Problem Set Week 8 Solutions

ETHZ Math Olympiad Club

7 April 2025

1 Problem 2 (IMC 1999)

Does there exist a bijective map $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

Answer:

Solution 1. No. For, let π be a permutation of $\mathbb{N}$ and let $N \in \mathbb{N}$. We shall argue that

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} > \frac{1}{9}.$$

In fact, of the $2N$ numbers $\pi(N+1), \ldots, \pi(3N)$ only N can be $\leq N$ so that at least N of them are $> N$. Hence

$$\sum_{n=N+1}^{3N} \frac{\pi(n)}{n^2} \geq \frac{1}{(3N)^2} \sum_{n=N+1}^{3N} \pi(n) > \frac{1}{9N^2} \cdot N \cdot N = \frac{1}{9}.$$

Solution 2. Let π be a permutation of $\mathbb{N}$. For any $n \in \mathbb{N}$, the numbers $\pi(1), \ldots, \pi(n)$ are distinct positive integers, thus $\pi(1) + \ldots + \pi(n) \geq 1 + \ldots + n = \frac{n(n+1)}{2}$. By this inequality,

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} = \sum_{n=1}^{\infty} (\pi(1) + \ldots + \pi(n)) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \geq$$

$$\geq \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \cdot \frac{2n+1}{n^2(n+1)^2} = \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \geq \sum_{n=1}^{\infty} \frac{1}{n+1} = +\infty.$$

2 Problem 2 (IMC 1994)

Let $f \in C^1(a,b)$, $\lim_{x\to a+} f(x) = +\infty$, $\lim_{x\to b-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in (a,b)$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

Answer:

From the inequality we get:

$$\frac{\mathrm{d}}{\mathrm{dx}}(\arctan(f(x)) + x) = \frac{f'(x)}{1+f^2(x)} + 1 \geq 0$$

for $x \in]a,b[$. Thus $\arctan(f(x)) + x$ is non-decreasing in the interval and using the limits we get

$$\frac{\pi}{2} + a \leq -\frac{\pi}{2} + b.$$

Hence $b - a \geq \pi$. One has equality for $f(x) = \cot(x)$, $a = 0$, $b = \pi$.

3 Problem B-3 (IMC 2005)

In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V, \quad \operatorname{tr}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

Answer:

If A is a nonzero symmetric matrix, then $\operatorname{tr}(A^2) = \operatorname{tr}(A^T A)$ is the sum of the squared entries of A, which is positive. So V cannot contain any symmetric matrix except 0.

Denote by S the linear space of all real $n \times n$ symmetric matrices; its dimension is $\frac{n(n+1)}{2}$. Since $V \cap S = \{0\}$, we have

$$\dim V + \dim S \leq n^2,$$

which gives

$$\dim V \leq n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the given condition.

Therefore, the maximum dimension of V is $\frac{n(n-1)}{2}$.

4 Problem 4 (Bernoulli Competition 2024)

Let n and m be positive integers, with $m \geq 3$. Suppose a matrix $A \in \mathbb{Z}^{n\times n}$ of finite order satisfies

$$A \equiv I_n \pmod{m}.$$

Prove that $A = I_n$.

Answer:

Solution 1. Write $A = I + mB$. Then

$$\det(XI - A) = \det((X-1)I - mB).$$

Therefore, α is an eigenvalue of A if and only if $\alpha - 1$ is an eigenvalue of mB — that is, if and only if $(\alpha-1)/m$ is an eigenvalue of B.

Since A has finite order, any of its eigenvalues α satisfies $|\alpha| = 1$. Then any eigenvalue β of B satisfies

$$|\beta| = \left|\frac{\alpha - 1}{m}\right| \leq \frac{2}{m} < 1.$$

Factor the characteristic polynomial P_B of B as a product $\prod P_i$ of monic irreducible polynomials in $\mathbb{Z}[x]$.

