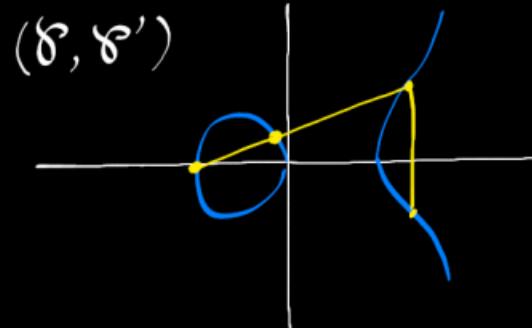


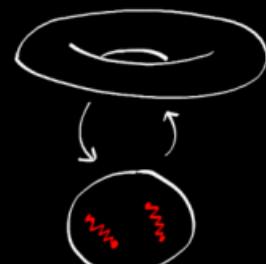
$$\int_0^a \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

## La Géométrie Révélée

Promenade en compagnie des fonctions elliptiques et des surfaces de Riemann



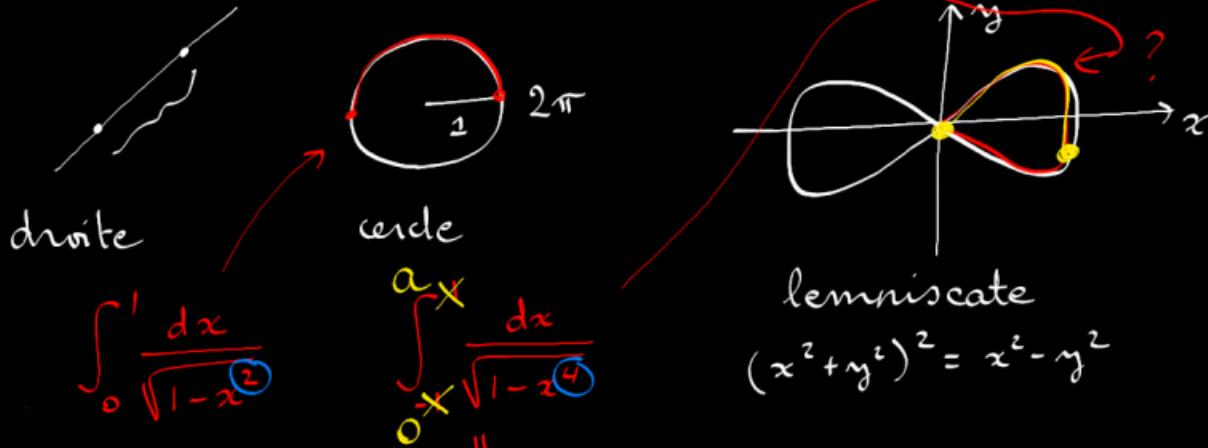
$$G = 0,8346268\dots$$



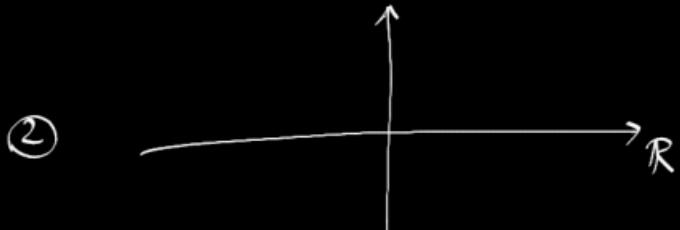
$$\frac{n}{2} = d + g - 1$$

## Introduction .

①



2,62205755429...



$$\delta(x) = \begin{cases} 0 & \text{si } x \in \mathbb{Q} \\ 1 & \text{si } x \notin \mathbb{Q} \end{cases}$$

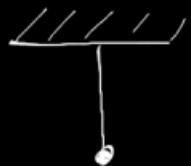
A red arrow points from the word "périodique" to the value 1 in the definition of  $\delta(x)$ .

périodique de période  $x \in \mathbb{Q}$

fonction lisse : ↗ périodique :  $T > 0$   
↗ non périodique

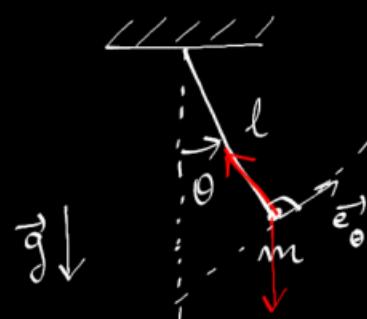
Q : 2 périodes ?

③



$$\ddot{\theta} + \omega^2 \underbrace{\sin \theta}_{\approx \theta} = 0 \rightarrow \ddot{\theta} + \omega^2 \theta = 0$$

# I) Pendule



$$m\vec{a} = \sum \vec{F}$$

$$\omega^2 = \frac{g}{l}$$

$$lm\ddot{\theta} = -mg \sin \theta \rightarrow \boxed{\ddot{\theta} + \frac{g}{l} \sin \theta = 0}$$

Supposons les oscillations petites :  $\boxed{\sin \theta \sim \theta}$

$$\boxed{\ddot{\theta} + \omega^2 \theta = 0}$$

$$\underline{\theta(t) = A \sin(\omega t) + B \cos(\omega t)}$$

$$\dot{\theta} \swarrow$$

$$\ddot{\theta}\dot{\theta} + \omega^2 \theta \dot{\theta} = 0 = \frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 + \frac{\omega^2}{2} \theta^2 \right)$$

$$\boxed{\frac{1}{2} \dot{\theta}^2 + \frac{1}{2} \theta^2 = E}$$

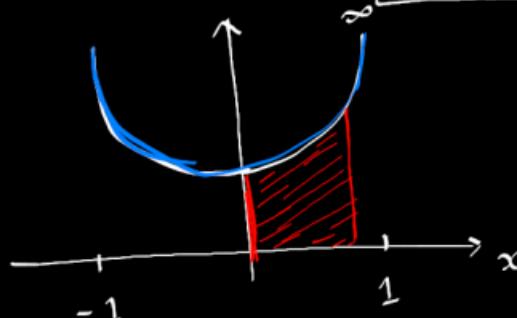
On prend  $\underline{\omega = 1}$

Possens:  $x = \frac{\theta}{\sqrt{2E}}$

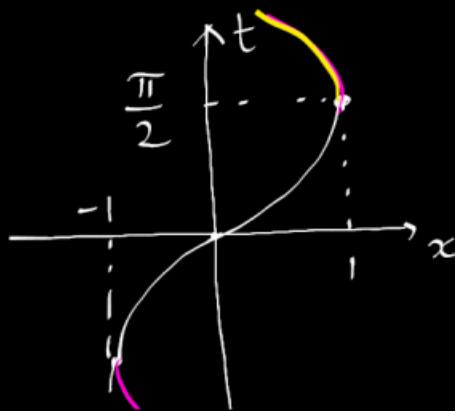
$$1 - x^2 = 1 - \frac{\theta^2}{2E} = \frac{1}{E} \left( E - \frac{\theta^2}{2} \right) = \frac{\dot{\theta}^2}{2E} = \dot{x}^2$$

