The KPT Correspondence

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Topological flows

Definition

Let G be a topological group.

A topological flow, or G-flow, is a compact space X, along with a continuous G-action $G \curvearrowright X$.

More precisely, there is a continuous map $\phi : G \times X \to X$.

Example : If T is a homeomorphism of X, $\mathbf{Z} \curvearrowright X$ by repeated applications of T.

$\mathbf{Q}$ and extreme amenability

We consider the structure $(\mathbf{Q}, <)$ and more precisely, its group of automorphism :

$$\mathrm{Aut}(\mathbf{Q}) := \{f : \mathbf{Q} \to \mathbf{Q} : f \text{ is an order-preserving bijection}\}.$$

We give it the pointwise convergence topology :

$$f_i \to f \iff f_i(x) \to f(x), \forall x \in \mathbf{Q}.$$

$\mathbf{Q}$ and extreme amenability

Surprisingly, the following property, called extreme amenability, holds :

Theorem (Extreme Amenability of $\mathrm{Aut}(\mathbf{Q})$, Pestov)

Every $\mathrm{Aut}(\mathbf{Q})$*-flow has a global fixed point.*

For example : $2^{\mathbf{Q}}$ is a compact space having a natural action $\mathrm{Aut}(\mathbf{Q}) \curvearrowright 2^{\mathbf{Q}}$. The identically 0 sequence is a global fixed point.

$\mathbf{Q}$ and Fraïssé limits

There is one interesting property to extract of $(\mathbf{Q}, <)$:

- If $L, L' \subset \mathbf{Q}$ are finite linear orders and there is an isomorphism between them, we can extend it to an automorphism on $\mathbf{Q}$.

We say that $\mathbf{Q}$ is **ultrahomogenous**. Also, $\mathbf{Q}$ is countable. This is how we define Fraïssé limits.

Age of a Fraïssé limit and the Ramsey property

The Age of a Fraïssé limit F is the class of finite substructures of F. Ages satisfy some nice amalgamation property.

For $\mathbf{Q}$, we have the class of finite linear orders. This also has the "Ramsey property" :

Theorem (Ramsey's theorem)

Fix $k \in \mathbf{N}$. If the k-elements subsets of $\mathbf{Q}$ are colored, for all $N \in \mathbf{N}$, there is a N-element subset of $\mathbf{Q}$ whose k-element subsets are all of the same color.

Ramsey Property : examples

Definition (Ramsey Property)

The class Age(F) has the **Ramsey Property** if whenever A is a fixed finite substructure of F and the copies of A in F are colored, we have that for each finite substructure $B \subset F$ there exists some substructure $B' \cong B$ such that all copies of A in B' have the same color.

For example, the class of finite triangle free graph has this property, as well as the class of all finite graphs.

The Rado graph

The Rado graph is constructed as such :

- Pick a countable set, like $\mathbf{N}$.
- For every pair of points, flip a coin. If it lands on heads, put an edge inbetween them. If tails, don't.

This will almost always give the same graph, up to isomorphism. This just so happens to be a Fraïssé limit, We are interested in the dynamics of $\mathrm{Aut}(R)$. More precisely, is $\mathrm{Aut}(R)$ extremely amenable ?

The KPT Correspondence

We recall that a structure is rigid if it has no non-trivial automorphisms. For example, well-ordering are rigid.

Theorem (Kechris-Pestov-Tordorčević, rephrased by Nguyen Van Thé)

Let F be a Fraïssé limit for $\mathrm{Age}(F)$*. Then, the following are equivalents :*

- $\mathrm{Aut}(F)$ *is extremely amenable.*
- $\mathrm{Age}(F)$ *has the Ramsey property and its elements are rigid.*

What about R ?

For the Rado graph R, its Age is not made out of rigid elements. Therefore, it must act continuously on a compact space without global fixed points.

Corollary

The KPT correspondence implies the followings :

- *The automorphism group of the Rado graph is not extremely amenable.*
- *There exists an ordering $<$ of the Rado graph such that the subgroup of* $\mathrm{Aut}(R)$ *of automorphisms respecting $<$ is extremely amenable.*

Bibliography

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Thank you !