Geometric algebra and Quaternions

A. Herrmann
Department of Physics, University of South Florida, Tampa, FL, United States

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1 Geometric Algebra

The geometric algebra is a way to understand the product of two vectors through the consideration of a vectorial space containing both vectors and trying to define a close space which contains all possible results of the product of these two vectors.

1.1 Inner Product

As a first observation, the definition of a vector necessitates a vector space $\mathscr{V}$ and a basis in which the vector will be expressed. Let's call this basis $\mathscr{B}$ in 3D.

$$\mathscr{B} = \{\hat{x}, \hat{y}, \hat{z}\} \tag{1.1.1}$$

The choice of the 3D limitation is due to the use of these notions for physical applications and the object in physics belongs to a 3D space plus a 1D time space.

The vector of this basis follows the rules :

$$\begin{cases} |\hat{x}| = |\hat{y}| = |\hat{z}| = 1 \\ \hat{x} \perp \hat{y} \perp \hat{z} \end{cases} \tag{1.1.2}$$

There is to define the notion of norm and perpendicularity.

To do so, let's introduce the inner product $(\cdot)$ from the space of vectors to numbers such that :

$$\forall \ \vec{v}, \vec{w} \in \mathscr{V}, \qquad \begin{cases} \vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \\ \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|cos(\vec{v}, \vec{w}) \\ |\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} \end{cases} \tag{1.1.3}$$

The presence of the inner product requires then to consider the total space as a larger space than only the space of the vector. Furthermore, this inner product is not sufficient to extract all the information contained in a set vector because it only returns the length of these vectors and their relative projection one on another. It's thus necessary to introduce an operation which returns an information about the orientation in the space between two vectors. Let's call it the outer product $(\wedge)$.

1.2 Outer Product

This outer product has to contain the information of the orientation of the two spins one with respect to the other and of the area enclosed by these two vectors. It's possible to combine these two informations in an oriented area $\hat{A}$ in return of this outer product. Combined with the inner product, these two operations then contains the whole information that can be extracted from two vectors.

The oriented area changes of "sens of rotation" with the swap of the order of the two vectors in input. So let's define it as :

$$\forall \ \vec{v}, \vec{w} \in \mathscr{V}, \qquad \begin{cases} \vec{v} \wedge \vec{w} = -\vec{w} \wedge \vec{v} \\ \vec{v} \wedge \vec{w} = |\vec{v}||\vec{w}|sin(\vec{v}, \vec{w})\hat{A} \end{cases} \tag{1.2.1}$$

The nature of the oriented area is then not a number, nor a vector. The description of the total space needs then to be extended by introducing a basis of these kind of objects. Let's call them *bi-vectors* because they are made of two vectors and behaves as vectors of higher dimension.

The product between two vectors $\vec{v}, \vec{w}$ belonging to the vector space $\mathscr{V}$ can also be rewritten as :

$$\vec{v}\vec{w} = \vec{v} \cdot \vec{w} + \vec{v} \wedge \vec{w} \tag{1.2.2}$$

Note that the product between two vector is not commutative because it is made of a symmetric part (inner product) and an anti-symmetric part (outer product). The order of multiplying is thus very important and has to be at the center of a particular attention.

This strong definition of the geometric product of two vectors requires then to creates different objects of dimension 0 for scalars, 1 for vectors, 2 for bi-vectors. However, the space is three dimensional so it can contain 3D objects which are so called tri-vectors and represents oriented volumes in the space.

1.3 Basis of the total space

In the case of multiplying three or more vectors together, the product of two of them will return a set of objects which can be scalars, vectors or bi-vectors. Multiplying this ensemble of objects requires to know how to multiply a vector and a bi-vector.

From definition 1.2.1, any $\hat{b}$ bi-vector can be expressed from two vectors $\vec{v}, \vec{w}$ as :

$$\breve{b} = \frac{1}{2}(\vec{v}\vec{w} - \vec{w}\vec{v}) \tag{1.3.1}$$

The multiplication of this bi-vector and a vector $\vec{r}$ is then :

$$\begin{aligned}\vec{r}\breve{b} &= \vec{r}\frac{1}{2}(\vec{v}\vec{w} - \vec{w}\vec{v}) \\ &= \frac{1}{2}(\vec{r}\vec{v}\vec{w} - \vec{r}\vec{w}\vec{v})\end{aligned} \tag{1.3.2}$$

This shows that the product of a vector and a bi-vector is an oriented volume in the space which requires a three dimensional basis to be described.

The different combinations of the three basis vectors $\{\hat{x}, \hat{y}, \hat{z}\}$ of the 3D space can however be derived from the commutations relations of the basis vectors. Thus, in 3D, there is only one relevant combination of $\{\hat{x}, \hat{y}, \hat{z}\}$ to describe completely the space of the tri-vectors.

