Two Pages on Matrix Models and Topological Strings

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These are reading notes on [1]. It includes extensive quotations.

1 Matrix Models

Matrix models are quantum gauge theories in zero dimensions. Consider an action $\frac{1}{g_s}W(M) = \frac{1}{2g_s}\mathrm{Tr}M^2 + \frac{1}{g_s}\sum_{p\geq 3}\frac{g_p}{p}\mathrm{Tr}M^p$ for a Hermitian $N \times N$ matrix. The partition function Z can be evaluated by perturbation theory around the Gaussian point as a power series in the g_p, using fatgraphs. The perturbative expansion of the free energy $F = \log Z$ will involve only connected vacuum bubbles and we can write

$$F(t) = \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} F_{g,h} g_s^{2g-2} t^h = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}\,, \qquad (1)$$

where g is the genus[1] of the fatgraphs, h is the number of holes and $t = Ng_s$ is the 't Hooft parameter. The right-hand side is the large N expansion at fixed t.

How to compute $F_g(t)$? There is a clever trick to tackle this problem. The matrix model has a gauge symmetry $M \to UMU^\dagger$, which can be used to diagonalize M. Using the Fadeev-Popov technique we can rewrite Z as an integral over the eigenvalues:

$$\begin{aligned} Z &= \frac{1}{\mathrm{Vol}U(N)}\int \mathrm{d}M e^{-\frac{1}{g_s}W(M)} \\ &= \frac{1}{N!}\int \prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i}{2\pi} e^{N^2 S_{\mathrm{eff}}(\lambda)}\,, \end{aligned} \qquad (2)$$

where the effective action is

$$S_{\mathrm{eff}}(\lambda) = -\frac{1}{tN}\sum_{i=1}^{N} W(\lambda_i) + \frac{2}{N^2}\sum_{i<j}\log|\lambda_i - \lambda_j|\,. \qquad (3)$$

In the large N limit, the eigenvalues can be described by the density function $\rho(\lambda)$ that can be computed[2] by variation of $S_{\mathrm{eff}}(\lambda)$. The effective action can be expressed in terms of ρ, and one can show that $F_0(t) = t^2 S_{\mathrm{eff}}(\rho)$. Higher-genus coefficients can also be obtained.

Note that a different strategy, involving orthogonal polynomials, can be used to compute the $F_g(t)$.

2 Topological Sigma Models

2.1 Cohomological TQFT

A cohomological TQFT is a QFT defined on a manifold M that has an underlying scalar symmetry δ (called *topological* symmetry) acting on the fields ϕ_i in such a way that the correlation functions don't depend on the background metric.

If the energy-momentum tensor $T_{\mu\nu}$ can be written as

$$T_{\mu\nu} = \delta G_{\mu\nu} \qquad (4)$$

for some tensor $G_{\mu\nu}$, then by a standard calculation a correlator of δ-invariant operators $\mathcal{O}$ doesn't depend on the metric.[3] Here we will assume that $\delta^2 = 0$ and we restrict the observables to the cohomology of δ. Another standard argument shows that in such cohomological theories, the semi-classical approximation for the computation of a correlation function is exact.

The *descent equations* are the equations $\mathrm{d}\phi^{(n)} = \delta\phi^{(n+1)}$ that, if solved for a scalar topological observable $\phi^{(0)}$, provide a family of topological non-local observables $\int_{\gamma_{i_n}}\phi^{(n)}$ for $i_n = 1, \cdots, b_n$ and $n = 1, \cdots, \dim M$.

2.2 Topological Twists

An $\mathcal{N} = 2$ sigma model, defined on a Riemann surface Σ_g, has four supercharges $Q_{\pm\pm}$, in addition to the spacetime generators (the translations P_μ and the rotation J) and internal $U(1)$ currents $F_{L,R}$. We define the vectorial current $F_V = F_L + F_R$ and the axial current $F_A = F_L - F_R$. We consider d chiral and d anti-chiral superfields $\Phi^I = (x^I, \psi^I, F^I)$ and $\Phi^{\bar{I}}$ and the action

$$S = \int_{\Sigma_g}\mathrm{d}^2 z\int \mathrm{d}^4\theta K(\Phi^I, \Phi^{\bar{I}})\,. \qquad (5)$$

This is a sigma model whose target is a Kähler manifold of complex dimension d and metric $G_{I\bar{J}} = \partial_I\partial_{\bar{J}}K(x^I, x^{\bar{J}})$.

This sigma model can be twisted in two different ways, with a redefinition of the spin current:

- A-twist : $\tilde{J} = J - F_V$.
- B-twist : $\tilde{J} = J + F_A$.

