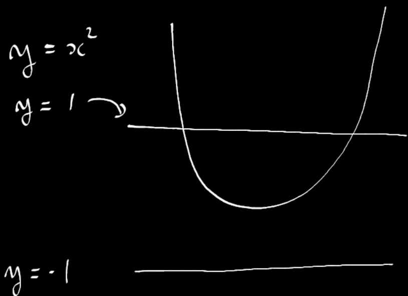
Vers la Géométrie Algébrique

EPISODE III : LES COMPLEXES

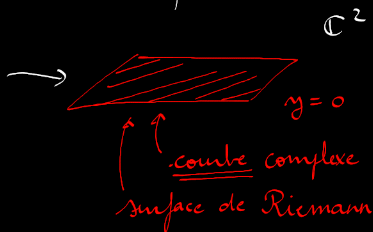


$x^2 = -1$

$x = \pm i = \pm\sqrt{-1}$

__Exemple__ : $\mathbb{R}^2$. $\quad C = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$

$\mathbb{C}^2$: $\quad C = \{(x,y) \in \mathbb{C}^2 \mid y = 0\}$

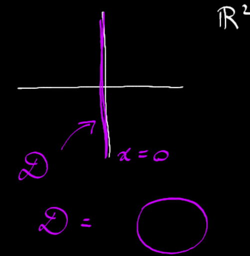


$$\mathbb{CP}^2 = \{(x,y,z) \in \mathbb{C}^2 - \{(0,0,0)\}\} \Big/ \begin{array}{l}(x,y,z) \sim (\lambda x, \lambda y, \lambda z) \\ \lambda \in \mathbb{C}^*\end{array}$$

Droite projective complexe :

Dans $\mathbb{RP}^2$, $x = 0$

$\mathcal{D}$

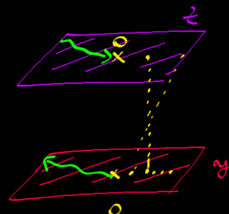


Dans $\mathbb{CP}^2$, on regarde la droite projective

$$\mathcal{D} = \{x = 0\}.$$

$$(x, y, z) = (0, y, z)$$

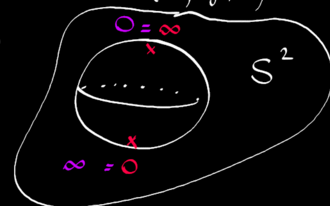
- $x = 1$ $\quad \emptyset$
- $y = 1$ $\quad (0, 1, z)$
- $z = 1$ $\quad (0, y, 1)$



Si z et y sont tous deux non nuls, alors

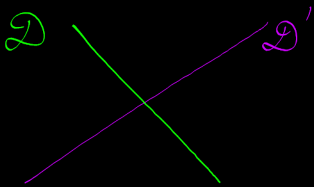
$$(0, 1, z) = \left(0, \frac{1}{z}, 1\right)$$
$$= (0, y, 1)$$

$\mathcal{D} =$

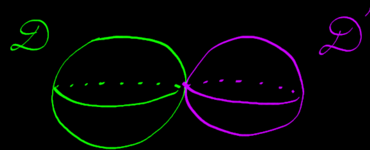


Droite projective complexe = $\mathbb{CP}^1$ = Sphère de Riemann $\cong S^2$.

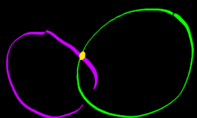
Deux droites (distinctes) (dans $\mathbb{RP}^2$ ou $\mathbb{CP}^2$) s'intersectent toujours en un unique point.



dans $\mathbb{RP}^2$



dans $\mathbb{CP}^2$

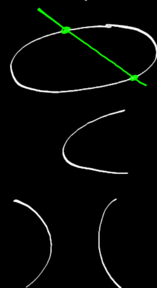


Coniques : Dans $\mathbb{R}^2$

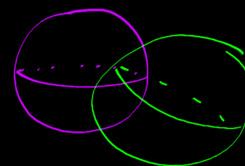
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = a x^2$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



topologiquement

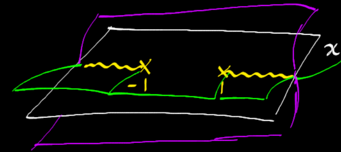


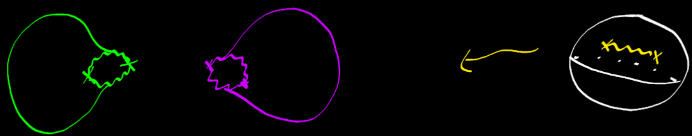
Vérification :

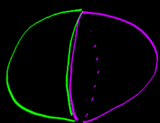
$$x^2 + y^2 = 1$$

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1 - x^2}$$



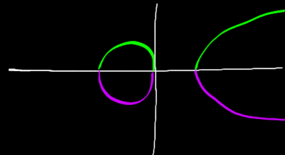


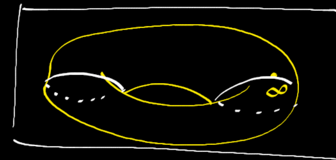


courbe conique !

