Nankai Symposium on Mathematical Dialogues

(August 4th 2021

BRANES, QUIVERS & AFFINE GRASSMANNIANS

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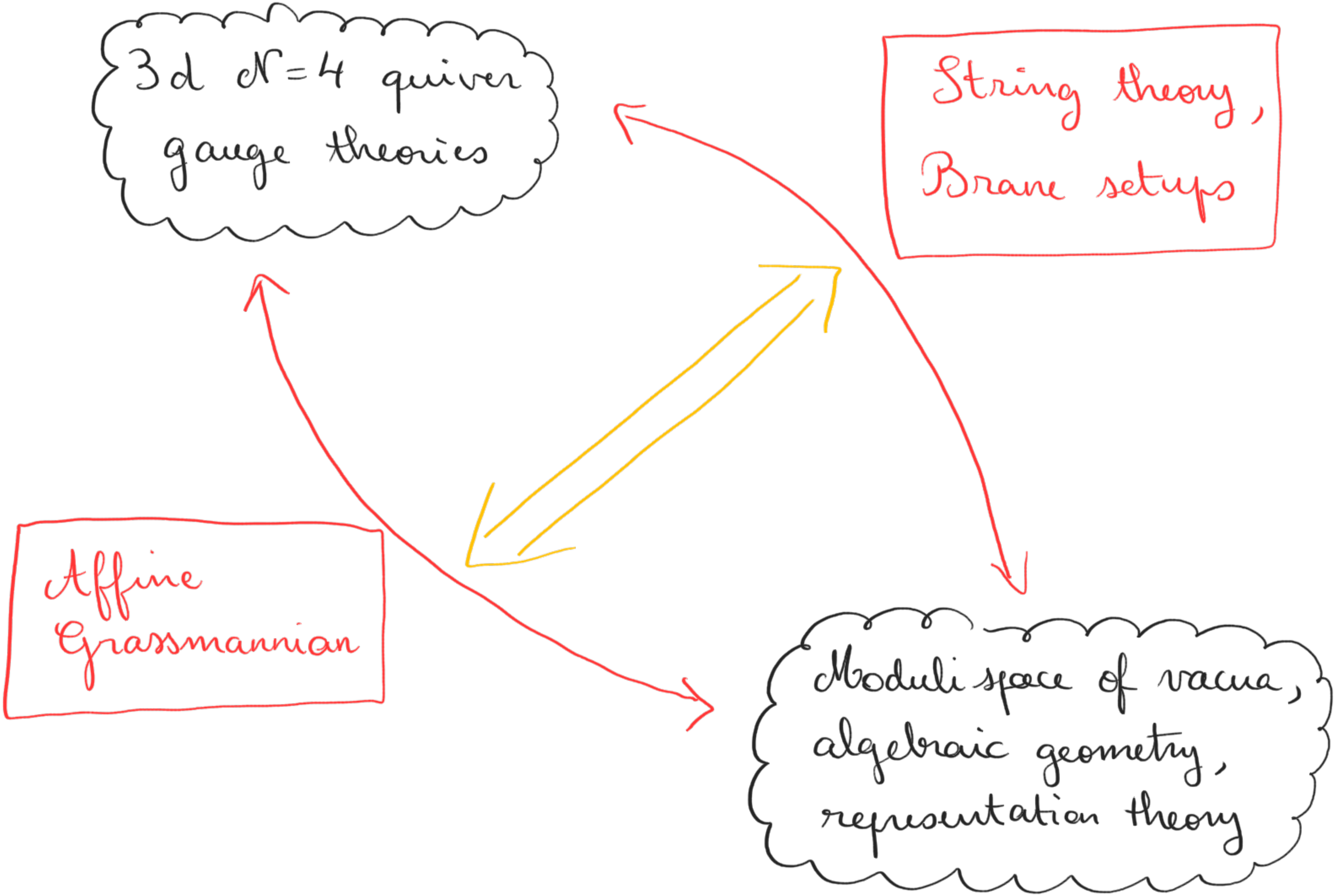
Based on [2102.06190] with J. Grimminger, A. Hanany, M. Sperling, Z. Zhong

OVERVIEW



3d $\mathcal{N}=4$ quiver gauge theories
String theory, Brane setups
Affine Grassmannian
Moduli space of vacua, algebraic geometry, representation theory

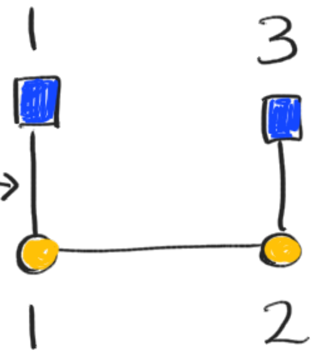
QUIVER GAUGE THEORY

3d $\mathcal{N}=4$ gauge theory :

hypermultiplets

flavor symmetry
("framing")

Gauge group
$U(1) \times U(2)$



quiver of type $G = A_2$

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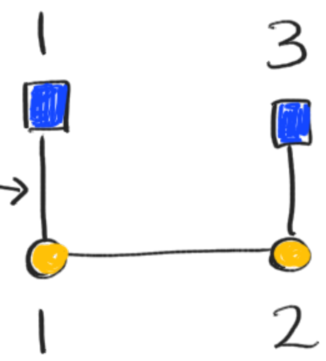
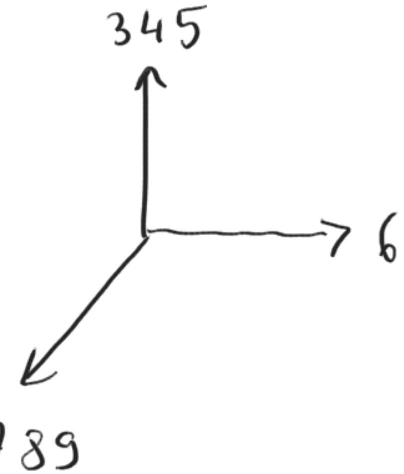


