Imperial College London

AE3-414 Computational Fluid Dynamics

Coursework Report

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February 1, 2016

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Question 1

We apply the two-step Lax-Friedrichs scheme for the linear advection equation $u_t + au_x = 0$. We want to perform a Von Neumann stability analysis of this scheme. One step of the scheme is:

$$u_i^{n+1} = \frac{1}{2}(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} + u_{i-\frac{1}{2}}^{n+\frac{1}{2}}) - \frac{\Delta t}{2\Delta x}(F_{i+\frac{1}{2}}^{n+\frac{1}{2}} - F_{i+\frac{1}{2}}^{n-\frac{1}{2}})$$

With the fact that for the linear advection equation $F_i^n = au_i^n$ this gives:

$$u_i^{n+1} = \frac{1}{2}(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} + u_{i-\frac{1}{2}}^{n+\frac{1}{2}}) - a\frac{\Delta t}{2\Delta x}(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} - u_{i+\frac{1}{2}}^{n-\frac{1}{2}}) \tag{1.1}$$

Let u_i^n for the m^{th} mode be $u_i^n = \hat{u}_m^n e^{I i \Phi_m}$. If we inject that in the (equation 1.1) and rearrange the terms this gives:

$$H = \frac{u_i^{n+1}}{u_i^{n+\frac{1}{2}}} = cos(\frac{\Phi_m}{2}) - Ia\frac{\Delta t}{\Delta x}sin(\frac{\Phi_m}{2})$$

Noting that the amplification factor G can be written:

$$G = \frac{u_i^{n+1}}{u_i^n} = \frac{u_i^{n+1}}{u_i^{n+\frac{1}{2}}}\frac{u_i^{n+\frac{1}{2}}}{u_i^n} = H^2 = [cos(\frac{\Phi_m}{2}) - Ia\frac{\Delta t}{\Delta x}sin(\frac{\Phi_m}{2})]^2$$

We then have:

$$|G| = |H^2| = |H|^2 = [cos(\frac{\Phi_m}{2})^2 + (a\frac{\Delta t}{\Delta x}sin(\frac{\Phi_m}{2}))^2]^2 \tag{1.2}$$

For this scheme to be stable, we must ensure that for every value of Φ_m we have $|G| \leq 1$. From the (equation 1.2) this means that:

$$cos(\frac{\Phi_m}{2})^2 + [\sigma sin(\frac{\Phi_m}{2})]^2 \leq 1 \tag{1.3}$$

With $\sigma = a\frac{\Delta t}{\Delta x}$. Since we have $cos(\frac{\Phi_m}{2})^2 + sin(\frac{\Phi_m}{2})^2 = 1$ we have by subtracting this equation to (equation 1.3) :

$$(\sigma^2 - 1)sin(\frac{\Phi_m}{2})^2 \leq 0 \tag{1.4}$$

Thus, this scheme is stable if we have $\sigma \leq 1$.

The amplification factor was plotted for different values of σ against the phase angle on (figure 1.1). From this figure we can clearly see that for a value of $\sigma \geq 1$ the scheme would not be stable. From the values of $\sigma \leq 1$ we can conclude that the higher frequencies are damped by this scheme. Finally, with a value of $\sigma = 1$ this scheme, we can see that the scheme is not distorting the frequencies throughout the iterations.

