Conjecture on the Sign of the Functional Equation

The functional equation greatly speeds up the computation of the L-functions. If E has a bad reduction at infinity, Theorem 2.2 [1] cannot be applied for odd characters. Computing an extra coefficient is time-expensive, so we give the following conjecture on what the formula should be in this case.

Conjecture. *Let ℓ be prime, $q \equiv 1 \bmod \ell$, and χ be an odd primitive Dirichlet character of order ℓ with conductor coprime to N_E. We define $\alpha_{E,\ell,q,\chi}$ by*

$$\omega_{E\otimes\chi} = \alpha_{E,\ell,q,\chi}\omega_\chi^2\, \omega_E\, \chi(N_E)$$

where $\omega_{E\otimes\chi}$ is the sign of the functional equation of $L(E\otimes\chi, u)$, ω_χ is the sign of the functional equation of $L(\chi, u)$, and ω_E is the sign of the functional equation of $L(E, u)$. If $\chi_1|_{\mathbb{F}_q} \equiv \sigma(\chi_2)|_{\mathbb{F}_q}$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$, then $\alpha_{E,\ell,q,\chi_1} = \sigma(\alpha_{E,\ell,q,\chi_2})$.

Remark. *If χ is even, then $\alpha_{E,\ell,q,\chi} = 1$ because the prime at infinity isn't ramified and Theorem 2.2 [1] can be applied.*

We give some examples for $\ell = 3$. Since $\alpha_{E,\ell,q,\chi}$ depends on the restriction of χ to $\mathbb{F}_q$, we omit χ in the index. Since χ is odd, there are two possible restrictions to $\mathbb{F}_q$.

For $E_1 : y^2 = x^3 + t$ we have

$$\alpha_{E_1,3,7} = e^{2\pi i/3}, \text{ or } e^{4\pi i/3}.$$

For $E_2 : y^2 = x(x+t)(x+t^2)$ we have

$$\alpha_{E_2,3,7} = -1.$$

For $E_3 : y^2 = x(x-1)(x-t)$ we have

$$\alpha_{E_3,3,7} = -\frac{3}{7} - \frac{8}{7}e^{2\pi i/3}, \text{ or } -\frac{3}{7} - \frac{8}{7}e^{4\pi i/3}.$$

References

[1] Antoine Comeau-Lapointe, Chantal David, Matilde Lalin, and Wanlin Li. On the vanishing of twisted L-function of elliptic curves over rational function fields. *Proceedings of the Fifteenth Algorithmic Number Theory Symposium*, 2022.