7)$). Sufficient data to reproduce them is provided in section 4.5.3. These multiplets exhibit points of monodromy, and the full duality diagram is captured in figure 4.7. It should be understood that we only represent points of monodromy that are inequivalent (where two monodromies are taken to be equivalent when they are equal up to conjugation by other elements of the duality group). For instance $S_2M_\tau S_2$ is the monodromy around $-1/(2\tau)$.

We draw attention to a few features of the diagram. There are 5 extrema that form a quintuplet under T-duality (around $\tau = i\infty$), labelled $5,6,7,8,9$. When we also turn around the point of monodromy, the quintuplet enhances to a septuplet. This is reminiscent of a feature of the duality diagram for the duodecuplet of $\mathfrak{so}(8)$.

Finally, we performed an exhaustivity check on the extrema by summing the extremal values of the potential. We found[5]

$$\begin{aligned} \sum_{i\in\mathbf{4}_1} V_i(\tau) &= -8\pi^2E_{2,2}(\tau) \\ \sum_{i\in\mathbf{4}_2} V_i(\tau) &= 2\pi^2E_{2,2}(\tau) \\ \sum_{i\in\mathbf{12}} V_i(\tau) &= -20\pi^2E_{2,2}(\tau) \\ \sum_{i\in\mathbf{14}} V_i(\tau) &= 19\pi^2E_{2,2}(\tau)\,, \end{aligned}$$

showing again that the sum of potentials over every multiplet is a modular form of $\Gamma_0(4)$.

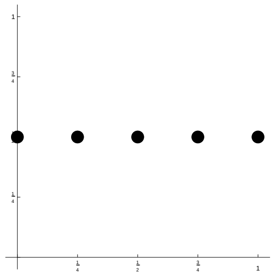
4.5.3 The List of Extrema for $\mathfrak{so}(7)$ and $\mathfrak{sp}(6)$

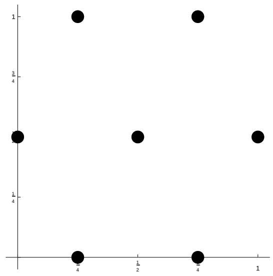
Finally, in the case of the algebras $B_3 = \mathfrak{so}(7)$ and $C_3 = \mathfrak{sp}(6)$, we only present diagrams of the extremal positions for the $\mathfrak{so}(7)$ root system, since the corresponding extrema for $\mathfrak{sp}(6)$ can be

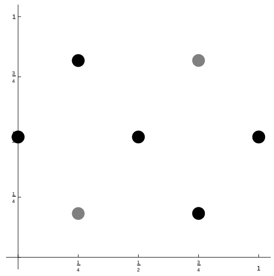
[5] We evaluated the sum of the extrema numerically at two different values of τ to identify the linear combination of $E_{2,2}$ and $E_{2,4}$ that equals the sum. We can then perform arbitrary many numerical checks at other values of τ, and these work out.

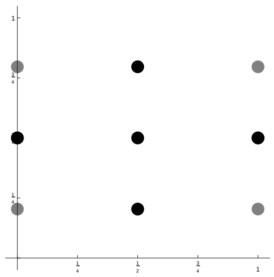
found by Langlands duality. We use the same conventions as for the $\mathfrak{so}(5)$ figures. Additional data, like the data we presented for $\mathfrak{so}(8)$ in the previous section, can be found.

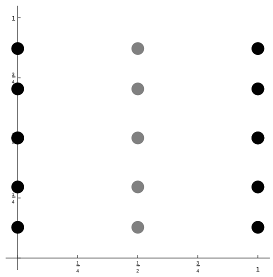
Extrema at $\tau = i$ for $\mathfrak{so}(7)$

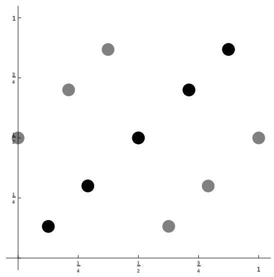

Extremum 1

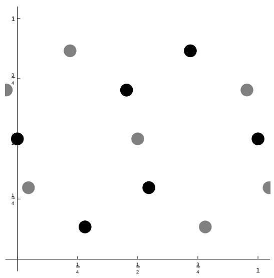

Extremum 2

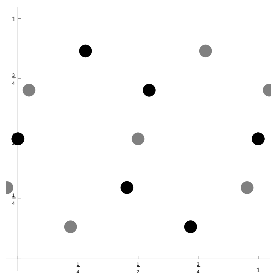

Extremum 3

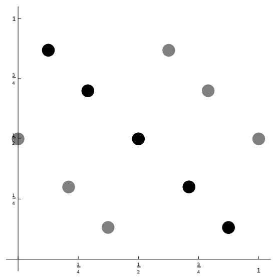

Extremum 4

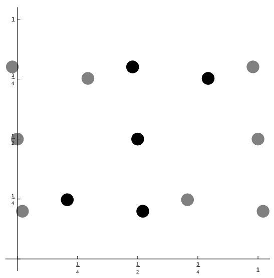

Extremum 5

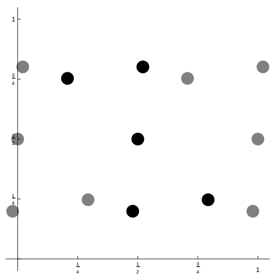

Extremum 6

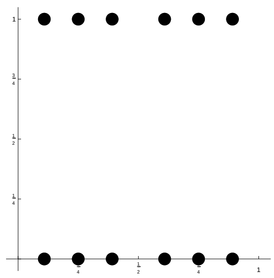

Extremum 7


Extremum 8

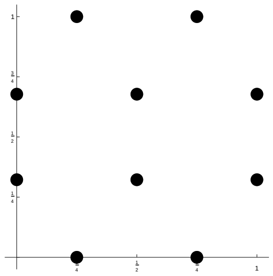

Extremum 9


Extremum 10

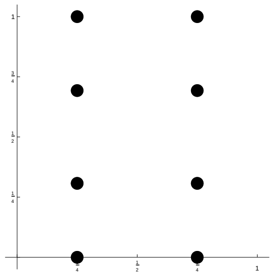

Extremum 11


Extremum 12

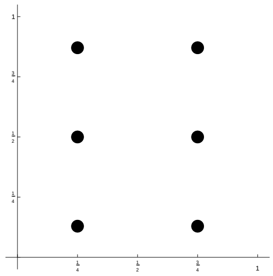
Extrema at $\tau = i$ for $\mathfrak{so}(7)$



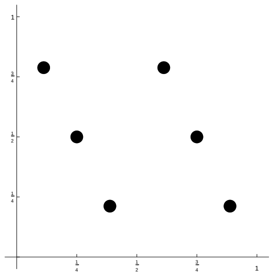
Extremum 13



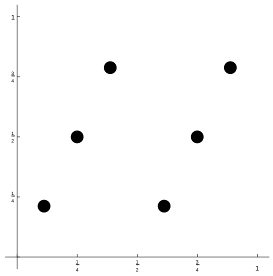
Extremum 14



Extremum 15



Extremum 16



Extremum 17

This concludes our systematic case-by-case discussion of the low rank B, C, D isolated extrema of (twisted) elliptic Calogero-Moser models. We finish the section with a few further remarks on general features of the problem of identifying isolated extrema.

4.6 Partial Results for Other Lie Algebras

In this subsection, we discuss very partial results for some higher rank Lie algebras. We think of the elliptic integrable model as a perturbation of the Sutherland model, with trigonometric potential. The Sutherland model has a ground state with all particles sprinkled on the real circle. We can perturb this traditional ground state by turning on the elliptic deformation by powers of the small parameter q, and follow the ground state under perturbation. In this way, we can reconstruct the extremum of the complexified elliptic potential associated to the Sutherland extremum on the real line. To take the limit from the elliptic integrable system towards the Sutherland model, it is sufficient to use the expansion formula:

$$\wp(x;\omega_1,\omega_2) \quad = \quad -\frac{\pi^2}{12\omega_1^2}E_2(q) + \frac{\pi^2}{4\omega_1^2}\csc^2\left(\frac{\pi x}{2\omega_1}\right) - \frac{2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos\frac{n\pi x}{\omega_1}\,, \qquad (4.33)$$

valid when the imaginary part of the modular parameter τ is sufficiently large. The first term in the formula (4.33) is constant from the perspective of the integrable system dynamics, while the second term gives rise to the leading Sutherland potential. The minimum at the equilibrium of the Sutherland potential on the real line can be computed analytically [108] – it is related to the norm of the Weyl vector of the Lie algebra. The positions of the equilibria are given in terms of zeroes of the Jacobi polynomials. We can perform perturbation theory around these extrema (numerically), and we find the following series in q for the potential at perturbed Sutherland extrema, for various gauge algebras:

$$
\begin{aligned}
\frac{V_{\mathfrak{so}(5)}}{\pi^2} &= \frac{26}{3}+\frac{112q}{3}+\frac{8656q^2}{81}+\frac{392128q^3}{2187}+\frac{5011568q^4}{19683}+\frac{319117472q^5}{1594323} \\
&\quad +\frac{12236034880q^6}{43046721}+\frac{112088197760q^7}{387420489}+\ldots \\
\frac{V_{\mathfrak{so}(6)}}{\pi^2} &= 8+64q+192q^2+256q^3+192q^4+384q^5+768q^6+512q^7+192q^8+\ldots \\
\frac{V_{\mathfrak{so}(7)}}{\pi^2} &= 25+\frac{408q}{5}+\frac{153816q^2}{625}+\frac{23730528q^3}{78125}+\frac{6103562136q^4}{9765625}+\frac{663346128528q^5}{1220703125} \\
&\quad +\frac{129316813943136q^6}{152587890625}+\frac{10819167546478272q^7}{19073486328125}+\ldots \\
\frac{V_{\mathfrak{so}(8)}}{\pi^2} &= 24+\frac{576q}{5}+\frac{212544q^2}{625}+\frac{39538944q^3}{78125}+\frac{6618263616q^4}{9765625}+\frac{909871629696q^5}{1220703125} \\
&\quad +\frac{171403608639744q^6}{152587890625}+\frac{8112643818471936q^7}{19073486328125}+\frac{1087819119225488448q^8}{2384185791015625}+\ldots \\
\frac{V_{\mathfrak{so}(9)}}{\pi^2} &= \frac{164}{3}+\frac{992q}{7}+\frac{5133728q^2}{12005}+\frac{2305844608q^3}{4117715}+\frac{168902799438112q^4}{176547030625} \\
&\quad +\frac{11307570247017024q^5}{12111126300875}+\frac{640315787843154194816q^6}{370903242964296875}+\frac{1106383118191321793331968q^7}{890538686357276796875} \\
&\quad +\frac{6992975426525938043543690368q^8}{381818461775682426660156 25}+\frac{176835032301731636090243294882 24q^9}{1309637323890590723444335937 5}+\ldots \\
\frac{V_{\mathfrak{so}(10)}}{\pi^2} &= \frac{160}{3}+\frac{1280q}{7}+\frac{1303808q^2}{2401}+\frac{616518656q^3}{823543}+\frac{365560247552q^4}{282475249} \\
&\quad +\frac{101140172889600q^5}{96889010407}+\frac{9869502718168064q^6}{4747561509943} \\
&\quad +\frac{1840112769746623897 6q^7}{11398895185373143}+\frac{6582207315175560008960q^8}{3909821048582988049}+\ldots \\
\frac{V_{\mathfrak{so}(11)}}{\pi^2} &= \frac{305}{3}+\frac{1960q}{9}+\frac{30141880q^2}{45927}+\frac{29034410080q^3}{33480783}+\frac{4243088924219480q^4}{2790589782267} \\
&\quad +\frac{7560807432828504560q^5}{6103019853817929}+\frac{4158609757083162994374880q^6}{1526041805387611692663} \\
&\quad +\frac{963487422865188667206742 40q^7}{529754512441699487595 87}+\frac{30488526503804116257966072492 4120q^8}{92724468600756742242419154123}+\ldots \\
\frac{V_{\mathfrak{so}(12)}}{\pi^2} &= 100+\frac{800q}{3}+\frac{4055200q^2}{5103}+\frac{1335804800q^3}{1240029}+\frac{63808646477600q^4}{34451725707} \\
&\quad +\frac{4294563385869280 0q^5}{25115308040403}+\frac{6332155765834649948800q^6}{2093335809859549647}+\ldots
\end{aligned}
$$

We see that at least for some extrema, it is fairly straightforward to generate interesting data on the value of the potential at these extrema at higher rank. We note a first pattern, valid at the order to which we have worked, in both the rank of the gauge group, and the power of the modular parameter q. Table 4.2 gives the conjectured smallest integer N such that for gauge algebra g, the potential $V_g(Nq)/\pi^2$ has a Fourier expansion with only integer coefficients in the following sense: the expansion can be written as $n_0(r+n_1q+n_2q^2+\ldots)$ where the n_i are integers, and the first term r is rational.

As an example of this pattern, let us quote the formula:

$$
\begin{aligned}
\frac{1}{66679200\pi^2} V_{\mathfrak{so}(12)}\left(63^3 \times q\right) \quad = \quad & \frac{1}{666792} + q + 745143q^2 + 252572301828q^3 + 108583732036588599q^4 \\
& +25066769592690393853446q^5 + 1108797393440320434232075 2348q^6 \\
& +196665218038734185416818286761 4728q^7 + ...
\end{aligned}
$$

As a final remark, we note that our numerical searches in this and previous subsections are far from exhausting the capabilities of present day computers.

k	5	6	7	8	9	10	11	12	13	14
N	3^3	1	5^3	5^3	$7^3 5^2$	7^3	$3^6 7^2$	$3^6 7^3$	$3^1 7^1 11^3$	$3^2 7^1 11^3$

Table 4.2: The integer N for gauge algebra $\mathfrak{so}(k)$ rendering the Fourier expansion integral

4.7 Conclusions

We studied the isolated extrema of complexified elliptic Calogero-Moser models, and encountered a plethora of beautiful phenomena. The values of the integrable interparticle potential at the extrema are true vector-valued modular forms in some cases, allowing for an analytic determination of the extrema in terms of modular forms of congruence subgroups of the modular group. This gives rise to webs of extrema that form representations under the duality group of the model. The latter can either be a modular or a Hecke group. A more intricate phenomenon is the appearance of monodromies amongst a second class of extrema as we loop around a point in the fundamental domain of the modular group. The duality group is then enlarged to include the monodromy generator. We determined the action of these generators on extrema. Moreover, we provided a wealth of Fourier coefficients of the extremal potential. These analyses can be viewed as a considerable widening of the observation of the integrality of observables in equilibria of integrable systems.

With this data about the Calogero-Moser systems, we now return to our original gauge theory problem. In the next chapter, we will see how the two questions are related, and what subtleties have to be taken into account in order for a precise agreement to be found.

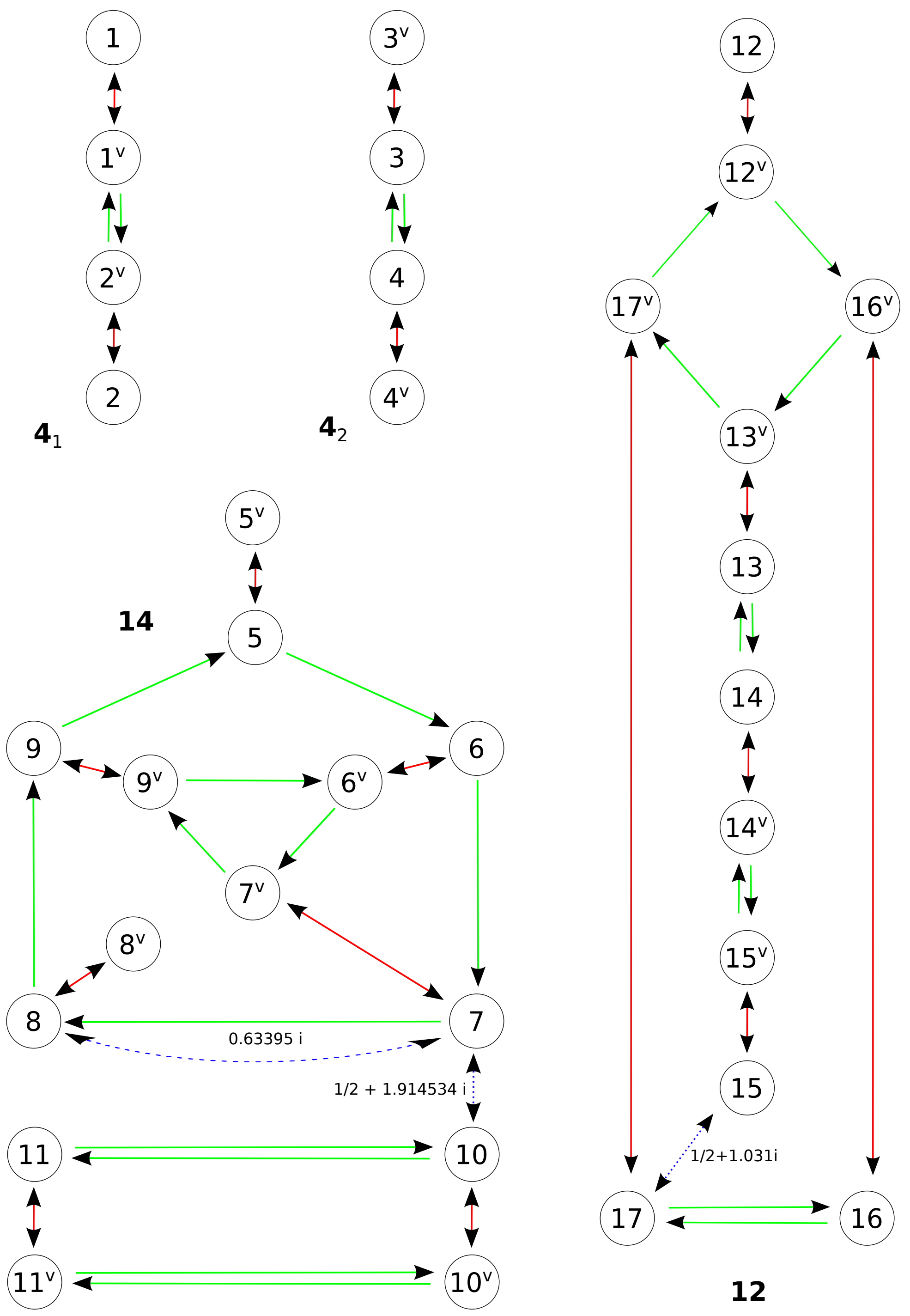


Figure 4.7: The diagram of dualities for $\mathfrak{so}(7)$ and $\mathfrak{sp}(6)$ extrema. In red, we show the action of Langlands S_2-duality on the extrema, in green, T-duality, and in dotted blue, monodromies, with the corresponding approximate values of the points of monodromy τ. As discussed in the text, monodromies relating $\mathfrak{sp}(6)$ extrema exist but are not represented here as they are equivalent to those already depicted.

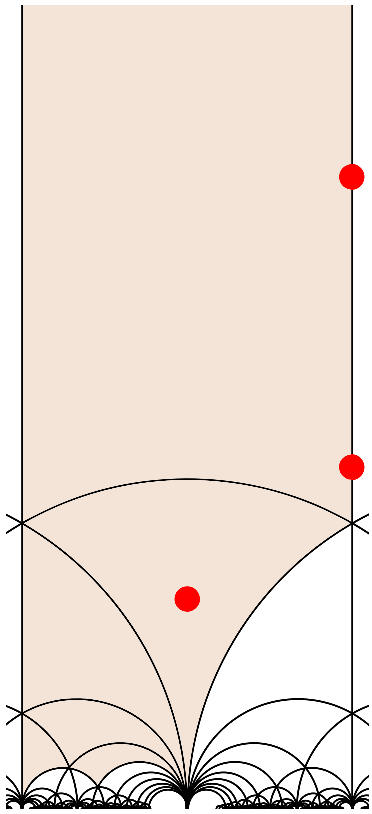


Figure 4.8: The positions of the monodromies (red dots) inside the fundamental domain of $\Gamma_0(4)$ (shaded).

Chapter 5

The $\mathcal{N} = 1^*$ Gauge Theories on $\mathbb{R}^3 \times S^1$

In this chapter, we reinterpret the results we obtained in chapter 4 in terms of the physics of massive vacua of $\mathcal{N} = 1^*$ theories. The first section is devoted to general facts about the intimate relation between integrable systems and supersymmetric gauge theories. We will depict this bridge in detail for the theory we are concerned with. In section 5.2.3, we review the properties of the infrared physics of the $\mathcal{N} = 1^*$ theory and in particular the way the effective superpotential on the Coulomb branch is determined.

We then show how the data we gathered on elliptic integrable systems in section 3.3 provides semiclassical inputs for the gauge theory and elucidates it further. Many semiclassical results will be confirmed in this comparison, but part of the observations made on the integrable system side will have to wait until the next chapters to be fully explained – most strikingly, the mismatch between the number of extrema of the Calogero-Moser potential and the number of vacua found semi-classically will be explained there. The electric-magnetic duality properties in the infrared under the modular group are then discussed, with various subtleties depending on the gauge algebra. For $\mathfrak{so}(5)$, the Hecke group will play a role, and especially its congruence subgroup $\Gamma_0(4)$ that we used to classify extrema in section 4.3.3, and which is a subgroup of the modular group $\Gamma_0(2)$ introduced back in section 1.2. On the other hand, for $\mathfrak{so}(8)$, we make explicit the action of the global triality symmetry on the massive vacua.

5.1 Supersymmetric gauge theories and integrable systems

In this section, we give an overview of the relations between supersymmetric gauge theories on $\mathbb{R}^4$ and integrable system, relations that underlie a large part of the work presented in this thesis. Perhaps the most noticeable point is the identification between the spectral curve of the integrable system, defined in equation (3.3), and the Seiberg-Witten curve of the gauge theory (at least for $\mathcal{N} = 2$ theories). In this section, we begin with a rapid survey of the Seiberg-Witten technology for $\mathcal{N} = 2$ gauge theories, and then review various results about the correspondence between these theories and integrable systems. Finally, we present how this story survives the partial breaking of supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

5.1.1 Seiberg-Witten curves for $\mathcal{N} = 2$ theories

In the seminal paper [19], Seiberg and Witten consider the pure $\mathcal{N} = 2$ theory with gauge group $SU(2)$, whose Lagrangian is given by (A.24). In a generic supersymmetric vacuum, the gauge field is broken to $U(1)$ and the low-energy physics is described by the vacuum expectation values of the scalar part of the vector multiplet and its dual, (a, a_D). The moduli space of vacua is the complex plane parametrized by u which is related to a by $u \sim \frac{1}{2}a^2$ in the semiclassical limit. This suggests that a might not be single-valued as a function of u. Indeed, when one moves along a closed curve in the moduli space, the functions $(a, a_D)(u)$ undergo monodromies, and the breakthrough of Seiberg and Witten was to realize these functions as periods of a differential one-form on a complex elliptic curve whose structure is given by u. Although our aim here is not to establish these facts, we give the equation of the curve to illustrate a few points: it is defined as the subspace of points $(z, x) \in \hat{\mathbb{C}}^2$, where $\hat{\mathbb{C}}$ is the Riemann sphere (C.5), that satisfy[1]

$$\Sigma_{\text{pure } SU(2)}: \qquad \Lambda^2\left(z + \frac{1}{z}\right) = x^2 - u\,. \tag{5.1}$$

This curve depends on the external parameter u, and we often want to study the whole family of such curves when u varies in the moduli space. Here $z \in \hat{\mathbb{C}}$ parametrizes the so-called ultraviolet curve [122, 22] that we mentioned already in section 1.2.2.

The original papers of Seiberg and Witten deal with the $SU(2)$ theories, and this seminal work has been generalized to various gauge groups and matter content [57, 56, 123, 124, 59, 60, 61, 62]. For instance, the curve for the pure $SU(N)$ theory which generalizes (5.1) is

$$\Sigma_{\text{pure } SU(N)}: \qquad \Lambda^N\left(z + \frac{1}{z}\right) = x^N + u_2 x^{N-2} + \cdots + u_N\,. \tag{5.2}$$

This curve is a N-sheeted cover of the ultraviolet Riemann sphere $\hat{\mathbb{C}}$. It is a Riemann surface of genus $N-1$, as can be seen from the Riemann-Hurwitz formula:

$$2g - 2 = N(2\cdot 0 - 2) + (2N-2)\cdot(2-1) + 2\cdot(N-1)\,. \tag{5.3}$$

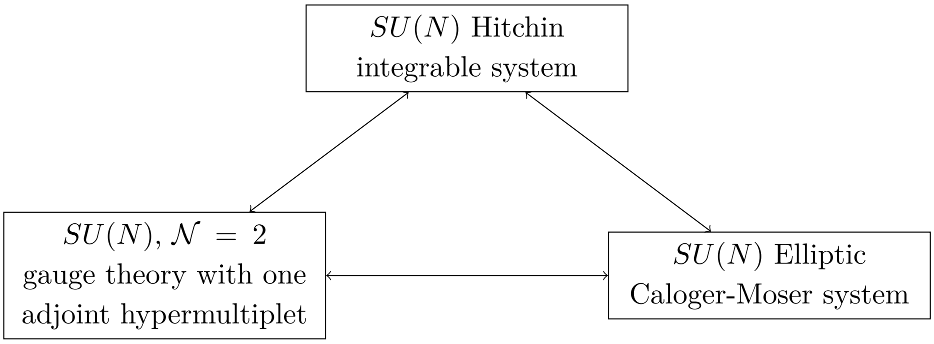
Note that for other gauge groups, the genus of the Seiberg-Witten curve may differ from the rank of the group; for instance if $G = SO(2N)$ then it is a $2N$-sheeted cover of $\hat{\mathbb{C}}$. One particularly interesting matter content is the one which is obtained from $\mathcal{N} = 4$ by a mass deformation, and that we can call the $\mathcal{N} = 2^*$ theory, namely the case of one adjoint hypermultiplet. In this case the Seiberg-Witten curve was soon recognized to be known from another context. In the next subsection we explain how all this can be put in that different perspective.

5.1.2 Integrable Systems and $\mathcal{N} = 2$ theories

In 1995, Martinec and Warner proved in [70] that the Seiberg-Witten curve for the *pure* $\mathcal{N} = 2$ super Yang-Mills theory with simple gauge group G is a spectral curve of the periodic Toda lattice for the dual gauge group $G^\vee$. A more general earlier discussion of the link between the Seiberg-Witten solution and integrable systems can be found in [69].

[1] Here Λ is defined by the one-loop running of the coupling constant g_{YM}, equation (A.23), and the fact that the angle θ_{YM} doesn't run. Calling $b = 3T(\text{adjoint}) - T(\text{chiral mult.})$, the running of τ is solved by $2\pi i\tau(\mu) = b\log(\Lambda/\mu)$. In other words, $\Lambda = \mu \exp 2\pi i\tau$ is an intrinsic scale of the theory, called the (holomorphic) *dynamical* scale.

Almost simultaneously, Donagi and Witten [77] identified the $SU(N)$ theory with a massive adjoint matter hypermultiplet, which is known as $\mathcal{N} = 2^*$ theory, with the $SU(N)$ Hitchin system, which is an integrable model arising from two-dimensional $SU(N)$ gauge theory. It appeared then that the spectral curve of this Hitchin integrable system is identical to the spectral curve of an Elliptic Caloger-Moser system, proving the following correspondence:



The general slogan is that the Coulomb branch of vacua of an $\mathcal{N} = 2$ theory has a geometric structure called *special geometry*. We have seen in section 3.2.5 that special Kähler manifolds appear naturally in complexified integrable systems. Using certain properties of BPS dyons in the gauge theory and the structure of the Coulomb branch, it is in general possible to construct action-angle variables of an integrable systems [125]. To identify an integrable system dual to a given supersymmetric field theory, a necessary condition is that the numbers of moduli on both sides have to match,namely the number of scalar fields that might acquire a vacuum expectation value has to be equal to the dimension of the moduli space of the spectral curve of the integrable system. Moreover, the matching between the special Kähler structures can be checked in the weak-coupling limit, and in many case from exact computations using the techniques recently developed in [126].

5.1.3 The Calogero-Moser poteneial and $\mathcal{N} = 2^*$

We will now focus on the particular example of Donagi and Witten and see how the Calogero-Moser systems are connected to the $\mathcal{N} = 1^*$ theories.

The solution of Donagi and Witten

Let us describe the solution [77] of the $\mathcal{N} = 2^*$ theory with adjoint matter for gauge group $SU(N)$. Note that this theory was already solved in ([49]), section 16.2, for gauge group $SU(2)$. In order for this presentation to be more concrete, we give a detailed description of this solution. We introduce a curve in the (t, x, y) space defined by its equations

$$\Sigma_{\mathcal{N}=2^*,SU(N)}: \qquad \begin{cases} y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)) \qquad (\mathcal{E}) \\ P_N(t,x,y) + A_2 P_{N-2}(t,x,y) + \cdots + A_N P_0(t,x,y) = 0\,. \end{cases} \tag{5.4}$$

with parameter τ given by the complexified coupling constant. Here the A_i are constants that should be identified with the order parameters on the Coulomb branch, and the P_i are polynomials in t, x and y that can be computed explicitly. This is a curve of genus N. In order to ensure modular covariance of the solution, we impose the following modular weights : $t \to 1$, $x \to 2$, $y \to 3$, $A_k \to k$. Alternatively we can parametrize the solution by λ and write (5.4) as

$$\Sigma_{\mathcal{N}=2^*,SU(N)} : \qquad \begin{cases} y^2 = x(x-1)(x-\lambda) \\ P_N(t,x,y) + A'_2 P_{N-2}(t,x,y) + \cdots + A'_N P_0(t,x,y) = 0\,. \end{cases} \tag{5.5}$$

The dictionary between the τ parametrization and the λ parametrization is[2]

$$\lambda = \frac{e_3(\tau) - e_2(\tau)}{e_1(\tau) - e_2(\tau)} = 16 q^{1/2} + O(q)\,. \tag{5.6}$$

Because of this last equation, the weak-coupling limit corresponds to $\lambda \to 0$. The results of [49] for $N = 2$ can be recovered in this more general setup, and moreover we can flow from $\mathcal{N} = 2^*$ to pure $\mathcal{N} = 2$ by sending the mass of the adjoint hypermultiplet to infinity, or equivalently through the renormalization group $\lambda \to 0$ with a suitable rescaling of the other parameters. These are important consistency checks, and are good indications that the curve (5.4) is correct. But the true novelty here resides in the way it is obtained, and it is the objective of the next paragraph to give an idea of how this is achieved.

The Hitchin and Calogero-Moser Systems

A crucial point in this analysis is that ultraviolet finite $\mathcal{N} = 2$ theories should have a spectral parameter living on an torus whose modulus is the complexified gauge coupling τ, and the pure $\mathcal{N} = 2$ theory arises from the degeneration of this torus to a punctured sphere. Therefore the integrable system should also degenerate to the system associated to the pure $\mathcal{N} = 2$ theory, which is the affine Toda system. It was noticed [127, 128] that the elliptic Calogero-Moser has this property. As we discussed in section 3.4.1, this system reduces to the affine Toda system in the limit $\omega_2 \to \infty$.

D'Hoker and Phong showed [129] that the Calogero-Moser integrable system captures the physics of the low energy dynamics of $\mathcal{N} = 2$ SYM with gauge algebra $SU(N)$ and one massive adjoint hypermultiplet. The Seiberg-Witten differential arises naturally from the construction, using the spectral parameter z. Decoupling the full hypermultiplet by letting the mass go to infinity while keeping the vacuum expectation value of the gauge scalar fixed reproduces the gauge theory without hypermultiplet [57, 56].

The appearance of Calogero-Moser systems can be understood from a more general construction by Hitchin [130]. Donagi and Witten [77] showed on general grounds that there should be a connection between the low energy effective action of any four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory and such an integrable system. For an introduction to Hitchin systems and their relation with Calogero-Moser systems, one can consult [99, 131, 132, 133] on which this paragraph is based. The main idea behind the relation between $\mathcal{N} = 2$ field theories and Hitchin systems is that the latter arise as the moduli space of the field theory compactified on a circle.

The Hitchin construction provides an integrable system for any given spectral curve, of arbitrary genus, with or without marked points. It involves a gauge group G. Most classical integrable systems can be obtained from this construction, and this is the case of the elliptic

[2] We send linearly $e_1 \mapsto 1$, $e_2 \mapsto 0$ and $e_3 \mapsto \lambda$ in order to have $\lambda \to 0$ in the semiclassical regime $q \to 0$.

Calogero-Moser systems. Let C be a Riemann surface, with marked points z_k, and let u_k be an element of the Lie algebra $\mathfrak{g}$ associated to each marked point. Let the letter A denote a generic $(0,1)$ connection and let ϕ denote a generic $(1,0)$-form on the curve C. The phase space of the Hitchin system is the space of solutions (A,ϕ) of

$$\bar{\nabla}_A \phi = 2\pi i \sum_k u_k \delta_{z_k} \tag{5.7}$$

divided by the *generalized gauge group*, where the generalized gauge group is the set of maps $h : C \to G$ such that $h(z_k)$ lies in the stabilizer of u_k; they act on A and ϕ as $A \to h^{-1}Ah + h^{-1}\bar{\partial}h$ and $\phi \to h^{-1}\phi h$. Let us now introduce the polynomials P_m, for $m = 1, \ldots, r$, that generate the ring of invariant polynomials on $\mathfrak{g}$ (their degrees are the exponents of $\mathfrak{g}$). Then for each m, we can decompose the meromorphic $(m,0)$-form $P_m(\phi)$ in a basis of holomorphic differentials of type $(m,0)$; the coefficients thereby obtained are in involution and define a Hamiltonian integrable system.

Let us consider a particularly useful example, where C is a torus with parameter τ with the point $z = 0$ marked and associated to $u \in \mathfrak{g}$. The equation (5.7) determines the behavior of ϕ around the origin, and the fact that it is defined on the torus allows to use the theory of elliptic functions to solve for ϕ in terms of Lamé functions (3.18). The Hamiltonien of the system can be read off from $P_2(\phi) = \operatorname{tr}\phi^2$. In this case one obtains the spin Calogero-Moser system, where the spin variables are related to u. The spin Calogero-Moser system is a generalization of the systems that we have studied under this name until now, and the latter can be obtained by Hamiltonian reduction from the former.

