A fast introduction to the theory of matching without transfers

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Math+Econ+Code June 2022 masterclass

Motivation

Matching algorithms such as Deferred Acceptance are used everywhere in the world, in some key elements of our lives:

- School admission
- College admission
- Assignment to a hospital room

But they can also be used to model some other, important enough, elements of our lives:

- Finding a life partner (marriage market)
- Finding a job

There exists several matching algorithms/mechanisms, but we will focus on the Deferred Acceptance (DA) one, presented by Gale and Shapley in their famous paper from 1962.

- How does it work ?
- What are (some of) its structural properties ?

References

This presentation is based on the following resources:

- Gale and Shapley (1962) *College Admissions and the Stability of Marriage*, The American Mathematical Monthly, Vol. 69, No. 1, pp. 9-15, Mathematical Association of America
- Diamantaras et al (2009), *A toolbox for Economic Design*, Chapter 9
- Roth and Sotomayor (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Econometric Society Monographs. Cambridge: Cambridge University Press.

Notation

- Let $X = \{x_1, x_2, ..., x_I\}$ denote a set of I men.
- Let $Y = \{y_1, y_2, ..., y_J\}$ denote a set of J women.
- We denote a typical matching with a tuple (x_i, y_j).
- Every man and woman forms **preferences** on the other side of the market, and includes celibacy as an option. These preferences take the form of an ordered ranking.
- Formally, we denote R^{x_i} the ranking of man x_i over the set $Y \cup \{x_i\}$, and R^{y_j} the ranking of woman y_j over $X \cup \{y_j\}$
- Every agent has a quota q, which includes the matching capacity of agents. In the simple marriage market we are modelling here, $q = 1$ for every agent. But if we model college admission, for example, the quota of a college would be $q > 1$

Matching and Stability

- A matching is a map $\mu : X \cup Y \to Y \cup X$
- $\mu(x_i) = \begin{cases} y_j & \text{if } x_i \text{ and } y_j \text{ are matched} \\ x_i & \text{if } x_i \text{ is single} \end{cases}$
- A matching μ is **blocked**:
  - by an individual agent k if $k \succ^{R^k} \mu(k)$
  - by a pair of agent (x_k, y_l) if $y_l \succ^{R^k} \mu(x_k)$ and $x_k \succ^{R^l} \mu(y_l)$

Definition

A matching is stable if it is not blocked

Existence Theorem (Gale and Shapley 1962)

In the Gale Shapley marriage market, there always exists a stable matching.

The Deferred Acceptance algorithm: first example

This algorithm rules out by construction any possible blocking pair. For simplicity and readability, we assume that all women and men consider each other acceptable.

$$R^{x_1} : y_2 \succ y_1$$
$$R^{x_2} : y_2 \succ y_1$$
$$R^{y_1} : x_1 \succ x_2$$
$$R^{y_2} : x_2 \succ x_1$$

In the DA algorithm, men propose, and women retain offers until they obtain a better offer or there are no more offers available.

1. Step 1: x_1 and x_2 both propose to y_2. y_2 retains the offer from x_2 and rejects the offer from x_1.
2. Step 2: x_1 proposes to y_1, who retains the offer.

No more proposals are made, and the two following pairs are formed: $(x_1, y_1), (x_2, y_2)$.

The Deferred Acceptance algorithm: second example

In the former example, you should note that, if women propose, the resulting stable matching is the same. But that is not systematically the case, and there are welfare consequences. Take the following example, where we only changed the preferences of x_2.

$$R^{x_1} : y_2 \succ y_1; \;\; R^{x_2} : \;\; y_1 \succ y_2$$
$$R^{y_1} : x_1 \succ x_2; \;\; R^{y_2} : \;\; x_2 \succ x_1$$

x_1 proposes to y_2, who retains the offer. x_2 proposes to y_1, who retains the offer. The result is: $(x_1, y_2), (x_2, y_1)$. The matching is stable and optimal for men: 'M-optimal'. What if women propose ? y_1 proposes to x_1, who retains the offer. y_2 proposes to x_2, who retains the offer. The result is $(x_1, y_1), (x_2, y_2)$. The matching is stable and optimal for women: 'W-optimal'

DA algorithm: M-optimal vs W-optimal

We generalize these notions: a matching is **W-optimal** if every woman likes it at least as well as any other stable matching. It is **M-optimal** if every man likes it at least as well as any other stable matching.

Theorem: Gale and Shapley

When all men and women have strict preferences, there always exists a M-optimal and a W-optimal stable matching. The matching μ_M produced by the deferred acceptance algorithm with men proposing is the M-optimal, and the matching μ_W obtained when women propose is the W-optimal stable matching.

Intuition

In the men-proposing DA algorithm, no man is ever rejected by an achievable woman. (full proof in RS p.33)

DA algorithm

All men "agree" on the best stable matching possible for them, and all women "agree" on the best stable matching for them as well.