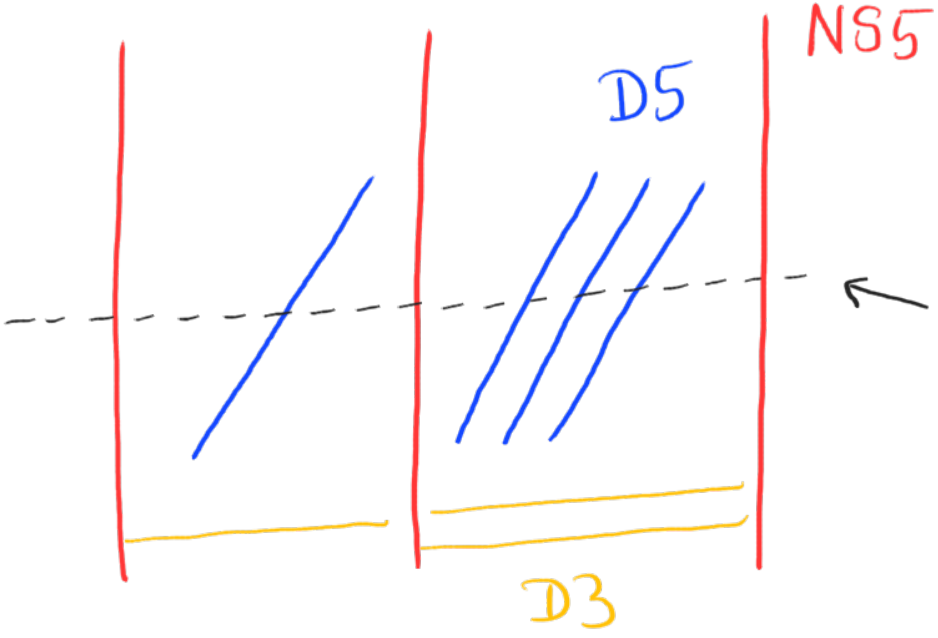
Gauge group $U(1) \times U(2)$

quiver of type $G = A_2$

Type IIB string:

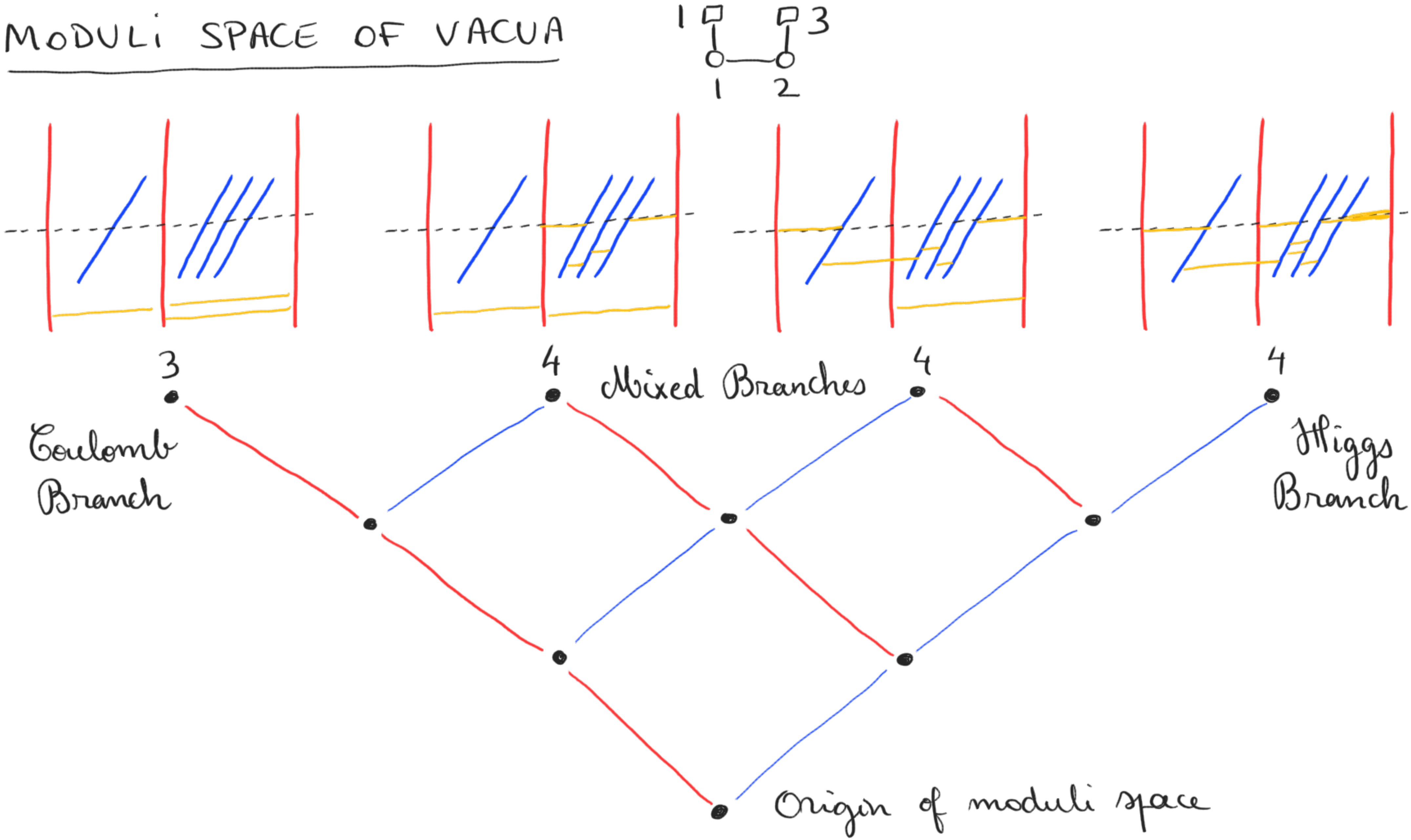
$\mathbb{R}^{1,2} \times$





$x^{345} = x^{789} = 0$

MODULI SPACE OF VACUA



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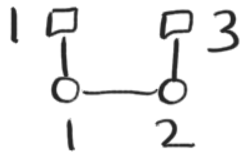
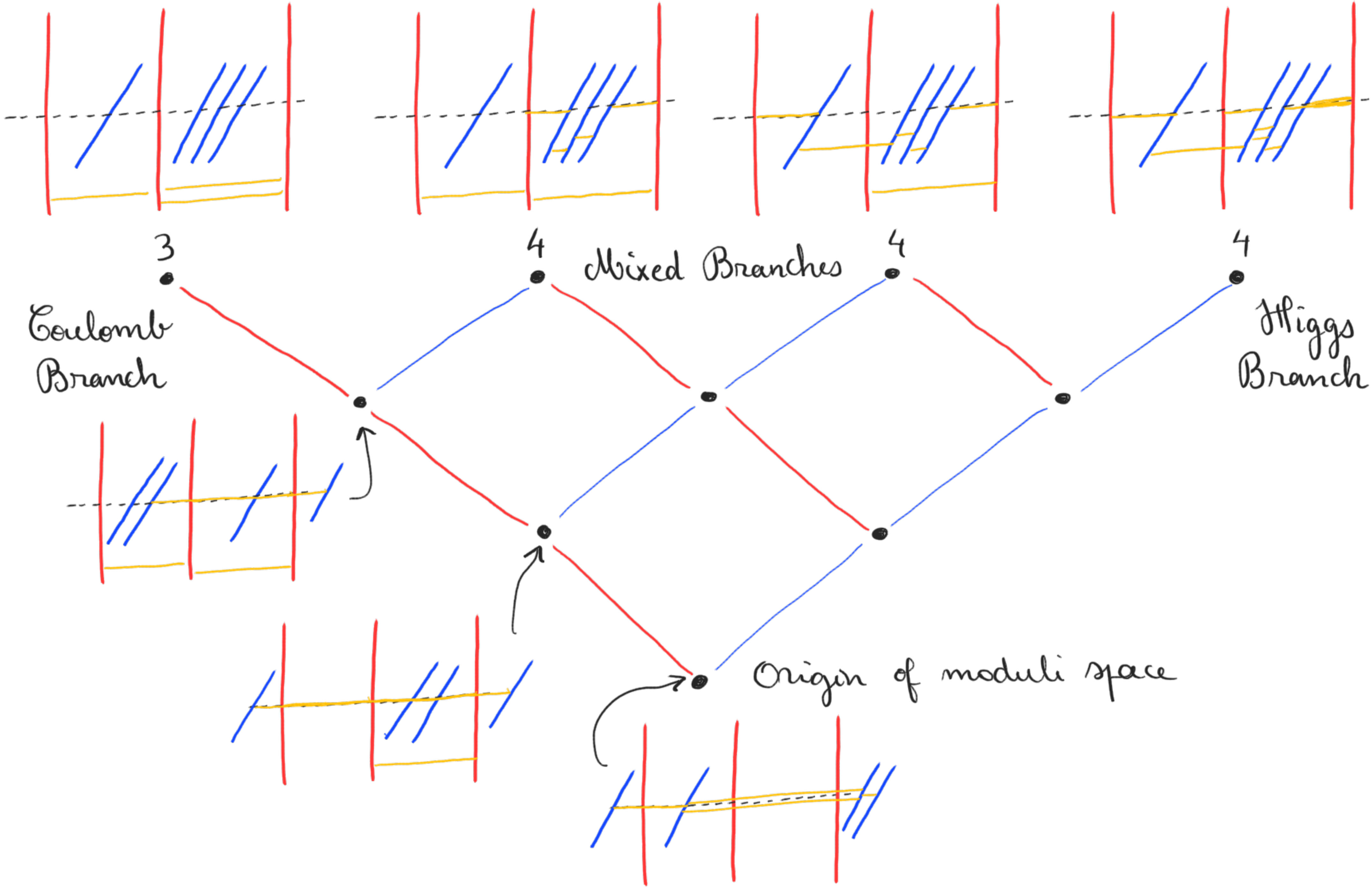
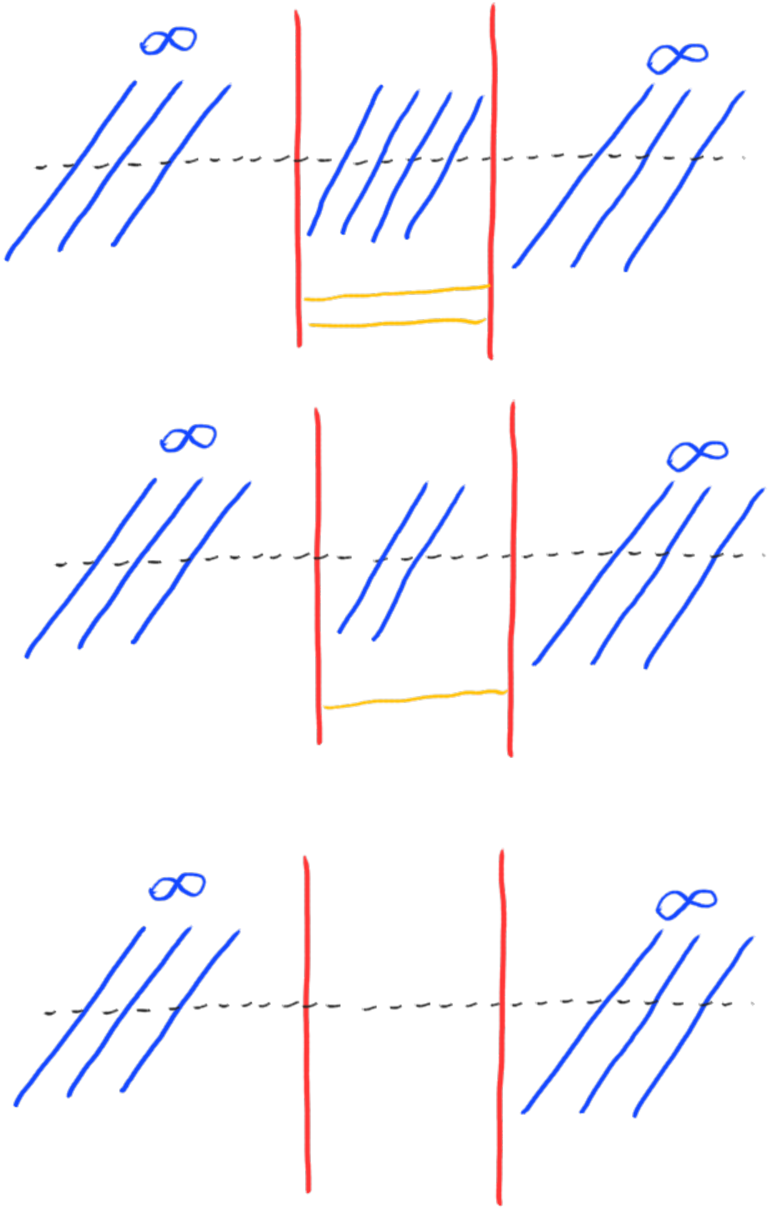




COULOMB BRANCH



$$\lim_{N\to\infty} \mathcal{C}\left(\begin{matrix}\square\, 2N\\ |\\ \circ\, N\end{matrix}\right)$$

$$|||$$

$$Gr_{SL(2,\mathbb{C})}$$

$$\longrightarrow \mathcal{C}\left(\begin{matrix}\square\, 4\\ |\\ \circ\, 2\end{matrix}\right)$$

$$\longrightarrow \mathcal{C}\left(\begin{matrix}\square\, 2\\ |\\ \circ\, 1\end{matrix}\right)$$

$$\longrightarrow \mathcal{C}\left(\begin{matrix}\square\, 0\\ |\\ \circ\, 0\end{matrix}\right)$$

