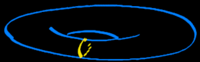
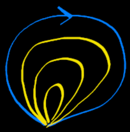
$\log(-1) = i\pi$?





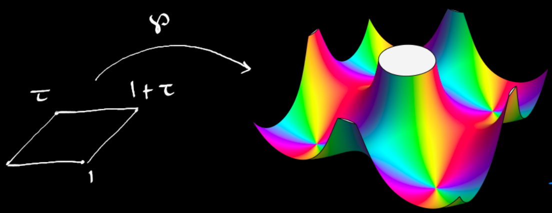
Analyse Complexe & Surfaces de Riemann

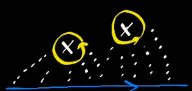
$\oint_{\mathcal{C}} f = 0$

$\sum_{n \in \mathbb{Z}} a_n z^n$

$\frac{\partial f}{\partial \bar{z}} = 0$

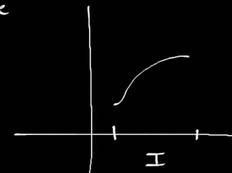
$\mathrm{Res}_0 f = a_{-1}$



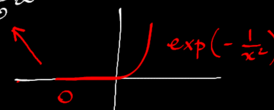


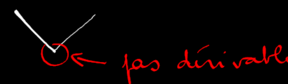
Analyse réelle.

$f : I \longrightarrow \mathbb{R}$ $\qquad$ I = intervalle



* fonctions polynomiales $f(x) = a_0 + a_1 x + \dots + a_N x^N$
* fonctions analytiques $\sum_{n=0}^{\infty} a_n x^n \overset{\not\Longleftarrow}{\Longrightarrow} \mathcal{C}^\infty$
* fonctions de classe $\mathcal{C}^1$ … $\mathcal{C}^2$ …
* fonctions dérivables
* fonctions continues (monstres)





Analyse complexe :

$$\boxed{\text{Rigidité} + \text{Géométrie 2d} = \underline{\text{Miracles}}!}$$

Fonctions $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$: très compliquées.

* fonctions conformes : préservent les angles orientés

expérimentalement : "$f(z)$ sans $\bar{z}$"

$$\underset{(x,y)}{\mathbb{R}^2} \approx \underset{(z,\bar{z})}{\mathbb{C}}$$

$$\begin{cases} z = x + iy \\ \bar{z} = x - iy \end{cases}$$

$$\begin{cases} x = \frac{1}{2}(z + \bar{z}) \\ y = \frac{1}{2i}(z - \bar{z}) \end{cases}$$

$$f(x,y) \Leftrightarrow f(z,\bar{z})$$

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0}$$

$$f(z) - f(w) \approx \alpha (z - w)$$

On préserve les angles si $\lim\limits_{z \to w} \dfrac{f(z)-f(w)}{z-w}$ existe.

$$f(z,\bar z) - f(w,\bar w) = \frac{\partial f}{\partial z}(z-w) + \overset{0}{\cancel{\frac{\partial f}{\partial \bar z}}}(\bar z - \bar w) + \dots$$

$$f(z,\bar z) = |z|^2 = z\bar z \qquad \frac{\partial f}{\partial z} = \bar z \qquad \frac{\partial f}{\partial \bar z} = z$$

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar z)$$

Plan : II) Fonctions holomorphes (= dérivable)

III) Fonctions analytiques

IV) Rigidité et topologie

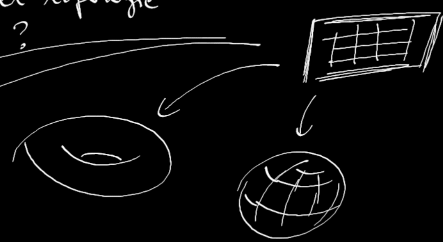
V) Périodique ?