Moreover, the 3D space cannot contain higher dimensional objects so adding this last combination is sufficient to span entirely the output space of the Geometric Product. The basis of the total space is then :

$$\mathscr{T} = \{1, \hat{x}, \hat{y}, \hat{z}, \hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}, \hat{x}\hat{y}\hat{z}\} \tag{1.3.3}$$

1.4 Commutations of the basis vectors

Let's see how do the basis vectors 1.3.3 behaves under the geometrical product defined above. To do this, the basis is assumed to be orthonormal such that :

$$\begin{cases}\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{x}\cdot\hat{z} = 0 \\ \hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1\end{cases} \tag{1.4.1}$$

Obviously, the scalar 1 is the identity element and commutes with every other vectors. Furthermore, the 2D and 3D basis vectors are composed by geometrical product of 1D basis vectors so it's only necessary to define the commutation relations for these ones.

It is immediate that due to the orthogonality of the basis vectors and the anti-symmetric nature of the outer product, the geometrical product of two vectors is anti-commutative :

$$\begin{cases}\hat{x}\hat{y} = \hat{x}\wedge\hat{y} = -\hat{y}\wedge\hat{x} = -\hat{y}\hat{x} \\ \hat{y}\hat{z} = \hat{y}\wedge\hat{z} = -\hat{z}\wedge\hat{y} = -\hat{z}\hat{y} \\ \hat{x}\hat{z} = \hat{x}\wedge\hat{z} = -\hat{z}\wedge\hat{x} = -\hat{z}\hat{x}\end{cases} \tag{1.4.2}$$

This anti-commutation can be understood as the reverse orientation of the oriented area when the order of the vectors defining this area is switched.

From this, it's possible to deduce the commutation relations of the bi-vectors :

$$\begin{cases} (\hat{x}\hat{y})(\hat{y}\hat{z}) = \hat{x}\hat{z} \\ (\hat{x}\hat{y})(\hat{z}\hat{x}) = \hat{y}\hat{z} \\ (\hat{y}\hat{z})(\hat{z}\hat{x}) = \hat{y}\hat{x} \end{cases} \tag{1.4.3}$$

The normalization of the basis vectors comes then naturally out as :

$$(\hat{x}\hat{y})^2 = (\hat{y}\hat{z})^2 = (\hat{z}\hat{x})^2 = (\hat{x}\hat{y}\hat{z})^2 = -1 \tag{1.4.4}$$

A further exploration of the product of the basis vectors shows that the product of a vector and a bi-vector returns either a vector if the bi-vector and the vector belongs to the same plane, or a tri-vector if the bi-vector and the vector are orthogonal. By the same way, the product of a vector and a tri-vector returns a bi-vector and the product of a bi-vector and a tri-vector returns a vector. This is shown in equation (1.7.2) for only one case.

$$\begin{cases} \hat{x}(\hat{x}\hat{y}) = \hat{y} \\ \hat{x}(\hat{y}\hat{z}) = \hat{x}\hat{y}\hat{z} \\ (\hat{x}\hat{y})(\hat{x}\hat{y}\hat{z}) = -\hat{z} \end{cases} \tag{1.4.5}$$

1.5 Complex conjugate

By the same way than for complex numbers, the complex conjugate of an object in geometric algebra is taken such that the sum of the original object and its complex conjugate is a real number. From the definition of the Geometric product (1.2.2), the point is to cancel the term of the wedge product. Then, a good choice is to define :

$$(\vec{v}\vec{w})^* = \vec{w}\vec{v} \tag{1.5.1}$$

This implies that the multiplication of an object on the right is equivalent to multiplying this object on the left by the complex conjugate such that :

$$\begin{aligned} \vec{u}(\vec{v}\vec{w}) &= \vec{u}(\vec{v}\cdot\vec{w} + \vec{v}\wedge\vec{w}) \\ &= \vec{u}(\vec{v}\cdot\vec{w}) + \vec{u}\cdot(\vec{v}\wedge\vec{w}) + \vec{u}\wedge\vec{v}\wedge\vec{w} \\ &= (\vec{v}\cdot\vec{w})\vec{u} + (\vec{v}\wedge\vec{w})\cdot\vec{u} + \vec{v}\wedge\vec{w}\wedge\vec{u} \\ &= (\vec{w}\cdot\vec{v})\vec{u} - (\vec{w}\wedge\vec{v})\cdot\vec{u} - \vec{w}\wedge\vec{v}\wedge\vec{u} \\ &= (\vec{w}\cdot\vec{v} - \vec{w}\wedge\vec{v})\vec{u} \\ &= (\vec{w}\vec{v})^*\vec{u} \end{aligned} \tag{1.5.2}$$

This operation leaves the vectors invariant but flips the sign of the bi-vectors and tri-vectors. This tends to show sort of an equivalence between the scalars and tri-vectors and the vectors and bi-vectors as if one is the dual component of the other one.

