Numerical integration of hillslope storage Boussinesq equations

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Introduction

Boussinesq equations are highly non linear and their integration is often done on a variable integration volume. This is due to the fact that free surface of water can going up and intersect the surface. Then the water level is fixed at the surface elevation.

In this paper we develop a new(?) formulation of the hillslope storage boussinesq equations enabling to take into account the interception of the free surface with the surface directly in the equation resolution.

This new formulation is based on the use of Differential Algebraic Equations (DAEs). Indeed use of DAE enable to change from one time step to another the dimension and the type of equation governing the problem.

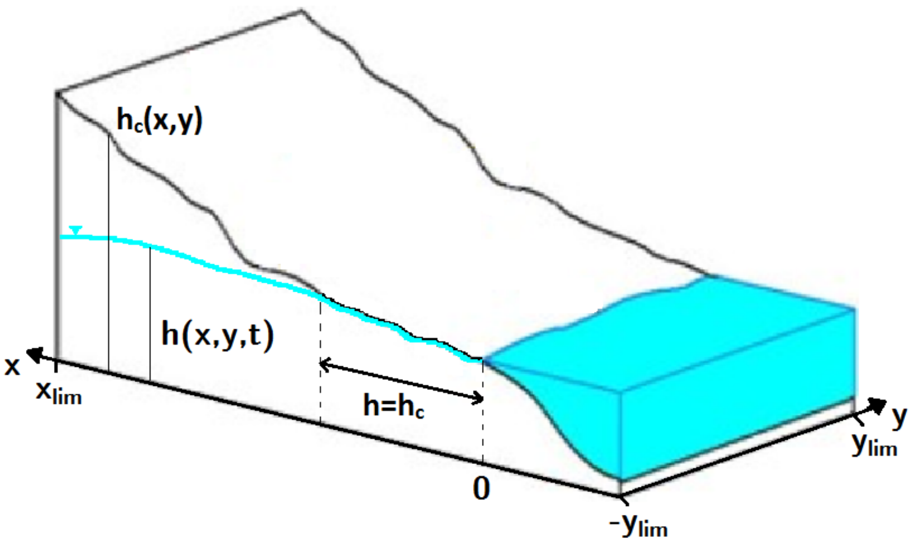
We solve both the soil storage moisture and the flux going in or out each cell in the same equation. This allows us to add a flux going out of the cell is the cell is fully saturated.

Problem Statement

Let's consider a sloping aquifer in two dimension (cf. figure 1). h denotes the hydraulic head, h_c is the soil/air interface. Considering the bottom of the aquifer flat, we have the following equations uder the Dupuits-Forchheimer assumption:

$$\forall t > 0, \forall x \in [0, x_{lim}], \forall y \in [-y_{lim}, y_{lim}], \quad \begin{cases} f\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}(kh\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(kh\frac{\partial h}{\partial y}) + N(t) \\ h(x,t) \leq h_c(x) = d(x) \end{cases} \tag{1}$$

where k is the hydraulic conductivity and f is the porosity of the aquifer and N is the source term of the equation (efficace rainfall, recharge, infiltration, etc.).

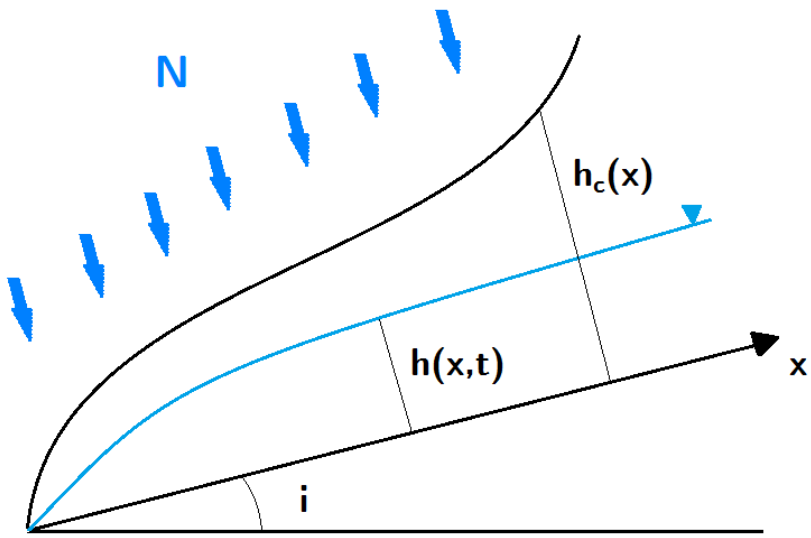


Scheme 1: 2D hillslope

In one dimension, the case where the bottom of the aquifer is not flat can be easily added (cf. figure 2). This leads to the following equations:

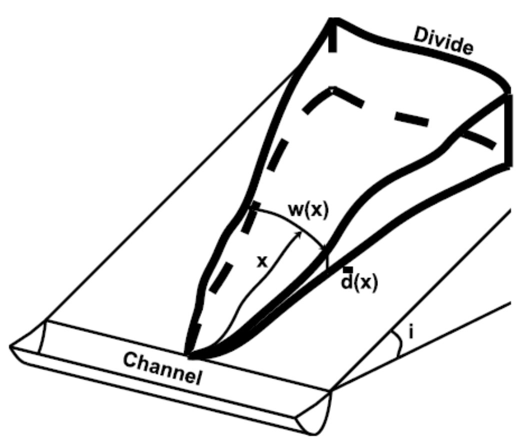
$$\forall t > 0, \forall x \in [0, x_{lim}], \quad \begin{cases} \frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x} + N(t) \\ Q = -kh(\cos i\frac{\partial h}{\partial x} + \sin i) \\ h(x,t) \leq h_c(x) = d(x) \end{cases} \tag{2}$$

where Q is the Darcy flux and d is the maximum potential thickness of the aquifer. If we are interested in subsurface flow, d is typically the soil depth, subsurface flow taking place in the first meter under ground. Here the reference for h is different from equation 1. The reference for h is set at the bottom of the aquifer, having an elevation varying with depth.



Scheme 2: 1D hillslope. Adapted from Troch et al.[1].

Following Troch et al.[1], we can rewrite this problem by considering the soil moisture storage $S = fwh$ instead of the hydrauilc head h. w represents the width function[2]. The width function w characterize for a given position x the width of one hillslope (cf. figure 3).



Scheme 3: Width function visualization. Taken from Troch et al.[1].

This enables us to aggregate the response of a 2 dimensional aquifer into one dimension by writing the following equation on S and Q. This leads to the so called hillslope storage Boussinesq equations:

$$\forall t > 0, \forall x \in [0, x_{lim}] \quad \begin{cases} \frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial x} + N(t)w(x) \\ Q = -\frac{kS}{f}(\cos i \frac{\partial}{\partial x}\frac{S}{fw} + \sin i) \\ S(x,t) \leq S_c(x) = fw(x)d(x) \end{cases} \tag{3}$$

Note that the angle i can vary with the distance to the stream ($x = 0$), i.e. $i = i(x)$.

We can add Dirichlet boundary condition at $x = 0$ and Neumann's at $x = x_{lim}$:

$$\begin{cases} S(x = 0, t) = 0 \\ Q(x = x_{lim}, t) = 0 \end{cases} \tag{4}$$

Finally, we can consider two types of initial conditions:

$$\begin{cases} S(x, t = 0) = 0 \\ Q(x, t = 0) = 0 \end{cases} \tag{5}$$

or

$$\begin{cases} S(x, t = 0) = R \cdot S_c(x) \\ Q(x, t = 0) = -\frac{k}{f} S_{t=0} \cdot (\cos i \frac{\partial}{\partial x}(\frac{S}{fw})\big|_{t=0} + \sin i) \end{cases} \tag{6}$$

where R represents the percentage of how much the hillslope is loaded.

Transforming the problem into a DAE

What is typically done with equation 3, is to replace Q by its Darcy expression. So the system reduces to the equation on S:

$$\begin{cases} f\frac{\partial S}{\partial t} = \frac{k\cos i}{f}\left(\frac{\partial}{\partial x}\left(\frac{S}{w}\left(\frac{\partial S}{\partial x} - \frac{S}{w}\frac{\partial w}{\partial x}\right)\right) + k\sin i\frac{\partial S}{\partial x} + fNw\right. \\ S(x,t) \leq S_c(x) = fw(x)d(x) \end{cases} \tag{7}$$

What we develop here is a code that computes both S and Q and stays with a formulation that separates conservation equation and Darcy flux equation. The major advantage of this formulation is to exactly conserve the mass in a finite difference scheme. The code also gains in clarity as it is obvious to write equations in a vectorized form.

The second thing is to transform the condition that prevent soil moisture storage to go over the maximal value S_c. To do this and in order to conserve mass in the equation we can consider the flux $q_S(x,t)$ which represents the flux going out of the soil if $S(x,t) = S_c(x)$.

Basically the idea is to partition the flux balance (flux entering minus flux exiting) of a given spatial domain between one part that goes to the variation of $\frac{\partial S}{\partial t}$ and another that go to the outflow flux q_S. Obviously, the partition's ratio depends on the value of S compared to S_c and on values of $\frac{\partial S}{\partial t}$.