∞ ———————— ∞ $\mathbb{R}$

$\mathbb{R}$ replié

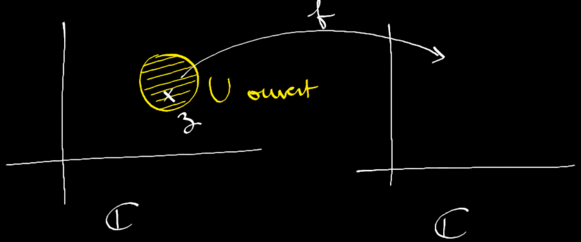
périodique $\downarrow$

S^1



. . . .

I) Fonctions holomorphes



Soit $f : U \longrightarrow \mathbb{C}$. Soit $z \in U$.

On dit que f est dérivable en z si $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ existe dans $\mathbb{C}$. Quand c'est le cas, on note $f'(z)$ cette limite.

Si f est dérivable en tout $z \in U$, on dit que f est holomorphe (ou dérivable) sur U.

$$\left(u_n \in \mathbb{C} \quad u_n \xrightarrow[n \to \infty]{} l \iff \lim_{n \to \infty} |u_n - l| = 0 \right)$$

⚠ : $\boxed{f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \ \mathcal{C}^\infty \text{ n'est pas forcément holomorphe !!}}$

Soit $f : \mathbb{R}^2 \simeq \mathbb{C} \longrightarrow \mathbb{R}^2 \simeq \mathbb{C}$ de classe $\mathcal{C}^1$

$$z = x + iy \longmapsto f(z) = u(x,y) + i\, v(x,y)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

$$f(z+h) - f(z) = h f'(z) + O(|h|^2).$$

$h = p + iq$ $\qquad$ $a + ib$

$$u(x+p, y+q) + i\, v(x+p, y+q) - u(x,y) - i\, v(x,y) = (p+iq)(a+ib) + \dots$$

$$p \frac{\partial u}{\partial x} + q \frac{\partial u}{\partial y} + i p \frac{\partial v}{\partial x} + i q \frac{\partial v}{\partial y} = (pa - qb) + i(pb + qa) + \dots$$

$$\boxed{a = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad b = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}}$$

Cauchy - Riemann.

$$\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \quad \checkmark \qquad (x+iy)^2 = (x^2 - y^2) + i(2xy)$$

$$f(x,y) = \underbrace{x^2 + y^2}_{u} \qquad v = 0 \qquad \left(|z|^2 = z\bar{z} \right)$$

$$f\,dz = (u+iv)(dx+i\,dy)$$
$$= (u\,dx - v\,dy) + i\,(v\,dx + u\,dy)$$

$d(\dots) = 0$

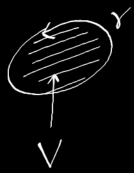
$$d(u\,dx - v\,dy) = \frac{\partial u}{\partial y}\,dy\wedge dx - \frac{\partial v}{\partial x}\,dx\wedge dy$$
$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dy\wedge dx = 0$$

$$\left(\int_{\partial V} (u\,dx - v\,dy) = \iint \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\,dy\right)$$

$$\left(\int_{\partial V} \vec{A}\cdot\vec{dl} = \iint_V \vec{\mathrm{rot}}\,\vec{A}\,dS\right)$$

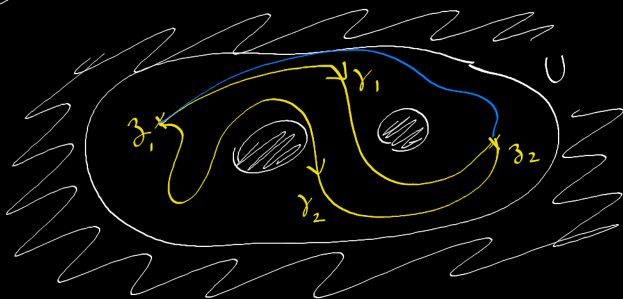
" $$d(f\,dz) = \frac{\partial f}{\partial z}\underbrace{dz\wedge dz}_{0} + \underbrace{\frac{\partial f}{\partial \bar{z}}}_{0}\,d\bar{z}\wedge dz = 0\ !$$ "

