A coincidence ?

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Let's consider the following integral :

$$\int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x = \frac{\pi}{2}\,.$$

It is easy to compute, either by expanding the trigonometric functions into exponentials. Now let's say we want to consider something slightly more complicated :

$$\int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} \cos\frac{x}{3} \ldots \cos\frac{x}{n}$$

where n is a positive integer. When n is large this is very cumbersome to compute exactly by expanding all the factors, so we could rely on a numerical computation to have a guess. We find

$$\begin{aligned}
\int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} &= \frac{\pi}{2} \\
\int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} \cos\frac{x}{3} &= \frac{\pi}{2} \\
\int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} \cos\frac{x}{3} \cos\frac{x}{4} &= \frac{\pi}{2}
\end{aligned}$$

It seems very likely that

$$\forall n \geq 1, \qquad \int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} \cos\frac{x}{3} \ldots \cos\frac{x}{n} = \frac{\pi}{2}\,.$$

However if you try $n = 31$ you find

$$\forall n \geq 1, \qquad \int_0^\infty \frac{\mathrm{d}x}{x} \sin 4x \cos x \cos\frac{x}{2} \cos\frac{x}{3} \ldots \cos\frac{x}{31} \neq \frac{\pi}{2}\,.$$

How can this be explained ??

This is a good example of a problem that seems inextricable but that becomes tractable once it is correctly generalized. Here we will prove that if $(a_j)_{j=1,\dots,n}$ and b are strictly positive real numbers, then

$$\boxed{\sum_{j=1}^{n} a_j < b \qquad \Longrightarrow \qquad \int_0^\infty \frac{\mathrm{d}x}{x} \sin bx \prod_{j=1}^{n} \cos a_j x = \frac{\pi}{2}}$$

We will prove this using the powerful residue theorem combined with a resolution of the pole at $x = 0$. Let's consider the integral

$$I(\eta) = \int_0^\infty \frac{\mathrm{d}x}{x - i\eta} \sin bx \prod_{j=1}^{n} \cos a_j x \,.$$

We compute

$$\begin{aligned}
I(\eta) &= \frac{1}{2^{n+2} i} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - i\eta} \left(e^{ibx} - e^{-ibx}\right) \prod_{j=1}^{n} \left(e^{ia_j x} + e^{-ia_j x}\right) \\
&= \frac{1}{2^{n+2} i} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - i\eta} \left(e^{ibx} - e^{-ibx}\right) \sum_{\epsilon_j = \pm 1} \exp\left(ix \sum_{j=1}^{n} \epsilon_j a_j\right) \\
&= \frac{1}{2^{n+2} i} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - i\eta} \sum_{\epsilon_j = \pm 1} \exp\left[ix \left(b + \sum_{j=1}^{n} \epsilon_j a_j\right)\right] \\
&= \frac{1}{2^{n+2} i} \sum_{\epsilon_j = \pm 1} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x - i\eta} \exp\left[ix \left(b + \sum_{j=1}^{n} \epsilon_j a_j\right)\right] \\
&= \frac{1}{2^{n+2} i} \sum_{\epsilon_j = \pm 1} 2\pi i \exp\left[-\eta \left(b + \sum_{j=1}^{n} \epsilon_j a_j\right)\right] \to \frac{\pi}{2} \,.
\end{aligned}$$

In the first step we use the parity of the integrand to integrate over all of $\mathbb{R}$, then in the second step we expand the product and see that half of the term disappear due to the residue theorem applied to a contour that closes in the lower half-plane. Finally for the remaining half of the terms, we close the contour in the upper half-plane, and compute the residue at $x = i\eta$. The last step is to send the regulator $\eta \to 0$. The crucial point is that for any choice of the $\epsilon_j = \pm 1$, we have

$$b + \sum_{j=1}^{n} \epsilon_j a_j \geq b - \sum_{j=1}^{n} a_j > 0$$

so that we always close the contour in the same half-plane (the upper for e^{ibx} and the lower for e^{-ibx}). This is the reason why the sum simplifies at the end.

Using this general property, we see that the strange appearance of a counter-example at $n = 31$ is justified by the fact that

$$1 + \frac{1}{2} + \cdots + \frac{1}{30} < 4 < 1 + \frac{1}{2} + \cdots + \frac{1}{31} \,.$$

As a consequence there is a simple method to make the coincidence even more impressive: just replace the 4 in the first integral by a higher number. The integer n from which the value will differ from $\frac{pi}{2}$ grows exponentially as a function of b, because of the logarithmic divergence of the harmonic series.