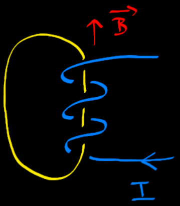
Théorie des noeuds, Topologie et Physique Quantique

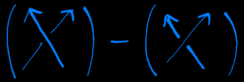
$$\langle \gamma_1 , \gamma_2 \rangle = \int_{\gamma_2} A_1$$

$$\int \mathcal{D}A \, e^{ik\, CS(A)}$$

$$F = dA + A \wedge A$$

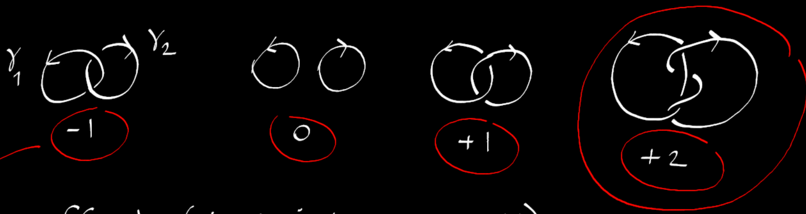
$$F \wedge F = d\left(A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A\right)$$





(I) Intro

Histoire : Gauss (1833)



$$\langle \gamma_1, \gamma_2 \rangle = \frac{1}{4\pi} \iint_{S^1 \times S^1} \frac{\det\left(\dot\gamma_1(t), \dot\gamma_2(s), \gamma_1(t) - \gamma_2(s)\right)}{\left|\gamma_1(t) - \gamma_2(s)\right|^3} \, ds \, dt$$

$\gamma_1 : S^1 \longrightarrow \mathbb{R}^3$

$\gamma_2 : S^1 \longrightarrow \mathbb{R}^3$

Listing : "topologie"

Mendeliev : Thomson (Kelvin) Tait Helmholtz

$0 \to 1$
$1 \to 0$
$2 \to 0$
$3 \to 1$
$4 \to 1$
$5 \to 2$
$6 \to 3$

$7 \to 7$
$8 \to 21$
$9 \to 49$
$10 \to 165$
$11 \to 552$
$\vdots$

$18 \to 48\,266\,466$

non premier

$J\Big($  $\Big)$ = nombre
polynôme
...

1984

$\tilde{V}\Big($  $\Big) = a^{-4} + a^{-12} - a^{-16}$

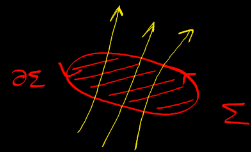
Atiyah : Pourquoi ? Physique ? → médaille Fields

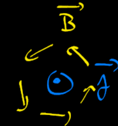
1988 : Witten : $\boxed{\text{OUI}}$ en physique quantique → Chern Simons

→ Champ de recherche en physique
en maths

(II) Électromagnétisme

$\vec{E}$, $\vec{B}$

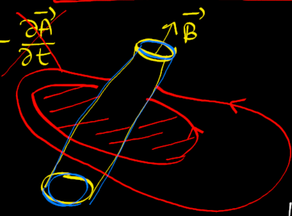




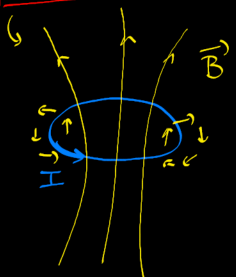
$$\mathrm{div}\,\vec{E} = \rho/\varepsilon_0$$

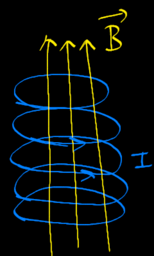
$$\mathrm{div}\,\vec{B} = 0 \longrightarrow \boxed{\vec{B} = \vec{\mathrm{rot}}\,\vec{A}} \iff \boxed{\overbrace{\iint_\Sigma \vec{B}\cdot\vec{dS}}^{\Phi} = \int_{\partial\Sigma} \vec{A}\cdot\vec{dl}}$$