Théorème (Cauchy). $f : D \to \mathbb{C}$ holomorphe. Soit γ un lacet contractile dans D. Alors $\oint_\gamma f(z)\,dz = 0$



$$\oint_\gamma f(z)\,dz = \oint_{\partial V} f(z)\,dz = \iint_V d(f(z)\,dz) = 0.$$

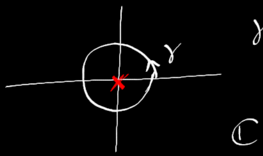
$$\int_{\gamma_1} f = \int_{\gamma_2} f$$



$$\left(\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt. \right)$$

$\gamma : [a,b] \longrightarrow U$

Exemples. $\gamma =$ cercle unité.



$\gamma(t) = e^{it}$

$$\int_\gamma 1\, dz = 0 = \int_0^{2\pi} i e^{it}\, dt = \left[e^{it}\right]_0^{2\pi}$$

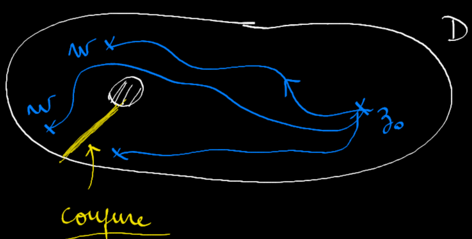
$$\int_\gamma z\, dz = 0 = \int_0^{2\pi} e^{it}\, i e^{it}\, dt = 0$$

$$\int_\gamma z^n\, dz = \int_0^{2\pi} e^{int}\, i e^{it}\, dt = i \int_0^{2\pi} e^{i(n+1)t}\, dt = \begin{cases} 0 & \text{si } n \neq -1 \\ 2\pi i & \text{si } n = -1 \end{cases}$$

$n \in \mathbb{Z}$.

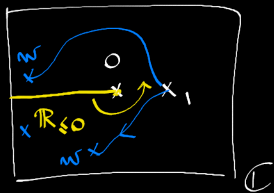
$$\boxed{\oint_0 z^n\, dz = \begin{cases} 0 & \text{si } n \neq -1 \\ 2\pi i & \text{si } n = -1 \end{cases}}$$

$$\boxed{\oint_0 \frac{dz}{z} = 2\pi i}$$



$$g(z) = \int_{z_0}^{z} f(w)\, dw$$

$$\boxed{g' = f}$$



Sur $\mathbb{C} - \mathbb{R}_{\leq 0}$, on définit $\log(z) = \int_1^z \frac{dw}{w}$

$\log(1) = 0 \ \ldots \overset{?}{=} 2\pi i$ fonction multivaluée.

$e^{2\pi i} = e^0 = 1$

Exemple. $\alpha \in \mathbb{C}^*$. $f(z) = z^\alpha := e^{\alpha \log z}$

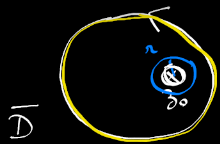
$\alpha \in \mathbb{Q} \longrightarrow$ multivaluation finie.

$\alpha = \frac{1}{2} \qquad z^\alpha = \sqrt{z}$

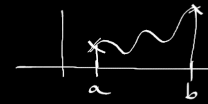
II) Fonctions analytiques

Théorème (Cauchy). Soit $\overline{D}$ un disque fermé, $U \supset \overline{D}$ ouvert. Soit $f : U \longrightarrow \mathbb{C}$ holomorphe. Alors :

$$\forall z_0 \in D, \quad f(z_0) = \frac{1}{2\pi i} \oint_{\partial \overline{D}} \frac{f(z)}{z - z_0}\, dz$$



Preuve.



$$g(z) = \frac{f(z)-f(z_0)}{z-z_0}$$

pour $z \in U-\{z_0\}$

* g est <u>holomorphe</u> sur $U-\{z_0\}$.
* g est <u>bornée</u> au voisinage de z_0.

