Elliptic Calogero-Moser and Gauge Theory

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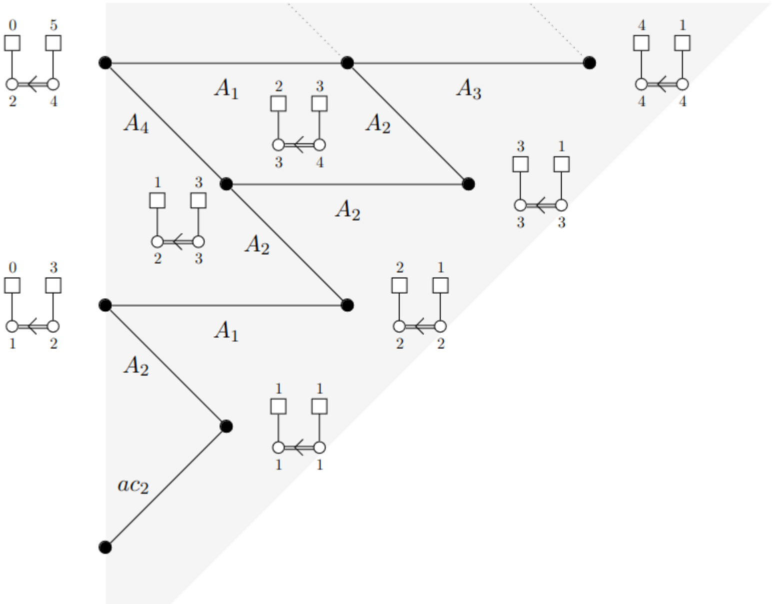
At the crossroads of physics and mathematics
The joy of integrable combinatorics

Based on work with Jan Troost [1501.05074], [1702.02102]
and ongoing work with
Romain Vandepopeliere, Valdo Tatitscheff, Riccardo Argurio...

Quivers?



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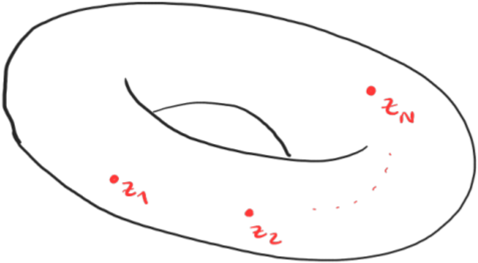
Particles on a torus

Consider N particles on a torus T^2 with a pairwise interaction. What are the stable configurations?



What kind of interaction on a torus?

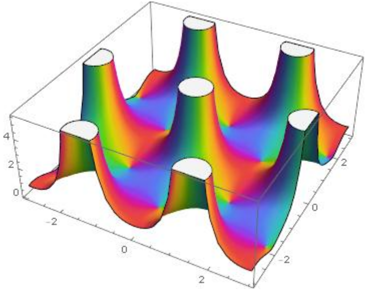
Particles on a torus

Consider the torus as $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The particles are

$$z_i \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})\,, \qquad i=1,\ldots,N$$

We need the potential to be doubly periodic: use

$$V(z_1,\ldots,z_N)=\sum_{i<j}\wp(z_i-z_j|\tau)\,.$$



Translation invariance : one can assume $z_N=0$.

Particles on a torus

Hamiltonian:

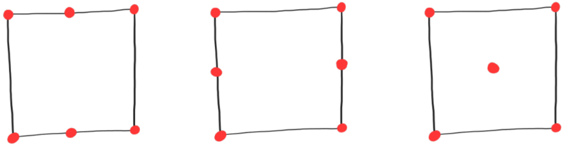
$$H = \sum_i \frac{p_i^2}{2} + V(z_1, \ldots, z_N)$$

Equilibrium positions:

$$p_i = 0\,, \qquad \frac{\partial}{\partial z_i} V(z_1, \ldots, z_N) = 0\,.$$

Example: $N = 2$. Then $V(z) = \wp(z|\tau)$.
Equilibrium positions

$$\wp'(z|\tau) = 0 \qquad \Longrightarrow \qquad z = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}\,.$$



Particles on a torus : $N = 3$

$$V(z_1, z_2) = \wp(z_1|\tau) + \wp(z_2|\tau) + \wp(z_1 - z_2|\tau)\,.$$

Equilibrium positions

$$\wp'(z_1|\tau) = -\wp'(z_1 - z_2|\tau) = \wp'(z_2|\tau)$$

Two types:

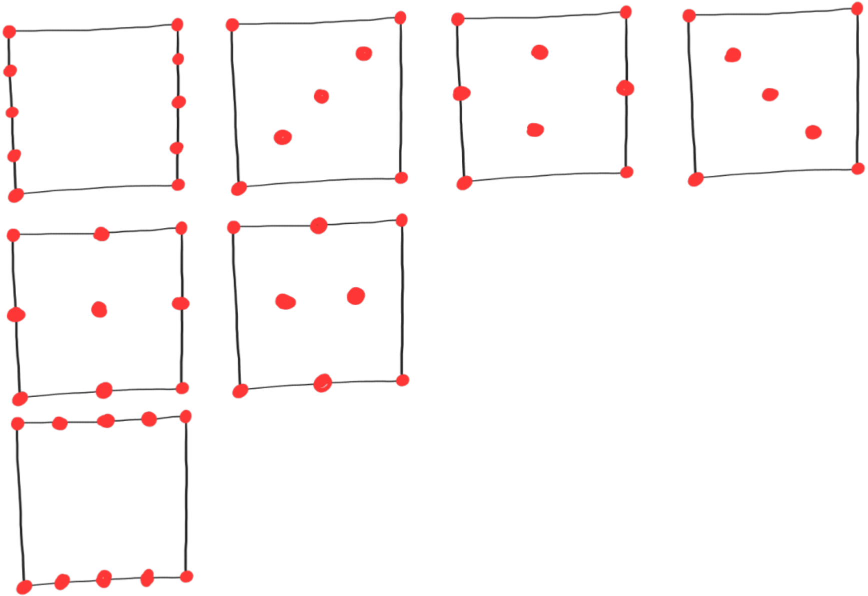
- Isolated equilibrium: $J > 0$.
- Non-isolated equilibrium: $J = 0$.

with

$$J = \left| \frac{\partial^2 V}{\partial z_1^2} \frac{\partial^2 V}{\partial z_2^2} - \left(\frac{\partial^2 V}{\partial z_1 \partial z_2} \right)^2 \right|$$

From now on, we focus on the *isolated* ones.

Particles on a torus : $N = 4$



Elliptic Calogero-Moser Potential

Let $\mathfrak{g}$ be a simple complex Lie algebra of type ADE.

$$V(\mathbf{z}|\tau) = \sum_{\alpha \in \mathrm{Roots}^+(\mathfrak{g})} \wp(\alpha \cdot \mathbf{z}|\tau)\,.$$

Isolated equilibria : $\mathbf{z}^a(\tau)$, $a \in \mathcal{A}$. Value of the potential at each isolated equilibrium:

$$V^a(\tau) := V(\mathbf{z}^a(\tau)|\tau)\,.$$

Recall that

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \qquad \wp\left(\frac{z}{c\tau+d} \mid \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 \wp(z, |\tau)$$

Questions:

- How many isolated equilibria?
- How do they evolve as τ is modified?

Elliptic Calogero-Moser Potential

Expectations:

1. There exist *permutations* $T, S : \mathcal{A} \to \mathcal{A}$ such that

$$V^a(\tau+1) = V^{T(a)}(\tau)\,, \qquad V^a(-1/\tau) = \tau^2 V^{S(a)}(\tau)$$

2. The vector

$$\begin{pmatrix} V^1(\tau) \\ \vdots \\ V^{|\mathcal{A}|}(\tau) \end{pmatrix}$$

   is a *vector-valued modular form of weight 2*.

3. The permutations S and T define a *permutation representation* of $\mathrm{PSL}(2,\mathbb{Z})$.

Elliptic Calogero-Moser Potential

Solution for $\mathfrak{g} = A_{N-1}$:

$$\boxed{\textbf{z} \text{ is an isolated equilibrium iff it lies on a sublattice of order } N \text{ of } \tfrac{1}{N}\mathbb{Z} + \tfrac{\tau}{N}\mathbb{Z}.}$$

These are parametrized by pairs (d, k) where $d|N$ and $0 \leq k < d$. So

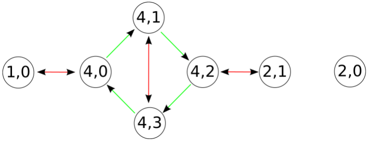
$$|\mathcal{A}| = \sum_{d|N} d\,.$$

Actually,

$$\mathcal{A} \cong \mathrm{SL}(2, \mathbb{Z})/\Gamma^0(N)$$

and

$$V^{(d,k)}(\tau) = E_2(\tau) - \frac{N}{d^2} E_2\left(\frac{N\tau + kd}{d^2}\right)\,.$$



What about $\mathfrak{g} = D_N$?

The $\mathcal{N} = 1^*$ gauge theories

Consider $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{g}$, and break supersymmetry to $\mathcal{N} = 1^*$ by giving a mass m to all three chiral multiplets.

The Calogero-Moser system arises as the complex integrable system associated to the theory on $\mathbb{R}^{1,2} \times S^1$. (This can be deduced from the *class S* realization).

The Calogero-Moser potential is the *non-perturbative superpotential* of the gauge theory. So there is a one-to-one correspondence between

1. Isolated extrema of the Calogero-Moser Hamiltonian (counted with appropriate multiplicity)
2. Massive (gapped) vacua of the $\mathcal{N} = 1^*$ gauge theories

Using the correspondence and *nilpotent orbit* theory, one can get their number.