1.6 Inverse of an object

The completion of the space requires the existence of an inverse element for each element of the space defined by (1.3.3). This inverse is defined such that :

$$\forall v \in \mathscr{T}, \quad vv^{-1} = 1 \tag{1.6.1}$$

where v^{-1} designates the inverse element and 1 is the real unit.

From this statement, it's obvious that the inverse of a scalar is the inverse element defined for real numbers.

$$(r)^{-1} = \frac{1}{r} \tag{1.6.2}$$

For vectors the only way to fall back on a real scalar form the geometric product is to make vanish the wedge product so :

$$(\vec{v})^{-1} = \frac{1}{|\vec{v}|^2}\vec{v} \tag{1.6.3}$$

Bi-vectors behave like vectors except that their norm is a defined negative application. Then the same rule applies than for vectors at a negative sign :

$$(\overset{\curvearrowleft}{b})^{-1} = -\frac{1}{|\hat{b}|^2}\overset{\curvearrowleft}{b} \tag{1.6.4}$$

where the negative sign can be interpreted as the reversion of the orientation of the area.

Finally, tri-vectors' inverse are defined as :

$$(\overset{\circlearrowleft}{t})^{-1} = -\frac{1}{|\hat{t}|^2}\overset{\circlearrowleft}{t} \tag{1.6.5}$$

The very similar definition of the inverses is a consequence of the choice of the inverse as a real unit which commutes with every elements of the space (1.3.3).

1.7 pseudo-quantities in 3D

There is a surprising coincidence in 3D which is that the basis of vectors and bi-vectors have the same size. Moreover, these two objects behaves the same except that the norm of a bi-vector is negative. In this case, it's interesting to try to find a mapping between these two basis such that it's possible to make calculations using the properties of only one of these.

To do so, note that an oriented surface can be assimilated to a "plane with a rotation around an axis". This axis is taken perpendicular to the plane and the "sens of rotation" is given by the order of consideration of the bi-vector. Therefore the sens of rotation can take only two values : counter-clockwise and clockwise. The analogy with complex numbers is the introduction of the imaginary number i.

From equation (1.7.2), the tri-vector commutes with every elements of the basis (1.3.3). Further more, it is normalized to a negative constant with equation (1.4.4) which makes it behave like a scalar. Finally, the relation (1.7.2) shows that there is a dual relation between vectors and bi-vectors which are related by the multiplication with a tri-vector.

It's then convenient to build :

$$\hat{x}\hat{y}\hat{z} = i \tag{1.7.1}$$

which introduces a mapping from scalar to tri-vector. From this there is a mapping from bi-vectors to vectors such that :

$$\begin{cases} \hat{x}\hat{y} = i\hat{z} \\ \hat{y}\hat{z} = i\hat{x} \\ \hat{z}\hat{x} = i\hat{y} \end{cases} \tag{1.7.2}$$

This new mapping introduces new objects such that the bi-vectors are mapped to complex vectors which are so called "*pseudo-vectors*" and the tri-vector which is equivalent to the imaginary number and then called "*pseudo-scalar*".

This similarity relation introduces the notion of *dual spaces* of conventional objects and *pseudo*-objects where the transition element is the imaginary number i.

1.8 Duality Wedge product / Cross product

The one to one mapping between bi-vectors and vectors and tri-vectors and scalars introduces a way to write each bi-vector as a vector at an imaginary unit. This helps to develop the geometrical product of more than two vectors.

Consider here $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$. The product of these three vectors returns :

$$\begin{aligned}\vec{a}\vec{b}\vec{c} &= \vec{a}(\vec{b}\vec{c}) \\ &= \vec{a}(\vec{b}\cdot\vec{c} + \vec{b}\wedge\vec{c}) \\ &= \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}\cdot(\vec{b}\wedge\vec{c}) + \vec{a}\wedge\vec{b}\wedge\vec{c}\end{aligned} \tag{1.8.1}$$

The first term is a vector by composition of a vector and a scalar. The second term is a vector dot a bi-vector which returns an oriented area projected along a complex bi-vector perpendicular to the vector $\vec{a}$ with the use of the mapping (1.7.2). This quantity behaves as a scalar due to the dot product but is of a very different nature due to the wedge product. The last term is a pure tri-vector made of composition of three vectors.