We define:

$$q_{in} - q_{out} = -\frac{\partial Q}{\partial x} + Nw \tag{8}$$

Mathematically we have

$$\begin{cases} \frac{\partial S}{\partial t} = \alpha \cdot (q_{in} - q_{out}) \\ q_S = (1-\alpha) \cdot (q_{in} - q_{out}) \end{cases} \tag{9}$$

where α is defined as

$$\alpha = \begin{cases} 1 & \text{if} \quad \frac{\partial S}{\partial t} < 0 \quad || \quad S < S_c \\ 0 & \text{if} \quad \frac{\partial S}{\partial t} \geqslant 0 \quad \& \quad S = S_c \end{cases} \tag{10}$$

To do this we introduce the sigmoid g (see figure 4):

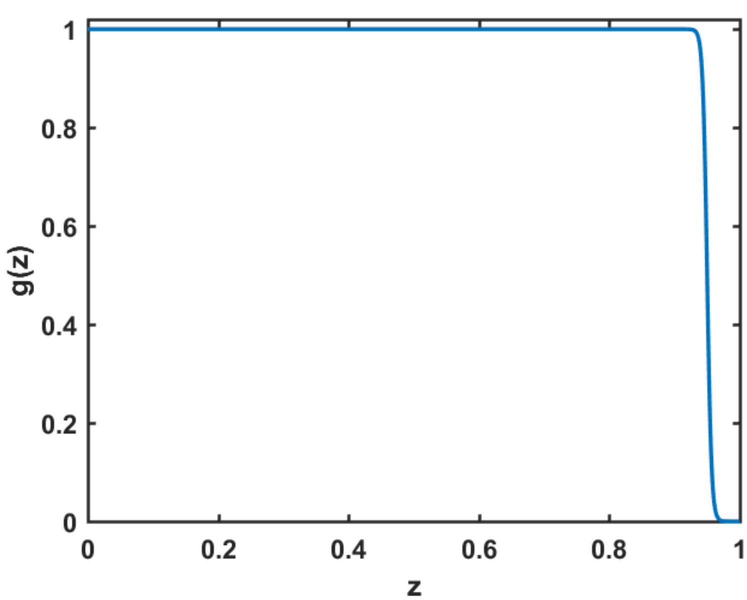
$$\forall z \in [0,1] \quad g(z) = 1 - \frac{1}{1 + \exp(-300 * (z - 0.95))} \tag{11}$$

and the function test h

$$\forall z \in [0,1] \quad h(z) = \begin{cases} 1 & \text{if} \quad z \geqslant 0 \\ 0 & \text{if} \quad z < 0 \end{cases} \tag{12}$$

α becomes

$$\alpha = g\left(\frac{S}{S_c}\right) \cdot h\left(\frac{\partial S}{\partial t}\right) + \left(1 - h\left(\frac{\partial S}{\partial t}\right)\right) \tag{13}$$



Scheme 4: Sigmoid

Finally, equation 3 becomes:

$$\begin{cases} \frac{\partial S}{\partial t} = \alpha \cdot (-\frac{\partial Q}{\partial x} + Nw) \\ Q = -\frac{kS}{f}(\cos i \frac{\partial}{\partial x}\frac{S}{fw} + \sin i) \\ q_S = (1-\alpha) \cdot (-\frac{\partial Q}{\partial x} + Nw) \end{cases} \tag{14}$$

After some arrangements, the system can be written like this:

$$\begin{cases} \frac{\partial S}{\partial t} & = & & - & \alpha \cdot \frac{\partial Q}{\partial x} & & & + & \alpha \cdot Nw \\ 0 & = & P(S) \cdot S & + & Q & & & & \\ 0 & = & & - & (1-\alpha) \cdot \frac{\partial Q}{\partial x} & - & q_s & + & (1-\alpha) \cdot Nw \end{cases} \tag{15}$$

where $P(S) = \frac{k}{f}(\cos i \frac{\partial}{\partial x}\frac{S}{fw} + \sin i)$.

Finally to prevent S to become negative, one can add a β coefficient to equation 15 so that if $\frac{\partial S}{\partial t} < 0$ & $S = 0$ then $\frac{\partial S}{\partial t} = 0$ for the next time step. We have

$$\beta = 1 - \delta\big(S\big) \cdot \Bigg(1 - h\big(\frac{\partial S}{\partial t}\big)\Bigg) \tag{16}$$

where δ is the dirac function.

