Homework

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1 Geodesics in the hyperbolic plane

Consider the hyperbolic plane $H = \{(x, y)|y > 0\}$ with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}\,.$$

We would like to find the geodesics of the hyperbolic plane.

1. Write down the geodesic equation using the Lagrangian formalism.
2. Find the quantities which are conserved along the geodesics.
3. Show that the Euclidean curvature[1] of a given geodesic is given by

$$\rho = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

   where the prime denotes the derivation with respect to a parameter τ.

4. Show that this curvature is constant along a given geodesic.
5. Describe the set of geodesics.

2 Static and weak field metric

We study some properties of the spacetime described by the metric

$$\mathrm{d}s^2 = -(1 + 2\Phi(\vec{x}))\mathrm{d}t^2 + (1 - 2\Phi(\vec{x}))(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{1}$$

where $\Phi(\vec{x}) << 1$.

2.1 Gravitational field of a star

In this section, we take

$$\Phi(\vec{x}) = -\frac{M}{r} \tag{2}$$

with $r >> M$.

1. Compute the Christoffel coefficients at first order in M/r.

[1] Note that there is no intrinsic meaning of curvature for a one-dimensional manifold. Here we deal with the extrinsic curvature of the geodesic seen as a curve in $\mathbf{R}^2$.

2. Consider a particle of mass m and 4-momentum p^μ falling freely in this spacetime. Using the geodesic equation, express $\mathrm{d}p^i/\mathrm{d}\tau$, where τ is the proper time.
3. The particle is non-relativistic. How can this hypothesis be written in terms of the components of p^μ?
4. Using the assumption above, write the dominant terms of the geodesic equation. What is the interpretation of M?
5. Are the approximations used in this section reasonable for studying the movement of the Earth around the Sun?

2.2 Bending of light

We would like to use the results of the subsection above to compute the deviation of a light ray by the Sun. We consider a light ray in the plane $z = 0$ approaching the Sun along the line of equation $y = b$ far from the Sun ($x \to -\infty$), where b is a constant. We use an affine parameter s along the trajectory, so that $p^\mu = \mathrm{d}x^\mu/\mathrm{d}s$.

1. Write down the equation for p^y.
2. Show that one can assume $p^y << p^x$, and deduce that

$$\frac{\mathrm{d}p^y}{\mathrm{d}s} = \frac{-2Mbp^x}{(x^2+b^2)^{3/2}}\frac{\mathrm{d}x}{\mathrm{d}s} \tag{3}$$

3. Express the value of p^y when $x \to +\infty$.
4. Deduce the value of the deflection angle $\Delta\varphi$ in terms of R/b, where R is the radius of the Sun.

2.3 A derivation

Now we want to prove that (1) is the most general static and weak-field solution of the linearised vacuum Einstein equations. In the first part, we study these linearised equations, and then in the second part we derive (1).

2.3.1 Linearised gravity

We consider a metric that is close to the flat metric, $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ with $h_{\mu\nu}(x) << 1$.

1. Show that the linearized Einstein equations in the vacuum are

$$-\Box h_{\mu\nu} + \partial_\mu V_\nu + \partial_\nu V_\mu = 0 \tag{4}$$

where V_μ is a vector that depends on $h_{\mu\nu}$. Give the expression of V_μ.

2. What does (4) become when applied to the metric (1)?
3. Consider a change in coordinates of the form $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$ where $\xi^\mu(x)$ are arbitrary functions of the same size as the metric perturbations $h_{\mu\nu}(x)$, so that the resulting metric is again close to the flat one. How do those perturbations $h_{\mu\nu}(x)$ transform under such a change in coordinates?

4. Show that it is possible to choose a system of coordinates such that the Einstein equations become
$$\Box h_{\mu\nu} = 0 \tag{5}$$
together with the so-called *Lorentz gauge* condition
$$V_\mu = 0. \tag{6}$$
5. Make the analogy with electromagnetism and explain why $V_\mu = 0$ is called the Lorentz gauge condition.

2.3.2 Deriving the static weak field metric

Now we assume that we have a solution of the linearised Einstein equations that is static (time-independent) and asymptotically flat ($h_{\mu\nu}(x) \to 0$ when $|\vec{x}| \to +\infty$).

1. Argue that $h_{tx} = h_{ty} = h_{tz} = 0$.
2. Show that the gauge condition (6) doesn't completely fix the coordinates that we use, and that it is possible to make a coordinate transformation that preserves the gauge *and* make the perturbation $h_{\mu\nu}(x)$ diagonal.
3. Using these coordinates, show that (1) is the most general solution.

3 Are wormholes physical?

We remind the metric of the wormhole that we already studied :
$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + (b^2 + r^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \,.$$
We would like to know whether a certain distribution of matter and energy can cause this geometry through Einstein's equations.

1. Compute the Einstein tensor in this geometry.
2. Is it possible to construct the wormhole with classical means?