$B \geq 0$

Il existe $r>0$ tel que $\forall z$,

$$|z-z_0|<r \implies z \in D \text{ et } |g(z)| \leq B.$$

$$\forall r'<r \quad \left|\oint_{C(z_0,r')} g(z)\,dz\right| \leq \oint_{r'} |g(z)|\,dz \leq \underline{2\pi r' B}$$

$$\|$$

$$\forall r'<r \quad \left|\oint_{C(z_0,r)} g(z)\,dz\right| \leq 2\pi r' B.$$

Donc $\boxed{\oint_{\partial\overline{D}} g(z)\,dz = 0}$

$$\oint_{\partial\overline{D}} \frac{f(z)-f(z_0)}{z-z_0}\,dz = 0$$

$$\downarrow$$

$$\oint \frac{f(z)}{z-z_0}\,dz = \oint \frac{f(z_0)}{z-z_0}\,dz$$

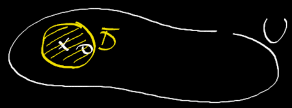
$$= f(z_0)\,2\pi i$$

Théorème : Soit U ouvert de $\mathbb{C}$ et $f : U \to \mathbb{C}$. Alors :

f holomorphe $\Longleftrightarrow$ f analytique ($\Longleftrightarrow f\ \mathcal{C}^1 \Longleftrightarrow f\ \mathcal{C}^2 \Longleftrightarrow \dots$)

Preuve. On suppose $0 \in U$ et on montre l'analyticité en 0.

Supposons f holomorphe. On prend un disque $\bar{D}$ centré en 0, $\bar{D} \subset U$.



$$\forall z \in D, \quad f(z) = \frac{1}{2\pi i} \oint \frac{f(w)}{w - z}\, dw$$

$$= \frac{1}{2\pi i} \oint \frac{f(w)}{w} \left(\frac{1}{1 - \frac{z}{w}} \right) dw$$

$$= \frac{1}{2\pi i} \oint \frac{f(w)}{w} \sum_{n=0}^{\infty} \left(\frac{z}{w} \right)^n dw$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \oint \frac{f(w)}{w^{n+1}}\, dw \right]}_{a_n} z^n$$

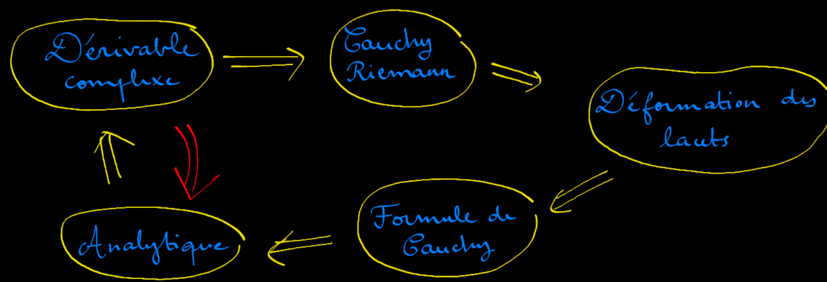
$$= \sum_{n=0}^{\infty} a_n z^n.$$

$f(z) = \sum a_n z^n$

Bonus :

$$\boxed{\frac{f^{(n)}(0)}{n!} = a_n = \frac{1}{2\pi i} \oint \frac{f(w)}{w^{n+1}}\, dw}$$

Résumé :



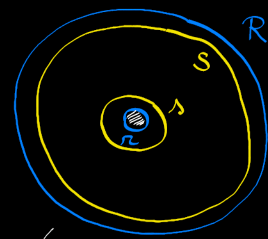
Analytique : $\displaystyle \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Définition : $f : U \to \mathbb{C}$. f analytique sur U si pour tout $z_0 \in U$, $\exists (a_n) \in \mathbb{C}^{\mathbb{N}}$ $\exists r > 0$ tq $\forall z \in U$, $|z - z_0| < r$, $\displaystyle f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ (avec $a_n \in \mathbb{C}$).