The DAE we consider is finally:

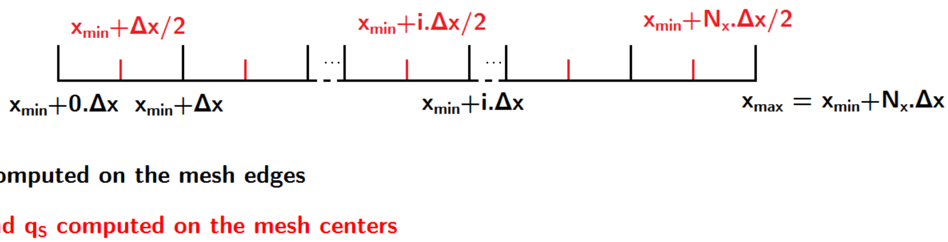
$$\begin{cases} \frac{\partial S}{\partial t} & = & & - & \alpha \cdot \beta \cdot \frac{\partial Q}{\partial x} & & & + & \alpha \cdot \beta \cdot Nw \\ 0 & = & P(S) \cdot S & + & Q & & & & \\ 0 & = & & - & (1-\alpha) \cdot \frac{\partial Q}{\partial x} & - & q_s & + & (1-\alpha) \cdot Nw \end{cases} \tag{17}$$

Space Discretization

Let's now discretize the system 17 in space. Consider $N_x \in \mathbb{N}^+$.

We define $\Delta x = (x_{max} - x_{min})/N_x$. $\forall i \in [\![0; N_x]\!]$ $Q_i(t) = Q(x_{edges_i}, t) = Q(x_{min} + i \cdot \Delta x, t)$.

For S and q_S, we discretize it a the center of the mesh. We get: $\forall i \in [\![1; N_x]\!]$ $S_i(t) = S(x_{centers i}, t) = S(i \cdot \Delta x/2, t)$. Similar notations are used for q_s. The scheme 5 clarifies the discretization.



Scheme 5: Discretization

Now we have:

$$S = \begin{bmatrix} S_1 \\ \vdots \\ S_{N_x} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{N_x} \end{bmatrix} \quad \& \quad q_s = \begin{bmatrix} q_{s1} \\ \vdots \\ q_{sN_x} \end{bmatrix} \tag{18}$$

Discretization in space leads to:

$$\frac{\partial Q}{\partial x} = \frac{A}{2\Delta x} \cdot Q, \ \ where \ \ A = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in \mathcal{M}_{N_x, N_x+1}(\mathbb{R}) \tag{19}$$

and

$$\frac{\partial S}{\partial x} = \frac{B}{\Delta x} \cdot S,\ \ where\ \ B = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \in \mathcal{M}_{N_x+1,N_x}(\mathbb{R}) \tag{20}$$

and

$$\frac{S(x) + S(x+\Delta x)}{2} = \Omega \cdot S,\ \ where\ \ \Omega = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 0.5 & 0.5 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0.5 & 0.5 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix} \in \mathcal{M}_{N_x+1,N_x}(\mathbb{R}) \tag{21}$$

Let y now denote the column vector: $\begin{bmatrix} S \\ Q \\ q_s \end{bmatrix}$. We have now a differential algebraic equation (DAE):

$$M \cdot \frac{\partial y}{\partial t} = C \cdot y + D \tag{22}$$

where M is the mass matrix defined by:

$$M = \left[\begin{array}{c|c|c} I_{N_x+1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{23}$$

C is defined by:

$$C = \left[\begin{array}{c|c|c} 0 & -\frac{\alpha\beta}{2\Delta x}A & 0 \\ \hline P(S) & I_{N_x+1} & 0 \\ \hline 0 & -\frac{1-\alpha}{2\Delta x}A & -I_{N_x+1} \end{array}\right] \tag{24}$$

where:

$$P(S) \;=\; \tfrac{k}{f}(\cos i \tfrac{B}{\Delta x}\tfrac{S}{fw} + \sin i)\cdot\Omega \tag{25}$$

and α (resp. β) are evaluated with equation 13 (resp. 16) at each point of the scheme mesh. and D by:

$$D = \begin{bmatrix} \alpha\cdot\beta\cdot Nw \\ 0 \\ (1-\alpha)\cdot Nw \end{bmatrix} \tag{26}$$

Some numerical results

We consider in these numerical experiments a straight hillslope (not convergent neither divergent) 100 m large so $[x_{min}; x_{max}] = [0; 100]$ m with a bedrock slope angle i of 5%. the hydraulic conductivity k is set at 1 m/d and the drainable porosity f at 0.3.