AFFINE GRASSMANNIAN for $G = SL(2,\mathbb{C})$

$$G((t)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{C}((t)) \text{ and } ad-bc=1 \right\} \qquad \left(a = \sum_{k=-N}^{\infty} a_k t^k \right)$$

$$G[[t]] = \left\{ \quad " \quad \middle| \quad " \in \mathbb{C}[[t]] \quad " \quad \right\} \qquad \left(a = \sum_{k=0}^{\infty} a_k t^k \right)$$

ring of integers

field

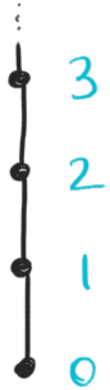
$$Gr_G = G((t)) / G[[t]] \qquad \leftarrow \text{quotient on the right}$$

Points in Gr_G :

$$\begin{pmatrix} t^{\lambda/2} & \sum_{k=-\lambda/2}^{\lambda/2 - 1} x_k t^k \\ 0 & t^{-\lambda/2} \end{pmatrix} \cdot G[[t]] \qquad (\lambda = 0, 2, 4, \dots)$$

Action of $G[[t]]$ on the left produces orbit $[Gr_G]^\lambda$

$$\overline{[Gr_G]^\lambda} = \bigcup_{\substack{\mu \in 2\mathbb{N} \\ \mu \leq \lambda}} [Gr_G]^\mu$$



$$\dim_{\mathbb{H}} [Gr_G]^\lambda = \frac{\lambda}{2}$$

SLICES IN Gr_G

Λ = Coweight lattice of G

Λ_0 = Coroot lattice of G

PARTIAL ORDER on Λ:

$$\lambda \leq \mu \iff \mu - \lambda \in \Lambda_0^+$$

Transverse slice:

$$\boxed{\overline{[\mathcal{W}_G]}^{\mu}_{\lambda}} = \mathcal{C}oul\left(\begin{array}{l}\text{Quiver given by } G \text{ with framing} \\ \mu \text{ and imbalance } \lambda\end{array}\right)$$

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$$\boxed{\overline{[W_G]}^{\mu}_{\lambda}} = \text{Coul}\left(\begin{array}{l}\text{Quiver given by } G \text{ with framing} \\ \mu \text{ and imbalance } \lambda\end{array}\right)$$

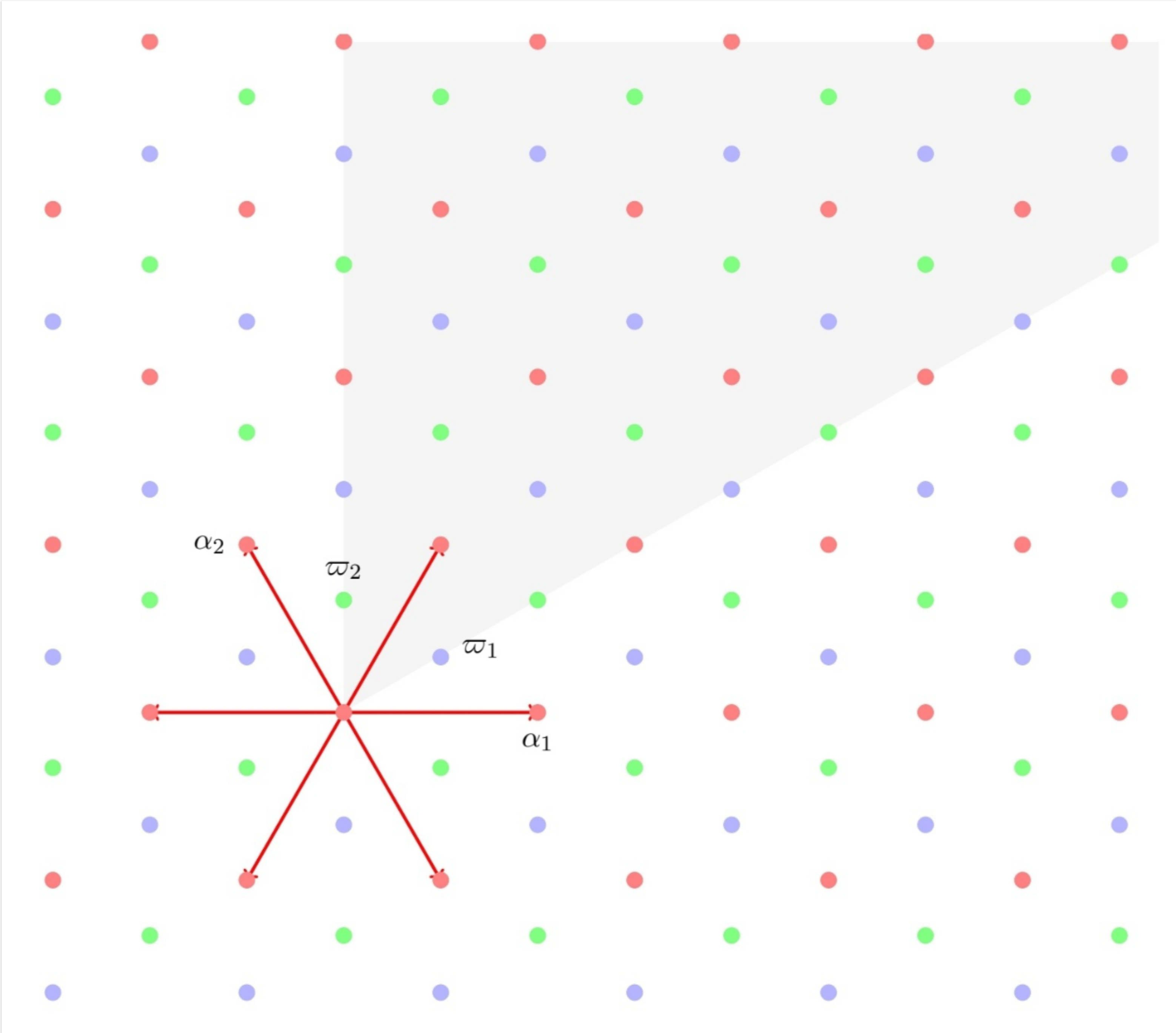
$$= \begin{cases} \mathbb{C}^2/\mathbb{Z}_k & \text{if } \mu - \lambda \text{ simple coroot} \\ \overline{\mathcal{O}}_{min} & \text{if } \mu - \lambda \text{ is a short dominant coroot } [\dots] \\ \text{"Quasiminimal" singularities} & \text{otherwise} \end{cases}$$

