AE3-414 Computational Fluid Dynamics

Coursework Report

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Question 1

The one-dimensional advection-diffusion equation we study in this coursework is linear :

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \alpha\frac{\partial^2 u}{\partial^2 x} \tag{1.1}$$

The exact solution being $u(x,t) = \sum_i e^{I\gamma_i(x-at)}e^{\delta_i t}$, we can thus determine a relationship between γ_i and δ_i by substituting into the equation (1.1) only the i-th term of the exact solution.
We get :

$$\frac{\partial e^{I\gamma_i(x-at)}e^{\delta_i t}}{\partial t} + a\frac{\partial e^{I\gamma_i(x-at)}e^{\delta_i t}}{\partial x} = \alpha\frac{\partial^2 e^{I\gamma_i(x-at)}e^{\delta_i t}}{\partial^2 x}$$

$$(-aI\gamma_i + \delta_i)e^{I\gamma_i(x-at)}e^{\delta_i t} + aI\gamma_i e^{I\gamma_i(x-at)}e^{\delta_i t} = -\alpha{\gamma_i}^2 e^{I\gamma_i(x-at)}e^{\delta_i t}$$

And by dividing by $e^{I\gamma_i(x-at)}e^{\delta_i t}$, which is different from zero we get :

$$\delta_i = -\alpha\gamma_i^2 \tag{1.2}$$

Question 2

Using an implicit formulation of the equation (1.1) centred in space for both the advection and diffusion terms and with a first order in time we get :

$$\frac{{u_i}^{n+1} - {u_i}^n}{\Delta t} + a\frac{{u_{i+1}}^{n+1} - {u_{i-1}}^{n+1}}{2\Delta x} = \alpha\frac{{u_{i+1}}^{n+1} - 2{u_i}^{n+1} + {u_{i-1}}^{n+1}}{\Delta x^2} \tag{2.1}$$

To study if this scheme is stable, and since the equation (1.1) is linear, we perform a Von-Neumann analysis. By considering only the m-th mode ${u_i}^n = \hat{u}_m^n e^{Ii\phi_m}$ we get from (2.1) :

$$\hat{u}_m^{n+1} - \hat{u}_m^n + \sigma I sin(\phi_m)\hat{u}_m^{n+1} = 2\beta\hat{u}_m^{n+1}(cos(\phi_m) - 1)$$

With $\sigma = a\frac{\Delta t}{\Delta x}$ and $\beta = \alpha\frac{\Delta t}{\Delta x^2}$. Thus we get :

$$G = \frac{\hat{u}_m^{n+1}}{\hat{u}_m^n} = \frac{1}{1 + 2\beta(1 - cos(\phi_m)) + I\sigma sin(\phi_m)} \tag{2.2}$$

$$\mid G \mid = \frac{1}{\sqrt{\left(1 + 2\beta(1 - cos(\phi_m))\right)^2 + \left(\sigma sin(\phi_m)\right)^2}}$$

Since the denominator is always greater than 1 we can conclude that we have for all modes $\mid G \mid \leq 1$ and thus that this scheme is unconditionally stable.

Question 3

We can now study the analytic amplification factor $\widetilde{G}$. Let $u(x,t) = \hat{v}_m^n(t)e^{Ik_m x}$ and substitute this expression in the equation (1.1) :

$$\frac{d\hat{v}_m(t)}{dt}e^{Ik_m x} + Iak_m\hat{v}_m(t)e^{Ik_m x} = -\alpha {k_m}^2\hat{v}_m(t)e^{Ik_m x}$$

Which gives us :

$$\frac{d\hat{v}_m(t)}{dt} = -(\alpha {k_m}^2 + Iak_m)\hat{v}_m(t)$$

Which means that :

$$\hat{v}_m(t) = A_m e^{-(\alpha {k_m}^2 + Iak_m)t}$$

$$\widetilde{G} = \frac{\hat{v}_m(t^{n+1})}{\hat{v}_m(t^n)} = \frac{e^{-(\alpha {k_m}^2 + Iak_m)(n+1)\Delta t}}{e^{-(\alpha {k_m}^2 + Iak_m)n\Delta t}} = e^{-(\alpha {k_m}^2 + Iak_m)\Delta t}$$

And by using the fact that $k_m = \phi\Delta x$ we finally get :

$$\widetilde{G} = e^{-\sigma\phi I}e^{-\beta\phi^2} \tag{3.1}$$

Question 4

In order to study how this numerical scheme behaves compared to the analytic one, we will plot the diffusion and dispersion errors over the frequency range ϕ [0:2π] for different values of σ and β. Compared to the simple advection equation, we are here studying numerically two physical phenomenon (advection and diffusion) with each their speed and order of magnitude. We thus need to find a balance between β and σ (which depends of Δx and Δt once a and α are fixed) to satisfy both phenomenon and save computational time.