The one-punctured torus used in the last paragraph should be reminiscent of the one-punctured torus that served as ultraviolet curve for the $\mathcal{N} = 2^*$ theory in chapter 1. This is no coincidence, as we explain now – let us rephrase the story in physical gauge theoretic language. The importance of the compactification is seen more clearly in a six-dimensional setup, described by the following diagram [132]:

$$\begin{array}{ccc}
\begin{array}{c}\text{Six-dimensional}\\ \mathcal{N} = (2,0)\ \text{SCFT}\\ \text{with algebra } \mathfrak{g}\end{array} & \longrightarrow & \begin{array}{c}\text{Five-dimensional}\\ \text{super Yang-Mills}\\ \text{with } g_{YM} = \sqrt{R}\end{array}\\
\downarrow & & \downarrow\\
\begin{array}{c}\text{Four-dimensional}\\ \mathcal{N} = 2\ \text{theory } T[\mathfrak{g}, C]\end{array} & \longrightarrow & \begin{array}{c}\text{Three-dimensional}\\ \text{theory } T\,[R]\end{array}
\end{array}$$

Here, the horizontal arrows represent compactification on a circle S^1 of radius R, while vertical arrows denote compactification on the ultraviolet curve C. The left-hand side is the usual construction of four-dimensional class S theories. The resulting three-dimensional theory $T\,[R]$ can be thought of as a sigma model (with $\mathcal{N} = 4$ in three-dimensional terms) with target space $\mathcal{M}$. Taking the point of view of the five-dimensional theory, the BPS equations combined with translation invariance in $\mathbb{R}^3$ give the Hitchin equations (5.7) on C. The moduli space of solutions of these equations can be seen as the target space $\mathcal{M}$ of the sigma model $T\,[R]$.

Generalization to any gauge algebra

The initial work for $\mathfrak{su}(N)$ has been extended to any gauge algebra (except G_2) in [134], the precise correspondence being with the ordinary elliptic Calogero-Moser system when the algebra is simply laced and with the *twisted* Calogero-Moser (3.26) system when it is not. In this configuration, the spectral curves constructed from the integrable systems generate the Seiberg-Witten curves for the SYM theory with one massive hypermultiplet. This has been checked by explicitly constructing the Lax pairs with spectral parameters in [105, 109] for all simple Lie algebras.

$$\begin{tikzcd} \boxed{\text{$\mathcal{N}=2^*$ gauge theory with gauge algebra $\mathfrak{g}$}} \arrow[r, leftrightarrow] \arrow[d] & \boxed{\text{Twisted Elliptic Caloger-Moser system associated to $\mathfrak{g}$}} \arrow[d] \\ \boxed{\text{Pure $\mathcal{N}=2$ gauge theory with gauge algebra $\mathfrak{g}$}} \arrow[r, leftrightarrow] & \boxed{\text{Toda system associated to } \left(\mathfrak{g}^{(1)}\right)^\vee} \end{tikzcd} \tag{5.8}$$

The associated elliptic Calogero-Moser system permits generalisations to any root system, and allows for twists, which were used to provide Seiberg-Witten curves and differentials for $\mathcal{N}=2^*$ theory with general gauge group G [111]. The generalisation was non-trivial since the elegant technique of lifting to M-theory [135] is difficult to implement in the presence of orientifold planes (see e.g. [136, 137]), while the relevant generalised Hitchin integrable system has a gauge group which is related to the gauge group of the Yang-Mills theory in an intricate manner [131]. For a review of part of the history, see the lectures [138].

5.1.4 Breaking $\mathcal{N}=2$ to $\mathcal{N}=1$

Now that we have described how integrable systems come into play for describing $\mathcal{N}=2$ theories, we want to break part of the supersymmetry and reach the $\mathcal{N}=1$ realm. We are primarily interested in the massive deformations path to enter it. The basic phenomenon, which is illustrated in [19] for pure $SU(2)$ theory, is that the $\mathcal{N}=2$ theory has a continuous manifold of quantum vacua which degenerates to a finite number of massive vacua when one adds a $\mathcal{N}=1$ preserving mass perturbation in the superpotential. The privileged points in the original moduli space are those where massless charged particles appear, and we know from the general features of the solution presented in section 5.1.1 that these correspond to singularities of the Seiberg-Witten curve. If we consider a general gauge group G, a generic point on the Coulomb branch corresponds to a low-energy theory where the residual gauge group at low energy is $U(1)^r$. In order to obtain a massive vacuum, we need to give a mass to the r photons through the Higgs mechanism, or to remove them from the low-energy spectrum through confinement.

In the case of the $\mathcal{N}=2^*$ theory with gauge group $SU(N)$, we have seen that the curve (5.4) has genus N, and needs to develop $N-1$ nodes to correspond to a massive vacuum of the deformed $\mathcal{N}=1^*$ theory. The resulting curve has genus one, and the Riemann-Hurwitz formula then shows that it is an *unramified* N-cover of the elliptic curve $\mathcal{E}$. Donagi and Witten then argue [77] that this provides a good classification of the vacua : there is a one-to-one correspondence between massive vacua of the $SU(N)$, $\mathcal{N}=1^*$ theory and the N-fold unramified covers of a torus. Seeing the torus $\mathcal{E}$ as $\mathbb{C}$ quotiented by a lattice Γ, an N-fold unramified cover is a sublattice Γ' of index N in Γ. It is not hard to see that these lattices correspond to subgroups

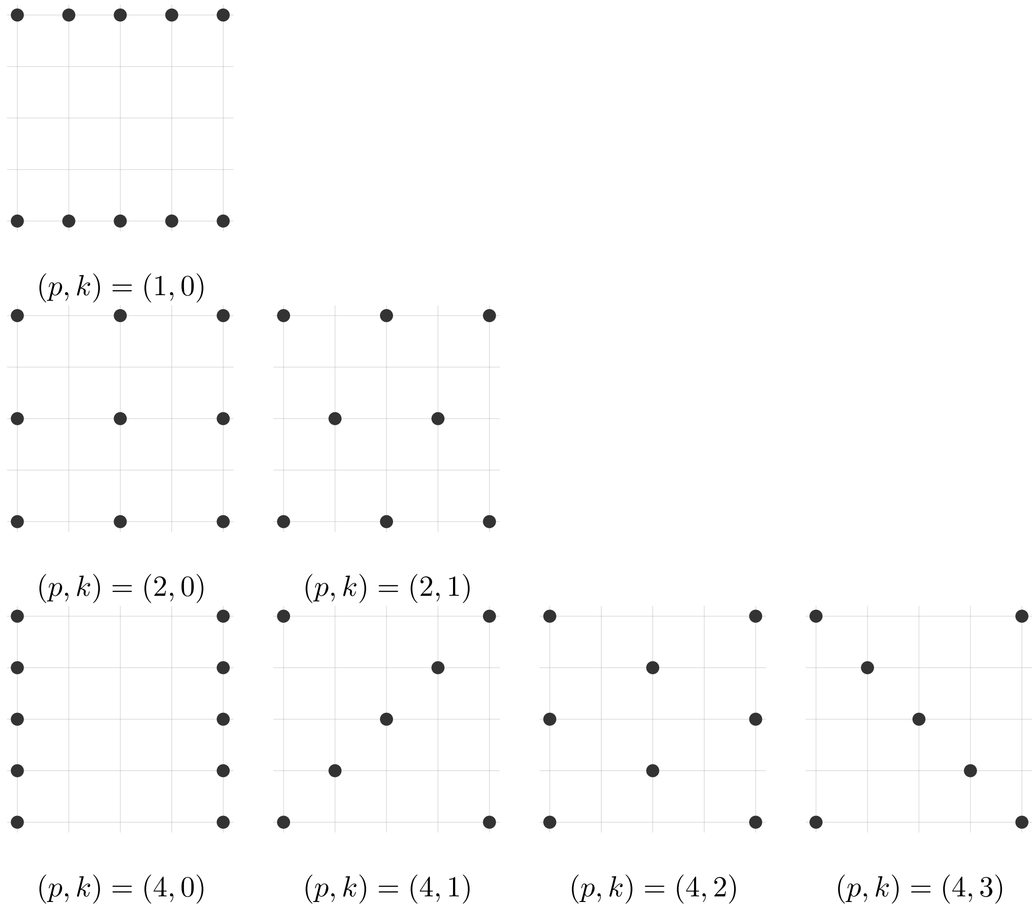
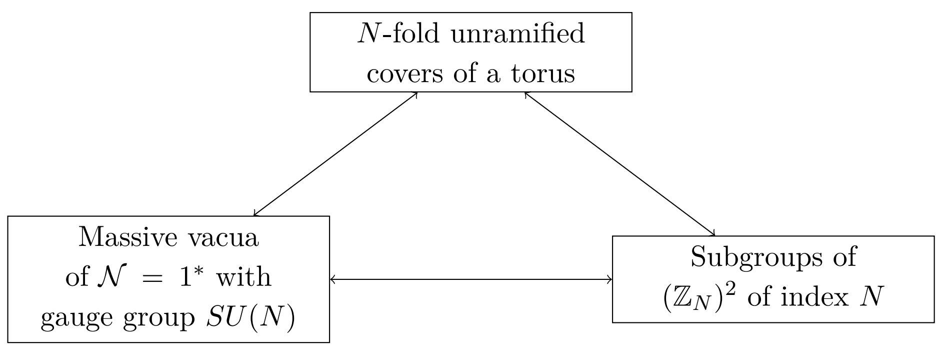


Figure 5.1: Symbolic representations of the 7 subgroups of $(\mathbb{Z}_N)^2$ of index N for $N = 4$.

of $(\mathbb{Z}_N)^2$ of index N. These subgroups are easily found, and correspond in an obvious manner to the extrema found while examining the Calogero-Moser potential in section 4.1. In figure 5.1, we have represented graphically these subgroups in the case $N = 4$, and used the (p, k) parametrization of equation (4.4). We can summarize this discussion by:



In $\mathcal{N} = 2$ gauge theories, a milestone of the correspondence with integrable systems is the computation of the prepotential. This is no longer available in $\mathcal{N} = 1$ theories, and one would like to find another object that could be computed exactly on both sides. This object will be the superpotential, and the general idea is that it has a very concrete interpretation on the integrable side as the classical potential. We make this idea more precise in the next section.

5.2 The non-perturbative superpotential

Let us stress three important points that we have made in the previous paragraphs:

- We have seen in section 5.1.4 that the quantum vacua found by Donagi and Witten for the $SU(N)$, $\mathcal{N}=1^*$ theory on $\mathbb{R}^4$ coincide with the extrema of the Calogero-Moser associated with the gauge algebra.
- The physics of $\mathcal{N}=2^*$ theories is described in a certain sense by Calogero-Moser systems, as indicated in (5.8). The Seiberg-Witten curve and the spectral curve coincide.
- The vacua of a $\mathcal{N}=1$ theory which is a massive deformation of a $\mathcal{N}=2$ theory can be obtained from the singularities of the Seiberg-Witten curve of the latter theory.

Recalling the general fact of supersymmetric gauge theories that vacua are extrema of the superpotential $\mathcal{W}$, these observations are hints that $\mathcal{W}$ is in fact proportional to the Elliptic Calogero-Moser quadratic Hamiltonian. In this section we give more arguments in favor of this conclusion. Note that if this is true, then the isolated extrema of the Elliptic Calogero-Moser system found in chapter 4 will get an immediate interpretation as massive vacua of the associated gauge theory.

In section 5.1, we have reviewed the fertile relation between $\mathcal{N}=2$ supersymmetric gauge theories and integrable systems. This holds in particular for $\mathcal{N}=2^*$ theories, which are mass deformations of $\mathcal{N}=4$ theories, and represent but an intermediary step towards $\mathcal{N}=1^*$ theories.

In [39], following the reasonings in [19, 29, 134, 38, 105, 111], an exact effective superpotential for $\mathcal{N}=1^*$ was proposed for any gauge group, generalizing the result (5.21). The bottom line is that the exact superpotential is equal to the potential (3.26) of the twisted elliptic Calogero-Moser model, that we repeat here:

$$\mathcal{W}(Z) \propto V_{\Delta,tw}(Z) = g_l \sum_{\alpha_l \in \Delta_l^+} \wp(\alpha_l(Z);\omega_1,\omega_2) + g_s \sum_{\alpha_s \in \Delta_s^+} \wp_n(\alpha_s(Z);\omega_1,\omega_2)\,, \tag{5.9}$$

The arguments were based on holomorphy, uniqueness of the deformation from $\mathcal{N}=2^*$, the form of non-perturbative contributions, and integrability. We have added to these reasonings the test of S-duality in subsection 3.3.1.

The effective potential on $\mathbb{R}^4$ can then be recuperated from the radius independent potential on $\mathbb{R}^{2,1}\times S^1$. However, in this procedure it is clear that one should be mindful about the global distinctions between the gauge theory on $\mathbb{R}^4$ and the theory on $\mathbb{R}^3\times S^1$. An example of such subtlety is provided by the supersymmetric index of pure $\mathcal{N}=1$ theories which indeed depends on those global properties. In that case, the choice of the center of the gauge group and the spectrum of line operators is crucial in computing the vacuum structure after compactification on S^1 [10, 139].

5.2.1 A Strategy : Compactification

A very useful tool that we will use extensively from now on is compactification of the theory from the topologically trivial spacetime $\mathbb{R}^4=\mathbb{R}^{1,3}$ to the cylinder $\mathbb{R}^3\times S^1=\mathbb{R}^{1,2}\times S^1$ where one spatial direction is compact, the circle S^1 being of radius R. In this situation, the complex variable Z used in the Calogero-Moser potential (3.25) acquires a very natural field theoretic interpretation, as we explain below.

In [29] the pure $\mathcal{N}=2$ theory of [19] is studied with one spacelike dimension compactified. The theory is formulated on $\mathbb{R}^3 \times S^1$, with S^1 being a circle of radius R. There are two important limits :

- For $R \to \infty$, the results of the analysis on $\mathbb{R}^4$ should be recovered.
- In the $R \to 0$ limit, one should obtain the dimensionally reduced theory (this theory is obtained by quantifying the classical theory where all the fields are independent of the compact dimension).

We then see that studying the theory on $\mathbb{R}^3 \times S^1$ can shed new light on gauge theories in three or four flat spacetime dimensions.

There is another reason why one would want to compactify the theory on the cylinder, as explained in [140, 65] : it allows to connect the confining phase continuously to a weakly coupled Coulomb phase. Indeed on the cylinder we will see that the now possible non-trivial Wilson loop can break the gauge group G to the abelian $U(1)^r$. This should be contrasted with the so-called Weak-Coupling Instanton approach where suitable matter fields are added so that the gauge group is completely broken, and the theory is in a weakly coupled Higgs phase.

In section 2.2, we have determined the vacuum structure of the $\mathcal{N}=1^*$ theory on $\mathbb{R}^4$. The gauge group is broken to a subgroup: this subgroup can be trivial, in which case the theory is said to be in the Higgs phase, it can have abelian factors, in which case we say that the theory is in the Coulomb phase, or it can have only non-abelian factors, in which case we have confinement. We now turn to the theory on the cylinder.

5.2.2 The gauge field on the cylinder

We first focus on the effects of compactification on the gauge field A_μ, which we is required to be periodic:

$$A_\mu(x^0+2\pi R, x^j) = A_\mu(x^0, x^j)\,. \tag{5.10}$$

In the compactified theory, smooth and finite energy configurations of periodic gauge field have been classified in [141] by Gross, Pisarski and Yaffe. As a result, they found three sets of invariants, which we review now. Of course, a fundamental ingredient of the compactified theory is the Wilson line (A.6) on the nontrivial circle,

$$W(\mathcal{C}) = \mathcal{P}\exp\left(i\oint_{\mathcal{C}} A\right) = e^{2\pi i a}\,. \tag{5.11}$$

Under a gauge transformation parametrized by a function $U(x)$ defined on our compact spacetime, it transforms as

$$W(\mathcal{C}) \to U(x)W(\mathcal{C})U^{-1}(x) \tag{5.12}$$

where x is any point on the path of integration. This means that the eigenvalues and the eigenspaces of $W(\mathcal{C})$, seen as a matrix in the adjoint representation of the gauge group, are gauge invariant. We define a as an element of the Lie algebra,

$$a = \frac{1}{2\pi}\oint_{S^1} A\,. \tag{5.13}$$

This is a scalar with respect to the three-dimensional Lorentz group, and may be viewed as a new scalar degree of freedom. Up to a global gauge transformation, we can assume that a belongs to the Cartan subalgebra $\mathfrak{h}$.

Let us now examine a subtlety related to the global choice of the gauge group (see [29] for a similar discussion in the case of gauge groups $SU(2)$ and $SO(3)$). We assume that the gauge group is connected. If there are no fields in the theory which are charged under its center, or if there is no nontrivial center at all, we may choose the gauge group such that we allow for gauge transformations that twist around the circle by an element of the center. This reasoning corresponds to a choice of gauge group $G_{ad} = \tilde{G}/Z(\tilde{G})$ where $\tilde{G}$ is the universal cover, and $Z(\tilde{G})$ its center. Indeed, let $U : \mathbb{R}^3 \times S^1 \to \tilde{G}$ be a gauge transformation as in section A.2.1, and let $U_{ad} : \mathbb{R}^3 \times S^1 \to G_{ad}$ be the map defined by $U_{ad}(x) = Z(\tilde{G})U(x)$. There are several U that are associated to the same U_{ad} if $Z(\tilde{G})$ is not trivial. This means that more gauge transformations are allowed in we take the gauge group to be G_{ad}, and we call the set of gauge transformations U (respectively U_{ad}) the *small* gauge group (respectively the *big* gauge group), despite the injection $G_{ad} \to G$. Note that the dual theory to the one with gauge group $\tilde{G}/Z(G)$ has gauge group $\tilde{G}$, for a simply laced group. The two scalars are interchanged under S-duality. Thus, the duality symmetries mix various global choices of gauge groups and also acts on the twist direction of the twisted elliptic potential.

We will take the assumption that the gauge group is G_{ad} for granted in this chapter, and come back to the issue in chapter 7. This lends a periodicity to the Wilson line under shifts taking values in the co-weight lattice $P^\vee$, defined in the appendix section B.2 dedicated to the various lattices associated to a gauge group. The classical moduli space for the infinitesimal Wilson line a is then [65]

$$\mathcal{M}_a = \frac{\mathbb{R}^r}{P^\vee \rtimes \text{Weyl group}} . \tag{5.14}$$

Generically, the gauge group is broken by a to its maximum abelian subalgebra $U(1)^r$, and all the eigenvalues are distinct.

Associated to each eigenspace is a magnetic charge, obtained by integrating on a large sphere S^2 the projection of the magnetic field (which is the three-dimensional Hodge dual of the field strength) on the eigenspace. Let us for simplicity assume that these spaces are all one-dimensional. Then all these magnetic charges can be assembled in a r-dimensional vector, which can be brought to the Cartan subalgebra using part of the global gauge symmetry. More precisely, let us define the magnetic field by

$$B_i = \frac{1}{2}\varepsilon_{ijk}F^{jk} , \tag{5.15}$$

where i, j and k denote the noncompact coordinates. Then we define the magnetic charge to be

$$\alpha^\vee_{\text{magnetic}} = -\frac{1}{2\pi}\int_{S^2} B_j \mathrm{d}S^j . \tag{5.16}$$

The notation reflects the fact that because of the Dirac quantization rule [142, 143], the magnetic charge belongs to the dual root lattice $Q^\vee$ of the Lie algebra.

Finally we have the four-dimensional instanton charge

$$m = \frac{1}{8\pi^2}\int \operatorname{Tr} F^2 . \tag{5.17}$$

With this notation, the θ-term in the Lagrangian, equation (A.13), is just $\mathcal{L}_\theta = m\theta_{YM}$. There is of course a strong connection with the Pontryagin index $\mathtt{p} \in \mathbb{Z}$, but not equality, as explained in detail in appendix B of [141], because of the twisting introduced by the magnetic charges. The relation between these quantities is

$$m = \mathtt{p} - \alpha^\vee_{\text{magnetic}} \cdot a \,. \tag{5.18}$$

We summarize this discussion by saying that to a given gauge configuration we can associate a unique triplet $(m, a, \alpha^\vee_{\text{magnetic}}) \in \mathbb{R} \times \mathcal{M}_a \times P^\vee$, which is gauge invariant and also invariant under local and continuous deformations of the gauge field.[3] Now we would like to characterize the remaining degrees of freedom. A generic value of a breaks the gauge group to $U(1)^r$, and we will assume that we stand at such a point from now on. This condition can also be written $\alpha \cdot a \notin \mathbb{Z}$ for all roots α. If this is the case, the three-dimensional remaining photon can be dualized to a scalar σ, introduced in the action as a Lagrange multiplier for the Bianchi identity before integrating out the gauge field. This field is a Lorentz scalar and can be seen as an element of the Cartan subalgebra (or its dual); more precisely, it belongs to the moduli space

$$\mathcal{M}_\sigma = \frac{\mathbb{R}^r}{P \rtimes \text{Weyl group}} \,, \tag{5.19}$$

where P is the weight lattice. This periodicity condition follows from the fact that σ is introduced as a Lagrange multiplier for a term which is quantized in integer units, namely the magnetic flux. After this change of variable, it appears that σ has a standard kinetic term with zero mass, and it can be combined with the Wilson line a into one single complex scalar field

$$Z = \sigma + \tau a \,. \tag{5.20}$$

This adjoint-valued scalar field parametrizes the *Coulomb branch* of the compactified theory. This branch is not compact, since there are special points where a non-abelian symmetry would be restored. Let us study the geometrical properties of this Coulomb branch. Note that we can write $2\omega_1 Z = 2\omega_1\sigma + 2\omega_2 a$, which is the version we use in the following discussion to make contact with section 3.2.6.

- As shown in equations (5.14) and (5.19), the periodicities of the variable Z is the weight lattice in the ω_1 direction and the co-weight lattice in the ω_2 direction.
- We also have discrete identifications, controlled by the Weyl group of the gauge algebra. This is the remnant of gauge invariance.

This classification of supersymmetric vacua agrees with the classification we did in section 3.2.6 in the elliptic integrable systems.

5.2.3 The Non-perturbative Superpotential for $\mathcal{N} = 1$ theories

In this section, we determine the superpotential of the $\mathcal{N} = 1^*$ theory. Let us choose a particular point Z on the Coulomb branch, and halt for a moment to see where we are. We started from a vacuum of the theory on $\mathbb{R}^4$ where the adjoint supermultiplets $\tilde{\Phi}_i$ took a vacuum expectation value that corresponds to an extremal point of the superpotential (2.7). When going to low energies, these three massive fields can be integrated out. In fact, all the fields except Z acquire

[3] Note that the triplet can distinguish between configurations with the same field strength which are not gauge equivalent.

a mass via the Higgs mechanism. Therefore at low energies, we have an effective $\mathcal{N} = 1$ theory on $\mathbb{R}^3 \times S^1$ with one adjoint valued massless field Z. The most general Lagrangian for such a theory is given in (A.21), and all we have to specify is the Kähler potential $\mathcal{K}(Z, \bar{Z})$ and the superpotential $\mathcal{W}(Z)$. Let us focus on the superpotential, because it is the superpotential that controls the vacuum structure – the Kähler potential can receive quantum corrections that are not constrained by holomorphy, and as a consequence are hard to compute.

In the classical theory, we just have $\mathcal{W}(Z) = 0$. In the next three paragraphs, we present various arguments that lead to the non-perturbative value of $\mathcal{W}(Z)$. First we show how the superpotential can be computed for gauge group $SU(2)$, then we show how this result can be generalized to any gauge group. Finally we link this result to the semi-classical analysis.

The superpotential from Analyticity

The first argument we present may be the most intuitive one for a supersymmetric gauge theorist, since it is strongly based on holomorphy of the superpotential. We present it for gauge group $G = SU(2)$, following the original derivation of [38]. In this case, since the gauge group has rank one, the variable Z can be thought as a complex number, $Z \in \mathbb{C}$. The periodicity properties derived in section 5.2.2 show that the function $\mathcal{W}(Z)$ is defined on an elliptic curve and has exactly one double pole at $Z = 0$. The theory of elliptic functions then implies that $\mathcal{W}(Z)$ is a rational function in $\wp(Z)$, where $\wp$ is the Weierstrass function with appropriate periodicity conditions. The pole structure then implies that this rational function is simply a polynomial of degree one. The constant term in this polynomial depends only on the parameter τ of the elliptic curve. We have then proved

$$\mathcal{W}(Z) \propto \wp(Z; \tau) + C(\tau) \,. \tag{5.21}$$

Although it may be subtle to say whether $C(\tau)$ vanishes or not, this term is inoffensive if we aim to find the vacuum structure of the theory, which is governed by derivatives of $\mathcal{W}$ with respect to Z.

The Superpotential from Instantons

There are three finite action configurations that can contribute to the superpotential:

- The four-dimensional instantons, with four-dimensional instantonic charge m which is not necessarily an integer because of the compactification, see equation (5.18), and with classical action $2\pi i m\tau$ for $m > 0$ and $-2i\pi m\bar{\tau}$ for $m < 0$. They give holomorphic contributions $\exp 2\pi i m\tau = q^m$.

- The three-dimensional instantons, with magnetic charge $\alpha^\vee$. They contribute $\exp\left(2\pi i n\alpha^\vee \cdot Z\right)$ for α a positive root.

- The Lee-Yi instantons [112], which contribute $\left[\exp\left(2\pi i\alpha^\vee \cdot Z\right) q^m\right]^n$ for α any root (positive or negative) and m a positive integer.

These three kinds of contribution are in bijection with the positive affine roots, which are decomposed into three sets in equation (3.40). The positive roots Δ^+ correspond to the three-dimensional instantons, the set $\{\alpha + m\delta | m \in \mathbb{N}^* \text{ and } \alpha \in \Delta\}$ to the Lee-Yi instantons and the purely imaginary roots $\{m\delta | m \in \mathbb{N}^*\}$ to the four-dimensional instantons. In each case, the contribution to the action is just $\exp\left(2\pi i n \hat{\alpha}^\vee \cdot \hat{Z}\right)$, with the notations of section 3.4.2.

It is important to stress that for an instanton to contribute to the superpotential, it must have exactly two fermionic zero-modes in the quantum theory – the classical argument in favor of this fact is reviewed in the lectures [144]. The number of zero-modes can be found using the Callias index theorem [145, 142, 143], but in general one should take into account that modes unprotected by supersymmetry may be lifted [146, 147], leaving exactly two zero-modes that are protected. This is why all the configurations listed above do contribute to the superpotential. Note that in the pure $\mathcal{N} = 1$ theory, this lifting does not occur because there is no matter to trigger it, and only simple roots bring a contribution, resulting in Toda potentials.

Hence we understand the result (5.9), and the expansion (3.53) obtained in the chapter dedicated to the integrable system from a semiclassical point of view. In the classical theory, we just have $\mathcal{W}(Z) = 0$. However the gauge theory compactified on a circle gets non-perturbative superpotential contributions from magnetic monopole configurations whose charges take values in the dual root lattice $Q^\vee$. It is possible to understand these configurations in an illuminating way using brane systems and orientifolds in string theory [113].

The scalar duals of the photons have as a result a smallest possible periodicity equal to the weight lattice P. We choose to classify extrema of the superpotential with respect to these identifications. We should mention that other choices would be physically relevant. Since in deriving the effective superpotential we compactified the theory on $\mathbb{R}^3 \times S^1$, the resulting effective theory is influenced by the choice of the spectrum of line operators that probe the phases of our four-dimensional theory [148, 77, 10]. These determine the set of allowed monopole operators in three dimensions, and this set may be larger than the collection allowed by the minimal periodicity relation chosen above. Depending on the choice of the spectrum of line operators, this can lead to an increase of the number of inequivalent solutions, and therefore to an increase in the Witten index. This was analyzed carefully in [10, 139], and we will see the mechanism at work in detail for the gauge algebra $\mathfrak{so}(5)$ in section 7.2.3.

Note that the purely four-dimensional instanton terms associated to the imaginary roots contribute a τ dependent, but position independent term in the superpotential. We wish to further strengthen the arguments for the superpotential by comparing the results for the exact quantum vacua for the theory on $\mathbb{R}^3 \times S^1$ with semi-classical results.

5.2.4 Electric-Magnetic Duality

Our $\mathcal{N} = 1^*$ theory is a deformation of $\mathcal{N} = 4$ theory, and it inherits some of its properties. In particular, the electric-magnetic duality group of $\mathcal{N} = 4$ gauge theories in four dimensions [149, 13, 14] plays a crucial role. The duality symmetry was determined to be the group $SL(2, \mathbb{Z})$ for simply laced gauge groups and $\Gamma_0(4)$ for the B and C type gauge groups [45, 46, 12]. Moreover, the S_2 generator of the Hecke group exchanges the B and C type systems. An infrared counterpart to these duality groups are present in our integrable systems, which allow for a (generalized) duality group action on the infrared modular parameter τ [84], inherited after mass deformation from the $\mathcal{N} = 4$ duality. Note in particular that the requirement of the B type and C type exchange is implemented in our integrable system by the Langlands duality we discussed in subsection 3.3.1. This duality provides a further consistency check on the relative weight of the short and long root contributions, fixed in [39] through consistency with the superpotential of the pure $\mathcal{N}=1$ super Yang-Mills theory.

5.3 The Massive Vacua of $\mathcal{N}=1^*$ gauge theories

In this section, we compare the analysis of integrable system extrema to the semi-classical analysis of massive vacua of $\mathcal{N}=1^*$ gauge theory on $\mathbb{R}^4$. To wrap up a loose end first, let us note that the minimal mass M_i of a given vacuum i can be computed using the equation

$$M_k^2 = \min \left[\mathrm{Spec}(\mathcal{M}_k^T \mathcal{M}_k) \right] , \tag{5.22}$$

where $\mathcal{M}_k$ is the matrix of second derivatives of the potential in vacuum k:

$$(\mathcal{M}_k)_{ij} = \frac{\partial^2 \mathcal{W}_k(Z)}{\partial Z_i \partial Z_j} .$$

This clarifies the logic behind our definition of isolated extrema of the integrable system, see equation (4.2).

We have computed the masses of the vacua. For a given algebra, they are all within few orders of magnitude, and much above the accuracy of our numerical approximations, thus guaranteeing that our vacua are indeed massive. Moreover, for a given massive vacuum, the values of the masses are all approximately within a factor of 100 from each other. Interesting patterns in the (ratios) of masses (of various vacua) exist – it should be fruitful to study them systematically.

5.3.1 Comparison with Vacua on $\mathbb{R}^4$

A semi-classical analysis of the massive vacua of $\mathcal{N}=1^*$ on $\mathbb{R}^4$ proceeds in several steps, that were explained in chapter 2.

For gauge algebra $\mathfrak{su}(n)$ the number of classical vacua was thus argued to be equal to the sum of the divisors of n [77], and this number coincides precisely with the number obtained from the exact superpotential [77, 38] for the theory on $\mathbb{R}^3 \times S^1$ (where one classifies vacua in the manner described above).

For other gauge algebras, the number of massive vacua on $\mathbb{R}^4$ was given for the classical algebras by the generating functions of chapter 2. Let us focus on an algebra of type different from A, and on which we have a good analytic control, namely $\mathfrak{so}(5)$. According to equation 2.71, there are six massive quantum vacua for the $\mathcal{N}=1^*$ theory on $\mathbb{R}^4$. Let's recall in a little more detail how this counting arises. We allow for various five-dimensional representations of $\mathfrak{su}(2)$ as vacuum expectation values for the three complex scalars of $\mathcal{N}=1^*$. Even-dimensional representations must appear in even numbers. They need to take values in the gauge Lie algebra, and we classify them up to gauge equivalence. One then finds the following allowed representations [79] – we indicate the dimensions of the $\mathfrak{su}(2)$ representations, the unbroken part of the gauge group, and then the number of massive vacua they give rise to in the infrared:

$$\begin{aligned} 5 \quad &: \quad 1 \quad : 1 \\ 3+1+1 \quad &: \quad \mathfrak{so}(2) : 0 \\ 2+2+1 \quad &: \quad \mathfrak{sp}(2) : 2 \\ 1+1+1+1+1 \quad &: \quad \mathfrak{so}(5) : 3 \, . \end{aligned} \tag{5.23}$$

For instance, the $2+2+1$ dimensional representation breaks the gauge algebra down to $\mathfrak{sp}(2)$. Classically, this gives rise to a pure $\mathcal{N} = 1$ theory with $\mathfrak{sp}(2)$ gauge algebra at low energies, which gives rise to two quantum vacua. Summing all the resulting numbers of semi-classical vacua, we find six massive vacua in total.

On the other hand, we have found seven extrema for the twisted elliptic Calogero-Moser Hamiltonian associated to $\mathfrak{so}(5)$. Does it mean that there is a mistake somewhere in the analysis? As we will see in the last two chapters, there is no mistake except for the assumption that compactification on $\mathbb{R}^3 \times S^1$ leaves the vacuum structure unchanged. This compactification was a key element for the introduction of the integrable system. However, we remark that we have a partial correspondence. In particular, there is one vacuum, on the real axis, that we can identify in the exact quantum regime as the fully Higgsed vacuum (corresponding to the 5-dimensional irreducible representation of $\mathfrak{su}(2)$ in the list (5.23)). Its S-dual we interpret as a confining vacuum, and it is a triplet under the T-transformation, agreeing neatly with the $\mathfrak{so}(5)$ confining vacua (corresponding to the trivial representation of $\mathfrak{su}(2)$ in (5.23)). We moreover found a doublet under T-transformation, again in agreement with the two vacua corresponding to the $\mathfrak{sp}(2)$ classical vacuum. Thus, at this level, we find excellent agreement. We note that the analysis of section 3.3 demonstrates that the six vacua are in a single $SL(2,\mathbb{Z})$ sextuplet and that their transformation properties are in correspondence with the transformation properties of sublattices of the torus lattice. Their (generalized) S-duality and T-duality properties are now entirely known.

The seventh extremum has no matching partner in the vacua on $\mathbb{R}^4$. We will explain its appearance in the compactified theory in chapter 7.

5.3.2 Tensionless Domain Walls, Colliding Quantum Vacua and Masslessness

The point in the fundamental domain around which we have found a monodromy in the case of the $\mathfrak{so}(8)$ gauge algebra, corresponds to a point at which two massive vacua have equal superpotential. At this point, a supersymmetric domain wall between the vacua becomes tensionless [150, 28], according to the general discussion of section 1.2.3. The physics associated to such a situation is hard to discuss in detail, because of the difficulty of controlling the Kähler potential in gauge theories with $\mathcal{N} = 1$ supersymmetry only. Explorations of the physics in this regime can be found in [151, 67, 152]. We note in particular that in a mass and cubically deformed $\mathcal{N} = 1$ $U(N)$ theory in [151, 152], an extension of the $\mathbb{Z}_N$ action associated to shifts in the θ angle of the gauge theory to $\mathbb{Z}_{2N}$ was observed due to the presence of a point of monodromy in an effective coupling. The T-operation (shifting the θ angle of the gauge theory) in our situation is also crucially influenced by the presence of the point of monodromy : above the point of monodromy (at weak effective coupling), we find a $\mathbb{Z}_{N-2}$ action, while below (at strong effective coupling), we find a $\mathbb{Z}_N$ action (for the case $N = 8$ as well as for the case of $N = 7$). We also note that the collision of the extrema of the superpotential indicates the existence of an effectively massless excitation since there will be a zero mode for the matrix of second derivatives. The physics, or at least the properties of the effective description, seem close to the discussion in e.g. [151]. It would be interesting to elucidate this point further.