Quand c'est le cas, f est $\mathcal{C}^\infty$ et $\displaystyle a_n = \frac{f^{(n)}(z_0)}{n!}$

IV Calculs et géométrie.

$$A = \{ z \in \mathbb{C} \mid r < |z| < R \}$$

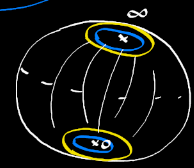


$f : A \to \mathbb{C}$ holomorphe. Pour tout

$s \le |z| \le S$, $\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n}$

avec $a_n = \begin{cases} \frac{1}{2\pi i}\oint_S \frac{f(w)}{w^{n+1}}\, dw & \text{si } n \ge 0 \\ \frac{1}{2\pi i}\oint_s \frac{f(w)}{w^{n+1}}\, dw & \text{si } n<0. \end{cases}$

série de Laurent.

$\mathbb{C}\cup\{\infty\} =$  $= \Sigma$

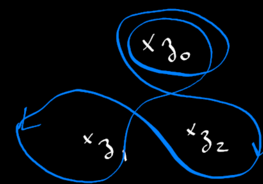
* Si $a_n = 0$ $\forall n<0$, alors f holomorphe en 0.
  (la singularité est effaçable) $\quad \left(f(z) = \frac{z}{z} = 1\right)$
* Si $a_n=0$ $\forall n < -N$ pour $N\in\mathbb{N}$, alors pôle. $\quad f(z)=\frac{1}{z^3}$
* Si $\forall N \in \mathbb{N}, \exists\, n<-N$ tq $a_n\neq 0$, alors singularité essentielle.
  $f(z) = e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!\, z^n}$

Définition. Soit U ouvert de $\mathbb{C}$ et $z_1, \dots, z_n \in U$.
$f : U - S \to \mathbb{C}$ holomorphe. On dit que f est *méromorphe* sur U si elle a des pôles en $z_1, \dots, z_n$.

Ex : $f(z) = \dfrac{z+3}{z^2-1}$ méromorphe sur $\mathbb{C}$.

Théorème (résidus). U ouvert de $\mathbb{C}$, γ lacet contractile, f méromorphe sur U, holomorphe sur $U - \{z_1, \dots, z_m\}$.
Alors
$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{m} W(\gamma, z_k)\,\mathrm{Res}_{z_k}(f)$$

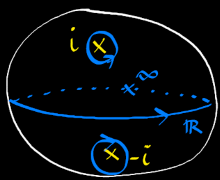
$$\boxed{W(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} \in \mathbb{Z}}$$



$$W(\gamma, z_0) = +2$$
$$W(\gamma, z_1) = +1$$
$$W(\gamma, z_2) = -1$$

$$\boxed{\mathrm{Res}_{z_0}(f) = a_{-1} \quad \text{dans} \quad f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n}$$

Exemple 1. Calculer $\displaystyle\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx \quad \left(= \left[\arctan(x)\right]_{-\infty}^{+\infty} = \pi\right)$



$$1+x^2=0 \iff x = \pm i$$

$$\int_{\mathbb{R}} \frac{1}{1+x^2}\,dx = \oint_i \frac{1}{1+x^2}\,dx = 2\pi i \left(\frac{1}{2i}\right) = \pi$$

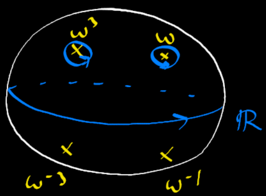
Résidu : $x = i+z$

$$\frac{1}{1+(i+z)^2} = \frac{1}{1+(-1)+2iz+z^2} \sim \frac{1}{2iz}$$

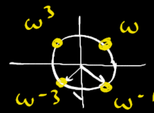
Exemple 2. Calculer $\int_{\mathbb{R}} \frac{1}{1+x^4}\,dx = 2\pi i\left(\frac{1}{4\omega^3}+\frac{1}{4\omega}\right) = \frac{\pi i}{2}\left(\omega^{-1}+\omega^{-3}\right)$

$$= \frac{\pi i}{2}\left(-i\sqrt{2}\right)$$

$$= \frac{\pi}{\sqrt{2}}$$



$1+x^4=0 \iff x =$



$x = \omega + z$. $\quad \frac{1}{1+x^4} = \frac{1}{1+(\omega+z)^4} = \frac{1}{4\omega^3 z + \dots}$

