OKUDA

Susy localization with boundaries in 2d

B-type boundary conditions

2d $\mathcal{N}=(2,2)$ theory on S^2 or hemisphere.

Consider $U(1)_V$ sugra background (reduction of new minimal 4d $\mathcal{N}=1$ sugra)

$$ds^2 = f(\theta)^2 d\theta^2 + \ell^2 \sin^2\theta \, d\varphi^2 \qquad \theta \in [0,\pi]$$

$$f(\theta) = \ell + \mathcal{O}(\theta^2) \quad \text{for } \theta \to 0 \qquad \varphi \in [0, 2\pi[$$

- vielbein : $e^{\hat{1}} = f(\theta) d\theta \qquad e^{\hat{2}} = \ell \sin\theta \, d\varphi$

<u>Ex</u> : $f(\theta) = \ell \rightarrow$ round S^2

$f(\theta) = \sqrt{\ell^2 \cos^2\theta + \tilde{\ell}^2 \sin^2\theta}$ other example.

- $U(1)_R$ symmetry $V^R = \frac{1}{2}\left(1 - \frac{\ell}{f(\theta)}\right) d\varphi$
- graviphoton $\mathcal{H} = \bar{\mathcal{H}} = \frac{\#}{f}$
- Generalized Killing spinor eq : $(\nabla_\mu - i V_\mu^R)\varepsilon = \frac{1}{2f}\gamma_\mu \gamma_3 \varepsilon$

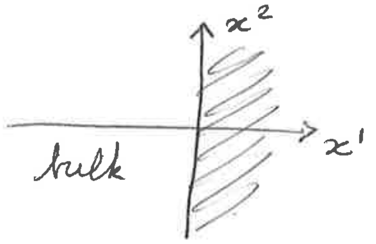
$$(\nabla_\mu + i V_\mu^R)\bar{\varepsilon} = -\frac{1}{2f}\gamma_\mu\gamma_3\bar{\varepsilon}$$

with $\gamma_{\hat{\theta}} = \gamma_{\hat{1}} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \qquad \gamma_{\hat{\varphi}} = \gamma_{\hat{2}} = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ SU(1|1)

Solutions :
$$\begin{cases} \varepsilon = \exp\left[-\frac{i}{2}\theta\gamma_{\hat{2}}\right]\begin{pmatrix} e^{i\varphi/2} \\ 0 \end{pmatrix} \\ \bar{\varepsilon} = \exp\left[+\frac{i}{2}\theta\gamma_{\hat{2}}\right]\begin{pmatrix} 0 \\ e^{-i\varphi/2} \end{pmatrix} \end{cases}$$

When $\theta \to \frac{\pi}{2}$, $\varepsilon \sim \bar{\varepsilon} \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ up to $U(1)_R$ sym.

Define $\begin{cases} x^1 = \tilde{\ell}\left(\theta - \frac{\pi}{2}\right) \\ x^2 = \ell \varphi \end{cases}$



Supercharges $\left\{ \begin{array}{l} Q_1 + Q_2 \\ \overline{Q}_1 + \overline{Q}_2 \end{array} \right. \rightarrow$ known as $\boxed{\text{B-type susy}}$

[remember at pole it's A-type!]

$$Z_{\text{hemisphere}} \overset{?}{\sim} \langle \text{B-brane} \,|\, \mathbb{1} \text{ in } (a,c)\text{-ring} \rangle$$

- Vector multiplet for gauge group G : $(A_\mu, \sigma_1, \sigma_2, \lambda_\alpha, \overline{\lambda}_\alpha, D)$ $\quad \alpha = 1,2$
- Chiral multiplet (ϕ, ψ_α, F)
- Susy transfo : $\delta A_\mu = -\frac{i}{2}(\overline{\varepsilon}\gamma_\mu \lambda + \overline{\lambda}\gamma_\mu \varepsilon)$

$\vdots$

$\delta\phi = \overline{\varepsilon}\psi$ | $\delta\psi = i\gamma^\mu \varepsilon D_\mu \phi + i\varepsilon\sigma_1 \phi + \gamma^3 \varepsilon \sigma_2 \phi + i\frac{q}{2f}\gamma_3 \varepsilon \phi + \overline{\varepsilon} F$

$\delta F = \ldots$

- Susy boundary conditions at $\theta = \pi/2$

Vector : $\sigma_1 = 0$, $D_1 \sigma_2 = 0$, $A_1 = 0$, $F_{12} = 0$,
$\overline{\varepsilon}\lambda = \varepsilon\overline{\lambda} = 0$, $D_1(\overline{\varepsilon}\gamma_3\lambda) = D_1(\varepsilon\gamma_3\overline{\lambda}) = 0$, $D_{\hat{1}}(D - iD_{\hat{1}}\sigma_1) = 0$

Chiral Neumann : $D_1\phi = 0$, $\overline{\varepsilon}\gamma_3\psi = D_1(\overline{\varepsilon}\psi) = 0$, $F = 0$

Dirichlet : $\phi = 0$, $\overline{\varepsilon}\psi = D_1(\overline{\varepsilon}\gamma_3\psi) = 0$, $D_{\hat{1}}(e^{-i\varphi}F + iD_{\hat{1}}\phi) = 0$

- Action

$$S_{\text{phys}} = S_{\text{vec}} + S_{\text{chiral}} + S_W + S_\Theta + S_{FI}$$

where $S_W = \int \#\left(F^i \partial_i W(\phi) - \frac{1}{2}\psi^i\psi^j \partial_i\partial_j W + cc\right)$, etc...

$$\delta_{\text{susy}} S_{\text{phys}} = 0 + 0 + \delta_{\text{susy}} S_W + 0 + 0 \sim \underbrace{\oint d\varphi\, (\varepsilon\gamma^\mu \psi^i \partial_i W + cc)}_{\text{Warner term}}$$

$$Z \sim \int_{b.c.} \mathcal{D}(\text{fields})\, e^{-S_{phys}}\, \underset{\substack{\uparrow \\ \text{Chan-Paton} \\ \text{vector space}}}{\mathrm{Tr}_V}\, \mathcal{P}\, e^{\underbrace{i\oint \mathcal{A}_\varphi d\varphi}_{\text{to cancel the Warner term}}}$$

"boundary interaction" $\mathcal{A}_\varphi \sim A_\varphi + i\sigma_2 + (\text{R-charge}) + (\text{twisted mass})$
$+ \# \{Q(\phi), \bar{Q}(\phi)\} + \#(\psi_1 - \psi_2)^i \partial_i Q(\phi)$
$+ cc$

with $Q(\phi) : V \longrightarrow V$. One can check that

$$\delta_{susy}\left(e^{-S_W} \mathrm{Tr}_V\, \mathcal{P}\, e^{i\oint \mathcal{A}_\varphi d\varphi} \right) = 0 \quad \text{if} \quad \boxed{Q(\phi)^2 = W(\phi)\, id_V}$$

Call ρ the representation of $G \times G_{Flavor}$ on Chan-Paton V.

Need $\rho(g)^{-1} Q(g\phi) \rho(g) = Q(\phi)$ $\qquad g \in G \times G_F$

$e^{ir_* \chi} Q(e^{iR\chi}\phi)\, e^{-ir_* \chi} = e^{i\chi} Q(\phi)$ with r_* is R-sym rep on V.

This is called "matrix factorization":

$V = V^{ev} \oplus V^{odd}$, Q is odd, $Q = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$.

$$\boxed{Q^2 = W \iff ab = ba = W \cdot id}$$

Hemisphere partition function

Localization on $F_{\mu\nu} = 0$, $\sigma_1 = 0$, $\sigma_2 = c^{ste}$, $\phi = 0$, $F = 0$

Compute the 1-loop det for chirals:

Neumann: $Z^{1\text{-loop}}_{Neu} = \text{divergent} \xrightarrow{\text{J-reg}} \prod_{w \in R} \Gamma\left(w \cdot \sigma + \frac{q}{2}\right)$

Dirichlet: $Z^{1\text{-loop}}_{Dir} = \dots \xrightarrow{\text{J-reg}} \text{WRONG!}$

Zeta reg does not always work:

$$Z^{1\text{-loop}}_{Dir} = \prod_{w \in R} \frac{-2\pi i \exp(\pi i (w \cdot \sigma + q/2))}{\Gamma(1 - w\sigma - q/2)} \longleftarrow \text{additional term}$$

The boundary data we use are :

$$\mathcal{B} = (\text{Neu}, \text{Dir}, (V, \rho, r_*), Q) \quad := \boxed{\mathcal{B}\text{-brane}}$$

For gauge group G,

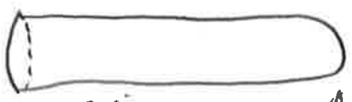
$$\boxed{Z_{\text{hem}}(\mathcal{B}, t) = \frac{1}{|\text{Weyl}(G)|} \int_{(i\mathbb{R})^{rk}} \frac{d^{\text{rank}(G)}\sigma}{(2\pi i)^{\text{rank}(G)}} e^{t_{\text{ren}}\sigma} \, \text{Str}_V \, e^{-2\pi i \sigma} \, Z_{1\text{-loop}}(\sigma)}$$

$t = 2\pi\xi - i\theta_{\text{top}}$

<u>Conjecture</u> (Honda - Okuda, Hori - Romo)

For CFT, Z_{hem} = unnormalized central charge of D-brane

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$



tt* amplitude

$\langle B | 0 \rangle_{RR}$

<u>Proved</u> by Bachas - Plencner.

<u>Ex</u>. CY hypersurface in $\mathbb{P}^{N-1}$

$$\begin{cases} N=3 & \text{torus} \\ N=4 & \text{K3} \\ N=5 & \text{quintic} \end{cases}$$

Gauge group $U(1)$

N chirals ϕ_i charge $+1$

1 chiral P charge $-N$

Superpot $W = P\, G_N(\phi)$

Choose Neu for everything.

Fermionic oscillators (Koszul construction) $\{\eta, \bar{\eta}\} = 1$

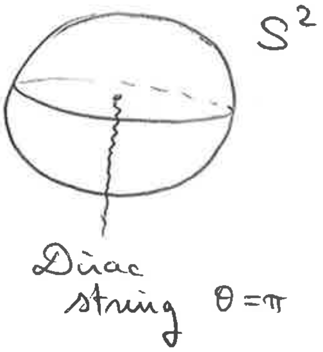
and choose Clifford vacuum $\eta|0\rangle = 0 = \bar{\eta}|0\rangle$

$$Q(\phi, P) := G_N(\phi)\eta + P\bar{\eta} \quad \longrightarrow \quad Q^2 = P\,G_N(\phi) = W$$

Assume gauge charge of $|0\rangle$ is $n + \frac{N}{2}$, $n \in \mathbb{Z}$, and $r_*|0\rangle = 0$.

and $\phi \sim i \frac{B}{2\pi}$, for $\theta_{top} = 0$. [to be consistent with Witten effect]

Dirac quantization condition on B



$$A \sim -\frac{B}{2}(1-\cos\theta)\,d\varphi$$

Condition $\langle B, w\rangle \in \mathbb{Z}$ for w = weight of matter representation. Call Λ_m the lattice of such B.

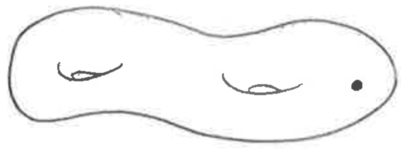
Dyonic operators specified by $(B,w) \in \Lambda_m \times \Lambda_w$ up to Weyl group action.

$\hookrightarrow$ $(\Lambda_m \times \Lambda_w)/\text{Weyl}$ (Kapustin)

When 2 line operators (B_1, w_1) and (B_2, w_2) are rotated around, pick a phase $\exp(2\pi i(\langle B_2, w_1\rangle - \langle B_1, w_2\rangle) \underset{\substack{\nearrow \\ \text{mutually local}}}{=} 1$.

AST: to specify a theory, need to choose a maximal set of mutually local operators.

A_1 theory of class S : two M5s on $\mathcal{C}_{g,n}$



Homotopy class of closed curves on $\mathcal{C}_{g,n}$

$\updownarrow$

Line operator charge

Tachikawa (after AGT) : choice of isotropic subgroup of $H^1(\mathcal{C}_{g,n}, \mathbb{Z}_2^{3g-3})$

Then $Q \longleftrightarrow$ Sheaf $\mathcal{O}_M(n)$ over the CY M.

$$Z_{hem}\left[\mathcal{O}_M(n)\right] = \int_{i\mathbb{R}} \frac{d\sigma}{2\pi i}\, e^{-2\pi i n\sigma}\left(e^{-N\pi i\sigma} - e^{N\pi i\sigma}\right) e^{t\sigma}\,\Gamma(\sigma)^N\,\Gamma(1-N\sigma)$$

$$\underset{\nearrow}{\sim} \int_M ch\left(\mathcal{O}_M(n)\right) e^{B+i\omega}\,\hat{\Gamma}(TM)\ \left(+\,\mathcal{O}(e^{-t})\right)$$

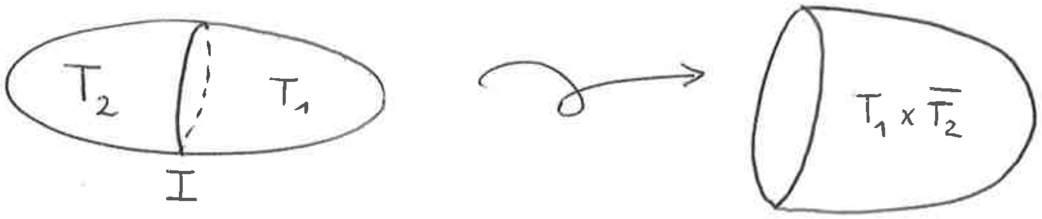
large $Re(t) = 2\pi\}$

with $B+i\omega = -\frac{t}{2\pi i} e$ $\qquad$ e is $H^2(M,\mathbb{Z})$ generator

and $\hat{\Gamma}(E) = \prod_j \Gamma\left(1+\frac{i x_j}{2\pi}\right)$ $\qquad$ "gamma class"

$ch(E) = \sum_j e^{x_j}$

Interfaces



Line operators and 2d-4d

G = gauge group

$\underline{t}$ = Cartan sub. $\qquad$ $\underline{t}^*$ = dual.

$$\boxed{\begin{tikzcd} \Lambda_r \arrow[r, phantom, "\subset"] \arrow[dr, leftrightarrow] & \Lambda_w \arrow[r, phantom, "\subset"] \arrow[dl, leftrightarrow] & \underline{t}^* \\ \Lambda_{cr} \arrow[r, phantom, "\subset"] & \Lambda_{cw} \arrow[r, phantom, "\subset"] & \underline{t} \end{tikzcd}}$$

$\frac{1}{2}$ BPS Wilson line operators in 4d $\mathcal{N}=2$:

$$W_R = Tr_R\, Pexp \underset{\nearrow}{\oint} \left(iA + Re\,\phi\, ds\right)$$

R = irrep, associated $w \in \Lambda_w$.

straight line
or circle
(to be $\frac{1}{2}$ BPS)

[Open Q: Which are all allowed curves?
(see Pestun for $\mathcal{N}=4$ case)]

't Hooft line operator $T(B)$ defined by singular boundary condition $F \sim \frac{-B}{2}\varepsilon_{ijk}\frac{x^i}{r}dx^k \wedge dx^j = -\frac{B}{2}\sin\theta\, d\theta \wedge d\varphi$.

Kronheimer's correspondence

- Bogomolny eq. $*_3 F = D\Phi$
- Anti self duality eq. $\mathcal{F} + *\mathcal{F} = 0$.

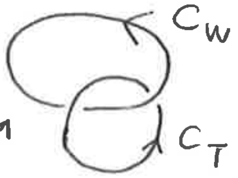
$\boxed{\text{Instantons on Taub-NUT space invariant under } U(1)} \Leftrightarrow \boxed{\text{Singular monopoles}}$

Multi-centered Taub-Nut : $ds^2 = V d\vec{x}^2 + V^{-1}(d\Psi + \omega)^2$

$$V = l + \sum_j \frac{1}{2|\vec{x}-\vec{x}_j|} \quad , \quad d\omega + *_3 dV = 0 \qquad \Psi \sim \Psi + 2\pi$$

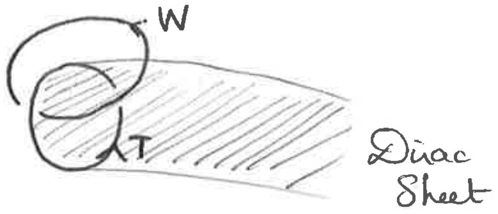
NB : 't Hooft algebra of loop operators

$\begin{cases} W \text{ on } C_W \\ T \text{ on } C_T \end{cases}$ Hopf linked at $t=0$ →



As operators on the Hilbert space, $\boxed{W \cdot T = T \cdot W e^{2\pi i/N}}$ for gauge group $SU(N)$.

Mutually non local → T is actually the boundary of topological surface operator, the Dirac Sheet



If gauge group is $SU(N)/\mathbb{Z}_N$,
then $\boxed{\text{W is boundary of topological surface operator}}$

indeed, $\mathrm{Tr}_{\square} P \exp \oint_{\mathcal{C}} A$ is still gauge invariant if $\mathcal{C}$ is contractible. Otherwise, might not be because of large gauge transformations.