Chapter 6

The $SU(N)$ theory

6.1 Introduction

We saw in the last chapter that compactification has an effect on the vacuum structure of four-dimensional supersymmetric gauge theories like the $\mathcal{N} = 1^*$ theory we are interested in. We mentioned in particular the fact that the number of vacua can change. One possible mechanism that makes this possible is that massless vacua in the semiclassical analysis can become massive upon compactification. We will then have to pay more attention to these massless vacua.

We will concentrate in this chapter on the $\mathcal{N} = 1^*$ theory with $su(N)$ gauge algebra on the cylinder $\mathbb{R}^3 \times S^1$. This theory has many simplifying features. In particular, the unbroken gauge group in all semi-classical vacua of the $SU(N)/\mathbb{Z}_N$ theory is connected, so that the component group (in the adjoint group) is trivial. Indeed, from the mathematical point of view we have that Bala-Carter theory coincides with Bala-Carter-Sommers theory – this refinement will be necessary for other types of Lie algebras, as we will see in chapter 7. Thus, in this theory, we can isolate new semi-classical limits of the integrable system, and the corresponding gauge theory physics from other interesting features of $\mathcal{N} = 1^*$ theories when compactified on S^1. We will find branches of massless vacua for low rank, characterize their equilibrium positions, analyze their superpotential and study how these vacua behave under duality.

To understand the fate of semi-classically massless and massive vacua in $su(N)$ $\mathcal{N} = 1^*$ theory, we again take the elliptic Calogero-Moser Hamiltonian as our starting point [38]. For starters, we analyze this effective superpotential in the semi-classical regime $\tau \to i\infty$ and classify extrema of the integrable system using the technique laid out in section 3.4. We will be able to promote parts of our limiting knowledge to exact statements at finite coupling.

6.2 Semi-Classical Preliminaries

As argued previously, a classification of extrema is governed by pseudo-Levi subalgebras, in turn determined by Weyl (i.e. permutation) inequivalent subsets of the affine root system (whose Dynkin diagram is a circle with N nodes). The number of inequivalent subsets of roots (except all of them) is the number of partitions of N. In more detail, we let $\Delta = \{\alpha_1, ..., \alpha_{N-1}\}$ be a set of simple roots of A_{N-1}. For any subset $\Theta \subset \Delta$ we construct a partition of N. We can write Θ uniquely as a disjoint union of sets of the form $\Delta_{k_i,d_i} = \{\alpha_{k_i}, ..., \alpha_{k_i+d_i-2}\}$ where $d_i \geq 2$. If our choice of subset Θ is

$$\Theta = \bigcup_i \Delta_{k_i,d_i} \tag{6.1}$$

then the partition is $1 + ... + 1 + \sum_i d_i = N$ with as many 1's as necessary to obtain a partition of N.

For each choice of subsystem, we know the corresponding centralizer subgroup in the complexification of $SU(N)$. We denote the latter by $GL(N)$, the group of size N invertible matrices with complex entries. The algebra of the centralizer is given in [93]. With the notation $r_i = |\{j|d_j = i\}|$ for the number of times a representation of dimension i occurs in the $sl(2)$ representation spanned by the adjoint scalars, so that

$$\sum_i ir_i = N\,, \tag{6.2}$$

the centralizer algebra is

$$\left(\prod_i A_{r_i-1}\right) \times u(1)^k\,, \tag{6.3}$$

where $k = |\{i|r_i > 0\}| - 1$ (i.e. the number of distinct dimensions minus one). Then the global structure of the centralizer group is [92]

$$S\left(\prod GL(r_i)^i_\Delta\right)\,, \tag{6.4}$$

where the Δ denotes the diagonal copy of $GL(r_i)$ inside $GL(r_i)^i$, and the S in front means that we keep only the matrices with unit determinant. This is the centralizer group in the complexification of $SU(N)$. The counting of the abelian factors in this group goes as follows: there is one abelian factor for each term in the product and the constraint of unit determinant reduces the total number of abelian factors by one. In terms of pseudo-Levi subalgebras, a group with no abelian factor is obtained from a set of roots Θ containing all the simple roots except d of them (where d divides N) equally spaced on the cyclic affine Dynkin diagram. In other words, one takes disconnected groups of $d-1$ roots on the affine Dynkin diagram, where d is a divisor. These give rise to the massive vacua of the $\mathcal{N} = 1^*$ theory that were described in [77, 38]. The corresponding semi-classical limits of the integrable system are well-understood. We wish to advance the more general case. We will do this on a case-by-case basis, working our way up in rank.

In the next subsections, we use the semi-classical limiting technique to gain information on the massless vacua of the first non-trivial low rank cases. For the $su(3)$ theory, we will complete the picture at finite coupling, while for the $su(4)$ algebra, we present a few features that will be typical of higher rank.

6.3 The Gauge Algebra $su(3)$, the Massless Branch and the Singularity

We remind the reader that the superpotential for the $su(3)$ algebra [38] can be parametrized in terms of the coordinates z_i with $i = 1, 2, 3$ where we can use a shift symmetry to put $z_3 = 0$:

$$\mathcal{W}_{A_2} \;=\; \wp(z_1 - z_2) + \wp(z_2 - z_3) + \wp(z_3 - z_1)$$

$$\begin{aligned} &= \wp(z_1 - z_2) + \wp(z_2) + \wp(z_1) \\ &= \wp(Z_1) + \wp(Z_2) + \wp(Z_1 + Z_2)\,. \end{aligned} \tag{6.5}$$

In the last line, we have used the more intrinsic parametrization in terms of the coordinates Z_i associated to the fundamental weights.

6.3.1 Semi-classical analysis

When we apply our program of identifying vacua in the semi-classical limit to the case of $su(3)$, we recuperate the known results for the massive vacua, and find new results for massless vacua.

- For the choice $J_0 = \emptyset$, which corresponds to the partition $1+1+1$, we set the leading behavior $Y_0 = Y_1 = Y_2 = \frac{1}{3}$. One finds three confined massive vacua (with $k = 0, 1, 2$) at

$$(z_1, z_2, z_3) = \left(\frac{k}{3} + \frac{2}{3}\tau, \frac{2k}{3} + \frac{1}{3}\tau, 0\right)\,. \tag{6.6}$$

  These extremal positions are exact and the superpotential in these vacua is known [38].

- For the pick $J_0 = \{1\}$, namely the partition $3 = 1 + 2$, we choose $Y_0 = Y_2 = \frac{1}{2}$, and at first order the A_1 trigonometric system fixes $X_1 = \frac{1}{2}$. We analyze the potential near this equilibrium by expanding $\mathcal{W}_{A_2}(\frac{1}{2} + \delta X_1, \frac{\tau}{2} + X_2)$ in perturbation theory in δX_1, and as a function of X_2. We find that the first coordinate is corrected as follows

$$\delta X_1 \;=\; -\frac{4i\left(e^{2i\pi X_2} - e^{-2i\pi X_2}\right)\sqrt{q}}{\pi} - \frac{16i\left(e^{4i\pi X_2} - e^{-4i\pi X_2}\right)q}{\pi} + \dots \tag{6.7}$$

  Plugging this correction into the superpotential leads to a superpotential which to the relevant order no longer depends on X_2, and in fact, is equal to zero. We have checked this to order q^4. These facts point towards the existence of a branch of massless vacua, with zero superpotential along the whole branch. We will obtain full analytic control of this branch below.

- Finally, for the choice $J_0 = \{1, 2\}$, namely the partition 3, one obtains the A_2 trigonometric potential. This potential has a real extremum, the fully Higgsed vacuum

$$(z_1, z_2, z_3) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)\,, \tag{6.8}$$

  as well as complex massless extrema which form a portion of the same branch of vacua with zero superpotential just mentioned.

6.3.2 The Massless Branch and the Singularity

Semi-classically, we have found evidence for the existence of a massless branch of vacua with zero superpotential. In the following, we will concentrate on describing the properties of this branch analytically, at any finite coupling τ. Together with the known results about massive vacua that our analysis also recovers, we thus obtain all the vacua of the $\mathcal{N} = 1^*$ theory with $su(3)$ gauge algebra exactly.

Firstly, we introduce some notation. We will denote the elliptic curve variables as

$$\mathcal{X}_i \;=\; \wp(Z_i)$$

$$\mathcal{Y}_i \;=\; \wp'(Z_i) \tag{6.9}$$

for $i = 1, 2, 3$, where $Z_3 = -Z_1 - Z_2$ by convention. The points $(\mathcal{X}_i, \mathcal{Y}_i)$ all lie on the same elliptic curve, parametrized by τ, and described by the equation

$$\mathcal{Y}^2 \;=\; 4\mathcal{X}^3 - g_2\mathcal{X} - g_3\,. \tag{6.10}$$

The equations for extremality of the superpotential then read

$$\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}_3\,. \tag{6.11}$$

Moreover, the addition theorem for the elliptic Weierstrass function implies

$$\mathcal{X}_i \;=\; \frac{1}{4}\left(\frac{\mathcal{Y}_j - \mathcal{Y}_k}{\mathcal{X}_j - \mathcal{X}_k}\right)^2 - \mathcal{X}_j - \mathcal{X}_k\,, \tag{6.12}$$

where i, j, k take three distinct values in the set $\{1, 2, 3\}$. Thus, we see that there are two possibilities: either the superpotential is zero

$$\mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 \;=\; 0\,, \tag{6.13}$$

or we must have that

$$\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}_3\,. \tag{6.14}$$

We split the analysis of the extrema according to these two cases. Firstly, we consider the case in which we have the equality (6.14). This equation, together with extremality shows that $Z_1 \equiv Z_2 \equiv Z_3$ modulo a period. This implies that all Z_i equal a non-trivial third of a period of the torus, and gives rise to 4 inequivalent vacuum solutions, which are the known massive vacua [38]. The superpotential is three times the Weierstrass function evaluated at a third period.

Let us return then to the first possibility, which is that the superpotential is zero, equation (6.13). By eliminating the variables $\mathcal{Y}_i$ through the curve equation and extremality, we obtain two equations characterizing the massless branch

$$\begin{aligned} \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 &= 0 \\ \mathcal{X}_1^2 + \mathcal{X}_2^2 + \mathcal{X}_3^2 &= \frac{g_2}{2}\,. \end{aligned} \tag{6.15}$$

These equations are gauge invariant. Solving for the variables $\mathcal{Y}_i$ will provide a further double cover of this space. Moreover, we mod out the space by the discrete gauge symmetry S_3, which exchanges the three indices $\{1, 2, 3\}$ of the variables $\mathcal{X}_i$ (and flips the sign of the variables $\mathcal{Y}_i$ if the permutation is odd, exchanging the two sheets of the cover). We can parametrize the curve more explicitly by eliminating more variables. A description of the curve in terms of two variables is

$$\mathcal{X}_1^2 + \mathcal{X}_1\mathcal{X}_2 + \mathcal{X}_2^2 \;=\; \frac{g_2}{4}\,. \tag{6.16}$$

This equation parametrizes a complex line. Equivalently, using the reparametrization $\mathcal{X}_1' = \mathcal{X}_1 + \mathcal{X}_2$ and $\mathcal{X}_2' = \mathcal{X}_1 - \mathcal{X}_2$, this equation reads

$$(\mathcal{X}_2')^2 = 3(\mathcal{X}_1')^2 - g_2\,. \tag{6.17}$$

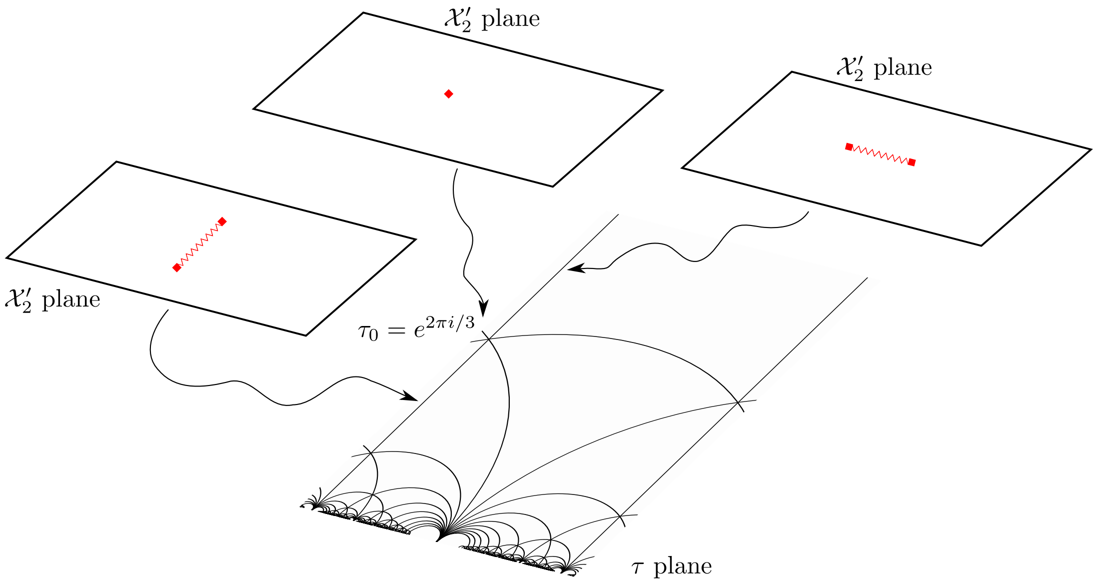


Figure 6.1: We illustrate the singularities in of the curve $\mathcal{X}_2' = \pm\sqrt{3(\mathcal{X}_1')^2 - g_2(\tau)}$ as a function of τ. The branch-cut shrinks to zero size when $\tau = \tau_0$.

Note that at the values τ_0 of the complexified gauge coupling where the fourth Eisenstein series g_2 is zero, the complex line has a singular point at $\mathcal{X}_1 = \mathcal{X}_2 = 0$. This is illustrated in figure 6.1. The singularity is a crucial feature of the massless branch. The zeros of g_2 in the τ upper half-plane are exactly the $SL(2,\mathbb{Z})$ images of $\tau_0 = e^{2\pi i/3}$, which is the only zero of g_2 in the fundamental domain.[1] Thus, at these couplings the massless branch develops a singularity. These are elliptic points of order three.

Finally, we note that the conditions that all $\mathcal{X}_i$ be equal (which is valid for the 4 vacua associated to third periods), and that the superpotential vanish can both be satisfied at the singular points. More precisely, for each given singular coupling in the $SL(2,\mathbb{Z})$ orbit of τ_0, one of the four formerly massive vacua becomes massless and joins the massless branch. The fact that a massive vacuum becomes massless at this coupling may indicate a higher order critical point, and the existence of an interacting $\mathcal{N} = 1$ superconformal field theory. The value of the critical coupling points towards a natural candidate for this theory, which is the Argyres-Douglas theory [41] broken to $\mathcal{N} = 1$ [153].

In fact, the analysis of $\mathcal{N} = 2$ $SU(3)$ theory with an adjoint hypermultiplet reveals that the Seiberg-Witten curve has eight cusps [77]. When we analyze the cusps at values of the moduli such that they coincide with vacua that would be massive at generic coupling, we find that the number of cusps reduces to four.[2] Of these four cusps, one is associated to a pure $SU(3)$ theory, and the other three correspond to a $SU(2)$ theory with a massless fundamental hypermultiplet at a (generalized) Argyres-Douglas point [68]. Since the $\mathcal{N} = 1^*$ massive vacua are invariant under $\Gamma_0(3)$, we can classify the singular couplings τ into $\Gamma_0(3)$ cosets of $SL(2,\mathbb{Z})$ according to which massive vacuum becomes massless at the given singular coupling. We find that at

[1] It is easy to show that $\tau_0 = e^{2\pi i/3}$ is a zero of $E_4 = \frac{3}{4\pi^4} g_2$ using $E_4(\tau_0) = E_4(\tau_0+1) = E_4(-1/\tau_0) = \tau_0^4 E_4(\tau_0) = \tau_0 E_4(\tau_0)$. There remains to show that there is no other zero. We use the formula $\mathrm{ord}_{i\infty} + \frac{1}{2}\mathrm{ord}_i + \frac{1}{3}\mathrm{ord}_{\tau_0} + \sum \mathrm{ord}_\tau = k/6$, valid for any modular form of weight $2k$. At weight $2k = 4$ the formula gives $\mathrm{ord}_{\tau_0} = 1$ and there can be no other zero.

[2] The operation S^2 discussed in [77] acts trivially in this circumstance.

$\tau = \frac{-1}{\tau_0+2}$ the Higgs vacuum becomes massless, while at $\tau = \tau_0 + 2$ the confined vacuum situated on the imaginary axis becomes massless, at $\tau = \tau_0 + 1 = e^{\pi i/3}$, its T-dual and at $\tau_0 = e^{2\pi i/3}$ the third confined vacuum. Thus, the $SL(2,\mathbb{Z})$ action on these Argyres-Douglas singularities coincides with the action of the duality group on the four massive vacua of the $\mathcal{N} = 1^*$ theory. From the action of the T-transformation, we can identify the confined vacua with the $SU(2)$ cusps and the Higgs vacuum with the pure $SU(3)$ cusp [77]. Our analysis provides a concrete picture for how the transformation properties of the massive phases are locked with the duality properties of the cusps.

At generic coupling τ, the duality properties of the massive vacua are well-known. We find that the massless branch, in the description in terms of elliptic curve variables, is invariant under the action of the T-transformation, since the fourth Eisenstein series is. Moreover, under the S-transformation, the variables $\mathcal{X}_i$ transform with weight two, as one expects from their definition in terms of the elliptic Weierstrass function. Thus, the branch is self-dual under the full modular group (or more precisely, is mapped to an equivalent, scaled branch at dual coupling).

6.3.3 The Massless Branch in the Toroidal Variables

The description of the massless branch was straightforward in terms of gauge invariant polynomials of the variables $\mathcal{X}_i$. Still, we can ask for the description of the massless branch of vacua in terms of the extrema of the integrable system, parametrized by the coordinates Z_i (namely, the complexified Wilson lines), at finite coupling τ. That description too can be obtained, but it demands further effort. We can for instance work with the following parameterization of the massless branch

$$\begin{aligned}
\mathcal{X}_1 &= \frac{i\sqrt{g_2}}{2\sqrt{3}}\left(\lambda - \frac{1}{\lambda}\right) \\
\mathcal{X}_2 &= \frac{\sqrt{g_2}}{4}\left[\left(\lambda + \frac{1}{\lambda}\right) - \frac{i}{\sqrt{3}}\left(\lambda - \frac{1}{\lambda}\right)\right] \\
\mathcal{X}_3 &= \frac{\sqrt{g_2}}{4}\left[-\left(\lambda + \frac{1}{\lambda}\right) - \frac{i}{\sqrt{3}}\left(\lambda - \frac{1}{\lambda}\right)\right] ,
\end{aligned} \tag{6.18}$$

for $\lambda \in \mathbb{C}^*$. However, we still have to take into account both the fact that we have a double cover (when we solve for $\mathcal{Y}_i$) as well as the action of the Weyl group to faithfully describe the branch of vacua. The Weyl group has generators that exchange two distinct coordinates, $Z_i \leftrightarrow -Z_j$ (while also changing the sign of the third coordinate). This translates into identifications on our parameter space:

- $Z_2 \leftrightarrow -Z_3$ corresponds to $\lambda \leftrightarrow -\frac{1}{\lambda}$.
- $Z_1 \leftrightarrow -Z_2$ corresponds to $\lambda \leftrightarrow e^{-2\pi i/3}\lambda$.
- $Z_1 \leftrightarrow -Z_3$ corresponds to $\lambda \leftrightarrow e^{2\pi i/3}\lambda$,

and each transformation exchanges the two sheets of the $\mathcal{Y}$ cover. Hence the massless branch is a double cover of the sphere parametrized by λ. We excise the points $\lambda = 0$ as well as the point $\lambda = \infty$, because the superpotential blows up in these points. This indicates the enhancement of gauge symmetry, and the breakdown of the effective superpotential description at these $\mathbb{Z}_3$ fixed points. A fundamental domain for λ is given by the following region: $|\lambda| \leq 1$

and $\pi/6 \leq \arg\lambda \leq 5\pi/6$ with a $\mathbb{Z}_2$ identification of the borders of the unit disk as well as of the two rays on the boundary.

We now wish to distinguish between two physically distinct sets of configurations. They are characterized by the way they behave under the charge conjugation symmetry of the gauge theory. Conjugation acts by exchanging $Z_1 \leftrightarrow Z_2$, which is a global symmetry of the gauge theory, inherited by the low-energy effective superpotential. When we have the equality $\mathcal{X}_1 = \mathcal{X}_2$ (or a permutation thereof), we can either have $Z_1 = -Z_2$ or $Z_1 = Z_2$, modulo the periodicity of these variables. The first case corresponds to a fixed point of the local Weyl symmetry group, and it leads to a singular term in the effective superpotential, indicating the enhancement of the gauge group (i.e. the fact that we leave the Coulomb branch). We exclude this singular configuration from our analysis. The second case indicates a fixed point of the charge conjugation symmetry. This occurs when $\lambda^6 = -1$. When there is no equality between any of the variables $\mathcal{X}_i$, we are at a less symmetric point on the massless branch. These two regimes will lead to a qualitatively different solution for the variables Z_i as we show in detail below.

We would like to solve equation (6.9) for the complexified Wilson lines Z_i. The solution relies on inverting the Weierstrass function. The techniques for performing this inversion were presented in [40] by Eichler and Zagier in their analysis of the zeros of the Weierstrass $\wp$ function. These authors also study the solutions to the equation $\wp(Z) = \mathcal{X}(\tau)$ where $\mathcal{X}(\tau)$ is a (e.g. meromorphic) modular form of weight 2. Our equation does not fit this mold – the Weierstrass function is equal to the square root of a modular form of weight 4. Still, we can apply the bulk of the Eichler-Zagier methods. The Eichler-Zagier technique for inverting the Weierstrass function consists of two parts. On the one hand, since the argument Z is multi-valued due to the periodicity of the Weierstrass function, it is useful to derive with respect to the modular parameter τ twice, to eliminate this ambiguity. The two integration constants that one subsequently needs can be determined by matching the semi-classical limits. On the other hand, one inverts the equation through integration of the defining equation for the elliptic curve

$$\wp(Z;\tau) = \mathcal{X}(\tau) \qquad \Longleftrightarrow \qquad Z = \pm\frac{\sqrt{3}}{2\pi}\int_{\frac{3}{\pi^2}\mathcal{X}(\tau)}^{\infty} \frac{dt}{\sqrt{t^3 - 3E_4(\tau)t - 2E_6(\tau)}}\,. \tag{6.19}$$

From this equation, we determine the second derivative with respect to τ, by multiple application of the Ramanujan identities for the derivatives of the Eisenstein series. The calculation is presented in pedagogical detail in [40] and results in the equality

$$\begin{aligned}\pm\frac{\mathrm{d}^2 Z}{\mathrm{d}\tau^2} \;=\;& \Big(4\pi^2(g_3 - 4\mathcal{X}^3 + g_2\mathcal{X})D_3D_6\mathcal{X} + 2\pi^2(12\mathcal{X}^2 - g_2)(D_6\mathcal{X})^2 + (6g_3\mathcal{X} + g_2^2/3)D_6\mathcal{X} \\ & + \frac{1}{72\pi^2}(12g_2\mathcal{X}^4 + 3g_2^2\mathcal{X}^2 + 6g_2g_3\mathcal{X} - g_2^3 + 27g_3^2)\Big)/(4\mathcal{X}^3 - g_2\mathcal{X} - g_3)^{\frac{3}{2}}\,,\end{aligned} \tag{6.20}$$

where the function $\mathcal{X}$ acts as a seed, and the modular covariant derivative is given by $D_n = q\partial_q - \frac{1}{n}E_2$. The integration constants are fixed by taking the semi-classical limit of the formula (6.19). Here, we will add a point to the analysis in [40], by exhibiting a special case of the limiting formula, which is also physically distinct. We define the variable $\mathcal{X}_{i\infty} = \mathcal{X}(\tau \to i\infty)$. If $\mathcal{X}_{i\infty} \neq -\frac{\pi^2}{3}$, the semi-classical limit is given by [40]

$$Z(\tau \to i\infty) = \frac{1}{2} \pm \frac{1}{2\pi i}\log\frac{1 + \sqrt{\frac{2}{3} - \frac{1}{\pi^2}\mathcal{X}_{i\infty}}}{1 - \sqrt{\frac{2}{3} - \frac{1}{\pi^2}\mathcal{X}_{i\infty}}} \tag{6.21}$$

while for the case $\mathcal{X}_{i\infty} = -\frac{\pi^2}{3}$ it is

$$Z(\tau \to i\infty) = \pm\frac{1}{4} + \frac{\tau}{2}\,. \tag{6.22}$$

The latter case occurs when the $\mathcal{X}_i$ are at a charge conjugation fixed point, i.e. a fixed point of the global $\mathbb{Z}_2$ symmetry. Note that the limit formula (3.47) shows that this case is common. Let us nevertheless first concentrate on the case in which all the variables $\mathcal{X}_i$ are different, and construct the solution for the variables Z_i. We then come back to the global $\mathbb{Z}_2$ fixed point.

6.3.4 The points $\lambda^6 \neq -1$

The point $\lambda = 1$, for instance, is representative for all λ not at a $\mathbb{Z}_2$ fixed point. In this case, we have

$$\begin{aligned} \mathcal{X}_1 &= 0 \\ \mathcal{X}_2 &= \frac{\sqrt{g_2}}{2} \\ \mathcal{X}_3 &= -\frac{\sqrt{g_2}}{2}\,, \end{aligned}$$

and the formulas (6.21) and (6.20) from [40] apply. We can for instance write the solution as a series expansion q at large imaginary τ

$$\pm\pi Z_1 = \frac{1}{2}\left(\pi - i\cosh^{-1}(5)\right) + 36i\sqrt{6}q + 6588i\sqrt{6}q^2 + \ldots \tag{6.23}$$

$$\begin{aligned} \pm\pi Z_2 = {} & \left(\frac{\pi}{2} - i\tanh^{-1}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)\right) + \left(-36i\sqrt{2} - 18i\sqrt{6}\right)q \\ & + \left(1188i\sqrt{2} - 3294i\sqrt{6}\right)q^2 + \ldots \end{aligned} \tag{6.24}$$

$$\begin{aligned} \pm\pi Z_3 = {} & -\pi - \frac{i}{2}\log\left(1 - \frac{6}{3+\sqrt{6+3\sqrt{3}}}\right) + \left(36i\sqrt{2} - 18i\sqrt{6}\right)q \\ & + \left(-1188i\sqrt{2} - 3294i\sqrt{6}\right)q^2 + \ldots \end{aligned} \tag{6.25}$$

The series that we obtain has a finite radius of convergence. The integration formula (6.19) is valid at any modular parameter τ. In this explicit solution (6.23), we can choose a sign for each Z_i, consistently with the constraint $Z_1 + Z_2 + Z_3 \equiv 0$. Thus, we see that we must pick the same sign for all Z_i – there are two solutions. The solutions are invariant under T-duality. This implies that they are also S-invariant, since $1 = (ST)^3 = S^3 = S$. The semi-classical limit of these vacua lies in the class $J_0 = \{1, 2\}$. The semi-classical limit of the massless branch that contains these vacua can be obtained by setting $Y_{1,2} = 0$ and taking the corresponding limit on the equations (6.15) parameterizing the branch.

6.3.5 The $\mathbb{Z}_2$ symmetric points

We return to the $\mathbb{Z}_2$ symmetric values of λ which lie at $\lambda^6 = -1$. Let us further concentrate on the case where the equality $\mathcal{X}_2 = \mathcal{X}_3$ holds. Note that the condition we impose is duality invariant. The solutions will therefore transform into each other under the $SL(2,\mathbb{Z})$ action of the duality group. We solve for the coordinates Z_i at these particular points. From equation

(6.18) we read that the equality $\mathcal{X}_2 = \mathcal{X}_3$ translates into $\lambda^2 = -1$, which implies that we can focus on the two points $\lambda = \pm i$.

At the value $\lambda = -i$, we have to solve the equations:

$$\mathcal{X}_1 = \sqrt{\frac{g_2}{3}} \tag{6.26}$$

$$\mathcal{X}_2 = -\frac{1}{2}\sqrt{\frac{g_2}{3}} \tag{6.27}$$

$$\mathcal{X}_3 = -\frac{1}{2}\sqrt{\frac{g_2}{3}}\,. \tag{6.28}$$

We begin with the second equation (6.27), for which the equality (6.22) gives the asymptotic value $Z_2(i\infty) = \pm\frac{1}{4} + \frac{\tau}{2}$. We define

$$Z_2 = \frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q) \tag{6.29}$$

to be the solution of (6.27) such that $\alpha(q)$ has semi-classical behavior

$$\alpha(q) = \frac{8}{\pi}q^{\frac{1}{2}} + O\left(q^{3/2}\right) \tag{6.30}$$

and is analytic along the line $i\mathbb{R}^*_+$. Note that from the equation, we can compute the Fourier expansion to arbitrary order.[3] Next, we consider the first equation (6.26). The asymptotic behavior of its solutions $\pm Z_1$ is now given by equation (6.21), and it is $Z_1(i\infty) = \frac{1}{2}$. The exact solution involves the function $\alpha(q)$ just defined, $Z_1 = \frac{1}{2} \pm \alpha(q)$ as a consequence of the doubling formula

$$\wp\left(\frac{1}{2} \pm \alpha\right) = -2\wp\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right) + \frac{6\wp\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right)^2 - \frac{1}{2}g_2}{4\wp'\left(\frac{1}{4} + \frac{\tau}{2} - \frac{1}{2}\alpha(q)\right)^2} = \sqrt{\frac{g_2}{3}}\,. \tag{6.32}$$

The relative sign is determined by the requirement that $Z_3 = -Z_1 - Z_2$ be a solution of (6.28). Therefore we have found two inequivalent vacua at $\lambda = -i$:

$$(Z_1, Z_2) = \left(\frac{1}{2} + \alpha, \frac{1}{4} + \frac{\tau}{2} - \frac{\alpha}{2}\right) \qquad (z_1, z_2, z_3) = \left(\frac{3}{4} + \frac{\tau}{2} + \frac{\alpha}{2}, \frac{1}{4} + \frac{\tau}{2} - \frac{\alpha}{2}, 0\right) \tag{6.33}$$

and

$$(Z_1, Z_2) = \left(\frac{1}{2} - \alpha, \frac{1}{4} + \frac{\tau}{2} + \frac{\alpha}{2}\right) \qquad (z_1, z_2, z_3) = \left(\frac{3}{4} + \frac{\tau}{2} - \frac{\alpha}{2}, \frac{1}{4} + \frac{\tau}{2} + \frac{\alpha}{2}, 0\right)\,. \tag{6.34}$$

We now turn to the value $\lambda = i$ and proceed similarly. Our task is to solve

$$\mathcal{X}_1 = -\sqrt{\frac{g_2}{3}} \tag{6.35}$$

[3] We have

$$\begin{aligned} \pi\alpha(q) &= 8q^{\frac{1}{2}} - \frac{1088q^{\frac{3}{2}}}{3} + \frac{198288q^{\frac{5}{2}}}{5} - \frac{39006080q^{\frac{7}{2}}}{7} + \frac{7975383560q^{\frac{9}{2}}}{9} \\ &\quad - \frac{1669600216512q^{\frac{11}{2}}}{11} + \frac{355119960987280q^{\frac{13}{2}}}{13} + \dots \end{aligned} \tag{6.31}$$

$$\begin{aligned} \mathcal{X}_2 &= \frac{1}{2}\sqrt{\frac{g_2}{3}} & (6.36) \\ \mathcal{X}_3 &= \frac{1}{2}\sqrt{\frac{g_2}{3}} \,. & (6.37) \end{aligned}$$

We define $\frac{1}{2} - \beta(\tau)$ to be the solution of (6.36) with semi-classical behavior

$$\beta(q) = \frac{i}{2\pi} \log\left(2+\sqrt{3}\right) + O(q) \tag{6.38}$$

and demand analyticity on $i\mathbb{R}_+^*$,[4] and again we find that $2\beta(q)$ is a solution of (6.35) using the duplication formula for the Weierstrass function. The signs are determined as previously, and we conclude that a solution is

$$(Z_1, Z_2) = (2\beta, \frac{1}{2} - \beta) \qquad (z_1, z_2, z_3) = \left(\frac{1}{2}+\beta, \frac{1}{2}-\beta, 0\right) . \tag{6.39}$$

As before, we could flip the sign in front of β in this expression, but this would lead to an equivalent vacuum. We have only one vacuum at $\lambda = i$.

While for generic λ the action of T-duality and as a consequence $SL(2,\mathbb{Z})$ duality on the vacua was trivial, here we see, e.g. from the expansion (6.30), that T-duality exchanges the two vacua (6.33) and (6.34). As a consequence S-duality will act as well. We devote the next paragraph to a detailed study of these dualities.

