Numerical integration of hillslope storage Boussinesq equations

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Introduction

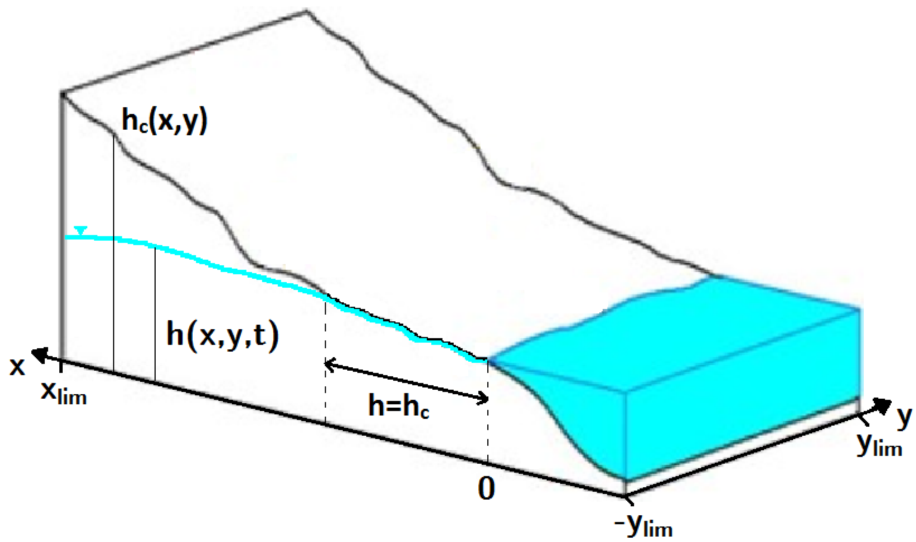
Boussinesq equations are highly non linear and their integration is often done on a variable integration volume. This is due to the fact that free surface of water can going up and intersect the surface. Then the water level is fixed at the surface elevation. In this paper we develop a new formulation of the hillslope storage boussinesq equations enabling to take into account the interception of the free surface with the surface. This new formulation is based on the use of Differential Algebraic Equations (DAEs). Indeed use of DAE enable to change from one time step to another the dimension and the type of equation governing the problem. This approach is, as far as we know, new.

Problem Statement

Let's consider a sloping aquifer in two dimension (cf. figure 1). h denotes the hydraulic head, h_c is the soil/air interface. Considering the bottom of the aquifer flat, we have the following equations uder the Dupuits-Forchheimer assumption:

$$\forall t > 0, \forall x \in [0, x_{lim}], \forall y \in [-y_{lim}, y_{lim}], \quad \begin{cases} f\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}(kh\frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(kh\frac{\partial h}{\partial y}) + N(t) \\ h(x,t) \leq h_c(x) = d(x) \end{cases} \tag{1}$$

where k is the hydraulic conductivity and f is the porosity of the aquifer and N is the source term of the equation (efficace rainfall, recharge, infiltration, etc.).



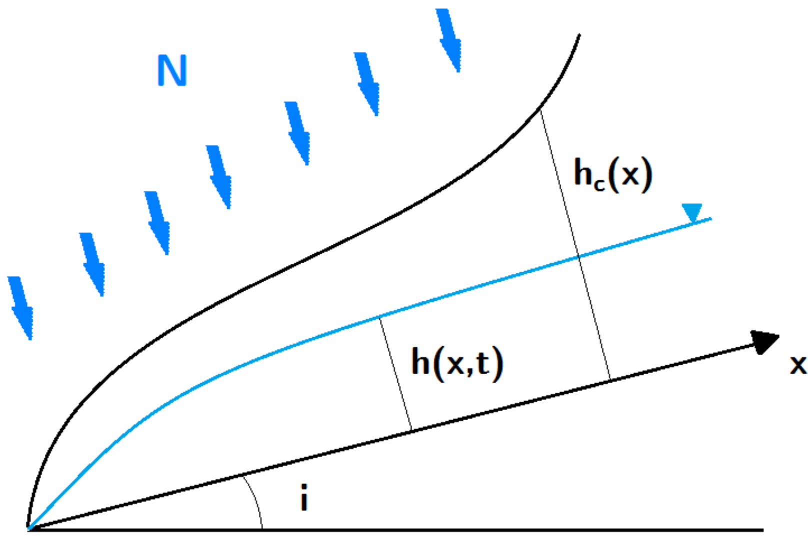
Scheme 1: 2D hillslope

In one dimension, the case where the bottom of the aquifer is not flat can be easily added (cf. figure 2). This leads to the following equations:

$$\forall t > 0, \forall x \in [0, x_{lim}], \quad \begin{cases} \frac{\partial h}{\partial t} = -\frac{\partial Q}{\partial x} + N(t) \\ Q = -kh(\cos i \frac{\partial h}{\partial x} + \sin i) \\ h(x,t) \leq h_c(x) = d(x) \end{cases} \tag{2}$$

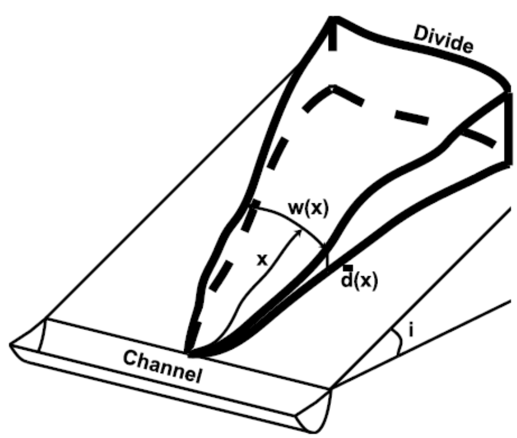
where Q is the Darcy flux and d is the maximum potential thickness of the aquifer. If we are interested in subsurface flow, d is typically the soil depth, subsurface flow taking place in the first meter under ground. Here the reference for h is different from equation 1. The reference for h is set at the bottom of the aquifer, having an elevation varying with depth.

Following Troch et al.[1], we can rewrite this problem by considering the soil moisture



Scheme 2: 1D hillslope. Adapted from Troch et al.[1].

storage $S = fwh$ instead of the hydrauilc head h. w represents the width function **?**. The width function w characterize for a given position x the width of one hillslope (cf. figure **??**).



Scheme 3: Width function visualization. Taken from Troch et al.[1].

This enables us to aggregate the response of a 2 dimensional aquifer into one dimension by writing the following equation on S and Q. This leads to the so called hillslope storage

Boussinesq equations:

$$\forall t > 0, \forall x \in [0, x_{lim}] \quad \begin{cases} \frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial x} + N(t)w(x) \\ Q = -\frac{kS}{f}(\cos i \frac{\partial}{\partial x}\frac{S}{fw} + \sin i) \\ S(x,t) \leq S_c(x) = fw(x)d(x) \end{cases} \tag{3}$$

Note that the angle i can vary with the distance to the stream ($x = 0$), i.e. $i = i(x)$.

We can add Dirichlet boundary condition at $x = 0$ and Neumann's at $x = x_{lim}$:

$$\begin{cases} S(x = 0, t) = 0 \\ Q(x = x_{lim}, t) = 0 \end{cases} \tag{4}$$

Finally, we can consider two types of initial conditions:

$$\begin{cases} S(x, t = 0) = 0 \\ Q(x, t = 0) = 0 \end{cases} \tag{5}$$

or

$$\begin{cases} S(x, t = 0) = R \cdot S_c(x) \\ Q(x, t = 0) = -\frac{k}{f} S_{t=0} \cdot (\cos i \frac{\partial}{\partial x}(\frac{S}{fw})\big|_{t=0} + \sin i) \end{cases} \tag{6}$$

where R represents the percentage of how much the hillslope is loaded.

Transforming the problem into a DAE

What is typically done with equation 3, is to replace Q by its Darcy expression. So the system reduces to the equation on S:

$$\begin{cases} f\frac{\partial S}{\partial t} = \frac{k \cos i}{f}(\frac{\partial}{\partial x}\big(\frac{S}{w}(\frac{\partial S}{\partial x} - \frac{S}{w}\frac{\partial w}{\partial x})\big) + k \sin i \frac{\partial S}{\partial x} + fNw \\ S(x,t) \leq S_c(x) = fw(x)d(x) \end{cases} \tag{7}$$

What we develop here is a code that computes both S and Q and stays with a formulation that separates conservation equation and Darcy flux equation. The major advantage of this formulation is to exactly conserve the mass in a finite difference scheme. The code also gains in clarity and computation's time with this formulation as it is obvious to write equations in a vectorized form.