[1] A fatgraph is characterized by its number of edges E, of vertices V, and closed loops h. The genus is defined by $2g - 2 = E - V - h$.

[2] One has to solve $\frac{1}{2t}W'(\lambda) = \mathrm{P}\int\frac{\rho(\lambda')\mathrm{d}\lambda'}{\lambda-\lambda'}$, which can be done by introducing the resolvent – there is a rich domain of research.

[3] We assume that δ is not anomalous, and we neglect boundary problems.

Note that this amounts to gauging one of the two $U(1)$ global currents by coupling it to the spin connection. Since the axial current has an anomaly given by the first Chern class of X, the B-model makes sense only on a Calabi-Yau space, where $c_1(X) = 0$. In each case, the four (fermionic) supercharges become two scalars (whose sum we call $\mathcal{Q}$) and one vector G_μ that satisfy

$$\mathcal{Q}^2 = 0 \quad \text{and} \quad \{\mathcal{Q}, G_\mu\} = P_\mu \,. \tag{6}$$

One can prove that the two twisted theories are cohomological TQFTs, by taking $\delta = \mathcal{Q}$ and finding an appropriate tensor that satisfies (4).

2.3 Correlation Functions

Let us focus on the A-model on a Calabi-Yau X. One finds that the $\mathcal{Q}$-cohomology is given by operators[4] $\mathcal{O}_\phi$ where $\phi \in H^p(X)$, so the $\mathcal{Q}$-cohomology is in one-to-one correspondence with the de Rham cohomology of the target X. Then one can prove that $\langle \mathcal{O}_{\phi_1} \cdots \mathcal{O}_{\phi_l} \rangle = 0$ unless

$$\sum_{k=1}^{l} \deg \phi_k = 2d(1-g)\,. \tag{7}$$

This implies that for $g > 1$ all correlation functions vanish. This problem will be addressed next, by coupling the theory with two-dimensional gravity.

3 Topological Strings

3.1 Closed strings

The twisted TQFTs of section 2 are very similar to the bosonic string, with $\mathcal{Q}$ playing the role of the BRST charge. This suggests the definition[5]

$$F_g = \int_{\bar{M}_g} \langle \prod_{k=1}^{6g-6} \int_{\Sigma_g} \mathrm{d}^2 z \left(G_{zz} (\mu_k)_{\bar{z}}{}^{z} + G_{\bar{z}\bar{z}} (\bar{\mu}_k)_{z}{}^{\bar{z}} \right) \rangle \,, \tag{8}$$

where μ_k are the Beltrami differentials and $\bar{M}_g$ is the moduli space of Riemann surfaces of genus g. We can decompose $F_g = \sum_{\beta \in H^2(X,\mathbb{Z})} N_{g,\beta} Q^\beta$, where $N_{g,\beta}$ are the Gromov-Witten invariants[6], with $Q^\beta = \exp\left(-\int_\beta \omega\right)$ and ω the complexified Kähler form on X.

There is a relation between topological string amplitudes and physical superstring amplitudes. For instance, type IIA/B compactified on X is $\mathcal{N} = 2$ supergravity in four dimensions. The low-energy effective action for the vector multiplets (up to two derivatives) is coded by the prepotential, which is F_0 of the A/B models of topological strings. The higher-genus F_g corresponds to other couplings in the supergravity theory.

3.2 Open strings

The previous discussion can be extended to open strings if we replace the Riemann surface Σ_g by $\Sigma_{g,h}$, with h holes. It is then necessary to specify boundary conditions in X: for the A model it turns out that the relevant boundary conditions are Dirichlet and given by Lagrangian[7] submanifolds of X.

4 String theories and Gauge theories

In equation (1), the middle term involves coefficients $F_{g,h}$ that could be seen as open string amplitudes on $\Sigma_{g,h}$. Is there such a string theory? In some cases, the answer is yes, and involves open topological strings whose target is a Calabi-Yau with topological D-branes. The identification is obtained using string field theory.

Now the right-hand side of (1) looks more like a closed string amplitude, which would be related to the open string theory by an open-closed duality. This kind of dualities are associated to *geometric transitions* that relate different geometric backgrounds.

References

[1] M. Marino. Chern-Simons theory, matrix models, and topological strings. *Int. Ser. Monogr. Phys.*, 131:1–197, 2005.

[2] J. Polchinski. *String theory. Vol. 1: An introduction to the bosonic string.* Cambridge University Press, 2007.

[4] We don't explain here how the operators are constructed.
[5] See for instance equation (5.4.19) in [2].
[6] These invariants are in general rational, and they can be written in terms of the integer Gopakumar-Vafa invariants.
[7] A Lagrangian submanifold is a cycle on which the Kähler form vanishes.