$x = \omega^3 + z \quad \frac{1}{1+x^4} = \frac{1}{1+(\omega^3+z)^4} = \frac{1}{4\omega^9 z + \dots}$

$$\boxed{\int_{-\infty}^{+\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}}$$

Théorème de Liouville : Soit $\boxed{f : \mathbb{C} \to \mathbb{C}}$ holomorphe.

Si f est bornée alors f est constante

Preuve. f holomorphe $\rightsquigarrow$ f analytique : $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

avec $a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}_R} \frac{f(w)}{w^{n+1}} dw$. Soit $B \geq 0$ tq $\forall z \in \mathbb{C}, |f(z)| \leq B$.

$$|a_n| \leq \frac{1}{2\pi} \oint_{\mathcal{C}_R} \frac{|f(w)|}{|w|^{n+1}} dw \leq \frac{B}{2\pi} \frac{2\pi R}{R^{n+1}} \leq \frac{B}{R^n} \quad \forall R > 0.$$

Si $n \geq 1$, $a_n = 0$.

Théorème : Soit P polynôme à coefficients dans $\mathbb{C}$. Si P n'est pas constant, P admet une racine dans $\mathbb{C}$.

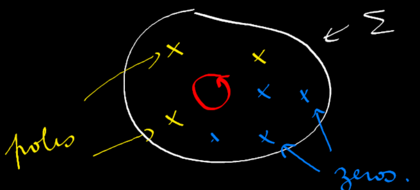
Preuve. Si P n'a pas de racine, $f(z) = \frac{1}{P(z)}$.

Si P non constant, f bornée et holomorphe. Donc f constante.

Si $f : \Sigma \longrightarrow \Sigma$

borné   méromorphe.

$$\boxed{\frac{1}{2\pi i}\oint_\gamma \frac{f'}{f} = (\#\,\text{zeros} - \#\,\text{poles}) \text{ dans } \gamma.}$$



> f méromorphe sur Σ, alors
> $\#$ zeros $=$ $\#$ poles.

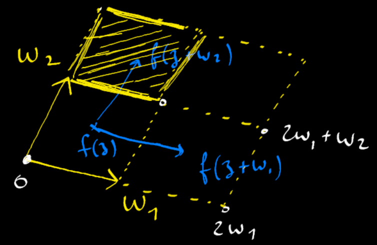
<u>Polynôme</u> : $f(z) = a_0 + a_1 z + \dots + a_N z^N \qquad a_N \neq 0.$

Il a N zeros et 0 pole sur $\mathbb{C}$.

$w = \frac{1}{z} \qquad f(w) = a_N w^{-N} + \dots$

pole d'ordre N en ∞.

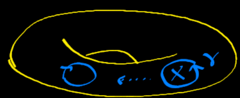
(V) Fonctions "périodiques".

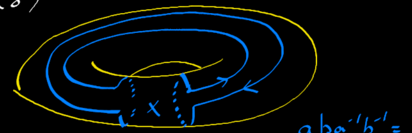


Soient $\omega_1, \omega_2 \in \mathbb{C}$ tels qu'ils ne soient pas colinéaires.

$$\Lambda = \{ n\,\omega_1 + m\,\omega_2 \mid (n,m) \in \mathbb{Z}^2 \}$$

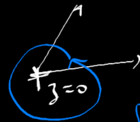
<u>Déf.</u> Soit f <u>méromorphe</u> sur $\mathbb{C}$. On dit que f est Λ-elliptique si $\forall\, \omega \in \Lambda,\ \forall\, z \in \mathbb{C},\ f(z+\omega) = f(z)$

$\mathbb{C}/\Lambda =$ 



<u>Q</u>: Y a-t-il des fonctions Λ-elliptiques holomorphes ? <u>Seulement les constantes !</u> $\longrightarrow$ il faut autoriser les pôles !