We now prove that 0 is the only eigenvalue of B. Let β be an eigenvalue of B. Say $P_i(\beta) = 0$. All roots $\beta_1, \ldots, \beta_s$ of P_i are eigenvalues of B; hence, $|\beta_j| < 1$ for each j. But also each β_j is an algebraic integer, as a root of P_B. Therefore, so is their product $\prod \beta_j = \pm P_i(0)$. Then

$$|P_i(0)| = \prod |\beta_j| < 1.$$

Therefore $P_i(0) = 0$. Since P_i is irreducible, we conclude $P_i(x) = x$, and therefore $\beta = 0$.

So B is similar to a matrix $B = I + N$, where N has zeroes on the diagonal and below (in particular N is nilpotent). The fact that B has finite order implies $N = 0$. One way to see this is to look at the entries immediately above the diagonal in the powers of $I + N$, then look at the entries along the second diagonal above the main diagonal, and so on, to deduce all entries of N are zero. Alternatively, $(I+N)^k = I$ for some k gives

$$N(k \cdot I + N_T(N)) = 0$$

for some polynomial r. But $k \cdot I + N_T(N)$ is a unit in $\mathrm{Mat}_{n\times n}(\mathbb{Q})$ (since $N_T(N)$ is nilpotent); therefore, $N = 0$.

Alternative proof of intermediate step. At the point where B has eigenvalues smaller than 1, notice that $\mathrm{tr}(B^n)$ tends to 0. This means that starting from some N, $\mathrm{tr}(B^n) = 0$ as $\mathrm{tr}(B^n) \in \mathbb{Z}$. But this implies that all the eigenvalues are exactly 0 (we can look at it as sums of powers of λ_i^N, which are thus zeros — use Newton's identities).

Solution 2. Assume that $A \neq I_n$. Since m is greater than 2, it must be divisible by some prime power $p^c > 2$. So, $A \equiv I_n \pmod{p^c}$. Next, let c' be the largest integer such that $A \equiv I_n$ $\pmod{p^{c'}}$ and let $B = (A - I_n)/p^{c'}$. By the definition of c', $B \not\equiv 0 \pmod{p}$.

Next, let k be the order of A. Then $A^k = (I_n + p^{c'}B)^k = I_n$. In other words,

$$\sum_{i=1}^{k} \binom{k}{i} p^{ic'} B^i = 0$$

Now, let c'' be the largest integer such that k is divisible by $p^{c''}$. For any $1 \leq i \leq k$, $i!$ $(\binom{k}{i})$ is divisible by k, and the largest power of p dividing $i!$ is $\sum_{j=1}^{\infty} \lfloor i/p^j \rfloor < \sum_{j=1}^{\infty} i/p^j = i/(p-1)$.

So, $\binom{k}{i} p^{ic'}$ is always divisible by $p^{ic'+c''-\lfloor i(i-1)/(p-1) \rfloor}$. Recalling that if $p = 2$ then $c' \geq 2$ we have that $ic' + c'' - \lfloor i(i-1)/(p-1) \rfloor \geq c' + c'' + \lfloor (i-1)/2 \rfloor \geq c' + c''$ for all $i > 1$. So, every term in $\sum_{i=1}^{k} \binom{k}{i} p^{ic'} B^k$ other than the first is divisible by $p^{c'+c''+1}$ but the first term is not. Therefore, this sum is nonzero and we reach a contradiction.

5 problem (unknown)

For this problem, we strongly encourage you to try to recreate the scenario in the real world with a sufficiently long rope and some nails/pins fixed to a surface. We have $n \geq 1$ nails fixed to a wall and a sufficiently long rope wrapped around these nails in a non-trivial configuration (i.e., the rope must be physically engaged with the nails in such a way that it does not fall off initially). For any fixed $n \geq 1$, find **all** wrapping configuration around the n nails such that:

1. The rope remains securely wrapped (i.e., it does not fall off when all nails are present).
2. When **any single nail** is removed (regardless of which one), the entire rope falls from the wall (in practice, some friction might prevent it from falling, but we consider it as falling if it is no longer securely wrapped).