6.3.6 Dualities at the $\mathbb{Z}_2$ symmetric points

In the course of our analysis we have found the four solutions of the equation

$$\wp(z)^2 = \frac{g_2}{2} \tag{6.40}$$

that we can gather in a vector

$$V(\tau) = \begin{pmatrix} \frac{1}{2}+\alpha(\tau) \\ \frac{1}{2}-\alpha(\tau) \\ 2\beta(\tau) \\ -2\beta(\tau) \end{pmatrix} , \tag{6.41}$$

which can be interpreted as a vector-valued and multi-valued modular form [40]. The word multi-valued here refers to the fact that these quantities are defined up to periods of the Weierstrass function. This vector transforms under $SL(2,\mathbb{Z})$ according to

$$V(\tau) \xrightarrow{T} V(\tau+1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V(\tau) \,, \tag{6.42}$$

[4] We have the further expansion

$$\begin{aligned} i\pi\beta &= -\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right) + 12\sqrt{3}(q - 87q^2 + 11080q^3 - 1671095q^4 + \frac{1384694994q^5}{5} \\ &\quad -48732765432q^6 + \frac{62575601740112q^7}{7} - 1690589139219255q^8 + 327268705474374265q^9 + \ldots) \,. \end{aligned}$$

and

$$V(\tau) \xrightarrow{S} V\left(\frac{-1}{\tau}\right) = \frac{-1}{\tau}\left[\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} V(\tau) - \begin{pmatrix} \tau \\ 0 \\ 0 \\ 1 \end{pmatrix}\right], \tag{6.43}$$

where we have used the analyticity along $i\mathbb{R}_+^*$ to fix the periodic dependence. Thus, we have a weight -1 modular form up to periodicity. Let's call T and S the matrices that appear in these equations and that are associated to the two generators of $SL(2,\mathbb{Z})$. The periodicity is linear in the modular parameter τ, such that again, if we take two derivatives with respect to τ, this ambiguity drops out, and we find a vector valued modular form of weight 3:

$$V''(\tau+1) = TV''(\tau)\,, \qquad V''\left(\frac{-1}{\tau}\right) = (-\tau)^3 SV''(\tau)\,. \tag{6.44}$$

The method of [40] gives the explicit solution

$$\psi(\tau) = V''(\tau) \times \left[\left(\frac{g_2(\tau)}{3}\right)^{\frac{3}{2}} - g_3(\tau)\right]^{\frac{3}{2}} \tag{6.45}$$

and each component of the vector V is given as

$$\begin{aligned} \pm V'' \quad = \quad & \frac{1}{(\quad \phi_\pm^3 - g_3)^{\frac{3}{2}}}\Big[4\pi^2(g_3 - \phi_\pm^3)D_3 D_6\phi_\pm + 18\pi^2\phi_\pm^2(D_6\phi_\pm)^2 \\ & +3\phi_\pm(2g_3 + \phi_\pm^3)D_6\phi + \frac{1}{8\pi^2}(4\phi_\pm^6 + 2g_3\phi_\pm^3 + 3g_3^2)\Big] \end{aligned} \tag{6.46}$$

where the seed $\phi_\pm$ is a branch of the square root of the Eisenstein series $\phi_\pm = \pm\sqrt{\frac{g_2}{3}}$, and the 4 components of V'' correspond to the 4 possible choices of signs (on the left, and on the right hand side independently). We give the first few terms (the first line is obtained from ϕ_+ and the second line from ϕ_-):

$$V''(\tau) = \pm 8\pi\begin{pmatrix} q^{\frac{1}{2}} - 408q^{\frac{3}{2}} + 123930q^{5/2} - 34130320q^{7/2} \\ 12i\sqrt{3}(q - 348q^2 + 99720q^3 - 26737520q^4 + ...) \end{pmatrix} = \pm\begin{pmatrix} \alpha'' \\ 2\beta'' \end{pmatrix} \tag{6.47}$$

and

$$\psi(\tau) \propto -q^2 + 336q^3 - 94824q^4 + 25238080q^5 - 6506938620q^6 + \ldots\,. \tag{6.48}$$

After double integration, this characterizes the q expansion of α, β, and therefore analytically completes the series we obtained previously. We further analytically continue the functions α and β in the double cover of the upper half plane. The triplet of solutions to the equation becomes degenerate at the zeros of E_4. Note that we can switch branch for the seed by rotating around the zero $\tau_0 = e^{2\pi i/3}$ of the weight 4 Eisenstein series. As a consequence, this operation flips α and β, and this introduces a monodromy amongst the sheets of massless vacua in the elliptic integrable system parameterization.

6.3.7 Summary Remarks

We recapitulate the duality diagram for both the massive and massless extrema of the $su(3)$ integrable system.[5] We have four massive vacua, of which two are self-S-dual, and two are

[5] We repeat that the global aspects of the gauge group can be taken into account by carefully treating the subgroup of $\mathbb{Z}_3$ which one chooses as center, and the possible electro-magnetic line operators in the theory, which have consequences on the periodic identifications of variables.

mutually S-dual. They form a singlet and a triplet under T-duality. We have one massless branch which is duality invariant in the elliptic curve variables.[6] We note that the semi-classical limit that allows for the Higgs vacuum, also sees the massless branch.

Partition	J_0	Unbroken	Vacua
$1+1+1$	$\emptyset$	$su(3)$	3 confining vacua of pure $\mathcal{N}=1$
$2+1$	$\{1\}$ or $\{2\}$	$u(1)$	the massless branch
3	$\{1,2\}$	1	1 Higgs vacuum + the massless branch

Table 6.1: Summary of vacua for $su(3)$

There is a more intricate description of the massless branch in terms of the elliptic integrable system variables, which allows to follow the duality map on the massless vacua point by point. For the extremal positions of the massless vacua in terms of the complexified Wilson lines, we have exhibited a point of monodromy on the boundary of the fundamental domain, and in particular, the elliptic point of order 3 of the $SL(2,\mathbb{Z})$ action on the upper half plane. This point is a singular point for the manifold of massless vacua. It is reminiscent of the point of monodromy in the interior of the fundamental domain for two massive vacua of the $so(8)$ theory [1].

6.4 The Gauge Algebra $su(4)$

We have obtained a complete picture of the massive and massless vacua of the $su(3)$ theory. In this subsection, dedicated to the gauge algebra $su(4)$, we will only perform a partial analysis. Recall that for $su(4)$, the partition $1+1+1+1$ gives rise to an affine Toda limit with four solutions, which correspond to the four confining vacua of pure $\mathcal{N}=1$. The partition $2+2$ corresponds to the choice $J_0=\{\alpha_1,\alpha_3\}$ which gives rise to two trigonometric A_1 systems with one solution, and the two remaining variables then form an A_2 affine Toda system which has two solutions, corresponding to the two confining vacua of the unbroken $su(2)$ gauge algebra. Finally, we have the partition 4 which corresponds to the trigonometric A_3 system. This gives rise to a real extremum which represents the fully Higgsed vacuum. We have a total of seven massive vacua.[7] Our focus in the following are massless vacua. A natural way to generate massless vacua is by exploring the partitions $2+1+1$ and $3+1$ which leave unbroken abelian gauge group factors. We will consider them in turn. Let us first remind the reader that the superpotential for the $su(4)$ gauge algebra is

$$\mathcal{W}_{A_3}(Z_1,Z_2,Z_3)=\wp(Z_1)+\wp(Z_2)+\wp(Z_3)+\wp(Z_1+Z_2)+\wp(Z_2+Z_3)+\wp(Z_1+Z_2+Z_3)\,, \quad (6.49)$$

in variables Z_i which are coefficients of fundamental weights.

6.4.1 The Partition $2+1+1$

The partition $2+1+1$ corresponds to a choice of simple root system $J_0=\{1\}$. The centralizer algebra is $su(2)\oplus u(1)$ in this case. We may intuit the existence of two massless branches on

[6] There is a point this branch which is S-duality and T-duality invariant. It is given by $(z_1,z_2,z_3)=(1/2,\tau/2,(1+\tau)/2)$, and is mentioned in [84].

[7] There are other complex extrema of the trigonometric integrable systems.

the basis of this centralizer algebra. We will approach them through the semi-classical limit.

In this limit, we have one leading trigonometric root that sets $Y_1 = 0$. To find the other shifts, we use a heuristic argument based on cancellations that happen in the superpotential at first order in perturbation theory, where $X_1 = \frac{1}{2}$. Such cancellations occur at level $n = 1$ in (3.64) in the sum over the roots that have a non-vanishing scalar product with $\hat{Y}$. As illustrated on the affine Dynkin diagram on figure 6.2, the contributions of α_0 and $\alpha_0 + \alpha_1$ will cancel each other in (3.66), as well as α_2 and $\alpha_1 + \alpha_2$, and all other roots involving α_0 and α_2 are suppressed in the semi-classical limit. Therefore in order to stabilize the system we use the next level $n = 2$ for these roots, which then contribute with factors of q^{2Y_0} and q^{2Y_2}. On the other hand α_3 contributes with a factor q^{Y_3}. Stabilization at leading order requires that these powers of q be equal, and we therefore propose the following ansatz:

$$\begin{cases} Y_1 = 0 \\ 2Y_0 = 2Y_2 = Y_3 \\ Y_0 + Y_1 + Y_2 + Y_3 = 1 \end{cases} \implies \begin{cases} Y_0 = \frac{1}{4} \\ Y_2 = \frac{1}{4} \\ Y_3 = \frac{1}{2} \end{cases} . \tag{6.50}$$

To obtain the subleading Toda potential, we need to take into account the non-perturbative corrections to the value $X_1 = \frac{1}{2}$. Firstly, we expand the superpotential (6.49) around the leading order values (6.50), assuming that the variation δX_1 of X_1 behaves as a power of q. The dominant terms are

$$\frac{1}{\pi^2}\mathcal{W}\left(\frac{1}{2} + \delta X_1, \frac{\tau}{4} + X_2, \frac{\tau}{2} + X_3\right) = -1 + \pi^2 \delta X_1^2 + 8i\pi \delta X_1 q^{\frac{1}{4}} \left(e^{2i\pi X_2} - e^{-2i\pi X_2 - 2i\pi X_3}\right) \tag{6.51}$$

There is a linear term in the non-perturbative correction δX_1 which determines its value at order $q^{\frac{1}{4}}$:

$$(\delta X_1)_{\frac{1}{4}} = \frac{4i}{\pi} q^{\frac{1}{4}} \left(e^{-2i\pi(X_2+X_3)} - e^{2i\pi X_2}\right) . \tag{6.52}$$

This confirms that the value $X_1 = 1/2$ has to be corrected, and that the superpotential should be expanded around the point shifted by $(\delta X_1)_{\frac{1}{4}}$:

$$\frac{1}{\pi^2}\mathcal{W}\left(\frac{1}{2} + (\delta X_1)_{\frac{1}{4}} + \delta X_1, \frac{\tau}{4} + X_2, \frac{\tau}{2} + X_3\right) = -1 - 4q^{\frac{1}{2}} \left(9e^{-2\pi i X_3} + e^{2\pi i X_3}\right) . \tag{6.53}$$

We conclude that X_3 can be determined at this step, and we find

$$X_3 = -\frac{i \log 3}{2\pi} + \frac{1}{2}\mathbb{Z} . \tag{6.54}$$

A longer calculation at higher order shows that X_3 in turn receives non-perturbative corrections, starting at order $q^{\frac{1}{2}}$. Taking into account this second step in our non-perturbative staircase, we find that the superpotential becomes independent of X_2, and equal to $-1 \mp 24\sqrt{q} - 24q + \cdots = \pi^2 E_{2,2}(\pm q^{\frac{1}{2}})$ where the upper sign is for the choice of an integer in equation (6.54) and the lower sign for a strictly half-integer choice. Thus, we have found semi-classical evidence for two one-dimensional complex manifolds of massless vacua characterized by these superpotentials. Again, numerical and analytical evidence can be amassed to argue that the superpotentials are exact.[8]

[8] One extra technique compared to those presented elsewhere is to find a special point on the branch, and then prove that at that point the superpotential takes the claimed value. For the case at hand, for instance, we can

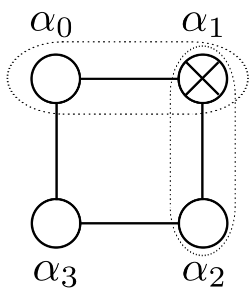


Figure 6.2: Affine Dynkin diagram for algebra A_3 and partition 2+1+1. The crossed simple root corresponds to the set $J_0 = \{1\}$, and the dotted lines encircle roots that cancel the contribution of the simple roots α_0 and α_2.

6.4.2 The Partition $3 + 1$

The partition $3+1$ corresponds to the choice of simple roots $J_0 = \{1, 2\}$. The unbroken gauge algebra is $u(1)$, and we expect one massless branch. Our ansatz for the linear behavior in τ is dictated by the choice of the partition which gives $Y_1 = Y_2 = 0$ and by the symmetry of the affine Dynkin diagram which leads to $Y_3 = Y_0$. Together with the normalization (3.58) we obtain $Y_3 = \frac{1}{2}$. The trigonometric A_2 system arises, and we consider the standard Higgs solution of this system. We thus have $X_1 = 1/3$ and $X_2 = 1/3$ to leading order. We obtain a series of non-perturbative corrections to both leading coordinates and find that when we take those into account, the third variable X_3 parametrizes a massless branch. We calculate the superpotential to order q^2 and it is consistent with the exact value we propose, namely $\mathcal{W} = -2\pi^2 E_{2,2}(q)$.

6.4.3 The Duality Diagram

We have gathered semi-classical and exact data on the $su(4)$ $\mathcal{N} = 1^*$ theory. The duality diagram for the massive states is essentially known, with our without the refinement due to the global choice of gauge group and line operator spectrum. The massless branches fit into the following scheme: we have two massless branches that arise from the partition 2+1+1 and they are T-dual. This is consistent with the confining dynamics of the summand $su(2)$ in the unbroken gauge group. The branch that we found for partition $3+1$ is self-T-dual. Moreover, the branch with superpotential $\pi^2 E_{2,2}(q^{\frac{1}{2}}) = -\pi^2 \left(\theta_2(q)^4 + \theta_3(q)^4\right)$ is S-dual to the manifold with superpotential $\pi^2(\theta_4(q)^4 + \theta_3(q)^4) = -2\pi^2 E_{2,2}(q)$. Similarly, the branch with superpotential $\pi^2 E_{2,2}(-q^{\frac{1}{2}})$ is self-S-dual. This is a familiar three-node permutation representation of the $SL(2,\mathbb{Z})$ duality group. The table below gives a summary of some of the data we laid bare.

---

concentrate on the point

$$(Z_1, Z_2, Z_3) = \left(\frac{1}{2}, \frac{\tau}{4} + \frac{\gamma}{2}, \frac{\tau}{2} - \gamma\right) . \tag{6.55}$$

One then shows that these positions are indeed extremal provided the function $\gamma(\tau)$ satisfies the equation

$$\wp'(2A) + \wp_2'(A) = 0 \qquad \text{with} \qquad A = \frac{\tau}{4} + \frac{\gamma}{2} . \tag{6.56}$$

One can then also analytically prove that this vacuum is massless and has the claimed superpotential. The result is then valid along the whole branch.

Partition	J_0	Unbroken	Vacua
$1+1+1+1$	$\emptyset$	$su(4)$	4 confining vacua of pure $\mathcal{N}=1$
$2+1+1$	$\{1\}$, $\{2\}$ or $\{3\}$	$su(2)\oplus u(1)$	branches $\pi^2 E_{2,2}(\pm q^{\frac{1}{2}})$
$3+1$	$\{1,2\}$ or $\{2,3\}$	$u(1)$	branch $-2\pi^2 E_{2,2}(q)$
$2+2$	$\{1,3\}$	$su(2)$	2 massive vacua
4	$\{1,2,3\}$	1	1 Higgs massive vacuum

Table 6.2: Summary of vacua for $su(4)$

6.4.4 Summary Remarks

We again found new features in the $su(4)$ analysis. These included a staircase structure for determining the positions, with each step corresponding to non-perturbative corrections of a given order. We also discovered an example in which massless manifolds fit into a non-trivial duality diagram. These features are expected to be generic. We moreover are bound to find higher dimensional vacuum manifolds when higher dimensional abelian centralizers are present.

It would be interesting to fully complete the analysis of the vacua of the $su(4)$ theory, in the spirit of the analysis we performed for $su(3)$. In particular, one can exploit the algebraic approach, and parametrize the extrema in terms of algebraic equations. This will allow to determine for instance potential singularities, and possible intersections of manifolds of massless vacua for given values of the complexified coupling τ.

6.5 A Word on the $su(N)$ Theory

It should be clear that in the $su(N)$ case there will be many extra solutions compared to the known sublattices of order N of the torus that represent the massive vacua. Below, we offer only one rudimentary observation on the massless vacua.

The number of massless directions at $\tau \to i\infty$ in the integrable system equals the number of $U(1)$ factors for the semi-classical vacuum in the Coulomb phase. Indeed, all directions that are stabilized by terms with leading behavior a power of q will become untethered when we take the semi-classical limit. In our problem, these are all directions associated to the (affine) Toda system(s). Thus, in the semi-classical limit, we obtain $|\bar{J}_0|-1$ flat directions. We can check that this matches the dimension of the semi-classically unbroken abelian factors, when we go to the Coulomb branch.

Recall that a partition (d_i) (which satisfied $\sum d_i = N$) corresponds to strands (d_i-1) in the set of simple roots of A_{N-1}. The number of coordinates we fix at leading order, using the trigonometric integrable system potential, is equal to $\sum_i (d_i-1)$. The number of coordinates that is unfixed then, in the semi-classical limit (taken on the low-energy effective action) is equal to $N-1-\sum_i (d_i-1) = |i|-1$ where $|i|$ counts the number of (non-zero) terms in the partition.

The number of abelian factors in the Coulomb phase is given by the rank of the centralizer of the nilpotent orbit. The rank is equal to $\sum_j (r_j-1)+k$ where the r_j are defined as the number of times the summand j appears in the partition and $k = |\{i|r_i>0\}|-1$ as in the discussion around (6.3). If we compute this sum, using $\sum_j r_j = |i|$ and $\sum_j 1 = k+1$, we find $|i|-1$, which matches the number of massless modes in the semi-classical limit.

Chapter 7

Topological properties of Groups and Lines

7.1 Introduction

This chapter contains advanced nilpotent orbit theory, complexified integrable system analysis, as well as intricate aspects of $\mathcal{N} = 1$ gauge theories in four dimensions upon circle compactification. We have therefore decided to first illustrate many features of the generic analysis in the example of $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$, where a lot of details can be worked through by hand. We include a description of the consequences of the choice of global gauge group and the spectrum of line operators, which neatly complements the analysis of [10, 139] in an example that is intermediate between $\mathcal{N} = 4$ and pure $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions. Section 7.2 serves to study a tree before exploring the forest. The finer features of the $so(5)$ example will motivate the later sections.

We also discuss the $\mathcal{N} = 1^*$ theory with gauge group of exceptional type G_2. A first reason to study this case is that G_2 is a gauge group of limited rank, allowing for an elaborate numerical analysis of the duality properties of the massive vacua. A second reason is that the group G_2 exhibits an orbit with an unbroken discrete gauge group. This will allow us to cleanly illustrate the role played by the discrete group in the identification of the extrema of the integrable system with massive gauge theory vacua on $\mathbb{R}^{2,1} \times S^1$. This aspect puts into focus the difference between the gauge theory on $\mathbb{R}^4$ and the gauge theory compactified on a circle.

In section 7.4, we thus provide a large amount of detail of the semi-classical analysis of the vacua of $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge group G_2, including a nilpotent orbit classification with their pertinent properties, and the low-energy quantum dynamics in the corresponding phases. Moreover, we perform an in-depth analysis of the associated twisted elliptic Calogero-Moser integrable system, and we make a comparison with the semi-classically predicted vacua. We also provide the duality diagram of the massive vacua and a first estimate of a point of monodromy. In section 7.5, we tie up a loose end, and analytically describe the branch of massless vacua for the $so(5)$ theory.

7.2 The $\mathcal{N} = 1^*$ Theory with Gauge Algebra $so(5)$

To illustrate finer points that crop up when analyzing $\mathcal{N} = 1^*$ gauge theories with generic gauge group upon circle compactification, we concentrate in this section on the study of $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$, and the associated twisted elliptic integrable system with root system B_2 [39].

7.2.1 The Semi-Classical Analysis and Nilpotent Orbit Theory

The $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $\mathbb{R}^4$ has fields in one vector and three chiral multiplet representations of the $\mathcal{N} = 1$ supersymmetry algebra. All fields transform in the adjoint representation of the gauge algebra. After triple mass deformation to $\mathcal{N} = 1^*$ gauge theory, the F-term equations of motion (divided by the complexified gauge group) for the three adjoint chiral scalars have solutions classified by embeddings of $sl(2)$ commutation relations inside the adjoint of the gauge algebra. By the Jacobson-Morozov theorem, these $sl(2)$ triples are in one-to-one correspondence with nilpotent orbits, which have been classified for simple algebraic groups [93, 92, 95, 94].

Nilpotent orbits of the classical groups can be enumerated by partitions that correspond to the dimensions of the $sl(2)$ representations that arise upon embedding in the gauge algebra.[1] The Lie algebra of the centralizer has been computed, and non-abelian centralizers give rise to effective pure $\mathcal{N} = 1$ gauge theories that have a number of quantum vacua equal to the dual Coxeter number of the unbroken gauge group. The partition, the unbroken gauge algebra, and the number of massive quantum vacua they give rise to on $\mathbb{R}^4$ for the gauge algebra $so(5)$ are enumerated in the first three columns in table 7.1. For instance, the $2+2+1$ partition of 5 corresponds to a configuration for the adjoint scalar expectation values that represent a particular orbit (via the correspondence between $sl(2)$ embeddings and nilpotent orbits), and these vacuum expectation values leave a $C_1 = A_1$ gauge algebra unbroken. The resulting pure $\mathcal{N} = 1$ gauge theory at low energy gives rise to two massive vacua. See [77, 79, 34, 1, 2].

Orbit Partition	Unbroken	Massive Vacua on $\mathbb{R}^4$	W-class	Levi
$1+1+1+1+1$	B_2	3	$\emptyset$	0
$2+2+1$	C_1	2	$\{\alpha_1\}$	C_1
$3+1+1$	$u(1)$	0	$\{\alpha_2\}$	$\tilde{A}_1$
5	1	1	$\{\alpha_1, \alpha_2\}$	B_2

Table 7.1: Nilpotent orbit data for $so(5)$.

The last two columns in table 7.1 are related to the Bala-Carter theory of nilpotent orbits [154, 155] that associates a Weyl group equivalence class of subsets of the set of simple roots to each Levi subalgebra of the gauge algebra. The reader is referred to section 2.4.3 for the necessary nilpotent orbit theory, and also section 7.4.2 for an example worked out in detail. When we compactify the gauge theory on $\mathbb{R}^{2,1} \times S^1$, properties of the centralizer beyond its Lie

[1] For the case of gauge algebra $so(2N)$ and the adjoint gauge group $SO(2N)$, the very even partitions (having only even parts with even multiplicity) give rise to two distinct nilpotent orbits. For this gauge algebra, each orbit gives rise to its own vacua. When the outer automorphism of $so(2N)$ is joined to the adjoint gauge group, we obtain the gauge group $O(2N)$ in which these orbits and the corresponding vacua are identified.

type become crucial. A refined classification of the nilpotent orbits, including the conjugacy classes of the component group[2] of the unbroken gauge group (by Bala, Carter and Sommers [154, 155, 35]) gives rise to table 7.2.

Orbit	Centr.	C. C.	Massive Vac.	W-classes	PLS
$1+1+1+1+1$	B_2	1	3	$\emptyset$	0
$2+2+1$	A_1	1	2	$\{\alpha_0\}$, $\{\alpha_1\}$	C_1
$3+1+1$	$u(1)$	1	0	$\{\alpha_2\}$	$\tilde{A}_1$
$3+1+1$	$u(1)$	(12)	1	$\{\alpha_0, \alpha_1\}$	D_2
5	0	1	1	$\{\alpha_0, \alpha_2\}$, $\{\alpha_1, \alpha_2\}$	B_2

Table 7.2: The Bala-Carter-Sommers classification of nilpotent orbits with their centralizers, including the conjugacy classes (C.C.) of the component group of the centralizer. The first column gives the partition labeling the orbit, the second the Lie type of the centralizer (i.e. the unbroken gauge algebra for given adjoint vacuum expectation values), the third the conjugacy class of the discrete part of the centralizer corresponding to the chosen pseudo-Levi subalgebra (PLS) in the last column, the fourth the number of massive vacua and the previous to last the Weyl conjugacy classes of subsystems of simple roots of the affine root system. In each case, there is only one distinguished parabolic subalgebra, which is the principal one. This analysis is valid for the adjoint group and will be further refined when we take into account the choice of global properties of the gauge group (see table 7.3).

At this stage, we wish to take away the elementary fact that the partition $3+1+1$ appears twice in the first column of table 7.2, because there is a discrete $\mathbb{Z}_2$ component subgroup of the centralizer. The $\mathbb{Z}_2$ component group has two conjugacy classes, namely the trivial one, and the non-trivial one (labeled by the cyclic permutation (12)). The importance of the second occurrence is the fact that we can turn on a Wilson line on the circle equal to this conjugacy class while still satisfying the equations of motion (as discussed in detail in section 7.4). Because the $\mathbb{Z}_2$ forms a semi-direct product with the $SO(2)$ unbroken gauge group for the $3+1+1$ partition, turning on the Wilson line breaks the abelian gauge group, and generates a new massive vacuum on $\mathbb{R}^{2,1} \times S^1$ [1]. Finally, we note that we also have a massless branch of rank one.

7.2.2 The Elliptic Integrable System

We turn to how the physics of the $\mathcal{N} = 1^*$ theory with gauge algebra $so(5)$ is coded in the twisted elliptic integrable system of type B_2 that was proposed to be the low-energy effective superpotential for the model [39]. In as far as this constitutes a review of the results presented in [1], we will again be concise, while new features will be emphasized.

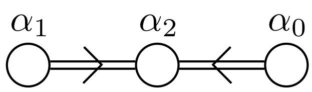


Figure 7.1: The Dynkin diagram of the affine algebra $\widehat{so(5)} = B_2^{(1)}$ with our convention for the numbering of long and short roots.

[2]The component group is the quotient of the group by its identity component.

The Dynkin diagram for the affine algebra $B_2^{(1)}$ (as well as its finite counterpart, upon deleting the zeroth node) can be read off from figure 7.1. The long simple root α_1 of B_2 can be parametrized as $\alpha_1 = \epsilon_1 - \epsilon_2$ and the short root α_2 as $\alpha_2 = \epsilon_2$, where the ϵ_i are orthonormal basis vectors in a two-dimensional Euclidean vector space.[3] The superpotential of the twisted elliptic Calogero-Moser model with root system B_2 is [105]

$$\begin{aligned} \mathcal{W}_{B_2,tw}(Z) &= \wp(z_1+z_2) + \wp(z_1-z_2) + \frac{1}{2}\left[\wp_2(z_1) + \wp_2(z_2)\right] && (7.1)\\ &= \wp(Z_1) + \wp(Z_1+Z_2) + \frac{1}{2}\left[\wp_2\left(\frac{Z_2}{2}\right) + \wp_2\left(Z_1 + \frac{Z_2}{2}\right)\right], && (7.2) \end{aligned}$$

where we combine the Wilson line a and dual photon σ of the low-energy theory in the Coulomb phase in a complex field $Z = \sigma + \tau a$ parametrized by

$$Z = Z_1\pi_1 + Z_2\pi_2 = z_1\epsilon_1 + z_2\epsilon_2 \,. \tag{7.3}$$

Throughout the chapter we use capital letters to denote the components of an element of the dual Cartan space decomposed on the basis of fundamental weights, and small letters to denote its components in the basis ϵ_i. For $so(5)$, the relation is

$$\begin{cases} Z_1 = z_1 - z_2 \\ Z_2 = 2z_2 \end{cases} \quad \text{or} \quad \begin{cases} z_1 = Z_1 + \frac{1}{2}Z_2 \\ z_2 = \frac{1}{2}Z_2 \end{cases} \,. \tag{7.4}$$

The superpotential $\mathcal{W}$ depends on the elliptic Weierstrass function $\wp$ with half-periods $\omega_1 = 1/2$ and $\omega_2 = \frac{\tau}{2}$ (where the complexified gauge coupling is $\tau = \omega_2/\omega_1$) and its twisted cousin $\wp_2$ which is defined to have half the period in the ω_1 direction, $\wp_2(z,\tau) = \wp(z,\tau) + \wp(z+\frac{1}{2},\tau)$. The ratio of the coupling constants for short and long roots was fixed in [39] and checked using Langlands duality in chapter 4. In this chapter, we established the existence of seven massive vacua (up to a given equivalence relation to be discussed shortly), determined their positions numerically, and provided analytic expressions for the value of the superpotential in each of these massive vacua. The extremal positions at $\tau = i$ are rendered in figure 4.2. We moreover established the duality diagram in figure 4.3 between the seven massive vacua.

In the present section, we wish to add to the analysis presented in chapter 4 in several ways. We analyze the semi-classical limits of the effective low-energy superpotential. We propose a list of such limits, and show that we obtain an analytic handle on each of the seven vacua, and on the massless vacua as well. Moreover, we will carefully exhibit the differences between various global choices of gauge group and spectra of line operators, and consequently a more refined duality diagram.

Importantly, our list of limits is based on table 7.2. Each nilpotent orbit and conjugacy class of the component group is associated, by Bala-Carter-Sommers theory to a choice of inequivalent[4] subsystem of simple roots of the affine root system. To each such subsystem, we

[3] Let us recall a few Lie algebra data for future reference. The root lattice is generated by $\epsilon_{1,2}$. The fundamental weights are $\pi_1 = \epsilon_1$ and $\pi_2 = (\epsilon_1+\epsilon_2)/2$. The dual simple roots are $\alpha_1^\vee = \alpha_1$ and $\alpha_2^\vee = 2\alpha_2 = 2\epsilon_2$. The dual weight lattice is spanned by the ϵ_i. The Weyl group allows for permutations of the ϵ_i, and all sign changes. We follow the conventions of [156].

[4] The equivalence relation is given precisely in [35], and can be technical in some cases. For the gauge theories we are concerned with, it can be stated as follows. In algebras of type A and G, two subsystems are equivalent if they have the same Lie algebra type and the same repartition of long and short roots. For type B, one should moreover distinguish between $A_1 + A_1$ and D_2, and between A_3 and D_3, using the index of the subsystem.

associate a limit of the integrable system as follows. We demand that for simple roots α_i in the subsystem, we have that the simple root is orthogonal to the vector of extremal positions Z, namely $(\alpha_i^\vee, Z) = 0$, to leading (linear) order in the complexified gauge coupling τ in the large imaginary τ limit. The simple root systems in the complement must have non-zero leading term.

We denote the part in Z that is linear in τ by $Y\tau$. We moreover introduce the redundant coordinate Y_0 which we constrain by the equation $Y_0 + Y_1 + Y_2 = 1$. It is a coordinate that is natural in treating this problem governed by affine algebra symmetry as will become more manifest in section 3.4. The list of semi-classical limits that we will consider are then labelled by the set J_0 which contains all i for which α_i is in the chosen simple root subsystem indicated in table 7.2. We therefore distinguish five limits, which we treat one-by-one below.

- The first limit corresponds to the empty set, $J_0 = \emptyset$. The arguments of the Weierstrass functions will all contain a linear term in τ. Therefore all terms are well-approximated by exponentials (see section 3.4, formulas (3.43) and (3.47)). The limiting procedure in this case is described in detail in [111], which in turn is a generalization of the Inozemtsev limit [36]. By demanding that all these exponentials have the same dependence on the instanton counting parameter $q = e^{2\pi i\tau}$, which is necessary in order to stabilize all variables, we determine that the linear behavior of the coordinates in τ is $Y_0 = Y_1 = Y_2 = 1/3$. We obtain a (fractional instanton) B_2 affine Toda system in the limit, with 3 extrema. The solutions (z_1, z_2) in the semi-classical limit are then $(\frac{\tau}{2}, \frac{\tau}{6})$, $\left(\frac{\tau}{2}, \frac{1}{3} + \frac{\tau}{6}\right)$ and $\left(\frac{\tau}{2}, \frac{2}{3} + \frac{\tau}{6}\right)$. One can check these solutions against the behavior of the numerical extrema labelled $2, 3$ and 4 in figures 4.2 and 4.3, and they match in the semi-classical limit. This codes the physics of the pure $\mathcal{N} = 1$ gauge theory with gauge algebra $so(5)$.[5] Indeed, the partition $1 + 1 + 1 + 1 + 1$ leaves the whole of the gauge group unbroken.

- The second case is the choice of subroot system $J_0 = \{0\}$. Note that this is completely equivalent to the choice $J_0 = \{1\}$, since the corresponding marked Dynkin diagrams are the same in both cases, so we concentrate on the first of these sets. Then we have $Y_0 = 0$ as a consequence, and to match powers of q in subleading terms, we choose $Y_1 = Y_2 = 1/2$. In the semi-classical limit, we then obtain a trigonometric A_1 system at leading order (associated to the long root α_0). At subleading order, we find a superpotential $\mathcal{W}(z_1, z_2)$ consisting of a sum of exponentials
$$\mathcal{W}_{B_2,tw}\left(\frac{3\tau}{4} + \frac{1}{4} + \frac{\delta x}{2}, \frac{\tau}{4} + \frac{1}{4} - \frac{\delta x}{2}\right) = \pi^2(12e^{-2i\pi\delta x} - 4e^{2i\pi\delta x})q^{\frac{1}{2}} + \dots . \tag{7.5}$$
The two extrema at large τ are $\left(\frac{1}{8} + \frac{3\tau}{4}, \frac{3}{8} + \frac{\tau}{4}\right)$ and $\left(-\frac{1}{8} + \frac{3\tau}{4}, \frac{5}{8} + \frac{\tau}{4}\right)$.[6] These match the behavior of the massive vacua number 5 and 6 in figures 4.2 and 4.3 at large τ. These are the two confining vacua of the unbroken pure $\mathcal{N} = 1$ $su(2)$ gauge theory. Note how this limit is intermediate in that one coordinate is fixed at leading order in the $q^{\frac{1}{2}}$ expansion, while a second is fixed at subleading order.

- Thirdly, we have the case $J_0 = \{1, 2\}$, which by the same token is equivalent to $J_0 = \{0, 2\}$. We find the trigonometric potential B_2 with a real extremum, which can be characterized

[5] We discuss the global choice of gauge group and line operators in subsection 7.2.3.

[6] The subleading behavior of the extrema in the large τ limit can easily be computed analytically as well. See later for more intricate explicit examples.

in terms of zeroes of orthogonal polynomials [108, 157]. This corresponds to the fully Higgsed vacuum, with label 1 in figure 4.2. Importantly, there are other, complex extrema of the trigonometric integrable system.[7] In the limit $\tau \to i\infty$, one example extremum is given by $(z_1, z_2) \sim (\frac{1}{4} + \frac{\log(1+\sqrt{2})}{2\pi} i, \frac{1}{4} - \frac{\log(1+\sqrt{2})}{2\pi} i)$. This is a massless extremum, part of a branch that we analyze in section 7.5.

- For our fourth pick, we take $J_0 = \{0, 1\}$ and obtain two trigonometric potentials, corresponding to the root system D_2. We find the one extremum $\left(\frac{\tau}{2}, \frac{1}{2} + \frac{\tau}{2}\right)$. This corresponds to extremum number 7. This is a massless vacuum lifted by the presence of a $\mathbb{Z}_2$ Wilson line. It is thus semi-classically massive on $\mathbb{R}^{2,1} \times S^1$. The $\mathbb{Z}_2$ Wilson line sits inside the non-trivial conjugacy class (12) of the component group $\mathbb{Z}_2$ of the unbroken gauge group. This is an occurrence of a general phenomenon that we analyze further in section 7.4.