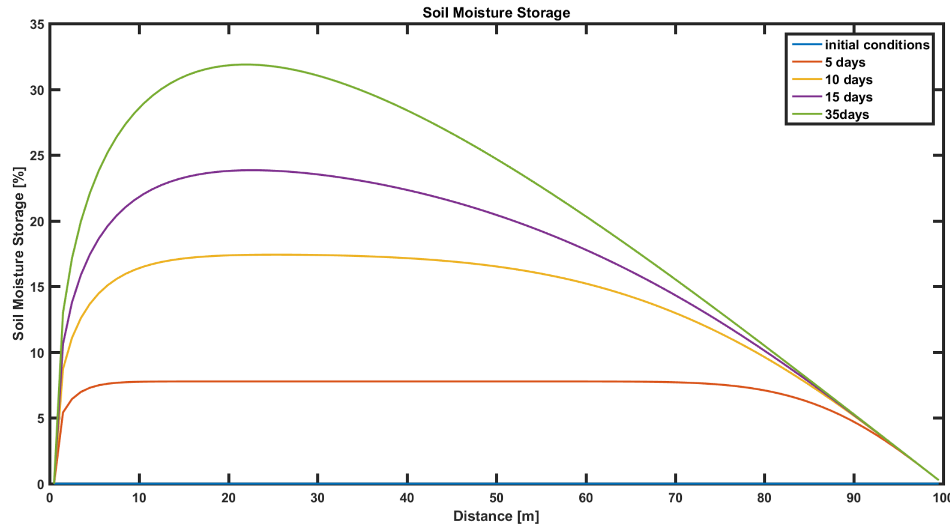
Time Resolution

Once the DAE has been discretized in space, the resolution in time is done with the MATLAB® solver ode15s. ode15s is a solver enabling DAE resolution.

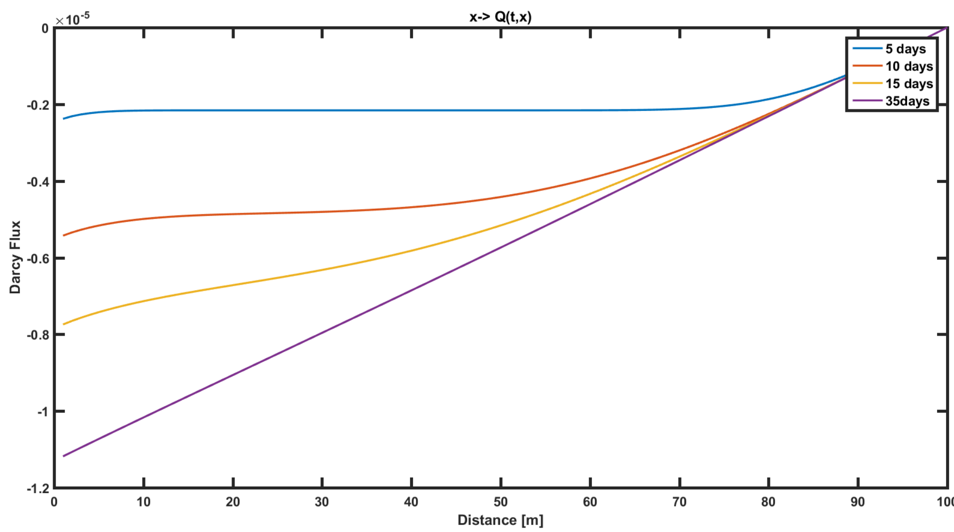
First experiment: the recharge case

Now we consider a first experiment: the recharge scenario. In this case the initial conditions are an initially dry hillslope (initial conditions of equation 5). The source term is set at a constant rate of 10 mm/d so $N = 11.1574 \times 10^{-7}$ m/s.

In this case the steady state is reached after 30 days approximately. This experiment was conducted in[1] and results are similar. Figure 6, 7 show the spatial behavior of the soil moisture storage and of the darcy flux at different times.



Scheme 6: The soil moisture content (in % of the total saturation) of the hillslope.

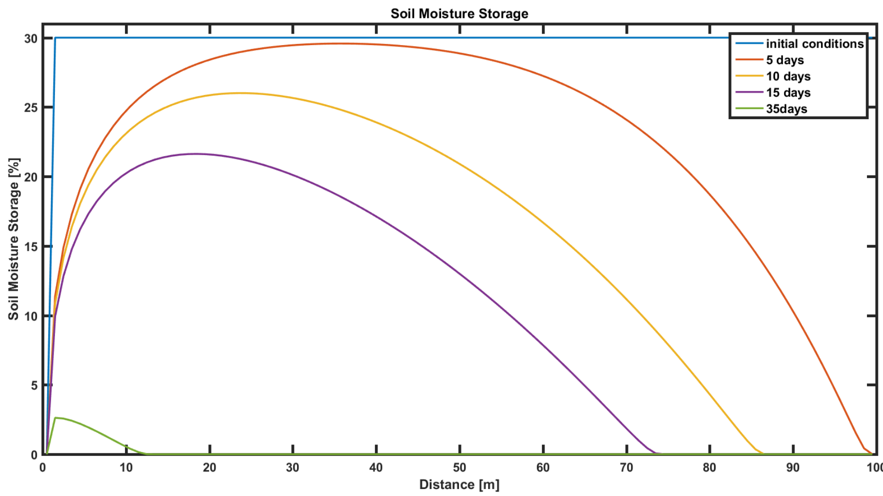


Scheme 7: The Darcy flux in the hillslope.