Here are the analytic expressions of ϵ_D and ϵ_Φ :

$\epsilon_D = \frac{\mid G \mid}{\mid \widetilde{G} \mid} = \frac{e^{\beta\phi^2}}{\sqrt{(1+2\beta(1-cos(\phi_m)))^2 + (\sigma sin(\phi_m))^2}}$ and $\epsilon_\Phi = \frac{\Phi}{\widetilde{\Phi}} = \frac{atan(\frac{\sigma sin(\phi)}{1+2\beta(1-cos(\phi))})}{\sigma\phi}$

The two quantities plotted are shown figure 4.2 and 4.1 :

We can first see that in both cases we have a large error in diffusion at high frequencies ($\phi \longrightarrow \pi$), especially in the case $\sigma = 0.2$ and β varying. This will cause a smearing of the numerical solution compared to the analytic one.

For both series of σ and β values we can see that the dispersion error grows as the frequencies increases, which will cause wriggles on the numerical solution for abrupt changes in the real solution.

For a fixed value of beta, we can see that the higher the value of σ, the larger the error (in both dispersion and diffusion). For high sigma values,the diffusion error even exists at low

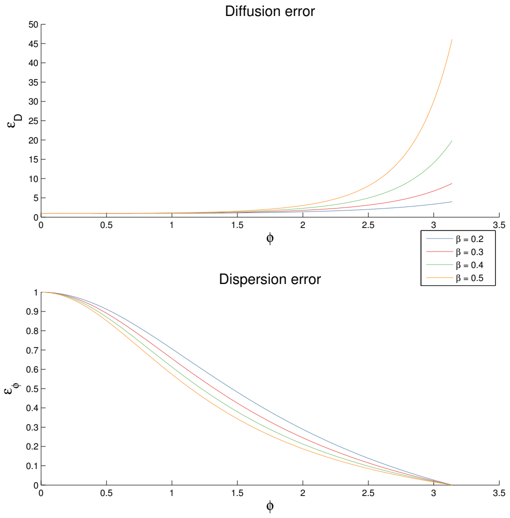

Figure 4.1: Evolution of phase and dispersion error for β varying and $\sigma = 0.2$

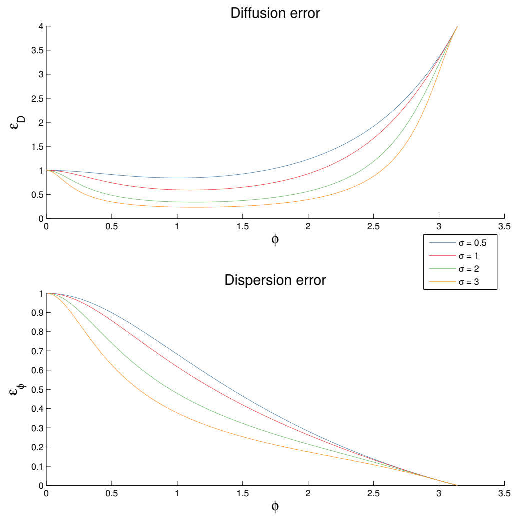

Figure 4.2: Evolution of phase and dispersion error for σ varying and $\beta = 0.2$

frequencies ($\phi \longrightarrow 0$), which means the numerical solution will not really look like the real solution. We can also note that since we are not studying an advection only equation, the diffusion error for the Courant Number $\sigma = 1$ is not error-less.

From those plot we can say that the best combination of parameters shown here is $\sigma = 0.5$ and $\beta = 0.2$.

Question 5

We now numerically solve the equation (1.1) using an explicit scheme.
When using a centred scheme for the advection term we have :

$$\frac{{u_i}^{n+1} - {u_i}^n}{\Delta t} + a\frac{{u_{i+1}}^n - {u_{i-1}}^n}{2\Delta x} = \alpha\frac{{u_{i+1}}^n - 2{u_i}^n + {u_{i-1}}^n}{\Delta x^2} \tag{5.1}$$

When using an upwind first order scheme for the advection term we get :

$$\frac{{u_i}^{n+1} - {u_i}^n}{\Delta t} + a\frac{{u_i}^n - {u_{i-1}}^n}{\Delta x} = \alpha\frac{{u_{i+1}}^n - 2{u_i}^n + {u_{i-1}}^n}{\Delta x^2} \tag{5.2}$$

We can in both case rearrange the terms to get ${u_i}^{n+1}$ function of ${u_i}^n$, ${u_{i+1}}^n$ and ${u_{i-1}}^n$. We then see we can represent the relation between $\mathbf{U}^n$ and $\mathbf{U}^{n+1}$ using a matrix $\mathbf{C}$:

$$\mathbf{U}^{n+1} = \mathbf{C}\mathbf{U}^n = \mathbf{C}^n\mathbf{U}^0$$

Taking into account the Dirichlet condition $u(0,t) = 0$ on the $x = 0$ boundary and the ${u_N}^{n+1} = {u_{N-1}}^n$ outflow boundary condition at $x = 1$ we have :