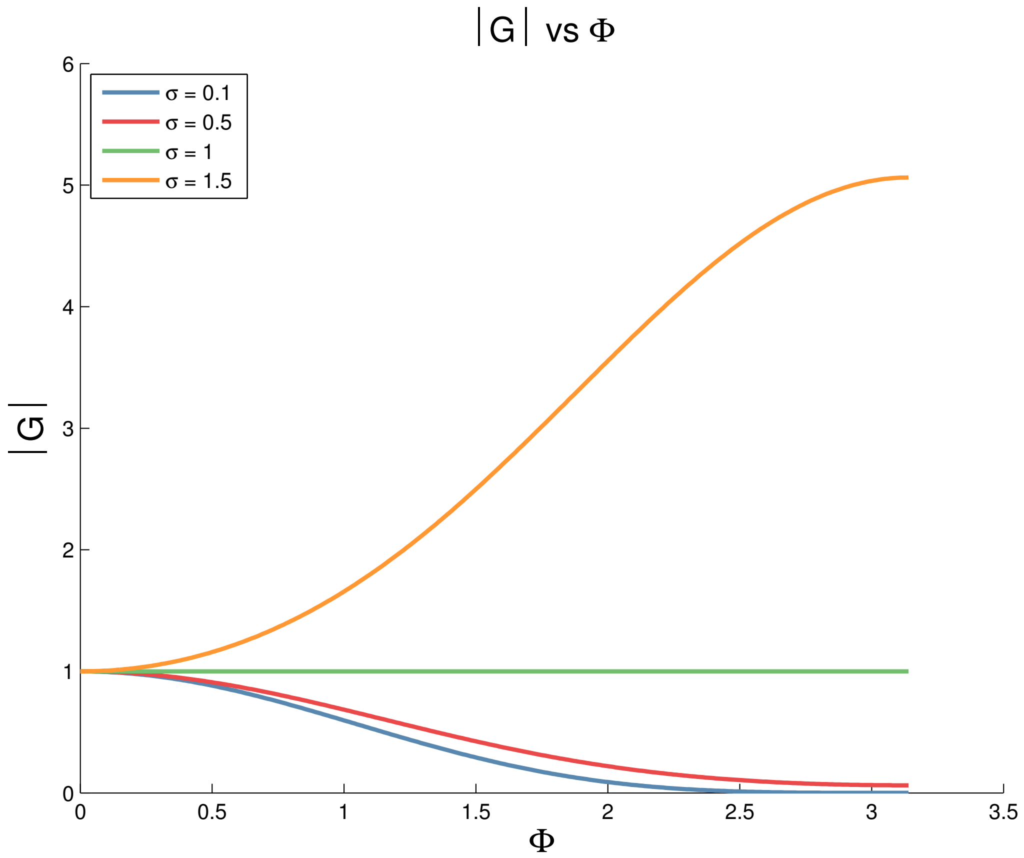

Figure 1.1: Plot of the amplification factor vs the phase angle for different σ values

Question 2

We use the given *Śhock_tube_analytic.m*´ Matlab file to create an analytic solution to the shock problem. This data will help us analyze the performances of the numerical scheme later.

Question 3

We now use the two-step Lax-Friedrichs scheme to solve the Euler equations in a shock tube problem. The first step to implement numerically the scheme is to write:

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix}$$

And then:

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} = \begin{bmatrix} u_2 \\ \frac{3-\gamma}{2}\frac{u^2}{u_1} + (\gamma - 1)u_3 \\ \gamma\frac{u_2 u_3}{u_1} - \frac{\gamma-1}{2}\frac{u_2^3}{u_1^2} \end{bmatrix}$$

We then implement this scheme under matlab, see the Matlab '.m' file handed out.

Question 4

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A}(\mathbf{U})\frac{\partial \mathbf{U}}{\partial x} = 0 \tag{4.1}$$

In order to choose a proper Δt to solve numerically the equations, we use the fact that, for the 1-D Euler equations used in this problem, the eigen-values of the Jacobian flux matrix in the conservation (equation 4.1) are the following:

$$\begin{cases} \lambda_1 = u \\ \lambda_2 = u + c \\ \lambda_3 = u - c \end{cases}$$

We saw in question 1 by performing a Von Neumann stability analysis on the two-steps Lax-Friedrichs scheme in the case of a linear advection equation that it was stable for $\sigma = a\frac{\Delta t}{\Delta x} \leq 1$.

This result is valid as seen in the course for the Euler equations, but replacing a by $\lambda_m = \mid u \mid + c$, the greatest value of the three eigen-values of $\mathbf{A}$. Hence we chose to solve numerically the scheme with $\Delta t = \frac{\Delta x}{\lambda_m}$ since as seen in the question 1 the high frequencies are not damped when we have $\sigma = 1$.