Expectation value of 't Hooft operator:

$$\langle T_B \rangle_{S^4_b} = \int_{\underline{t}} da \sum_{v} \underbrace{Z^{S^1 \times \mathbb{R}^3}_{1\text{-loop}}}_{\text{equator contrib}} \times Z_{\text{mono}} \times \underbrace{Z^{S^4_b}_{1\text{-loop}}}_{\text{poles contrib.}} \times |Z_{\text{inst}}|^2$$

$$\langle T_B \rangle_{S^1 \times_\lambda \mathbb{R}^3} = \mathrm{Tr}_{\mathcal{H}(T_B)} (-1)^F e^{-2\pi R H} e^{2\pi i \lambda (J_3 + I_3)} e^{-2\pi i m_f F_f}$$

Maxime ZABZINE

Index theorems and 5d loc

Smooth manifold with U(1) action : $U(1)\times M \longrightarrow M$, vector field $v^\mu(x)\partial_\mu = v(x)$

Super-manifold: $\underset{\nearrow \atop \psi^\mu}{T[1]}\ \underset{\nwarrow \atop x^\mu}{M}$ ψ^μ transforms as dx^μ

$$\Downarrow$$

Fact : $\int d^n x^\mu d^n \psi^\mu$ is well defined

Grassmann integrals : $\int d\psi = 0$ $\int d\psi\, \psi = 1$

if $\boxed{\tilde\psi = a\psi}$ then $\boxed{d\tilde\psi = \frac{1}{a}d\psi}$ because $\int d\tilde\psi\, \tilde\psi = 1$

$$\boxed{\text{Susy : } \begin{cases} \delta x^\mu = \psi^\mu \\ \delta\psi^\mu = v^\mu(x) \end{cases}}$$

Function $\alpha(x,\psi) = \sum \alpha_{\mu_1 \dots \mu_k} \psi^{\mu_1} .. \psi^{\mu_k}$ so identification with differential forms : $\boxed{\mathcal{C}^\infty(T[1]M) = \Omega^\bullet(M)}$

$$\delta\alpha(x,\psi) = \sum \partial_\rho \alpha_{\dots} \psi^\rho \psi^{\mu_1} \dots + \sum \alpha_{\dots} v^{\mu_1}\psi^{\mu_2}\dots\psi^{\mu_k} + \dots$$

We see that $\boxed{\delta \longleftrightarrow \underset{\substack{\uparrow \\ \text{de Rham}}}{d} + \underset{\substack{\uparrow \\ \text{contraction with} \\ \text{vector field}}}{i_v} = d_v}$ ← equivariant diff.

$$d_v^2 = d i_v + i_v d = \mathcal{L}_v \qquad \begin{cases} d : \Omega^p \to \Omega^{p+1} \\ i_v : \Omega^p \to \Omega^{p-1} \end{cases}$$

Introduce formal parameter ξ , $\deg \xi = 2$ and consider

$d_v = d + \xi i_v$ $\Omega^\bullet(M)[\xi]$

and $\Omega^\bullet_{inv}(M)[\xi] = \{\alpha \mid \mathcal{L}_v \alpha(\xi) = 0\}$

↳ Complex $(\Omega^\bullet_{inv}(M)[\xi], d_v) \rightsquigarrow H^\bullet_{U(1)}(M)$ equivariant cohomology

This is called the Cartan model of EQ. COH.

Idea is that $H^\bullet_{U(1)}(M) = H^\bullet(M/U(1))$

if this exists !

Equivariantly closed form $\delta\alpha(x,\Psi) = 0$. We will set $\zeta = 1$ as physicists do (but ζ is important for grading ...).

Necessarily α involves all forms, $\alpha = \alpha_0 + \dots + \alpha_{top}$.

Goal is to calculate $\int d^n x\, d^n\Psi\, \alpha(x,\Psi) := Z[0]$ with $\delta\alpha = 0$

> Define $Z[t] = \int d^n x\, d^n\Psi\, \alpha(x,\Psi)\, e^{-\delta w(x,\Psi)\, t}$.
>
> Fact: if $\delta^2 w = 0$ then $\frac{dZ}{dt} = 0$

If choose $\delta^2 w = 0$, $Z[0] = \lim_{t\to\infty} Z[t]$.

Example. $w = v^\mu g_{\mu\nu}\psi^\nu$. Then we have (on compact space)

$$\delta^2 w = 0 \iff \mathcal{L}_v g = 0 \quad \text{(ie } v \text{ is Killing vector).}$$

$$\delta w = v^\mu g_{\mu\nu} v^\nu + \psi^\mu\psi^\nu (g_{\mu\rho} v^\rho)_{,\nu}$$ Then in the limit

$$\lim_{t\to\infty} \int d^n x\, d^n\psi\, \alpha(x,\psi)\, e^{-t\|v\|^2 - t\psi^\mu\psi^\nu (dv)_{\mu\rho}}$$ only $\|v\| = 0$ contributes (fixed points of $U(1)$ action). Let's look at one fixed point, $v(0) = 0$, which is assumed to be isolated. Then

$$\delta w = \underset{\substack{\uparrow \\ \text{not degenerate}}}{H_{\mu\nu}} x^\mu x^\nu + O(x^3) + \underset{\substack{\uparrow \\ \text{antisymmetric}}}{S_{\mu\nu}} \psi^\mu\psi^\nu + O(\psi^3 x)$$

$$\boxed{\begin{array}{l} \tilde{x} = \sqrt{t}\, x \\ \tilde{\Psi} = \sqrt{t}\, \psi \end{array}}$$

$$Z[0] = \lim_{t\to\infty} \int \underbrace{d^n\tilde{x}\, d^n\tilde{\Psi}}_{\text{no transfo !!}}\ \alpha\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\Psi}}{\sqrt{t}}\right) \exp\left[-H_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu - S_{\mu\nu}\tilde{\Psi}^\mu\tilde{\Psi}^\nu + O\left(\frac{1}{\sqrt{t}}\right)\right]$$

$$= (\alpha_0(0) + 0 + \dots + 0) \int d^n\tilde{x}\, d^n\tilde{\Psi}\ \exp\underbrace{\left[-H_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu - S_{\mu\nu}\tilde{\Psi}^\mu\tilde{\Psi}^\nu\right]}_{:=(\delta w)_2 \ \leftarrow \text{order } 2}$$

$$\sim \alpha_0(0)\, \frac{Pf(S)}{\sqrt{\det H}}$$

Linearized susy : $\begin{cases} \delta_\ell \tilde{x}^\mu = \tilde{\psi}^\mu \\ \delta_\ell \tilde{\psi}^\mu = \partial_\rho v^\mu(0)\, \tilde{x}^\rho \end{cases}$

$\delta_\ell (\delta w)_2 = 0 \Rightarrow H_{\mu\nu} = \partial_\mu v^\rho(0)\, S_{\rho\nu}$. Then the formula becomes $\alpha_0(0) \frac{1}{\sqrt{\det \partial_\mu v^\rho(0)}}$.

$$\boxed{\text{Th (Atiyah-Bott)} : \int d^n x\, d^n \psi\, \alpha(x,\psi) = \sum_{fp} \frac{\pi^{\dim M/2}\, \alpha_0(x_{fp})}{\sqrt{\det\left[\partial_\mu v^\rho(x_{fp})\right]}}}$$

Exercices What happens if fp not isolated?

$$\int_M \alpha = \int_{M_{U(1)}} \frac{\alpha|_{..}}{\mathcal{E}(NM_{U(1)})} \qquad \text{(difficult exercise)}$$

Now, go to ∞-dim setting. Take vector bundle $E \longrightarrow M$.

$E[1]$ coords : $\begin{cases} \chi^a & \text{odd, expanded in basis of sections of } E \\ x^\mu & \text{on base.} \end{cases}$

$T[1]E[1]$ has coords : $\begin{cases} x^\mu & \text{even} \\ \psi^\mu & \text{odd, transforms like } dx^\mu \\ \chi^a & \text{odd, sections of } E \\ H^a & \text{even, like } d\chi^a \end{cases}$

⚠ Vector bundle over Vector bundle is not a vector bundle (over the same base). Here, the H^a don't transform as sections.

$\tilde{\chi}^a = g^a{}_b(x)\, \chi^b$ and $\underset{\tilde{H}^a}{\underset{\wr}{d\tilde{\chi}^a}} = g^a{}_b(x)\, \underset{H^b}{\underset{\wr}{d\chi^b}} + \underbrace{g^a{}_{b,\rho}\, dx^\rho \chi^b}_{\text{additional term !}}$

So a term like $H^a h_{ab} H^b$ is bad !

Equivariant bundle:

$$\begin{tikzcd} E \arrow[d, "\pi"] & \circlearrowleft U(1) \\ M & \circlearrowleft U(1) \end{tikzcd}$$

when π and $U(1)$ commute.

[Mathai - Quillen formalism]

Susy: $$\begin{cases} \delta x^\mu = \psi^\mu \\ \delta \psi^\mu = v^\mu(x) \\ \delta \chi^a = H^a \\ \delta H^a = \mathcal{L}_v \chi^a \end{cases}$$

$$\underbrace{\begin{cases} \delta_\nabla \chi^a = H^a + \overbrace{A^a{}_{b\mu}}^{\text{connection}} \psi^\mu \chi^b \\ \delta_\nabla H^a = \mathcal{L}_v \chi^a + \text{extra terms} \end{cases}}_{}$$

introducing that (non canonical) might be necessary to write w with $H^a h_{ab} H^b$.

Linearized version (dropping also indices):

$$\begin{cases} \delta x = \psi \\ \delta \psi = R_0 x \\ \delta \chi = H \\ \delta H = R_1 \chi \end{cases} \qquad R_0 = \text{matrix (linear op)}$$

$$\delta w = \delta\left(\chi^a h_{ab} (H^b - \underset{\substack{\uparrow \\ \text{section of } E}}{s^b_{(x)}}) + \psi^\mu g_{\mu\nu} v^\nu \right)$$

$\delta^2 w = 0 \implies \mathcal{L}_v g = 0$ and s equivariant section wrt $U(1)$

Define $D = s^b{}_{,\rho}(0)$.

Then $w = \langle \psi, R_0 x \rangle + \langle \chi, H - iDx \rangle$.

$\delta^2 w = 0 \rightarrow \boxed{R_1 D - D R_0 = 0}$ and $\boxed{R_i^\dagger = - R_i}$ $(i=0,1)$

$$\delta w = \langle R_0 x, R_0 x \rangle - \langle \psi, R_0 \psi \rangle + \langle H - iDx, H \rangle - \langle \chi, R_1 \chi + iD\psi \rangle$$
$$= \langle x, (-R_0^2 + \tfrac{1}{4} D^\dagger D) x \rangle - \langle \psi, R_0 \psi \rangle - \langle \chi, R_1 \chi \rangle - \langle \chi, \tfrac{i}{2} D\psi \rangle + \langle \psi, \tfrac{i}{2} D^\dagger \chi \rangle$$

The result is: $$\frac{\det^{1/2} \begin{pmatrix} R_0 & \frac{i}{2} D^\dagger \\ -\frac{i}{2} D & R_1 \end{pmatrix}}{\det^{1/2} (-R_0^2 + \frac{1}{4} D^\dagger D)}$$

$-R_0^2 + \frac{1}{4} D^\dagger D$ should be second order elliptic operator, with no big kernel.

$$\begin{pmatrix} R_0 & \frac{1}{2}D^\dagger \\ -\frac{1}{2}D & R_1 \end{pmatrix} \cdot \begin{pmatrix} & \\ & \end{pmatrix}^\dagger = \begin{pmatrix} -R_0^2+\frac{1}{4}D^\dagger D & \overbrace{\frac{1}{2}(R_0 D^\dagger - D^\dagger R_1)}^{0} \\ \underbrace{\text{"}}_{0} & -R_1^2+\frac{1}{4}DD^\dagger \end{pmatrix}$$

Then $$\boxed{Z \sim \frac{\det^{1/4}(-R_1^2+\frac{1}{4}DD^\dagger)}{\det^{1/4}(-R_0^2+\frac{1}{4}D^\dagger D)}}$$ When does this make sense?

Using $R_1 D - DR_0 = 0$, there are huge cancellations outside the $\mathrm{Ker}\,(DD^\dagger)$:

$$\boxed{Z \sim \frac{\det_{\mathrm{Ker}\,DD^\dagger}(-R_1^2)}{\det_{\mathrm{Ker}\,D^\dagger D}(-R_0^2)}}$$

The susy transfo (linearized) can also be written

$$\begin{cases} \delta x = \psi \\ \delta \psi = R_0 x \\ \delta \chi = H - iDx \\ \delta H = R_1 x + iD\psi \end{cases}$$

$$\begin{tikzcd} & x & \chi & \\ 0 \arrow[r] & E_0 \arrow[r, "-iD"] \arrow[d, "R_0^{-1}"'] & E_{-1} \arrow[r] \arrow[d, "R_1^{-1}"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "iD"] & F_0 \arrow[r] & 0 \\ & \psi & H & \end{tikzcd}$$

In ∞-dim setting, R_0, R_1 and D are <u>differential operators</u>.

Elliptic Operators

Local operator $$Du = \sum_{|\alpha|\leq m} a_\alpha(x)\, \partial^{\alpha} u$$ multi-derivative

$$\Big\downarrow \text{symbol}$$

$$\sigma(D) = \sum_{|\alpha| = m} a_\alpha(x)\, \xi^\alpha \quad \text{with } \xi \in \mathbb{R}^n.$$

Operator is <u>elliptic</u> if $\sigma(D)$ is non degenerate away from origin.

<u>Ex</u> $\Delta u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i^2}$ $\qquad$ $\sigma(\Delta) = \sum_{i=1}^{n} \xi_i^2 \rightarrow$ elliptic

Ex
$$\begin{tikzcd} \Omega^k(M) \arrow[r, shift left, "d"] & \Omega^{k+1}(M) \arrow[l, shift left, "d^\dagger"] \end{tikzcd} \qquad \Delta = dd^\dagger + d^\dagger d .$$

Prove that Δ is elliptic.

Ex Prove ellipticity of Dirac operator $\not{D}$.

Linear operator D on M is Fredholm if $\begin{cases} \dim \operatorname{Ker} D < \infty \\ \dim \operatorname{coker} D < \infty \end{cases}$

$\boxed{\text{Th. On } M \text{ compact, Elliptic} \Longrightarrow \text{Fredholm.}}$

Ex. $S = \int_M d\phi \wedge * d\phi = \langle \phi, \Delta \phi \rangle$ $\qquad \phi$ = scalar field.

$\rightsquigarrow \dfrac{1}{\sqrt{\det \Delta}}$ makes sense.

NB. Susy is perfect because the action can often be written in terms of elliptic operators.

Ex. (Abelian gauge theory)

$\int F \wedge * F = \int dA \wedge * dA = \langle A, d^\dagger d A \rangle$ not elliptic!

$\int F \wedge * F + d^\dagger A \wedge * d^\dagger A = \langle A, \Delta A \rangle$ now it's elliptic.

Ex
- In 2d, $\begin{cases} F = 0 \\ d^\dagger A = 0 \end{cases}$ elliptic problem.

  Indeed, $\begin{cases} \partial_2 A_1 - \partial_1 A_2 = 0 \\ \partial_1 A_1 + \partial_2 A_2 = 0 \end{cases} \longrightarrow$ Symbol $\begin{pmatrix} \xi_2 & -\xi_1 \\ \xi_1 & \xi_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$

  $\downarrow$

  $\det = \xi_1^2 + \xi_2^2 \geqslant 0 .$

- In 3d, with A and adjoint scalar σ,

  $\begin{cases} F + * d_A \sigma = 0 \\ d^\dagger A = 0 \end{cases}$ Prove it's elliptic.

- In 4d $\begin{cases} F^+ = \frac{1}{2}(1+*)F = 0 \\ d^\dagger A = 0 \end{cases}$ (instanton) → Prove elliptic.

- Consider S^2 with coords $(z, \bar{z})$ and ∂ elliptic. Extend to 3d on $S^1_t \times S^2$ and define $\Delta = -\partial_t^2 + \partial\bar{\partial}$.

$\Omega^{p,q}_H \xrightarrow{\partial_H} \Omega^{p+1,q}_H$ on horizontal forms. It's non elliptic but on each Fourier mode $(\mathcal{L}_t \omega_n = in\,\omega_n)$, $(\Omega^{p,q}_H)_n \xrightarrow{\partial} (\Omega^{p+1,q}_H)$ is elliptic. Then $\bigoplus_n (\Omega^{p,q}_H)_n = \Omega^{p,q}_H$ and operator is transversally elliptic.

- Hopf fibration

$$\begin{tikzcd} S^3 \arrow[d] & S^1 \arrow[l] \\ S^2 & \end{tikzcd}$$

Def. Group action $G \times M \longrightarrow M$, $T^*_G M = \{\xi \in T^*M, \langle \xi, \underset{\substack{\uparrow \\ \text{action of } G}}{v} \rangle = 0\}$

Then if $\sigma(D) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ on $\xi \in T^*_G M \setminus \{0\}$ is invertible, then D is transversally elliptic.

Second order Elliptic op $\Delta = \underset{\substack{\uparrow \\ -\mathcal{L}_v^2 \text{ elliptic}}}{-R_0^2} + \underset{\nwarrow \text{ transverse elliptic}}{DD^\dagger}$

$\dim \operatorname{Ker} D = \infty$

$\operatorname{Ker} D = \bigoplus_{\text{reps}} \underset{\substack{\uparrow \\ \text{finite dim!}}}{H_{\text{reps}}}$

Ex. Take $\mathbb{C}$ and $\bar{\partial}$. What is $\operatorname{Ker}\bar{\partial}$?