- Finally, we turn to the fifth possibility, $J_0 = \{2\}$. The leading τ behavior of the second coordinate is $Y_2 = 0$. As a first stab at the semi-classics in this regime, we choose the values $Y_0 = Y_1 = 1/2$, which is a natural ansatz given the symmetry of the Dynkin diagram about α_2. In any event, we obtain the trigonometric $\tilde{A}_1$ system (where the tilde stands for a short root) at leading order. The extremization of the superpotential at order 0 gives $Z_2 = \frac{1}{2}$. The value of Z_2 gets corrected non-perturbatively, namely at order $q^{\frac{1}{2}}$, order $q^{\frac{3}{2}}$ and higher strictly half-integer orders, by terms depending on δZ_1 exponentially. For the particular value of Y_1 that we chose we find

$$\begin{aligned} \delta Z_2 &= \frac{1}{\pi}\Big(- e^{-2i\pi\delta Z_1} q^{\frac{1}{2}} - e^{2i\pi\delta Z_1} q^{\frac{1}{2}} \\ &\quad + \frac{1}{3} e^{-6i\pi\delta Z_1} q^{\frac{3}{2}} + 5 e^{-2i\pi\delta Z_1} q^{\frac{3}{2}} + 5 e^{2i\pi\delta Z_1} q^{\frac{3}{2}} + \frac{1}{3} e^{6i\pi\delta Z_1} q^{\frac{3}{2}} + \dots \Big) . \end{aligned} \tag{7.6}$$

Injecting this value for Z_2 in the superpotential $\mathcal{W}(Z_1, Z_2)$ finally gives

$$\begin{aligned} \mathcal{W}_{B_2,tw}\left(\frac{\tau}{2} + \delta Z_1, \frac{1}{2} + \delta Z_2(\delta Z_1)\right) &= \pi^2 \left(\frac{2}{3} + 16q + 16q^2 + 64q^3 + 16q^4 + \dots\right) \\ &= -\frac{2\pi^2}{3} E_{2,2}(q) , \end{aligned} \tag{7.7}$$

to order q^4. We observe that non-perturbatively correcting the leading coordinate Z_2 leads to a vanishing potential for Z_1, in perturbation theory in q. The value of the coordinate Z_1 determines the non-perturbative correction to the leading coordinate Z_2. For instance, for the special value $\delta Z_1 = 1/2$, the non-perturbative correction is zero. See equation (7.6). Thus, we find a one-dimensional complex branch of massless vacua to which we return in section 7.5. The value of the superpotential in these vacua can be determined by a combination of numerics, and analytical expectations to be $\mathcal{W} = -\frac{2\pi^2}{3} E_{2,2}(q)$. The Einstein series $E_{2,2}$ is the modular form of weight 2 of $\Gamma_0(2)$ that has a q-expansion that starts out with -1.

We have made a list of semi-classical limits for the $so(5)$ integrable system. In particular, we have analytically recuperated all the numerical results of chapter 4, in the large imaginary τ limit. We have moreover made inroads into extra vacua, which are massless. Before discussing

[7] Complex extrema of integrable systems are rarely discussed. The observation we make here on the trigonometric B_2 integrable system, for instance, appears to be new.

the particular features of the $so(5)$ analysis that we will concentrate on in the rest of the chapter, we pause to discuss global aspects of the gauge theory at hand.

7.2.3 Global Properties of the Gauge Group and Line Operators

Up to now, we have implemented a concept of equivalence on the configuration space in which we identify the variables proportional to ω_1 by shifts in the weight lattice and the variables in the ω_2 direction by shifts in the dual weight lattice. These are natural identifications when one is concerned with analyzing the elliptic integrable potential, as we saw in chapter 3. However, from the gauge theory perspective, the global and local symmetries are fixed a priori, and in this subsection we will carefully track how they influence both the counting of vacua and their duality relations.

In other words, we give an example of how to generalize the analysis of the global choice of gauge group and the spectrum of line operators, performed for pure $\mathcal{N} = 1$ gauge theories and $\mathcal{N} = 4$ theories in [10, 139] to $\mathcal{N} = 1^*$ theories. Recall that $\mathcal{N} = 4$ gauge theories with $so(5)$ gauge algebra come in three varieties which satisfy Dirac quantization and maximality of the operator algebra. We first distinguish between the choice of gauge group $SO(5)$ and $Spin(5)$.[8] The $Spin(5)$ theory is unique. The $SO(5)$ theories come in two versions, depending on whether they include a 't Hooft operator which transforms in the fundamental of the dual gauge group, or a Wilson-'t Hooft operator that transforms in the fundamental of both the electric and the magnetic gauge group. The first can be denoted $SO(5)_+$ theory, and the second $SO(5)_-$ theory. The refined duality map of $\mathcal{N} = 4$ theories described in [10, 139] states that the $SO(5)_+$ theory is S_2-dual to the $Spin(5)$ theory.[9] The $SO(5)_-$ theory is self-S_2-dual. The goal of this subsection is to carefully examine the global electric and magnetic identifications of the extrema of the low-energy effective superpotential to show that the refined classification of vacua of the $\mathcal{N} = 1^*$ theory is consistent with the duality imparted by the $\mathcal{N} = 4$ theory.

To make contact with our set-up, we first analyze the periodicity of the Wilson line, which follows from the global choice of gauge group and line operators. In the case where we work with the adjoint gauge group $Spin(5)/\mathbb{Z}_2 = SO(5)$ and the spectrum of line operators corresponding to the $SO(5)_+$ theory, we allow gauge parameters that close only up to an element in the center of the covering group. The Wilson line periodicity is then the dual weight lattice. The dual weight lattice if spanned by the ϵ_i and therefore the two variables on the Coulomb branch will each have periodicity $2\omega_2$. When the gauge group is the covering group $Spin(5)$, gauge parameters are strictly periodic, and the periodicity of the Wilson line is the dual root lattice. In this case, Wilson lines are equivalent under shifts by $\epsilon_1 - \epsilon_2$ and $2\epsilon_2$. Thus, both coordinates are periodic with periodicity $4\omega_2$, and we can further divide by simultaneous shifts by $2\omega_2$.

For the magnetic line operator spectrum for the $Spin(5)$ and $SO(5)_+$ theories, it suffices to Langlands S_2-dualize the above reasoning. We thus obtain that for $SO(5)_+$ we can shift by $2\omega_1$ separately each coordinate (i.e. by the root lattice), and for $Spin(5)$ we add on top of this the simultaneous shift by ω_1 (i.e. the weight lattice). The factor of two difference in the lattice spacing is due to the mechanics of the Langlands S_2 duality. For the $SO(5)_-$ theory, the story is more subtle. There is a 't Hooft-Wilson line operator in the spectrum which is in

[8]The nomenclature is fixed by demanding that a choice of electric gauge group implies that all possible purely electric charges for Wilson line operators corresponding to the electric gauge group must be realized.

[9]We denote by S_2 the Langlands duality transformation $\tau \to -\frac{1}{2\tau}$.

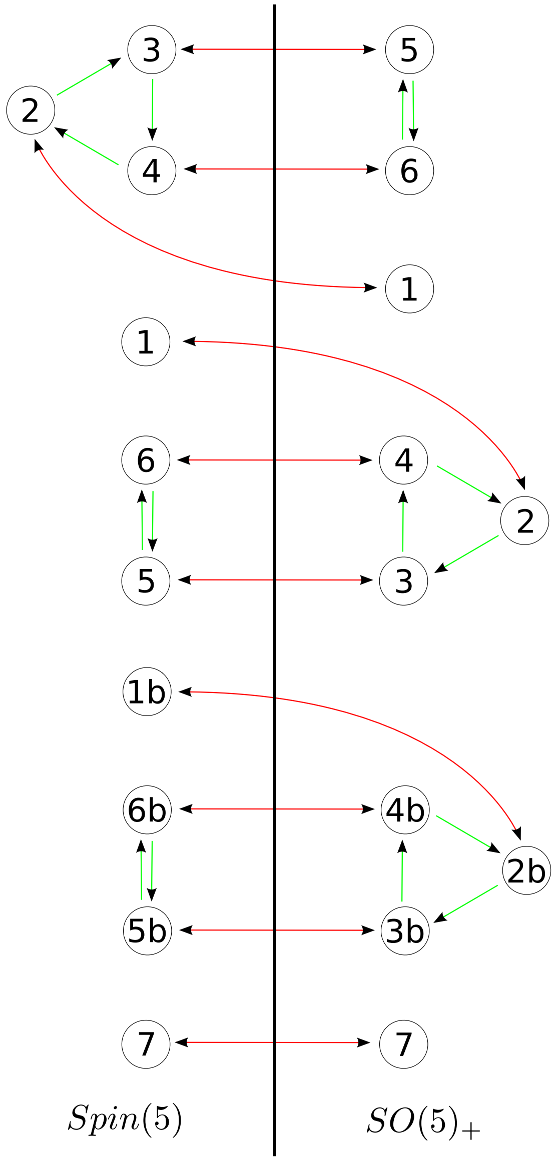


Figure 7.2: The diagram of the action of dualities on the massive vacua for the different B_2 theories. In red, we draw the action of Langlands S_2-duality, and in green, T-duality (when the action is non-trivial). On the left we represent the 10 vacua of the $Spin(5)$ theory, and on the right the 10 vacua of the $SO(5)_+$ theory. The diagram of dualities for the self-dual $SO(5)_-$ theory is identical to figure 4.3.

the fundamental of both the dual gauge group and the ordinary gauge group. We allow for the identifications common to $Spin(5)$ and $SO(5)_+$, and add the identification that shifts an individual coordinate by $2\omega_2$ and both coordinates simultaneously by ω_1. This is the diagonal $\mathbb{Z}_2$ in the magnetic and electric weight lattices divided by the magnetic and electric root lattices respectively.

$SO(5)_+$ vacua

Given the more limited identifications above, we obtain a longer list of extrema. The list of massive extrema for the $SO(5)_+$ theory is $(1, 2, 3, 4, 5, 6, 7, 2b, 3b, 4b)$ where the extrema $(2b, 3b, 4b)$ are obtained from $(2, 3, 4)$ by shifting by ω_1 (see figure 7.3 in the appendix). The doubling of the number of massive vacua arising from pure $\mathcal{N} = 1$ super Yang-Mills theory with $SO(5)_+$ gauge group is as expected from [10, 139]. We thus have ten massive vacua.

$SO(5)_-$ vacua

In this case, we remain with seven massive vacua. For the vacua $(2, 3, 4)$, this is as for the pure $\mathcal{N} = 1$ theory. By self-S_2-duality, this is expected for the vacua $(1, 5, 6)$ as well.

$Spin(5)$ vacua

For the $Spin(5)$ theory, we again find ten massive vacua. The doubling of vacua is S_2-dual to the duplication for $SO(5)_+$, and extrema $(1, 5, 6)$ obtain partner vacua $(1b, 5b, 6b)$ (see figure 7.4). The duality diagrams for the massive vacua are drawn in figure 7.2. The analysis in [10, 139] shows that the pure $\mathcal{N} = 1$ $Spin(5)$ theory on $\mathbb{R}^{2,1} \times S^1$ has 3 vacua, which is consistent with the one triplet under T-duality that we find on the left in figure 7.2. To explain the doubling of the singlet and doublet in the $Spin(5)$ theory, we refine our analysis of the unbroken gauge group further, and adapt it to include the differences between the adjoint group $SO(5)$ and the covering group $Spin(5)$. The results are in table 7.3.

Partition	Centralizers		Massive vacua on $\mathbb{R}^{2,1} \times S^1$		
B_2	$SO(5)$	$Spin(5)$	$SO(5)_+$	$SO(5)_-$	$Spin(5)$
$1+1+1+1+1$	$SO(5)$	$Spin(5)$	6	3	3
$2+2+1$	$SU(2)$	$SU(2) \times \mathbb{Z}_2$	2	2	4
$3+1+1$	$\mathbb{Z}_2 \ltimes U(1)$	$\mathbb{Z}_2 \ltimes U(1)$	1	1	1
5	1	$\mathbb{Z}_2$	1	1	2
Total			10	7	10

Table 7.3: For each B_2 partition we use the Springer-Steinberg theorem (see section 2.5.2) to compute the centralizer inside $SO(5)$ and $Spin(5) = Sp(4)$. Finally, we compute the number of massive vacua on $\mathbb{R}^{2,1} \times S^1$ in the different theories.

For the $Spin(5)$ gauge group, we find that the centralizer for the $2+2+1$ partition and the 5 partition, contains an extra $\mathbb{Z}_2$ discrete factor. We can turn on a Wilson line in this $\mathbb{Z}_2$ group, which doubles the number of massive vacua on $\mathbb{R}^{2,1} \times S^1$ corresponding to these partitions. This matches perfectly with the doubling of the T-duality doublet and singlet extrema of the integrable system that we witness on the left of figure 7.2.

Summary of the Global Analysis

Thus, we have checked the duality inherited from $\mathcal{N} = 4$, including the choice of the center of the gauge group as well as the spectrum of line operators, in the case of the Lie gauge algebra $so(5)$. The $\mathcal{N} = 1^*$ theory neatly illustrates both the features of the pure $\mathcal{N} = 1$ theory as well as those of the $\mathcal{N} = 4$ theory discussed in [10, 139]. The global refinement of the analysis of all vacua can be performed for $\mathcal{N} = 1^*$ theories with *any* gauge group, but we will refrain from belaboring this particular point in the rest of this chapter.

7.2.4 Summary and Motivation

By now, the reader may be convinced that the $\mathcal{N} = 1^*$ theory, even in the rank two case of the $so(5)$ gauge algebra, exhibits interesting elementary physical phenomena hiding in a maze governed by modularity and ellipticity. We will isolate a subset of these interesting phenomena, and clarify the mathematical structures relevant to each. We will show that they are general, and that they can often be understood in algebraic, modular or elliptic terms. The points we will concentrate on are the following.

- We used semi-classical limits of elliptic integrable systems to render an analytic exploration of the vacuum structure coded in the low-energy effective superpotential possible. In the process, we uncovered limits of integrable systems that generalize the Inozemtsev limit [36, 111]. From the gauge theory perspective, these limits are intermediate between the confining and the Higgs regimes. In section 3.4 we describe these limits in more detail, and show that they are closely related to the semi-classical analysis of the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge algebra $\mathfrak{g}$.

- We saw that a branch of massless vacua appeared as semi-classical limiting solutions, for the gauge algebra $so(5)$. The appearance of massless vacua as limiting solutions is again generic and also occurs for $su(N)$ theories, as we will show in sections 3.4 and 6. We will be able to analytically characterize the manifold of massless vacua for the $su(3)$ theory, including its duality properties. For the $su(4)$ theory, an analogous picture will be developed. Finally, the massless manifold of the $so(5)$ gauge theory will be scrutinized in section 7.5.

- We claimed that one vacuum of the $so(5)$ theory arises from turning on a $\mathbb{Z}_2$ Wilson line that breaks the abelian gauge group factor such as to render the vacuum massive on $\mathbb{R}^{2,1} \times S^1$. We will show that this phenomenon as well is rather generic and that we can characterize the discrete gauge group, and the Wilson line in terms of the Lie algebra data associated to the corresponding semi-classical limit. This will be demonstrated in sections 3.4 and 7.4.

The clarification of these points will occupy us for the rest of this chapter.

7.2.5 Representations of the Vacua for B_2 Theories

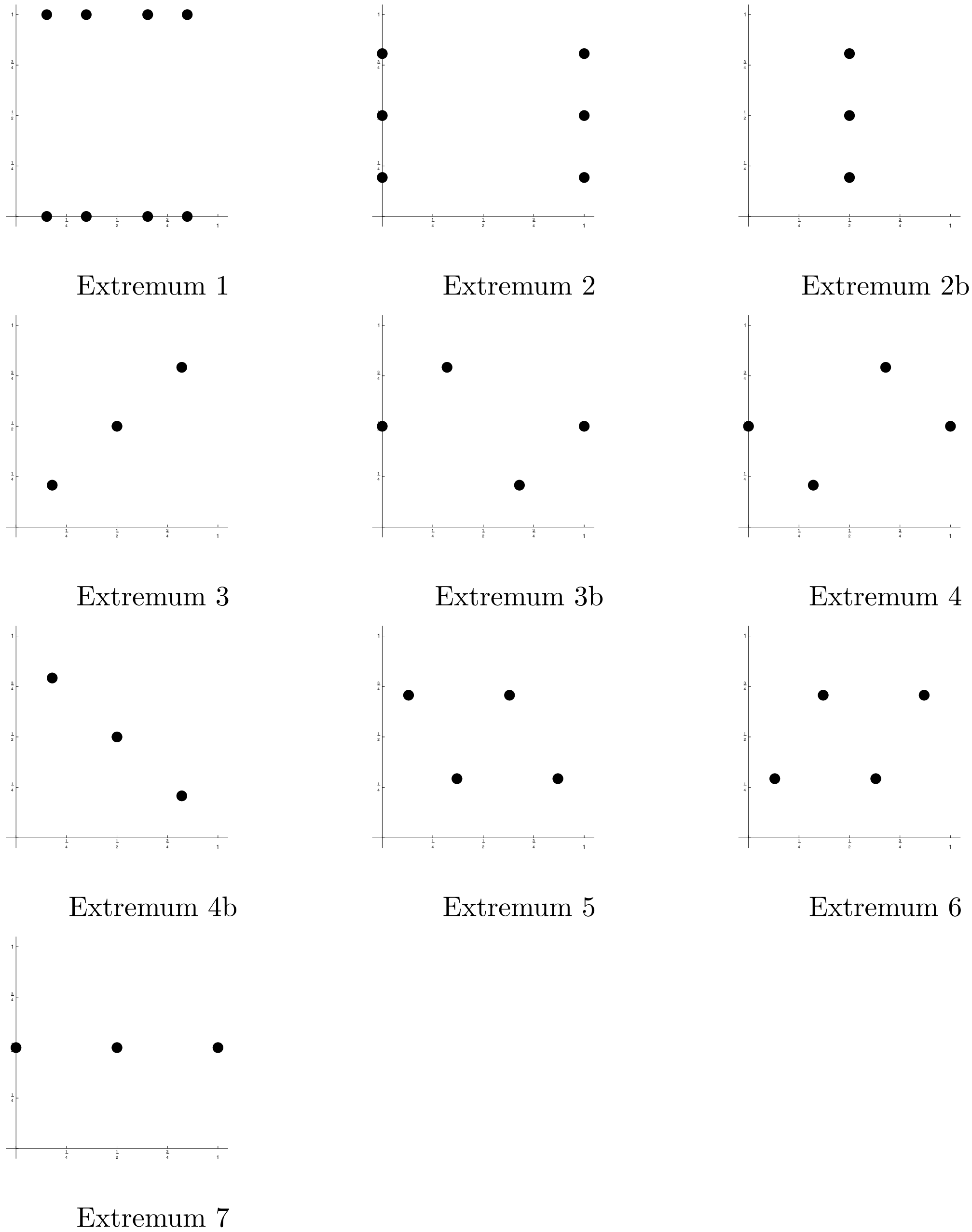
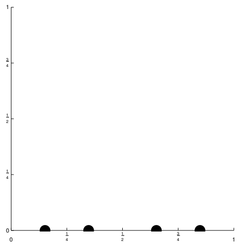
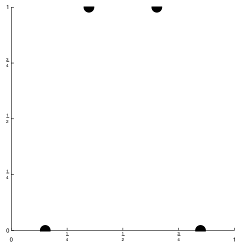


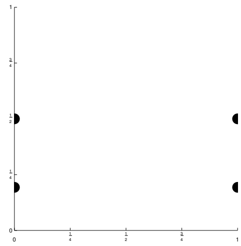
Figure 7.3: The extremal positions of the vacua for the $SO(5)_+$ theory.



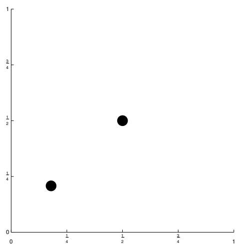
Extremum 1



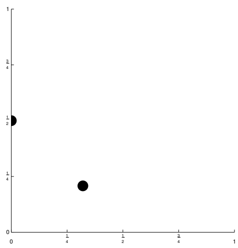
Extremum 1b



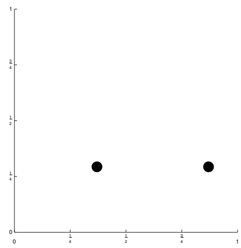
Extremum 2



Extremum 3



Extremum 4



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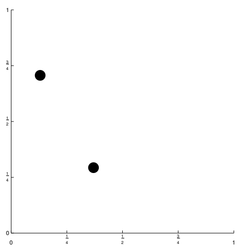
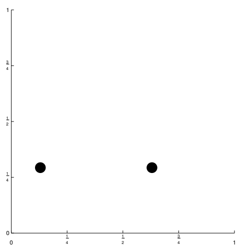
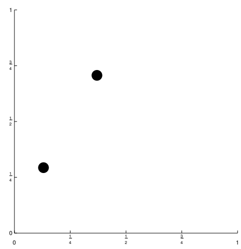


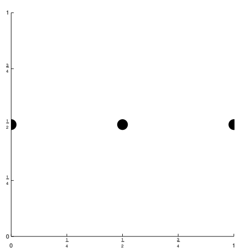
Extremum 5b



Extremum 6



Extremum 6b



Extremum 7

Figure 7.4: The extremal positions of the vacua for the $Spin(5)$ theory.

7.3 More on Nilpotent Orbit Theory

In this section, we review how subsets of simple roots of affine root systems enter in the theory of nilpotent orbits. Thus, we will be able to associate semi-classical limits of the elliptic integrable system, and therefore the low-energy superpotential of $\mathcal{N} = 1^*$ theory, to a detailed description of nilpotent orbits and the component group of their centralizer. We will exploit this map in the following sections.

7.3.1 The Nilpotent Orbit Theory of Bala-Carter and Sommers

From section 3.4, we conclude that we can associate semi-classical extrema of the elliptic integrable system to subsets of the (dual) affine simple root system. In this section, we show that there is another way to understand the relevance of these subsets, in terms of nilpotent orbit theory and the physics of $\mathcal{N}=1^*$ theory on $\mathbb{R}^{2,1}\times S^1$.

Firstly, let us briefly review highlights of nilpotent orbit theory that we have presented in some detail in section 2.4.3. The Bala-Carter classification of nilpotent orbits of simple algebraic groups goes as follows. Each nilpotent orbit of a Lie algebra $\mathfrak{g}$ of a connected, simple algebraic adjoint group G is a distinguished nilpotent orbit of a Levi subalgebra. Levi subalgebras of $\mathfrak{g}$ correspond to subsets of simple roots of $\mathfrak{g}$ up to conjugation by the Weyl group via the map (2.33).

As we have seen in section 3.4.2, the elliptic integrable system we are interested in is more naturally formulated using the language of affine Lie algebras, as reflected for instance in equation (3.52). Moreover, the fact that the integrable system appears in the gauge theory when compactified on a circle gives another indication that the loop algebra might be the right language to be used, and also begs for the introduction of affine algebras. And it appears that there is a very natural generalization of the notion of standard Levi subalgebra (2.33) that sheds new light on the effects of this compactification.

Sommers introduces in [35] the notion of *pseudo-Levi* subalgebra, as we describe now. Let $\hat{\Delta}^s$ be a basis of simple roots of the affine algebra $\hat{\mathfrak{g}}$, and let J be a *proper* subset of $\hat{\Delta}^s$. We now define the *standard pseudo-Levi* subalgebra $\mathfrak{l}_J$ by

$$\mathfrak{l}_J = \mathfrak{h} \oplus \sum_{\alpha\in\langle J\rangle} \mathfrak{g}_\alpha\,, \tag{7.8}$$

where again $\langle J\rangle$ is the abstract root system generated by J (which can be defined as the intersection between the full root system of $\hat{\mathfrak{g}}$ and the lattice generated by J). Any subalgebra of $\hat{\mathfrak{g}}$ which is a G-conjugate of some $\mathfrak{l}_J$ is called a pseudo-Levi subalgebra.

Now that we have this useful definition at hand, we quote the fundamental theorem of Sommers [35]. There is a one-to-one correspondence between the two following sets:

(i) The set of conjugacy classes of pairs (n, C), where n is a nilpotent element and C is a conjugacy class in the component group of the centralizer of n;

(ii) The set of conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ where $\mathfrak{l}$ is a pseudo-Levi subalgebra and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of $[\mathfrak{l},\mathfrak{l}]$.

This classification allows for the unified calculation of all the component groups of nilpotent orbits of simple Lie algebras. In the A-type case, all pseudo-Levi subalgebras are equivalent to Levi subalgebras, since the lowest root is Weyl equivalent to any other simple root.

7.3.2 The Bridge between Gauge Theory and Integrable System

Semi-classical solutions to the F-term equations of motion for $\mathcal{N}=1^*$ theory on $\mathbb{R}^4$ are classified by matching them onto nilpotent orbits [2]. When we compactify the gauge theory on S^1, there are further aspects of nilpotent orbits that come into play. In particular, we will allow for Wilson lines in the unbroken gauge group. If the latter contains topologically non-trivial conjugacy classes, i.e. conjugacy classes in the component group of the centralizer, then we need to consider each of these configurations separately.

As we saw, there is a one-to-one correspondence between the pair (nilpotent orbit, conjugacy class of component group) and pseudo-Levi subalgebras. The trivial conjugacy classes will correspond to a collection of non-affine simple roots. Each inequivalent choice of subset that necessarily includes the affine root will correspond to a non-trivial conjugacy class of a component group. These are classified by Bala-Carter-Sommers theory, which therefore is crucial in classifying semi-classical configurations for $\mathcal{N} = 1^*$ theory compactified on a circle. The example of the $\mathcal{N} = 1^*$ gauge theory with G_2 gauge group discussed in section 7.4 will neatly illustrate our reasoning.

Before we turn to this application, we demonstrate the use of the semi-classical limit in example systems. In particular, the techniques developed in this section allow for the analysis of the physics and duality properties of the massless vacua of $su(N)$ theories.

7.4 Discrete Gauge Groups and Wilson Lines

In section 6 we described features of semi-classical limits and massless vacua for the $\mathcal{N} = 1^*$ theory with $su(N)$ gauge algebra. We now wish to examine more closely another feature that we encountered in the example of $so(5)$ put forward in section 7.2. We study the appearance of extra massive vacua that occur on $\mathbb{R}^{2,1} \times S^1$, coded in advanced nilpotent orbit theory. We classified classical vacua of $\mathcal{N} = 1^*$ theory on $\mathbb{R}^4$ using nilpotent orbit theory in chapter 2. In this subsection, we wish to stress an important extra feature that comes into play after compactifying the theory on a circle, namely the multiplication of massive vacua through the existence of discrete gauge groups and Wilson lines.

7.4.1 Discrete Gauge Groups and Wilson Lines

We have a space-time equal to $\mathbb{R}^{2,1} \times S^1$, and parametrize the circle by the coordinate θ with period 2π, and we denote by R the radius of the circle. Suppose we fix constant vacuum expectation values $\tilde{\Phi}_j(0)$ for the three $\mathcal{N} = 1$ adjoint chiral multiplets ($j = 1, 2, 3$). Moreover, we have them satisfy $su(2)$ commutation relations, as required for constant scalar field configurations to obey the F-term equations of motion. Let us further suppose that the chosen $su(2)$ algebra has a discrete centralizer (equal by assumption to the component group of the centralizer). We therefore have a discrete unbroken gauge group.

It should be clear that a discrete component group permits discrete Wilson lines upon compactification. Suppose that a discrete centralizer of the $sl(2)$ triple contains a non-trivial element $e^{2\pi i a}$ with a an element in the Lie algebra $\mathfrak{g}$ of the gauge group. Then we can propose semi-classical configurations that are new to the theory compactified on the circle, compared to the theory on $\mathbb{R}^4$. These configurations are:[10]

$$\tilde{\Phi}_j(\theta) = \exp(ia\theta)\tilde{\Phi}_j(0)\exp(-ia\theta)\,. \tag{7.9}$$

The gauge field component along the circle is fixed to be the constant $A_\theta = \frac{1}{R}a$. These configurations are covariantly constant, since the gauge covariant derivative is given by $D_\theta\tilde{\Phi}_j =$

[10]Note that we satisfy ordinary boundary conditions. Interesting boundary conditions twisted by outer automorphisms can be imposed for gauge algebras of type A, D and E_6. See e.g. [113, 76].

$\partial_\theta \tilde{\Phi}_j - i[a, \tilde{\Phi}_j]$, and they are periodic by the fact that the group element $e^{2\pi i a}$ belongs to the centralizer of $\tilde{\Phi}_j(0)$:

$$\begin{aligned} D_\theta \tilde{\Phi}_j(\theta) &= 0 \\ \tilde{\Phi}_j(\theta + 2\pi) &= \tilde{\Phi}_j(\theta)\,. \end{aligned} \tag{7.10}$$

By construction these configurations satisfy the F-term equations of motion. Thus, on $\mathbb{R}^{2,1} \times S^1$, the unbroken discrete gauge group gives rise to a larger set of semi-classical configurations. A formal, non-periodic gauge transformation transforms the solutions $\tilde{\Phi}_j(\theta)$ with non-zero Wilson line into the configurations $\tilde{\Phi}_j(0)$ with zero Wilson line. Needless to say, these configurations remain physically distinct on S^1. Furthermore, true gauge transformations with constant parameter transform the constant gauge field a within a given conjugacy class. Thus, for each non-trivial conjugacy class in the discrete gauge group, we find a new semi-classical configuration on the circle.

For a purely discrete centralizer, the above discussion is complete. When there are both continuous identity components and a discrete component group, the analysis requires more care. Note for instance that the role of the component group can also be to exchange continuous factors in the centralizer, as discussed in detail in chapter 2, or to break an abelian factor in the centralizer as illustrated there and in section 7.2.

7.4.2 The Semi-Classical Vacua for G_2

We will discuss in greater detail an example theory that illustrates the above configurations neatly, namely $\mathcal{N} = 1^*$ theory with gauge algebra G_2. We start out with a description of our semi-classical expectations. We will see that G_2 is a good testing ground for the above general discussion. We perform semi-classical limits on the low-energy effective potential, and compare the results to our semi-classical expectations for the gauge theory. The extra configurations described above will indeed appear as solutions. We conclude with a duality diagram for the vacua, a point of monodromy, and other findings on the gauge theory and integrable system that are of interest.

Firstly, let's recall the classification of semi-classical configurations for $\mathcal{N} = 1^*$ theory with gauge group G_2 on $\mathbb{R}^4$. The group G_2 is both connected and simply-connected. For the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^4$, we classify semi-classical configurations by enumerating embeddings $\tilde{\Phi}_j : \mathfrak{sl}_2 \to G_2$, which are in one-to-one correspondence with nilpotent orbits of the Lie algebra G_2. Again, we apply the classification theory of Bala-Carter [154, 155] and Sommers [35]. We pause for a while to explain how this classification is obtained.

Bala-Carter Theory for Nilpotent Orbits

Suppose we want to find the nilpotent orbits of a Lie algebra $\mathfrak{g}$. The Bala-Carter theorem states that this is equivalent to finding the pairs $(\mathfrak{l}, \mathfrak{p}_\mathfrak{l})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_\mathfrak{l}$ is a distinguished parabolic subalgebra of $[\mathfrak{l}, \mathfrak{l}]$. In order to fully understand this statement, we recall three useful definitions and properties:

- A parabolic subalgebra of $\mathfrak{g}$ is a subalgebra which is conjugate to a subalgebra of the form $\mathfrak{p}_J$ where J is a set of simple roots, and where $\mathfrak{p}_J$ is generated by

(a) The Cartan subalgebra ;

(b) The root spaces corresponding to the root system $\langle J \rangle$ created by J ;

(c) The root spaces corresponding to all other positive roots.

We have that $\mathfrak{p}_J$ and $\mathfrak{p}_{J'}$ are conjugate if and only if $J = J'$.

- We can decompose a parabolic subalgebra $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{n}_J$, where the part generated by points (a) and (b) above is the Levi subalgebra $\mathfrak{l}_J$, and the part generated by point (c) is the nilradical $\mathfrak{n}_J$. The algebras $\mathfrak{l}_J$ and $\mathfrak{l}_{J'}$ are conjugate if and only if $\langle J \rangle$ and $\langle J' \rangle$ are Weyl-conjugate.

- A parabolic subalgebra $\mathfrak{p}_J = \mathfrak{l}_J \oplus \mathfrak{n}_J$ is *distinguished* if and only if $\dim \mathfrak{l}_J = \dim \mathfrak{n}_J / [\mathfrak{n}_J, \mathfrak{n}_J]$.

Let's apply this to G_2. In table 7.4 we compute, for the 4 conjugacy classes of parabolic subalgebras, the dimensions of the corresponding Levi subalgebra and of the nilradical. This gives the list of Levi subalgebras, which is the first step of the classification. The second step is to find, for each Levi subalgebra, the distinguished parabolic subalgebras of $[\mathfrak{l}_J, \mathfrak{l}_J]$. This is trivial for 0, $\tilde{A}_1$ and A_1, in which one can check that there is exactly one distinguished parabolic subalgebra, while for G_2 we use again table 7.4 in which we read that there are two distinguished parabolic subalgebras.

J	dim $\mathfrak{l}_J$	dim $\mathfrak{n}_J$	dim $\mathfrak{n}_J/[\mathfrak{n}_J, \mathfrak{n}_J]$	$[\mathfrak{l}_J, \mathfrak{l}_J]$
$\emptyset$	2	6	2	0
$\{\alpha_1\}$	4	5	4	$\tilde{A}_1$
$\{\alpha_2\}$	4	5	2	A_1
$\{\alpha_1, \alpha_2\}$	14	0	0	G_2

Table 7.4: The 4 (conjugacy classes of) parabolic subalgebras of G_2, which are in one-to-one correspondence with subsets of the set of simple roots. We read that a given parabolic subalgebra is distinguished if and only if the numbers in the second and fourth columns are equal.

The first classification results in five nilpotent orbits, exhibited in table 7.5. They correspond to the four Levi subalgebras of G_2 (classified up to Weyl group equivalence). One of these Levi subalgebras, namely G_2, has two distinguished parabolic subalgebras and gives birth to two orbits. We conclude that there are 5 nilpotent orbits in the G_2 Lie algebra, which are summarized in table 7.5.

J	Name and Number
$\emptyset$	$0 \to 1$ orbit
$\{\alpha_1\}$	$\tilde{A}_1 \to 1$ orbit
$\{\alpha_2\}$	$A_1 \to 1$ orbit
$\{\alpha_1, \alpha_2\}$	$G_2 \to 2$ orbits called G_2 and $G_2(a_1)$

Table 7.5: The 4 (conjugacy classes of) parabolic subalgebras of G_2, which are in one-to-one correspondence with subsets of the set of simple roots.