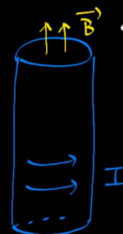
$$\vec{\mathrm{rot}}\,\vec{E} = -\cancel{\frac{\partial\vec{B}}{\partial t}} \longrightarrow \vec{E} = -\vec{\mathrm{grad}}\,V - \cancel{\frac{\partial\vec{A}}{\partial t}}$$



$$\boxed{\vec{\mathrm{rot}}\,\vec{B} = \mu_0\vec{j}} + \cancel{\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}}$$

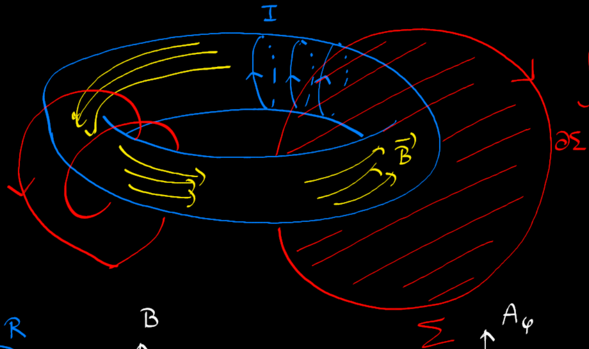






$\longleftarrow \boxed{\vec{B} = \mu_0\, n\, I\, \vec{e_z}}$

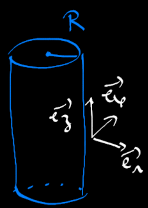
$\longleftarrow \vec{B} = \vec{0}$

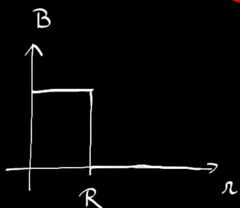


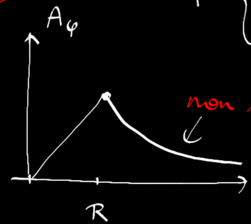
$$\int \vec{A} \cdot \vec{d\ell}$$

$$\vec{A} = A_\varphi \, \vec{e_\varphi}$$

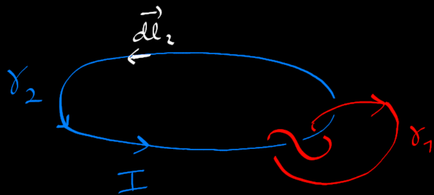
$$A_\varphi = \begin{cases} \dfrac{BR^2}{2r} & \text{si } r \geqslant R \\[2ex] \dfrac{Br}{2} & \text{si } r \leqslant R \end{cases}$$







non nul !



$$\oint_{\gamma_1} \underbrace{\vec{B}}_{\vec{B}(\vec{r})} \cdot \vec{dl} = \mu_0 I \langle \gamma_1, \gamma_2 \rangle$$

$$d\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \cdot \frac{\vec{dl}_2 \wedge \vec{r}}{r^3}$$ Biot - Savart

$$\vec{B}(\vec{r}) = \oint_{\gamma_2} \frac{\mu_0 I}{4\pi} \frac{\vec{dl}_2 \wedge \vec{r}}{r^3}$$

$$(\vec{A} \wedge \vec{B}) \cdot \vec{C} = \begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix}$$

$$\oint_{\gamma_1} \oint_{\gamma_2} \frac{\mu_0 I}{4\pi} \left(\frac{\vec{dl}_2 \wedge (\vec{r}_1 - \vec{r}_2)}{\| \vec{r}_1 - \vec{r}_2 \|^3} \right) \cdot \vec{dl}_1 = \mu_0 I \langle \gamma_1, \gamma_2 \rangle$$

Gauss (1833)