$\operatorname{Ker}\bar{\partial} = \operatorname{Vect}(1, z, z^2, \dots)$. Use $U(1)$ action on $\mathbb{C}$.

$\operatorname{ind}\bar{\partial} = 1 + t + t^2 + \dots = \frac{1}{1-t}$.

Ex. Map $S^2 \longrightarrow \mathbb{C}^n$

$$\begin{cases} \delta x^\mu = \psi^\mu \\ \delta \psi^\mu = 0 \\ \delta \chi^i_{\bar z} = H^i_{\bar z} - \bar\partial x^i \\ \delta H^i_{\bar z} = i\, \bar\partial \psi^i \end{cases}$$

$M = \{\text{maps } S^2 \to \mathbb{C}^n\}$, $T[1]M$,

$\chi^i_{\bar z} \in \Omega^{0,1}(S^2, x^*(T^{1,0}\mathbb{C}^n))$

$\rightsquigarrow$ Gromov-Witten.

Here, $R_0 = 0$, $R_1 = 0$.

Go to equivariant version under $U(1) \times S^2 \to S^2$ by replacing

$R_0 = \mathcal{L}_v$ on Ω^0

$R_1 = \mathcal{L}_v$ on $\Omega^{1,0}$ or $\Omega^{0,1}$.

$$\longrightarrow \frac{\det_{\Omega^{1,0}}(\mathcal{L}_v)}{\det^{1/2}_{\Omega^{0,0}}(\mathcal{L}_v)} = \cdots$$

Ex. 3d: maps $S^3 \longrightarrow \mathbb{C}^n$, with $U(1) \times S^3 \longrightarrow S^3$ (Hopf).

Decompose $\Omega = \Omega_V + \Omega_H$.

$$\begin{cases} \delta X^\mu = \psi^\mu \\ \delta \psi^\mu = \mathcal{L}_v X^\mu \\ \delta \chi^i_{\bar z} = H^i_{\bar z} - \bar\partial_H X^i \\ \delta H^i_{\bar z} = \mathcal{L}_v \chi^i_{\bar z} + \bar\partial_H \psi^i \end{cases}$$

$\chi^i_{\bar z}, H^i_{\bar z} \in \Omega^{0,1}_H(X^*(T^{1,0}M))$

Index theorems.

D = elliptic operator on compact M (hence D is Fredholm)

$\boxed{\text{ind } D = \dim \operatorname{Ker} D - \dim \operatorname{coker} D} < \infty$ (analytical index)

Atiyah-Bott: $(\bar\partial, E)$ ie $\bar\partial$ + connection.

$$\text{ind}(\bar\partial, E) = \sum (-1)^k \dim H^k(M, E)$$

$$\text{ind}(\bar\partial, E) = \frac{1}{(-2\pi i)^k} \int_M \text{td}(T^{1,0}_M)\, \text{ch}(E)$$

Exercise On S^1, $D = \frac{d}{dx} + \lambda$ $\lambda \in \mathbb{C}$.

Compute $\underbrace{\operatorname{ind} D}_{\text{doesn't depend on } \lambda}$, $\underbrace{\dim \operatorname{Ker} D, \quad \dim \operatorname{Ker} D^\dagger}_{\text{depends on } \lambda}$

Atiyah-~~Bott~~ Singer: $$\boxed{\underset{\substack{\nearrow \\ \text{analytic index}}}{\operatorname{ind}(D)} = \int_M ch(D)\, \underset{\substack{\uparrow \\ \text{topological index}}}{Td(M)}}$$

Characteristic classes

Invariant polynomials $\mathbb{R}[\mathfrak{g}]^G$, vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$

Then $\mathbb{R}[\mathfrak{g}]^G \ni P \longmapsto P(F) \in H^\bullet(M, \mathbb{R})$

Elliptic complex

In most previous examples, $E \xrightarrow{D} F$. Now take

$$0 \longrightarrow E_1 \xrightarrow{D_1} E_2 \xrightarrow{D_2} \cdots \longrightarrow E_n \longrightarrow 0$$

compute the symbols $\sigma(D_i)$ and require the sequence be exact with respect to the symbols.

Ex For the 3d problem $\begin{cases} F^+ = 0 \\ d^\dagger A = 0 \end{cases}$ show that the complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega^{2+}(M) \longrightarrow 0$$

is elliptic, where $d^+ = \frac{1}{2}(1 + *)d$

We want an <u>equivariant</u> version of the index theorem.

$$\begin{tikzcd} \mathcal{G} \circlearrowright E \arrow[d] \\ \mathcal{G} \circlearrowright M \end{tikzcd}$$

On the eq. bundle, eq. connection $D_{A,\mathcal{G}} = D_A + \varepsilon^a i_{v_a}$

and $F_{A,\mathcal{G}} = D^2_{A,\mathcal{G}} - \varepsilon^a \mathcal{L}_{v_a}$ curvature.

$$\underset{\substack{\uparrow \\ \text{invariant} \\ \text{polynomial}}}{P} \longrightarrow P(F_{A,\mathfrak{g}}) \in H^\bullet_{\mathfrak{g}}(M) \quad \text{equivariant cohomology}$$

Since $(d+i_v)P(F_{A,\mathfrak{g}})=0$, use Atiyah-Bott :

$$\int_M P(F_{A,\mathfrak{g}}) = \sum_{fp} \frac{P_0(F_{A,\mathfrak{g}})}{\sqrt{\cdots}}$$

Ex For $(\bar\partial, E)$ complex,

$$\operatorname{ind}(\bar\partial,E)(e^{\cdots}) = \frac{1}{(2\pi i)^{\#}} \int Td_{\mathfrak{g}}(T_M)\, ch_{\mathfrak{g}}(E)$$

$$\rightsquigarrow$$

$$\operatorname{ind}(\bar\partial,E) = \sum_{fp} \frac{\cdots}{\sqrt{\cdots}} = \sum_{fp} \frac{Tr_{E_x}(g)}{\det_{T_x^{1,0}}(1-g^{-1})}$$

Application to $\mathbb{P}^1$

$\mathbb{P}^1 = \{(z_1,z_2) \sim (\lambda z_1, \lambda z_2),\ \lambda \in \mathbb{C}^*\}$

$\mathcal{U}_1 = \{\zeta = \frac{z_2}{z_1},\ z_1 \neq 0\}$ $\quad \mathcal{U}_2 = \{\lambda = \frac{z_1}{z_2},\ z_2 \neq 0\}$

on the intersection, $\zeta = \frac{1}{\lambda}$.

$U(1)$ action on $\mathbb{P}^1$ by rotation, with 2 fixed points :

$$\begin{cases} \zeta \longrightarrow t\zeta \\ \lambda \longrightarrow t^{-1}\lambda \end{cases} \quad \text{for } |t|=1.$$

Choose $E = \Omega^{0,0}(\mathbb{P}^1)$, $F = \Omega^{0,1}(\mathbb{P}^1)$, $\bar\partial : E \longrightarrow F$.

$$\operatorname{ind}(\bar\partial) = \frac{1-t^{-1}}{(1-t)(1-t^{-1})} + \frac{1-t}{(1-t)(1-t^{-1})} \underset{\substack{\uparrow \\ t=1}}{=} 1$$

$$\operatorname{ind}(\bar\partial) = \sum_{fp} \frac{Tr_E\, t - Tr_F\, t}{\det(1-\rho(t))}$$

$\mathcal{O}(n)$ bundle for $\mathbb{P}^1$

$(z_1, z_2, z_3) \in (\mathbb{C}^2 \setminus \{0,0\}) \times \mathbb{C}$ with identification

$(z_1, z_2, z_3) \sim (\lambda z_1, \lambda z_2, \lambda^n z_3)$

$\mathcal{U}_1 = \{ \xi_1 = \frac{z_2}{z_1} \quad \xi_2 = \frac{z_3}{z_1^n} \quad z_1 \neq 0 \}$

$\mathcal{U}_2 = \{ \lambda_1 = \frac{z_1}{z_2} \quad \lambda_2 = \frac{z_3}{z_2^n} \quad z_2 \neq 0 \}$

Transition $\xi_1 = \frac{1}{\lambda_1} \qquad \xi_2 = \frac{\lambda_2}{\lambda_1^n}$

$U(1)$ action $\xi_1 \to t\,\xi_1 \qquad \xi_2 \to \xi_2$

$\lambda_1 \to t^{-1}\lambda \qquad \lambda_2 \to t^{-n}\lambda_2$

$E = \Omega^{0,0} \otimes \mathcal{O}(n) \qquad F = \Omega^{0,1} \otimes \mathcal{O}(n) \qquad \bar\partial : E \to F$

$$\text{ind}\,\bar\partial = \frac{1-t^{-1}}{(1-t)(1-t^{-1})} + \frac{t^{-n} - t^{1-n}}{(1-t)(1-t^{-1})} = 1-n$$

Hopf fibration

$S^3 : \quad |z_1|^2 + |z_2|^2 = 1$

$T^2 \times S^3 \longrightarrow S^3 \qquad z_1 \to e^{i\alpha} z_1, \quad z_2 \to e^{i\beta} z_2$

Adapted coords: $z_1 = \frac{e^{i\theta}}{\sqrt{1+\xi\bar\xi}} \qquad z_2 = \frac{e^{i\theta}\xi}{\sqrt{1+\xi\bar\xi}} \qquad \mathbb{C} \times S^1$

$\{ \lambda = \frac{1}{\xi}, \; \tilde\theta = \dots$

other patch.

$$\begin{tikzcd} S^3 \arrow[d] & U(1) \arrow[l] \\ S^2 & \end{tikzcd}$$

Prove $\boxed{\Omega_H^{\uparrow}(S^3) = \bigoplus_n \Omega_H^{\uparrow}(S^2, \mathcal{O}(n))}$

This is the "Fourier decomposition" on S^3

Physical calculation

Goal: compute 3d Chern-Simons ff on S^3.

$$S_{CS} = \frac{k}{4\pi}\int \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) \qquad \mathcal{A} = \{\text{space of connections}\}$$

$$\begin{tikzcd} P \arrow[d] & \mathcal{G} = \text{gauge group} \arrow[l] \\ U(1) \circlearrowleft S^3 & \end{tikzcd}$$

$$\begin{cases} \delta A = \Psi \\ \delta\Psi = \mathcal{L}_v A + d_A\varphi \\ \delta\varphi = 0 \end{cases}$$

$$\Updownarrow$$

for $i\sigma - i_v A = \varphi$.

$\sigma \in \Omega^0(S^3, \underline{g})$

$$\begin{cases} \delta A = \Psi \\ \delta\Psi = i_v F + i\, d_A\sigma \\ \delta\sigma = -i\, i_v\Psi \end{cases}$$

Susy action $S_{SCS} = S_{CS}(A - i\kappa\sigma) - \frac{k}{4\pi}\int \kappa\wedge\Psi\wedge\Psi$

and $\delta S_{SCS} = 0 \Leftrightarrow \kappa \in \Omega^1(S^3)\,,\ i_v\kappa = 1\,,\ i_v d\kappa = 0$

$$\begin{cases} \delta A = \Psi \\ \delta\Psi = i_v F + i\, d_A\sigma \\ \delta\sigma = -i\, i_v\Psi \\ \delta\chi = H \\ \delta H = \mathcal{L}_v^A\chi - i[\sigma,\chi] \end{cases}$$

χ odd, H even.

$$\Omega^0 = \underset{\overset{\uparrow}{\kappa\wedge i_v}}{\Omega^0_V} + \underset{\overset{\uparrow}{1-\kappa\wedge i_v}}{\Omega^0_H}$$

$$\Omega^2 = \underset{\text{dim } 2}{\Omega^2_V} + \underset{\text{dim } 1}{\Omega^2_H}$$

$$\delta w = \delta\left(\Psi\wedge *\,\overline{\delta\Psi} + \chi\wedge *(H - F_H)\right) = F\wedge *F + d_A\sigma\wedge *d_A\sigma + \dots$$

$$\downarrow$$

$F = 0$, $\sigma = $ const.

1 isolated point

$$\longrightarrow \int d\sigma\, e^{-\#\,\mathrm{tr}(\sigma^2)}\left(\frac{\det^{1/2}_{\Omega^0}(\mathcal{L}_v + \mathrm{ad}_\sigma)\ \det_{\Omega_0}(\mathcal{L}_v + \mathrm{ad}_\sigma)}{\det^{1/2}_{\Omega^1}(\mathcal{L}_v + \mathrm{ad}_\sigma)}\right)$$

$$= \int d\sigma \, e^{-\# \operatorname{tr}\sigma^2} \frac{\det_{\Omega^0}(\quad)}{\det_{\Omega_H^{1,0}}(\quad)} \sim \prod_n (2\pi i n + \mathrm{ad})^{n-1}$$

Summary Chern-Simons [Ref: Källén 2012]

Transverse elliptic problem

$$\begin{tikzcd}
{[c]} \arrow[r, "d"] & {[A]} \arrow[r, "d_H"] & {[\chi]} \arrow[r, phantom, "\otimes"] & {[\bar{c}]} & {} \\
{[\sigma]} \arrow[u] \arrow[r] & {[\Psi]} \arrow[u] \arrow[r] & {[H]} \arrow[u] \arrow[r, phantom, "\otimes"] & {[b]} \arrow[u] & {} \arrow[u, "R"']
\end{tikzcd}$$

$$\xrightarrow[D]{}$$

More generally,

$$\boxed{\begin{tikzcd}
0 \arrow[r] & E_0 \arrow[r] & E_1 \arrow[r] & E_2 \arrow[r] & 0 & {} \\
0 \arrow[r] & F_0 \arrow[u] \arrow[r] & F_1 \arrow[u] \arrow[r] & F_2 \arrow[u] \arrow[r] & 0 & {} \arrow[u, "R"'] \\
& & \xrightarrow[D]{} & & &
\end{tikzcd}}$$

- Fields = $\mathcal{A}$, bundle, ...
- Fixed pts = $F=0$, $\sigma = \mathrm{cst}$.
- Index theorem gives $Z_{S^3} = \int d\sigma \, e^{-\# \operatorname{tr}\sigma^2} \; \mathrm{Sdet}_{\Omega_H^{0,0}(S^3, \underline{g})}(\mathcal{L}_v + \mathrm{ad}_\sigma)$

$$\Rightarrow Z_{S^3} = \int_t dt \, e^{-\# \operatorname{tr}\sigma^2} \prod_{\beta \neq 0} \sin(i \langle \beta, \sigma \rangle)$$

Coincides with result by Witten, Mariño.

5d theory

2d	3d	4d	5d	6d	7d
$\mathcal{N}=2$	$\mathcal{N}=2$	$\mathcal{N}=2$			
$F=0 \rightsquigarrow$	$F_H=0$	$F^+=0 \rightsquigarrow$	$F_H^+=0$		
Elliptic	Transv. Elliptic		↑ Here, focus on that.		

On M_4, natural $F^+=0$ problem. Then it's natural to lift to $\underset{\uparrow t}{S^1} \times M_4$ with $v=\partial_t$ and $\begin{cases} i_v F = 0 \\ F_H^+ = 0 \end{cases}$

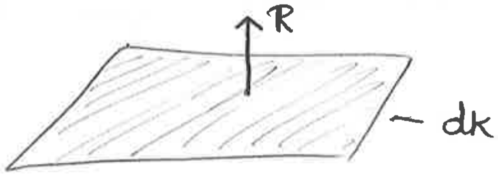
Some geometry $\quad M_{2n-1}$ is called contact manifold if $\exists \kappa \in \Omega^1(M)$ such that $\kappa (d\kappa)^n \neq 0$.

* Then $\exists!$ Reeb vector field v such that $\begin{cases} i_v \kappa = 1 \\ i_v d\kappa = 0 \end{cases}$

NB : "most" of 5d manifold are contact.

* $\exists\, g$ such that $g(v) = \kappa$

↑ "compatible metric".



* Given (v, κ, g), $\Omega^\bullet = \underset{\underset{\kappa i_v}{\uparrow}}{\Omega^\bullet_V} \overset{\perp}{\oplus} \underset{\underset{(1-\kappa i_v)}{\uparrow}}{\Omega^\bullet_H}$ projectors.

In 5d, $\Omega^2(M_5) = \Omega^2_V(M_5) + \Omega^2_H(M_5)$

$$= \Omega_V \overset{\perp}{\oplus} \underset{\underset{\frac{1}{2}(1+i_v *)}{\uparrow}}{\Omega^{2+}_H} \overset{\perp}{\oplus} \underset{\underset{\frac{1}{2}(1-i_v *)}{\uparrow}}{\Omega^{2-}_H} \quad \leftarrow \text{projectors on } \Omega^2_H.$$

Look at problem $\begin{cases} F_H^+ = 0 \\ F_V = 0 \end{cases} \iff \boxed{*F = -\kappa \wedge F}$

"Contact instanton"

NB : not (trans). elliptic pb, because 1st eq is 3 and 2nd is 4 eq.

Symplectization : $M_{2n-1} \times \underset{\overset{\uparrow}{r}}{\mathbb{R}_+}$ with $\underbrace{\omega = d(r^2\kappa)}_{\text{symplectic form}}$

This is a cone, $g = dr^2 + r^2 g_{M_{2n-1}}$.

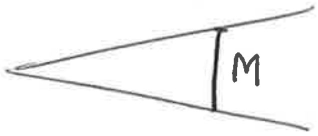
If the cone is Kähler, then M_{2n-1} is said to be Sasakian

———— CY ———— Sasaki-Einstein

Covariantly constant spinors in CY gives Killing spinors on SE.

