Computational Finance, Fall 2018, Dr. M. Schürle

Semester Exercise – Pricing a Bermudan Basket Option with Monte Carlo

Motivation

The Monte Carlo method cannot immediately be applied to the pricing of American and other options with early exercise rights since their value depends on the exercise strategy. In order to exercise at a certain point in time, the attainable payoff must be greater than the expectation of the discounted future cash flows from a later exercise. This expected value itself depends if and when the option will be exercised in the future.

If a large number of paths for the price of the underlying is generated by Monte Carlo simulation, one must not determine the "optimal" exercise strategy along each (single) path. This would imply a "perfect foresight" of the future price evolution which is in reality not given. On the other hand, in the binomial method applied to options with early exercise rights a decision (on exercise) is made in each node, except for the final stage, by comparing the immediately attainable payoff with the expectation of the holding strategy for a "favorable" and an "unfavorable" future state (up vs. down).

A possible way to apply the Monte Carlo method to options with early exercise rights was suggested by Longstaff and Schwartz (2001). The expectation of the "no-exercise strategy" is estimated from cross-sectional information from all simulated paths. This approach is considered the most promising method to apply Monte Carlo to options with early exercise features. Its practical importance results from the fact that other techniques cannot be used for more general valuation problems in the following situations:

- Additional state variables: Monte Carlo can also be used when not only the price of the underlying but also other quantities (interest rate, volatility etc.) are stochastic and interdependent.
- Curse of dimensionality: Although one additional stochastic factor could be taken into account by using a trinomial instead of a binomial tree, in higher dimensions the construction of a lattice is no longer practicable.
- Convergence: Without variance reduction methods, Monte Carlo requires for an improvement in the accuracy (width of a confidence interval) by a factor 10 an increase in the computational effort by a factor 100. On the other hand, this effort is at least independent of the dimension of the underlying distribution.
- Flexibility in modeling the price dynamics: General types of stochastic processes may be used to model the dynamics of the stochastic factors and dependencies between them.
- Path-dependent payoffs can be taken into account.

In order to keep things simple, most of these aspects play no role in this exercise. Here the Longstaff/Schwartz method should be applied to price a Bermudan best-of-basket option (recall: a Bermudan option may be exercised only at certain defined dates before the expiry date, while an American option can be exercised anytime). For illustrative purposes, the approach is first outlined for an American put on a non-dividend paying stock.

Longstaff-Schwartz method for a single stock

First the paths for the prices of the stock are simulated for a pre-defined number of steps. Then the payoff is determined at the end of each path at time T (expiry of the option). For all earlier points in time $t = T - 1, \ldots, 1$ the following procedure is repeated: Assume that for a given stock price S_t the value of the no-exercise strategy can be expressed by a nonlinear function that depends on S_t. The coefficients of this function are estimated by a regression over all paths with the stock prices at time t as independent (explanatory) variables and the observed values of the no-exercise strategy in $t + 1$, discounted over one period, as dependent variables. In this estimation only paths with payoff greater zero are considered.

For these paths the present value of the future cash flows (if not exercised), given by the assumed function with the estimated coefficients, is compared with the payoff from an early exercise. If the latter is higher, the option is exercised. In this way the exercise strategies are determined for all paths and points in time. The mean of the discounted payoffs is then equivalent to the option price.

Example: American put on single stock

Consider the case of an American put with expiry in 3 periods and strike price $K = 1.1$. The risk-free rate is $r = 0.06$ and, thus, the one-period discount factor is $e^{-0.06} = 0.9418$. Starting from an initial stock price $S_0 = 1.0$ a simulation of 8 paths over 3 periods leads to the following future prices:

price paths				
	0	1	2	3
1	1.000	1.090	1.080	1.340
2	1.000	1.160	1.260	1.540
3	1.000	1.220	1.070	1.030
4	1.000	0.930	0.970	0.920
5	1.000	1.110	1.560	1.520
6	1.000	0.760	0.770	0.900
7	1.000	0.920	0.840	1.010
8	1.000	0.880	1.220	1.340

cash flows in t = 3				
	0	1	2	3
1				0.000
2				0.000
3				0.070
4				0.180
5				0.000
6				0.200
7				0.090
8				0.000

The payoffs from an exercise at the expiry date $t = 3$ are shown in the table on the right.