Finally we take λ_m such that $\lambda_m = u_{max} + \sqrt{\frac{p_{max}}{\gamma \rho_{max}}}$, with the max values retrieved from the analytic solutions provided with this coursework, so as to catch the quickest part of the flow on the numerical scheme and make it converge.

Question 5

The quantities relevant to study the shock problem (velocity, pressure, density, mach number and entropy) have been plotted over the domain for the analytic solution and the three numerical solutions.

As expected for the shock problem, we can notice on the figures the formation of shock waves, as well as rarefaction waves. Though while there is only one shock for pressure and velocity at $x = 1$, we can see that there is a second discontinuity at $x = 0.5$ for the density (and hence a second shock as well for the Mach number and the entropy since $M = \sqrt{\frac{\gamma p}{\rho}}$ and $s = log(\frac{p}{\gamma \rho})$).

When analyzing the figures, we can see there are no oscillations near discontinuities. This comes from the fact that the Lax-Friedrichs scheme is first-order accurate in both time and space. A consequence of that first order accuracy is really pronounced diffusive properties around discontinuities.

Though, as expected, when we rise the number of points N, we see that the error of the numerical scheme reduces, but which causes in the meantime the computational cost of the simulation to increase.

We can also notice that while the first shock at x=1 is well simulated, the second shock at x=0.5 and the transition domain between $-0.75 \leq x \leq 0$ are not that well approximated.

A fact to account for regarding the stability analysis performed in the first question is that we do not have the same flow speed throughout the domain, hence the fact that high frequencies are damped, which means that the vicinity of discontinuities are not well simulated.

To conclude, we can say that this two-step scheme is really accurate and satisfying to study the shock problem. However we can remember the fact that we used results of the real problem solutions to find the right Δt for this scheme to work. Without this data it would have been much more based on an heuristic approach to make this scheme accurate.