Since in CY_3 (ie 6d) there are a lot of examples, this means 5d is very rich.

[in 4d, it's opposite situation].



Examples • S^5 $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$, $\text{cone}(S^5) = \mathbb{C}^3$

- Conifold $\mathbb{C}^4 /\!/ (1,1,-1,-1)$ ⇝ base $T^{1,1} \sim S^2 \times S^3$
- $\mathbb{C}^4 /\!/ (p-q, p+q, - , \ )$ $\gcd(p,q) = 1$

  ⇝ base $Y^{p,q} \sim S^2 \times S^3$

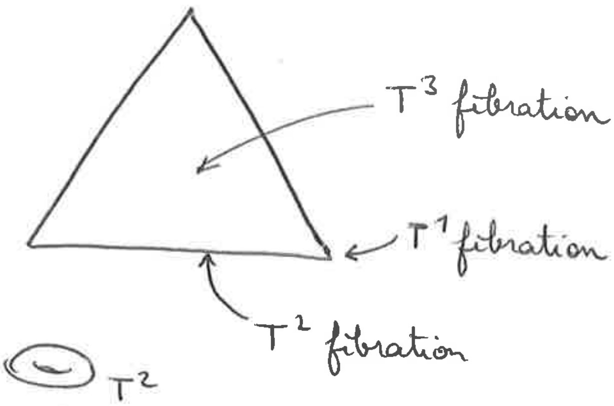
Take $T^3 \times S^5 \longrightarrow S^5$, $z_i \longmapsto e^{i\alpha_i} z_i$

① $v = e_1 + e_2 + e_3$: write κ. Show Hopf fibration $\begin{matrix} S^5 \leftarrow S^1 \\ \downarrow \\ \mathbb{CP}^2 \end{matrix}$

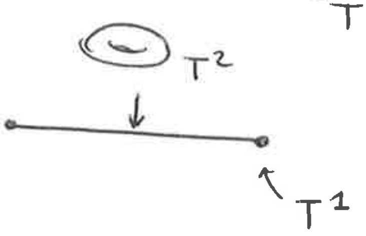
② Choose $w_1, w_2, w_3 \in \mathbb{R}_+$, $v = \sum w_i e_i$ This is "toric contact geom" because Reeb is related to actions of S^1.

$$|z_1|^2+|z_2|^2+|z_3|^2=1$$

$S^5 =$



Similarly $S^3 =$



$$\begin{cases} \delta A = \Psi \\ \delta\Psi = i_v F + i\, d_A \sigma \\ \delta\sigma = -i\; i_v \Psi \\ \delta H^{*}_{H} = \mathcal{L}^{A}_{v} \chi^{+}_{H} - i[\sigma - \chi^{+}_{H}] \\ \delta \chi^{+}_{H} = H^{+}_{H} \end{cases}$$

A = connection

Ψ = odd 1-form in adj

σ = even 0-form in adj

$H^{+}_{H} \in \Omega^{2+}_{H}$ even in adj

$\chi^{+}_{H} \in \Omega^{2+}_{H}$ odd in adj.

On SE manifolds, $\boxed{\begin{array}{c}\mathcal{N}=1\\ \text{vect. mult}\end{array}} \longleftrightarrow \boxed{\begin{array}{c}\text{Coh. field}\\ \text{theory}\end{array}}$

Missing fields are $\underset{\text{ghost}}{c}\,,\ \underset{\text{anti ghost}}{\bar{c}}\,,\ \underset{\text{lagrangian mult.}}{b}$

$$\begin{tikzcd} \begin{array}{c}\Omega^0_{odd}\\ {[c]}\end{array} \arrow[r, "d"] & \begin{array}{c}\Omega^1_{even}\\ {[A]}\end{array} \arrow[r] & \begin{array}{c}\Omega^{2+}_{H_{odd}} \otimes\\ {[\chi^+_H]} \otimes\end{array} & \begin{array}{c}\Omega^0_{odd}\\ {[\bar{c}]}\end{array} & \text{manifold} \\ {[\sigma]} \arrow[u] \arrow[r] & {[\Psi]} \arrow[u] \arrow[r] & {[H]} \otimes \arrow[u] & {[b]} \arrow[u] & \text{tangent bundle} \end{tikzcd}$$

$$\frac{\det^{1/2}_{\Omega^{2+}_H}(\ )\ \det^{1/2}_{\Omega^0}(\ )\ \det^{1/2}_{\Omega^0}(\ )}{\det^{1/2}_{\Omega^1}(\ )} \quad \begin{array}{l}\leftarrow \text{odd}\\[2ex] \leftarrow \text{even}\end{array}$$

$$\delta \mathcal{V} = \delta\left(\Psi \wedge * \overline{\delta \Psi} + \chi_H^+ \wedge * (H_H^+ - F_H^+) \right)$$

$$= \underbrace{F_V \wedge * F_V + F_H^+ \wedge * F_H^+}_{\parallel} + d_A \sigma \wedge * d_A \sigma$$

$$F \wedge * F + \kappa \wedge F \wedge F$$

We see $\begin{cases} \begin{cases} F_V = 0 \\ F_H^+ = 0 \end{cases} \Longrightarrow \; * F = - \kappa F \\ \text{and} \quad d_A \sigma = 0 \end{cases}$

We have the isolated point $\underline{\underline{F = 0}}$ (hence $A = 0$) and $\underline{\underline{\sigma = cst}}$

$$\int d\sigma \, e^{-\# \sigma^2} \frac{\det^{1/2}_{\Omega_H^{2+}}(\;)\, \det_{\Omega_0}(\;)}{\det^{1/2}_{\Omega_1}(\;)} + \left(\begin{array}{c} \text{Non trivial solutions} \\ \text{of } * F = - \kappa F \end{array} \right)$$

Decompose $\begin{cases} \Omega_H^{2+} = \Omega_H^{2,0} + \Omega_H^{0,2} + \Omega^0 \cdot \omega \leftarrow \text{Kähler form} \\ \Omega^1 = \Omega_H^{1,0} + \Omega_H^{0,1} + \Omega^0 \end{cases}$

Det becomes

$$\frac{\det_{\Omega^{0,2}}(\;)\, \det_{\Omega^0}(\;)}{\det_{\Omega_H^{0,1}}(\;)} = \operatorname{Sdet}_{\Omega_H^{0,\bullet}} (\mathcal{L}_v + \mathrm{ad}_\sigma)$$

$\downarrow$ using

$$\Omega_H^{0,0} \xrightarrow[\bar\partial_H]{} \Omega_H^{0,1} \xrightarrow[\bar\partial_H]{} \Omega_H^{0,2}$$

Now use the trick : $\boxed{\Omega_H^{0,p}(S^5) = \bigoplus_n \Omega^{0,p}(\mathbb{CP}^2, \mathcal{O}(n))}$

↪ $\operatorname{ind}(\bar\partial, \mathcal{O}(n)) = 1 + \frac{3}{2} n + \frac{1}{2} n^2 .$

↪ $\boxed{\prod_{n \neq 0} (2 \pi i n + \mathrm{ad}_\sigma)^{1 + \frac{3}{2} n + \frac{1}{2} n^2}}$

The previous calculation corresponds to $v = \sum \omega_i e_i$ with $\omega_i = 1$. What about general case?

$\uparrow$ Round S^5.

$$\int_t d\sigma \, e^{-\#\sigma^2} \prod_{\beta \neq 0} S_3(i \langle \sigma, \beta \rangle | \omega_1, \omega_2, \omega_3)$$

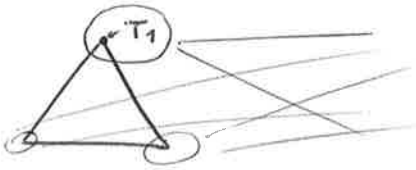
$\uparrow$ triple sine

$$S_3(x, \vec{\omega}) = \prod_{m_1, m_2, m_3 \geq 0} (x + \vec{n} \cdot \vec{\omega}) \prod_{m_1, m_2, m_3 \geq 1} (-x + \vec{n} \cdot \vec{\omega})$$

$$= e^{B} \underbrace{\left(e^{2\pi i \frac{x}{\omega_1}} \middle| e^{2\pi i \frac{\omega_2}{\omega_1}} ; e^{2\pi i \frac{\omega_3}{\omega_1}} \right)_\infty}_{} \cdots \text{cyclic} \cdots$$

if $\omega_i \in \mathbb{C}$ and other cond.

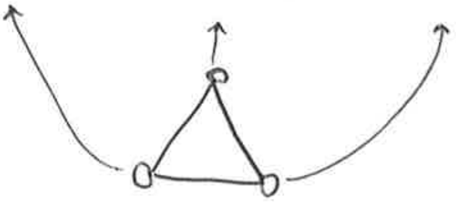
Perturbative answer for Nekrasov part. fn. on $\mathbb{R}^4_{\frac{\omega_2}{\omega_1}, \frac{\omega_3}{\omega_1}} \times S^1_{\frac{1}{\omega_1}}$

$S^5 =$ 

$\mathbb{R}^4 \times S^1$ at 3 points!

The analytic continuation agrees with the geometry!

$$Z_{S^5} = \int d\sigma \, e^{-\#\sigma^2} Z^{\text{Nek}}_{\mathbb{R}^4 \times S^1} \times Z^{\text{Nek}}_{\mathbb{R}^4 \times S^1} \times Z^{\text{Nek}}_{\mathbb{R}^4 \times S^1}$$



NB: in 5d instantons are particles, they sit in the S^1 at each vertex.

Nekrasov

Instantons

Path integral in Euclidean QFT : Space-time is Riemann, but space of fields might be complexified.

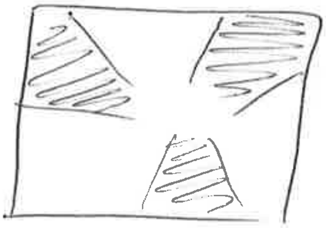
Finite dim analogy   Manifold $\mathcal{X}$, volume form Ω, $s : \mathcal{X} \to \mathbb{R}$.

$\mathcal{X}^{\mathbb{C}} \supset \mathcal{X}$, holomorphic volume form, $s : \mathcal{X}^{\mathbb{C}} \to \mathbb{C}$.

$$\int_{\mathcal{X}} e^{-s/\hbar}\,\Omega \rightsquigarrow \int_{\Gamma} e^{-s/\hbar}\,\Omega = Z_{\Gamma}(\hbar)$$

$\uparrow$
middle-dim contour
in $\mathcal{X}^{\mathbb{C}}$

$$\mathcal{X}_{\infty} = \mathrm{Re}(s/\hbar)^{-1}([c, \infty[) \subset \mathcal{X}_{\mathbb{C}}$$



$[\Gamma] \in H_n(\mathcal{X}^{\mathbb{C}}, \mathcal{X}_{\infty})$

$Z_{\Gamma}(\hbar)$ = fund solution of system of Picard-Fuchs equations.

When $\hbar \to 0$, integral dominated by saddle points $ds(p) = 0$.

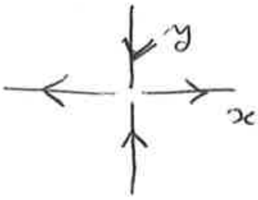
Basis of Lefschetz thimbles L_p = basis of special contours, tailored to the critical p.

So $\Gamma = \sum n_p L_p$.

Pick a generic hermitian metric on $\mathcal{X}^{\mathbb{C}}$ and look at gradient flow of $\mathrm{Re}(s/\hbar)$: $\dot{x} = \nabla \mathrm{Re}(s/\hbar)$ (starts and stops at critical points)

Near p, $s = s(p) + \sum_{i=1}^{n} \frac{1}{2} z_i^2$ $n = \dim_{\mathbb{C}} \mathcal{X}^{\mathbb{C}}$, z_i hol. coords.

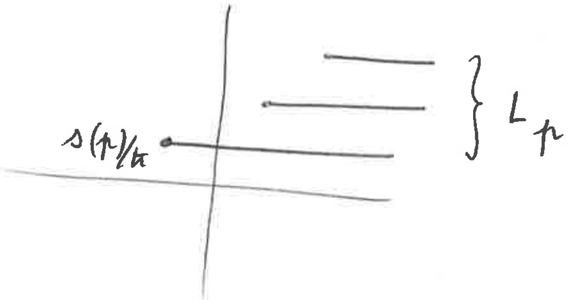
For $\hbar \in \mathbb{R}$, $\mathrm{Re}(s/\hbar) = \mathrm{Re}(s(p)/\hbar) + \frac{1}{2\hbar} \sum (x_i^2 - y_i^2)$



Take the union of outgoing trajectories (x-lines) $:= L_p$.

The $\mathrm{Im}(s/\hbar)$ is constant

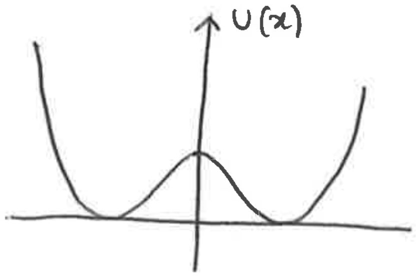
Im $s/\hbar$ plane :



Simplest ∞-dim case : Path integral in QM.

Non-relativistic particle in double-well potential.

$$U(x) = \frac{\lambda}{4}(x^2-v^2)^2 \qquad \hat{H} = -\frac{\hbar^2}{2}\partial_x^2 + U(x)$$



$\hat{H}\Psi = E\Psi$ find low energy states in $\mathcal{H} = L^2(\mathbb{R})$.

$$Z_\hbar(T) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\frac{T}{\hbar}\hat{H}}, \quad \mathrm{Re}\left(\frac{T}{\hbar}\right) \gg 0$$

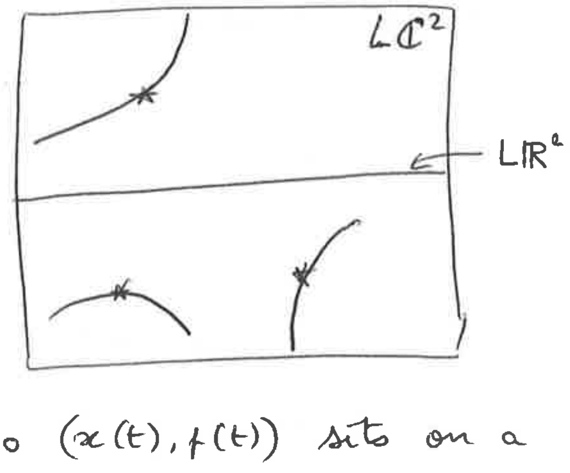
$$Z_\hbar(T) = \int_{(p,x):S^1\to\mathbb{R}^2} \mathcal{D}p\,\mathcal{D}x\, \exp\Big[\overbrace{\frac{i}{\hbar}\oint p\,dx - \frac{T}{\hbar}\oint H(p,x)\,dt}^{-S/\hbar}\Big]$$

$$H(p,x) = \frac{p^2}{2} + U(x)$$

$\nearrow$

See this as <u>one</u> contour in $L\mathbb{C}^2$, complexified loop space.

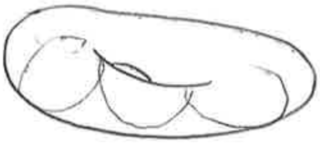
$$L\mathbb{C}^2 = \{(p(t),x(t)) / \text{maps } S^1 \to \mathbb{C}^2\}$$



Critical points of S are $*$.

$$\begin{cases} -i\dot{x} + Tp = 0 \\ i\dot{p} + T\,U'(x) = 0 \end{cases}$$

The value of H is constant on trajectory so $(x(t), p(t))$ sits on a complex curve $H(p,x) = \varepsilon$ for some constant ε. This is an <u>elliptic curve</u> $\frac{p^2}{2} + \frac{\lambda}{4}(x^2-v^2)^2 = \varepsilon$ with uniformizing coord $z = \int \frac{dx}{p}$.



Solution: $z = z(0) + v_0 t$

with $v_0 = n\omega_1 + m\omega_2$, $(n,m)\in\mathbb{Z}^2$

$$\omega_{1,2} = \int_{A,B} \frac{dx}{p}.$$

Critical points are labeled by (n, m).

When $T \to \infty$, one can safely expect $\varepsilon \to 0$, meaning the elliptic curve is nearly degenerate

$$n + \frac{m}{\pi} \log \frac{\varepsilon}{\varepsilon_0} \sim \frac{T}{i T_0} \longrightarrow \boxed{\varepsilon_{n,m} \sim e^{-\pi \frac{T}{m T_0}} e^{\frac{2\pi i n}{2m}}}$$

For most of t-time, $\frac{p^2}{2} + U \approx 0$, so $\frac{i \dot{x}}{T} = p \simeq \pm\sqrt{2U}$

$$\begin{cases} + = \text{instanton} \\ - = \text{anti-instanton} \end{cases}$$

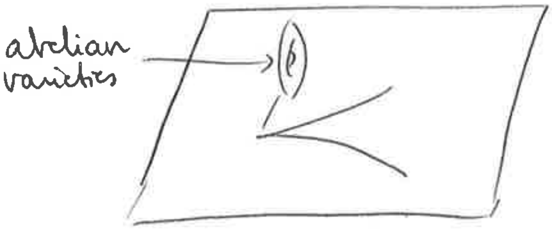
$$\begin{cases} m = \# \text{ of inst, } \overline{\text{inst}} \\ n = \# \text{ of perturbative fluctuations} \end{cases}$$

[NB: the topology (torus) emerges only at fixed ε]

Many-body systems (Algebraic integrable syst)

Action-angle variables : $\sum d\phi_a \wedge dx^a = \sum da_a \wedge d\phi^a$

H depends only on the a variables.