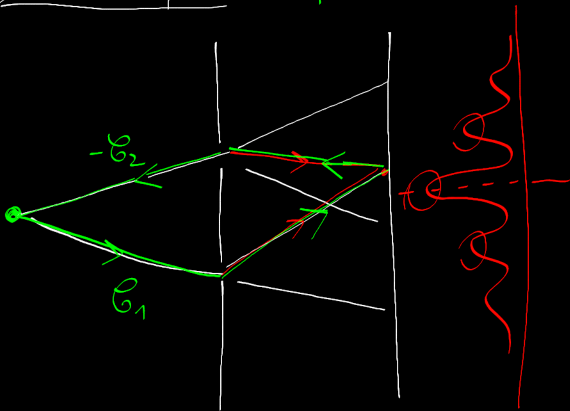
$$\boxed{\langle \gamma_1, \gamma_2 \rangle = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\det(\vec{dl}_1, \vec{dl}_2, \vec{r}_1 - \vec{r}_2)}{\| \vec{r}_1 - \vec{r}_2 \|^3}}$$

$$S = \int \mathcal{L}\, dt$$

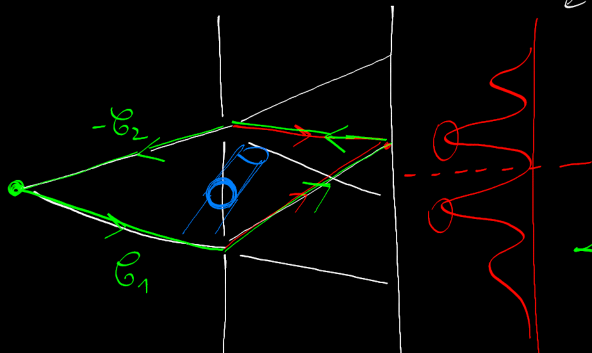
III Quantique

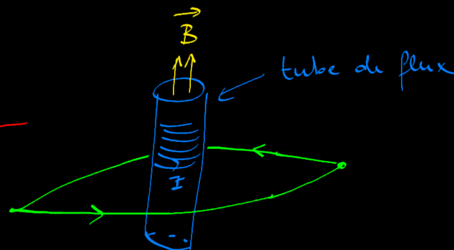
$\mathcal{C}_1 - \mathcal{C}_2 =$ noeud.

$$\sum_{\text{chemins}} e^{iS/\hbar} \sim e^{ikL_1} + e^{ikL_2}$$



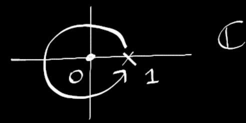
$$\sum_{\text{chemins}} e^{iS/\hbar} \sim e^{ikL_1 + q\int_{\mathcal{C}_1} \vec{A}\cdot\vec{dl}} + e^{ikL_2 + q\int_{\mathcal{C}_2} \vec{A}\cdot\vec{dl}}$$





$\vec{A}$ est défini à un gradient près, pas de réalité physique.

$$\oint \vec{A}\cdot d\vec{\ell} \longrightarrow \oint \vec{A}\cdot d\vec{\ell} + \underbrace{\oint \vec{\mathrm{grad}}\,\phi \cdot d\vec{\ell}}_{0} \qquad e^{2\pi i} = 1 = e^{0}$$



$$\boxed{\oint_{\mathcal{C}} \frac{dz}{z} = 2\pi i} = \oint_{\mathcal{C}} d(\log z)$$

$$\mathcal{L} = E_c - E_p = \frac{1}{2} m \vec{v}^{\,2} + \underset{\text{charge}}{q}\left(-\cancel{V} + \vec{A}\cdot\vec{v}\right)$$

$$S = \int \mathcal{L}\, dt = \int \frac{1}{2} m \vec{v}^{\,2}\, dt + q \int \vec{A}\cdot d\vec{\ell}$$

Différence de phase : $k(L_1 - L_2) + q\int_{\mathcal{C}_1} \vec{A}\cdot d\vec{\ell} - q\int_{\mathcal{C}_2} \vec{A}\cdot d\vec{\ell}$

$$= k(L_1 - L_2) + q \oint \vec{A}\cdot d\vec{\ell}$$

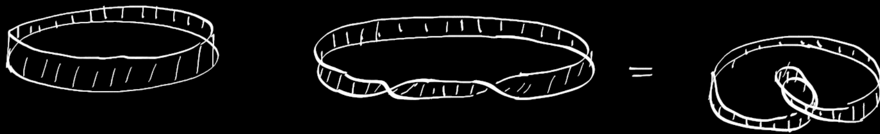
$$\exp\left[i\frac{S}{\hbar}\right] \qquad \exp\left[i\frac{q}{\hbar}\oint \vec{A}\cdot d\vec{\ell}\right] = \exp\left[2\pi i \left(\frac{q}{2\pi\hbar}\oint \vec{A}\cdot d\vec{\ell}\right)\right]$$