Now the attainable cash flows from early exercise in $t = 2$ must be compared with the present values of the no-exercise strategy (holding the option). To this end, it is assumed that the latter can be estimated from a "continuation function" of the type

$$E[Y \mid S] = c_0 + c_1 * S + c_2 * S^2$$

where Y represents the discounted cash flow realized after one period ($t = 3$), if the option is not exercised at the present time ($t = 2$) for the observed price S. c_0, c_1 and c_2 are coefficients that are estimated with linear regression. The situation is now

regression in t = 2				
	0	1	2	3
1			1.080	0.000
2			1.260	0.000
3			1.070	0.066
4			0.970	0.170
5			1.560	0.000
6			0.770	0.188
7			0.840	0.085
8			1.220	0.000

value of strategies		
	exercise	hold
1	0.020	0.037
2		
3	0.030	0.046
4	0.130	0.118
5		
6	0.330	0.152
7	0.260	0.156
8		

The table on the left shows in the column for t = 3 the option values at expiry, discounted at the factor 0.9418, that will be used as dependent variables in the regression. The stock prices in t = 2 serve as independent variables, but only those paths are taken into account where the option is in the money ("moneyness criterion", see underlined values in the table). Otherwise an early exercise would not make sense anyway. The estimation yields

$$E[Y \mid S] = -1.070 + 2.983 * S - 1.813 * S^2$$

The table on the right displays in the column "hold" the results when the continuation function is applied to the stock prices in the relevant paths. The attainable cash flows from early exercise in the paths 4, 6 and 7 are higher, therefore it is exercised. In the paths 1 and 3 the option is hold. The payoffs from exercise are shown in the next table. Obviously, in t = 3 the values on such paths where the option was exercised already at an earlier point in time must be zero.

cash flows in t = 2				
	0	1	2	3
1			0.000	0.000
2			0.000	0.000
3			0.000	0.070
4			0.130	0.000
5			0.000	0.000
6			0.330	0.000
7			0.260	0.000
8			0.000	0.000

Now the procedure is repeated to determine the exercise strategy at time t = 1. The following table contains in t = 2 again the values of the option if it is not exercised in t = 1. In the case of the paths 4, 6 and 7 these follow from the payoffs of an exercise in t = 2, discounted at the factor 0.9418. In the case of path 3 the value 0.062 corresponds to the discounted value of the no-exercise strategy, i.e., the payoff at expiry in t = 3, discounted over two periods: $0.07 \cdot e^{-0.12}$. These values serve again as dependent variables and the stock prices observed in t = 1 as explanatory variables in the estimation of the above function by regression. As before, only paths are considered where the payoff is greater zero (underlined).

regression in t = 1				
	0	1	2	3
1		1.090	0.000	
2		1.160	0.000	
3		1.220	0.066	
4		0.930	0.122	
5		1.110	0.000	
6		0.760	0.311	
7		0.920	0.245	
8		0.880	0.000	

value of strategies		
	exercise	hold
1	0.010	0.014
2		
3		
4	0.170	0.109
5		
6	0.340	0.287
7	0.180	0.118
8	0.220	0.153

It results from the estimation by linear regression:

$$E[Y \mid S] = 2.038 - 3.335 * S + 1.356 * S^2$$

The table on the right compares again the payoff from early exercise with the expected values of the decision "hold" that are obtained from this function. In path 1 the option is hold while in the paths 4, 6, 7 and 8 it is exercised. One obtains the following payoffs from the exercise strategy where again on paths with exercise all later cash flows are set to zero in the matrix:

cash flows in t = 1				
	0	1	2	3
1		0.000	0.000	0.000
2		0.000	0.000	0.000
3		0.000	0.000	0.070
4		0.170	0.000	0.000
5		0.000	0.000	0.000
6		0.340	0.000	0.000
7		0.180	0.000	0.000
8		0.220	0.000	0.000

Now the first stage (t = 0) is reached, and the option price is calculated as the mean of the discounted payoffs over all paths:

$$V_0 = [\,(0.17 + 0.34 + 0.18 + 0.22)\, e^{-0.06} + 0.07\, e^{-.018}\,] / 8 = 0.1144$$

Programming assignment

Consider now a call on the best performing asset of a basket of stocks. In case of up to three assets, this problem can also be solved with the finite difference method. Beyond three dimensions, the latter becomes impractical and Monte Carlo must be applied since here the computational effort is independent of the dimension.