Now we use the generalization of the Bala-Carter theorem by Sommers on this example. As a result, we find that for G_2 there are 7 such conjugacy classes, captured in table 7.6. The centralizer of orbit $G_2(a_1)$ (which is a discrete group) has 3 conjugacy classes, and is in fact S_3, while the other orbits have trivial component group.

W-classes of J	$[\mathfrak{l}_J, \mathfrak{l}_J]$	Distinguished	Orbit	C. C.	Comm.
$\emptyset$	0	1	1	1	G_2
$\{\alpha_0\}$, $\{\alpha_2\}$	A_1	Principal A_1	A_1	1	$\tilde{A}_1$
$\{\alpha_1\}$	$\tilde{A}_1$	Principal $\tilde{A}_1$	$\tilde{A}_1$	1	A_1
$\{\alpha_0, \alpha_1\}$	$A_1 + \tilde{A}_1$	Principal $A_1 + \tilde{A}_1$	$G_2(a_1)$	(12)	1
$\{\alpha_0, \alpha_2\}$	A_2	Principal A_2	$G_2(a_1)$	(123)	1
$\{\alpha_1, \alpha_2\}$	G_2	G_2	G_2	1	1
		$G_2(a_1)$	$G_2(a_1)$	1	1

Table 7.6: The 7 classes of pairs (X, C) where O is a nilpotent orbit and C is a conjugacy class in the component group of the centralizer of O. We tabulate the (derived algebra of the) pseudo-Levi subalgebra, its distinguished orbits, their name, the conjugacy class and the reductive part of the Lie algebra commutant. The discrete centralizer for the orbit $G_2(a_1)$ is the group S_3.

Finally, we can apply our reasoning on the multiplication of semi-classical vacua when we compactify the $\mathcal{N} = 1^*$ theory on $\mathbb{R}^{2,1} \times S^1$ with gauge group G_2. For each nilpotent orbit for which we have a single conjugacy class in the component group, we apply the same reasoning as on $\mathbb{R}^4$ based on the idea that we obtain pure $\mathcal{N} = 1$ super Yang-Mills theories with a number of massive vacua equal to the dual Coxeter number of the gauge group. We find $4+2+2+1 = 9$ massive vacua in this manner. Moreover, for the nilpotent orbit $G_2(a_1)$, we have three conjugacy classes in the discrete centralizer S_3, and we therefore expect 3 vacua. We therefore find a total of

$$4 + 2 + 2 + 1 + 3 = 12 \tag{7.11}$$

massive vacua for G_2.

Similarly, for the semi-classical limits, we have the following expectations. The trigonometric system will be determined by the gauge group breaking pattern, and more specifically by the root system associated to the breaking. We therefore may expect a trigonometric system of type $G_2, A_2, A_1 + \tilde{A}_1, \tilde{A}_1, A_1$ and none at all in the above cases (read from bottom to top). The A_1 and $\tilde{A}_1$ cases are cases of oblique confinement, and the case of trivial orbit, where the full G_2 gauge group remains unbroken, corresponds to confining vacua. The first three cases are Higgsed vacua, possibly with non-trivial Wilson lines corresponding to the non-trivial conjugacy classes in the component group. We distinguish three different cases, namely zero Wilson line, a Wilson line 2-cycle and a Wilson line 3-cycle since these are the conjugacy classes of the S_3 component group. Note that the two Higgs vacua associated to the G_2 orbit and $G_2(a_1)$ orbit with trivial conjugacy class share the same symmetry breaking pattern. Although the vacuum expectation values $\tilde{\Phi}_j$ are different, the integrable system may not distinguish them. Taking this last subtlety into account, we can predict that the integrable system has 11 extrema, which are recovered from the semi-classical limit in section 7.4.3 and found numerically in subsection 7.4.4.

7.4.3 The Elliptic Integrable System and the Semi-classical Limits

In this subsection, we explicitly calculate the semi-classical limits on the low-energy effective superpotential and compare the results to our expectations. The effective superpotential that we work with is

$$\mathcal{W}_{G_2,tw}(Z) = \sum_{\alpha\in\Delta^+_{\text{long}}} \wp(\alpha\cdot Z) + \frac{1}{3}\sum_{\alpha\in\Delta^+_{\text{short}}} \wp_3(\alpha\cdot Z)\,. \tag{7.12}$$

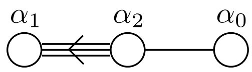


Figure 7.5: The Dynkin diagram of the affine algebra $G_2^{(1)}$.

The co-roots $\alpha_1^\vee = -3\epsilon_2$ and $\alpha_2^\vee = \epsilon_1 + 2\epsilon_2$ have length squared equal to 6 and 2 respectively. We deduce the fundamental co-weights $\pi_1^\vee = 3\epsilon_1$ and $\pi_2^\vee = 2\epsilon_1 + \epsilon_2$ and the fundamental weights $\pi_1 = \epsilon_1$ and $\pi_2 = 2\epsilon_1 + \epsilon_2$. Finally, the Weyl group has 12 elements, it is precisely

$$\left\{(r_1r_2)^n(r_1)^\epsilon | 0 \le n \le 5 \quad \text{and} \quad \epsilon = 0,1\right\}\,, \tag{7.13}$$

where r_i are simple Weyl reflections. One of those elements, $(r_1r_2)^2r_1$, exchanges ϵ_1 and ϵ_2, meaning that extrema with z_1 and z_2 exchanged are considered equivalent. A global sign flip is also allowed by $(r_1r_2)^3 = -1$. Finally $(r_1r_2)^3r_1$ acts as

$$\begin{cases} \epsilon_1 \to \epsilon_1 + \epsilon_2 \\ \epsilon_2 \to -\epsilon_2\,. \end{cases} \tag{7.14}$$

The final litany of useful facts includes that both the center of G_2 and its group of outer automorphisms are trivial. The algebra G_2 is its own Langlands dual. The dual of the (non-twisted) affine algebra $G_2^{(1)}$ on the other hand is $\left(G_2^{(1)}\right)^\vee = D_4^{(3)}$. This last algebra has two short simple roots and one long simple root whose length squared is three times larger. The co-marks of $g_2^{(1)}$ are $(1,2,1)$.

We can parametrize $Z = 3(z_1\epsilon_1 + z_2\epsilon_2)$ such that

$$\begin{aligned} \mathcal{W}_{G_2,tw}(z_1,z_2) &= \wp\left(3z_1 - 3z_2\right) + \wp\left(3z_1\right) + \wp\left(3z_2\right) \\ &\quad + \frac{1}{3}\left[\wp_3\left(z_1+z_2\right) + \wp_3\left(2z_1 - z_2\right) + \wp_3\left(-z_1 + 2z_2\right)\right]\,, \end{aligned} \tag{7.15}$$

or use the alternative parameterization $Z = Z_1\pi_1 + Z_2\pi_2 = (Z_1 + 2Z_2)\epsilon_1 + Z_2\epsilon_2$. The link is

$$\begin{cases} z_1 = \frac{1}{3}(Z_1 + 2Z_2) \\ z_2 = \frac{1}{3}Z_2 \end{cases} \qquad \begin{cases} Z_1 = 3z_1 - 6z_2 \\ Z_2 = 3z_2\,, \end{cases} \tag{7.16}$$

and the explicit form of the superpotential is then

$$\begin{aligned} \mathcal{W}_{g2,tw}(Z_1,Z_2) &= \wp(Z_2) + \wp(Z_1+Z_2) + \wp(Z_1+2Z_2) \\ &\quad + \frac{1}{3}\left[\wp_3(Z_1/3) + \wp_3(Z_1/3 + Z_2) + \wp_3(2Z_1/3 + Z_2)\right]\,, \end{aligned} \tag{7.17}$$

with $Z_i = X_i + \tau Y_i$. We still have to specify the periodicities and identifications. In the ω_1 direction, we identify by shifts by the weight lattice, and in the ω_2 direction by the co-weight lattice. This implies :

$$(z_1, z_2) \sim \left(z_1 + \frac{2}{3}\omega_1, z_2\right) \sim \left(z_1, z_2 + \frac{2}{3}\omega_1\right) \tag{7.18}$$

and

$$(z_1, z_2) \sim (z_1 + 2\omega_2, z_2) \sim (z_1, z_2 + 2\omega_2) \sim \left(z_1 - \frac{2}{3}\omega_2, z_2 + \frac{2}{3}\omega_2\right) . \tag{7.19}$$

The Weyl group action yields the further equivalences

$$(z_1, z_2) \sim (z_2, z_1) \sim (-z_1, -z_2) \sim (z_1, z_1 - z_2) . \tag{7.20}$$

We note that the group G_2 has trivial center.

Below, we distinguish between the trigonometric or Higgs limits, in which the leading trigonometric system is of rank two, the oblique limits, in which it is of rank one, and the affine Toda, or confining limit.

The Higgsed Limits

Firstly, we describe the limit for the Higgs vacuum, the 2-cycle vacuum and the 3-cycle vacuum. The superpotential becomes the trigonometric system corresponding to the pseudo-Levi subalgebra.

- **The trigonometric G_2 limit**

  In the first $\tau \to i\infty$ limit, where we take the Wilson line to be $a = (0,0)$ and consequently $Y_1 = 0 = Y_2$, we find the trigonometric G_2 model for the choice of simple roots $J_0 = \{\alpha_1, \alpha_2\}$:

  $$\mathcal{W}_{G_2,tw}(X) \to \mathcal{W}_{\text{trig},G_2} . \tag{7.21}$$

  We find a real extremum. It can be described through zeroes of an orthogonal polynomial [108, 157].

- **The trigonometric A_2 limit**

  In the second limit, corresponding to the 3-cycle conjugacy class and Wilson line $a = (1/3, 0)$ we find the trigonometric A_2 system corresponding to the choice of simple root system $J_0 = \{\alpha_0, \alpha_2\}$. The co-marks give the constraint $1 = Y_0 + 2Y_2 + Y_1$. If we impose $Y_0 = 0 = Y_2$, we find $Y_1 = 1$. We are left with a trigonometric A_2 system corresponding to all the long roots. The extremal positions are therefore given by the equilibria of the trigonometric A_2 integrable system. There is a massive extremum at $(z_1, z_2) = (1/3 + \tau/3, 2/9)$.[11]

[11] The A_2 trigonometric model also allows for massless complexified extrema at zeroth order. However, these extrema do not survive the order q perturbation.

- **Trigonometric** $A_1 + \tilde{A}_1$

  The third Higgs vacuum is associated to the Wilson line $a = (1/2, 0)$, with the choice $J_0 = \{\alpha_0, \alpha_1\}$, and gives rise to the trigonometric $A_1 + \tilde{A}_1$ system (with differing coupling constants). One finds a unique extremum up to equivalences, namely $(z_1, z_2) = (\frac{1}{6} + \frac{\tau}{2}, \frac{1}{6})$.

Remark

We remark that it is the centralizer of the Wilson line group element that determines the non-perturbative contributions to the superpotential in the semi-classical limit. Namely, the allowed monopole charges corresponds precisely to positive roots of the pseudo-Levi subalgebra. We recall that we have two configurations in which the full gauge group is broken, namely the orbit labelled G_2 and the orbit labelled $G_2(a_1)$ with zero Wilson line. In the elliptic integrable system, we only identified one real extremum. The two orbits are distinguished through their scalar adjoint vacuum expectation values.

The Confining Limit

If we pick zero simple roots, we obtain the affine Toda potential for the algebra $D_4^{(3)} = (G_2^{(1)})^\vee$

$$\mathcal{W}_{G_2,tw}(x_1 + \frac{\tau}{4}, x_2 + \frac{\tau}{12}) = q^{\frac{1}{4}} \left(e^{-6\pi i x_1} + e^{6\pi i x_2} + 3e^{6\pi i (x_1 - 2x_2)} \right) + \dots . \tag{7.22}$$

The associated simple roots are α_0, α_2 and $3\alpha_1$. The extrema of the affine Toda potential can be obtained analytically (see e.g. [65]).

The Oblique Limits

Let's turn to the limits with partial breaking of the gauge algebra through adjoint vacuum expectation values.

- **The Oblique Limit** $J_0 = \{1\}$

  The limit $J_0 = \{1\}$ corresponds to the orbit $\tilde{A}_1$ with unbroken gauge group A_1. We first determine the non-perturbative corrections to the leading coordinate Z_1, and find

$$(Z_1, Z_2) = \left(\frac{1}{2} - \frac{4i}{3\pi} e^{2i\pi\delta_2} q^{\frac{1}{4}} + \delta_1, \frac{\tau}{4} + \delta_2 \right) , \tag{7.23}$$

  and a final stabilized potential (at $\delta_1 = 0$)

$$\mathcal{W}_{G_2,tw}(\delta_2) \quad = \quad \pi^2 (1 + 4e^{-4i\pi\delta_2} q^{\frac{1}{2}} - \frac{20}{3} e^{4i\pi\delta_2} q^{\frac{1}{2}} + O(q)) . \tag{7.24}$$

  We can solve for the fluctuation δ_2 using this superpotential, and then find the superpotential at equilibrium to order $q^{\frac{1}{2}}$

$$\mathcal{W}_{G_2,tw} \quad = \quad \pi^2 (1 - 8i\sqrt{\frac{5}{3}}\sqrt{q} + \dots) . \tag{7.25}$$

- **The Oblique Limit** $J_0 = \{2\}$

  If we put $Y_1 = \frac{3}{4}, Y_2 = 0$, we get stabilization at level $q^{\frac{3}{2}}$. The first orders in the expansion of the coordinate Z_2 are given by

$$Z_2 \;=\; \frac{1}{2} - \frac{4ie^{-2i\pi\delta X_1}\sqrt[4]{q}}{\pi} + \frac{48ie^{-4i\pi\delta X_1}q^{\frac{1}{2}}}{\pi} + \dots\,, \tag{7.26}$$

  to finally find stabilization for the coordinate Z_1 at

$$Z_1 \;=\; \frac{3\tau}{4} - \frac{i\log 3}{4\pi} \quad \text{or} \quad \frac{3\tau}{4} - \frac{1}{4} - \frac{i\log 3}{4\pi}\,. \tag{7.27}$$

  The resulting superpotentials in the two inequivalent vacua are

$$\mathcal{W}_{G_2,tw} \;=\; \pi^2(-1 + 312q \pm 5832q^{\frac{3}{2}} + \dots)\,. \tag{7.28}$$

  The stabilizing potential for Z_1 arises at sixth order in the non-perturbative expansion parameter $q^{\frac{1}{4}}$.

Remark

One can ask about the oblique limit $J_0 = \{0\}$. We have found no choice of Y_i consistent with the condition $Y_0 = 0$ such that the second coordinate stabilizes. We note that the choices $J_0 = \{0\}$ and $J_0 = \{2\}$ are Weyl equivalent in the horizontal algebra, but inequivalent in the affine algebra. They also are inequivalent as limiting choices. In this example, using the pseudo-Levi subalgebra classification scheme as a starting point for the semi-classical limits works, if only because another, inequivalent limit, does not stabilize.

7.4.4 Results Based on Numerics

In this subsection, we present results based on numerical analyses performed at finite coupling τ. The main strategy is to combine a random exploration of the parameter space with the requirement that vacua should form closed multiplets under S_3 and T dualities. Our numerics is in essence based on the FindMinimum procedure of Mathematica, applied to the logarithm of the positive definite real potential of the gauge theory. Let us first explain how these dualities can be implemented numerically on a vacuum that we know at large τ (by which we always mean the semi-classical regime $\tau \to i\infty$):

- T-duality is performed by taking the vacuum at large τ and changing continuously $\tau \mapsto \tau + 1$ on a straight line.

- For S_3-duality, we first track the vacuum to the self-dual point $\tau_{sd} = i/\sqrt{3}$, then use the exact Langlands S_3-duality formula (see later, equation (7.32)) to S_3-dualize it, and finally bring it back to large τ.

Note that it is crucial that τ be large to T-dualize, because of potential points of monodromy at finite gauge coupling.

Finding the Vacua

Using our numerical procedure, it is easy to find the Higgs vacuum on the real axis; we label it H. Taking its S_3-dual as explained above, one obtains the confining vacuum dubbed C_0. When we T-dualize the confining vacua we obtain three more vacua, C_1, C_4, C_5, for a quadruplet of confining vacua at large τ. More subtle is the following fact. Consider these vacua brought down to the self-dual value of the gauge coupling τ_{sd}. We call T_{sd} duality the operation

$$T_{sd} : \tau_{sd} = i/\sqrt{3} \mapsto \tau_{sd} + 1 \tag{7.29}$$

continuously along a straight line in the upper-half plane $\mathcal{H}$. If we apply this transformation to the confining vacuum C_0, we find that we need to repeat it six times before falling on this confining vacuum once more. We thus find a sextuplet of T_{sd}-duality that we denote $(C_0, C_1, C_2, C_3, C_4, C_5)$. This indicates a point of monodromy that lies above the self-dual point.[12] The point of monodromy is located around $\tau_M \sim 1.440672920416i$, and all of these digits are significant. At the self-dual point, we can analytically check that S_3-duality acts as $S_3(C_1) = C_4, S_3(C_2) = C_5$ and $S_3(C_3) = C_3$. Moreover, if we bring up the two extra vacua (C_2, C_3) that complete the sextuplet to larger τ, they behave as a doublet under T-duality. These seven vacua obtained from the Higgs are represented on the right of figure 7.6.

In addition to these, we also find two extrema which are S_3-duality and T-duality singlets, and also two S_3-singlets (labelled J_1, J_2) which are T-dual (and T_{sd}-dual) to each other. This completes the duality web summarized in figure 7.6.

Identification with the Semi-classical Limits

We have obtained a total of eleven extrema, as expected from section 7.4.3. We can be more precise and match each T-multiplet with its corresponding limiting integrable system, using the value of the superpotential when necessary.

The singlets correspond to the 2- and 3-cycle semi-classical vacua, while the doublet of the duality group matches the semi-classical $J_0 = \{1\}$ extrema. The confining quadruplet is easily matched to the semi-classical solutions. The semi-classical origin of (C_2, C_3) is the choice $J_0 = \{2\}$. The numerical evidence we obtained for this last identification is limited to the first two coefficients in the superpotential (7.28).

Numerical values

Finally, let us provide a few concrete numbers of our simulations for easier reproducibility. The (z_1, z_2) positions of the numerical extrema are approximately given in the tables below, where the first entry is real part of z_1 and the second entry is the imaginary part of z_1 expressed in

[12] See [1] for a more gentle introduction to points of monodromy.

units of the purely imaginary value of τ.

Vacuum	Positions at $\frac{i}{\sqrt{3}}$	Positions at $\frac{5i}{2}$
H	$\{0.22754, 0., 0.03944, 0.\}$	$\{0.22738, 0., 0.03954, 0.\}$
H2	$\{0.16667, 0.5, 0.16667, 0.\}$	$\{0.16667, 0.5, 0.16667, 0.\}$
H3	$\{0.11111, 0.33333, 0.22222, 0.33333\}$	$\{0.11111, 0.33333, 0.22222, 0.33333\}$
C0	$\{0., 0.26698, 0., 0.41565\}$	$\{0., 0.26222, 0., 0.41795\}$
C1	$\{0.2727, 0.2679, 0.4226, 0.4103\}$	$\{0.25257, 0.2631, 0.41731, 0.41817\}$
C2	$\{0.5594, 0.3047, 0.8509, 0.4275\}$	$\{0.5, 0.33139, 0.83333, 0.42738\}$
C3	$\{0.86305, 0.33333, 1.26486, 0.46124\}$	$\{0.83812, 0.33333, 1.25239, 0.42829\}$
C4	$\{0.22607, 0.36197, 0.7085, 0.45614\}$	$\{0.16667, 0.40236, 0.66667, 0.4875\}$
C5	$\{0.60603, 0.39877, 0.18344, 0.47574\}$	$\{0.58591, 0.40356, 0.1686, 0.4884\}$
J1	$\{0.89497, 0.60134, 0.64682, 0.45821\}$	$\{0.87587, 0.58097, 0.62587, 0.41972\}$
J2	$\{0.98015, 0.20845, 0.56164, 0.73199\}$	$\{0.9592, 0.24694, 0.54253, 0.75236\}$

(7.30)

The superpotentials in these vacua are

Vacuum	Superpotential at $\frac{i}{\sqrt{3}}$	Superpotential at $\frac{5i}{2}$
H	271.5202972	256.6097930
H2	26.54254786	19.73924450
H3	26.54254786	19.73924450
C0	-218.4352014	-22.81452733
C1	$42.47856497 - 33.32941024i$	$-19.56246124 - 2.892724428i$
C2	$10.60653076 + 33.32941024i$	-9.869136924
C3	26.54254786	-9.869143660
C4	$10.60653076 - 33.32941024i$	-17.01826338
C5	$42.47856497 + 33.32941024i$	$-19.56246124 + 2.892724428i$
J1	$26.54254786 + 13.36027231i$	$9.869700650 + 0.03957060700i$
J2	$26.54254786 - 13.36027231i$	$9.869700650 - 0.03957060700i$

(7.31)

7.4.5 Langlands Duality and the Duality Diagram

Aside from the simply laced Lie algebras of A, D and E type, there are three more algebras that are mapped to themselves under Langlands duality. These are B_2, G_2 and F_4. The twisted elliptic integrable systems with appropriate couplings are indeed Langlands self-dual [1], namely, they permit the symmetry $S_\nu : \tau \to -\frac{1}{\nu\tau}$, where ν is the ratio of the length squared of the long versus the short roots. The invariance under S_ν translates into a relation involving the superpotentials evaluated at different positions X_i, including a shift. Explicitly, the fact that G_2 is invariant under $S_3 : \tau \to -\frac{1}{3\tau}$ duality reads

$$\mathcal{W}_{G_2,tw}(X_1, X_2; \tau) = \frac{1}{3\tau^2}\mathcal{W}_{G_2,tw}\left(\frac{X_1+X_2}{3\tau}, \frac{2X_1-X_2}{3\tau}; -\frac{1}{3\tau}\right) + 2\pi^2\left[3E_2(3\tau) - E_2(\tau)\right] \, . \quad (7.32)$$

As was the case for the $so(5)$ integrable system (see section 4), the shift resulting from the S_ν duality transformation can be identified with the superpotential in one of the vacua. The latter property allows for the realization of duality symmetries as permutations on the list of extremal superpotential values.

We have determined these permutations numerically (as reviewed above), and sum up the action of S_3, T and T_{sd} in the diagram shown in figure 7.6. This diagram demonstrates the importance of specifying the path followed in the moduli space while performing a duality: note for instance that in the diagram, S_3T_{sd} has order 7 while the order of the more standard operation S_3T is 6, as a consequence of monodromies. In [1] one can find other examples of generalized duality groups that are generated by points of monodromy.

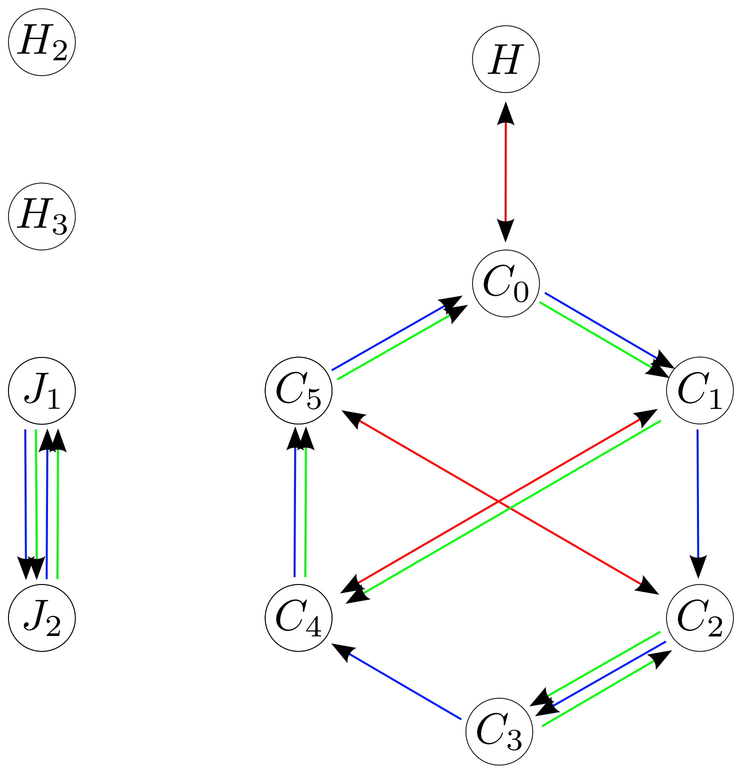


Figure 7.6: A diagram of dualities of G_2, below the point of monodromy. The blue arrows represent the transformation $T_{sd} : \tau \to \tau + 1$ below the point of monodromy, while the green arrows represent the transformation $T : \tau \to \tau + 1$ above the point of monodromy. To identify vacua at different τ we use the convention that the branch cut is on the left of the monodromy point ($\mathbb{R}_-$ direction). The red arrows indicate the action of S_3-duality (7.32) at the self-dual point $\tau = \frac{i}{\sqrt{3}}$. The absence of a given arrow indicates invariance of the vacuum under the corresponding transformation. Note that the order of S_3T_{sd} is 7 while the order of S_3T is 6.

Finally, we make a few remarks on the exact values of the superpotential in a number of vacua. The superpotentials in the Higgs vacua with non-zero Wilson line are identical. They are equal to

$$\mathcal{W}_{H2} \quad = \quad 2\pi^2(\theta_3(q^2)\theta_3(q^6) + \theta_2(q^2)\theta_2(q^6))^2 \,. \tag{7.33}$$

This is the theta series of the direct sum of 2 copies of a hexagonal lattice. It generates the (1-dimensional) space of modular forms of weight 2 for the congruence subgroup $\Gamma_0(3)$. Many further analytical statements can be made about the exact values of the superpotential. As an example, we have the closed form expression

$$26.54254786... = \frac{9\Gamma\left(\frac{1}{3}\right)^6}{8 \times 2^{\frac{2}{3}}\pi^2}\,, \tag{7.34}$$

for this particular entry in table (7.31) of values of the superpotential. It will be interesting to classify the superpotential values into (vector valued) modular forms (potentially with non-analyticity in the upper half plane) of $\Gamma_0(3)$ or the full Hecke group.

7.5 The $so(5)$ Massless Branch

In this section, we tie up a loose end. In section 7.2 we analyzed semi-classical limits for the B_2 twisted elliptic integrable system, and we found a single massless branch of complex dimension one. We wish to characterize this branch more precisely, including at finite coupling τ. We also exhibit its duality and global properties in the different theories associated to the gauge algebra $so(5)$.

7.5.1 The Local Description of the Massless Branch

We show that a massless branch exists at all couplings by a brute force analysis. We postulate that the superpotential on the massless branch is equal to $e_1(q)$ (as we found in perturbation theory in section 7.2). We will also consider the two equations that follow from the fact that we are studying an extremum of the superpotential. These equations give rise to three constraint equations in terms of two unknowns, for a single massless branch of complex dimension one. This doubly overdetermined system will have a simple solution which is the description of the massless branch. Before we get to the simple end result, we plough through some elliptic function identities. Firstly, we recall the definition of the Weierstrass function evaluated at half-periods

$$\wp(\omega_i;\tau) = e_i(q) \tag{7.35}$$

and note that we have the equality

$$e_1(q) = -\frac{2\pi^2}{3}E_{2,2}(q) = -\frac{2\pi^2}{3}(E_2(q) - 2E_2(q^2))\,, \tag{7.36}$$

as well as the identities

$$\begin{aligned} \wp_2(z;\tau) &= 4\wp(2z;2\tau) + e_1 \\ &= -e_1 + \frac{1}{4}\left(\frac{\wp'(z;\tau)}{\wp(z;\tau) - e_1}\right)^2 \\ \wp_2'(z;\tau) &= 8\wp'(2z;2\tau)\,. \end{aligned} \tag{7.37}$$

We again describe the superpotential and its derivatives algebraically using the variables[13]

$$\mathcal{X}_i = \wp(z_i;\tau) \qquad \text{and} \qquad \mathcal{Y}_i = \wp'(z_i;\tau)\,, \tag{7.38}$$

[13]These variables describe faithfully the vacua of the $SO(5)_+$ theory, by which we mean that for any $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2) \in \mathbb{C}^4$ there is exactly one vacuum of the $SO(5)_+$ theory that satisfies (7.38). In the $Spin(5)$ and $SO(5)_-$ theories there are two such vacua, namely (z_1, z_2) and $(z_1 + 2\omega_2, z_2 + 2\omega_2)$. Moreover, both in the $Spin(5)$ and in the $SO(5)_-$ theory (7.38) is not a well defined functional of a given vacuum because of the identification $(z_1, z_2) \sim (z_1 + \omega_1, z_2 + \omega_1)$ in $Spin(5)$ and $(z_1, z_2) \sim (z_1 + \omega_1 + 2\omega_2, z_2 + \omega_1)$ in $SO(5)_-$. These subtleties will be taken care of in subsection 7.5.3.

for $i = 1, 2$. The value of the superpotential, $\mathcal{W}_{B_2,tw}(Z;\tau) = e_1(\tau)$, translates into the equation

$$2\left(\frac{\mathcal{Y}_1-\mathcal{Y}_2}{\mathcal{X}_1-\mathcal{X}_2}\right)^2+2\left(\frac{\mathcal{Y}_1+\mathcal{Y}_2}{\mathcal{X}_1-\mathcal{X}_2}\right)^2+\left(\frac{\mathcal{Y}_1}{\mathcal{X}_1-e_1}\right)^2+\left(\frac{\mathcal{Y}_2}{\mathcal{X}_2-e_1}\right)^2=16(e_1+\mathcal{X}_1+\mathcal{X}_2)\,. \tag{7.39}$$

We also need the addition formula for the derivative of the Weierstrass function:

$$\wp'(u+v)=\frac{\left[\frac{1}{2}g_3(\wp'(u)-\wp'(v))+(\wp'(v)\wp(u)^2+\frac{1}{4}g_2\wp'(u))(\wp(u)+3\wp(v))\right]-[u\leftrightarrow v]}{(\wp(u)-\wp(v))^3}\,. \tag{7.40}$$

It is convenient to write this in a more symmetric way, where all the derivatives are isolated on the left-hand side :

$$\frac{\wp'(u+v)+\wp'(u-v)}{\wp'(u)}=\frac{g_2(\wp(u)+3\wp(v))+4g_3-4\wp(v)^2(3\wp(u)+\wp(v))}{2(\wp(u)-\wp(v))^3}\,, \tag{7.41}$$

which we can use to express the derivative of the twisted Weierstrass function as

$$\frac{\wp_2'(u)}{\wp'(u)}=\frac{8e_1^3-3e_1\left(g_2-4\wp(u)^2\right)-\wp(u)\left(g_2+4\wp(u)^2\right)-4g_3}{4(e_1-\wp(u))^3}\,. \tag{7.42}$$

Using these relations, the three equations describing the massless branch become

$$\begin{aligned}\frac{4(4\mathcal{X}_1^3+4\mathcal{X}_2^3-g_2(\mathcal{X}_1+\mathcal{X}_2)-2g_3)}{(\mathcal{X}_1-\mathcal{X}_2)^2}+\frac{4\mathcal{X}_1^3-g_2\mathcal{X}_1-g_3}{(\mathcal{X}_1-e_1)^2}& \\ +\frac{4\mathcal{X}_2^3-g_2\mathcal{X}_2-g_3}{(\mathcal{X}_2-e_1)^2}-16(e_1+\mathcal{X}_1+\mathcal{X}_2)&=0\end{aligned} \tag{7.43}$$

$$\begin{aligned}\frac{g_2(\mathcal{X}_1+3\mathcal{X}_2)+4g_3-4\mathcal{X}_2^2(3\mathcal{X}_1+\mathcal{X}_2)}{(\mathcal{X}_1-\mathcal{X}_2)^3}& \\ +\frac{8e_1^3-3e_1\left(g_2-4\mathcal{X}_1^2\right)-\mathcal{X}_1\left(g_2+4\mathcal{X}_1^2\right)-4g_3}{4(e_1-\mathcal{X}_1)^3}&=0\end{aligned} \tag{7.44}$$

$$\begin{aligned}\frac{g_2(\mathcal{X}_2+3\mathcal{X}_1)+4g_3-4\mathcal{X}_1^2(3\mathcal{X}_2+\mathcal{X}_1)}{(\mathcal{X}_2-\mathcal{X}_1)^3}& \\ +\frac{8e_1^3-3e_1\left(g_2-4\mathcal{X}_2^2\right)-\mathcal{X}_2\left(g_2+4\mathcal{X}_2^2\right)-4g_3}{4(e_1-\mathcal{X}_2)^3}&=0\,.\end{aligned} \tag{7.45}$$

Finally, we express the Eisenstein series g_2 and g_3 of weight 4 and 6 in terms of the half-period values e_i using the relations $g_2 = 2(e_1^2 + e_2^2 + e_3^2)$ and $g_3 = 4e_1e_2e_3$ to obtain

$$\frac{(2e_1-\mathcal{X}_1-\mathcal{X}_2)^2}{(e_1-\mathcal{X}_1)(e_1-\mathcal{X}_2)(\mathcal{X}_1-\mathcal{X}_2)^2}P_3(\mathcal{X}_1,\mathcal{X}_2,e_1,e_2)=0 \tag{7.46}$$

$$\frac{(2e_1-\mathcal{X}_1-\mathcal{X}_2)}{(e_1-\mathcal{X}_1)^2(\mathcal{X}_1-\mathcal{X}_2)^3}P_4(\mathcal{X}_1,\mathcal{X}_2,e_1,e_2)=0 \tag{7.47}$$

$$\frac{(2e_1-\mathcal{X}_1-\mathcal{X}_2)}{(e_1-\mathcal{X}_2)^2(\mathcal{X}_2-\mathcal{X}_1)^3}P_4(\mathcal{X}_2,\mathcal{X}_1,e_1,e_2)=0\,, \tag{7.48}$$

where P_3 and P_4 are homogeneous polynomials of degree 3 and 4 respectively. We see that $\mathcal{X}_1 + \mathcal{X}_2 = 2e_1$ is a sufficient condition to be on the massless branch of vacua. Restricting to these solutions – except at special points in the space of couplings, these are the only solutions –, we can parametrize the line by a single complex number $\lambda \in \mathbb{C}^*$ as

$$\mathcal{X}_1 = e_1(\tau) + \lambda\,, \qquad \mathcal{X}_2 = e_1(\tau) - \lambda\,. \tag{7.49}$$

7.5.2 Duality and the Massless Branch

T-duality manifestly leaves the description of the massless branch in terms of the elliptic curve variables invariant, as can be seen from equation (7.49). We analyze Langlands S_2 duality next. In the notation of equation (7.1), the duality formula for $so(5)$ reads [1]

$$\mathcal{W}_{B_2,tw}(z_1,z_2,\tau) = \frac{1}{2\tau^2}\mathcal{W}_{B_2,tw}\left(\frac{z_1+z_2}{2\tau},\frac{z_1-z_2}{2\tau},-\frac{1}{2\tau}\right) + 2e_1(\tau)\,. \tag{7.50}$$

Using the identity

$$e_1\left(-\frac{1}{2\tau}\right) = -2\tau^2 e_1(\tau)\,, \tag{7.51}$$

it can be written in the more symmetric form

$$\mathcal{W}_{B_2,tw}(z_1,z_2,\tau) - e_1(\tau) = \frac{1}{2\tau^2}\left[\mathcal{W}_{B_2,tw}\left(\frac{z_1+z_2}{2\tau},\frac{z_1-z_2}{2\tau},-\frac{1}{2\tau}\right) - e_1\left(-\frac{1}{2\tau}\right)\right]. \tag{7.52}$$

We define the dual elliptic curve variables

$$\begin{aligned}
\mathcal{X}_1' &= \wp\left(\frac{z_1+z_2}{2\tau};-\frac{1}{2\tau}\right) && (7.53)\\
\mathcal{X}_2' &= \wp\left(\frac{z_1-z_2}{2\tau};-\frac{1}{2\tau}\right) && (7.54)\\
\mathcal{Y}_1' &= \wp'\left(\frac{z_1+z_2}{2\tau};-\frac{1}{2\tau}\right) && (7.55)\\
\mathcal{Y}_2' &= \wp'\left(\frac{z_1-z_2}{2\tau};-\frac{1}{2\tau}\right), && (7.56)
\end{aligned}$$

which are related to the original elliptic curve variables (7.38) by

$$\begin{aligned}
-e_1(\tau) - \mathcal{X}_1 - \mathcal{X}_2 + \frac{1}{4}\left(\frac{\mathcal{Y}_1-\mathcal{Y}_2}{\mathcal{X}_1-\mathcal{X}_2}\right)^2 &= \frac{1}{16\tau^2}\left(\frac{\mathcal{Y}_1'}{\mathcal{X}_1'-e_1(-\frac{1}{2\tau})}\right)^2 && (7.57)\\
-e_1(\tau) - \mathcal{X}_1 - \mathcal{X}_2 + \frac{1}{4}\left(\frac{\mathcal{Y}_1+\mathcal{Y}_2}{\mathcal{X}_1-\mathcal{X}_2}\right)^2 &= \frac{1}{16\tau^2}\left(\frac{\mathcal{Y}_2'}{\mathcal{X}_2'-e_1(-\frac{1}{2\tau})}\right)^2. && (7.58)
\end{aligned}$$

The sum of these relations is

$$-2e_1(\tau) - 2\mathcal{X}_1 - 2\mathcal{X}_2 + \frac{\mathcal{Y}_1^2+\mathcal{Y}_2^2}{2(\mathcal{X}_1-\mathcal{X}_2)^2} = \frac{1}{16\tau^2}\left(\frac{\mathcal{Y}_1'}{\mathcal{X}_1'-e_1(-\frac{1}{2\tau})}\right)^2 + \frac{1}{16\tau^2}\left(\frac{\mathcal{Y}_2'}{\mathcal{X}_2'-e_1(-\frac{1}{2\tau})}\right)^2.$$

After these preparations, we will now show that if we choose a point on the dual of the massless branch, namely a point satisfying the equation $\mathcal{X}_1'+\mathcal{X}_2' = 2e_1(-\frac{1}{2\tau})$, that this is consistent with the original variables lying on the original massless branch. Indeed, this equality implies that the sum of the relations becomes

$$2(e_1(\tau)+\mathcal{X}_1+\mathcal{X}_2) - \frac{\mathcal{Y}_1^2+\mathcal{Y}_2^2}{2(\mathcal{X}_1-\mathcal{X}_2)^2} + \frac{(\mathcal{Y}_1')^2+(\mathcal{Y}_2')^2}{4\tau^2(\mathcal{X}_1'-\mathcal{X}_2')^2} = 0\,. \tag{7.59}$$

Taking into account the elliptic curve equation, we can simplify this to

$$\frac{(\mathcal{Y}_1')^2+(\mathcal{Y}_2')^2}{4\tau^2(\mathcal{X}_1'-\mathcal{X}_2')^2} = \frac{3}{4\tau^2}(\mathcal{X}_1'+\mathcal{X}_2') = \frac{3}{2\tau^2}e_1\left(-\frac{1}{2\tau}\right) = -3e_1(\tau)\,, \tag{7.60}$$

and we end up with

$$\frac{(-2e_1+\mathcal{X}_1+\mathcal{X}_2)\,(-2e_1e_2+e_1\mathcal{X}_1+e_1\mathcal{X}_2-2e_2^2+2\mathcal{X}_1\mathcal{X}_2)}{(\mathcal{X}_1-\mathcal{X}_2)^2}=0\,. \tag{7.61}$$

Finally, we see that this equality is implied by the original point being on the original branch $-2e_1+\mathcal{X}_1+\mathcal{X}_2=0$, and we have therefore obtained a non-trivial check of the statement that the massless branch is invariant under S_2-duality.