Le décalage observable est $\boxed{\dfrac{q\Phi}{2\pi\hbar} \text{ modulo } 1}$

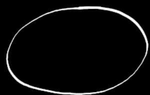
pas détectable $\Longleftrightarrow$ $\boxed{\dfrac{\text{“électrique”} \times \text{“magnétique”}}{h} \in \mathbb{Z}}$

$$\text{electrique} \in \frac{h}{\text{magn}} \mathbb{Z}$$

IV Entortillement



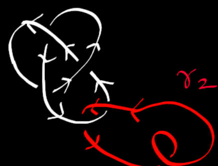
noeud trivial   noeud trivial

$w=0$  $\underset{\text{noeuds}}{\overset{\text{rubans.}}{\neq \atop =}}$  $w=+1$  $w=+3$

$w(\gamma) = \#\big($  $\big) - \#\big($  $\big)$

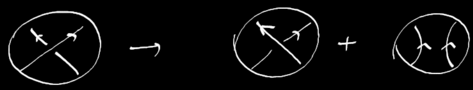
Je prétends que

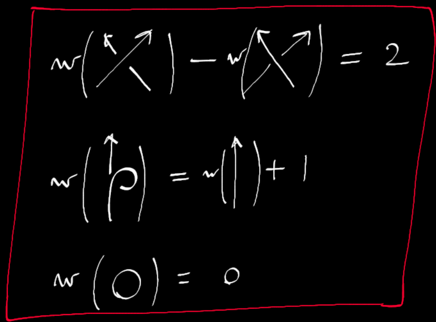
$$w(\gamma) = \langle \gamma, \gamma \rangle$$



$\boxed{w = 2}$

$$\begin{aligned} w(\gamma_1+\gamma_2) &= \langle \gamma_1+\gamma_2, \gamma_1+\gamma_2 \rangle \\ &= \langle \gamma_1,\gamma_1\rangle + \langle \gamma_2,\gamma_2\rangle + 2\langle \gamma_1,\gamma_2\rangle \\ &= \underset{3}{w(\gamma_1)} + \underset{1}{w(\gamma_2)} + 2\underbrace{\langle \gamma_1,\gamma_2\rangle}_{-1} \end{aligned}$$

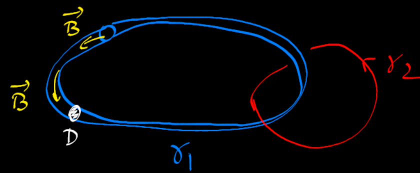




$B = dA$

1-forme 2-forme

$$w(\gamma) = \iiint_{\mathbb{R}^3} A \wedge B$$



<u>Idée de preuve</u> Si $\gamma = \gamma_1 + \gamma_2 + \dots$

$$w(\gamma) = w(\gamma_1 + \gamma_2 + \dots) = \langle \gamma_1 + \gamma_2 + \dots, \gamma_1 + \gamma_2 + \dots \rangle$$

$$= w(\gamma_1) + w(\gamma_2) + \dots + 2 \underbrace{\langle \gamma_1, \gamma_2 \rangle} + \dots$$

$$\tilde{\gamma}_1 : \underline{S}^1 \times D \longrightarrow \mathbb{R}^3$$
$$(t, r, \theta) \longrightarrow \dots$$

$$B_1 = f(r,\theta)\, r\, dr \wedge d\theta$$

$$\int_0^1 \int_0^{2\pi} B_1 = 1$$

$$\iiint_{\mathbb{R}^3} A_2 \wedge B_1 = \int_{t=0}^{1} \int_{r=0}^{1} \int_{\theta=0}^{2\pi} (A_2)_t\, dt \wedge f(r,\theta)\, r\, dr \wedge d\theta$$