The mapping (1.7.2) can then be inserted in the definition of the geometrical product (1.2.2) to work only with vectors and scalars. Then the definition is modified as :

$$\vec{v}\vec{w} = \vec{v}\cdot\vec{w} + i\vec{v}\times\vec{w} \tag{1.8.2}$$

where $(\times)$ is the outer product for vectors only and returns a vector with the condition :

$$\forall\ \vec{v}, \vec{w} \in \mathscr{V}, \qquad \begin{cases} \vec{v}\times\vec{w} = -\vec{w}\times\vec{v} \\ \vec{v}\times\vec{w} = |\vec{v}||\vec{w}|sin(\vec{v},\vec{w})\vec{n} \\ \vec{n}\cdot\vec{v} = \vec{n}\cdot\vec{w} = 0 \end{cases} \tag{1.8.3}$$

With this new outer product, the triple product (1.8.4) becomes :

$$\begin{aligned}\vec{a}\vec{b}\vec{c} &= \vec{a}(\vec{b}\vec{c}) \\ &= \vec{a}(\vec{b}\cdot\vec{c} + i\vec{b}\times\vec{c}) \\ &= \vec{a}(\vec{b}\cdot\vec{c}) + i\vec{a}\cdot(\vec{b}\times\vec{c}) + i^2\vec{a}\times\vec{b}\times\vec{c}\end{aligned} \tag{1.8.4}$$

Proof : Consider the three vectors as $\vec{a} = a_1\hat{x} + a_2\hat{y} + a_3\hat{z}$, $\vec{b} = b_1\hat{x} + b_2\hat{y} + b_3\hat{z}$, $\vec{c} = c_1\hat{x} + c_2\hat{y} + c_3\hat{z}$:

$$
\begin{aligned}
\vec{a}\vec{b}\vec{c} &= (a_1\hat{x} + a_2\hat{y} + a_3\hat{z})(b_1\hat{x} + b_2\hat{y} + b_3\hat{z})(c_1\hat{x} + c_2\hat{y} + c_3\hat{z}) \\
&= (a_1\hat{x} + a_2\hat{y} + a_3\hat{z})(b_1c_1\hat{x}\hat{x} + b_1c_2\hat{x}\hat{y} + b_1c_3\hat{x}\hat{z} \\
&\qquad + b_2c_2\hat{y}\hat{y} + b_2c_1\hat{y}\hat{x} + b_2c_3\hat{y}\hat{z} \\
&\qquad + b_3c_3\hat{z}\hat{z} + b_3c_1\hat{z}\hat{x} + b_3c_2\hat{z}\hat{y}) \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + (a_1\hat{x} + a_2\hat{y} + a_3\hat{z})\big[(b_1c_2 - b_2c_1)\hat{x}\hat{y} + (b_2c_3 - b_3c_2)\hat{y}\hat{z} + (b_3c_1 - b_1c_3)\hat{z}\hat{x}\big] \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + a_1(b_1c_2 - b_2c_1)\hat{x}\hat{x}\hat{y} + a_1(b_3c_1 - b_1c_3)\hat{x}\hat{z}\hat{x} + a_1(b_2c_3 - b_3c_2)\hat{x}\hat{y}\hat{z} \\
&\qquad + a_2(b_2c_3 - b_3c_2)\hat{y}\hat{y}\hat{z} + a_2(b_1c_2 - b_2c_1)\hat{y}\hat{x}\hat{y} + a_2(b_3c_1 - b_1c_3)\hat{y}\hat{z}\hat{x} \\
&\qquad + a_3(b_3c_1 - b_1c_3)\hat{z}\hat{z}\hat{x} + a_3(b_2c_3 - b_3c_2)\hat{z}\hat{y}\hat{z} + a_3(b_1c_2 - b_2c_1)\hat{z}\hat{x}\hat{y} \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{c})\hat{x}\hat{y}\hat{z} + \big[a_3(b_3c_1 - b_1c_3) - a_2(b_1c_2 - b_2c_1)\big]\hat{x} \\
&\qquad + \big[a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)\big]\hat{y} \\
&\qquad + \big[a_2(b_2c_3 - b_3c_2) - a_1(b_3c_1 - b_1c_3)\big]\hat{z} \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{c})\hat{x}\hat{y}\hat{z} + (\vec{b}\times\vec{c})\times\vec{a} \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}\cdot(\vec{b}\times\vec{c})\hat{x}\hat{y}\hat{z} - \vec{a}\times(\vec{b}\times\vec{c}) \\
&= \vec{a}(\vec{b}\cdot\vec{c}) + i\vec{a}\cdot(\vec{b}\times\vec{c}) + i^2\vec{a}\times\vec{b}\times\vec{c}
\end{aligned} \tag{1.8.5}
$$

2 Reflections and Rotations

The great advantage of the geometric algebra is the new objects of bi-vectors which, as oriented areas are understandable as a rotating plane where the sens of rotation is the orientation of the area and the speed of rotation, the amplitude of the bi-vector.