$$\left( \frac{dx}{dt} \right)^2 = 1 - x^2 \quad \frac{dx}{\sqrt{1-x^2}} = dt$$

Conclusion :  $A = \int_0^{x_0} \frac{dx}{\sqrt{1-x^2}}$  ←  $\arcsin(x_0)$

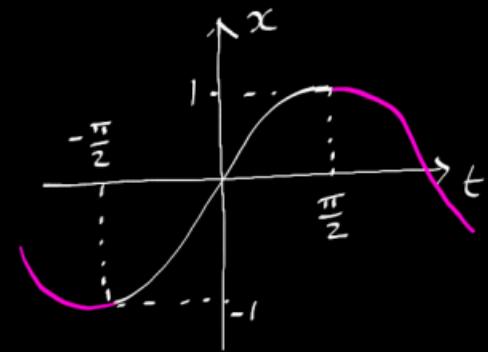


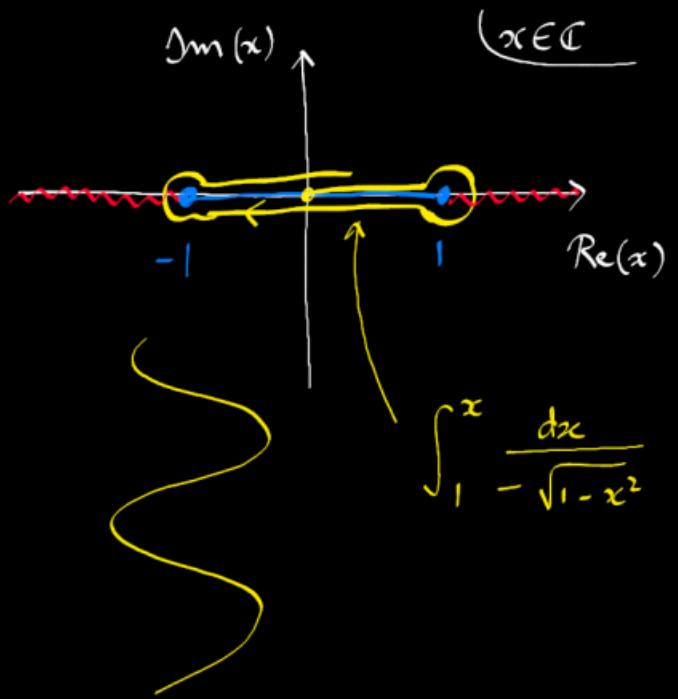
$(x \in \mathbb{C})$



On définit

$$\pi = \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$





$$\sqrt{z} \rightarrow e^{i\pi} z = -z$$

$$z \rightarrow e^{2i\pi} z$$

$$\int_1^x \frac{dx}{-\sqrt{1-x^2}}$$

Tans approximation  $\sin \theta \sim \theta$   
(Euler 1738)

$$\ddot{\theta} + \sin \theta = 0$$
$$x\dot{\theta} \left( \frac{d}{dt} \left( \frac{1}{2} \dot{\theta}^2 - \cos \theta \right) \right) = 0$$

$$E = \frac{1}{2} \dot{\theta}^2 + \underbrace{1 - \cos \theta}_{\approx \frac{1}{2} \dot{\theta}^2}$$

Possens

$$x = \sqrt{\frac{2}{E}} \sin \frac{\theta}{2}$$
$$= \frac{1}{k} \sin \frac{\theta}{2}$$

$$k = \sqrt{\frac{E}{2}}$$

$$1 - x^2 = 1 - \frac{2}{E} \sin^2 \frac{\theta}{2} = \frac{1}{E} \left( \frac{1}{2} \dot{\theta}^2 + \underbrace{1 - \cos \theta - 2 \sin^2 \frac{\theta}{2}}_0 \right) = \frac{\dot{\theta}^2}{2E}$$

$$1 - k^2 x^2 = 1 - \sin^2 \frac{\theta}{2} = \cos^2 \frac{\theta}{2}$$

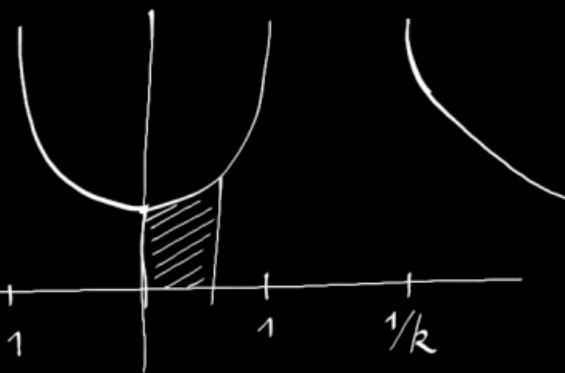
$$(1 - x^2)(1 - k^2 x^2) = \frac{\dot{\theta}^2}{2E} \cos^2 \frac{\theta}{2} = \frac{\dot{\theta}^2}{4k^2} \cos^2 \frac{\theta}{2} = \dot{x}^2$$

$$\dot{x} = \frac{1}{k} \frac{\dot{\theta}}{2} \cos \frac{\theta}{2}$$

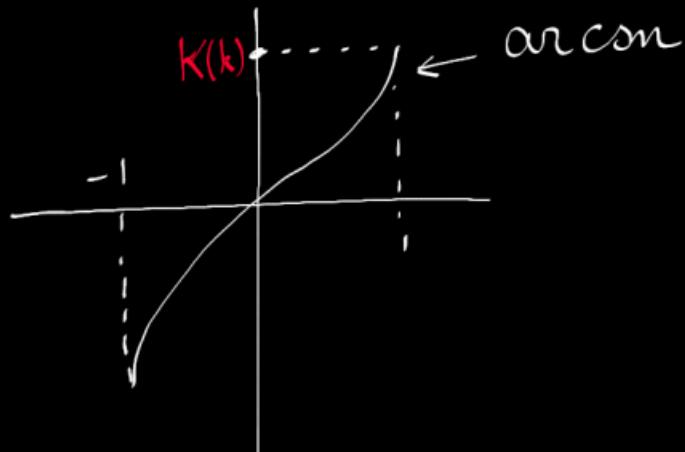
$$\dot{x}^2 = \frac{\dot{\theta}^2}{4k^2} \cos^2 \frac{\theta}{2}$$