The second thing is to transform the condition that prevent soil moisture storage to go over the maximal value S_c. To do this and in order to conserve mass in the equation we can consider the flux $q_S(x,t)$ which represents the flux going out of the soil if $S(x,t) = S_c(x)$.

Basically the idea is to partition the flux balance (flux entering minus flux exiting) of a given spatial domain between one part that goes to the variation of $\frac{\partial S}{\partial t}$ and another that go to the outflow flux q_S. Obviously, the partition's ratio depends on the value of S compared to S_c and on values of $\frac{\partial S}{\partial t}$.

We define:

$$q_{in} - q_{out} = -\frac{\partial Q}{\partial x} + Nw \tag{8}$$

Mathematically we have

$$\begin{cases} \frac{\partial S}{\partial t} = \alpha \cdot (q_{in} - q_{out}) \\ q_S = (1-\alpha) \cdot (q_{in} - q_{out}) \end{cases} \tag{9}$$

where α is defined as

$$\alpha = \begin{cases} 1 \quad \text{if} \quad \frac{\partial S}{\partial t} < 0 \quad || \quad S < S_c \\ 0 \quad \text{if} \quad \frac{\partial S}{\partial t} \geqslant 0 \quad \& \quad S = S_c \end{cases} \tag{10}$$

To do this we introduce the sigmoid g (see figure 4):

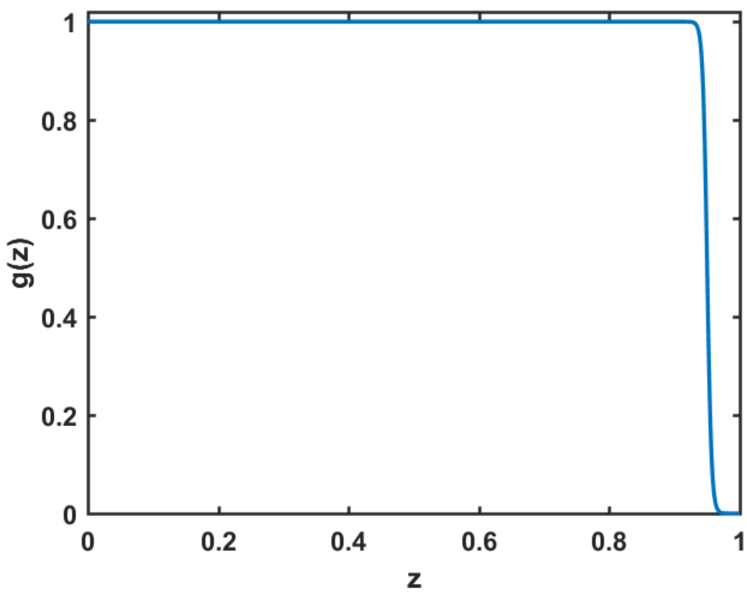
$$\forall z \in [0,1] \quad g(z) = 1 - \frac{1}{1 + \exp(-300 * (z - 0.95))} \tag{11}$$

and the function test h

$$\forall z \in [0,1] \quad h(z) = \begin{cases} 1 \quad \text{if} \quad z \geqslant 0 \\ \\ 0 \quad \text{if} \quad z < 0 \end{cases} \tag{12}$$

α becomes

$$\alpha = g(\frac{S}{S_c}) \cdot h(\frac{\partial S}{\partial t}) + \left(1 - h(\frac{\partial S}{\partial t})\right) \tag{13}$$



Scheme 4: Sigmoid

Finally, equation 3 becomes:

$$\begin{cases} \frac{\partial S}{\partial t} = \alpha \cdot (-\frac{\partial Q}{\partial x} + Nw) \\ \\ Q = -\frac{kS}{f}(\cos i \frac{\partial}{\partial x}\frac{S}{fw} + \sin i) \\ \\ q_S = (1-\alpha) \cdot (-\frac{\partial Q}{\partial x} + Nw) \end{cases} \tag{14}$$

After some arrangements, the system can be written like this:

$$\begin{cases} \frac{\partial S}{\partial t} & = & & - & \alpha \cdot \frac{\partial Q}{\partial x} & & & + & \alpha \cdot Nw \\ 0 & = & P(S) \cdot S & + & Q & & & & \\ 0 & = & & - & (1-\alpha) \cdot \frac{\partial Q}{\partial x} & - & q_s & + & (1-\alpha) \cdot Nw \end{cases} \tag{15}$$

where $P(S) = \frac{k}{f}(\cos i \frac{\partial}{\partial x} \frac{S}{fw} + \sin i)$.

