McGill University
Winter 2022 — MECH309
Project

Deadline April 8th, 2022 at 6pm on MyCourses.

Instructions Write a final report (LateX, LibreOffice, Word...) with relevant explanations, comments, equations and figures. Save it as a .pdf file which should exceed neither 3 Mb nor 8 pages. No title page. No appendix. Indicate all names at the top of the first page only. Fontsize should be 11pt. Excel plots are not accepted. Organize the plots in an effective manner. Submit all nicely commented and organized Matlab/Python scripts in a .zip archive into which you will also include your pdf report. The name of the archive should be a concatenation of the family names of the team members. Submit only one archive per team on MyCourses. Unless stated otherwise, the use of advanced Matlab/Python commands is prohibited. Equally important is the prohibited use in the provided scripts of the Matlab/Python symbolic toolbox.

Grading scheme Below is the grading scheme used for the final report. It is designed on the following attributes with their respective weight:

- Comprehension of information and concepts of fundamental engineering sciences (10%)
- Critical evaluation of the validity and accuracy of solution methods (30%)
- Understanding of the problems and appropriate definition of the objectives (14%)
- Appropriate selection of solution techniques and resources (13%)
- High quality written engineering report (13%)
- Engagement in self-study on topics of interest (10%)

For each student, 10% of the grade is based on the Peer Evaluations submitted by the other team members.

Peer evaluation Every student must fill out and submit the peer evaluation form by the above deadline. The form can be found in the "Project" folder on MyCourses. A student failing to do so will be assigned a 0/100 grade for the project. All students are asked to read the content of this form before starting the project.

Plagiarism and Cheating at McGill It is never too late to read (again) McGill's code on students' rights and responsibilities: https://www.mcgill.ca/students/srr/academicrights/integrity/cheating

1 Errors and stability in numerical methods [15pts]

Write an essay of 700 words on the notions of errors and stability in Numerical Methods. Plain text only is allowed. Lists and mathematical characters/symbols are prohibited. Cover all possible topics related to forward error, backward error, conditioning, and stability. Explain the associated issues and the possible solution strategies. Indicate the number of words.

2 Inverted flags [85 pts]

Conventional flags are thin rectangular *plates* flapping due to the action of a surrounding air flow, also called *wind*, circulating from the fixed end to the free end (think of a flag attached to a post: the clamped end is the part of the flag attached to the post while the free end is the other one). Inverted flags are similar systems where the air flow circulates from the free end to the fixed end instead (see a video on this system). The present and highly simplified formulation is focused on the time-independent static solution of an inverted flag subject to a transverse force $F_N(x)$ induced by the action of the wind on the plate. First, the plate, which is commonly a two-dimensional system, is reduced to a simpler and somehow equivalent one-dimensional system called a *beam* of length L. In the present exercise, the main quantity of interest is the angle of rotation, denoted $\psi(x)$, of a straight cross-section, as assumed in the Euler-Bernoulli beam theory. Once the angle

$\psi(x)$ is known, the corresponding transverse deflection $w(x)$ of the beam is recovered using the relationship

$$w(x) = \int_0^x \sin\psi(s)\mathrm{d}s. \tag{1}$$

1. Investigate on *plates* and *beams* from the standpoint of Continuum Mechanics and sketch the system of interest with all relevant quantities.

The equation governing $\psi(x)$ is

$$D\psi''(x) + \int_x^L F_N(x)\cos[\psi(x) - \psi(s)]\mathrm{d}s = 0, \quad x \in [0, L] \tag{2}$$

where D is the transverse stiffness of the beam. Unless stated otherwise, we assume that $L = 1$, $D = 1$ and $F_N(x) = 100$. This governing equation comes with the two boundary conditions

$$\psi(0) = 0 \quad \text{and} \quad \psi'(L) = 0. \tag{3}$$

Given D and $F_N(x)$, the question is thus to find a function $\psi(x)$ which satisfies Equations (2) and (3).

2. Explain why finding the expression of $\psi(x)$ is not an easy task.

In the remainder, the following family of functions, called *shape functions*, will be used:

$$\phi_i(x) = \sin(r_i x) + \sinh(r_i x) + \frac{\cos(r_i) + \cosh(r_i)}{\sin(r_i) + \sinh(r_i)}[\cos(r_i x) - \cosh(r_i x)], \quad i = 1, 2, \ldots \tag{4}$$

where r_i is the ith root of the transcendental equation $\cos(x)\cosh(x) + 1 = 0$.