$$\dot{x} = \frac{dx}{dt} = \sqrt{(1-x^2)(1-k^2x^2)}$$

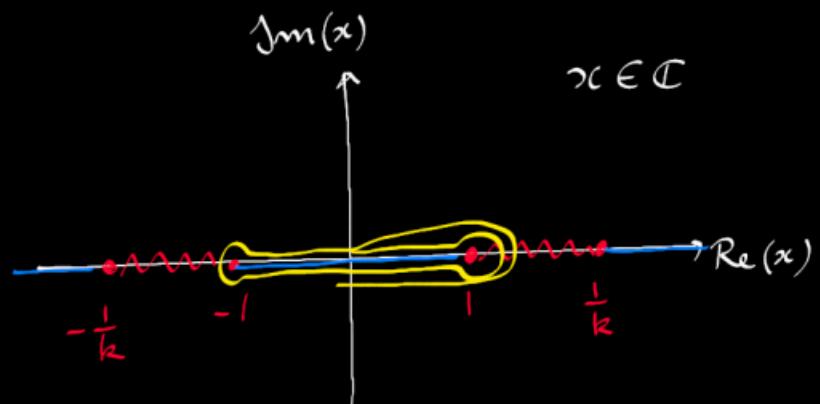
$$k < 1$$



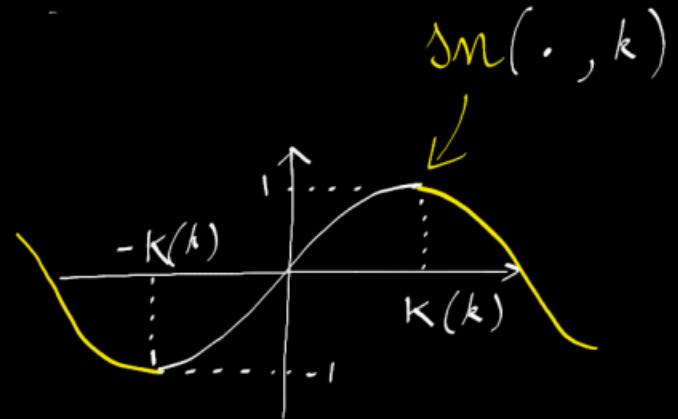
$$t = \int_0^{x_0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$



On définit  $K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$



$$\begin{aligned} t &= \arcsin(x) \\ x &= \sin(t) \end{aligned}$$



$$\sin \frac{\theta}{2} = k \operatorname{sn}(t, k)$$

$$\boxed{\theta(t) = 2 \arcsin(k \operatorname{sn}(t, k))}$$

Petites oscillations :  $\theta \approx 2k \sin(t)$

période  $4K(k) \approx 2\pi$

## Résumé :

Équation

$$\ddot{\Theta} + \Theta = 0 \iff \ddot{\Theta} + \sin \Theta = 0$$

Fonction multivaluée

↓ réciproque

Fonction "périodique"

Période

$k$

$$\int \frac{dx}{\sqrt{1-x^2}} \iff \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \iff \int \frac{dx}{1-x^2}$$

↑      "arcsin"      ↓      "arcsn"      ↑  
"arctanh"

$\sin(\cdot)$

$\leftarrow\rightarrow$

$sn(\cdot, k)$   
Jacobi: sinus amplitudinus (1830)

$tanh(\cdot)$

$2\pi$

$\leftarrow\rightarrow$

$4K(k)$

$\leftarrow\rightarrow \infty$

$k=0$

$\leftarrow\rightarrow$

$0 < k < 1$

$\leftarrow\rightarrow k=1$

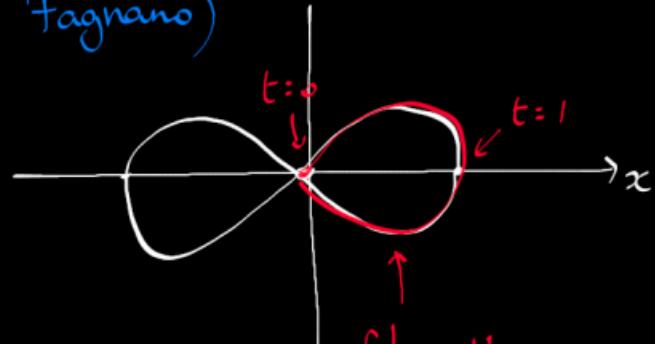
Lemniscate  $(x^2 + y^2)^2 = x^2 - y^2$  (1718 Fagnano)

On veut calculer la longueur du lemniscate.

$$x = \sqrt{\frac{t^2}{2} (1+t^2)}$$

$$y = \sqrt{\frac{t^2}{2} (1-t^2)}$$

$$\int_0^{t_0} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt$$



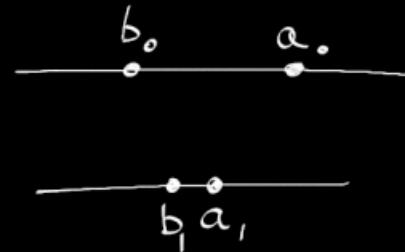
$$\int_{-1}^1 \frac{dt}{\sqrt{1-t^4}} = 2K(i)$$

//  
2,622057554...

Ellipses :  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$        $k^2 = 1 - \frac{b^2}{a^2}$       ( $b < a$ )

Périmètre de l'ellipse :  $4 \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}} dx$       (avec  $a=1$   
pour simplifier)

Jeu : moyenne arithmétique - géométrique



$$a, b \in \mathbb{R}.$$

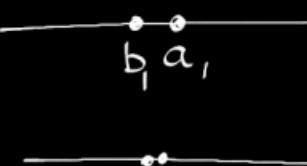
Moyenne arithmétique

$$\frac{a+b}{2}$$

$$0 < b < a < 1$$

Moyenne géométrique

$$\sqrt{ab}$$



$$0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{ab} = 2 \left( \underbrace{\frac{a+b}{2}}_{\geq 0} - \sqrt{ab} \right)$$

$$0 < b_0 < a_0 < 1$$

$$\begin{cases} b_1 = \sqrt{a_0 b_0} \\ a_1 = \frac{a_0 + b_0}{2} \end{cases}$$

$$\dots \begin{cases} b_{n+1} = \sqrt{a_n b_n} \\ a_{n+1} = \frac{a_n + b_n}{2} \end{cases}$$