7.5.3 The Moduli Space of Vacua for the Different Gauge Theories

In this subsection we obtain the global structure of the moduli space of massless vacua for the different theories with $so(5)$ gauge algebra, taking into account various discrete identifications. We also perform a consistency check on these global properties by showing how S_2-duality acts on these moduli spaces, thus completing the results in subsection 7.2.3.

We wish to characterize the branch by extracting the positions z_i from the elliptic curve variables (7.38), and this should be done up to Weyl equivalence. The Weyl group is generated by two reflections: the reflection about α_1 leads to the identification

$$(\mathcal{X}_1,\mathcal{Y}_1,\mathcal{X}_2,\mathcal{Y}_2)\equiv(\mathcal{X}_2,-\mathcal{Y}_2,\mathcal{X}_1,-\mathcal{Y}_1)\,, \tag{7.62}$$

while the reflection about α_2 gives the identification

$$(\mathcal{X}_1,\mathcal{Y}_1,\mathcal{X}_2,\mathcal{Y}_2)\equiv(\mathcal{X}_1,-\mathcal{Y}_1,\mathcal{X}_2,\mathcal{Y}_2)\,. \tag{7.63}$$

This shows that the sign of the variables $\mathcal{Y}_i$ is irrelevant, and we no longer need to keep track of them. The Weyl symmetry therefore implies that we can study the manifold described by the variables $(\mathcal{X}_1,\mathcal{X}_2)$ subject to the identification $(\mathcal{X}_1,\mathcal{X}_2)\equiv(\mathcal{X}_2,\mathcal{X}_1)$. The branch of massless vacua of the $SO(5)_+$ theory, for which there is no additional identification, is described by $\lambda\in\mathbb{C}^*/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is $\lambda\leftrightarrow-\lambda$. This is a sphere with two excised points.

In the $SO(5)_-$ theory we have the additional identification $(z_1,z_2)\equiv(z_1+\omega_1+2\omega_2,z_2+\omega_1)$. On the manifold parametrized by λ it corresponds to $\lambda\equiv\lambda'=\pi^4\theta_4^8(2\tau)/\lambda$. This follows from the observation that if $\wp(z_1)=e_1+\lambda$, then

$$\begin{aligned}\wp(z_1+\omega_1) &= -e_1-(e_1+\lambda)+\frac{\wp'(z_1)^2}{4\lambda^2} &(7.64)\\ &= e_1+\frac{3e_1^2-\frac{1}{4}g_2}{\lambda}+\frac{4e_1^3-g_2e_1-g_3}{\lambda^2} &(7.65)\\ &= e_1+\lambda'\,, &(7.66)\end{aligned}$$

and similarly if $\wp(z_2)=e_1-\lambda$ then $\wp(z_2+\omega_1)=e_1-\lambda'$. Note that the function $\theta_4(\tau)$ doesn't vanish on the upper-half plane,[14] so that $\lambda\mapsto\lambda'$ is a well-defined involution everywhere in the moduli space.

For a given $\lambda\in\mathbb{C}^*$, the $SO(5)_-$ theory has two non-equivalent vacua (z_1,z_2) and $(z_1+2\omega_2,z_2)$ which correspond to this λ. These two vacua are respectively equivalent to $(z_1+\omega_1+2\omega_2,z_2+\omega_1)$ and $(z_1+\omega_1,z_2+\omega_1)$, which are associated to the same λ'. Therefore the branch of massless vacua is a double cover of $\mathbb{C}^*/\mathbb{Z}_2$.

[14] The zeros of $\theta_4(z,\tau)$ are given by $z=n+(m+1/2)\tau$ with $n,m\in\mathbb{Z}$

As for the $Spin(5)$ theory, we also need to take a double cover of the quotiented sphere $\mathbb{C}^*/\mathbb{Z}_2$. For a generic $\lambda \in \mathbb{C}^*$, the two vacua λ and λ' in the $SO(5)_+$ theory are inequivalent. They are mapped by S_2 to inequivalent vacua that share the same dual λ_D, or equivalently the same λ'_D. We see that S_2-duality cancels the cover and the quotient to recover the manifold for $SO(5)_+$ which is just $\mathbb{C}^*$.

Conclusions and Outlook

Summary of the results

At the end of this journey let us recap briefly the results we have obtained. On $\mathbb{R}^4$, we have computed exactly the number of massive vacua for the $\mathcal{N}=1^*$ gauge theory for any gauge group. This result in encapsulated into the generating functions for the Witten index disseminated in chapter 2, namely

$$\begin{aligned}
I_{SU(n)} &= \sum_{n=1}^{\infty} \sigma_1(n) q^n \\
I_{O(n)}(q) &= \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{2k-1})^2(1-q^{4k})^2} \\
I_{SO(n)}(q) &= \prod_{k=1}^{\infty} \frac{P_0(q^{2k-1})}{(1-q^{4k})^2(1-q^{2k-1})^2} + \prod_{k=1}^{\infty} \frac{1+q^{8k-4}}{(1-q^{4k})^2}\,, \\
I_{Sp(2n)}(q) &= q^{-1} I_{SO(2n+1)}(q)\,.
\end{aligned} \tag{7.67}$$

In these expressions, P_0 is a polynomial defined [34] by equation (2.61). This confirms various calculation already present in the literature and brings up a precision concerning the difference between gauge groups $O(n)$ and $SO(n)$. For the exceptional gauge groups, the result is

$$\begin{aligned}
I_{G_2} &= 10\,, \\
I_{F_4} &= 45\,, \\
I_{E_6} &= 44\,, \\
I_{E_7} &= 174\,, \\
I_{E_8} &= 301\,.
\end{aligned} \tag{7.68}$$

The main technique was to use the bijection between the classical vacuum expectation values for the three massive adjoint chiral multiplets and the nilpotent orbits in the gauge algebra. The centralizer of a given nilpotent orbit is a precious piece of data that allows to reduce the computation of the number of massive vacua to the supersymmetric index in pure $\mathcal{N}=1$ super Yang-Mills theories. Aspects of the global structure of Lie groups are crucial in obtaining the correct result, since the centralizer may have a component group that acts non-trivially on the summands of the residual gauge algebra.

The importance of the global structure of the gauge group is magnified when the theory is compactified on $\mathbb{R}^3 \times S^1$, where it can affect the correlation functions of local operators. It is therefore a exciting task to unveil the vacuum structure on this compact space. Although it appears that the number of vacua for gauge algebras of type A remains the same upon compactification, it turns our that this very number is modified for other types of gauge algebras. At least three mechanisms can be hold responsible for that, all involving line operators:

1. The global structure of the gauge group has to be explicitly chosen, and in fact a more precise definition of the theory in terms of the possible line operators [10] is needed.

2. A massive vacuum on $\mathbb{R}^4$ can be split into several massive vacua on the compact space, because Wilson lines can be turned on in the component group.

3. A massless configuration on $\mathbb{R}^4$ can flow to a massive vacuum after compactification, because Wilson lines that commute with the expectation values of the three adjoint multiplets can break an abelian residual gauge group to a non-abelian one.

These mechanisms are illustrated in one of the more elementary algebras available that is not of type A, namely B_2. In this case, the index on $\mathbb{R}^4$ is six, but after compactification the number of vacua becomes seven (because of a massless vacuum has acquired a mass) or ten (three massive vacua have split) depending on the line content, as explained in table 7.3.

To go further in the understanding of the vacuum structure, we use abundantly the correspondence with integrable systems, that provides an exact superpotential [38, 39] that can be analyzed numerically. Modularity lends a helping hand in classifying the vacua, giving the opportunity to perform extensive consistency checks, and tightly constraining the value of the potential at extrema. This allowed us to present detailed maps of the landscape of extrema of the twisted complexified elliptic Calogero-Moser potential – and taking into account the subtleties of gauge theories on the cylinder, these maps can be translated into an accurate cartography of the vacua of the $\mathcal{N} = 1^*$ theory. The relevant modular group acts on the vacua and extrema, and this action is represented as a graph on the map.

The set of extrema of the Calogero-Moser Hamiltonian for Lie algebras of type A has been known for a long time, and we focused on other types of algebras. We discovered that in this case, extrema have many surprises in store, such as points of monodromies in the interior of the fundamental domain.

On our road to fully realize the correspondence between integrable systems and the semiclassical limit of gauge theory on the cylinder, with all the aforementioned subtleties, we relied on structures from the field of affine Lie algebras. The pseudo-Levi subalgebras that are involved in the refinement of the Bala-Carter classification of nilpotent orbits play a corresponding role in defining semiclassical limits of the elliptic integrable system. A staircase structure can be identified in these limits, with more and more degrees of freedom being fixed at various powers of the nome q. On the gauge theory side, this can be seen as instanton effects responsible for fixing vacuum expectation values of fields. Note that the fact that we can rely on this semiclassical analysis on the gauge theory side gives an algorithm to find the number of extrema of the twisted elliptic Calogero-Moser potential for any gauge algebra, supplemented with much data about the duality transformations.

This semiclassical approach was also the starting point of an exploration of branches of massless vacua of the $\mathcal{N} = 1^*$ theory. Even in such elementary frameworks as the $su(3)$ gauge theory, such branches appear, and we were able to give analytical equations for it and to identify the Argyres-Douglas point, where mutually non-local massless degrees of freedom coexist. With such control on the gauge invariant variables that parametrize this Coulomb branch, it was possible to go back to the Wilson line and dual photon – or equivalently the integrable system complex variable. The key technical ingredient here was the Eichler-Zagier formulas which solve the inversion problem for a class of elliptic functions. We then complete the task of determining potential massless branches for the other rank two simple gauge algebras – B_2 and G_2. In the case of B_2, we find the equation on which the action of the $S_2 : \tau \mapsto -1/(2\tau)$ is explicit, and show how the branch depends on the line content of the theory. For G_2, there are no massless vacua.

Future Directions

In this thesis we only scratched the surface of the field of physics at a crossroad between different domains, which presage several avenues for future research. One of these domains is the world of integrable systems, and a pivotal ingredient is modularity. These gracefully blend in elliptic integrable systems, where the integrality properties of the models at extrema takes the form of integers in the Fourier expansion of functions with modular properties. These functions are not limited to modular or automorphic forms in their most basic incarnation, but realize modularity in polymorphous ways. One of the ways, to which we resorted at several occasions, was vector-valued modular forms for congruence subgroups of $SL(2, \mathbb{Z})$. We remind however that the full duality group appeared in the guise of the Hecke group, and the associated modular objects are necessarily more constrained. It would be interesting to study what these constraints add to our understanding of the extrema, and specifically there should be relations with the geometric Langlands program.

The first property that one arguably craves to know is the number of these extrema. As we indicate in the last section, the semiclassical analysis of the gauge theory provides an algorithm to find it, but a general formula would be more satisfying. The positions of the extrema may be associated to a generalization of zeroes of orthogonal polynomials [115]. As we pointed out in chapter 3, the Calogero-Moser systems are of central significance in the compass of integrable systems, and answering these questions would rain down upon the whole family of systems that can be reached by limits. It is conceivable that considering instead the Ruijsenaars model with spin or taking different coupling constants than what we have chosen to reproduce the superpotential of gauge theories prove to be rewarding. Increasing the rank would be especially exciting, as it could then be possible to connect to holographic dual backgrounds.

Perhaps more intriguingly, we have seen the appearance of points of monodromy. In the case of $\mathfrak{so}(8)$, there is a critical complex coupling $\tau_M \approx 2.41i$ that satisfies

$$\frac{1}{1728} j(\tau_M) = \frac{7626496}{3375} \,. \tag{7.69}$$

This number begs for an explanation, and beyond that, one could ask about the topology change in the fundamental domain as a Riemann surface that this point entails. On the gauge theory side, a geometric interpretation could be found by embedding the field theory into string theory. This is certainly one of the most enthralling questions raised by our exploration of elliptic Calogero-Moser potentials. The phenomenon is quite generic, based on our detecting it in algebras of type B, C, D and G_2.

Massless vacua on the gauge theory side have been less studied, and one can wonder for instance about the dimension of the branches beyond rank two. More Argyres-Douglas points are suspected to appear, although characterizing them by the Seiberg-Witten curve singularities may be a hard row to hoe because the equations increase in order. Again, the integrable complexified Hamiltonian could be the right weapon to attack the question.

In this thesis, we focused on $\mathcal{N} = 1^*$ theories in relation with the elliptic Calogero-Moser systems in part because modularity inherited from the conformal $\mathcal{N} = 4$ gives a wealth of constraints. It would be fascinating to understand how a similar story can be written starting with other ultraviolet-complete supersymmetric gauge theories. One example alluded to in the bulk of the thesis is the $\mathcal{N} = 2$ theory with gauge algebra $\mathfrak{su}(N)$ and $N_f = 2N$ flavors of quarks. The associated integrable system indubitably possesses modular properties matching those of the Seiberg-Witten curve. Recollecting the blossoming of beautiful phenomena encountered along the first part of the road, the future is appealing.

Appendix A

Supersymmetric gauge theories

This appendix takes a (non-exhaustive) inventory of supersymmetric gauge theories that are used or mentioned in the bulk of the thesis. The aim is to fix notations and conventions, and to write down the various Lagrangians in a unified way. We follow the excellent textbooks [23, 158, 159].

A.1 Supersymmetry in various dimensions

There are several ways to classify supersymmetric gauge theories. One way is to first specify the spacetime dimension D, and then specify the number of supersymmetries, that is always called $\mathcal{N}$. More precisely, supercharge generators are spinorial objects, and $\mathcal{N}$ is the number of minimal spinors that are used. It has to be distinguished from the number of real supercharges, which is equal to $\mathcal{N}$ multiplied by a factor that depends on the dimension, and which is equal to the number of *real* components of the *minimal* spinor in dimension D. This number is obtained by taking the number of real components of a Dirac spinor, which is $2^{[D/2]+1}$, and dividing it by 2 if the Majorana spinors exist, which is the case if $D \equiv 2, 3, 4 \bmod 8$, or if Weyl spinors exist, which is when D is even. When $D \equiv 2 \bmod 8$ there exist Majorana-Weyl spinors, and $2^{[D/2]+1}$ has to be divided by 4. This is summarized in the table below for the cases we will be interested in.

D	Number of real components of the minimal spinor	Maximal $\mathcal{N}$
2	$4_{\text{Dirac}}/(2_{\text{Majorana}} \times 2_{\text{Weyl}}) = 1$	16
3	$4_{\text{Dirac}}/2_{\text{Majorana}} = 2$	8
4	$8_{\text{Dirac}}/2_{\text{Majorana}} = 4$	4
6	$16_{\text{Dirac}}/2_{\text{Weyl}} = 8$	2
10	$64_{\text{Dirac}}/(2_{\text{Majorana}} \times 2_{\text{Weyl}}) = 16$	1

Number of real supercharges	$D=2$	$D=3$	$D=4$	$D=6$	$D=10$
1	$\mathcal{N}=1$				
2	$\mathcal{N}=2$	$\mathcal{N}=1$			
4	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=1$		
8	$\mathcal{N}=8$	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=1$	
16	$\mathcal{N}=16$	$\mathcal{N}=8$	$\mathcal{N}=4$	$\mathcal{N}=2$	$\mathcal{N}=1$

A.2 Supersymmetric gauge theories in four dimensions

A.2.1 The non supersymmetric gauge theories

Pure Yang-Mills theory

The kinetic term for the gauge field is

$$\mathcal{L}_{YM} = -\frac{1}{4g_{YM}^2} F_{\mu\nu}^a F^{a,\mu\nu} = -\frac{1}{2g_{YM}^2} \mathrm{Tr}\left(F^2\right) . \tag{A.1}$$

The coupling constant is g_{YM}. One can also define the coupling α_{YM} defined by $4\pi\alpha_{YM} = g_{YM}^2$. We then have the relation

$$\mathrm{d}\left(\frac{4\pi}{g_{YM}^2}\right) = -8\pi \frac{\mathrm{d}g_{YM}}{g_{YM}^3} = -\frac{\mathrm{d}\alpha_{YM}}{\alpha_{YM}^2} , \tag{A.2}$$

which is useful to relate the β-functions written in terms of one coupling or the other. We have the usual expression

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c , \tag{A.3}$$

or equivalently $F = \mathrm{d}A - iA^2$. The covariant derivative is $\nabla_\mu = \partial_\mu - iA_\mu$. A gauge transformation is parametrized by a function U defined on spacetime and valued in the gauge group G. In the following lines we use x, y to denote points in spacetime, and when no such point is indicated it is understood that all fields are evaluated at x. Under such a gauge transformation we then have

$$A_\mu \to A'_\mu = U\left[A_\mu + i\partial_\mu\right] U^{-1} = UA_\mu U^{-1} - i(\partial_\mu U)U^{-1} . \tag{A.4}$$

One can then check that

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1} . \tag{A.5}$$

The Wilson line is

$$W(y,x) = \mathcal{P}\exp\left(i\int_x^y A\right) , \tag{A.6}$$

and under the gauge transformation given above,

$$W(y,x) \to U(y)W(y,x)U^{-1}(x) . \tag{A.7}$$

Matter sector

We can add matter to the pure gauge theory described by (A.1). For instance one can add a Lorentz scalar field ϕ transforming in a given representation $\mathbf{R}$ of the gauge group. In this case, the gauge transformation is

$$\phi(x) \to \phi'(x) = U(x)\cdot\phi(x) \tag{A.8}$$

where the dot denotes the action of the gauge group in the representation $\mathbf{R}$. The Lagrangian corresponding to this matter field is

$$\mathcal{L}_{\text{Scalar}} = \nabla_\mu\bar{\phi}\nabla^\mu\phi - V(\phi) \tag{A.9}$$

where $V(\phi)$ is the potential energy term, and ∇_μ is the covariant derivative. The gauge transformation is the same for a spin $\frac{1}{2}$ field ψ, and the Lagrangian is

$$\mathcal{L}_{\text{Spin }\frac{1}{2}} = i\bar{\psi}\not{\nabla}\psi - m\bar{\psi}\psi \tag{A.10}$$

where we have only included a (bare) mass term.

Renormalization of the coupling

The one-loop renormalization of the gauge coupling is

$$\beta_{\text{1-loop}} = \left[\frac{\mathrm{d}g_{YM}}{\mathrm{d}\log(\mu/\Lambda)}\right]_{\text{1-loop}} = -\frac{g_{YM}^3}{(4\pi)^2}\left[\frac{11}{3}T(\text{adjoint}) - \frac{2}{3}T(\text{fermions}) - \frac{1}{3}T(\text{scalars})\right] . \quad \text{(A.11)}$$

where we have used the half Dynkin indices defined in (B.9), the representations labeled "fermions" are those in which the left-handed Weyl fermions transform (all the fermions of the theory being written in terms of left-handed Weyl fermions), and the ones labeled "scalars" correspond to complex scalars. In this formula μ denotes the energy scale of the theory. Note that this formula is valid for the abelian theory $G = U(1)$: in this case, one should put $T(\text{adjoint}) = 0$.

It should be emphasized that equation (A.11) is true for our normalization of the fields given by (A.1). If instead we use canonically normalized fields, the exact β-function is given by the Novikov-Shifman-Vainshtein-Zakharov formula [50].

The Georgi-Glashow model

One particular useful model is given by the Georgi-Glashow Lagrangian for gauge group $G = SU(2)$:

$$\mathcal{L}_{GG} = -\frac{1}{4g_{YM}^2}F^a_{\mu\nu}F^{a,\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \lambda(\phi^a\phi^a - v^2)^2 \quad \text{(A.12)}$$

In this Lagrangian, ϕ^a is a real scalar in the adjoint representation of the $SU(2)$ gauge group, as the Latin index a indicates, and v is a constant which is used to define the potential. We also use the covariant derivative $D_\mu\phi = \partial_\mu\phi - i[A,\phi]$. This Lagrangian can also be used with an arbitrary gauge group, with maybe a more general potential for the scalar field. It should be noted, however, that the construction of monopoles in these generalized models adds no essential features to the more basic $SU(2)$ monopole constructed from (A.12).

The theta angle

A term can be added to the Lagrangian (A.1), with a new parameter θ_{YM}:

$$\mathcal{L}_\theta = \frac{\theta_{YM}}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a,\mu\nu} = \frac{\theta_{YM}}{8\pi^2}\mathrm{Tr}\,(F\wedge F) \,. \quad \text{(A.13)}$$

Here $\tilde{F}$ is the Hodge dual of the 2-form F. In components, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. This term is purely topological, and does not affect the classical equations of motion. It has only non-perturbative effects and the physics of monopoles and instantons strongly depends on the value of θ.

A.2.2 The $\mathcal{N}=1$ gauge theory

Pure $\mathcal{N}=1$ gauge theory

The Lagrangian of a pure $\mathcal{N}=1$ super-Yang-Mills theory in four space-time dimensions, written in the superspace formalism, is

$$\mathcal{L}_{\text{pure }\mathcal{N}=1} = \left(\frac{\tau}{16\pi i}\int \mathrm{d}^2\theta\, W^{a\alpha}W^a_\alpha + h.c.\right) = \Im\mathrm{Tr}\left(\frac{\tau}{4\pi}\int \mathrm{d}^2\theta\, W^2\right) . \quad \text{(A.14)}$$

In this expression, W_α is the field strength tensor superfield constructed out of the *vector superfield* $V(x,\theta,\bar\theta)$ which is an element of the Lie algebra of the gauge group. Let $(T^a)_{a=1,\dots,r}$ be Hermitian generators of the Lie algebra in a given representation, satisfying

$$[T^a, T^b] = i f^{abc} T^c \,. \tag{A.15}$$

We can then write $V(x,\theta,\bar\theta) = V^a(x,\theta,\bar\theta)T^a$, and the field strength defined by

$$W_\alpha = \frac{1}{8}\bar D^2 \left[e^{-V}\left(D_\alpha e^V\right)\right] \tag{A.16}$$

can again be decomposed on the generator basis $W_\alpha = W_\alpha^a T^a$. Here D_α and $\bar D_{\dot\alpha}$ are the superderivatives, $\alpha = 1,2$ and $\dot\alpha = 1,2$ are a spinor indices, and $D^2 = D^\alpha D_\alpha = \epsilon^{\alpha\beta} D_\beta D_\alpha$. We also recall that the Grassmann variables θ have dimension $(\text{length})^{1/2}$, and that the rule of integration $\int \theta \mathrm{d}\theta = 1$ imposes that $\mathrm{d}\theta$ have dimension $(\text{length})^{-1/2}$.

The coupling constant is

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta_{YM}}{2\pi}\,, \tag{A.17}$$

where g_{YM} is the usual Yang-Mills coupling constant and θ is the θ-angle. Using this decomposition into real and imaginary part, we can expand

$$\mathcal{L}_{\text{pure }\mathcal{N}=1} = -\frac{1}{4g_{YM}^2} F^a_{\mu\nu} F^{a,\mu\nu} + \frac{i}{g_{YM}^2}\lambda^{a\alpha}\mathcal{D}_{\alpha\dot\beta}\bar\lambda^{a\dot\beta} + \frac{\theta_{YM}}{32\pi^2} F^a_{\mu\nu}\tilde F^{a,\mu\nu}\,. \tag{A.18}$$

In the expression above, the part corresponding to the D-component of the vector multiplet[1] has been integrated out, and it will contribute to the scalar potential under the name of *D-terms*.

The matter sector in $\mathcal{N} = 1$ gauge theory

We want to add a complex scalar field Φ which transforms in a representation $\mathbf{R}$ of the gauge group (and we are mostly concerned with the cases where $\mathbf{R}$ is the fundamental or the adjoint representation). This complex scalar is associated by supersymmetry to a Weyl fermion in the same representation of the gauge group, and together they form a *chiral multiplet*. A gauge invariant kinetic term is then $\bar\Phi e^V \Phi$, where V has to be expressed[2] in the representation $\mathbf{R}$. We then write

$$\mathcal{L}^{\text{kinetic}}_{\text{matter }\mathcal{N}=1} = \sum_{\text{flavors}} \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar\theta\, \bar\Phi e^V \Phi\,. \tag{A.19}$$

This expression should of course be a scalar, which is it if the representation $\mathbf{R}$ is the fundamental representation. If Φ transforms in the adjoint representation, then it is understood that a trace $\mathrm{Tr}\,\bar\Phi e^V \Phi e^{-V}$ should be taken. The fact that the Lagrangian should have dimension $(\text{length})^{-4}$ shows that our chiral field Φ has dimension $(\text{length})^{-1}$.

[1] Recall that the off-shell massless vector multiplet is composed of the gauge field, the gauginos and an auxiliary real scalar D, which accounts for $3+1$ bosonic degrees of freedom and $2+2$ fermionic degrees of freedom, making supersymmetry manifest. The on-shell content reduces to the gauge boson ($3-1=2$ degrees of freedom) and the gauginos ($(2+2)/2 = 2$ degrees of freedom).

[2] In practice, one chooses generators T^a in $\mathbf{R}$ satisfying (A.15), and just writes $V = V^a T^a$, where the superfields V^a don't depend on the chosen representation.

Now that we have accounted for the kinetic terms, we add interactions. This is realized through a *holomorphic* function $\mathcal{W}(\Phi)$ called the *superpotential*. Then the corresponding Lagrangian is just

$$\mathcal{L}^{\text{interactions}}_{\text{matter } \mathcal{N}=1} = \left(\int \mathrm{d}^2\theta\, \mathcal{W}(\Phi) + h.c. \right) . \tag{A.20}$$

Note that this has the same structure as (A.14), which is why the kinetic terms of the vector multiplet are sometimes included into the superpotential. We will refrain to do so in this thesis.

Effective action for $\mathcal{N} = 1$ matter

We follow closely section 49.7 from [158]. The most general Lagrangian in four dimensions compatible with supersymmetry and containing no more that two spacetime derivatives is

$$\mathcal{L}_{WZ} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, \mathcal{K}(\Phi^i, \bar{\Phi}^j) + \left(\int \mathrm{d}^2\theta\, \mathcal{W}(\Phi^i) + h.c. \right) , \tag{A.21}$$

where the Φ^i are a set of chiral superfields, the superpotential $\mathcal{W}$ is again a holomorphic function of the Φ^i and the *Kähler potential* $\mathcal{K}$ is a real function which generalizes the kinetic term (A.19). The subscript WZ stands for Wess-Zumino, and $\mathcal{L}_{WZ}$ is the Wess-Zumino Lagrangian. This kind of Lagrangian often appears as effective theories, and that is why the model need not be renormalizable, the kinetic term need not be the canonical one and the superpotential need not be polynomial. Note that the kinetic term for the lowest components of the superfields is

$$\frac{\partial^2 \mathcal{K}}{\partial \phi^i \partial \bar{\phi}^j} \partial_\mu \phi^i \partial^\mu \bar{\phi}^j \, . \tag{A.22}$$

The mass term for the fermions involve second derivatives of the superpotential.

Renormalization of the coupling

The one-loop renormalization of the gauge coupling is obtained from the expression for non supersymmetric theories (A.11) by imposing that the fermions transform in the same representations than the bosons, so that $T(\text{fermions}) = T(\text{adjoint}) + T(\text{scalars})$. The result is

$$\left[\frac{\mathrm{d} g_{YM}}{\mathrm{d} \log \mu} \right]_{1 \text{ loop}} = - \frac{g^3_{YM}}{(4\pi)^2} \left[3T(\text{adjoint}) - T(\text{chiral mult.}) \right] . \tag{A.23}$$

A.2.3 The $\mathcal{N} = 2$ gauge theory

Now we move on to $\mathcal{N} = 2$ theories. We will consider only theories where the two supersymmetries are equivalent, in the sense that the $SU(2)_R$ group which rotates them, called the R-symmetry group, has to be a symmetry of the Lagrangian. Although the supermultiplets are now bigger than in $\mathcal{N} = 1$ theories, we keep the language of these $\mathcal{N} = 1$ theories to write the Lagrangian. The additional supersymmetry then imposes constraints on the Lagrangian. Let us start with the pure super Yang-Mills theory. The $\mathcal{N} = 2$ vector multiplet consists of one $\mathcal{N} = 1$ vector multiplet, with Lagrangian given by (A.14), and one $\mathcal{N} = 1$ chiral multiplet in the adjoint representation of the gauge group, with Lagrangian (A.19), where there is only one flavor and V is in the adjoint representation. The matter representation is a necessary condition, but to enforce the extended supersymmetry and the R-symmetry we also need to tune the coupling constant. The resulting Lagrangian is

$$\mathcal{L}_{\text{pure } \mathcal{N}=2} = \left(\frac{\tau}{16\pi i} \int \mathrm{d}^2\theta \, W^{a\alpha} W^a_\alpha + h.c. \right) + \frac{\tau}{4\pi i} \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, \bar{\Phi} e^V \Phi \,. \tag{A.24}$$

Renormalization of the coupling

The renormalization of the coupling constant is again a simplification of (A.23), where one chiral multiplet has to transform in the adjoint representation, giving

$$\left[\frac{\mathrm{d} g_{YM}}{\mathrm{d} \log \mu} \right]_{1 \text{ loop}} = -\frac{g_{YM}^3}{(4\pi)^2} \left[2T(\text{adjoint}) - T(\text{chiral mult.}) \right] \,. \tag{A.25}$$

Effective action for $\mathcal{N} = 2$ gauge fields

We want to see what constraints extended supersymmetry imposes on (A.21) for the gauge sector of the $\mathcal{N} = 2$ theory. The superpotential and the Kähler potential have the forms given by (A.24) for each simple factor in the gauge group. There are two equivalent ways to reach the result. One is to impose $SU(2)_R$ symmetry on the fermions (gauginos from the vector multiplet and fermions from the chiral multiplet, in the minimal supersymmetry language), and deduce that a relation exists between the gauge couplings τ^{ij} and the Kähler potential. Another method is to work in extended superspace, and derive the effective Lagrangian in the form (A.21) from the *prepotential*, which is simply the Lagrangian density in the $\mathcal{N} = 2$ superfield formalism.