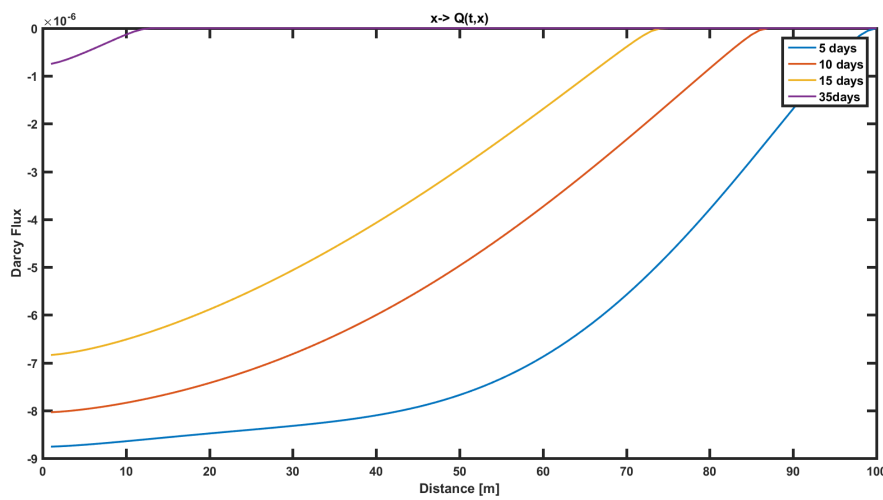
Second experiment: the drainage case

In the second experiment, the drainage scenario, we study the behaviour of a hillslope initially filled at 30%, that drains freely. Initial conditions are those of equation 6 with a ratio R of 0.3.

Figure 8, 9 show the spatial behavior of the soil moisture storage and of the darcy flux at different times.



Scheme 8: The soil moisture content (in % of the total saturation) of the hillslope.



Scheme 9: The Darcy flux in the hillslope.

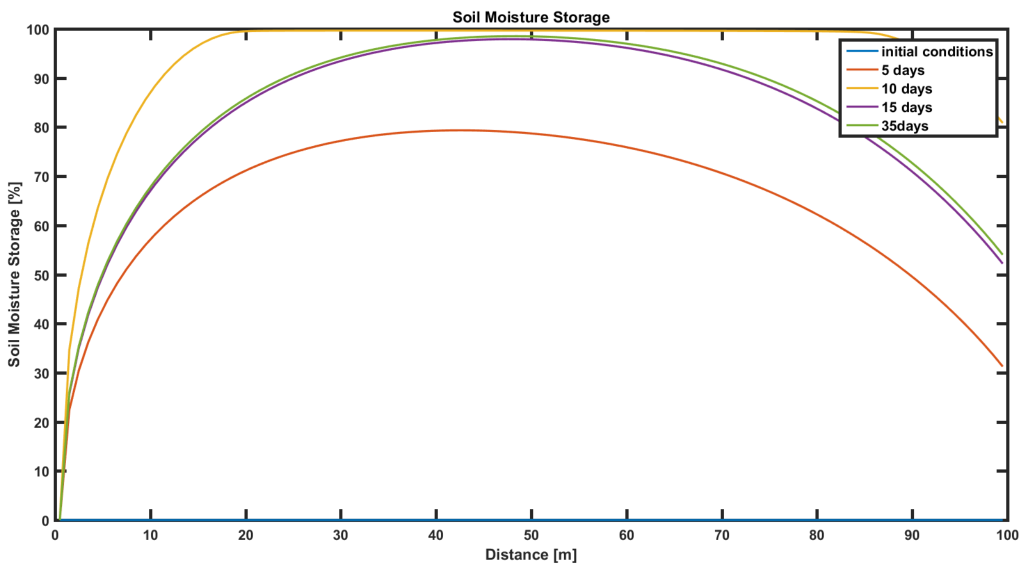
Third experiment: the sinusoidal recharge case

In this experiment, we start with a dry hillslope (initial conditions of equation 5) and force the system with a sinusoidal rainfall which is