Question :  $\lim_{n \rightarrow \infty} a_n = ?$

Gauss (179.)

$$\lim a_n = \lim b_n = \frac{\pi a}{2 K(k)}$$

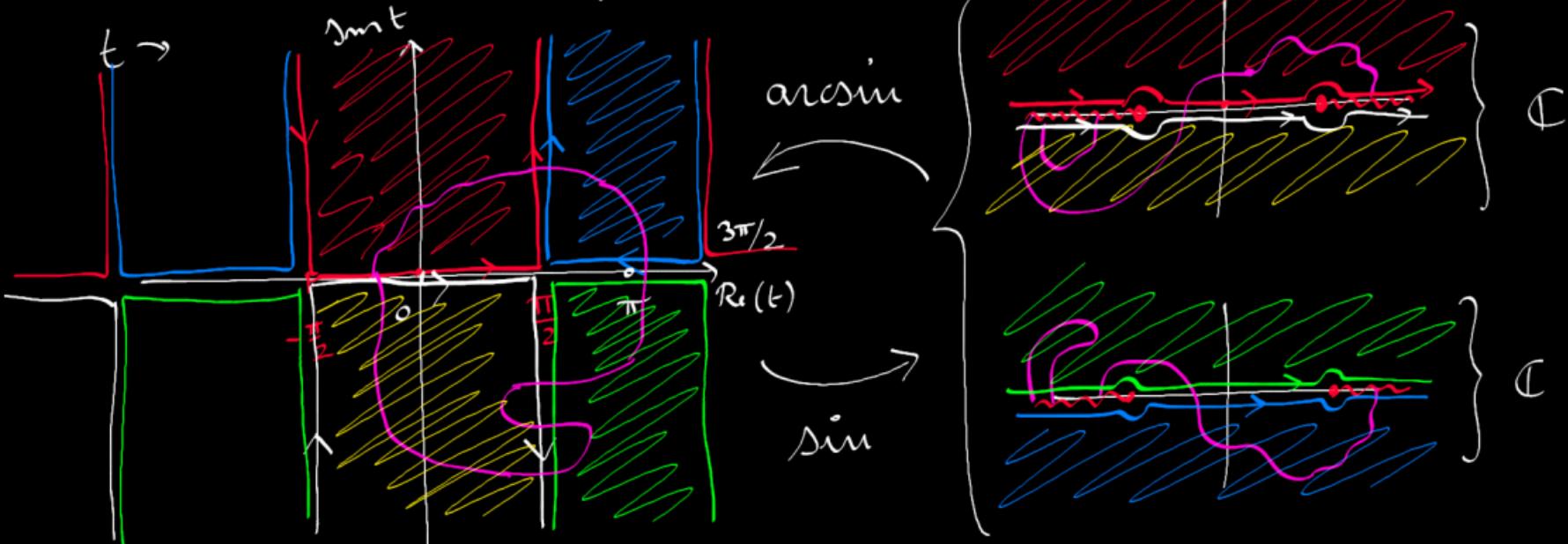
avec  $k = 1 - \frac{b^2}{a^2}$

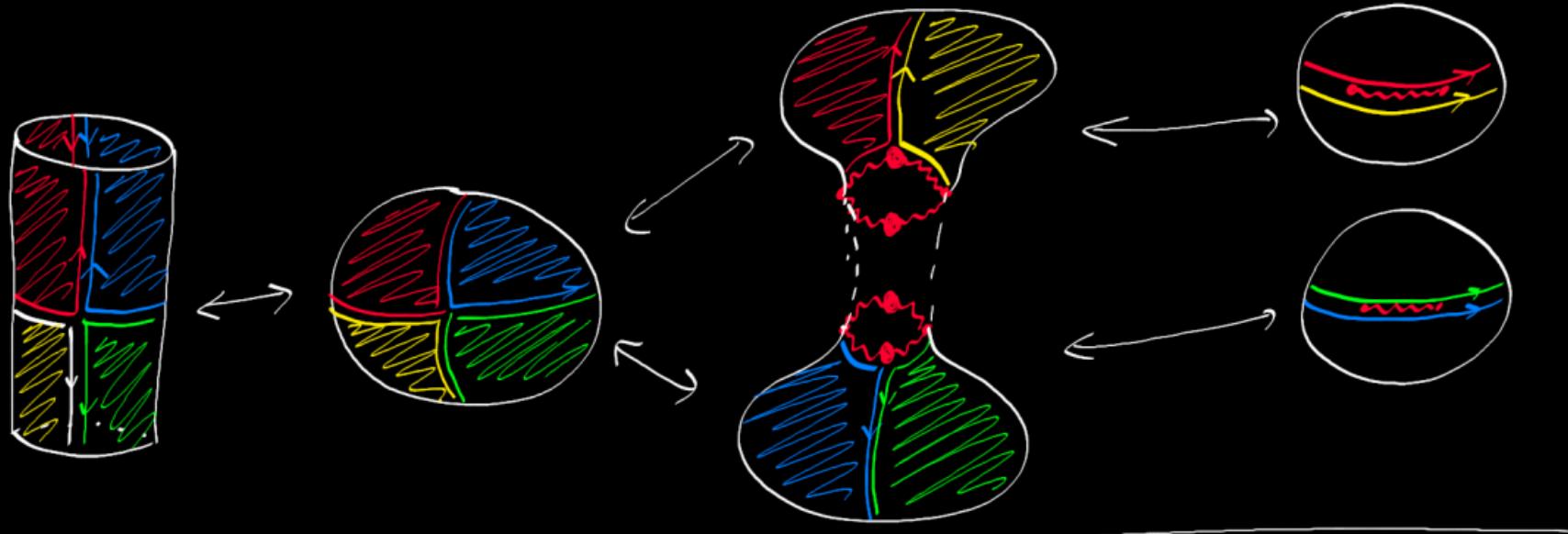
# Géométrie cachée

sin ?

$$\int_1^{x_0} \frac{dx}{\sqrt{1-x^2}} = i \int_1^{x_0} \frac{dx}{\sqrt{x^2-1}}$$

$$t = \arcsin(x_0) = \int_0^{x_0} \frac{dx}{\sqrt{1-x^2}} \quad \leftrightarrow \quad \sin(t) = x_0$$