Find critical points :

$$0 = d\left(\sum n^a a_a + m_a a_D^a - \sum T_k H_k(a)\right)$$

1-dim SQM

$(x(t), p(t), \psi(t), \bar{\psi}(t))$

$\delta x = \psi$ $\quad \delta\bar{\psi} = p$

$\delta\psi = 0$ $\quad \delta p = 0$

$\delta^2 = 0$.

Action: $S = \delta \int \bar{\psi}\left(\dot{x} + \frac{i}{2} p + V'(x)\right)$

V = morse function on $\mathbb{R}$

$V'' \neq 0$ when $V' = 0$.

$\delta = 0 \Rightarrow$ instantons = gradient trajectories $\dot{x} = V'$.

In the large volume limit (in target space) one can recover Morse theory.

Generalize :
- 2d QFT (σ-models)
- 4d QFT (gauge theories)

σ-model $X : \underset{\text{Riemann Surface}}{\Sigma} \longrightarrow \mathcal{X}$ where $\mathcal{X}$ has metric g and 2-form B.

$$S = \frac{1}{2}\int_{\Sigma} g_{mn}\, dX^m \wedge * dX^n + \frac{i}{2}\int_{\Sigma} X^* B$$

(Ex $\mathcal{X}$ = Kähler, $B = \omega$.)

Bogomolny trick:

$$S = \left\| \left(\frac{1-iJ}{2}\right) \bar{\partial} X \right\|^2 + i\int X^* \omega \quad \longrightarrow$$

instantons correspond to pseudo-holomorphic maps $(1-iJ)\bar{\partial}X = 0$

J = pseudo-complex structure.

$\omega = B + i(g\cdot J)$

NB : other Bogo trick for anti-instantons

4d Yang-Mills

$$S = \dots = \| F_A^+ \|^2 + 2\pi i \tau \int \frac{\mathrm{Tr}\, F_A^2}{2(2\pi i)^2}$$

$F_A^+ = \frac{1}{2}(F_A + * F_A)$ $\quad *^2 = +1$ on $\Omega^2(M^4)$.

$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$

$\rightsquigarrow$ instantons $F_A = - * F_A \dots$

Gauge theory $\underset{\substack{\uparrow \\ \text{Compact}}}{M^3} \times \mathbb{R}^1_t$.

Ansatz for $A = f(t)\,\Theta$ with Θ flat connection on M^3.

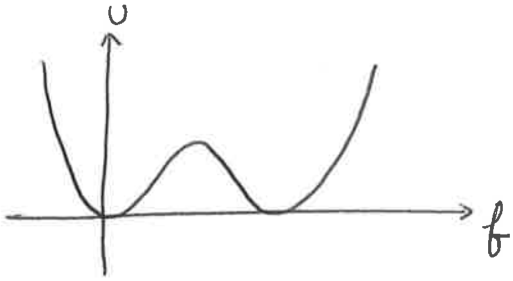
$$d\Theta + \Theta^2 = 0.$$

$$F_A = \dot{f}\, dt \wedge \Theta + (-f + f^2)\, \Theta \wedge \Theta .$$

$$S_{YM} = \int_{\mathbb{R}} dt \left[c_1 \dot{f}^2 + c_2 f^2 (f-1)^2 \right] \qquad c_1 = \int_{M^3} \mathrm{Tr}\left[\textcircled{H} \wedge * \textcircled{H} \right]$$

$$c_2 = \int_{M_3} \cdots$$

Reduces to classical action of particle



Minima: $f=0 \rightarrow A=0$

$f=1 \rightarrow A$ flat

$\dot{f} \propto f(f-1)$ instanton in QM sense

BPST instanton in radial coord.

Describe all instantons on $\mathbb{R}^4 \subset S^4$ for $SU(N)$,

$$-\frac{1}{8\pi^2}\int \mathrm{Tr}\, F^2 = k \geq 0.$$

Instead of solving PDE $F_A^+ = 0 \iff$ Algebraic equations on matrices: ADHM construction.

$E =$ rank N complex vector bundle over $\mathbb{R}^4$ (trivial). Solve $\not{D}_A \Psi = 0$ with $\Psi \in L^2(S_\pm \otimes E)$ when $S_- \simeq \Omega^{0,1}$, $S_+ \simeq \Omega^{0,0} \oplus \Omega^{0,2}$ under $\mathbb{R}^4 \simeq \mathbb{C}^2$

$$\not{D}_A = \bar{\partial}_A + \bar{\partial}_A^*, \qquad \bar{\partial}_A : \Omega^{0,i} \longrightarrow \Omega^{0,i+1}$$

$$\bar{\partial}_A^* : \Omega^{0,i} \longrightarrow \Omega^{0,i-1}$$

$F_A^+ = 0$ are $\frac{1}{2}$ rank $\Omega^2 = \frac{1}{2}\binom{4}{2} = 3$ real $\underline{g}$-valued equations

$\Updownarrow$

$F_A^{2,0} = 0 \qquad (F_A^{1,1})_\omega = 0 \quad \leftarrow$ these are $2+1$ real eq.

$\Updownarrow$

$\bar{\partial}_A^2 = 0$

Cohomology $\bar{\partial}_A \tilde{\Psi} = 0$ eq to $\tilde{\Psi} \sim \tilde{\Psi} + \bar{\partial}_A \cdots$

Study on S_+ ... find no solution. But index thm says

$$\underbrace{\dim \ker_{L^2(S_-\otimes E)} \not{D}_A}_{K} - \underbrace{\dim \mathrm{Ker}_{L^2(S_+\otimes E)} \not{D}_A}_{0} = k$$

$$\Psi \in K \Leftrightarrow \Psi \in L^2(\Omega^{0,1}\otimes E) \text{ s.t. } \begin{cases} \bar\partial_A \Psi = 0 \\ \bar\partial_A^{*} \Psi = 0 . \end{cases}$$

Multiply by coords and project on K (orthogonally):

$$\underset{\underset{B_1\Psi}{\shortparallel}}{\Pi(z_1\Psi)}, \quad \underset{\underset{B_2\Psi}{\shortparallel}}{\Pi(z_2\Psi)}, \quad \underset{\underset{B_1^\dagger\Psi}{\shortparallel}}{\Pi(\bar z_1\Psi)}, \quad \underset{\underset{B_2^\dagger\Psi}{\shortparallel}}{\Pi(\bar z_2\Psi)} \in K$$

At $r\to\infty$ $\quad (r^2 = |z_1|^2 + |z_2|^2)$, $\quad A \longrightarrow g^{-1}dg$ pure gauge

4 operators $\quad B_1, B_2 : K \longrightarrow K$

$I : N \longrightarrow K$

$J : K \longrightarrow N$

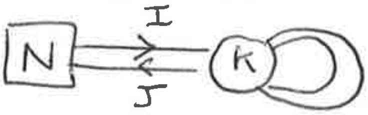
$$\boxed{\begin{aligned} &[B_1, B_2] + IJ = 0 \\ &\Sigma[B_i, B_i^\dagger] + II^\dagger - JJ^\dagger = 0 \end{aligned}} \ \text{mod } U(k)$$

ADHM eq.

$\updownarrow$ ∞-dim Fourier Transform

$$\boxed{F_A^{0,2} = 0 \quad \text{and} \quad (F_A^{1,1})_\omega = 0}$$

→ Quiver



Given (B, I, J), can construct A and Ψ.

$$\mathcal{D}^\dagger : K\otimes\mathbb{C}^2 \oplus N \longrightarrow K\otimes\mathbb{C}^2$$

$$\mathcal{D}^\dagger = \begin{pmatrix} B_1 - z_1 & B_2 - z_2 & I \\ -B_2^\dagger + \bar z_2 & B_1^\dagger - \bar z_1 & -J^\dagger \end{pmatrix}$$

$$\mathcal{D}^\dagger\mathcal{D} : K\otimes\mathbb{C}^2 \circlearrowleft \qquad \underset{\underset{\text{ADHM}}{\uparrow}}{\mathcal{D}^\dagger\mathcal{D} =} \ \overset{\overset{\Delta>0 \text{ for good } (I,J,B)}{\swarrow}}{\Delta} \otimes \mathbb{1}_{\mathbb{C}^2}$$

$E = \operatorname{Ker} \mathcal{D}^\dagger$

$A = \Xi^\dagger d \Xi \quad , \quad \mathcal{D}^\dagger \Xi = 0$

$\Xi^\dagger\Xi = \mathbb{1}$

$\downarrow$

$$\boxed{\text{Solves } F_A^+ = 0 \text{ and gives all solutions !}}$$

Application: 4d $\mathcal{N}=2$ SYM

Bosons $(A_m, \sigma, \bar{\sigma})$ Fermions $(\underset{\substack{\nearrow \\ \text{1-form}}}{\Psi_m}, \underset{\substack{\uparrow \\ \text{scalar}}}{\eta}, \underset{\substack{\nwarrow \\ \text{self-dual} \\ \text{2-form}}}{\chi^+_{mn}})$

$$\delta A_m = \Psi_m$$
$$\delta \Psi_m = D_m \sigma$$
$$\delta \sigma = 0$$
$$\delta \bar{\sigma} = \eta$$
$$\delta \eta = [\sigma, \bar{\sigma}]$$
$$\delta \chi^+_{mn} = H^+_{mn} \quad \text{(auxiliary bosonic)}$$
$$\delta H^+_{mn} = [\sigma, \chi^+_{mn}]$$

$$S_{SYM} = \tau \int \mathrm{Tr}\, F_A^2 + \delta \int \mathrm{Tr}(\dots)$$

δ-localization reduces to integral over instanton moduli space

$\mathcal{M}_k^+ = \{F_A^+ = 0\}/\mathcal{G}$. $\dim \mathcal{M}_k^+ = 4Nk$.

$\langle\sigma\rangle = \mathrm{diag}(a_1, \dots a_N)$ $SU(N) \longrightarrow U(1)^{N-1}$

$$S_{eff} = \int \tau_{ij}(a)\, F^{(i)} \wedge F^{(j)} + \text{fermions}.$$

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j}$$

Ω-deformation

Deform $\delta \longrightarrow \delta_\varepsilon$ where $(\varepsilon_1, \varepsilon_2)$ = params of $SO(4)$ in $\mathbb{R}^4$

$$\begin{cases} \delta_\varepsilon A = \Psi \\ \delta_\varepsilon \Psi = D_A \sigma + i_{V_\varepsilon} F_A \\ \delta_\varepsilon \sigma = i_{V_\varepsilon} \Psi \\ \vdots \end{cases} \qquad V_\varepsilon = i\varepsilon_1 (z_1 \partial_{z_1} - \bar{z}_1 \partial_{\bar{z}_1}) + (1 \leftrightarrow 2)$$

$\Longrightarrow$ Cohomology has been "reduced":

$$\delta_\varepsilon \mathcal{O} = 0 \Rightarrow \mathcal{O} = P(\sigma(\underset{\substack{\uparrow \\ \text{origin}}}{0}))$$

Now there are almost no massless fields $\longrightarrow$ we can integrate out everything.

Rotations in $\mathbb{R}^4$ act on :

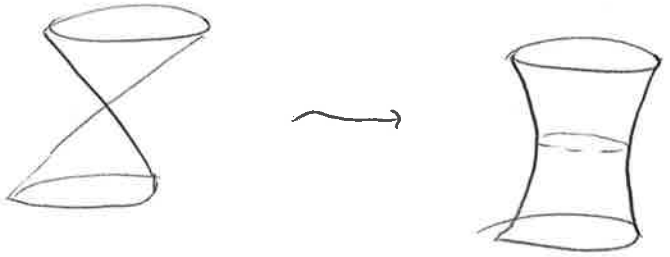
$$(B_1, B_2, I, J) \longrightarrow (e^{i\varepsilon_1}B_1 , e^{i\varepsilon_2} B_2 , I , e^{i(\varepsilon_1+\varepsilon_2)} J)$$

Defines V_ε vector field on $\mathcal{M}_k^+$. Let g denote the HK metric on $\mathcal{M}_k^+$ $\rightsquigarrow$ $g(V_\varepsilon, \cdot)$ = 1-form on $\mathcal{M}_k^+$.

$$Z_k = \int_{\mathcal{M}_k^+} \exp\left[-\delta_\varepsilon\left(g(V_\varepsilon, \cdot)\right)\right] = \sum_{\substack{\text{fixed pts} \\ p}} \frac{1}{\det A(\varepsilon)}$$

where $A(\varepsilon) = \partial V_\varepsilon(p)$.

Technical difficulty when $\mathcal{M}_k^+$ is singular and there is no good tangent space. To solve, introduce other deformation ζ :



When $\zeta > 0$, ADHM $\Leftrightarrow$ $\begin{cases} [B_1, B_2] + IJ = 0 \\ \mathbb{C}[B_1, B_2]\, I(N) = K \quad \text{mod } GL(K) \end{cases}$

"stability".

Fixed points are N-tuples of Young diagrams of total size $= k$.

ADHM equations describe moduli space of framed instantons.

$\mathcal{M}_{k,N}^+$ has $SU(N)$ sym

$$A \longmapsto h^{-1} A h \qquad h \in SU(N)$$

The deformation ζ breaks

$Spin(4) \supset U(2)$ $\searrow$ (B_1, B_2) is a doublet

$\uparrow$

rotations of $\mathbb{R}^4 = \mathbb{C}^2$

I invariant

$J \sim \det$ $\qquad J \to \det(t) J \qquad t \in U(2)$

$\swarrow$

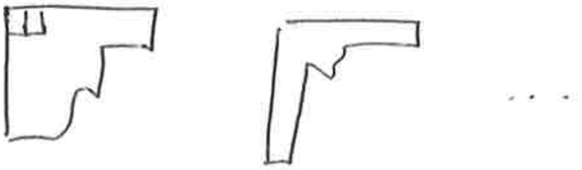
$$A \sim A^g = g^{-1} A g + g^{-1} dg$$

$$F_A^+ = 0$$

$$g(x) \to 1 \qquad x \to \infty$$

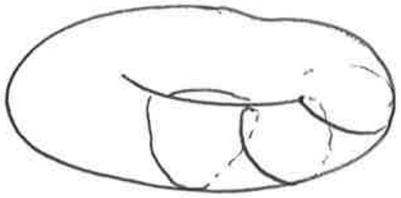
Look at maximal torus $T_{rot} \subset U(2)$ and $T_{gauge} \subset SU(N)$.

The $T_{rot} \times T_{gauge}$ fixed pts are N-tuples of partitions $\lambda^{(1)} \dots \lambda^{(N)}$ with $\sum |\lambda^{(\alpha)}| = k$.



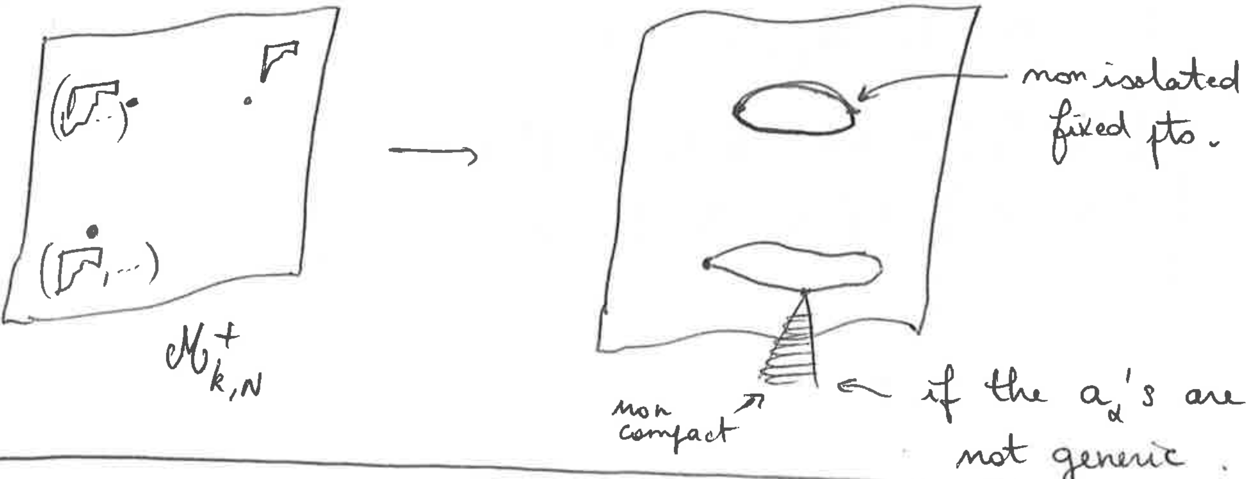
Each square is a vector in k-dim vector space K.

In some cases, if $t = \mathrm{diag}(e^{i\varepsilon_1}, e^{i\varepsilon_2})$ is non generic,



$\frac{\varepsilon_1}{\varepsilon_2} \in \mathbb{Q}$ then t^s $(s \in \mathbb{R})$ does not fill the torus and the sym group is actually of <u>dim 1</u>. (instead of 2)

$\Rightarrow$ in that case the space of fixed points will be strictly larger:



$$\boxed{\text{If } \underset{(\alpha\neq\beta)}{a_\alpha - a_\beta} \notin \underset{(\varepsilon_1\neq 0,\ \varepsilon_2\neq 0)}{\varepsilon_1 \mathbb{Z}_{>0} + \varepsilon_2 \mathbb{Z}_{>0}}, \text{ fixed pt set is compact.}}$$

<u>Compactness Theorem</u>

<u>Proof</u>: see notes.