2.1 Projections and Rejections

From the definition of inverse (1.6.1), it's always possible to describe an object by multiplying on the left by the real unit such that two quantities appears :

$$\vec{a} = \vec{a}mm^{-1} = (\vec{a}\cdot m + \vec{a}\wedge m)m^{-1} = \vec{a_\parallel} + \vec{a_\perp} \tag{2.1.1}$$

$\vec{a_\parallel}$ is called the projection and corresponds to the projection of the vector $\vec{a}$ on the object m with the alignment along the direction m^{-1} which is equivalent to the direction of m, according to (1.6.1). $\vec{a_\perp}$ is called the rejection and correspond to the rejection of the vector $\vec{a}$ from the object m e.g the component of $\vec{a}$ which does not belong to the sub-space spanned by m.

2.2 Reflection with respect to a vector

The decomposition of a vector with respect to a normalized vector $\vec{m}$ as a sum of a projection and a rejection allows to define the reflection of a vector $\vec{a_r}$ with respect to this object such that :

$$\vec{a_m}^r = \vec{a_\parallel} - \vec{a_\perp} \tag{2.2.1}$$

Definition (2.2.1) leads to introduce the notion of double multiplication on the left and on the right such that it takes the form of a similarity relation :

$$\vec{a_m}^r = (\vec{a}\cdot\vec{m} - \vec{a}\wedge\vec{m})\vec{m}^{-1} = (\vec{m}\cdot\vec{a} + \vec{m}\wedge\vec{a})\vec{m}^{-1} = \vec{m}\vec{a}\vec{m}^{-1} \tag{2.2.2}$$

In the case of m is a vector $\vec{m}$, the projection is the component of $\vec{a}$ along $\vec{m}$ and the rejection is the component of $\vec{a}$ perpendicular to $\vec{m}$ and can be simply expressed as :

$$\begin{cases} \vec{a_\parallel} = (\vec{a} \cdot \vec{m})\vec{m} \\ \vec{a_\perp} = (\vec{a} \times \vec{m}) \times \vec{m} \end{cases} \tag{2.2.3}$$

The reflection is associative and does not need a special attention to the order of the operation. The product of two reflected vectors is the reflection of the product of the original vectors.

$$\vec{a_m}^r \vec{b_m}^r = \vec{m}\vec{a}\vec{m}^{-1}\vec{m}\vec{b}\vec{m}^{-1} = \vec{m}\vec{a}\vec{b}\vec{m}^{-1} = (\vec{a}\vec{b})^r \tag{2.2.4}$$

2.3 Reflections with respect to scalars or tri-vectors

The reflection with respect to a scalar makes sens in the case of considering the scalar as the contribution of the metric to the description of an object. Under this hypothesis, the reflection with respect to a scalar r can be seen as an extension/constriction of the vector $\vec{a}$ of a scale r. In the case of a flat space described by a coordinate-independent inner product as discussed in this paper, this operation returns the vector itself.

$$\vec{a}rr^{-1} = r\vec{a}r^{-1} = \vec{a} \tag{2.3.1}$$

The reflection with respect to a tri-vector acts as a rotation of the vector and of the space itself in the same time such that the description of the reflection inside of the space does not change with respect to the space itself, but the entire space is changed. The normalization (1.6.5) in addition to the mapping (1.7.1) indicates that this operation transform the space from the conventional space to the dual space where the metric properties should be conserved. In the case of a flat space described by a coordinate-independent inner product as discussed in this paper, this operation returns the vector itself.

$$\vec{a}\,\overset{\circlearrowleft}{t}\,\overset{\circlearrowleft}{t}^{-1} = \overset{\circlearrowleft}{t}\,\vec{a}\,\overset{\circlearrowleft}{t}^{-1} = \vec{a} \tag{2.3.2}$$

This relation can however be broken if the double transformation is not equivalent to identity or if the tri-vector does not commutes anymore with the vector e.g relation (1.7.1) is no more valid (self interpretation → interesting researches on Gauss spaces and Hamilton discs could be fun).

2.4 Rotations

The rotation of a vector can be seen as a double reflection with respect to two different vectors forming a plane with an orientation. This nature of the reflection generator is then a bi-vector such that the sens of rotation is contained in the order of the two vectors when performing the double reflection. This description is then equivalent to taking the reflection with respect to a non-closed oriented plane described by the bi-vector.

The natural object describing a rotation is then a bi-vector forming the plane of rotation where the two vectors of this plane form an angle $\frac{\phi}{2}$.

From the definition of the reflection, a rotation is written as :

$$\vec{a}^R_{m_1 m_2} = \vec{m}_2\vec{m}_1\vec{a}\vec{m}_1^{-1}\vec{m}_2^{-1} = \overset{\curvearrowleft}{I}\vec{a}\overset{\curvearrowleft}{I}^{-1} = \vec{a_I}^R \tag{2.4.1}$$

where $\overset{\curvearrowleft}{I} = \vec{m}_2\vec{m}_1$ is the bi-vector of the rotation.