By "wrapping," we mean the rope can make multiple loops around the nails in non-trivial ways (e.g., making loops around single nails or multiple nails, or passing under/over certain nails). Here are examples where we've labeled the nails c_0, c_1, and c_2:

Figure 1: *Nontrivial rope wrappings for $n = 1$ satisfying the property. The first configuration has a single loop around c_0 (clockwise or counterclockwise), while the second has at least two loops around c_0 (clockwise or counterclockwise).*

Figure 2: *Nontrivial rope wrappings for $n = 2$ that do not satisfy the property. The first configuration has no loop around c_0 but at least two loops (clockwise or counterclockwise) around c_1. The second configuration has at least two loops (clockwise or counterclockwise) around both c_0 and c_1.*

Figure 3: *Nontrivial rope wrappings (performed from left to right) for $n = 3$ not satisfying the property. The first configuration has a single clockwise loop around c_0, no loop around c_1, and a single clockwise loop around c_2. The second configuration has at least two clockwise loops around c_0, no loop around c_1, and at least two counter-clockwise loops around c_2.*

Answer:

Let $n \geq 1$. To formalize the mathematical model of wrappings, we consider a set of n distinct symbols $\{c_i \mid i \in n\}$, interpreted as n nails. For simplicity, we imagine the nails pinned from left to right, with c_0 on the far left and c_{n-1} on the far right. What **only** matters in the wrapping are the loops' orientations, counts, and adjacency relationships. (Visualization is simpler in a physical simulation!)

Fix an orientation convention (clockwise as 1, counterclockwise as -1). We observe that there is a simple recursive procedure that, given a wrapping process, outputs a word over the alphabet $C_n := \{c_i^1 \mid i < n\} \cup \left\{c_i^{-1} \mid i < n\right\}$ by recursively recording loop orientations and positions from left to right. One key observation is that a "loop" need not form a closed circle as shown in the examples in the problem statement. Passing **above** a nail from left to right counts as a clockwise loop; passing from right to left counts as counterclockwise. Thus, we want to encode each action (c_i, θ), where $i < n$ and $\theta \in \{1, -1\}$, and their adjacency by writing them from left to right in the same order as the execution of the wrapping.

More formally, if one has partially constructed a word $w \in C_n^{<\omega}$ (including the empty word ε) from a partial wrapping process, then we update the word as follows. If no action is taken, w remains unchanged. Else, performing a loop at c_i with orientation θ updates the word to $w \frown \left\langle c_i^\theta \right\rangle$. This procedure defines a word for any wrapping process: each time performing an action updates the word. If the wrappings in the examples of the problem statement are performed from left to right, then they produce (for some $r, s \geq 2$) the words:

$$\left\langle {c_0}^1 \right\rangle, \left\langle {c_0}^1 \right\rangle^r, \left\langle {c_1}^1 \right\rangle^r, \left\langle {c_0}^1, {c_1}^1 \right\rangle^r, \left\langle {c_0}^1, {c_2}^1 \right\rangle^r, \left\langle {c_0}^1 \right\rangle^r \frown \left\langle {c_2}^{-1} \right\rangle^s \text{ respectively.}$$

Conversely, every word corresponds to a wrapping by executing its symbols from left to right recursively. This gives us a surjection from the set of words to the set of wrappings. For example, if $i < j < n$, the word $\langle c_i^1 \rangle^3 \frown \left\langle c_j^{-1} \right\rangle^4$ represents:

- 3 clockwise loops at c_i,
- Moving **below** intermediate nails $c_{i+1}, \ldots, c_{j-1}$ (to avoid creating unintended loops),
- 4 counterclockwise loops at c_j.