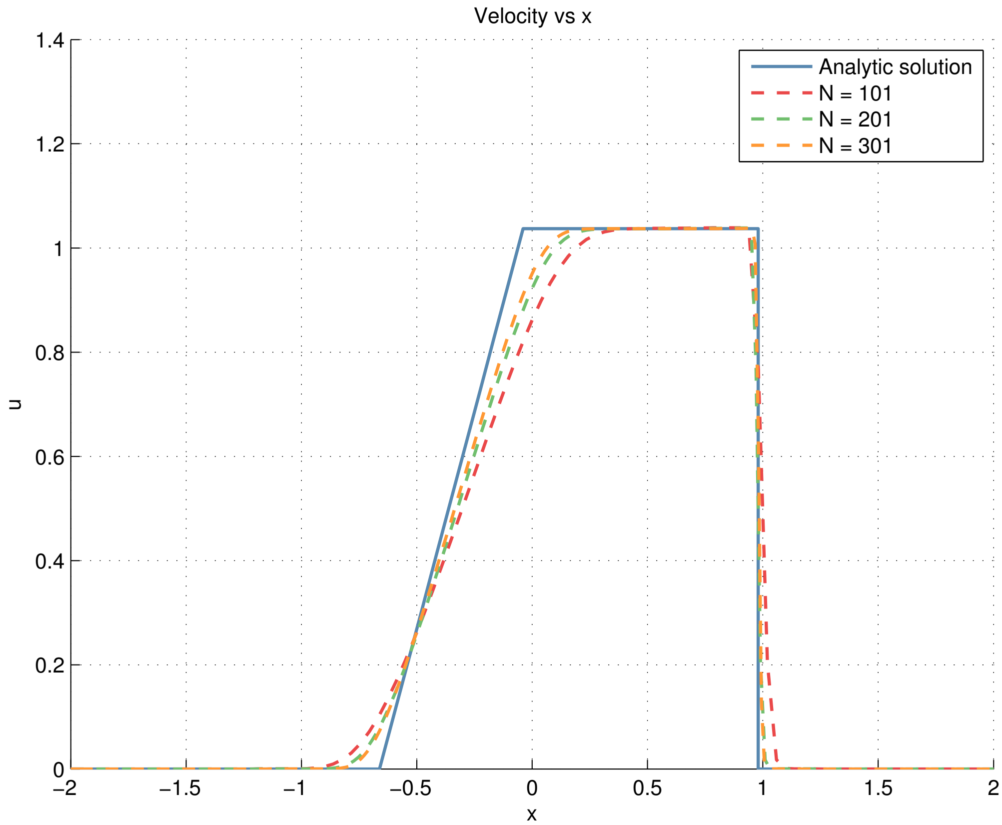

Figure 5.1: Plot of the velocity for the shock problem over the domain at time T=0.5

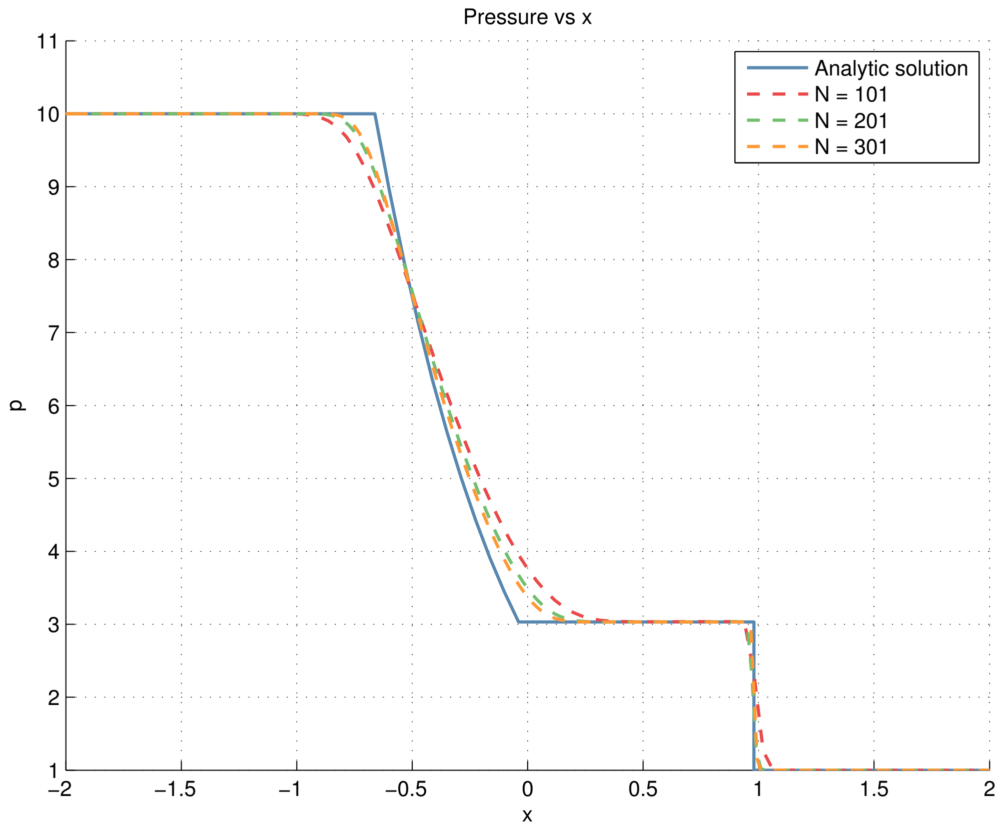

Figure 5.2: Plot of the pressure for the shock problem over the domain at time T=0.5

