Exercice n°3 :

A : Additif
C : coupLant

Problème 1 :

$$\min \frac{1}{2} \langle U, QU \rangle$$

$$\text{s.t} \begin{cases} \sum_{i=1}^{N} U_i e_i \geq R_e & \longrightarrow p_1^{(k)} \\ \sum_{i=1}^{N} U_i = 1 & \longrightarrow p_2^{(k)} \\ U_i \geq 0 \quad \forall i = 1, \dots, N \end{cases}$$

$\rightarrow J(U) = J^A(U) + J^C(U)$

$\rightarrow \Theta = \Theta^A + \cancel{\Theta^C}$

$$\Theta_i(U_i) = \begin{pmatrix} -e_i \\ 1 \end{pmatrix} U_i \quad \text{et} \quad V = \begin{pmatrix} -R_e \\ 1 \end{pmatrix}$$

$$U^{ad} = \prod_{i=1}^{N} U_i^{ad} \quad \text{avec } \forall i = 1, \dots, N \quad U_i^{ad} = \{ U \in \mathbb{R} \mid U \geq 0 \} = \mathbb{R}_+$$

$$Q_C = \begin{pmatrix} 0 & & (*) \\ & \ddots & \\ (*) & & 0 \end{pmatrix} \qquad Q_A = \begin{pmatrix} * & & (0) \\ & \ddots & \\ (0) & & * \end{pmatrix} \qquad Q = Q_C + Q_A$$

$$J(U) = \frac{1}{2} \langle U, QU \rangle = \frac{1}{2} \langle U, Q_A U \rangle + \frac{1}{2} \langle U, Q_C U \rangle$$

- Sous problèmes :

$$\inf_{U \in U^{ad}} A^{(k)}(U) + \varepsilon^{(k)} J^A(U) + \langle \varepsilon^{(k)} Q_C U^k, U \rangle + \varepsilon^{(k)} \langle \Theta_A(U), p^{(k)} \rangle$$

$$\text{avec} \quad A^{(k)}(U) = \frac{\alpha^k}{2} \|U\|^2 - \alpha^k \langle U^k, U \rangle$$

- Actualisation :

$$p^{(k+1)} \longrightarrow \begin{cases} p_1^{(k+1)} = \max\left(0, p_1^{(k)} + \varepsilon^{(k)} \alpha \left(\sum_{i=1}^{N} -e_i U_i^{k+1} + R_e \right)\right) \\ p_2^{(k+1)} = p_2^{(k)} + \varepsilon^{(k)} \alpha \left(\sum_{i=1}^{N} U_i - 1 \right) \end{cases}$$

Problème 2 :

$$\max \sum_{i=1}^{N} U_i e_i \quad \longrightarrow J(U) = J^A(U) + \cancel{J^C(U)}$$

$$\text{s.t} \begin{cases} \langle U, QU \rangle \leq De & \longrightarrow p_1^{(k)} \\ \sum_{i=1}^{N} U_i = 1 & \longrightarrow p_2^{(k)} \\ U_i \geq 0 \quad \forall i = 1, \dots, N \end{cases} \quad \longrightarrow \Theta = \Theta^A + \Theta^C$$

$$\iff \quad -\min - \sum_{i=1}^{N} U_i e_i$$

$$\text{s.t} \begin{cases} \langle U, QU \rangle \leq De \\ \sum_{i=1}^{N} U_i = 1 \\ U_i \geq 0 \quad \forall i = 1, \dots, N \end{cases}$$

$$Q = [Q_{ij}]_{1 \leq i,j \leq N} \in \mathbb{R}^{N \times N}$$

$$Q = Q_A + Q_C$$

$$\sum_{i=1}^{N} \Theta_i^A(U_i) + \Theta^C(U) \quad \begin{cases} \Theta_i^A(U_i) = \begin{pmatrix} Q_{ii} U_i^2 \\ U_i \end{pmatrix}, \\ \Theta^C(U) = \begin{pmatrix} \langle U, Q_C U \rangle \\ 0 \end{pmatrix} \end{cases} \quad \Theta^A(U) = \sum_{i=1}^{N} \begin{pmatrix} Q_{ii} U_i^2 \\ U_i \end{pmatrix} \quad V = \begin{pmatrix} De \\ 1 \end{pmatrix}$$

- Sous problèmes :

$$\inf_{U \in U^{ad}} A^{(k)}(U) + \varepsilon^{(k)} \left\langle \begin{pmatrix} \langle 2 Q_C U^k, U \rangle \\ 0 \end{pmatrix} + \sum_{i=1}^{N} \begin{pmatrix} Q_{ii} U_i^2 \\ U_i \end{pmatrix}, p^{(k)} \right\rangle - \varepsilon^{(k)} \langle e, U \rangle$$

avec $A^{(k)}(U) = \frac{\alpha^k}{2} \|U\|^2 - \alpha^k \langle U^k, U \rangle$

- Actualisation :

$$p^{(k+1)} \longrightarrow \begin{cases} p_1^{(k+1)} = \max\left(0, p_1^{(k)} + \varepsilon^{(k)} \alpha \left(\langle U^{k+1}, 2 Q U^{k+1} \rangle - De \right)\right) \\ p_2^{(k+1)} = p_2^{(k)} + \varepsilon^{(k)} \alpha \left(\sum_{i=1}^{N} U_i^{k+1} - 1 \right) \end{cases}$$