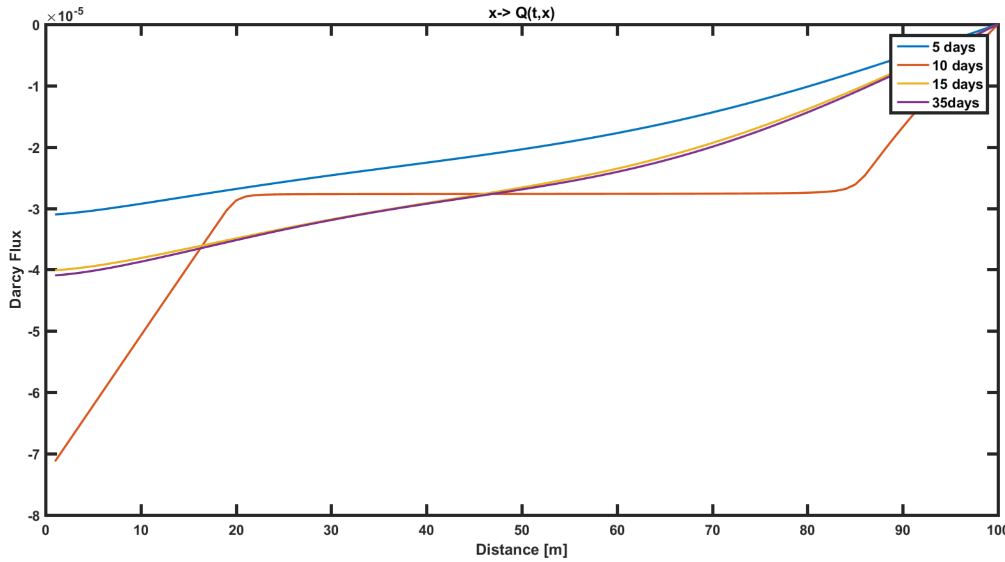
$$N(t) = N_0 \cdot \left(\cos(2\pi \frac{t}{T}) + 1 \right) \tag{27}$$

with $N_0 = 100$ mm/d (!!) and $T = 10$ days.

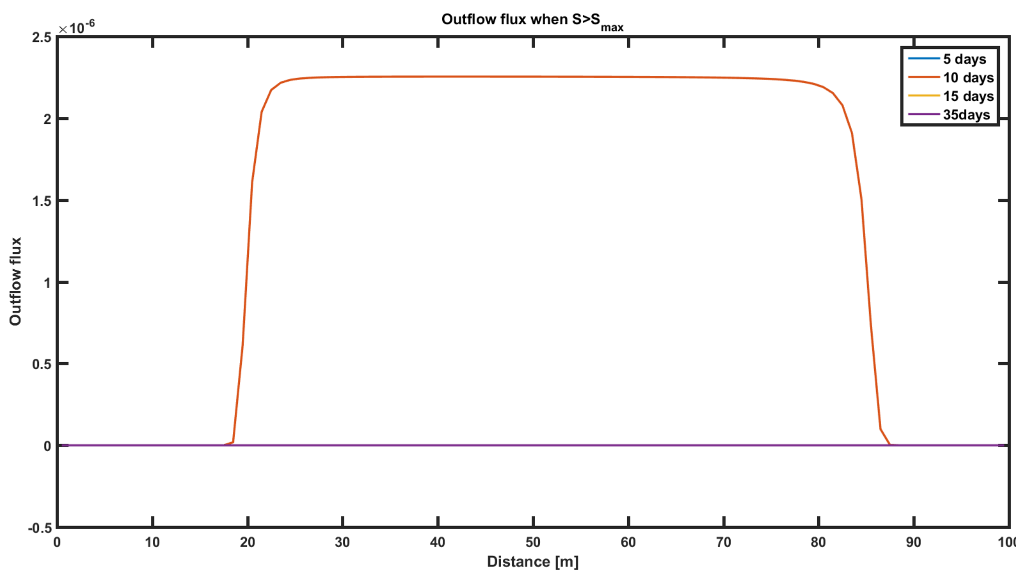
With this rainfall, the hillslope is fully saturated at some locations for some time steps (for $t = 10$ days for example, cf. figure 12).



Scheme 10: The soil moisture content (in % of the total saturation) of the hillslope.



Scheme 11: The Darcy flux in the hillslope.



Scheme 12: The outflow flux (seeping flux out of the soil matrix) in the hillslope.

Convergence...