So far we discussed pure $\mathcal{N}=2$ theory. Let's add matter.

↳ work with quivers (finite or affine of ADE type).

They all can be gotten from orbifolds of $\mathcal{N}=4$ SYM.

$\mathcal{N}=4$ SYM

$$A_\mu + 6 \text{ scalars}$$

$$\swarrow \qquad \searrow$$

$\sigma, \bar{\sigma}$ 4 remaining get twisted (Vafa) $\longrightarrow B^+, C$

self dual 2-form ↗ ↑ scalar

$$\begin{cases} \Omega^{2+} \otimes \underset{=}{g} \ni F_A^+ + [B^+, C] + [B, B]^+ = 0 \\ \Omega^1 \otimes g \ni D_A^* B^+ + D_A C = 0 \end{cases}$$

The eq. imply $B^+ = 0$, $C = 0$.

[cf Dijkgraaf and Moore, "balanced topological theory"]

The ADHM data is supplemented by 2 more matrices:

$$(B_1, B_2, B_3, B_4, I, J)$$

$$\left\{ \begin{array}{l} [B_1, B_2] + IJ + [B_3, B_4]^+ = 0 \\ [B_1, B_3] + [B_4, B_2]^+ = 0 \\ [B_1, B_4] + [B_2, B_3]^+ = 0 \\ \sum_{a=1}^{4} [B_a, B_a^\dagger] + II^\dagger - J^\dagger J = \zeta \mathbb{1}_k \\ B_3 I + (J B_4)^+ = 0 \\ B_4 I - (J B_3)^+ = 0 \end{array} \right\}$$

↳ can be obtained as $\left(\frac{\delta W}{\delta X}\right)^\dagger = \delta^{\text{gauge}}_{B_4} X$ with

$$W = \operatorname{Tr} B_3 ([B_1, B_2] + IJ)$$

Now instead of 2 params $\varepsilon_1, \varepsilon_2$, we have 3 params

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 = -(\varepsilon_1+\varepsilon_2+\varepsilon_3)$

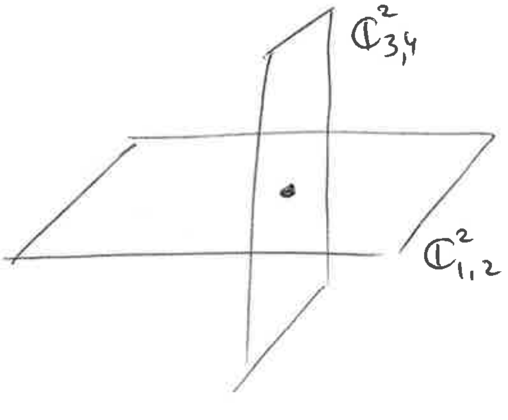
The symmetry is $U(1)^3 \subset SU(4)$.

$$Z_k = \sum_{\substack{N\text{-tuples of}\\ \text{partitions}}} \frac{\prod (\lambda + \varepsilon_3)}{\prod \substack{\text{eigenvalues of } \varphi \\ \text{on } T\mathcal{M}^+_{k,N}(\lambda^{(1)}\dots)}}$$

$$\begin{array}{ccc} SU(2) \times U(1) \times & SU(2) & \\ (B_1, B_2) & (B_3, B_4) & \\ \cup & \cup & \\ \Gamma_1 \text{ discrete} & \Gamma_2 \text{ discrete} & \rightsquigarrow \boxed{\Gamma_2 \text{ quiver on } \mathbb{R}^4/\Gamma_1} \end{array}$$

Symmetry $\Gamma_1 \leftrightarrow \Gamma_2$? Possible to symmetrize the ADHM eq.

It needs more data! $\begin{cases} \tilde{I} : \tilde{N} \to K \\ \tilde{J} : K \to \tilde{N} \end{cases}$



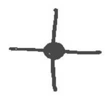
$$\text{eq} \Rightarrow \begin{cases} [B_1, B_2] + IJ = 0 & U(N) \text{ inst } \mathbb{C}^2_{12} \\ [B_3, B_4] + \tilde{I}\tilde{J} = 0 & U(\tilde{N}) \text{ inst } \mathbb{C}^2_{34} \end{cases}$$

and $k_{12} + k_{34} \geq k$

$K_{12} = \mathbb{C}[B_1, B_2]\, I(N)$

$K_{34} = \dots$

Stability condition $\to K_{12} + K_{34} = K$

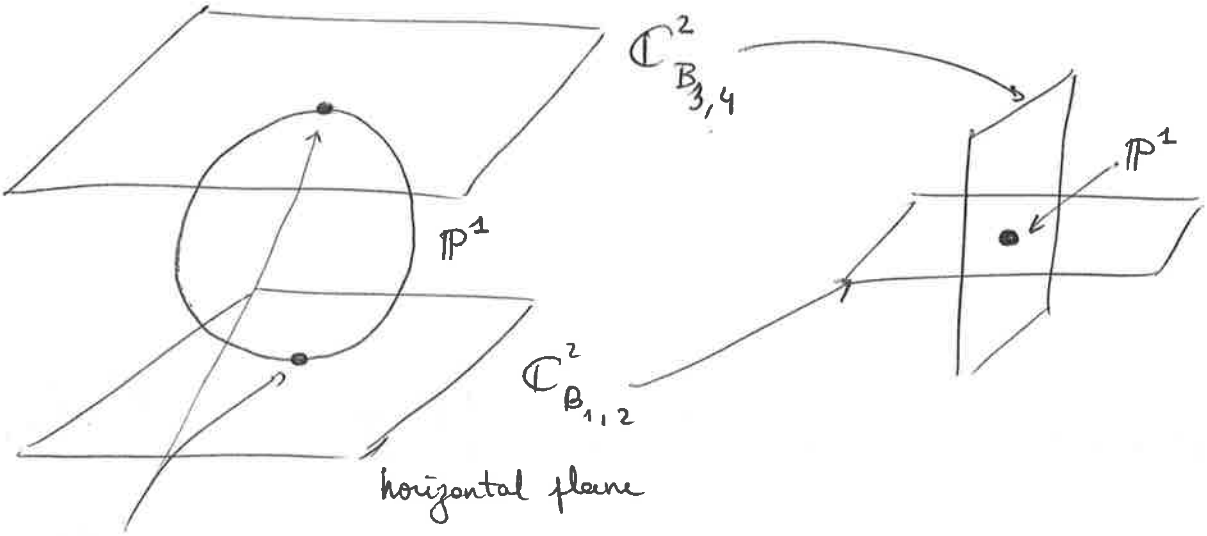
But they intersect, $\dim(K_{12} \cap K_{34}) = \#$ inst at 

Compactness thm $\Rightarrow$ regularity of cor. functions $\Rightarrow$ Ward identities of CFT.

Particular case $k = n = \tilde{n} = 1$ → all matrices are numbers

$$J = \tilde{J} = 0 \quad \text{and} \quad \left\{ \begin{array}{l} B_3 I = 0 \\ B_4 I = 0 \end{array} \right. \qquad \left\{ \begin{array}{l} B_4 \tilde{I} = 0 \\ B_2 \tilde{I} = 0 \end{array} \right.$$

$$\text{and} \quad |I|^2 + |\tilde{I}|^2 = \zeta .$$



Two fixed points $(\square, \emptyset)$, $(\emptyset, \square)$, one visible from horizontal plane, and one invisible.

KOMARGODSKI

Chiral Ring

①

Motivation & history

1998 : $\mathcal{N}=4$ SYM in 4d . SU(N) gauge group , ϕ^i $i=1\cdots 6$.

$$\mathcal{O}^I = C^I_{i_1\cdots i_k} \, \mathrm{Tr}(\phi^{i_1}\cdots\phi^{i_k}) \leftarrow \tfrac{1}{2} \text{ BPS operators}$$

$\uparrow$ symmetric traceless

Choose normalization $C^{I_1}_{i_1\cdots} C^{I_2 i_1\cdots} = 1\,\delta^{I_1 I_2}$.

and action $S = \frac{1}{2g^2_{YM}} \mathrm{Tr}\, F^2 + \ldots$

$$\langle \phi^i_a(x)\,\phi^j_b(y)\rangle = \frac{g^2_{YM}}{(2\pi)^2}\,\delta^{ij}\,\delta_{ab}\,\frac{1}{|x-y|^2}$$

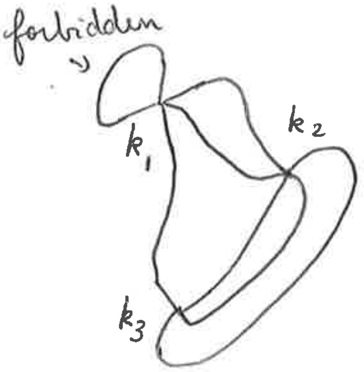
Using those formulas, we can compute at zero coupling :

$$g(x,y) = \langle \mathrm{Tr}(\phi^{i_1}\cdots\phi^{i_k})_{(x)}\mathrm{Tr}(\phi^{j_1}\cdots\phi^{j_k})_{(y)}\rangle$$

$$= \frac{N^k g^{2k}}{(2\pi)^{2k}}\left(\delta^{i_1 j_2}\cdots\delta^{i_k j_k} + \text{cyclic}\right)\frac{1}{|x-y|^{2k}}$$

Planar (large N) limit $\nearrow$

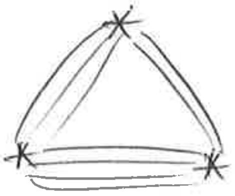
Then $\langle \mathcal{O}^I(x)\,\mathcal{O}^J(y)\rangle = \lambda^k \frac{k\,\delta^{IJ}}{(2\pi)^{2k}\,|x-y|^{2k}}$ $\quad(\lambda = g^2N)$



For 3-pt function to be $\neq 0$, we need $k_1 \leqslant k_2 + k_3$ + cyclic conditions

$$\langle \mathcal{O}^{I_1}(x)\,\mathcal{O}^{I_2}(y)\,\mathcal{O}^{I_3}(z)\rangle = \frac{\lambda^{\Sigma/2}}{N(2\pi)^{\Sigma}}\,\frac{k_1k_2k_3\langle C^{I_1}C^{I_2}C^{I_3}\rangle}{|x-y|^{2\alpha_3}|x-z|^{2\alpha_2}|y-z|^{2\alpha_1}}$$

where $\Sigma = k_1+k_2+k_3$, $\alpha_3 = \frac{k_1+k_2-k_3}{2}$, ...



Observations from 1998 - 2000

1) $\Delta(\mathcal{O}^I) = k$ is coupling invariant (because $\frac{1}{2}$ BPS)

2) From AdS, it seemed that $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$ is λ-independent. The coupling can be made at low λ and is valid at $\lambda = \infty$.

[this has been proved later, but very non-trivial proof]

3) For 3-pt fn, the x, y, z dependence is fixed by conformal invariance. For higher-pt fn, use extremality.

Label operators by $\Delta = k$. Extremal correlators have one saturated inequality : $\boxed{\Delta = \sum \Delta_i}$

$$\boxed{\langle \mathcal{O}^{\Delta_1}(x_1) \cdots \mathcal{O}^{\Delta_N}(x_N)\, \mathcal{O}^{\Delta}(y) \rangle = \mathcal{A}(\Delta, N)\, \lambda^{\#} \prod_{i=1}^{N} \frac{1}{|y-x_i|^{2\Delta_i}}}$$

The position dependence is fixed and the coupling dependence is trivial.

<u>Ref</u> : TASI lectures by d'Hoker et al.

This was believed to be a special property of $\mathcal{N}=4$. But we now know that some things survive in $\mathcal{N}=2$.

$\mathcal{N}=2$ overview

① We can define <u>extremal</u> correlators of <u>chiral ring</u> operators.
It remains true that position $\sim \prod \frac{1}{|y-x_i|^{2\Delta_i}}$.
However now $\mathcal{A}(\Delta, N, g_{YM})$ and the coupling dependence is exacty computable in any $\mathcal{N}=2$ theory.
Writing $\mathcal{A}(\Delta, N, \tau, \bar{\tau})$, it's <u>non</u>-holomorphic ! And still computable !

Start with 2-pt : $\langle \mathcal{O}_1^\Delta(x)\, \mathcal{O}_2^\Delta(y) \rangle \sim \frac{1}{|x-y|^{2\Delta}}$.

Then 3-pt $\langle \mathcal{O}_1^{\Delta_1} \mathcal{O}_2^{\Delta_2} \mathcal{O}_3^{\Delta_3} \rangle \sim \frac{1}{|x-y|^{\Delta_1+\Delta_2-\Delta_3} |y-z|^{\cdots} |x-z|^{\cdots}}$

Using conformal transfo, any operator can formally be put at ∞.
We <u>define</u> $\mathcal{O}^\Delta(\infty) = \lim_{y\to\infty} y^{2\Delta} \mathcal{O}^\Delta(y)$. (and this is reversible)

Following the prescription, $\langle \mathcal{O}_1^\Delta(0)\, \mathcal{O}_2^\Delta(\infty) \rangle = 1$.
and $\langle \mathcal{O}_1^{\Delta_1}(x)\, \mathcal{O}_2^{\Delta_2}(y)\, \mathcal{O}^{\Delta_3}(\infty) \rangle = \frac{1}{|x-y|^{\Delta_1+\Delta_2-\Delta_3}}$.

More generally, all correlators behave nicely. For extremal correlators, $\boxed{\langle \mathcal{O}_1^{\Delta_1}(x_1) \cdots \mathcal{O}_N^{\Delta_N}(x_N)\, \mathcal{O}^\Delta(\infty) \rangle = 1}$ <u>Independent</u> of distances !

$\Longrightarrow$ the operators at x_i can be brought together and we can OPE them. Only the regular pieces matter. Still it's not trivial, because depends on g_{YM}, etc.

4d $\mathcal{N}=2$ SCFT

$Q^i_\alpha \quad \bar{Q}^i_{\dot\alpha}$ SO(5,1) algebra + $SU(2)_R \times U(1)_R$

$S^i_\alpha \quad \bar{S}^i_{\dot\alpha}$ $i=1,2$ $\alpha=1,2$

Superconf primaries : $[S, \mathcal{O}] = [\bar{S}, \mathcal{O}] = 0$.

Their quantum numbers are : $(\Delta, \underbrace{j_\ell, j_r}_{SO(4)\text{ spin}}, \underset{\overset{\uparrow}{SU(2)_R}}{S}, \overset{\swarrow U(1)_R}{R})$

Huge work on <u>special classes</u> of such operators. Here, look at $\frac{1}{2}$ BPS operators.

↳ annihilated by 4 Q's.

- **Chiral operators**: $[\bar{Q}^i_\alpha, \mathcal{O}] = 0$ (= Coulomb-branch ops = Chiral ring ops)

SCA : $\{\bar{Q}, \bar{S}\} = \dots (\Delta + R) + \text{Lorentz} + SU(2)_R$

$\rightsquigarrow$ condition $\boxed{j_r = 0 \qquad \mathcal{S} = 0 \qquad \Delta = |R/2|}$

NB: It seems there is no restriction on j_ℓ, but on all Lagrangian examples, we also have $j_\ell = 0$.
Proving this is always true is an open question.

$$\begin{cases} \Delta = +R/2 & \text{for chiral ops} \quad [\bar{Q}, \mathcal{O}] = 0 \\ \Delta = -R/2 & \text{for anti-chiral ops} \quad [Q, \mathcal{O}] = 0 . \end{cases}$$

Product of operators: $\mathcal{O}_I(x)\,\mathcal{O}_J(y) = \underset{\substack{\uparrow \\ \text{depends on} \\ (x,y)}}{C^K_{IJ}}\ \underset{\substack{\uparrow \\ \text{not necessarily} \\ \text{chiral}}}{\mathcal{O}_K}(x)$

Unitarity bound $\Delta \geq |R/2|$ implies $C^K_{IJ}(x,y)$ has no singular term. The order 1 term (with no x,y dependence) corresponds to another chiral ring operator:

$$\boxed{\mathcal{O}_I \mathcal{O}_J = \sum_K \underset{\substack{\uparrow \\ \text{coupling constant} \\ \text{dependent}}}{C^K_{IJ}} \mathcal{O}_K}$$

(pert and non-pert. corrections) ← they can not be extracted from SW curve, they are non holomorphic !

Special case $R = 4$, $\Delta = 2$. Can add $\delta\mathcal{L} = \int d^4\theta\, \mathcal{O}$, which is a susy deformation, exactly marginal. Then $\langle \mathcal{O}\mathcal{O}^\dagger \rangle$ measures distances in theory spaces.

Conjecture $\forall$ Lagrangian SCFTs, ring is freely generated (and finitely gen ?)

- Higgs branch operators $[Q^1_\alpha, \mathcal{H}] = [\bar{Q}^1_{\dot\alpha}, \mathcal{H}] = 0$

Condition : $\boxed{j_\ell = j_r = R = 0 \quad \text{and} \quad \Delta = 2S}$

For Lagrangian theories, C_{IJ}^{K} for Higgs is tree-level exact.

Most interesting in non-Lagrangian theories.

[Reason is that (g_{YM}, θ) deformations are Coulomb branch operators, and they don't talk to Higgs branch]

⚠ Don't confuse space of theories and space of vacua.

For $\mathcal{N}=4$ SYM, huge space of vacua $\mathbb{R}^6/\Gamma$.

$\langle O_{CB} \rangle \rightsquigarrow$ U(1) order parameter

$\langle O_{HB} \rangle \rightsquigarrow$ SU(2) order parameter.