Finally to prevent S to become negative, one can add a β coefficient to equation 15 so that if $\frac{\partial S}{\partial t} < 0$ & $S = 0$ then $\frac{\partial S}{\partial t} = 0$ for the next time step. We have

$$\beta = 1 - \delta\big(S\big) \cdot \left(1 - h\big(\frac{\partial S}{\partial t}\big)\right) \tag{16}$$

where δ is the dirac function.

The DAE we consider is finally:

$$\begin{cases} \frac{\partial S}{\partial t} & = & & - & \alpha \cdot \beta \cdot \frac{\partial Q}{\partial x} & & & + & \alpha \cdot \beta \cdot Nw \\ 0 & = & P(S) \cdot S & + & Q & & & & \\ 0 & = & & - & (1-\alpha) \cdot \frac{\partial Q}{\partial x} & - & q_s & + & (1-\alpha) \cdot Nw \end{cases} \tag{17}$$

Space Discretization

Let's now discretize the system 17 in space. Consider $N_x \in \mathbb{N}^+$. We define $\Delta x = x_{lim}/N_x$ and $\forall i \in \mathbb{N}$ $S_i(t) = S(x_i, t) = S(i \cdot \Delta x, t)$. Similar notations are used for Q and q_s.

Now we have:

$$S = \begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N_x} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{N_x} \end{bmatrix} \quad \& \quad q_s = \begin{bmatrix} q_{s0} \\ q_{s1} \\ \vdots \\ q_{sN_x} \end{bmatrix} \tag{18}$$

Discretization in space leads to:

$$\frac{\partial Q}{\partial x} = \frac{A}{2\Delta x} \cdot Q, \;\; where \;\; A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & -1 & 0 \end{pmatrix} \tag{19}$$

and

$$\frac{\partial S}{\partial x} = \frac{B}{\Delta x} \cdot S, \;\; where \;\; B = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & -1 \end{pmatrix} \tag{20}$$

and

$$\frac{S(x) + S(x + \Delta x)}{2} = \Omega \cdot S, \;\; where \;\; \Omega = \begin{pmatrix} 0.5 & 0.5 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0.5 \\ 0 & \dots & \dots & 0 & 0.5 \end{pmatrix} \tag{21}$$

Let y now denote the column vector: $\begin{bmatrix} S \\ Q \\ q_s \end{bmatrix}$. We have now a differential algebraic equation (DAE):

$$M \cdot \frac{\partial y}{\partial t} = C \cdot y + D \tag{22}$$

where M is the mass matrix defined by:

$$M = \left[\begin{array}{c|c|c} I_{N_x+1} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \tag{23}$$

C is defined by:

$$C = \left[\begin{array}{c|c|c} 0 & -\frac{A}{2\Delta x} & -I_{N_x+1} \\ \hline P(S) & \Omega & 0 \\ \hline 0 & R(S) & I_{N_x+1} \end{array}\right] \tag{24}$$

where:

$$\begin{array}{rrcl} & R(S) & = & \frac{1}{2\Delta x} \cdot diag(1 - g(\frac{S}{fdw})) \cdot A \\ and & P(S) & = & \frac{k}{f}(\cos i \frac{B}{\Delta x} \frac{S}{fw} + \sin i) \cdot \Omega \end{array} \tag{25}$$

and D by:

$$D = \left[\begin{array}{c} Nw \\ 0 \\ -diag(1 - g(\frac{S}{fdw})) \cdot Nw \end{array}\right] \tag{26}$$

References

(1) Troch, P. A.; Paniconi, C.; van Loon, E. E. *Water Resources Research* **2003**, *39*.