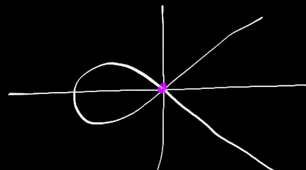
Exemple : courbes cubiques.

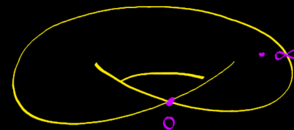
- $y^2 = x(x+1)(x-1)$



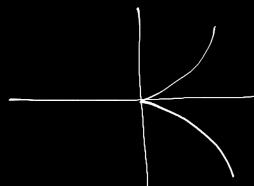


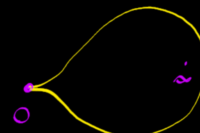
- $y^2 = x^2(x+1)$





- $y^2 = x^3$

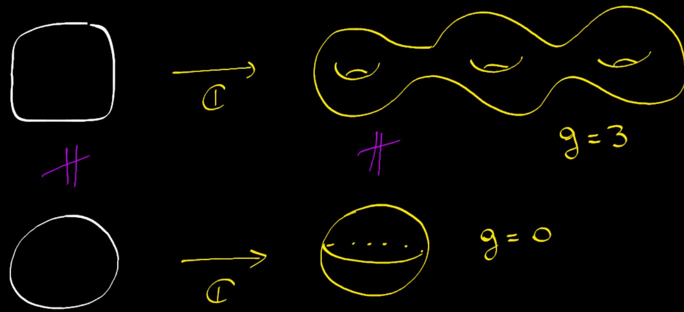




Exemple :

$x^4+y^4=1$

$x^2+y^2=1$



Résumé :

- Les courbes algébriques complexes sont des surfaces de Riemann
- On évite les pathologies du type $x^2 + y^2 = 0$
- Les courbes algébriques complexes planes projectives (dans $\mathbb{CP}^2$) lisses correspondent à des surfaces de Riemann compactes, topologiquement classifiées par leur genre $g \in \mathbb{N}$.

$g = 0$ ↑ droites, coniques

$g = 1$ ↑ cubiques

$g = 2$

. . .

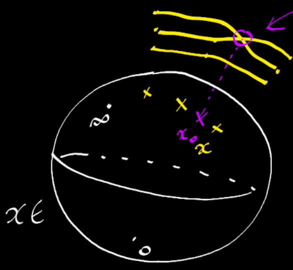
$g = 3$ ↑ $x^4 + y^4 = 1$

Relation genre - degré

Soit C une courbe plane (dans $\mathbb{CP}^2$) définie par $f(x,y)=0$,

avec f polynôme de degré $d \geq 1$.

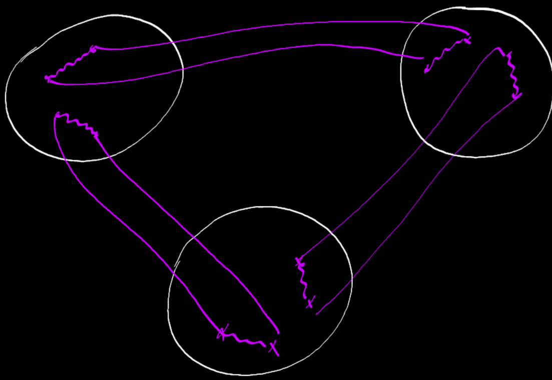
$\left(\begin{array}{c} \wr \\ f(x,y,z)=0 \end{array} \right)$

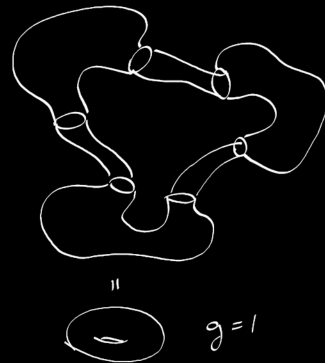


deux feuillets se croisent quand x_0 est tel que le polynome en y $\quad y \mapsto f(x_0,y)$ possède une racine double :

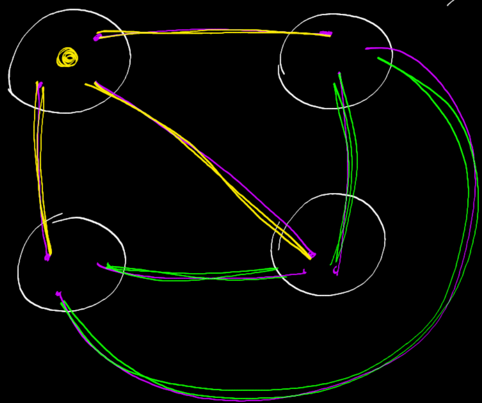
$$\begin{cases} f(x_0,y)=0 & \leftarrow \text{ équation de degré } d \\ \dfrac{\partial f}{\partial y}(x_0,y)=0 & \leftarrow \text{ équation de degré } d-1 \text{ (en } y) \end{cases}$$

Exemple $d=3$ $\qquad d(d-1)=6$





Cas d arbitraire :



$\rightarrow$ graphe complet à d éléments.