EXAMPLE : PSL(3, $\mathbb{C}$)

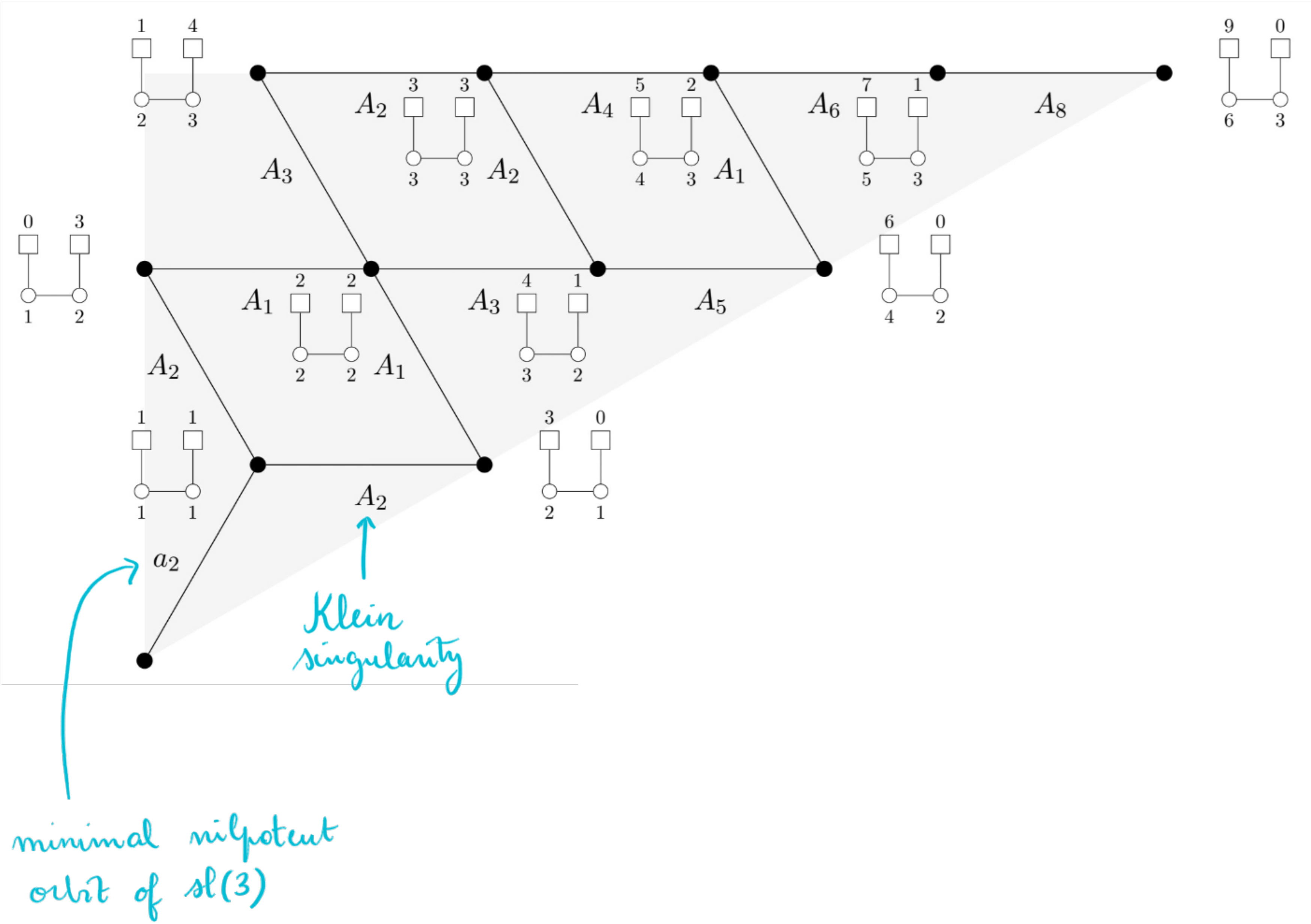
Coweight lattice

$\cup$

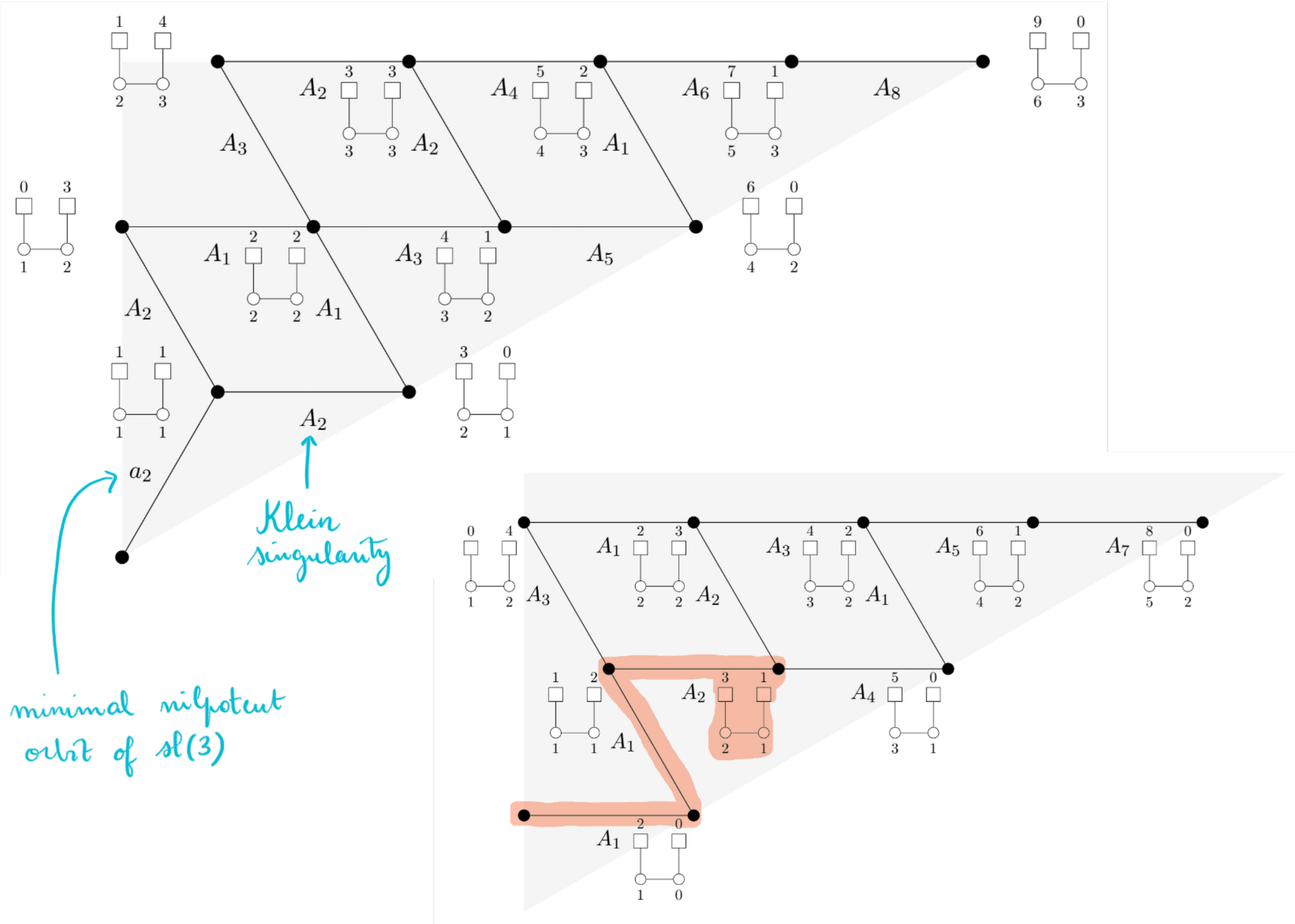
Coroot lattice



$$\pi_0\left(\mathrm{Gr}_{PSL(3,\mathbb{C})}\right) = \pi_1(PSL(3,\mathbb{C})) = \mathbb{Z}/3\mathbb{Z}$$



A_2
A_4
A_6
A_8
A_3
A_2
A_1
A_1
A_3
A_5
A_2
A_1
A_2
a_2
Klein singularity
minimal nilpotent orbit of sl(3)