Rq In QM in potential  there is 1 vacuum $\Psi_1 + \Psi_2$. The other $\Psi_1 - \Psi_2$ has energy level in $\exp(-S_{inst})$.

In QFT$_{(d>2)}$ there is a volume factor (infinite) → 2 vacua

Then there are superselection rules.

Extremal correlators : defined analogously:

$$\langle O_1(x_1) \cdots O_n(x_n) \underset{\substack{\uparrow \\ \text{anti-chiral}}}{\mathcal{O}^\dagger(y)} \rangle \quad \text{with} \quad \Delta_{\mathcal{O}} = \sum_{i=1}^{n} \Delta_i$$

We want to show that

$$\boxed{\langle \cdots \rangle = \mathcal{A}(\{g_{YM}, \theta\}) \prod_{i=1}^{n} |x_i - y|^{-2\Delta_i}}$$

Proof $\bar{Q}\mathcal{O} = 0$ so $Q\mathcal{O}^\dagger = 0$

and $\{Q,\bar{Q}\} \sim \partial$. We use the trick to send $y \to \infty$:

$$\mathcal{O}^\dagger(\infty) = \lim_{y\to\infty} y^{2\Delta_\mathcal{O}} \mathcal{O}^\dagger(y).$$

We then want to prove that $\langle \mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)\mathcal{O}^\dagger(\infty)\rangle$ is independent of coordinates:

$$\frac{\partial}{\partial x_1}\langle\cdots\rangle \sim \{Q,Q^\dagger\}\langle\cdots\rangle$$

$$= \langle \bar{Q}\,[Q,\mathcal{O}_1]\,\mathcal{O}_2\cdots\mathcal{O}_n\,\mathcal{O}^\dagger(\infty)\rangle$$

$$= \langle [Q,\mathcal{O}_1]\,\mathcal{O}_2\cdots\mathcal{O}_n\,\underbrace{[\bar{Q},\mathcal{O}^\dagger(\infty)]}_{:=\,\Psi_{\mathcal{O}^\dagger}(\infty)}\rangle$$

$= 0$ because $\Delta(\Psi_{\mathcal{O}^\dagger}) = \Delta_\mathcal{O} + \frac{1}{2}$

and the power of y in lim was not high enough. ■

Look at simple case: 2-pt functions: $\langle \mathcal{O}_i(x)\mathcal{O}_j^\dagger(y)\rangle$ with $\Delta_i = \Delta_j = 2$. In that case we can deform the Lagrangian with marginal operators $\delta\mathcal{L} = \int d^4\theta\, \lambda^i \mathcal{O}_i + cc$.

$$\langle \mathcal{O}_i(x)\,\mathcal{O}_j(y)\rangle = \underbrace{G_{ij}(g_{YM},\theta)}_{\boxed{\text{Zamolodchikov metric in theory space.}}}\,|x-y|^{-4}$$

← Goal: compute it!

Forget about susy for a moment. Suppose we have a CFT_d and suppose it has operators $\mathcal{O}_i$ with $\boxed{\Delta_i = d}$ (Not $d/2$!)

It's a tempting idea to deform the action:

$$\delta S = \sum_i \lambda^i \int \mathcal{O}_i(x)\, d^d x.$$

We assume the $\mathcal{O}_i$ are Hermitian. Question: is it still a CFT?

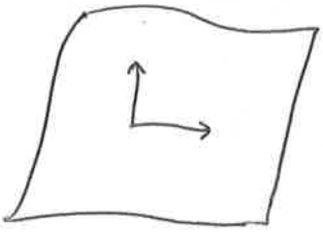
Ex: Take free field theory in 4d, $\int (\partial\phi)^2 d^4x$.
We can add $\lambda \int \phi^4 d^4x$.
But the operator is marginally irrelevant, $\beta_\lambda \neq 0$ and $\lambda \longrightarrow 0$, flows to the same CFT.

Ex: $\mathcal{N}=4$ SYM
$\mathcal{N}=2$ SU(N) + 2N hypers
$\mathcal{N}=1$ β-deformed
$\mathcal{N}=0$ Ashkin-Teller model

These are all exactly marginally deformable CFTs.

Call $\beta^i = \beta_{\lambda^i}$. Then $\beta^l = 0\,\lambda^l + C^l_{ij}\lambda^i\lambda^j + C^l_{ijk}\lambda^i\lambda^j\lambda^k + \dots$
For a (large) $\mathcal{M}_{conf}$ to exist, all the $C^l_{ij\dots}$ have to vanish.

Suppose $\mathcal{M}_{conf}$ does exist, with coords λ^i are coupling constants.



Zamolodchikov metric $\langle \mathcal{O}_i(0)\mathcal{O}_j(\infty)\rangle_\lambda = G_{ij}(\lambda)$

Locally, we can put $G_{ij} = \delta_{ij}$ and $\partial_i G_{jk} = 0$
[for that, use $C^l_{ij} = 0 \dots$]

The physically relevant quantity is the curvature R_{ijkl}.
This can be computed from correlators! For instance, in 2d

$$R_{ijkl} = \int d\eta \, \langle \mathcal{O}_i(0)\,\mathcal{O}_j(\eta)\,\mathcal{O}_k(1)\,\mathcal{O}_l(\infty)\rangle \log|\eta|^2 .$$

Anomaly on $\mathcal{M}_{conf}$

Recall $\langle \mathcal{O}_i(x)\, \mathcal{O}_j(y) \rangle = G_{ij}(\lambda) \dfrac{1}{(x-y)^{2d}}$ and go to momentum space : $\langle \tilde{\mathcal{O}}_i(p)\, \mathcal{O}_j(-p) \rangle = G_{ij}(\lambda)\, p^d \log(p^2/\Lambda^2)$ for d even

using
$$\int e^{ipx} \frac{d^dx}{x^{2d}} = \begin{cases} p^d \log p^2 & d \text{ even} \\ p^d & d \text{ odd} \end{cases}$$

[NB p^d is a polynomial when d even so the FT can not be $\frac{1}{x^{2d}}$. For d odd p^d is not polynomial !]

We have a scale Λ ! This seems to contradict CFT...
The point is that when $p \to \mu p$, difference is $p^d \log(\mu^2/\Lambda^2)$ and is only supported at coincident points. Ward identities only apply at separated points, so CFT not violated.
Actually, $\log^2(p^2/\Lambda^2)$ would be forbidden !

The conformal anomaly is a way to handle $\lambda^i \to \lambda^i(x)$.

$$Z[\lambda(x)] = \int \mathcal{D}X \exp\left(S_{CFT} + \int \lambda^i(x)\, \mathcal{O}_i(x)\, dx \right)$$

$$\delta_{\sigma(x)} \log Z[\lambda(x), g_{\mu\nu}(x)] = \int d^dx\, \mathcal{L}(\lambda^i(x), g_{\mu\nu}(x))$$

Result in 4d : $\delta_{\sigma(x)}(..) = \dfrac{1}{192\pi^2} \displaystyle\int d^4x \sqrt{g}\, \delta\sigma \left[G_{ij}\, \hat{\Box} \lambda^i \hat{\Box} \lambda^j + \ldots \right]$

(Paurietz - Fradkin - Tseytlin)

NB : independent of susy.

$\mathcal{N}=2$ susy implies various restrictions on the anomaly:

$$\boxed{\delta_\sigma \log Z \supset \frac{1}{2}\int d^4x \, K(\lambda^i, \bar{\lambda}^i)\, \Box^2(\delta\sigma)}$$

$\mathcal{N}=2 \Rightarrow \lambda^i, \bar{\lambda}^i$ come in pairs $\Rightarrow \mathcal{M}_{conf}$ is even-dim.

$\Downarrow$ $\mathcal{M}_{conf}$ is Kähler, $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(\lambda, \bar{\lambda})$

NB. Geometry of $\mathcal{M}_{conf}$.

$\mathcal{N}=0 \Longrightarrow \mathcal{M}_{conf}$ is Riemannian

	d=4		d=3
$\mathcal{N}=4$	Constant curvature Kähler space $\mathbb{H}/SL(2,\mathbb{Z})$	$\mathcal{N}\geq 4$	No $\mathcal{M}_{conf}$
$\mathcal{N}=2$	Kähler	$\mathcal{N}=2$	Kähler
$\mathcal{N}=1$	Kähler	$\mathcal{N}=1$	Riemannian

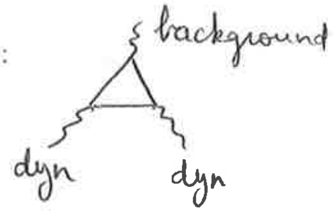
Conjectures: volume always finite?

non-compactness always fine throat?

NB: 2 types of anomalies:

- ABJ. This is an explicit violation of the symmetry.

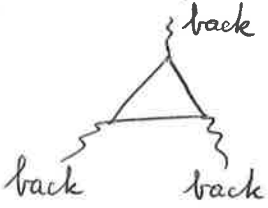
Ex:



(as bad as adding a mass)

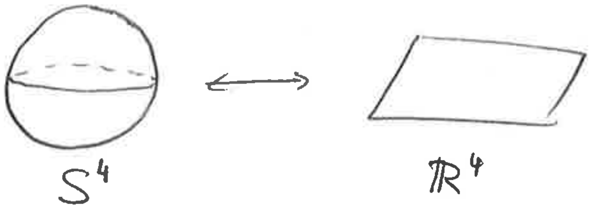
Do not need to match!

- 't Hooft These need to match!



Conformal anomaly is in this class.

Well-known stereographic map



[They are connected by a finite $\delta\sigma$.]

Integrating the anomaly polynomial, $\boxed{Z_{S^4} = e^{\frac{1}{12}K(\lambda^i;\bar{\lambda}^i)}}$

When Pestun computed Z_{S^4}, he did not give the physical meaning ... this is the meaning!

<u>Conclusion</u>: in $\mathcal{N}=2$ theories

$$\boxed{\begin{aligned} G_{ij} &= \langle \mathcal{O}_i(0)\,\mathcal{O}_j^{\dagger}(\infty)\rangle_{\mathbb{R}^4} = 16\,\frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\bar{\lambda}^i}\log Z_{S^4} \\ &= \frac{1}{(Z_{S^4})^2}\det\begin{pmatrix} Z_{S^4} & \frac{\partial}{\partial\lambda^i}Z_{S^4} \\ \frac{\partial}{\partial\bar{\lambda}^i}Z_{S^4} & \frac{\partial}{\partial\lambda^i}\frac{\partial}{\partial\bar{\lambda}^i}Z_{S^4}\end{pmatrix}\end{aligned}}$$

<u>NB</u> K can be redefined $K \longrightarrow K + F(\lambda) + \bar{F}(\bar{\lambda})$ so there is also an abiguity in Pestun's computation of Z_{S^4}. He made an arbitrary choice, which corresponds to a choice of Kähler frame.

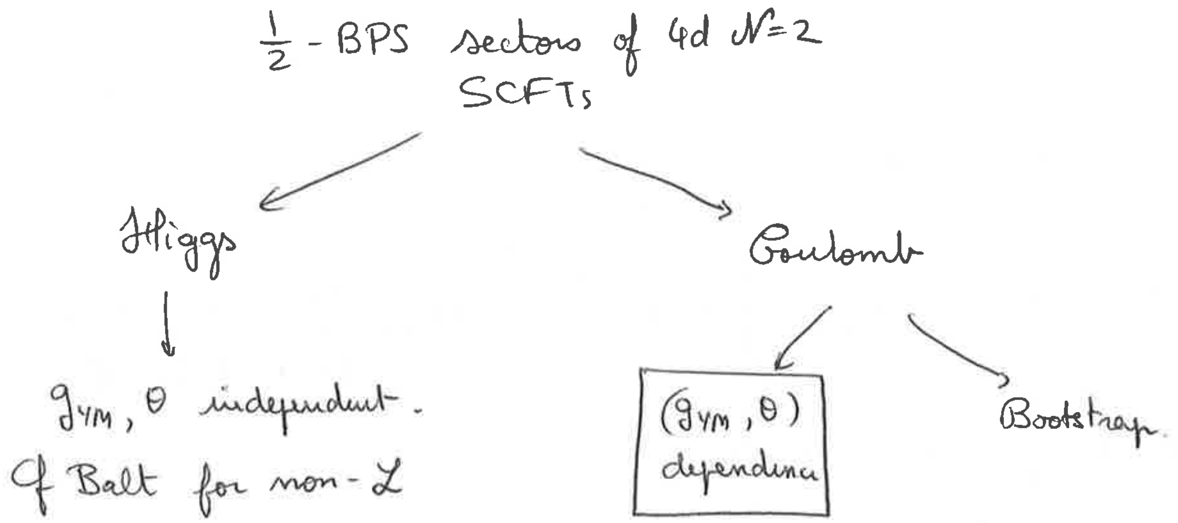
<u>NB</u> $Z_{S^4} \sim e^{K}$ is obtained by integrating the anomaly polynomial. This can be done to relate any conformally equivalent manifolds.

<u>NB</u>. In 2d, $Z_{S^2} = e^{-K}$. In 3d, $Z_{S^3} = e^{-F}$ [see F-theorem]. For other manifolds, not known $Z_{S^3\times S^1} \underset{??}{=} e^{L} \leftarrow$ Neitzke Line bundle?

See also Bachas et al, Calabi Diastasis ...

In 3d, the F coefficient is a number, independent from the point in the conformal manifold
↪ open question to compute Zam. metric in 3d.

Review



SU(2) SCFTs

- $\mathcal{N}=4$ theory, (g_{YM},θ) marginal params.
- 4 hypers, (g_{YM},θ) also.

Chiral ring : $\Phi_2 = -4\pi i \,\mathrm{tr}\,\varphi^2$
↖ scalar in vector mult.

Chiral ring given by $\mathcal{O}_n = (\Phi_2)^n$.

$$\Delta(\mathcal{O}_n) = 2n$$

$\mathcal{O}_1$ is exactly marginal, coupling $\tau \int \mathcal{O}_1 \, d^4\theta \, d^4x$.

Algebra is $\boxed{\mathcal{O}_n \mathcal{O}_m = \mathcal{O}_{n+m}}$ with no coupling cst dep!

The dependence is in the 2-pt function

$\langle \mathcal{O}_n(0) \mathcal{O}_n^\dagger(\infty) \rangle = G_{2n}(\tau,\bar{\tau})$. Then all correlators extremal are $\langle \mathcal{O}_{n_1}(x_1) \cdots \mathcal{O}_{m_k}(x_k)\, \mathcal{O}^\dagger_{m_1+\dots+m_k}(y) \rangle = G_{2(m_1+\dots+m_k)}(\tau,\bar\tau) \dfrac{1}{\prod (y-x_i)^{4n_i}}$

Goal: <u>compute G_{2n}</u>

We know $G_0 = 1$, $\quad G_2(\tau,\bar\tau) = 16 \dfrac{1}{Z^2_{S^4}} \det \begin{pmatrix} Z & \partial Z \\ \bar\partial Z & \partial\bar\partial Z \end{pmatrix}$

$$= 16\, \partial\bar\partial \log Z .$$

For $\mathcal{N}=4$ SYM, $\quad Z_{S^4} = \int da\, e^{-4\pi (\operatorname{Im}\tau) a^2} (2a)^2$

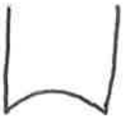
For SQCD, $\quad Z_{S^4} = \int da\, e^{-4\pi (\operatorname{Im}\tau) a^2} (2a)^2 \dfrac{H(2ia)H(-2ia)}{|H(ia)H(-ia)|^4} |Z_{inst}|^2$

For $\mathcal{N}=4$, $\quad G_2 = 6 \dfrac{g^2_{YM}}{4\pi}$

For SQCD, $\quad G_2 = 6 \dfrac{g^2_{YM}}{4\pi} - \dfrac{135\,\zeta(3)}{2\pi^2} \dfrac{1}{(\operatorname{Im}\tau)^4} + \cdots \quad \Big\}$ perturbative

$\quad + \cos\theta\, e^{-8\pi^2/g^2} \left(\dfrac{6}{(\operatorname{Im}\tau)^2} + \dfrac{3}{\pi (\operatorname{Im}\tau)^3} + \cdots \right) \Big\}$ instantons

These G_2 are two different metrics on the fundamental domain



: $\boxed{ds^2_{\mathcal{N}=4} = 6 \dfrac{d\tau\, d\bar\tau}{(\operatorname{Im}\tau)^2}}$ (Poincaré Disk)

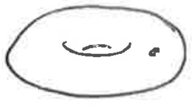
↳ tree level is exact!

In both cases, <u>finite volume</u>, and weak coupling limit is at log (∞) distance.

<u>Using AGT</u>: ds^2_{SQCD} is metric on Teichmuller space of sphere with 4 punctures

<u>Not standard WP metric</u>

$ds^2_{\mathcal{N}=4}$ is standard Weyl-Peterson metric on



OPen Quest : Is G_2^{SQCD} invariant under modular transform ?

Connection to resurgence and QCD

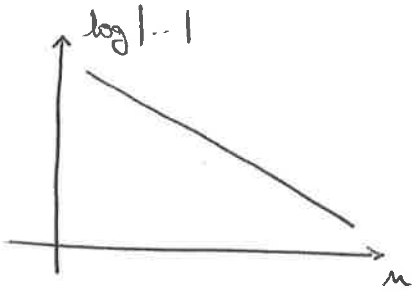
$$\text{Quantity} = (a_1\lambda + a_2\lambda^2 + \dots) + e^{-1/\lambda}(\lambda + \lambda^2 + \dots) + \dots$$

Dyson conjecture : $\left|\frac{a_{n+1}}{a_n}\right| \sim n$ This is true for G_2

Other conj . Assume n loops have been computed, match it with $\frac{\sum^{n/2} c_i \lambda^i}{\sum^{n/2} d_i \lambda^i}$ (this is the $(n/2, n/2)$ Padé approximation). This makes a prediction for a_{n+1}.