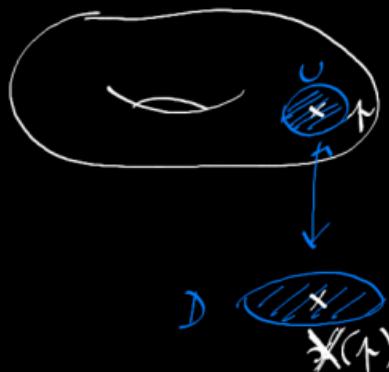


## Interlude : Surfaces de Riemann.

2d

- Une surface est un espace topologique qui peut étre couvert par des ouverts  $\cup$  diffeomorphes au disque  $D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ ,

$$\begin{array}{l} \varphi_X: U \rightarrow D \\ p \mapsto X(p) \end{array}$$



tels que si  $U_1 \cap U_2 \neq \emptyset$  alors si

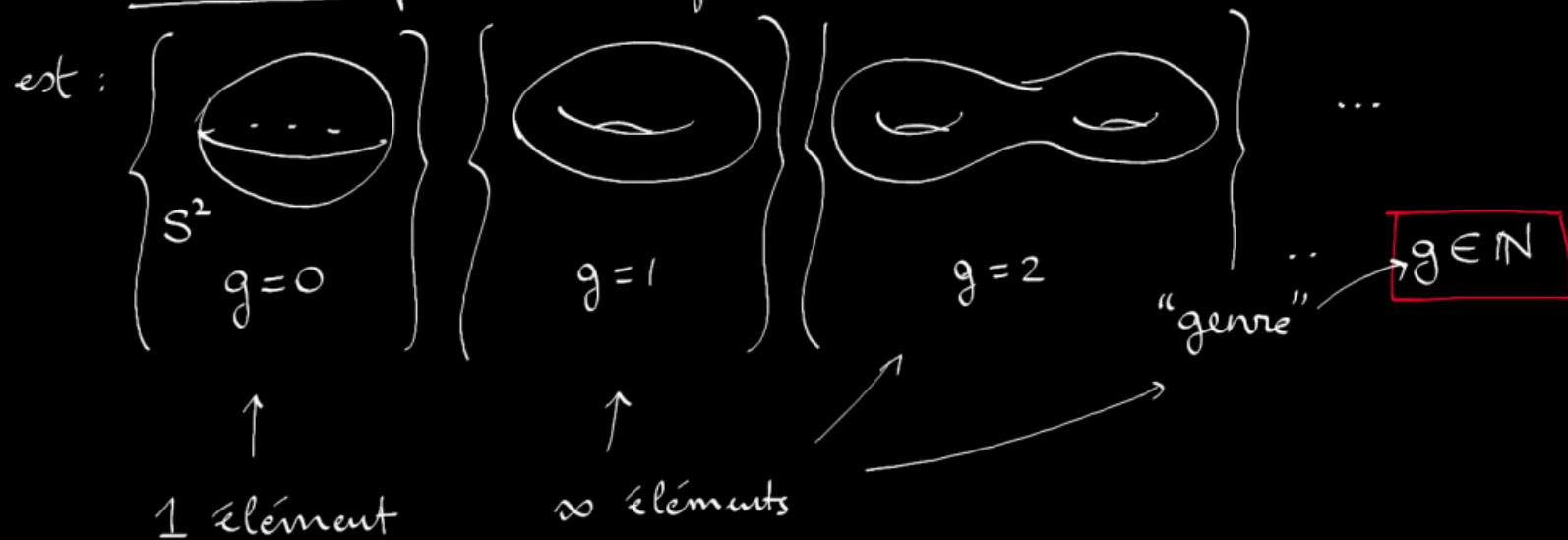
$$X_1: U_1 \rightarrow D$$

$$X_2: U_2 \rightarrow D$$

$$\text{alors } \varphi_{X_2}^{-1} X_1(U_1 \cap U_2) \longrightarrow X_2(U_1 \cap U_2)$$

est  $C^\infty$  et bijective.

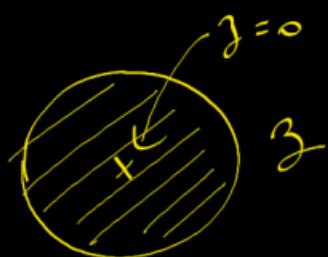
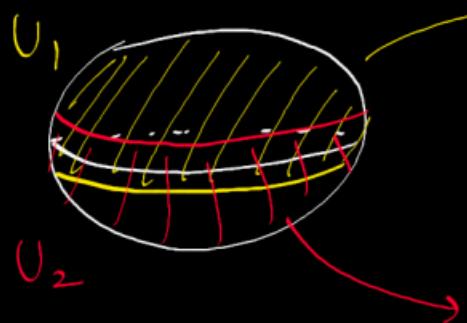
- Une surface de Riemann est une surface où les applications induites  $\varphi_{12}$  sont holomorphes  
(avec  $z = x + iy$ )
- La liste complète des surfaces de Riemann compactes (orientables) est :



- La raison d'être d'une surface de Riemann est de traiter les fonctions multivaluées.

### Exemples :

- Sphère de Riemann =  $\mathbb{P}^1$

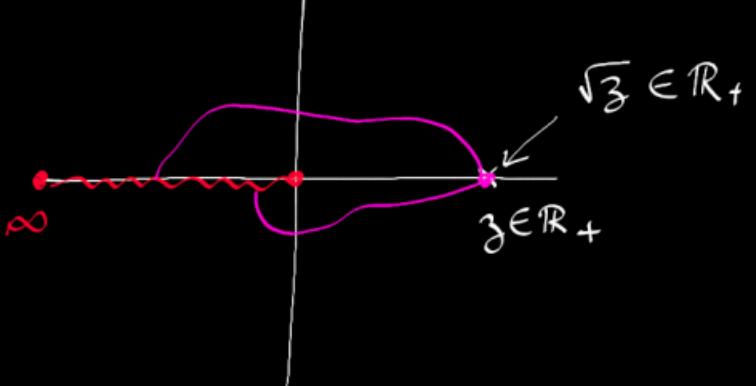


Sur  $U_1 \cap U_2 \neq \emptyset$ ,  
 $z \neq 0$  et  $w \neq 0$ .