A_2
A_4
A_6
A_8
A_3
A_2
A_1
A_1
A_3
A_5
A_2
A_1
A_2
a_2
Klein singularity
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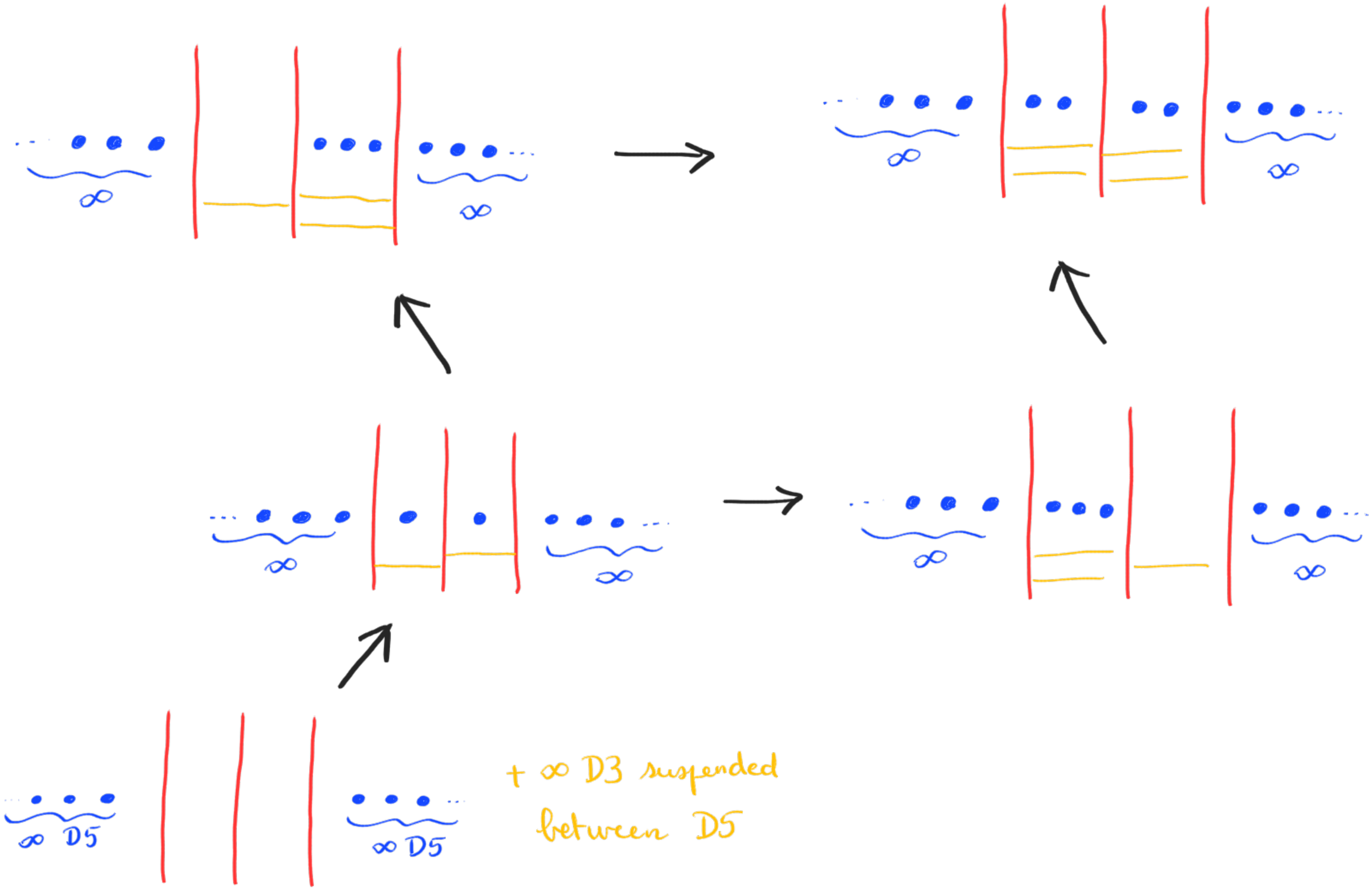
A_1
A_3
A_5
A_7
A_3
A_2
A_1
A_2
A_4
A_1
A_1



∞
∞ D5
+ ∞ D3 suspended
between D5

The Coulomb branch of ANY "good" framed quiver of G = ADE type appears as a transverse slice in Gr_G.