The $\mathcal{N} = 1^*$ gauge theories

Semiclassical analysis:

$$\mathcal{W} \sim \mathrm{Tr}(\Phi_1[\Phi_2, \Phi_3])$$

Vacua correspond to embeddings

$$\mathfrak{sl}(2) \to \mathfrak{g}\,.$$

These correspond to *nilpotent orbits* in $\mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{sl}(N)$ these are classified by partitions of N. The residual gauge group is non-abelian only for *rectangular* partitions $[(N/d)^d]$. One deduces

$$|\mathcal{A}| = \sum_{d|N} d\,.$$

Type D_4 Calogero-Moser

Turn to $\mathfrak{g} = \mathfrak{so}(8)$.

1. There exist *permutations* $T, S : \mathcal{A} \to \mathcal{A}$ such that

$$V^a(\tau+1) = V^{T(a)}(\tau)\,, \qquad V^a(-1/\tau) = \tau^2 V^{S(a)}(\tau)$$

2. The vector

$$\begin{pmatrix} V^1(\tau) \\ \vdots \\ V^{|\mathcal{A}|}(\tau) \end{pmatrix}$$

is NOT a *vector-valued modular form of weight 2*.

3. The permutations S and T DO NOT define a *permutation representation* of $\mathrm{PSL}(2,\mathbb{Z})$.

Type D_4 Calogero-Moser

Strategy:

1. Find approximate extrema numerically.
2. Exploit the fact that

$$\sum_{a\in\mathcal{A}} (V^a(\tau))^k \in \mathcal{M}_{2k}(\mathrm{SL}(2,\mathbb{Z})) \qquad \text{and} \qquad \dim \mathcal{M}_{2k}(\mathrm{SL}(2,\mathbb{Z})) \leq 1 + \frac{k}{6}$$

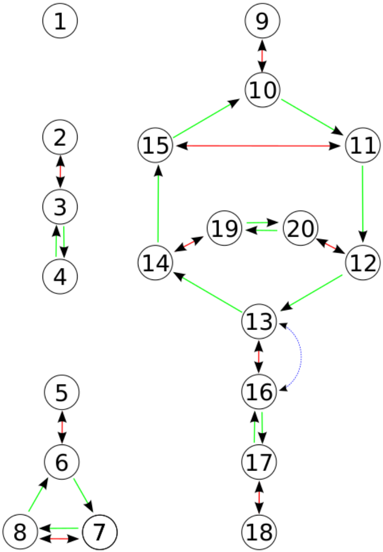
   to find the exact polynomial

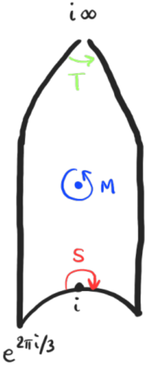
$$P(v|\tau) = \prod_{a\in\mathcal{A}} (v - V^a(\tau)) \in \mathbb{Z}[E_4(\tau), E_6(\tau)][v]$$

3. Study how the roots of P are permuted as τ moves in the complement of the discriminant locus Δ:

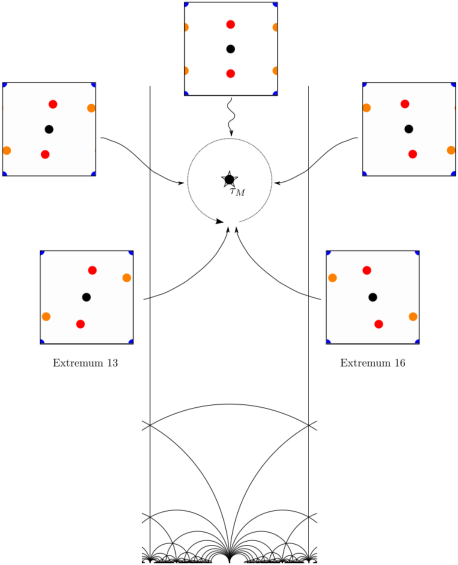
$$\pi_1(\mathcal{H}\backslash\Delta) \to \mathrm{Bij}(\mathcal{A})\,.$$

Type D_4 Calogero-Moser





Type D_4 Calogero-Moser



Type D_4 Calogero-Moser

Phase transition at critical value

$$\tau_M = i\,\frac{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{1}{2}+\frac{2761}{992\sqrt{31}}\right)}{{}_2F_1\left(\frac{1}{6},\frac{5}{6};1;\frac{1}{2}-\frac{2761}{992\sqrt{31}}\right)} \approx 2.4155769875\ldots i\,.$$

such that

$$j(\tau_M) = \frac{488095744}{125}.$$

This messes up with everything:

- Now $(ST)^3 \neq 1$, but instead $(STM)^3 = 1$. New kind of modularity.
- The q-expansions have finite radius of convergence $\exp(2\pi i\tau_M) < 1$.

Conclusion

Many open questions for a problem which should be elementary!

- What happens for other $\mathfrak{g}$?
- What is the explicit correspondence between extrema and massive vacua?
- What happens in the gauge theory at critical couplings?
- What about non-simply laced algebras?

Thank you for your attention!
And Happy Birthday Philippe!