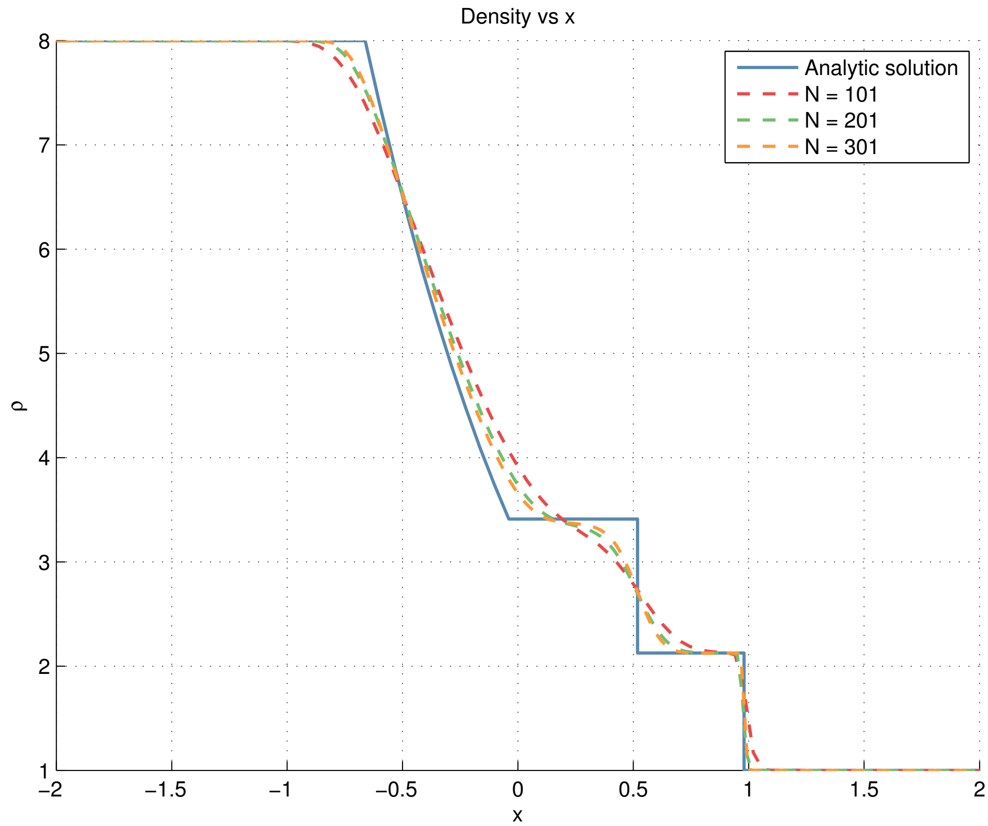

Figure 5.3: Plot of the density for the shock problem over the domain at time T=0.5