This correspondence, although being a surjective functional relation on one side, is not injective as one may notice: Fix any word $w \in C_n^{<\omega}$, perform its corresponding wrapping as above, then making a loop at c_i with orientation θ and then performing another loop at c_i with orientation $-\theta$ *leaves* the physical state of the wrapping unchanged while resulting from two different words:

$$w \frown \left\langle c_i^\theta \right\rangle \frown \left\langle c_i^{-\theta} \right\rangle \neq w.$$

This discrepancy (the obstruction of injectivity) necessitates defining equivalence classes of words (the Axiom of Choice is not needed to reverse the correspondence - gravity dictates the choice here!), where words are identified if they differ by cancellations of adjacent $\left\langle c_i^\theta \right\rangle \frown \left\langle c_i^{-\theta} \right\rangle$ pairs. To distinguish them from generic words, we call these equivalence classes *wrapping words*. *Wrapping words* over the alphabet C_n can thus be uniquely identified with elements of the free group $F(\{c_i \mid i < n\})$ (see the complete construction in Appendix 1). Conversely, each equivalence class of a word w over C_n corresponds to a unique physical wrapping of the rope around the nails via the retrieval procedure described above.

With this correspondence, we see that for each $i < n$, removing the nail c_i from a wrapping corresponding to an element $w \in F(\{c_i \mid i < n\})$ gives us obviously another wrapping around the remaining nails $\{c_j \mid j < n, j \neq i\}$, which again corresponds (by our correspondence) to an equivalence class of a word written on the very same alphabet without $\left\{c_i^1, c_i^{-1}\right\}$. That is, an element in $F(\{c_j \mid j < n\} \setminus \{c_i\})$. It is then easy to see that the unique function $\tilde{\pi}_i$ defined such that the following diagram commute:

$$\begin{tikzcd}
\{\text{wrappings on } \{c_j \mid j < n\}\} \arrow[r, "\text{removing the nail } c_i"] \arrow[d, equal] & \{\text{wrappings on } \{c_j \mid i \neq j < n\}\} \arrow[d, equal] \\
F(\{c_j \mid j < n\}) \arrow[r, "\tilde{\pi}_i"'] & F(\{c_j \mid i \neq j < n\})
\end{tikzcd}$$

is the one induced from the function π_i over $C_n^{<\omega}$ that removes every occurrence of c_i^1 and c_i^{-1} in any word $w \in C_n^{<\omega}$ and leaves the rest intact. That is:

$$\pi_i : C_n^{<\omega} \to \left(C_n \setminus \left\{c_i^1, c_i^{-1}\right\}\right)^{<\omega}$$

defined by:

$$\pi_i(w) = \frown_{\substack{j \in \mathrm{dom}(w) \\ w_j \notin \left\{c_i^1, c_i^{-1}\right\}}} \langle w_j \rangle$$

where the concatenation is in the order of the natural order of the indices in:

$$\left\{j \in \mathrm{dom}(w) \mid w_j \notin \left\{c_i^1, c_i^{-1}\right\}\right\}.$$

We now search the word that satisfies the property as stated in the problem statement. Notice that the rope falling or being initially trivially wrapped corresponds to any "wrapping" word $w \sim \varepsilon$ (the empty sequence). So we search for a word $w \in C_n^{<\omega}$ such that $w \not\sim \epsilon$. Moreover, each c_i must occur at least once in any equivalent word to w; otherwise, if there is a certain c_i not occurring in an equivalent word to w, then removing the nail c_i would not affect the underlying wrapping, and so the wrapping doesn't have the property that removing any single nail makes the entire rope fall. So we search for a word $w \in C_n^{<\omega}$ such that:

$$\forall v \in [w]_\sim \, \forall i < n \quad v \notin \left(C_n \setminus \left\{c_i^1, c_i^{-1}\right\}\right)^{<\omega}.$$