$$= \int_{t=0}^{1} (A_2)_t\, dt = \langle \gamma_1, \gamma_2 \rangle$$

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

relat

$$A = (V, \vec{A})$$

relat

$$\boxed{F = dA}$$

$$\boxed{w(\gamma) = \int_{\mathbb{R}^3} A \wedge dA}$$

forme de Chern-Simons

(V) Chern - Simons Abélien

$$U(1) = \{ e^{i\theta} \,/\, \theta \in \mathbb{R} \}$$

Théorie de jauge $U(1)$ sur $\boxed{\mathbb{R}^3}$ $\leftarrow$ $\boxed{\text{2 dimensions d'espace, 1 dimension de temps}}$

1 - forme $\quad A = A_\mu \, dx^\mu$

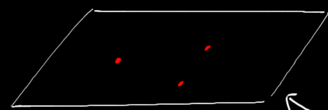
2 - forme $\quad F = dA \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$F = F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

3dim :

$$A = (\underbrace{A_0}_{V}, \underbrace{A_1, A_2}_{\vec{A}})$$

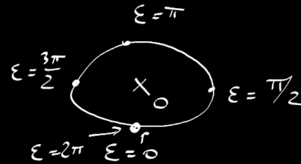
$$F = \begin{pmatrix} 0 & E_x & E_y \\ & 0 & B \\ & & 0 \end{pmatrix}$$



espace

(condition : $\displaystyle\int_{\Sigma_2} F \in 2\pi\mathbb{Z}$ pour tout 2-cycle Σ_2 fermé orienté)

Transfo de jauge : $\boxed{g : \mathbb{R}^3 \setminus \{\dots\} \longrightarrow U(1)}$



• Petite transformation : $g(x) = e^{i\varepsilon(x)}$.

$$g^{-1}dg = e^{-i\varepsilon}\, i d\varepsilon\, e^{i\varepsilon} = i\, d\varepsilon$$

$$A \longrightarrow A + d\varepsilon$$

$$\oint A \longrightarrow \oint A + \oint d\varepsilon = \oint A$$

$$F = dA \longrightarrow F = dA + dd\varepsilon = F$$

• Grandes transformations : on ne peut pas trouver de $\varepsilon(x)$ globalement.

$$A \longrightarrow A - i g^{-1}dg = A + \omega$$

$$\oint A \longrightarrow \oint A + \oint \omega$$

$$F \longrightarrow F + d\omega$$

2-forme n-2

$$S_{YM} = \# \int_{\mathbb{R}^n} dA \wedge {*dA}$$

1-forme 2forme

$$\boxed{S_{CS}[A] = \frac{k}{2\pi} \int_{\mathbb{R}^3} A \wedge dA}$$

Jauge : $A \wedge dA \longrightarrow A \wedge dA + \omega \wedge F$

$$S_{cs}[A] \longrightarrow S_{cs}[A] + \frac{k}{2\pi}\int_{\mathbb{R}^3} \omega \wedge F$$

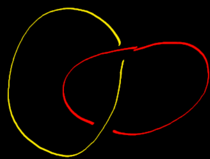
$$\int e^{i S_{cs}[A]} \mathcal{D}A = \text{intégrale de chemin}$$

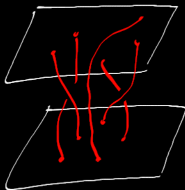
$$\exp\left[i \frac{k}{2\pi} \int \omega \wedge F\right] = \exp\left[2\pi i \underset{\in \mathbb{Z}}{k} \underbrace{\int \frac{\omega}{2\pi} \wedge \frac{F}{2\pi}}_{\mathbb{Z}}\right] = 1$$

Condition $\boxed{k \in \mathbb{Z}}$

$$\boxed{\begin{gathered}\int \mathcal{D}A \, e^{i S_{cs}[A]} W(\gamma_1) W(\gamma_2) = \exp\left(i \frac{\pi}{k} \langle \gamma_1, \gamma_2 \rangle\right) \\ W(\gamma_1) = \exp\left(i \oint_{\gamma_1} A\right)\end{gathered}}$$

$$\mathcal{L} = \frac{\kappa}{2\pi} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho - J^\mu A_\mu \qquad \underline{J^\mu} = \frac{\kappa}{\pi} \varepsilon^{\mu\nu\rho} \underline{\partial_\nu A_\rho}$$





<u>Espace des configurations</u> de n particules dans $\mathbb{R}^d$.