Q : Y a-t-il des fonctions Λ-elliptiques avec un <u>unique</u> pôle sur le domaine fondamental ? NON !



$$\oint_\gamma \wp = 0$$

<u>Définition</u> : $$\boxed{\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right]}$$

Séries d'Eisenstein

Formes modulaires

$$\boxed{\sum_{\omega\in\Lambda^*}\frac{1}{\omega^k} = G_k}$$

$G_k = 0$ si k impair

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\frac{1}{\omega^2}\left[\frac{1}{(1-z/\omega)^2} - 1\right]$$

$$= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\frac{1}{\omega^2}\left[\sum_{n=1}^{\infty}(n+1)\left(\frac{z}{\omega}\right)^n\right]$$

$$= \frac{1}{z^2} + \sum_{n=1}^{\infty}(n+1)z^n \underbrace{\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{2+n}}}_{G_{2+n}}$$

$$\left(\frac{1}{1-u}\right)^2 = \sum (n+1)u^n$$

$$\boxed{\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty}(2n+1)z^{2n}G_{2n+2}}$$

$$\begin{cases} \wp_\Lambda(z) = \dfrac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + \cdots \\[2ex] \wp'_\Lambda(z) = \dfrac{-2}{z^3} + 6G_4 z + 20 G_6 z^3 + \cdots \end{cases}$$

$$(\wp'_\Lambda)^2 - 4(\wp_\Lambda)^3 = \cancel{\frac{4}{z^6}} - \cancel{\frac{4}{z^6}} - \frac{24G_4}{z^2} - 4\times 3\times\frac{3G_4}{z^2} - 4\times 3\times 5G_6 - 80G_6 + O(z)$$

$$= \frac{-60G_4}{z^2} - 140G_6 + O(z) \leftarrow$$

$$(\wp'_\Lambda)^2 - 4(\wp_\Lambda)^3 + \underbrace{60G_4}_{=g_2}\wp_\Lambda + \underbrace{140G_6}_{=g_3} = O(z) \boxed{= 0}$$

holomorphe + elliptique = constante.

$$\boxed{(\wp'_\Lambda)^2 = 4(\wp_\Lambda)^3 - g_2\wp_\Lambda - g_3}$$

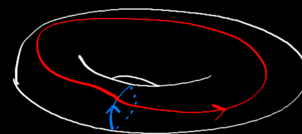
$$\mathcal{C} = \{ (x,y) \in \mathbb{C}^2 \, / \, y^2 = 4x^3 - g_2 x - g_3 \} \cup \{\infty\}.$$

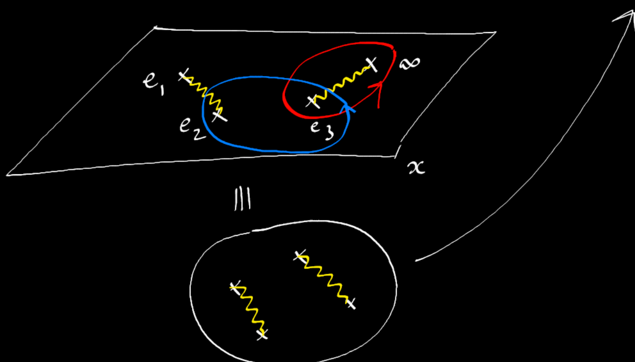
$$\int \frac{1}{\sqrt{x^3 + \cdots}}$$

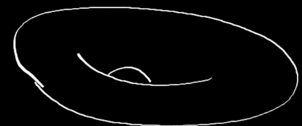
$$\varphi : \mathbb{C}/\Lambda \longrightarrow \mathcal{C}$$

$$z \longmapsto (\wp_\Lambda(z), \wp_\Lambda'(z))$$

$$y = \pm\sqrt{4x^3 - g_2 x - g_3}$$





Fonctions méromorphes sur  et .

Que se passe-t-il sur