- for the centred advection term scheme (case 1) :

$$\begin{bmatrix} {u_0}^{n+1} \\ {u_1}^{n+1} \\ \vdots \\ \vdots \\ {u_{N-1}}^{n+1} \\ {u_N}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \beta+\frac{\sigma}{2} & 1-2\beta & \beta-\frac{\sigma}{2} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \beta+\frac{\sigma}{2} & 1-2\beta & \beta-\frac{\sigma}{2} \\ 0 & \dots & \dots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} {u_0}^{n} \\ {u_1}^{n} \\ \vdots \\ \vdots \\ {u_{N-1}}^{n} \\ {u_N}^{n} \end{bmatrix}$$

- for the upwind advection term scheme (case 2) :

$$\begin{bmatrix} {u_0}^{n+1} \\ {u_1}^{n+1} \\ \vdots \\ \vdots \\ {u_{N-1}}^{n+1} \\ {u_N}^{n+1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ \beta+\sigma & 1-2\beta-\sigma & \beta & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \beta+\sigma & 1-2\beta-\sigma & \beta \\ 0 & \dots & \dots & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} {u_0}^{n} \\ {u_1}^{n} \\ \vdots \\ \vdots \\ {u_{N-1}}^{n} \\ {u_N}^{n} \end{bmatrix}$$

Here is what the two numerical solutions look like at a final time of $T = 0.5$:

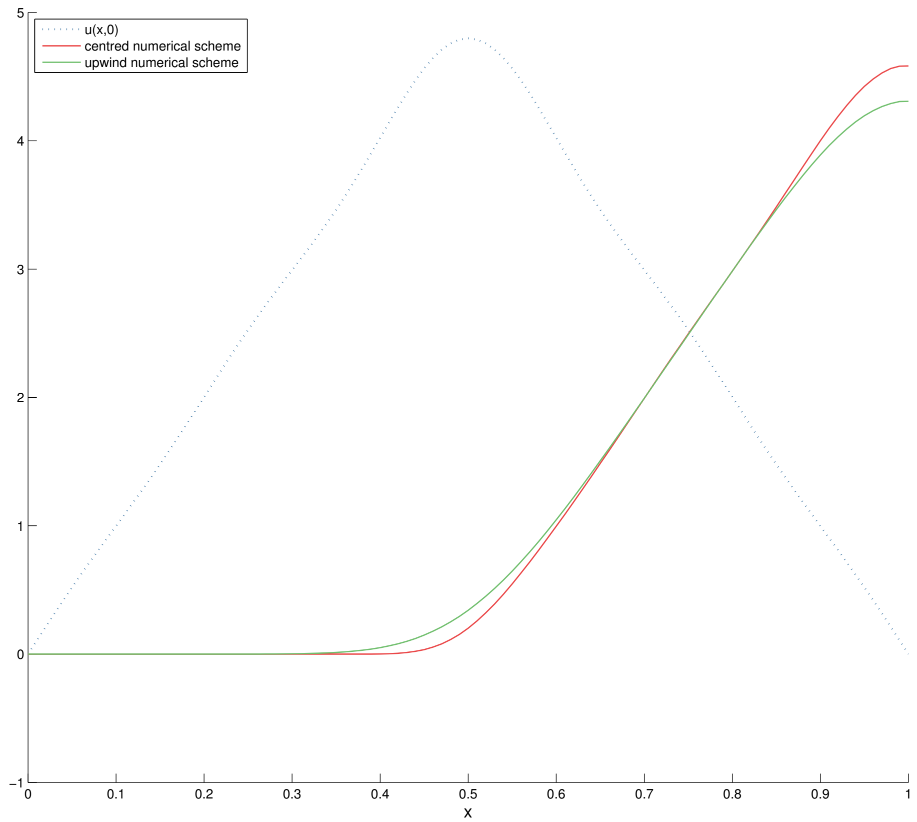

Figure 5.1: Numerical solutions of equation (1.1) using two different schemes for the advection term

Both schemes seem stable for the given σ and β values. First we can notice that at time T, we have a solution that has moved to the right of its initial position, which seems logical since the advection speed a is positive.

The right-hand side boundary condition seems suitable to represent a flow going out of our domain of observation $0 < x < 1$ since we know from $a > 0$ that the flow is moving to the right. It then seems logical to give a numerical continuity of the flow in $x = 1$ using the value on the grid point before the boundary.

We can see that the two numerical solutions are a bit different in term of magnitude (max of ~ 4.6 for the centred scheme and max of ~ 4.3 for the upwind scheme). We can also see in the vicinity of $x = 0.5$ that the upwind scheme seems to have left more "matter" behind than the centred one through diffusion during the propagation to the right. Those two facts help us to conclude that the upwind numerical scheme seems more respondent to diffusion properties than the centred one.

When $\alpha = 0$ in (1.1), we retrieve the simple advection equation. With the given values of a, Δt and Δx we have a Courant number of $\sigma = 0.1 \leq 1$. We saw in the lectures that for the linear advection equation, the only scheme stable with $\sigma \leq 1$ is the equation with the upwind advection scheme (CFL condition).