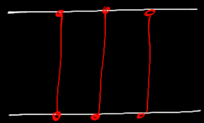
$$\mathrm{Conf}_n(\mathbb{R}^d) = \{ (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \;/\; x_i \neq x_j \text{ pour } i \neq j \}$$

$$\pi_1\left(\mathrm{Conf}_n(\mathbb{R}^d) / S_n \right) = \begin{cases} \{1\} & \text{si } d = 1 \\ \mathrm{Tresses}_n & \text{si } d = 2 \\ S_n & \text{si } d \geq 3 \end{cases}$$

si $d = 2$ → phase $e^{i\theta}$ ANYONS

si $d \geq 3$ → triviale BOSONS

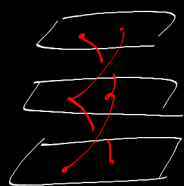
si $d \geq 3$ → signature FERMIONS



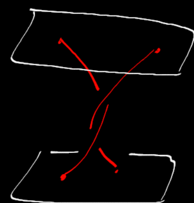
Statistique $\longleftrightarrow$ Représentations de dimension 1 de $\pi_1\left(\mathrm{Conf}/S_n\right)$

Dans S_n, $\quad (12)(12) = 1$

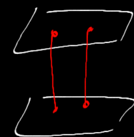
Dans tresses :



$=$



$\neq 1$



VI Chern-Simons

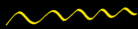
Chern Weil (1940...) $\longrightarrow$ Chern Simons (1974) $\longrightarrow$ A. Schwarz (1978) (TQFT)

$$d\,CS = F \wedge F$$

Yang Mills (1954)

G = groupe de Lie simple, connexe, simplement connexe

$$S_{YM} = -\frac{1}{2g^2}\underbrace{\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge * F)} + \underbrace{\frac{\theta}{16\pi^2}\int \mathrm{tr}(F \wedge F)}$$



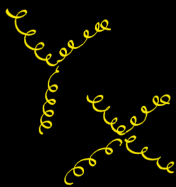
Yang-Mills

très compliqué

dépend de la métrique

$\Updownarrow$?

décrit le modèle standard



très simple.

topologique.

pas de dépendance en la métrique

$$\boxed{\mathrm{tr}(F \wedge F) = d\left(\mathrm{tr}\left(A \wedge dA + \frac{2}{3} A^3\right)\right)}$$

$$\int_{\mathbb{R}^4} \mathrm{tr}(F \wedge F) = \int_{\mathbb{R}^4} d(\ldots) = \int_{S^3} \mathrm{tr}\left(A\, dA + \frac{2}{3}A^3\right)$$

Théorème (Witten 1988)

Soit G groupe (...). Soit R une représentation de G.

Soit γ un noeud.

$$S_{cs}[A] = \frac{1}{4\pi}\int_{\mathbb{R}^3} \mathrm{tr}\left(A\,dA + \frac{2}{3}A^3\right).$$

$$W_R(\gamma) = \mathrm{Tr}_R\left(\mathrm{Pexp}\left(\oint A\right)\right).$$

Alors pour tout $k \in \mathbb{Z}$

$$\boxed{Z_k(\gamma, R) = \frac{1}{\mathrm{vol}(U)}\int_U \mathcal{D}A\; e^{ikS_{cs}[A]}\, W_R(\gamma)}$$

est un invariant de noeud.

Exemples :

- $G = U(1)$, on retrouve $w(\gamma)$
- $G = SU(2)$ et $R =$ fondamentale,
  on trouve $\widetilde{V}_\gamma\left(\exp\left(\frac{2\pi i}{k+2}\right)\right)$
- $G = SU(N)$ et $R =$ fondamentale, HOMFLY.
- $G = SO(N)$ et $R =$ fondamentale, Kauffman
- etc...