A.2.4 The $\mathcal{N} = 4$ gauge theory

Written in $\mathcal{N} = 1$ language, the Lagrangian is obtained by adding the gauge sector (A.14) and the matter sector (A.19) and (A.20). The extended supersymmetry implies that the matter sector is made of three complex chiral multiplets in the adjoint representation of the gauge group, which determines the kinetic term (A.19), interacting according to the superpotential

$$\mathcal{W}_{\mathcal{N}=4} = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr} \left(\Phi_1 [\Phi_2, \Phi_3] \right) \,. \tag{A.26}$$

Appendix B

Lie Algebra

B.1 Basic definitions

Let $\mathfrak{g}$ be a simple Lie algebra. Its Cartan subalgebra is denoted $\mathfrak{h}$, and we have $\dim \mathfrak{h} = r$, where r is the rank of $\mathfrak{g}$. The roots are elements of the dual of the Cartan algebra $\mathfrak{h}^*$ and are generically denoted by the letter α. The sets of all roots, all positive roots and simple roots are denoted Δ, Δ^+ and Δ^s respectively. There are precisely r simple roots, called $\alpha_1, \ldots, \alpha_r$. If $\mathfrak{g}$ is not simply laced, Δ_s stands for the set of short roots and Δ_l for the set of long roots.

The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is defined by

$$\kappa(x, y) = \frac{1}{2h^\vee} \mathrm{tr}\left(\mathrm{ad}x\, \mathrm{ad}y\right) , \tag{B.1}$$

where $h^\vee$ is the dual Coxeter number (B.6). The restriction of the Killing form to $\mathfrak{h} \times \mathfrak{h}$ allows to define a scalar product on the dual space $\mathfrak{h}^*$. For two roots $\alpha, \beta \in \mathfrak{h}^*$, we denote (α, β) their scalar product. The Cartan matrix is defined by

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{|\alpha_j|^2} = (\alpha_i, \alpha_j^\vee) , \tag{B.2}$$

where

$$\alpha^\vee = 2\frac{\alpha}{|\alpha_j|^2} \tag{B.3}$$

is the coroot associated to the root α. The highest root ϑ defines the marks a_i and the comarks $a_i^\vee$ via the relation

$$\vartheta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee . \tag{B.4}$$

The Coxeter number is

$$h = 1 + \sum_{i=1}^{r} a_i . \tag{B.5}$$

and the dual Coxeter number is

$$h^\vee = 1 + \sum_{i=1}^{r} a_i^\vee . \tag{B.6}$$

We also define the Weyl vector

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha , \tag{B.7}$$

and the dual Weyl vector, sometimes called the level vector

$$\rho^\vee = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha^\vee \,. \tag{B.8}$$

Let λ be a dominant weight; it is the highest weight of a representation that we can label by λ as well. We define the half Dynkin index of this representation to be

$$T(\text{Representation}) = \frac{\dim(\text{Representation})}{2\dim\mathfrak{g}}(\lambda, \lambda+2\rho)\,. \tag{B.9}$$

The quantity $\frac{1}{2}(\lambda,\lambda+2\rho)$ is the quadratic Casimir[1]. When evaluated in the adjoint representation, the quadratic Casimir is $h^\vee$, and the half Dynkin index is $T(\text{adjoint}) = h^\vee$. In the fundamental representation of $SU(N)$, we have $T(\text{fundamental}) = \frac{1}{2}$. We use the following normalization of the generators of the Lie algebra:

$$\mathrm{Tr}\, T^a T^b = T(\text{Representation})\delta^{ab}\,. \tag{B.10}$$

The fundamental weights basis $(\pi_i)_{i=1,\dots,r}$ is defined to be the dual to the simple coroot basis. This means that $(\pi_i, \alpha_j^\vee) = \delta_{ij}$. Let $Z \in \mathfrak{h}^*$ be a weight. When we decompose Z in the fundamental weights basis, its components are the Dynkin labels Z_i (note the capital letter) of Z.

We now reproduce here the root systems of the finite-dimensional simple complex Lie algebras. For algebras of type B, C, D and F we use an orthonormal basis $(\epsilon_1,\dots,\epsilon_r)$ of $\mathfrak{h}^*$, and for algebras of type A, E and G we use vectors $(\epsilon_1,\dots,\epsilon_{r+1})$ such that

$$\sum_{i=1}^{r}\epsilon_i = 0 \qquad\qquad (\epsilon_i,\epsilon_j) = \delta_{ij} - \frac{1}{r+1}\,. \tag{B.11}$$

Then the roots and simple roots are given by

Type	Roots	Simple roots
A_r	$\epsilon_i - \epsilon_j$	$\alpha_i = \epsilon_i - \epsilon_{i+1}$
B_r	$\pm\epsilon_i \pm \epsilon_j, \pm\epsilon_i$	$\alpha_i = \epsilon_i - \epsilon_{i+1}, \alpha_r = \epsilon_r$
C_r	$\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i$	$\alpha_i = \epsilon_i - \epsilon_{i+1}, \alpha_r = 2\epsilon_r$
D_r	$\pm\epsilon_i \pm \epsilon_j$	$\alpha_i = \epsilon_i - \epsilon_{i+1}, \alpha_r = \epsilon_{r-1} + \epsilon_r$
G_2	$\epsilon_i - \epsilon_j, \pm\epsilon_i$	$\alpha_1 = -\epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3$

(B.12)

In this table, $i \neq j$ vary between 1 and r or $r+1$ depending on the case, and in the last column $i < r$ except for the case of A_r. When the weight Z is expressed in the family (ϵ_i), its components are denoted z_i. In the cases where this family is a basis, there is no ambiguity in the definition of the z_i. In the other cases (for type A, E and G), we have an additional degree of freedom that we use to set $z_{r+1} = 0$. This can be summed up as

$$Z = \sum_{i=1}^{r} Z_i \pi_i = \sum_{i=1}^{r} z_i \epsilon_i\,. \tag{B.13}$$

[1] Sometimes a different normalization is used. For instance in [160], there is an additional factor of 2. Our convention is consistent with [159, 158, 23].

B.2 Lattices

We briefly review Lie algebra concepts that are useful to us in discussing the symmetries of both the integrable systems and gauge theories we discuss in the bulk of the text. We mostly follow [156] for our conventions. We will discuss in this section concepts that depend not only on the Lie algebra $\mathfrak{g}$, but also on the global structure of the group G.

Let us then consider a (compact simple) Lie group G with maximal torus T. They have corresponding tangent algebras $\mathfrak{g}$ and $\mathfrak{t}$. We can then identify T as a linear group, and its space of characters $\chi(T)$ is in bijection with a lattice in the space $\mathfrak{t}^*(\mathbb{R})$ dual to the tangent algebra $\mathfrak{t}$, and defined over the real numbers $\mathbb{R}$. To the Lie algebra $\mathfrak{g}$, we can associate its set of roots Δ. Again, these roots are elements of the Euclidean space $\mathfrak{t}^*(\mathbb{R})$. The space $\mathfrak{t}(\mathbb{R})$ comes equipped with a non-degenerate scalar product, which we will denote again by $(\cdot,\cdot)$. This scalar product allows us to identify a function λ on the space $\mathfrak{t}(\mathbb{R})$ with an element $u_\lambda \in \mathfrak{t}(\mathbb{R})$ through the relation:

$$\lambda(x) = (u_\lambda, x)\,, \tag{B.14}$$

valid for all elements x of $\mathfrak{t}(\mathbb{R})$. We will occasionally abuse notation and write $\lambda(x) = (\lambda, x)$, and also $(u_\lambda, u_{\lambda'}) = (\lambda, \lambda')$, which defines a dual scalar product on $\mathfrak{t}^*(\mathbb{R})$. The bijection between the space generated by the roots and its dual allows us to define the dual roots (i.e. the co-roots) through the relation:

$$\alpha^\vee = \frac{2u_\alpha}{(\alpha,\alpha)}\,. \tag{B.15}$$

The root lattice Q is the lattice generated by the roots. Any set of simple roots α_i spans the space $\mathfrak{t}^*(\mathbb{R})$. The weight lattice P also sits inside $\mathfrak{t}^*(\mathbb{R})$ and is defined to be generated by a basis π_j such that:

$$2\frac{(\alpha_i,\pi_j)}{(\alpha_i,\alpha_i)} = \delta_{ij}\,, \tag{B.16}$$

for all i and j that run from 1 to the rank of the group G. We moreover define the dual root lattice $Q^\vee$ to be the lattice generated by the dual roots, and the dual weight lattice $P^\vee$ to be the weight lattice corresponding to the dual root lattice. The dual of the lattice generated by the characters of a given group G will be denoted $\mathfrak{t}(\mathbb{Z})$. We have the following properties. The center $C(G)$ of the group G is given by:

$$C(G) \equiv P^\vee/\mathfrak{t}(\mathbb{Z}) \equiv \chi(T)/Q\,. \tag{B.17}$$

Moreover, when G is simply connected it is equal to its universal cover $\tilde{G}$. We then have that the space of characters is bijective to the whole of the weight lattice $\chi(T) = P$, and that $\mathfrak{t}(\mathbb{Z}) = Q^\vee$, such that $C(G)$ is maximal and $C(\tilde{G}) = P/Q = P^\vee/Q^\vee$. The group with minimal center $C(G) = 1$ is the universal cover $\tilde{G}$ divided by its center $C(\tilde{G})$. In this case we have that the set of weights is the set of roots $\chi(T) = Q$ and that $\mathfrak{t}(\mathbb{Z}) = P^\vee$.

Our definitions imply that the fundamental weights $\pi_j^\vee$ that generate the dual weight lattice $P^\vee$ satisfy:

$$(\pi_j^\vee, u_{\alpha_i}) = \delta_{ij}\,, \tag{B.18}$$

and therefore that:

$$\alpha(Z) = (u_\alpha, Z) = (\alpha, Z) \tag{B.19}$$

is integer for Z in the dual weight lattice, i.e. for Z a co-weight.

We summarize inclusions and dualities in the diagram below. The arrows indicate that the lattices are dual, i.e. that the contractions give integers.

$$\begin{array}{ccccccccc} \mathfrak{t}^*(\mathbb{R}) & \supset & P & \supset & \chi(T) & \supset & Q & & \\ & & \updownarrow & & \updownarrow & & \updownarrow & & \\ & & Q^\vee = P^* & \subset & \mathfrak{t}(\mathbb{Z}) & \subset & P^\vee = Q^* & \subset & \mathfrak{t}(\mathbb{R}) \end{array} \tag{B.20}$$

B.3 Lie algebra data

We end this review on Lie group and Lie algebra theory with table B.1 which exhibits useful data on the Weyl group, the outer automorphisms, the dual Coxeter number and the center of the universal covering group corresponding to the classical Lie algebras.

Algebra	Weyl group	OA	Dual Coxeter number	Center Univ. Cover
A_r, $r > 1$	S_{r+1}	$\mathbb{Z}_2$	$r+1$	$\mathbb{Z}_{r+1}$
A_1	$\mathbb{Z}_2$	1	2	$\mathbb{Z}_2$
B_r	$S_r \ltimes \mathbb{Z}_2^r$	1	$2r-1$	$\mathbb{Z}_2$
C_r	$S_r \ltimes \mathbb{Z}_2^r$	1	$r+1$	$\mathbb{Z}_2$
D_r, odd r	$S_r \ltimes \mathbb{Z}_2^{r-1}$	$\mathbb{Z}_2$	$2r-2$	$\mathbb{Z}_4$
D_r, even $r > 4$	$S_r \ltimes \mathbb{Z}_2^{r-1}$	$\mathbb{Z}_2$	$2r-2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
D_4	$S_4 \ltimes \mathbb{Z}_2^3$	S_3	6	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Table B.1: Lie Algebra Data. OA stands for Outer Automorphisms

We provide a table of dual Coxeter numbers for the simple complex exceptional Lie algebras.

E_6	E_7	E_8	F_4	G_2
12	18	30	9	4

For the various definitions about affine Lie algebras, see section 3.4.2.

Appendix C

Modular Forms and Elliptic Functions

This appendix presents various definitions and theorems concerning modular forms and elliptic functions that we use in bulk of the text. Throughout all the thesis, we denote the upper half-plane of complex numbers with strictly positive imaginary part by $\mathcal{H}$, and τ is a generic element of $\mathcal{H}$.

C.1 Modular and Automorphic forms

C.1.1 Definitions

Let N be a positive integer. The *principal congruence subgroup* of level N is the subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\} .$$

Note that $\Gamma(1) = SL_2(\mathbb{Z})$. More generally, a subgroup $\Gamma \in SL_2(\mathbb{Z})$ is called a *congruence subgroup* if it contains $\Gamma(N)$ for some $N \in \mathbb{N}^*$. A particularly important class of congruence subgroups is the class of $\Gamma_0(N)$ groups, defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \bmod N \right\} . \tag{C.1}$$

This group can be described in an abstract way as follows. It is generated by the three elements C, T and ST^NS, where the relations satisfied by C, S and T are

$$C^2 = 1 \qquad S^2 = C \qquad (ST)^3 = C\,. \tag{C.2}$$

In the case $N = 1$, one can check that S belongs to the group, because $S = C(STS)T(STS)$.

We now introduce a useful notation. For any matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{C.3}$$

and any function $f : \mathcal{H} \to \mathbb{C}$ we define the function $f[\gamma]_k : \mathcal{H} \to \mathbb{C}$ by the expression

$$(f[\gamma]_k)(\tau) = \frac{1}{(c\tau + d)^k} f\left(\frac{a\tau + b}{c\tau + d}\right) . \tag{C.4}$$

Let Γ be a congruence subgroup, and let $k \in \mathbb{Z}$. A function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form* of weight k with respect to Γ if

1. f is holomorphic on $\mathcal{H}$ and the function $f[\gamma]_k$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z})$,
2. f is weight-k invariant under Γ, meaning that $f[\gamma]_k = f$ for all $\gamma \in \Gamma$.

If in addition the constant term in the q-expansion of $f[\gamma]_k$ vanishes for all $\gamma \in SL_2(\mathbb{Z})$, then we say that f is a *cusp form* of weight k with respect to Γ. We denote by $\mathcal{M}_k(\Gamma)$ the set of such modular forms, and $\mathcal{S}_k(\Gamma)$ the set of cusp forms.

We also define the slightly more general concept of automorphic form as follows. Let

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \tag{C.5}$$

be the Riemann sphere. A function $f : \mathcal{H} \to \hat{\mathbb{C}}$ is an *automorphic form* of weight k with respect to Γ is it satisfies the same conditions 1 and 2 of the definition above, with the word “holomorphic” replaced by the word “meromorphic”. We denote by $\mathcal{A}_k(\Gamma)$ the set of such automorphic forms.

Let Γ be a congruence subgroup. The quotient space $\Gamma\backslash\mathcal{H}$ is a complex curve called the *modular curve* associated to Γ. In general it is not compact, because of the *cusps*, which constitute the Γ-equivalence class of $\mathbb{Q}\cup\{\infty\}$. We can compactify the modular curve by adjoining the cusps:

$$\mathcal{X}(\Gamma) = \Gamma\backslash\left(\mathcal{H} \cup \mathbb{Q} \cup \{\infty\}\right) \tag{C.6}$$

is the *compact* modular curve of Γ. We note that $\mathcal{X}(\Gamma(1)) \equiv \mathcal{X}(1)$ is the usual fundamental domain of the modular group. This domain has one elliptic point of order 2, namely $SL_2(\mathbb{Z}) \cdot i$, one elliptic point of order 3, namely $SL_2(\mathbb{Z}) \cdot e^{2\pi i/3}$, and one cusp $SL_2(\mathbb{Z}) \cdot \infty$. Now let us consider the natural projection $f : \mathcal{X}(\Gamma) \to \mathcal{X}(1)$. Let ε_2 be the number of points in $\mathcal{X}(\Gamma)$ that are sent by f on the elliptic point of order 2, and define similarly ε_3 and ε_∞. Then the genus of $\mathcal{X}(\Gamma)$ is

$$g = 1 + \frac{d}{12} - \frac{\varepsilon_2}{4} - \frac{\varepsilon_3}{3} - \frac{\varepsilon_\infty}{2}\,, \tag{C.7}$$

where d is the degree of f. Moreover, let $k \in \mathbb{Z}$ be an even integer. If $k \leq 0$, then $\dim \mathcal{M}_k(\Gamma) = \delta_{k,0}$. If $k \geq 2$, then

$$\boxed{\dim \mathcal{M}_k(\Gamma) = (k-1)(g-1) + \left[\frac{k}{4}\right]\varepsilon_2 + \left[\frac{k}{3}\right]\varepsilon_3 + \frac{k}{2}\varepsilon_\infty}\,. \tag{C.8}$$

Now we give the dimension of the space of cusp forms. For $k \leq 2$ we have $\dim \mathcal{S}_k(\Gamma) = g\delta_{k,2}$ and for $k \geq 4$, $\dim \mathcal{S}_k(\Gamma) = \dim \mathcal{M}_k(\Gamma) - \varepsilon_\infty$.

Full Modular Group

Let k be an even integer. For the full modular group $SL_2(\mathbb{Z})$, we have $\varepsilon_2 = \varepsilon_3 = \varepsilon_\infty = d = 1$. This gives $g = 0$ and:

- If $k < 0$ or $k = 2$ then $\mathcal{M}_k\left(SL_2(\mathbb{Z})\right) = \mathcal{S}_k\left(SL_2(\mathbb{Z})\right) = \{0\}$.
- $\mathcal{M}_0\left(SL_2(\mathbb{Z})\right) = \mathbb{C}$ and $\mathcal{S}_0\left(SL_2(\mathbb{Z})\right) = \{0\}$.
- For $k \geq 4$, we have $\mathcal{M}_k\left(SL_2(\mathbb{Z})\right) = \mathcal{S}_k\left(SL_2(\mathbb{Z})\right) \oplus \mathbb{C}E_k$ and

$$\dim \mathcal{S}_k\left(SL_2(\mathbb{Z})\right) = \begin{cases} \left[\frac{k}{12}\right] - 1 & \text{if } k \equiv 2 \bmod 12 \\ \left[\frac{k}{12}\right] & \text{otherwise.} \end{cases} \tag{C.9}$$

The congruence subgroup $\Gamma_0(4)$

One can prove that $\varepsilon_2 = \varepsilon_3 = 0$ and that $\varepsilon_\infty = 3$. Moreover, $d = 6$. Then we find that $g = 0$, and therefore for $k \geq 2$ even,

$$\dim \mathcal{M}_k(\Gamma_0(4)) = 1 + \frac{k}{2}\,. \tag{C.10}$$

and if $k \geq 4$,

$$\dim \mathcal{S}_k(\Gamma_0(4)) = \frac{k-4}{2}\,. \tag{C.11}$$

C.1.2 Eisenstein Series

Let $k \geq 4$ be an integer. We define the classical Eisenstein series of weight k to be the function defined by the following absolutely convergent series:[1]

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(c\tau+d)^k} = \frac{1}{2}\sum_{c\wedge d=1} \frac{1}{(c\tau+d)^k}\,. \tag{C.12}$$

The function E_k is a modular form of weight k for $SL_2(\mathbb{Z})$, and has a q-expansion that starts with 1:

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\,, \tag{C.13}$$

where B_k is the k-th Bernoulli number[2] and $\sigma_{k-1}(n)$ is the sum

$$\sigma_{k-1}(n) = \sum_{m>0,\, m|n} m^{k-1}\,. \tag{C.15}$$

For $k = 2$, the sum (C.12) is not absolutely convergent. Its terms can be arranged so that equation (C.13) remains true; however conditional convergence spoils modularity. A detailed calculation shows that, using the notations (C.3) and (C.4)

$$(E_2[\gamma]_2)(\tau) = E_2(\tau) - \frac{6i}{\pi}\left(\frac{c}{c\tau+d}\right)\,. \tag{C.16}$$

Since these classical Eisenstein series are used extensively throughout, we give here the first few terms in their q-expansion:

$$E_2(q) = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - \ldots \tag{C.17}$$
$$E_4(q) = 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + \ldots \tag{C.18}$$
$$E_6(q) = 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - \ldots \tag{C.19}$$

[1] In this expression $\zeta(k) = \sum_{d=1}^{\infty} d^{-k}$ is the usual Riemann zeta function.

[2] This number can be defined by either one of the two following formulas for even $k \geq 2$:

$$2\zeta(k) = -\frac{(2\pi i)^k}{k!}B_k \qquad \frac{t}{e^t-1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \tag{C.14}$$

C.2 Elliptic functions

C.2.1 The Weierstrass function

Our conventions for the elliptic Weierstrass function are:

$$\begin{aligned}\wp(x;\omega_1,\omega_2) &= \frac{1}{x^2}+\sum_{(m,n)\neq(0,0)}\left[\frac{1}{(x+2m\omega_1+2n\omega_2)^2}-\frac{1}{(2m\omega_1+2n\omega_2)^2}\right]\\ \wp(z;\tau) &= \frac{1}{z^2}+\sum_{(m,n)\neq(0,0)}\left[\frac{1}{(z+m+n\tau)^2}-\frac{1}{(m+n\tau)^2}\right]\end{aligned} \tag{C.20}$$

which entails the equality

$$\wp(z;\tau)=4\omega_1^2\,\wp(2\omega_1 z;\omega_1,\omega_2)\,. \tag{C.21}$$

with

$$\tau=\frac{\omega_2}{\omega_1}\,. \tag{C.22}$$

We impose the convention that $\Im(\omega_2/\omega_1)=\Im(\tau)>0$. The Weierstrass function is a Jacobi form of level 2 and index 0 :

$$\wp\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^2\wp(z;\tau)\,. \tag{C.23}$$

It has the following expansion for large imaginary part of τ:

$$\wp(x;\omega_1,\omega_2)=-\frac{\pi^2}{12\omega_1^2}E_2(q)+\frac{\pi^2}{4\omega_1^2}\csc^2\left(\frac{\pi x}{2\omega_1}\right)-\frac{2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos\frac{n\pi x}{\omega_1}\,. \tag{C.24}$$

This expression is valid when the series es absolutely convergent, which requires $|\Im x|<\Im\tau$. In this expression, csc stands for the cosecant, which is just

$$\csc x=\frac{1}{\sin x}\,. \tag{C.25}$$

We will almost always take $\omega_1=\frac{1}{2}$ and $\omega_2=\frac{\tau}{2}$, in which case the expansion simplifies to

$$\wp(x;\tau)=-\frac{\pi^2}{3}E_2(q)+\frac{\pi^2}{\sin^2(\pi x)}-8\pi^2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos 2n\pi x\,. \tag{C.26}$$

From this we deduce the limit

$$\lim_{\tau\to i\infty}\wp(x;\tau)=-\frac{\pi^2}{3}+\frac{\pi^2}{\sin^2(\pi x)}\,. \tag{C.27}$$

Note also that we can take the opposite limit $\omega_1\to\infty$. A useful result is

$$\sum_{n\in\mathbb{Z}}\frac{1}{(x+2n\omega)^2}-\sum_{n\in\mathbb{Z}-\{0\}}\frac{1}{(2n\omega)^2}=\left(\frac{\pi}{2\omega}\right)^2\left(\frac{1}{3}+\frac{1}{\sinh^2(\frac{\pi}{2\omega}x)}\right)\,. \tag{C.28}$$

C.2.2 The twisted Weierstrass functions

The twisted Weierstrass function is defined, for any integer $n \geq 1$, by

$$\wp_n(x;\omega_1,\omega_2) = \sum_{k\in\mathbb{Z}_n} \wp\left(x + \frac{k}{n}2\omega_1;\omega_1,\omega_2\right) . \tag{C.29}$$

For $n = 1$ one recovers the usual Weierstrass function $\wp$. The periodic properties are

$$\wp_n\left(x;\omega_1,\omega_2\right) = \wp_n\left(x + \frac{2\omega_1}{n};\omega_1,\omega_2\right) = \wp_n\left(x + 2\omega_2;\omega_1,\omega_2\right) . \tag{C.30}$$

For the $n = 2$ twisted Weierstrass function, we can derive the equality:

$$\wp_2\left(x;\omega_1,\omega_2\right) = \wp(x;\omega_1,\omega_2) + \wp(x+\omega_1;\omega_1,\omega_2) = \wp(x;\frac{\omega_1}{2},\omega_2) + \frac{\pi^2}{6\omega_1^2}\left[2E_2(2\frac{\omega_2}{\omega_1}) - E_2(\frac{\omega_2}{\omega_1})\right] \tag{C.31}$$

as well as the dual

$$\wp(x;\omega_1,\omega_2) + \wp(x+\omega_2;\omega_1,\omega_2) = \wp(x;\omega_1,\frac{\omega_2}{2}) - \frac{\pi^2}{6\omega_1^2}\left[E_2(\frac{\omega_2}{\omega_1}) - \frac{1}{2}E_2(\frac{\omega_2}{2\omega_1})\right] . \tag{C.32}$$

These can be proven using the definition of the Weierstrass function $\wp$, as well as the definition of the second Eisenstein series E_2.

C.3 Theta and Eta Functions

We first fix our conventions for the theta-functions with characteristics:

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \sum_{n\in\mathbb{Z}} \exp\left[i\pi(n+\alpha)^2\tau + 2\pi i\beta(n+\alpha)\right] . \tag{C.33}$$

Particular examples of these theta-functions include:

$$\begin{aligned}
\theta_2(q) &= \theta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(q) = 2q^{1/8} + 2q^{9/8} + 2q^{25/8} + 2q^{49/8} + ...\\
\theta_3(q) &= \theta\begin{bmatrix}0\\ 0\end{bmatrix}(q) = 1 + 2q^{1/2} + 2q^2 + 2q^{9/2} + 2q^8 + ...\\
\theta_4(q) &= \theta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}(q) = 1 - 2q^{1/2} + 2q^2 - 2q^{9/2} + 2q^8 + ...
\end{aligned}$$

We also make use of the Dedekind eta-function:

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) , \tag{C.34}$$

and the Klein invariant

$$j(q) = 1728\frac{E_4^3(q)}{E_4^3(q) - E_6^2(q)} = \frac{1}{q} + 744 + 196884q + ... \tag{C.35}$$

The function j is invariant under the modular group, while η satisfies

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau) \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau) . \tag{C.36}$$

C.4 Modular Forms and Sublattices

In this subsection we recall how to find a basis of the space of modular forms of weight k for the congruence subgroup $\Gamma(N)$ ([121]). First we note that the cusps of $\Gamma(N)$ can be identified with the pairs $\pm v \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N. This makes it possible to count the number of such cusps :

$$\epsilon_\infty(\Gamma(N)) = \begin{cases} 3 & N=2 \\ \frac{N^2}{2}\prod_{p|N}\left(1-\frac{1}{p^2}\right) & N\geq 3\,. \end{cases}$$

For any congruence subgroup Γ the space of modular forms $\mathcal{M}_k(\Gamma)$ of weight k decomposes into the subspace of cusp forms and the Eisenstein space: $\mathcal{M}_k(\Gamma) = \mathcal{S}_k(\Gamma) \oplus \mathcal{E}_k(\Gamma)$. For $N = 2$ we have

$$\dim \mathcal{S}_2(\Gamma(2)) = 0$$

and for $N \geq 3$,

$$\dim \mathcal{S}_2(\Gamma(N)) = 1 + \frac{N^2(N-6)}{24}\prod_{p|N}\left(1-\frac{1}{p^2}\right).$$

In particular, $\dim \mathcal{S}_2(\Gamma(3)) = 0$ and $\dim \mathcal{S}_2(\Gamma(6)) = 1$.

We will also use the fact that $\dim \mathcal{S}_{2k}(\Gamma_0(4)) = k-2$ for every integer $k \geq 2$, while $\dim \mathcal{S}_2(\Gamma_0(4)) = 0$.

We want an explicit basis of the Eisenstein space. For any vector $v = \begin{bmatrix} c \\ d \end{bmatrix} \in (\mathbb{Z}/N\mathbb{Z})^2$ of order N, and for $k \geq 3$, we define the (non-normalized) Eisenstein series

$$G_{k,N}\left[v\right](\tau) = G_{k,N}\begin{bmatrix} c \\ d \end{bmatrix}(\tau) = \sum_{v' \equiv v(N)}{}' \frac{1}{(c'\tau + d')^k}\,,$$

and for weight two

$$G_{2,N}\left[v\right](\tau) = G_{2,N}\begin{bmatrix} c \\ d \end{bmatrix}(\tau) = \frac{1}{N^2}\left[\wp\left(\frac{c\tau+d}{N},\tau\right) + G_2(\tau)\right]\,,$$

where the primed sum runs over those non-vanishing vectors $v' = \begin{bmatrix} c' \\ d' \end{bmatrix}$ that equal v modulo N. One can show that the Fourier expansion of these functions in terms of $q = e^{2i\pi\tau}$ is:

$$G_{k,N}\begin{bmatrix} c \\ d \end{bmatrix}(q) = \delta(c)\zeta_N^d(k) + \frac{(-2\pi i)^k}{N^2(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1,N}\begin{bmatrix} c \\ d \end{bmatrix}(n)q^{n/N}$$

where

$$\sigma_{k-1,N}\begin{bmatrix} c \\ d \end{bmatrix}(n) = \sum_{m|n \text{ and } \frac{n}{m}\equiv c(N)} \operatorname{sgn}(m)m^{k-1}\exp\left(2\pi i\frac{dm}{N}\right)$$

and

$$\zeta_N^d(k) = \sum_{d'\equiv d\,(N)}{}' \frac{1}{(d')^k}\,.$$

This Fourier expansion is valid for all $k \geq 2$, including $k = 2$ which is the case we are mostly interested in.

For $k \geq 3$, any set $\{G_{k,N}[v]\}$ with one v corresponding to each cusp of $\Gamma(N)$ represents a basis of the space $\mathcal{E}_k(\Gamma(N))$ of Eisenstein series of weight k on $\Gamma(N)$ (and in particular $\dim \mathcal{E}_k(\Gamma(N)) = \epsilon_\infty$). For the case $k = 2$ these statements have to be modified, because of the lack of modularity of the (ordinary) weight 2 Eisenstein series. It turns out that $\dim \mathcal{E}_2(\Gamma(N)) = \epsilon_\infty - 1$, and that $\mathcal{E}_2(\Gamma(N))$ is the set of linear combinations of the $\{G_{k,N}[v]\}$ (where $v \in (\mathbb{Z}/N\mathbb{Z})^2$ is of order N) whose coefficients sum to 0.[3]

The Eisenstein series $G_{k,N}[v]$ have good transformation properties under $SL(2,\mathbb{Z})$ for $k \geq 3$ and $N \in \{2,3\}$ provided the vector v is transformed accordingly:[4]

$$\frac{1}{(c\tau+d)^k} G_{k,N}[v]\left(\frac{a\tau+b}{c\tau+d}\right) = G_{k,N}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} v\right](\tau)\,.$$

For $k = 2$, we have to take into account a non-holomorphic term, except for linear combinations where the sum of the coefficients vanishes, as is the case for the potentials considered in the bulk of the thesis.

Finally, we also define

$$E_{2,N}(\tau) \quad = \quad E_2(\tau) - N E_2(N\tau)\,. \tag{C.37}$$

These are weight 2 modular forms of $\Gamma_0(N)$. We use extensively the fact that $\mathcal{M}_2(\Gamma_0(4))$ has dimension 2 and is generated by

$$\begin{aligned} -E_{2,2}(q) &= 1 + 24q + 24q^2 + 96q^3 + 24q^4 + 144q^5 + \ldots \\ -E_{2,4}(q) &= 3 + 24q + 72q^2 + 96q^3 + 72q^4 + 144q^5 + \ldots \end{aligned}$$

We note the transformation property of the form $E_{2,2}$ under $S_2 : \tau \to -1/(2\tau)$:

$$E_{2,2}(-1/(2\tau)) \quad = \quad E_2(-1/(2\tau)) - 2E_2(-1/\tau) = -2\tau^2 E_{2,2}(\tau)\,. \tag{C.38}$$

[3]Theorem 4.6.1 in [121]

[4]For generic N the relation between the normalized Eisenstein series, which enjoy these good transformation properties, and the series $G_{k,N}[v]$ is not simply a proportionality relation (see formula (4.5) in [121]), but it is a simple rescaling for $N = 2$ and $N = 3$.

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Antoine BOURGET **01/07/2016**

Sujet : Vides et Modularité dans les théories de jauge supersymétriques $\mathcal{N} = 1^*$

Résumé : Nous explorons la structure des vides dans une déformation massive de la théorie de Yang-Mills maximalement supersymétrique en quatre dimensions. Sur un espace-temps topologiquement trivial, la théorie des orbites nilpotentes dans les algèbres de Lie rend possible le calcul exact de l'indice de Witten. Nous en donnons les fonctions génératrices pour les algèbres classiques, et recourons à un calcul explicite pour les exceptionnelles. Après compactification sur un cercle, un lien entre les théories de jauge supersymétriques et les systèmes intégrables est exploitable pour réduire la chasse aux vides à une extrémisation du hamiltonien de Calogero-Moser elliptique twisté. Une analyse soigneuse des propriétés globales du groupe de jauge et des opérateurs de ligne est nécessaire pour obtenir un accord parfait. En combinant exploration numérique sur ordinateur et contrôle analytique grâce à la théorie des formes modulaires, nous exhibons la structure des vides massifs pour des algèbres de rang petit, et mettons en évidence de nouvelles propriétés modulaires. Nous montrons que des branches de vides de masse nulle existent, et nous en donnons la structure exacte pour les algèbres de rang deux.

Mots clés : Théories de jauge supersymétriques, Systèmes intégrables, Modularité

Subject : Modularity and Vacua in $N = 1^*$ Supersymmetric Gauge Theory

Résumé : We investigate the vacuum structure of a massive deformation of the maximally supersymmetric Yang-Mills gauge theory in four dimensions. When the topology of spacetime is trivial, the Witten index can be computed exactly for any gauge group using the theory of nilpotent orbits in Lie algebras. We provide generating functions for classical algebras and an explicit calculation for the exceptional ones. Upon compactification on a circle, one can use a bridge between supersymmetric gauge theories and complex integrable systems to reduce the analysis of vacua to the search of extrema of the twisted elliptic Calogero-Moser Hamiltonian. A careful inspection of global properties of the gauge group and line operators are needed to reach total agreement. Using a combination of numerical exploration on a computer and analytical control through the theory of modular forms, we determine the structure of massive vacua for low-rank gauge algebras and exhibit new modular properties. We also show that massless branches of vacua can exist, and provide an analytic description for rank two gauge algebras.

Keywords : Supersymmetric gauge theories, Integrable systems, Modularity