Empirical IO: Homework 1

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Problem 1

Part (a)

The individual choice probability for individual h to choose option j at period t is given by:

$$P_{hjt} = \frac{\exp(\beta_j^h + \eta^h x_{jt}^h)}{1 + \sum_{j=1}^{J} \exp(\beta_j^h + \eta^h x_{jt}^h)}$$

For each individual h, we observe a sequence of choices $y_{ht} = (y_{h1}, ..., y_{hT})$. The likelihood function for an individual's sequence of choices is given by:

The individual-level log-likelihood function is given by:

$$\begin{aligned}
\mathcal{L}_h(\Theta^h) = \log\left(L_h(\Theta^h)\right) &= \sum_{t=1}^{T} \sum_{j \in J} y_{jt}^h \log(P_{hjt}) \\
&= \sum_{t=1}^{T} \sum_{j=1}^{J} y_{jt}^h \log\left(\frac{\exp(\beta_j^h + \eta^h x_{jt}^h)}{1 + \sum_{k=1}^{J} \exp(\beta_k^h + \eta^h x_{kt}^h)}\right)
\end{aligned}$$

Where y_{jt}^h is to be correctly interpreted as an indicator for which alternative individual h has chosen. Given that the choice set does not change over time, the denominator of the probability remains a simple expression.

The score function for individual h is given by:

$$\nabla_{\Theta^h}\mathcal{L}_h(\Theta^h) = \begin{bmatrix} \frac{\partial \mathcal{L}_h(\Theta^h)}{\partial \beta_1^h} \\ \vdots \\ \frac{\partial \mathcal{L}_h(\Theta^h)}{\partial \beta_J^h} \\ \frac{\partial \mathcal{L}_h(\Theta^h)}{\partial \eta^h} \end{bmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial \beta_k^h} = \sum_j \sum_t y_{jt}^h \frac{\partial}{\partial \beta_k^h} \log(P_{hjt})$$

For easy notation, I denote $v_k = \beta_k^h + \eta^h x_{kt}^h$

$$\begin{aligned} \frac{\partial P_{hjt}}{\partial \beta_k^h} &= \frac{1 + \sum_k e^{v_k}}{e^{v_j}} \frac{e^{v_j}(1 + \sum_k e^{v_k}) - e^{v_j} e^{v_j}}{(1 + \sum_k e^{v_k})^2} \\ &= \frac{1 + \sum_k e^{v_k} - e^{v_j}}{1 + \sum_k e^{v_k}} \\ &= (1 - P_{hjt}) \end{aligned}$$