On définit

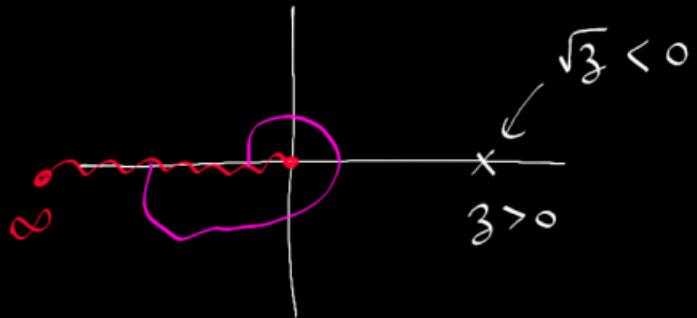
$$\varphi_{12}(z) = \boxed{\frac{1}{z} = w}$$

• Surface de  $z \mapsto \sqrt{z}$

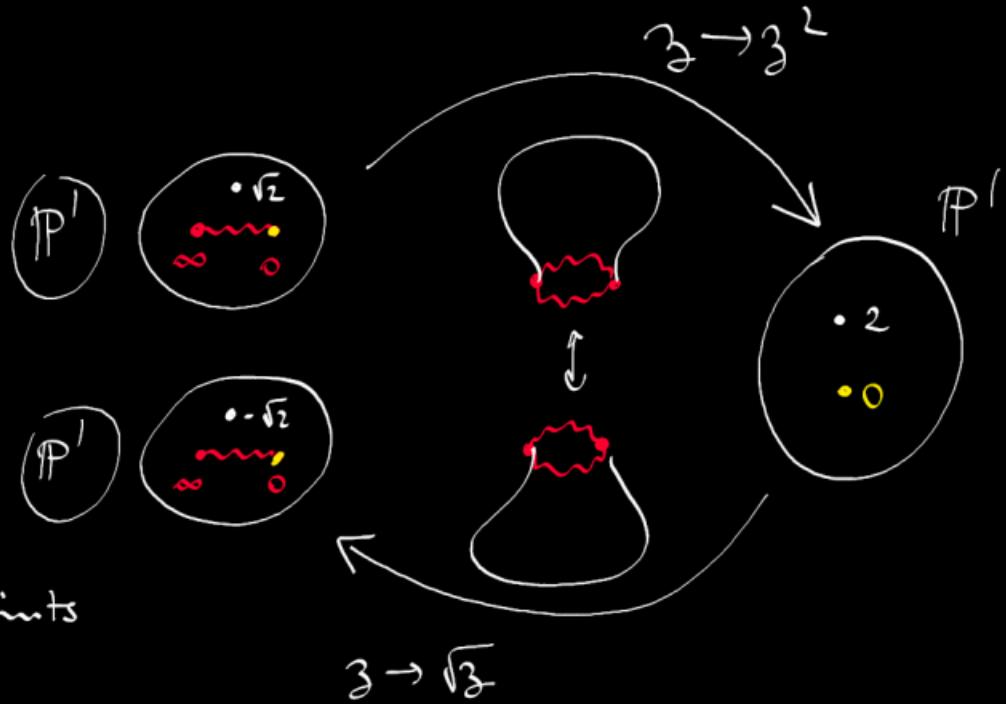


$$\begin{cases} z \rightarrow e^{2\pi i} z \\ \sqrt{z} \rightarrow -\sqrt{z} \end{cases}$$

$$\begin{cases} z \rightarrow e^{4\pi i} z \\ \sqrt{z} \rightarrow \sqrt{z} \end{cases}$$



à 2 feuillets  
Revêtement ramifié en 2 points



## Généralisation (Riemann-Hurwitz)

point où  $n$  feuilles  
se croisent, l'indice  
est  $n-1$

Si on a un revêtement  $\Sigma$  à  $d$  feuilles, d'indice de  
ramification  $n$ , de la sphère de Riemann  $\mathbb{P}^1$ , alors

$\sum$  est de genre

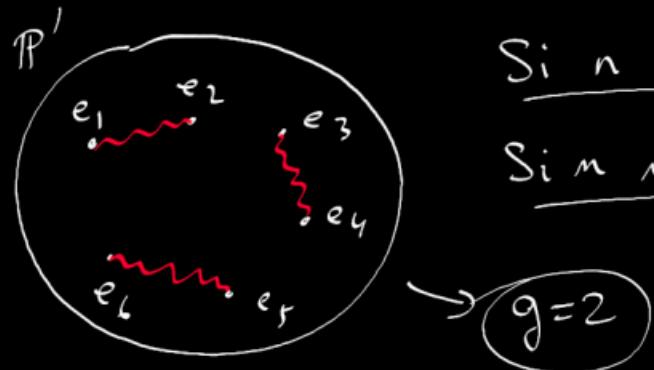
$$g = 1 + \frac{n}{2} - d$$

Exemple  $\sqrt[3]{z}$

$$\begin{aligned}d &= 2 \\n &= 2 = 1+1\end{aligned}$$

$$g = 1 + \frac{2}{2} - 2 = 0$$

Example :  $f(z) = \sqrt{(z - e_1)(z - e_2) \cdots (z - e_n)}$   $e_1 \neq e_2 \neq z \cdots$