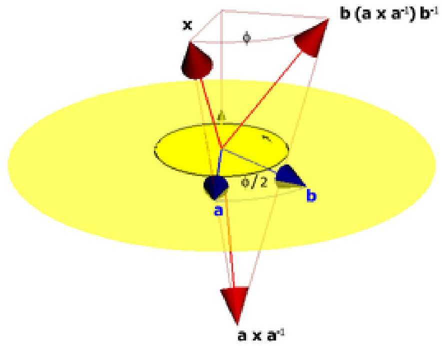

Figure 1: Rotation as a double reflection with respect to two planar vectors forming a bi-vector. The order of reflections determines the sens of rotation

Note that the two reflection vectors are switched in the product so the bi-vector comes then out with a reversed sens of rotation. In the case of $\hat{I}$ is normalized, the rotation can be rewritten in an exponential form such as :

$$\vec{a_I}^R = e^{\frac{\phi}{2}\overset{\frown}{I}}\vec{a}e^{-\frac{\phi}{2}\overset{\frown}{I}} \tag{2.4.2}$$

Using the mapping 1.7.2, the rotation takes the form similar to the multiplication by a complex plane normal to a vector $\vec{n}$ defining classically a plane in euclidean geometry :

$$\vec{a_n}^R = e^{i\frac{\phi}{2}\vec{n}}\vec{a}e^{-i\frac{\phi}{2}\vec{n}} \tag{2.4.3}$$

This definition of a rotation requires to use the half angle $\frac{\phi}{2}$ as shows Figure 1 which can be understood as a consequence of the multiplication on the left and on the right. This comes from the double mapping between the orthogonal group of 3D matrices and the unitary matrices of positive determinant $O_3 \sim SU_2$. This double mapping is visible in the definition of the bi-vector as a generator of the rotation where the bi-vector $\hat{I}$ is normalized to $-1 = e^{\pi\hat{\mathbb{1}}}$.

3 Quaternions Algebra

The complex numbers are the concatenation of a real number and an imaginary number. This closes the field of the polynomials such that every roots of polynomials belong to the space of complex numbers.

Complex numbers are then generators of the polynomial space and can be understood as the smallest necessary space to describe analytic scalar fields using the constraint that a scalar field is a field which can be extended in Taylor series over the polynomial space.

The great interest of quaternions is to try to extend this property to more complex objects such as vectors. An analogy with tensors can be made to understand this purpose as trying to extend the analysis from zeroth order tensors to first order tensors.

3.1 Quaternions algebra

The quaternions comes from an extension of the complex numbers by extending the basis of complex numbers. To complete this goal, William Rowan Hamilton built a new object made of the concatenation of two complex numbers, one "real" complex number and one "complex" complex number such that both of them form a new basis of dimension 4.

Therefore, it's necessary to define the relation between the basis vectors of this new space. Then, let's introduce two quantities : the imaginary number I and the imaginary number J which generates the space of complex complex numbers. Then every element of this space belongs to the basis :

$$\mathscr{Q} = \{1, I, J, JI\} \tag{3.1.1}$$

This definition necessitates to precise the product JI. From the construction of the complex numbers, it's obvious that the normalization sets $I^2 = J^2 = -1$. Note that the product of the two complex numbers is not commutative :

$$IJ = R_I(\pi/2)R_J(\pi/2) = -R_J(\pi/2)R_I(\pi/2) = -JI \tag{3.1.2}$$

with $R_z(\theta)$ the matrix representation of the element z with mapping $e^{2i\theta}$.

From this statement, the double product can be expanded as :

$$1 = -II = (IJ)(JI) = -(IJ)^2 \tag{3.1.3}$$

The third basis vector then verifies the same normalization than the complex numbers I and J. Let's call it $IJ = K$.

Obviously, 1 commutes with every basis vectors and the above definition sets the relations :

$$IJ = K, \qquad JK = I, \qquad KI = J \tag{3.1.4}$$

from which we deduce the triple product :

$$IJK = K^2 = -1 \tag{3.1.5}$$

With the above properties, the left multiplication of a quaternions is different than the right multiplication. The complex conjugation can then be taken as :

$$I^* = -I, \qquad J^* = -J, \qquad K^* = -K \tag{3.1.6}$$

3.2 Field description of quaternions

With the above structure, the quaternions can be expressed as a one-dimensional field concatenated with a three-dimensional complex vector. From this, let's expressed is as a 4-field :

$$q = r + \hat{\imath}\vec{v} \tag{3.2.1}$$

where r is the scalar field part, $\vec{v}$ is the vectorial field part and $\hat{\imath}$ is the complex unit.