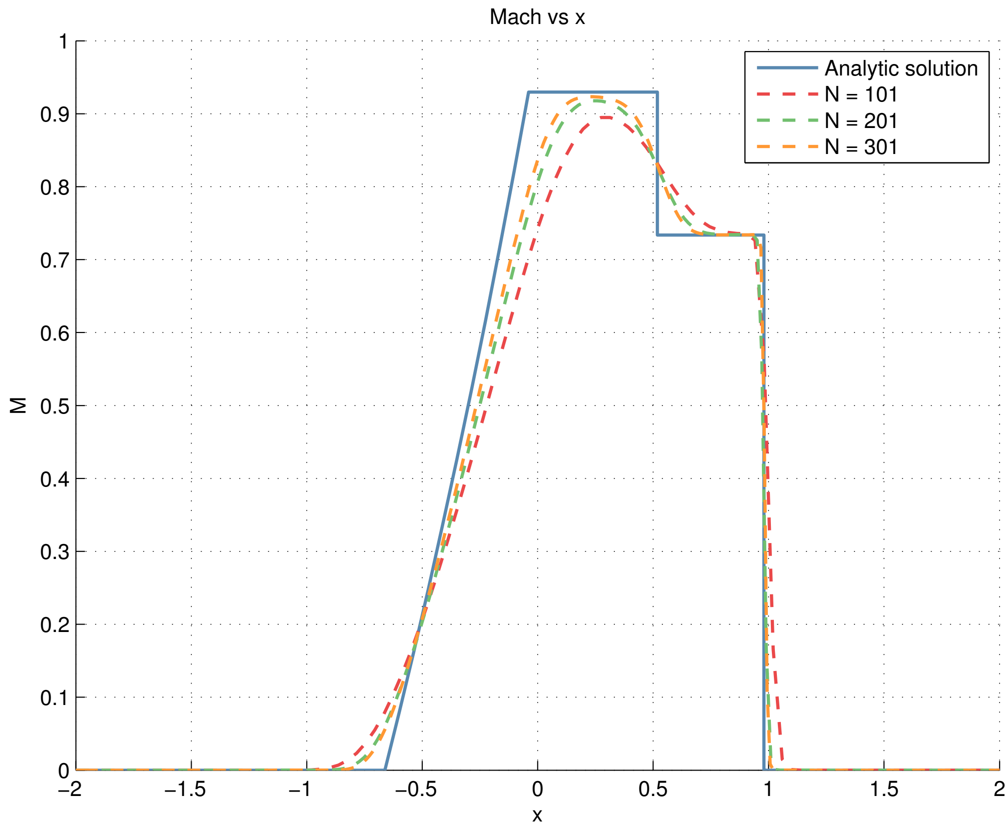

Figure 5.4: Plot of the Mach number for the shock problem over the domain at time T=0.5

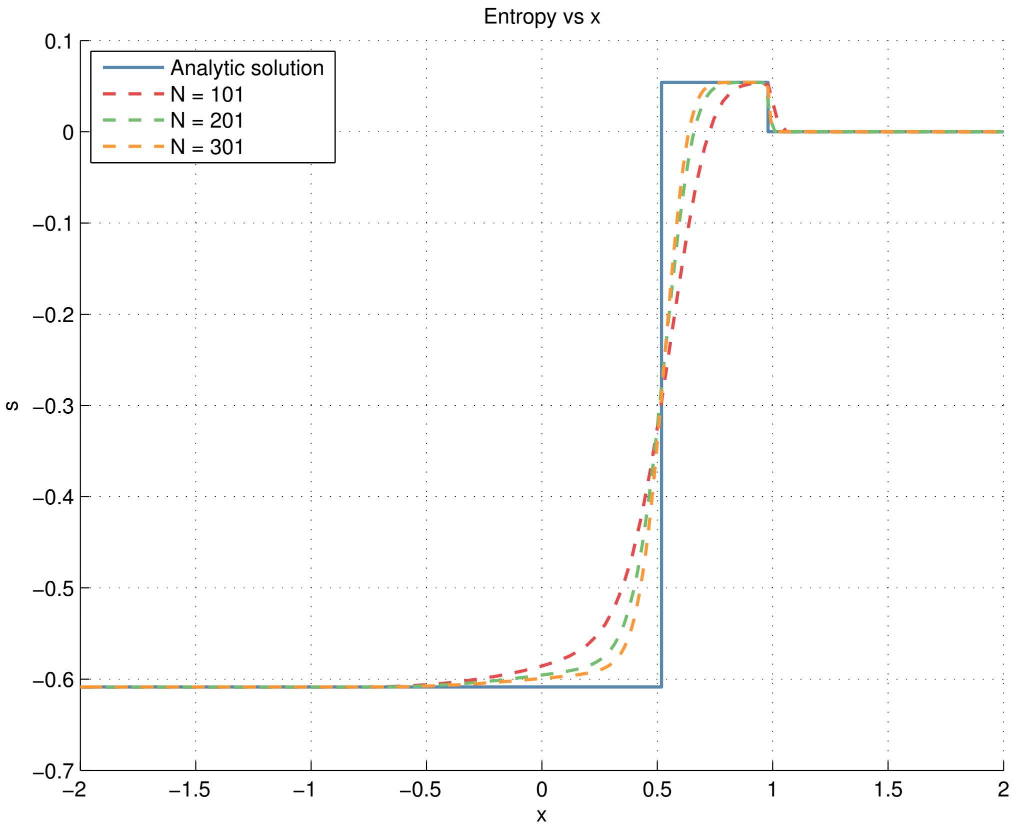

Figure 5.5: Plot of the entropy for the shock problem over the domain at time T=0.5