Orbits / slices in Gr_G

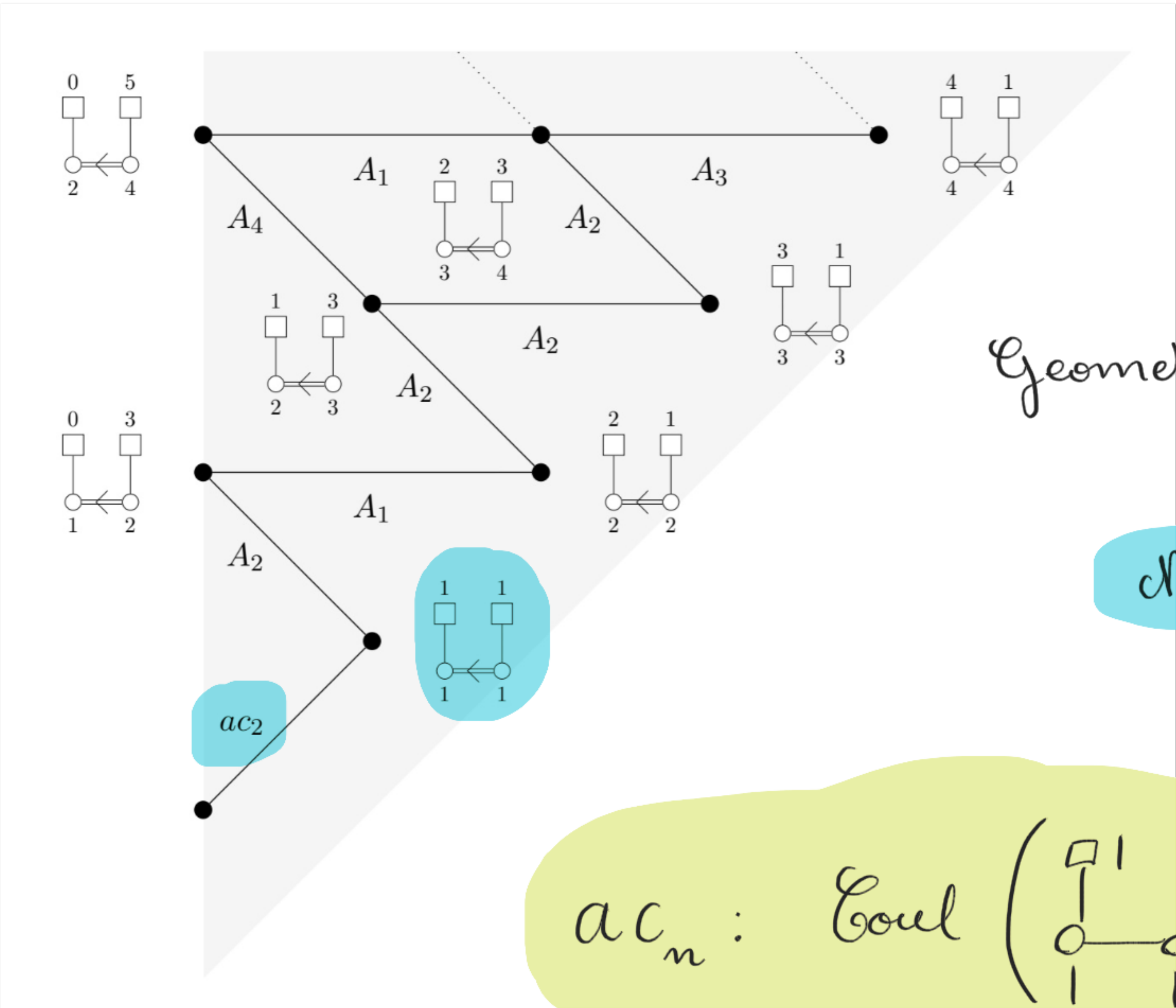
Hasse diagram
(phases of gauge theories, Higgs mechanism)

Brane systems

Kraft-Procesi transitions
(for nilpotent orbits)

What about non simply laced?

NON-SIMPLY LACED GROUPS

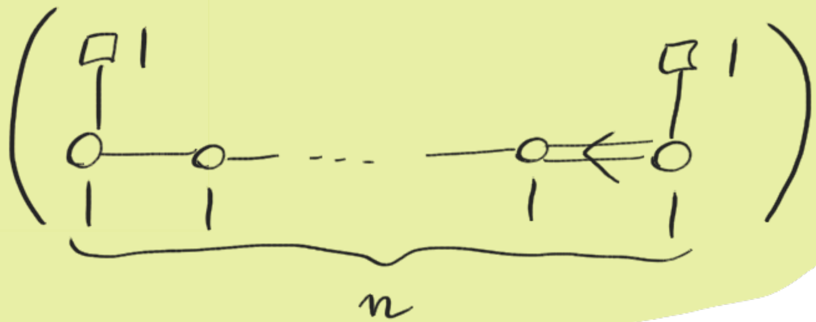


Geometry of Weyl chamber

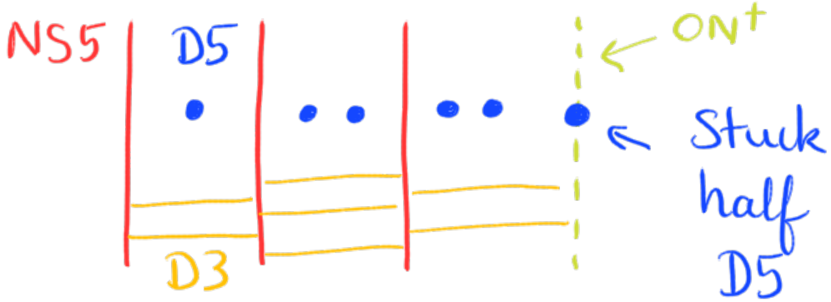
$\Downarrow$

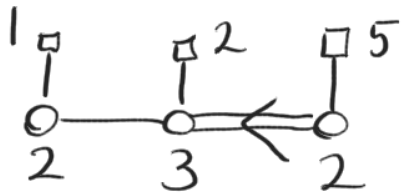
New slices beyond $\mathcal{O}_{min}$ and $\mathbb{C}^2/\Gamma$

ac_n : Coul



BRANES + ORIENTIFOLDS





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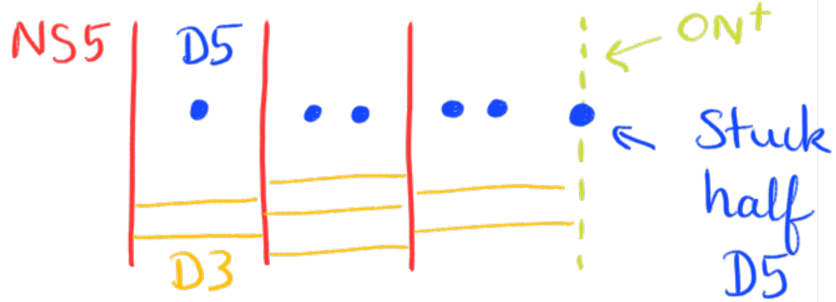