Let n be the number of assets in the basket. Each individual stock follows a Geometric Brownian motion

$$dS_i = (r - y_i)\, S_i\, dt + \sigma_i\, S_i\, dW_i, \qquad i = 1,\ldots,n,$$

where y_i is the yield of a continuously paid dividend. Notice that the subscript is used to denote different assets. The Wiener processes for two different assets may be correlated, i.e., $dW_i\, dW_j = \rho_{ij}\, dt$, $i \neq j$. The payoff function of the Bermudan call at exercise time τ is

$$\text{payoff} = \max\left\{\max S_i(\tau) - K, 0\right\}$$

i.e., relevant for the payoff is the highest asset price in the basket. It is assumed that all assets have the same initial price at the valuation date. This may be achieved, e.g., by standardization of all asset prices to an initial value of 100.

Write a function in Matlab that implements the Longstaff/Schwartz (2001) method for a Bermudan call with the above specified payoff function which depends on the best performing asset in a basket of dividend paying stocks. The function should receive the following parameters:

- Initial stock prices $S_1(0), \ldots, S_n(0)$,
- strike price level K as a percentage of the initial asset prices,
- time to expiry T,
- riskless rate r (annualized, continuously compounded),
- dividend yields $y_1, \ldots, y_n$,
- volatilities $\sigma_1, \ldots, \sigma_n$,
- correlation matrix of the Wiener processes in the above process equation and
- number of exercise opportunities which are by assumption equidistant during the time interval $[0, T]$.

Beside these option characteristics, also technical parameters like the number of paths n should be passed to the function so that the impact of different values can be studied easily. The function should return the option value and the sample standard deviation of the estimator. It should in principle work for an arbitrary number of basket components.

Generate graphs that show the option value and the end points of a 95%-confidence interval for various values for the number of paths $n \in \{10, 50, 100, 500, 1000, 5000, 10000, \ldots\}$ in combination with 10, 15 and 30 exercise opportunities. Document your results for a Bermudan call on a best-of-three basket with the parameters

$$\begin{aligned} S_1(0) = S_2(0) = S_3(0) &= 100 \\ y_1 = y_2 = y_3 &= 0.1 \\ \sigma_1 = \sigma_2 = \sigma_3 &= 0.2, \end{aligned}$$

the correlation matrix

$$\begin{pmatrix} 1 & -0.25 & 0.25 \\ -0.25 & 1 & 0.3 \\ 0.25 & 0.3 & 1 \end{pmatrix}$$

and strike price level $K = 100$, risk-free rate $r = 0.05$ and $T = 3$ years. For these parameters, an option value of 18.082 for 30 exercise opportunities can be derived with the finite difference method. Of course, your result may deviate slightly, depending on the seed of the random number generator.

Use the idea of the earlier described "moneyness criterion" for your implementation, i.e., only paths where the option is in the money should be considered for the estimation of the continuation value. As a variation to the single-asset case, it is recommended to regress on the prices of the two highest assets. Assume that S_{I} and S_{II} are the two highest asset prices at a given time step, then the regression may be performed on

$$E[Y \mid S_{\mathrm{I}}, S_{\mathrm{II}}] = c_0 + c_1 S_{\mathrm{I}} + c_2 S_{\mathrm{I}}^2 + c_3 S_{\mathrm{II}} + c_4 S_{\mathrm{II}}^2 + c_5 S_{\mathrm{I}} S_{\mathrm{II}},$$

where again Y represents the discounted cash flow realized after one period. You may also experiment with variations of this continuation function which take into account also products of higher powers of S_{I} and S_{II} and document your insights. Think also of variance reduction methods.

Literature

Francis A. Longstaff, Eduardo S. Schwartz: Valuing American Options by Simulation – A Simple Least-Squares Approach. *Review of Financial Studies*, 14 (2001) 113 – 147