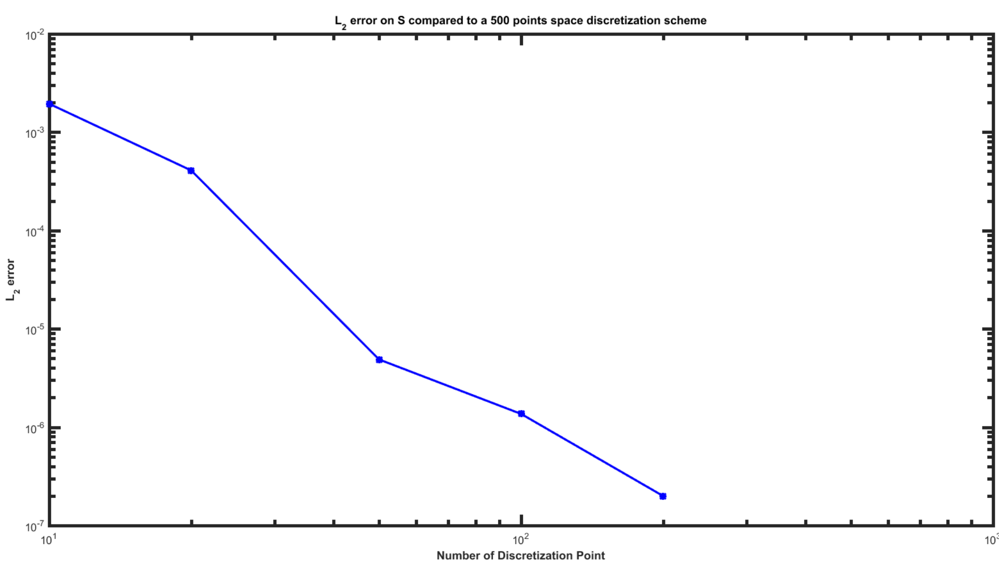
Finally, we "check" the convergence by studying how the $\mathcal{L}^2$ error decreases by refining our space discretization scheme. To do this we take the first numerical experiment described in . We discretized the 100 m large hillslope with N_x varying between 10 and 500.

We take as the reference the numerical solution with $N_x = 500$ as no analytical solutions of this problem exist. Then we define the $\mathcal{L}^2$ error as:

$$\epsilon_{S_{N_x}} = \frac{1}{10} \sum_{i=1}^{10} \Big(S_{ref}(i\Delta x) - S_{N_x}(i\Delta x) \Big)^2 \tag{28}$$

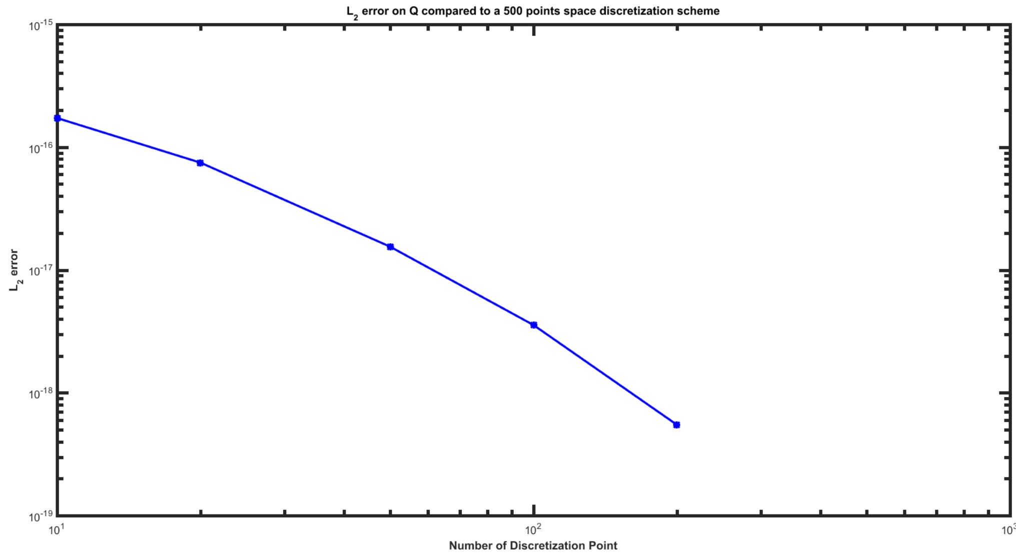
where S_{ref} is the numerical solution with $N_x = 500$ and S_{N_x} is the solution with a space discretization degraded. We take i varying from 1 to 10 since the most degraded numerical solution is for $N_x = 10$.

We define the same error for Q and see how it evolves as the space discretization gets finer (figure 13 and 14). It seems to converge?



Scheme 13: The $\mathcal{L}^2$ error for S solutions (soil moisture storage) in a log/log scale.

For a comparison the $\mathcal{L}^2$ norm for the S (resp. Q) solution (with $N_x = 500$) is approximately 0.018 (resp. 3.2×10^{-11}).



Scheme 14: The $\mathcal{L}^2$ error for Q solutions (soil moisture storage) in a log/log scale.

References

(1) Troch, P. A.; Paniconi, C.; van Loon, E. E. *Water Resources Research* **2003**, *39*.

(2) Fan, Y.; Bras, R. L. *Water Resources Research* **1998**, *34*, 921–927.