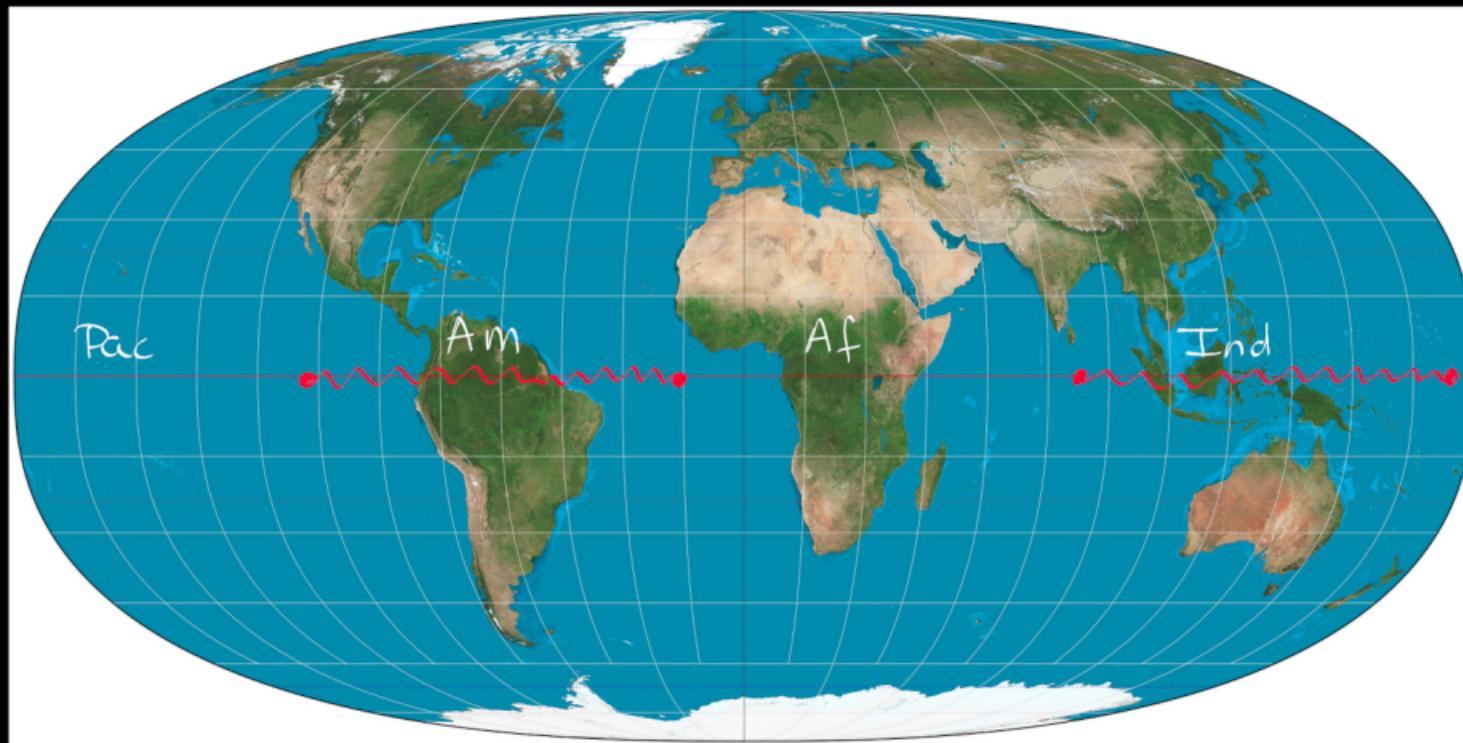
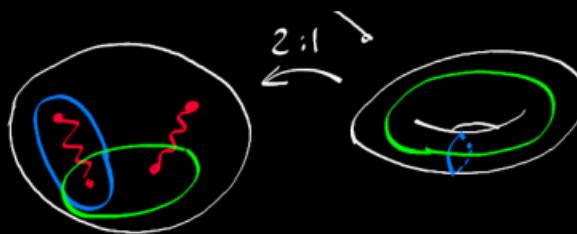


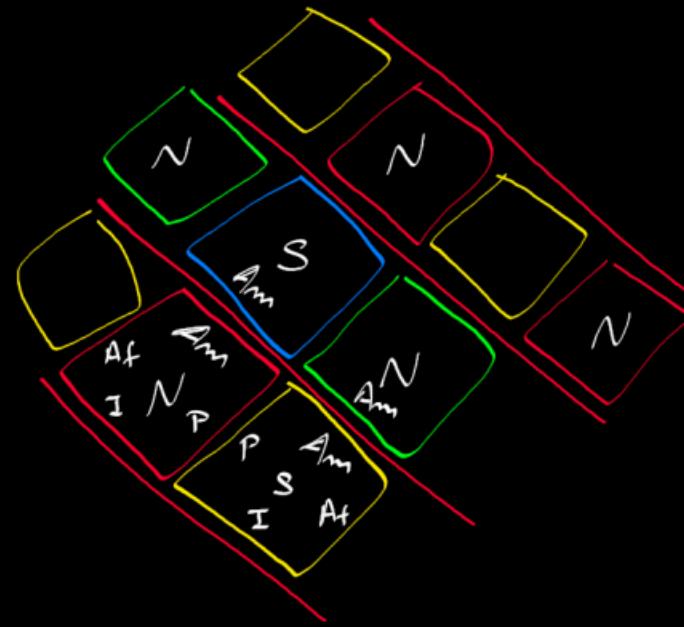
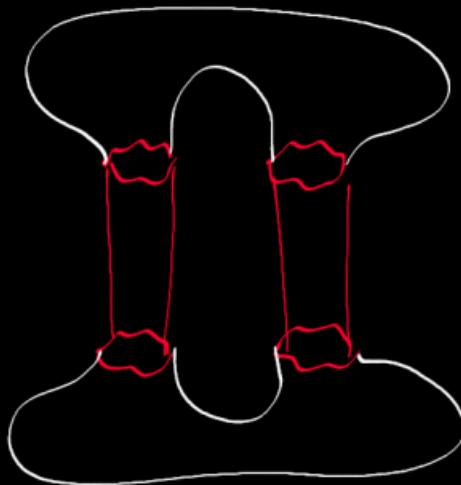
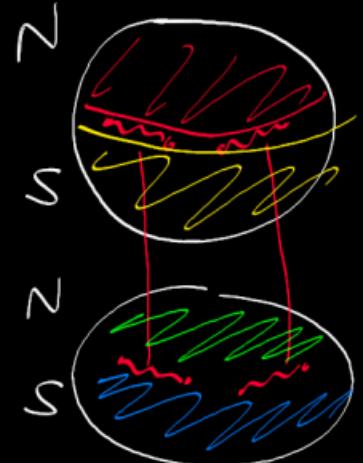
Si n pair :  $g = 1 + \frac{n}{2} - 2 = \frac{n}{2} - 1$

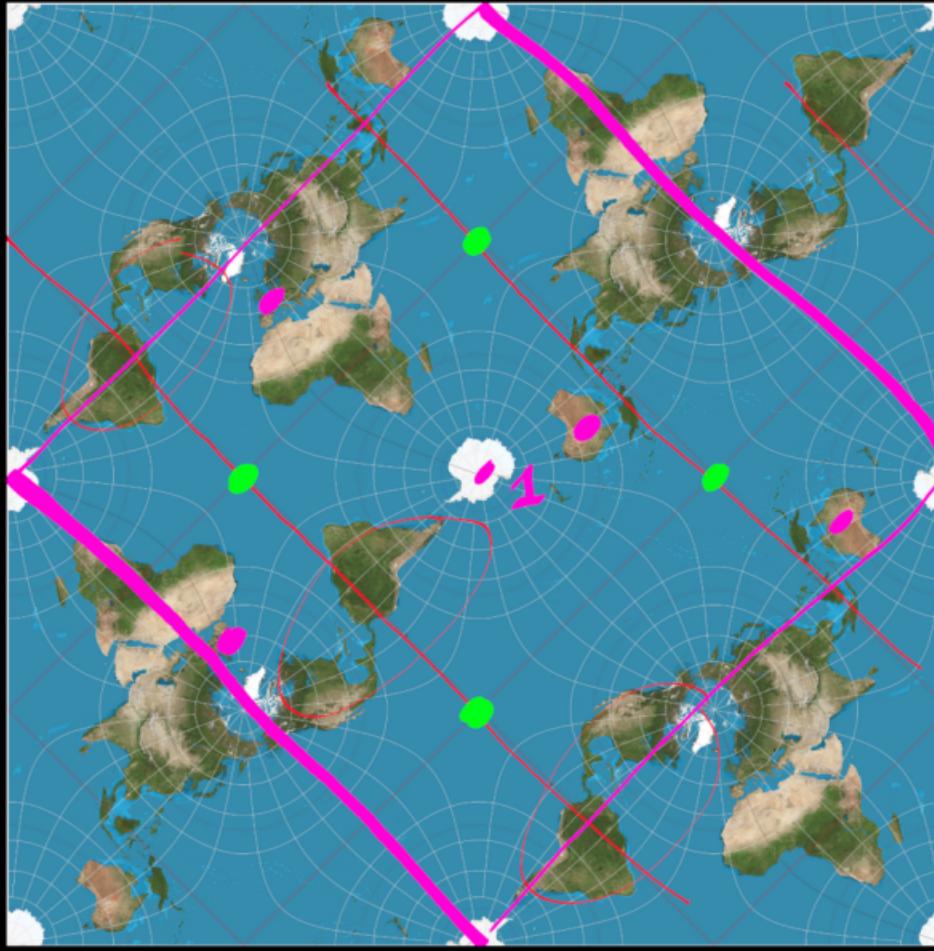
Si n impair :  $g = 1 + \frac{n+1}{2} - 2 = \frac{n}{2} - \frac{1}{2}$