Consider two pure complex quaternions $\hat{\imath}\vec{v} = aI + bJ + cK$ and $\hat{\imath}\vec{w} = \alpha I + \beta J + \gamma K$. The product of these two vectors under the quaternions algebra is :

$$
\begin{aligned}
\hat{i}\vec{v}\hat{i}\vec{w} &= (aI + bJ + cK)(\alpha I + \beta J + \gamma K) \\
&= a\alpha I^2 + a\beta IJ + a\gamma IK + b\alpha JI + b\beta J^2 + b\gamma JK + c\alpha KI + c\beta KJ + c\gamma K^2 \\
&= -(a\alpha + b\beta + c\gamma) + (a\beta - b\alpha)IJ + (b\gamma - c\beta)JK + (c\alpha - a\gamma)KI \\
&= -(a\alpha + b\beta + c\gamma) + (a\beta - b\alpha)K + (b\gamma - c\beta)I + (c\alpha - a\gamma)J \\
&= -\vec{v}\cdot\vec{w} + \hat{i}\vec{v}\times\vec{w}
\end{aligned}
\tag{3.2.2}
$$

Therefore, the geometrical product is completely accurate to describe the quaternions algebra by considering the subspace :

$$\mathscr{S} \in \mathscr{T} = \{1, \hat{x}\hat{y}, \hat{y}\hat{z}, \hat{z}\hat{x}\}$$

Including that the scalar part of the quaternion commutes with the imaginary part, the product of two quaternions $q_1 = r_1 + \hat{i}\vec{v}_1$ and $q_2 = r_2 + \hat{i}\vec{v}_2$ becomes :

$$q_1 q_2 = r_1 r_2 - \vec{v}_1\vec{v}_2 + \hat{i}(r_1\vec{v}_2 + r_2\vec{v}_1 + \vec{v}_1\times\vec{v}_2) \tag{3.2.3}$$

The vectorial field part is then anti-symmetric with respect to the complex conjugation due to the complex unit :

$$q^* = (r + \hat{i}\vec{v})^* = (r + aI + bJ + cK)^* = r - aI - bJ - cK = r - \hat{i}\vec{v} \tag{3.2.4}$$

with this complex conjugation and the inner product (3.2.3), the norm of a quaternion is :

$$|q|^2 = qq^* = q^*q \tag{3.2.5}$$

from what we deduce the inverse quaternion :

$$q^{-1} = \frac{1}{|q|^2}q^* \tag{3.2.6}$$

There is then a mapping from the exterior of the quaternion unit sphere and the interior of the quaternion unit sphere. On the quaternion unit sphere, there are multi-inverses for rotational degenerate points which creates issues in defining unique quantities with quaternions.

Therefore, each quaternion can be rewritten from a unit quaternion as :

$$q = |q|\hat{q} \tag{3.2.7}$$

where $\hat{q}$ is a unit quaternion and $|q|$ is the norm of the quaternion.

From this statement, the quaternion can be splitted in two parts :

$$q = |q|\left(\frac{r}{|q|} + \frac{|v|}{|q|}\hat{i}\hat{v}\right) \tag{3.2.8}$$

where $\hat{v}$ is a unit vector describing the complex part of the quaternion.

The two components of the unit quaternion are then made of quantities in $[-1, 1]$ such that $\left(\frac{r}{|q|}\right)^2 + \left(\frac{|v|}{|q|}\right)^2 = 1$. Therefore, it's possible to associate an angle θ to rewrite the unit quaternion as :

$$\hat{q} = cos(\theta) + sin(\theta)\hat{i}\hat{v} \tag{3.2.9}$$

which induces a mapping of any quaternion q to the exponential form :

$$q = |q|e^{\hat{i}\theta\hat{v}} \tag{3.2.10}$$

Note that in this case, $\theta \in [0, \pi]$. This half mapping translates a lack of coverage of the phase space with quaternions and has the consequence the non commutation of the quaternions and the double mapping (scalars, vectors) $\leftrightarrow$ (*pseudo*-scalars, *pseudo*-vectors).

3.3 Associativity and Commutativity of quaternions

The product of quaternions sets the problem of the associations and the commutations rules. If the sum of quaternions is obviously associative and commutative, it's not the case of the product.

3.3.1 Commutativity : Laminar fields and Turbulent fields

Let's consider two quaternions $q_1 = r_1 + \vec{v_1}$ and $q_2 = r_2 + \vec{v_2}$.

$$\begin{aligned} q_1 q_2 &= (r_1 + \vec{v_1})(r_2 + \vec{v_2}) \\ &= r_1 r_2 - \vec{v_1} \cdot \vec{v_2} + r_1 \vec{v_2} + r_2 \vec{v_1} + \vec{v_1} \times \vec{v_2} \end{aligned} \tag{3.3.1}$$

$$\begin{aligned} q_2 q_1 &= (r_2 + \vec{v_2})(r_1 + \vec{v_1}) \\ &= r_1 r_2 - \vec{v_1} \cdot \vec{v_2} + r_1 \vec{v_2} + r_2 \vec{v_1} - \vec{v_1} \times \vec{v_2} \end{aligned} \tag{3.3.2}$$

Then, commutator of these two objects exhibits the turbulent term and the anti-commutator exhibits the linear quaternion fluid :

$$\begin{cases} [q_1, q_2] = 2\vec{v_1} \times \vec{v_2} \\ \{q_1, q_2\} = 2\big(r_1 r_2 - \vec{v_1} \cdot \vec{v_2} + r_1 \vec{v_2} + r_2 \vec{v_1}\big) \end{cases} \tag{3.3.3}$$

The commutator isolates then a term that could be interpreted as the rotational of the quaternion flux which looks like a turbulence term of the quaternion fluid. The anti-commutator represents only the modification of the flux in the direction of each vector of the quaternions and rescales the scalar part so it looks more like a laminar term where the flux carried by each quaternion just superposes.