3. Find approximations of r_1 and r_2 with three digits of accuracy. Plot $\phi_1(x)$ and $\phi_2(x)$.

As a first approach, we assume that the exact $\psi(x)$ is approximated by $\psi_h(x) = \phi_1(x)q_1$, where $\phi_1(x)$ is defined in Equation (4). The expression can be differentiated twice in space so that $\psi_h''(x) = \phi_1''(x)q_1$. As a consequence, the unknown of the problem is now q_1. Galerkin's procedure is then applied. It reads: find q_1 such that

$$f(q_1) = D\int_0^1 \phi_1(x)\psi_h''(x)\mathrm{d}x + F_N \int_0^1 \int_x^L \phi_1(x)\cos[\psi_h(x) - \psi_h(s)]\mathrm{d}s\,\mathrm{d}x = 0. \tag{5}$$

4. Briefly explain Galerkin's procedure.
5. Plot the function $f(q_1)$ on an interval of interest.
6. Suggest a fixed-point iteration equivalent to solving Equation (5). Does it converge? Explain which quadrature scheme is used.
7. Using an algorithm of your choice, find a numerical approximation of q_1 with three digits of accuracy.
8. Plot the corresponding deflection of the beam.
9. For this question only, F_N is still constant in space and its magnitude is varied in the interval [0, 100]. Plot the corresponding $\psi_h(L)$ as a function of F_N.

The accuracy of the approximation can be improved by relying on the assumption $\psi_h(x) = \phi_1(x)q_1 + \phi_2(x)q_2$. As a consequence, the unknowns of the problem are now q_1 and q_2. Galerkin's procedure then generates the system of equations:

$$\begin{aligned} f_1(q_1, q_2) &= D\int_0^1 \phi_1\psi_h(x)dx + F_N\int_0^1\int_x^L \phi_1(x)\cos(\psi_h(x) - \psi_h(s))\mathrm{d}s\,\mathrm{d}x = 0 \\ f_2(q_1, q_2) &= D\int_0^1 \phi_2\psi_h(x)dx + F_N\int_0^1\int_x^L \phi_2(x)\cos(\psi_h(x) - \psi_h(s))\mathrm{d}s\,\mathrm{d}x = 0 \end{aligned} \tag{6}$$

10. Plot the functions $f_1(q_1, q_2)$ and $f_2(q_1, q_2)$ on a relevant domain.
11. Suggest a fixed-point iteration equivalent to solving system (6). Does it converge?
12. Using an algorithm of your choice, find a numerical approximation of q_1 and q_2 with three digits of accuracy.
13. Plot the corresponding deflection of the beam and compare to the previous one. Is the contribution of the term $\phi_2(x)q_2$ critical in the approximation?

The accuracy of the approximation is again improved via the expression $\psi_h(x) = \phi_1(x)q_1 + \phi_2(x)q_2 + \phi_3(x)q_3$. As a consequence, the unknowns of the problem are now q_1, q_2 and q_3.

14. Find a numerical approximation of q_1, q_2 and q_3 with three digits of accuracy.
15. Plot the corresponding deflection of the beam and compare to the previous one.
16. Does the proposed scheme seem to converge towards the true solution $\psi(x)$?

The convergence analysis of the proposed scheme can achieved by inserting the computed approximation into the governing Equation (2) (which is then only approximately satisfied) and compute the residual function in order to take its magnitude in the sense of the 2-norm.

17. Does the proposed scheme seem to converge towards the true solution $\psi(x)$ as the number of shape functions in the approximation is increased? It is possible to find an estimate of the rate fo convergence?

A non-constant function $F_N(x)$ is now considered.

18. Briefly explain how you would solve the above questions.

In the above questions, it is strongly advised to simplify by hand all quantities or expression that can be simplified by hand. Numerical methods should be implemented only when the latter step is completed.

Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm

An extension of Broyden's algorithm seen in class is the BFGS algorithm. It follows the same steps but differs in the update of the approximate Jacobian matrix, which now reads:

$$\mathbf{A}_{k+1} \leftarrow \mathbf{A}_k - \frac{\mathbf{A}_k \Delta\mathbf{x}_k \Delta\mathbf{x}_k^\top \mathbf{A}_k}{\Delta\mathbf{x}_k^\top \mathbf{A}_k \Delta\mathbf{x}_k} + \frac{\Delta\mathbf{f}_k \Delta\mathbf{f}_k^\top}{\Delta\mathbf{x}_k^\top \Delta\mathbf{f}_k} \tag{7}$$