$n$	1	2	3	4	5	6	7	$\cdots$
$g$	0	0	1	1	2	2	3	$\cdots$

Interlude : Double recouvrement de

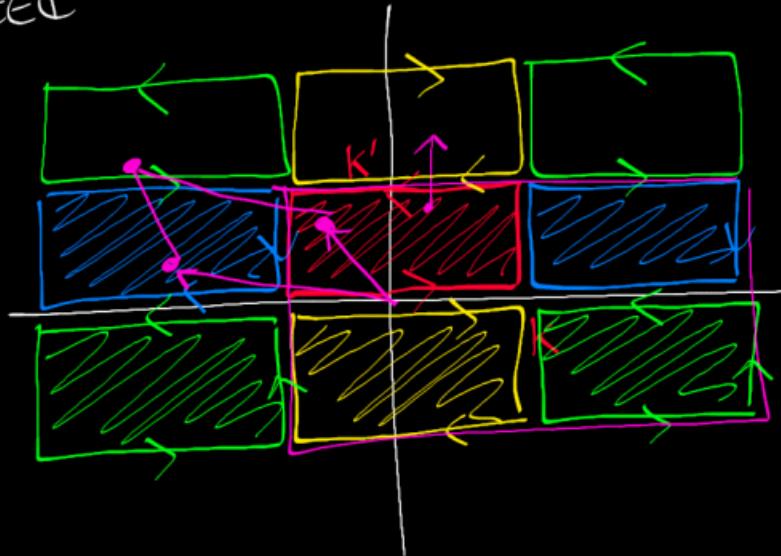






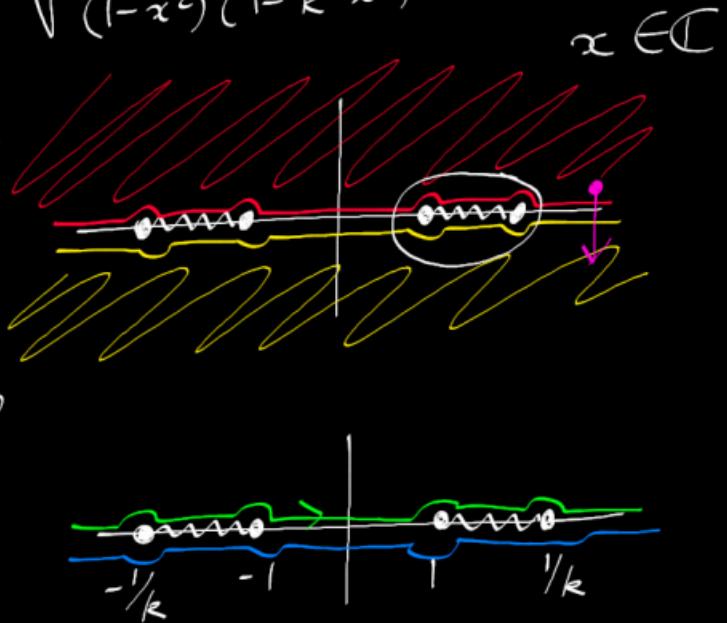
Retour au sn

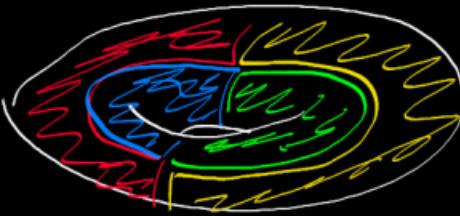
$t \in \mathbb{C}$



$$\operatorname{arcsn}(x_0) = \int_0^{x_0} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$\operatorname{arcsn}$





$$\left\{ \begin{array}{l} K(k) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \\ K'(k) = \int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}} \end{array} \right.$$

La fonction  $\sin(\cdot, k)$   
est doublument périodique

$$\left\{ \begin{array}{l} \omega_1 = 4K(k) \\ \omega_2 = 2iK'(k) \end{array} \right.$$

La structure complexe du tore est définie par le rapport

$$\boxed{\frac{\omega_2}{\omega_1} = \frac{i K'(k)}{2 K(k)} = \tau}$$

Définition: Une fonction méromorphe doubllement périodique est appelée fonction elliptique.

La surface de Riemann associée



est appelée courbe elliptique.

Fait. "L'ensemble des fonctions elliptiques de période  
 $(\omega_1, \omega_2)$  est un corps généré par 1 fonction"

$$= \mathbb{C}(\wp)[\wp']$$

## Conclusion

1736 : Euler et le pendule

1797 : Gauss trouve la double périodicité

$$\int \frac{dx}{\sqrt{1-x^4}}$$

Analyse  
complexe

182... : Abel / Jacobi inversent les intégrales

185... : Riemann  $\rightarrow$  géométrie cachée

⋮

2001 : Théorème de Modularité  $\Rightarrow$  Théorème de Fermat

## Trichotomie :

$g = 0$  courbure > 0

GROUPES

$g = 1$  courbure = 0

Surface de Riemann = Variété Abélienne

$g \geq 2$  courbure < 0  
hyperbolique

Variété Abélienne dim > 1



← géométric hyperbolique

6 angles droits