Theorem: Knuth 1976

When all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings. Let μ and μ' be stable matchings.
$\mu \succ^M \mu' \Leftrightarrow \mu' \succ^W \mu$

Proof.

Suppose it is not true that $\mu' \succ^W \mu$. Then some woman y prefers $x = \mu(y)$ to $x' = \mu'(y)$. By assumption, $\mu \succ^x \mu'$. So, x and y form a blocking pair to μ', which contradicts stability of μ'. $\square$

Reminder: Lattices

Definition

A partially ordered set $(X, \succ)$ is a set of alternatives X and a partial order relation $\succ$. $\succ$ is reflexive, transitive, and antisymmetric.

Definition

$x \in X$ is an upper bound for $A \subseteq X$ if $x \succ a \ \forall a \in A$. The least upper bound for $A \subseteq X$ is an element $x \in X$ that is dominated by every upper bound in A.

Definition

A lattice L is a partially ordered set in which every pair of elements has a least upper bound and a greatest lower bound.

DA algorithm: lattice structure

Let R^M and R^W denote respectively the "aggregate" preferences of men and women.

Theorem: Conway

The set of stable matchings ordered by R^M and the set of stable matchings ordered by R^W are lattices.

Let us take two (arbitrary) matchings μ and μ'. We define the least upper bound of these matchings as $\lambda = \mu \vee_M \mu'$, and the greatest lower bound as $\nu = \mu \wedge_M \mu'$. To obtain $\lambda(\mu, \mu')$, we ask every man x to "point" to its favorite choice between $\mu(x)$ and $\mu'(x)$.

$$\lambda(x) = \begin{cases} \mu(x) & \text{if } \mu(x) \succ^x \mu'(x) \\ \mu'(x) & \text{if } \mu'(x) \succ^x \mu(x) \end{cases}$$

DA algorithm: lattice structure

Conway:

If μ and μ' are stable matchings, then λ and ν are stable matchings.

Proof.

Suppose (x, y) blocks λ. Then $y \succ^x \mu(x)$ and $y \succ^x \mu'(x)$. Also, $x \succ^y \lambda(y)$.
$\lambda(y)$ can be $\mu(y)$ or $\mu'(y)$
If $\lambda(y) = \mu(y)$, (x, y) blocks μ. If $\lambda(y) = \mu'(y)$, (x, y) blocks μ'.
Either of these two cases contradict the stability of μ and μ'. □

Lattice theorem: Illustration (RS p.38)

Example 2.17: The lattice of stable matchings (Knuth)

$P(m_1) = w_1, w_2, w_3, w_4 \qquad P(w_1) = m_4, m_3, m_2, m_1$

$P(m_2) = w_2, w_1, w_4, w_3 \qquad P(w_2) = m_3, m_4, m_1, m_2$

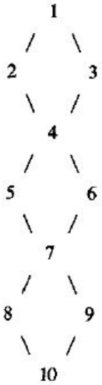
$P(m_3) = w_3, w_4, w_1, w_2 \qquad P(w_3) = m_2, m_1, m_4, m_3$

$P(m_4) = w_4, w_3, w_2, w_1 \qquad P(w_4) = m_1, m_2, m_3, m_4$

There are ten stable matchings where w_1, w_2, w_3, and w_4 are matched respectively to

$$m_1 \; m_2 \; m_3 \; m_4 \qquad (1)$$
$$m_2 \; m_1 \; m_3 \; m_4 \qquad (2)$$
$$m_1 \; m_2 \; m_4 \; m_3 \qquad (3)$$
$$m_2 \; m_1 \; m_4 \; m_3 \qquad (4)$$
$$m_3 \; m_1 \; m_4 \; m_2 \qquad (5)$$
$$m_2 \; m_4 \; m_1 \; m_3 \qquad (6)$$
$$m_3 \; m_4 \; m_1 \; m_2 \qquad (7)$$
$$m_4 \; m_3 \; m_1 \; m_2 \qquad (8)$$
$$m_3 \; m_4 \; m_2 \; m_1 \qquad (9)$$
$$m_4 \; m_3 \; m_2 \; m_1. \qquad (10)$$

(a)



(b) The lattice structure

Many-to-one DA

College admission problem: here the capacity becomes important. We denote $q_1 = 2$, $q_2 = 1$.

$$
\begin{aligned}
x_1 &: y_1 \succ y_2 & \quad y_1 &: x_1, x_2, x_4, x_3 \\
x_2 &: y_2 \succ y_1 & \quad y_2 &: x_3, x_1, x_2, x_4 \\
x_3 &: y_1 \succ y_2 \\
x_4 &: y_1 \succ y_2
\end{aligned}
$$

1. Step 1: x_1 and x_4 are matched to y_1, x_2 is matched to y_2, x_3 is rejected
2. Step 2: x_3 is matched to y_2, x_2 is rejected
3. Step 3: x_2 is matched to y_1, x_4 is rejected.