EXEMPLES

Exercice n°1 :

$$J(U) = \frac{1}{2}\langle AU, U\rangle - \langle b, U\rangle$$

$A = I_N$

$b = (1,-1,1,-1,\dots)^T \in \mathbb{R}^N$

$U_i + 2U_{i+1} \leq 0 \quad \forall i = 1,\dots,N-1$

$U_N \leq 0$

- $N=2$: $U = (U_1, U_2) \in \mathbb{R}^2$

$$\min J(U) = \frac{1}{2}(U_1^2 + U_2^2) - U_1 + U_2$$

$$\text{s.t} \begin{cases} U_1 + 2U_2 \leq 0 \\ U_2 \leq 0 \end{cases}$$

$U^* = (1,-1)$

$J(U^*) = -1$

((c'est aussi le minimum global !))

- $N=3$: $U = (U_1, U_2, U_3) \in \mathbb{R}^3$

$$\min J(U) = \frac{1}{2}(U_1^2 + U_2^2 + U_3^2) - U_1 + U_2 - U_3$$

$$\text{s.t} \begin{cases} U_1 + 2U_2 \leq 0 \\ U_2 + 2U_3 \leq 0 \\ U_3 \leq 0 \end{cases}$$

$U^* = (1,-1,0)$

$J(U^*) = -1$

- $N=4$: $U = (U_1, U_2, U_3, U_4) \in \mathbb{R}^4$

$$\min J(U) = \frac{1}{2}(U_1^2 + U_2^2 + U_3^2 + U_4^2) - U_1 + U_2 - U_3 + U_4$$

$$\text{s.t} \begin{cases} U_1 + 2U_2 \leq 0 \\ U_2 + 2U_3 \leq 0 \\ U_3 + 2U_4 \leq 0 \\ U_4 \leq 0 \end{cases}$$

$U^* = \left(1, -\frac{6}{5}, \frac{3}{5}, -1\right)$

$J(U^*) = -\frac{19}{10}$

Exercice n°2 :

$$\min_{\substack{U_i=(U_i^1,U_i^2)\\ \in \mathbb{R}^2\\ i=1,\dots,N+1}} \sum_{i=1}^{N+1} J_i(U_i) \qquad A_i = \mathrm{diag}(\alpha_i,\beta_i)\in\mathbb{R}^{2,2}$$

$$\text{s.t } \sum_{i=1}^{N} U_i - U_{N+1} = 0$$

$$U_i \in [0,a_i]\times[0,b_i-a_i] \quad \forall i=1,\dots,N$$

$$\min_{\substack{V_i=(V_i^1,V_i^2)\\ \in\mathbb{R}^2}} J_i(V_i) \quad \text{s.t } \begin{cases} -V_i^1 - U_i^1 + a_i \leq 0 \\ V_i^1 + V_i^2 + U_i^1 + U_i^2 - b_i = 0 \end{cases}$$

$$\min_{\substack{U_{N+1}\\ \in\mathbb{R}^2}} J_{N+1}(U_{N+1})$$

- $N=1$ $\qquad U_1=(U_1^1,U_1^2)\in\mathbb{R}^2 \qquad U_2=(U_2^1,U_2^2)\in\mathbb{R}^2$

$a_1 = 1 \qquad b_1 = 2 \qquad A_1 = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$a_2 = 1 \qquad b_2 = 3$

$$\min \frac{1}{2}\left(4{V_1^1}^2 + 5{V_1^2}^2\right) + \frac{1}{2}\left({U_2^1}^2 + 2{U_2^2}^2\right) + 0\times U_1^1 + 0\times U_1^2$$

s.t

$$\left.\begin{aligned} U_1^1 - U_2^1 = 0 \\ U_1^2 - U_2^2 = 0 \end{aligned}\right\} \longrightarrow \begin{aligned} U_1^1 = U_2^1 = \tilde{U}_1 \\ U_1^2 = U_2^2 = \tilde{U}_2 \end{aligned}$$

$$0 \leq U_1^1 \leq 1$$
$$0 \leq U_1^2 \leq 1$$
$$0 \leq U_2^1 \leq 1$$
$$0 \leq U_2^2 \leq 2$$
$$-V_1^1 - U_1^1 + 1 \leq 0$$
$$V_1^1 + V_1^2 + U_1^1 + U_1^2 - 2 = 0$$

$$\min \frac{1}{2}\left(4{V_1^1}^2 + 5{V_1^2}^2\right) + \frac{1}{2}\left(\tilde{U}_1^2 + 2\tilde{U}_2^2\right)$$

$$\text{s.t} \begin{cases} 0 \leq \tilde{U}_1 \leq 1 \\ 0 \leq \tilde{U}_2 \leq 1 \\ -V_1^1 - \tilde{U}_1 + 1 \leq 0 \\ V_1^1 + V_1^2 + \tilde{U}_1 + \tilde{U}_2 - 2 = 0 \end{cases}$$

$$J^*(U) = \frac{40}{9}$$

$${V_1^1}^* = \frac{10}{9}$$

$${V_1^2}^* = \frac{8}{9}$$

$${U_1^1}^* = 0$$

$${U_2^1}^* = 0$$

$${U_1^2}^* = 0$$

$${U_2^2}^* = 0$$

Exercice n°3 :