Genre :

$$g = \frac{d(d-1)}{2} - (d-1)$$

$$g = \frac{(d-1)(d-2)}{2}.$$

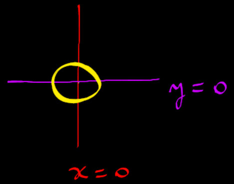
$\boxed{\text{Théorème : Si } C \text{ est une courbe projective complexe plane lisse de degré } d \text{, alors } g(C) = \frac{(d-1)(d-2)}{2}}$

d	1	2	3	4	5	$\cdots$
g	0	0	1	3	6	$\cdots$

A quoi ressemblent vraiment les courbes complexes ?

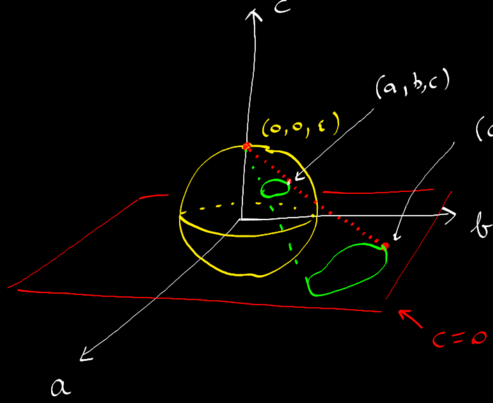
- Si la courbe est lisse :

- On s'intéresse aux points singuliers.



$$xy = 0$$

Projection stéréographique :



$$\lambda(a,b,c-\varepsilon) = (a',b',-\varepsilon)$$

$$\lambda = \frac{-\varepsilon}{c-\varepsilon} = \frac{\varepsilon}{\varepsilon - c}$$

$$\boxed{(a',b') = \left(\frac{\varepsilon a}{\varepsilon - c}, \frac{\varepsilon b}{\varepsilon - c}\right)}.$$

Soit C la courbe $\{xy=0\}$.

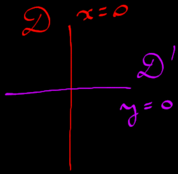
On intersecte C avec $S_\varepsilon = \{(x,y)\in\mathbb{C}^2 \,/\, |x|^2+|y|^2=\varepsilon^2\} \simeq S^3$.

$$\begin{gathered} \Downarrow \\ (a,b,c,d) \\ a^2+b^2+c^2+d^2=\varepsilon^2. \end{gathered} \qquad \left(\begin{aligned} a &= \mathrm{Re}(x) \\ b &= \mathrm{Im}(x) \\ c &= \mathrm{Re}(y) \\ d &= \mathrm{Im}(y) \end{aligned}\right)$$

Projection stéréographique :

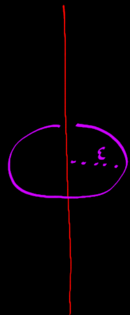
$$S_\varepsilon - \{(0,0,\varepsilon,0)\} \longrightarrow \mathbb{R}^3$$

$$(a,b,c,d) \longmapsto \left(\frac{\varepsilon a}{\varepsilon-c}, \frac{\varepsilon b}{\varepsilon-c}, \frac{\varepsilon d}{\varepsilon-c}\right)$$

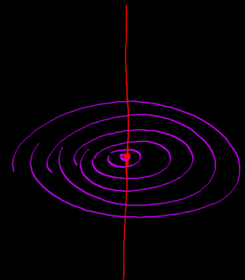


$\mathcal{D}$: $\left(0,\ 0,\ \frac{\varepsilon d}{\varepsilon-c}\right)$ avec $d^2+c^2=\varepsilon^2$

$\mathcal{D}'$: $(a, b, 0)$ avec $a^2+b^2=\varepsilon^2$



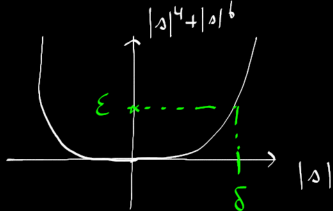
On fait varier $\varepsilon \in \,]0,+\infty[$:



Exemple.

$y^2 = x^3$ la courbe est paramétrée par $s \in \mathbb{C}$: $\begin{array}{l} y = s^3 \\ x = s^2 \end{array}$

L'intersection avec S_ε : $|x|^2+|y|^2 = \varepsilon^2 = |s|^4 + |s|^6$.



$|s| = \delta$.

$s = \delta e^{it}$ avec $t \in [0, 2\pi[$.

$$\begin{cases} y = \delta^3 e^{3it} \\ x = \delta^2 e^{2it} \end{cases}$$