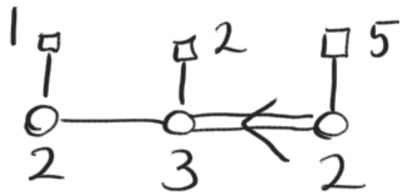

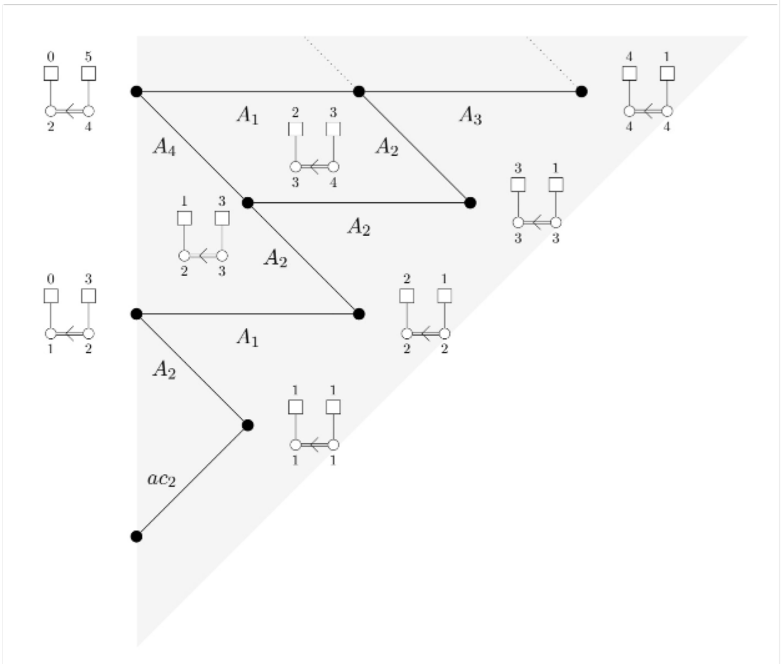

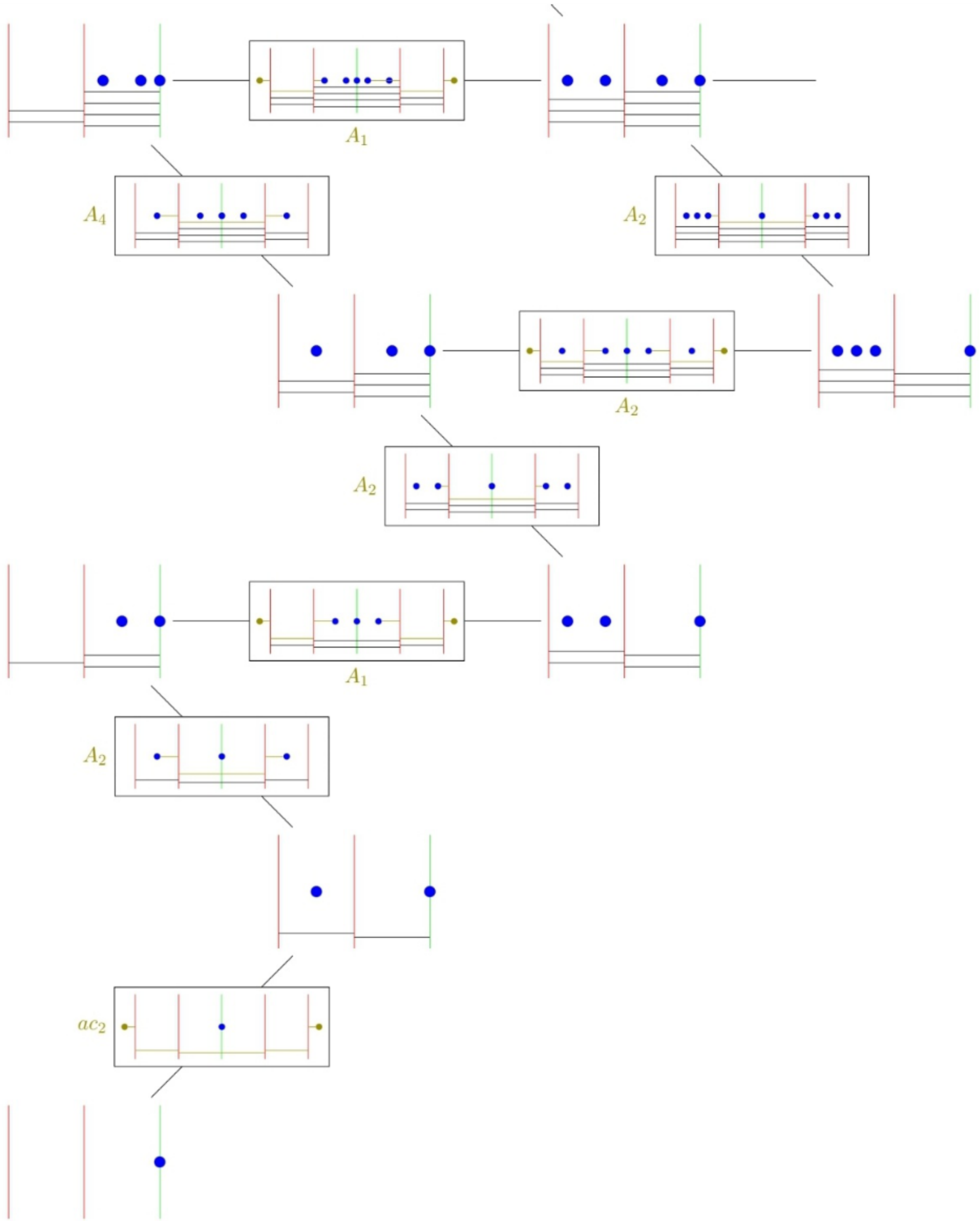

CONCLUSION

- $Gr_G \Longleftrightarrow$ a "universal" geometric analog of certain brane systems

- Slices in $Gr_G \rightsquigarrow$ known & new elementary slices.

QUIVER $\begin{cases} \text{SUBTRACTION} \\ \text{ADDITION} \end{cases}$ algorithms

Study theories in 3d, 4d, 5d, 6d ...

- Open problems: double affine grassmannian, moduli space of instantons exceptional groups, ...