- Problème 1 :

$$\min_{U \in \mathbb{R}_+^N} \frac{1}{2} \langle U, QU \rangle$$

$$\text{s.t} \quad \sum_{i=1}^{N} U_i e_i \geq R_e$$

$$\sum_{i=1}^{N} U_i = 1$$

- $N = 2$:

$\rightarrow$ $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad U = (U_1, U_2)^T \in \mathbb{R}^2 \quad e = (e_1, e_2)^T \in \mathbb{R}^2$

$e_1 = 4 \quad e_2 = 5 \quad R_e = \frac{9}{2}$

$$\min U_1^2 + U_1 U_2 + U_2^2$$

$$\text{s.t} \begin{cases} 4U_1 + 5U_2 \geq \frac{9}{2} \\ U_1 + U_2 = 1 \\ U_1 \geq 0 \\ U_2 \geq 0 \end{cases}$$

$U_1^* = \frac{1}{2}$

$U_2^* = \frac{1}{2}$

$J(U^*) = \frac{3}{4}$

$\rightarrow$ $Q = \begin{pmatrix} 3/5 & 3 \\ 3 & \frac{10}{3} \end{pmatrix}$ $\quad U = (U_1, U_2)^T \in \mathbb{R}^2 \quad e = (e_1, e_2)^T \in \mathbb{R}^2$

$e_1 = 5/4 \quad e_2 = 3 \quad R_e = 3/2$

$$\min \frac{3}{10} U_1^2 + 3 U_1 U_2 + \frac{10}{6} U_2^2$$

$$\text{s.t} \begin{cases} \frac{5}{4} U_1 + 3 U_2 \geq 3/2 \\ U_1 + U_2 = 1 \\ U_1 \geq 0 \\ U_2 \geq 0 \end{cases}$$

$U_1^* = \frac{6}{7}$

$U_2^* = 1/7$

$J(U^*) = \frac{457}{735}$

• Problème 2 :

$$\max_{U \in \mathbb{R}_+^N} \sum_{i=1}^{N} U_i e_i$$

$$\text{s.t } \langle U, QU \rangle \leq De$$

$$\sum_{i=1}^{N} U_i = 1$$

• $N=2$:

$\rightarrow$ $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad U = (U_1, U_2)^T \in \mathbb{R}^2 \quad e = (e_1, e_2)^T \in \mathbb{R}^2$

$e_1 = 4 \quad e_2 = 5 \quad De = \frac{3}{2}$

$$\max \; 4U_1 + 5U_2$$

$$\text{s.t} \begin{cases} 2U_1^2 + 2U_1U_2 + 2U_2^2 \leq \frac{3}{2} \\ U_1 + U_2 = 1 \\ U_1 \geq 0 \\ U_2 \geq 0 \end{cases}$$

$U_1^* = \frac{1}{2}$

$U_2^* = \frac{1}{2}$

$J(U^*) = \frac{9}{2}$

$\rightarrow$ $Q = \begin{pmatrix} \frac{3}{5} & 3 \\ 3 & \frac{10}{3} \end{pmatrix}$ $\quad U = (U_1, U_2)^T \in \mathbb{R}^2 \quad e = (e_1, e_2)^T \in \mathbb{R}^2$

$e_1 = \frac{5}{4} \quad e_2 = 3 \quad De = \frac{5}{3}$

$$\max \; \frac{5}{4}U_1 + 3U_2$$

$$\text{s.t} \begin{cases} \frac{3}{5}U_1^2 + 6U_1U_2 + \frac{10}{3}U_2^2 \leq \frac{5}{3} \\ U_1 + U_2 = 1 \\ U_1 \geq 0 \\ U_2 \geq 0 \end{cases}$$

$U_1^* = \frac{20\sqrt{2}}{31} - \frac{5}{31}$

$U_2^* = \frac{36}{31} - \frac{20\sqrt{2}}{31}$

$J(U^*) = \frac{1}{124}\left(407 - 140\sqrt{2}\right)$