Some algorithms for expectile computation in R

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1 Introduction

Expectiles, introduced in Newey and Powell (1987), are an alternative of quantiles to measure the risk of a random variable $Y \in \mathbb{R}$. For all $\alpha \in (0,1)$, the expectile of Y, denoted $e_\alpha(Y)$, is the solution of the following minimization problem :

$$e_\alpha(Y) = \underset{t\in\mathbb{R}}{\arg\min}\ \mathbb{E}\left[\eta_\alpha(Y-t) - \eta_\alpha(Y)\right], . \tag{1.1}$$

where $\eta_\alpha(y) = |\alpha - \mathbb{1}_{\{y\leq 0\}}|y^2$. Unlike quantiles, expressed as the generalized inverse of the cumulative distribution function of Y, expectiles do not have an obvious formulation, and their computation is thus a challenging problem. This note gives some algorithms to compute theoretical expectiles of several distributions. These algorithms have been implemented in R functions, and their performances are compared with some existing functions.

2 Fixed point algorithms

A sequence $(y_k)_{k\in\mathbb{N}}$ is said to converge to y^* with order $p \in [1, +\infty)$ if for all k, $|y_k - y^*| \leq \epsilon_k$, where $(\epsilon_k)_{k\in\mathbb{N}}$ is a positive sequence such that

$$\exists\ c > 0 : \lim_{k\to\infty} \frac{\epsilon_{k+1}}{\epsilon_k^p} = c. \tag{2.1}$$

In particular, the convergence is said linear if $p = 1$ and $c < 1$, and quadratic if $p = 2$. The coefficient c is called asymptotic factor. Now, let us consider the case of fixed point algorithms $y_{k+1} = g(x_k)$, where $g : \mathbb{R} \to \mathbb{R}$ is a continuous and differentiable function. It is proved in Frontini and Sormani (2003) that if g is $p-$times differentiable with $g^{(m)}(y^*) = 0$ for all $m < p$ and $g^{(p)}(y^*) \neq 0$, then the convergence is of order p with an asymptotic factor

$$c = \frac{g^{(p)}(y^*)}{p!}.$$

In Maume-Deschamps et al. (2018), some fixed point algorithms are introduced to compute expectiles in the case of elliptical distributions. The next section gives several examples of elliptical distributions.

3 Some examples on elliptical distributions

A random variable $Y \in \mathbb{R}$ is said to be elliptical if it admits the stochastic representation $Y \stackrel{d}{=} \mu + \sigma RU$, where $\mu, \sigma \in \mathbb{R}$ and R and U are two independant random variables such that R is non negative and U is -1 of 1 with probability 1/2. Elliptical distributions are thus symmetric distributions. Furthermore, using the properties of expectiles, it is straightforward to notice that for all $\alpha \in (0,1)$, $e_\alpha(Y) = \mu + \sigma e_\alpha(Y_0)$, where $Y_0 \stackrel{d}{=} RU$. Therefore, expectiles may easily be deduced from the centered and reduced case $\mu = 0$ and σ. This is why, for sake of simplicity, we do this assumption in the sequel. All the proposed functions require the libraries `Rcpp` and `RcppArmadillo`.

3.1 Normal distribution

We start by the classical normal distribution. The gaussian quantiles may be obtained by the function `qnorm`. In order to compute the expectile of level α $(0,1)$, we proposed the following algorithm :

$$\left\{\begin{array}{lcc} y_0 & = & 0 \\ y_{k+1} & = & \frac{(1-2\alpha)\varphi(y_k)}{(2\alpha-1)\Phi(y_k)-\alpha} \end{array}\right. , \tag{3.1}$$

where $\varphi(.)$ and $\Phi(.)$ are respectively the p.d.f. and c.d.f. of the reduced-centered Gaussian distribution. It is proved in Maume-Deschamps et al. (2018) that the sequence $(y_k)_{k\in\mathbb{N}}$ has a quadratic convergence towards $e_\alpha(Y)$. We thus propose the function `expnorm`, based on (3.1).

3.2 Laplace distribution

The second elliptical distribution we consider is the Laplace distribution, which has the following p.d.f.:

$$f(x,\mu,\sigma,\lambda) = \frac{1}{\sqrt{2\lambda}\sigma} \exp\left(-\sqrt{\frac{2}{\lambda}}\left|\frac{x-\mu}{\sigma}\right|\right),$$

where $\lambda > 0$. We proposed the following fixed point algorithm in Maume-Deschamps et al. (2018) for the Laplace expectiles.

$$\left\{\begin{array}{lcc} y_0 & = & 0 \\ y_{k+1} & = & \operatorname{sgn}\left(\alpha-\frac{1}{2}\right)\frac{|2\alpha-1|\exp\left(-\sqrt{\frac{\lambda}{2}}y_k\right)\left(\sqrt{\frac{\lambda}{2}}+y_k\right)}{1-|2\alpha-1|\left(1-\exp\left(-\sqrt{\frac{\lambda}{2}}y_k\right)\right)} \end{array}\right. . \tag{3.2}$$

Algorithm (3.2) being derived from Proposition 3.7 in Maume-Deschamps et al. (2018), it also has a quadratic convergence. We implemented it in the function `explaplace`. Their performances are therefore similar to those of `expnorm`.

3.3 Logistic distribution

Another usual elliptical distribution is the logistic distribution, given by the p.d.f.:

$$f(x,\mu,\sigma) = \frac{\exp\left(-\frac{x-\mu}{\sigma}\right)}{\sigma\left(1+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)^2}.$$

By taking $\mu = 0$ and $\sigma = 1$, the following algorithm with a quadratic convergence is proposed in the function `explog` following Maume-Deschamps et al. (2018):

$$\left\{\begin{array}{lcc} y_0 & = & 0 \\ y_{k+1} & = & (1-2\alpha)\frac{(\exp(y_k)+1)\ln(\exp(y_k)+1)-y_k\exp(y_k)}{(\alpha-1)\exp(y_k)-\alpha} \end{array}\right. . \tag{3.3}$$

4 Numerical performances

Several functions already propose the computation of gaussian expectiles : `enorm` in the package `expectreg` or `qenorm` in `VGAM`. We thus propose to compare our functions (in particular `expnorm`) with these three ones. For that purpose, we replicate 1,000 times the computation of $e_\alpha(Y)$ (and $q_\alpha(Y)$ for `qnorm`) for each function and use the function `microbenchmark` to compare them. Note that all the computations are done on RStudio Cloud (`https://rstudio.cloud`).
We notice that our functions seem to be more than 10 times faster (in terms of median) than `qenorm`, and more than 30 times faster than `enorm`. However, these functions are obviously a bit slower than the quantile function `qnorm`.

Listing 1: Computation of expectiles

```
Unit: nanoseconds
            expr    min       lq       mean   median       uq     max neval
     qnorm(0.95)    813   1285.0   1703.830   1513.0   1768.5   58816  1000
    qenorm(0.95)  48079  55748.0  65373.487  59896.0  70684.0  244682  1000
     enorm(0.95) 138355 152267.0 183074.531 161839.5 197947.5 4423293  1000
   expnorm(0.95)   2770   4149.0   5486.236   5004.0   5900.5   60636  1000
 explaplace(0.95)  3037   4245.5   5757.221   5235.5   6131.0  128740  1000
    explog(0.95)   3224   4615.0   6093.529   5531.0   6470.5   75131  1000
```

References

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