3.3.2 Associativity

Consider now three quaternions $q_1 = r_1 + \vec{v_1}$, $q_2 = r_2 + \vec{v_2}$ and $q_3 = r_3 + \vec{v_3}$.

$$\begin{aligned} q_1(q_2 q_3) =& q_1(r_2 r_3 - \vec{v_2} \cdot \vec{v_3} + r_3 \vec{v_2} + r_2 \vec{v_3} + \vec{v_2} \times \vec{v_3}) \\ =& r_1 r_2 r_3 - r_1 \vec{v_2} \cdot \vec{v_3} - r_3 \vec{v_1} \cdot \vec{v_2} - r_2 \vec{v_1} \cdot \vec{v_3} - \vec{v_1} \cdot (\vec{v_1} \times \vec{v_3}) \\ &+ r_1 r_3 \vec{v_2} + r_1 r_2 \vec{v_3} + r_1(\vec{v_2} \times \vec{v_3}) + r_2 r_3 \vec{v_1} - (\vec{v_2} \cdot \vec{v_3})\vec{v_1} + r_3(\vec{v_1} \times \vec{v_2}) + r_2(\vec{v_1} \times \vec{v_3}) + \vec{v_1} \times (\vec{v_2} \times \vec{v_3}) \end{aligned} \tag{3.3.4}$$

$$\begin{aligned} (q_1 q_2) q_3 =& (r_1 r_2 - \vec{v_1} \cdot \vec{v_2} + r_1 \vec{v_2} + r_2 \vec{v_1} + \vec{v_1} \times \vec{v_2}) q_3 \\ =& r_1 r_2 r_3 - r_3 \vec{v_1} \cdot \vec{v_2} - r_1 \vec{v_2} \cdot \vec{v_3} - r_2 \vec{v_1} \cdot \vec{v_3} - (\vec{v_1} \times \vec{v_2}) \cdot \vec{v_3} \\ &+ r_1 r_3 \vec{v_2} + r_2 r_3 \vec{v_1} + r_3(\vec{v_1} \times \vec{v_2}) + r_1 r_2 \vec{v_3} - (\vec{v_1} \cdot \vec{v_2})\vec{v_3} + r_1(\vec{v_2} \times \vec{v_3}) + r_2(\vec{v_1} \times \vec{v_3}) + (\vec{v_1} \times \vec{v_2}) \times \vec{v_3} \end{aligned} \tag{3.3.5}$$

Then, the difference of these two terms is :

$$
\begin{aligned}
(q_1q_2)q_3 - q_1(q_2q_3) &= (\vec{v_1}\times\vec{v_2})\cdot\vec{v_3} - \vec{v_1}\cdot(\vec{v_1}\times\vec{v_3}) + (\vec{v_1}\cdot\vec{v_2})\vec{v_3} - (\vec{v_2}\cdot\vec{v_3})\vec{v_1} \\
&= (\vec{v_1}\cdot\vec{v_2})\vec{v_3} - (\vec{v_2}\cdot\vec{v_3})\vec{v_1} \\
&= \vec{v_2}\times(\vec{v_3}\times\vec{v_1})
\end{aligned}
\tag{3.3.6}
$$

So the quaternion product is only associative over the real numbers, but is not in general. The difference of the associativity product is called the anisotropic flux representing the favored direction induced by the order of the quaternion product. The condition for this flux to vanish is $\vec{v_1} = \vec{v_3}$.

3.4 Rotations from quaternions

Rotations are described from geometric algebra by (2.4.1) and necessitates an oriented plane to perform the rotation.

For quaternions, the rotation of a vector $\vec{w}$ to ${\vec{w}_v}^R$ is represented by a unit quaternion q as :

$$
{\vec{w}_v}^R = q\vec{w}q^{-1} \tag{3.4.1}
$$

where the rotation formula (2.4.3) is recovered using the exponential form (3.2.10) of the unit quaternion :

$$
{\vec{w}_v}^R = e^{\hat{i}\frac{\theta}{2}\hat{v}}\vec{w}e^{-\hat{i}\frac{\theta}{2}\hat{v}} \tag{3.4.2}
$$

References

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