Conjecture is $\left|\frac{a_{n+1}^{\text{Padé}}}{a_{n+1}} - 1\right| \leq C e^{-\sigma n}$ for large n, for any QFT.

Compute it here :



similar to QCD !!

find $\sigma \sim 0{,}7$.

Q: $\sigma = \log 2$??

Borel summability? Yes !

Q: what do the poles on negative axis mean?

Some refs $Z = e^{-K}$ 1405.7271 from susy
1509.08511 from trace anomaly

extremal corr. 1602.05971

global properties 1803.04366
1805.04202

Heavy operators

General extremal correlators

$$M_{nm} = \frac{1}{Z}\partial^n\bar{\partial}^m Z$$ Infinite matrix

$$\boxed{G_{2n} = (16)^n \frac{D_n}{D_{n-1}}}$$

Exercise: Show G_{2n} is invariant under $Z \to e^{F+\bar{F}} Z$.

Recursion relation between the G_{2n} for any $\mathcal{N}=2$ SCFT.

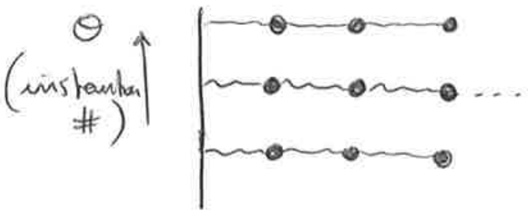
- tt^* geometry
- integrable system

Show that $$\boxed{\partial\bar{\partial}\log D_n = \frac{D_{n+1}D_{n-1}}{D_n^2} - (n+1)D_n}$$

Change variables: $D_n = 16^n \exp(q_n - \log Z)$

$$\boxed{\partial\bar{\partial} q_n = e^{q_{n+1}-q_n} - e^{-q_{n-1}+q_n}}$$ Toda equation (Half infinite chain)

$$(\sim \ddot{q}_n = V(q_n, q_{n+1}, q_{n-1}))$$



Exactly solvable given the boundary conditions.

Ratios of determinants "⟷" integrable systems.

$\mathcal{N}=4$ and SQCD differ only by the boundary condition.

Open Question: large charge limit of Toda chain?
Effective field theory?

SU(N) + 2N hypers

Chiral ring $\mathrm{tr}\,\varphi^2, \ldots, \mathrm{tr}\,\varphi^N$, one (g_{YM}, θ).

Exactly marginal $\mathrm{tr}\,\varphi^2$ has $\Delta = 2$.

There is still a $\frac{\det}{\det}$ formula ⟶ what is the associated integrable system? $\boxed{OQ}$.

In general, $\delta \mathcal{L}_{\mathcal{N}=2} = \lambda \int d^4\sigma\, \mathcal{O}_n$ is irrelevant deformation $(\Delta > 2)$

To compute det, need $\frac{\partial Z}{\partial \lambda}$ as well, not just $\frac{\partial Z}{\partial \tau}$.

For that, need $\int da_1 \cdots da_{..}\, e^{\lambda a^n}\, \mathrm{Perturb}(\ldots)\, |\underset{\substack{\uparrow \\ \text{not known}}}{Z_{inst}(\lambda)}|^2$

$\frac{\det}{\det}$ is known at all orders in g_{YM}^2 but not the instanton contribution.

Open Quest: Higher Casimirs in Ω-background?

Even in SU(3) case, details not known.

Higher Casimirs ⟷ W_N - symmetry

Infinite chiral symmetry in 4d

1312.5344

Context

4d $\mathcal{N}=2$ SCFTs and stay at superconformal point. We don't refer to a Lagrangian, and take an algebraic approach.

Operators $\mathcal{O}_I(x)$ transform in

$$su(2,2|2) \supset \underbrace{so(4,2)}_{\text{conformal algebra}} \oplus su(2)_R \oplus u(1)_r$$

Some representations of the conformal algebra

In radial quantization, Hilbert space of states on sphere S^{d-1}.

$\exists$ State-operator correspondence: $|\mathcal{O}_I\rangle = \mathcal{O}_I(0)|\Omega\rangle$ $\uparrow$ vacuum.

$\hat{\mathcal{O}}_I(x)$ is primary if

$$D\mathcal{O}_I(0)|\Omega\rangle = \Delta\, \mathcal{O}_I(0)|\Omega\rangle$$

$$K^{\alpha\dot\alpha}\mathcal{O}_I(0)|\Omega\rangle = 0$$

$$\begin{tikzcd} \mathcal{O}_I(0)|\Omega\rangle \arrow[d, "P_\mu"] & \text{Primary} \\ \partial_\mu \mathcal{O}_I(0)|\Omega\rangle \arrow[d, "P_\mu"] & \\ \partial_\mu\partial_\nu \mathcal{O}_I(0)|\Omega\rangle \arrow[d] & \text{descendants} \\ \vdots & \end{tikzcd}$$

[Note that $\hat{\mathcal{O}}_I(x) = e^{x\cdot \hat{P}}\,\hat{\mathcal{O}}_I(0)\, e^{-x\cdot P}$

so $[K^\mu, \hat{\mathcal{O}}_I(x)] \neq 0$

unless $x=0$ and $\hat{\mathcal{O}}_I$ prim.

So the $x=0$ is important]

Example Free massless boson $\phi(x)$ satisfying $\Box\phi(x)=0$

Primaries: $\phi(x)$, $:\phi^n:(x)$, $T_{\mu\nu}$, ...

Descendants: $\partial_\mu\phi(x)$, $\phi\partial_\mu\phi = \frac{1}{2}\partial_\mu :\phi^2:$

Sometimes a descendant is null and decouple, the representation is short. Examples: $\Box\phi=0$, $\partial^\mu T_{\mu\nu}=0$

Representations of su(2,2|2)

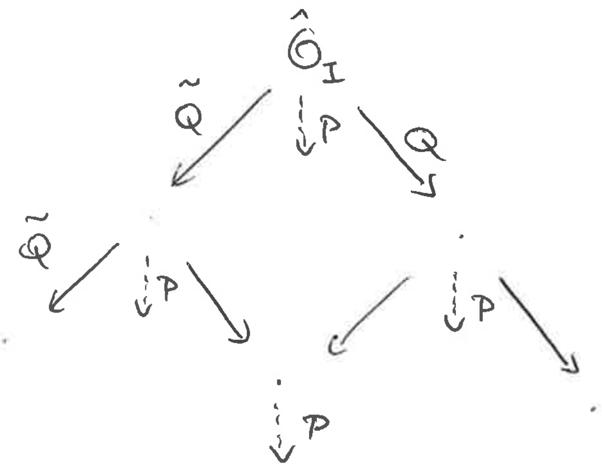
$\hat{\mathcal{O}}_I(x)$ superconformal primary if $[D,\hat{\mathcal{O}}_I(0)] = \Delta\,\hat{\mathcal{O}}_I(0)$

↑ multi-index (Δ, j_1, j_2, R, r)

and $[K^{\dot\alpha\alpha}, \mathcal{O}_I(0)] = 0$

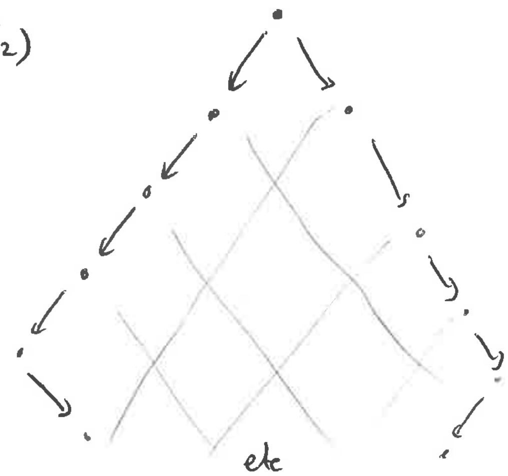
$[S, \mathcal{O}_I(0)\} = 0$

$[\tilde{S}, \mathcal{O}_I(0)\} = 0$

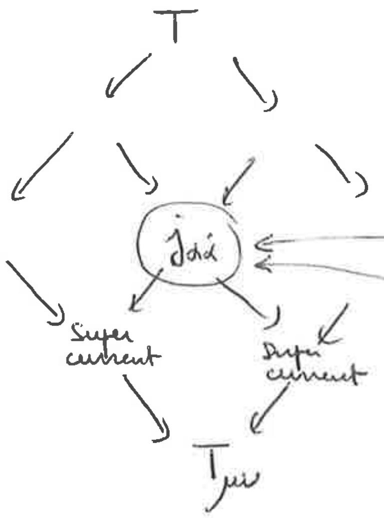


Some representations

① $\mathcal{A}^{\Delta}_{R,r,(j_1,j_2)}$



② $\hat{C}_{0(0,0)}$

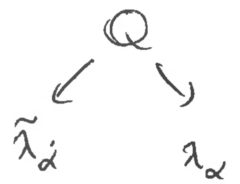


$SU(2)_R \times U(1)_r$ symm. currents

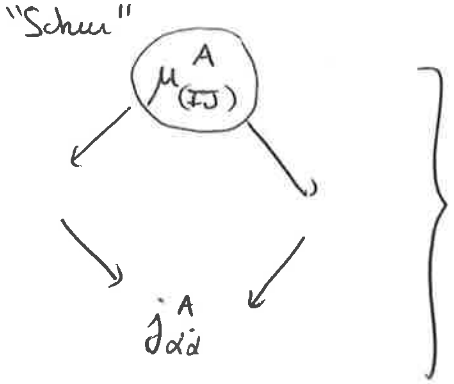
It's "Schur"

③ $\hat{B}_R$ $\Delta = 2R$, $j_1 = j_2 = r = 0$

Ex • $\hat{B}_{1/2}$ is hypermultiplet



• $\hat{B}_1$



in adjoint of flavor symmetry

• If N_f hypermultiplets in rep $\mathcal{R}$ of algebra $\underline{g}$, then

$$\mu^A_{(IJ)} = \underbrace{\tilde{Q}^{\bar a}_{I\bar m} \, Q^b_{Jn}}_{\text{gauge}} \, \delta^{\bar m n} \, T^A_{\underbrace{b\bar a}_{\text{flavor}}} \longleftarrow \text{adjoint flavor}$$

generates $\hat{B}_1$ representation.

Chiral algebras

Consider n-point function

- Of Schur operators : $\Delta = j_1 + j_2 + 2R$, which are:
  - ⋆ Always in short reps
  - ⋆ Always highest weight of $SU(2)_R \times SU(2)_{\text{Lor}\,L} \times SU(2)_{\text{Lor}\,R}$

* Always $R > 0$

* Written as $\mathcal{O}^{1\cdots\cdots 1}_{+\cdots+\dot{+}\cdots\dot{+}}$ ← one or more indices ← zero or more indices

and full R-sym multiplet is $\mathcal{O}^{I_1\cdots I_{2R}}_{+\cdots+\dot{+}\cdots+}(x)$

- Restricted to two-plane $x_1 = x_2 = 0$. We set $z = x_3 + i x_4$.
- With R-sym indices contracted with position dependent vector $u_I(\bar{z}) = \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}$

Then: $\bar{u}_I(\bar{z}_1)\, u_J(\bar{z}_2)\cdots \langle \mathcal{O}^I_{+\dot{+}}(z_1,\bar{z}_1)\, \mathcal{O}^J_{+\dot{+}}(z_2,\bar{z}_2)\cdots \mathcal{O}^{(LM)}(z_n,\bar{z}_n)\rangle\, u_L(\bar{z}_n)\, u_M(\bar{z}_n)$

The correlator is meromorphic! → can gain information by looking at poles, ...

Example Free hypers $\hat{B}_{1/2}$

$$\langle Q_I(z,\bar{z})\, \tilde{Q}_J(w,\bar{w})\rangle = \frac{\varepsilon_{IJ}}{|z-w|^2}$$

$$u^I(\bar{z})\, u^J(\bar{w}) \langle \cdots \rangle = \frac{\bar{w}-\bar{z}}{|z-w|^2} = \frac{-1}{z-w} \quad \text{meromorphic.}$$

If $q(z) = \left[u_I(\bar{z})\, Q^I(z,\bar{z})\right]_\chi$ ← in twisted correlators of Schur operators

Then we have a chiral algebra,

generated by $q(z)$, $\tilde{q}(z)$, ie all operators are normal ordered products like $:q\partial\tilde{q}:(z) = \lim_{w\to z} q(w)\,\partial\tilde{q}(z)$ − singular OPE terms

with singular OPEs:

$$\begin{cases} q(z)\,\tilde{q}(w) \sim \dfrac{-1}{z-w} \\ q(z)\,q(w) \sim 0 \\ \tilde{q}(z)\,\tilde{q}(w) \sim 0 \end{cases}$$

This is symplectic bosons with $h_q = h_{\tilde{q}} = 1/2$

Chiral algebras — "proper" approach

Let $\mathbb{Q} = Q^1_- + \tilde{S}^{2\dot{-}}$. Then $\mathbb{Q}^2 = 0$, $\{\mathbb{Q}, \mathbb{Q}^\dagger\} = D - M_+{}^+ - M^{\dot{+}}{}_{\dot{+}} - 2R$

This shows $\Delta - j_1 - j_2 - 2R \geqslant 0$ for all states, and

$\mathcal{O}(0)$ is Schur $\iff$ $\mathbb{Q}\,\mathcal{O}(0)|\Omega\rangle = 0$

We have $[\mathbb{Q}, P_1] \neq 0$, $[\mathbb{Q}, P_2] \neq 0$

but $[\mathbb{Q}, P_{\bar{z}}] = 0$.

$$\Downarrow$$

$$\mathbb{Q}\, e^{\bar{z} P_{\bar{z}}}\, \mathcal{O}(0)|\Omega\rangle = 0$$

and $\{\mathbb{Q}, \dots\} = P_{\bar{z}} + R^-$

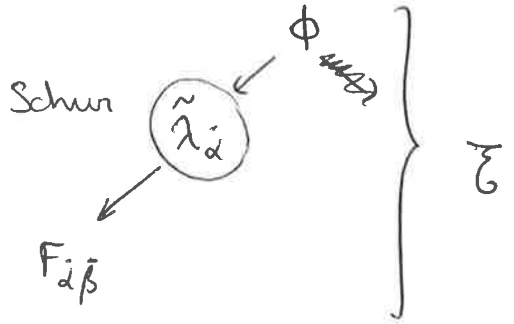
$\Rightarrow$ chirality of "twisted - translated" Schur operators.

$sl(2)_z$ is $\mathbb{Q}$ - closed and $\widehat{sl}(2)_{\bar{z}}$ is $\mathbb{Q}$ exact.

To any 4d $\mathcal{N}=2$, can associate chiral algebra. The OPE algebra is single - valued.

Chiral algebras for gauge theories

Free vector multiplet is in $\mathcal{E}_{1(0,0)}$, $\bar{\mathcal{E}}_{1,(0,0)}$



The gaugino is Schur

$j_2 = \frac{1}{2}$ $\quad$ $j_1 = 0$

$R = \frac{1}{2}$ $\quad$ $\Delta = 3/2$

$h = 1$

So if $b(z) = [u_I(\bar{z})\,\tilde{\lambda}_{\dot{+}}(z,\bar{z})]_{Q}$

$\partial c(z) = [u_I(\bar{z})\,\lambda(\quad)]_{Q}$

Then chiral algebra is $\boxed{b(z)c(w) \sim \frac{1}{z-w}}$.

It is generated by b, c without zero-mode for c.

Now we want to gauge:

T_{gauge} = vectors in adj of $\underline{g}$ coupled to T with global symmetry $\underline{g}$; and $\underset{\substack{\nearrow \\ \text{level of} \ldots}}{K_{4d}} = 4h^\vee_{\underline{g}}$

Can we combine the two chiral algebras? Find a prescription for $\mathcal{X}[T_{gauge}]$ from $\mathcal{X}[T] \supset \underline{g}$ affine-Kac-Moody at level $K_{2d} = -2h^\vee_g$

and $\mathcal{X}[\text{free.vect}]$

Define $Q_{BRST} = \oint \frac{dz}{2\pi i}\, j_{BRST}(z)$

$$j_{BRST} = c_A\left[j^A - \frac{i}{2} f^{ABC} c_B b_C\right].$$

Claim: $\boxed{\mathcal{X}[T_{gauge}] = H^*_{BRST}\left[\Psi \in \mathcal{X}[T] \otimes \mathcal{X}[\text{free vect}]\right]\Big|_{\text{zero-mode}}}$

NB: $[Q_{BRST}, \{Q_{BRST}, \cdot\}\} = 0$ only if $K_{4d} = 4h^\vee_g$.

Conjectures of chiral algebras

- $N_f = 2N_c$ SQCD.

  $\longrightarrow$ $SU(N_f)_{-N} \oplus U(1)$ AKM + baryons

- $MN(E_6) \longrightarrow (E_6)_{-3}$ AKM
- $\mathcal{N}=4$ SYM. The $\chi[\mathcal{N}=4]$ has small $N=4$ susy
- $\mathcal{N}=3$; $\chi[\mathcal{N}=3]$ has $N=2$ susy

It would be interesting to solve the cohomology problem, and prove the above conjectures.

Checking "S-duality" would be nice.