Problem Set Week 8

ETHZ Math Olympiad Club

7 April 2025

1 Problem A-2 (IMC 1999)

Does there exist a bijective map $\pi\colon \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

2 Problem 2 (IMC 1994)

Let $f \in C^1(]a,b[\,,\mathbb{R})$ with $\lim_{x\to a^+} f(x) = +\infty$, $\lim_{x\to b^-} f(x) = -\infty$, and $f'(x) + f^2(x) \geq -1$ for all $x \in \,]a,b[$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

3 Problem B-3 (IMC 2005)

In the linear space of all real $n \times n$ matrices, find the maximum possible $\mathbb{R}$-dimension of an $\mathbb{R}$-linear subspace V such that

$$\forall X, Y \in V, \quad \operatorname{tr}(XY) = 0.$$

(The trace of a matrix is the sum of its diagonal entries.)

4 Problem 4 (Bernoulli Competition 2024)

Let $n, m \in \mathbb{N}_{>0}$ be positive integers, with $m \geq 3$, and let $A \in \mathbb{Z}^{n\times n}$. Suppose A has finite order ($\exists k \in \mathbb{N}^*$, $A^k = I_n$) and satisfies

$$A \equiv I_n \pmod{m}^{1}.$$

Prove that $A = I_n$, and find counterexamples when $m = 2$.

[1] For an integer $u \in \mathbb{Z}$, $\equiv \pmod{u}$ is the equivalence relation on the set of integer matrices $\bigcup_{r,l\in\mathbb{N}^*} \mathbb{Z}^{r\times l}$, where for $C, D \in \mathbb{Z}^{r\times l}$, $C \equiv D \pmod{u} \Leftrightarrow \forall\,(i,j) \in r \times l,\ u \mid (C - D)(i,j)$.