We require the additional property that removing any nail makes the rope fall, i.e., every projection of the word is equivalent to the empty sequence $\pi_i(w) \sim \varepsilon$. Thus, the searched set of *wrapping words* is:

$$\mathcal{S}_n := \left\{[w]_\sim \in \bigcap_{i<n} \ker(\tilde{\pi}_i) \mid \forall v \in [w]_\sim \, \forall i < n \quad v \notin \left(C_n \setminus \left\{c_i^1, c_i^{-1}\right\}\right)^{<\omega}\right\}.$$

For $n = 1$, we have that $\left(C_1 \setminus \left\{c_0^1, c_0^{-1}\right\}\right)^{<\omega} = \varnothing^{<\omega} = \{\varepsilon\}$, so that:

$$\mathcal{S}_1 = \{[w]_\sim \in \ker(\tilde{\pi}_0) \mid \forall v \in [w]_\sim \quad v \neq \varepsilon\} = \ker(\tilde{\pi}_0) \setminus \{[\varepsilon]_\sim\}.$$

Indeed, this is quite obvious (in regard to the example given in the problem statement for $n = 1$) that all wrappings have the property as long as they are not the trivial one.

Now suppose $n \geq 2$. It is (though quite long) to prove by induction on n that $\bigcap_{i<n} \ker(\tilde{\pi}_i)$ is the subgroup generated by all k-th commutators for $2 \leq k \leq n$ of elements in $\left\{\left[c_i^1\right]_\sim \mid i < n\right\}$:

$$\bigcap_{i<n} \ker(\tilde{\pi}_i) = \left\langle \left\{ \left[\left[c_{j_0}^1\right]_\sim, \ldots, \left[c_{j_{k-1}}^1\right]_\sim \right] \mid \exists k \exists j \quad 2 \leq k \leq n \wedge j : k \to n \right\} \right\rangle,$$

where we define recursively for $k = 2$ and any $j : 2 \to n$:

$$\left[\left[c_{j_0}^1\right]_\sim, \left[c_{j_1}^1\right]_\sim\right] := \left[c_{j_0}^1\right]_\sim^{-1\frown} \left[c_{j_1}^1\right]_\sim^{-1\frown} \left[c_{j_0}^1\right]_\sim^{\frown} \left[c_{j_1}^1\right]_\sim,$$

and for $3 \leq k$ and any $i : k \to n$:

$$\left[\left[c_{j_0}^1\right]_\sim, \ldots, \left[c_{j_{k-1}}^1\right]_\sim\right] = \left[\left[\left[c_{j_0}^1\right]_\sim, \ldots, \left[c_{j_{k-2}}^1\right]_\sim\right], \left[c_{i_{k-1}}^1\right]_\sim\right].$$

Since we must have at least each occurrence of c_i in our word, we must restrict ourselves to the set of all n-th commutators with distinct elements (that is, the indexation must be injective). Hence, for $n \geq 2$:

$$\mathcal{S}_n = \left\langle \left\{ \left[\left[c_{j_0}^1\right]_\sim, \ldots, \left[c_{j_{k-1}}^1\right]_\sim\right] \mid \exists j \quad j : n \hookrightarrow n \right\} \right\rangle \setminus \{[\varepsilon]_\sim\}.$$

For example:

- When $n = 2$, the *wrapping words* are generated by the two generators:

  $$\left[\left[{c_0}^1\right]_\sim, \left[{c_1}^1\right]_\sim\right] := \left[{c_0}^1\right]_\sim^{-1\frown} \left[{c_1}^1\right]_\sim^{-1\frown} \left[{c_0}^1\right]_\sim^{\frown} \left[{c_1}^1\right]_\sim = \left[\left\langle {c_0}^{-1}, {c_1}^{-1}, {c_0}^1, {c_1}^1 \right\rangle\right]_\sim,$$

  and

  $$\left[\left[{c_1}^1\right]_\sim, \left[{c_0}^1\right]_\sim\right] := \left[{c_1}^1\right]_\sim^{-1\frown} \left[{c_0}^1\right]_\sim^{-1\frown} \left[{c_1}^1\right]_\sim^{\frown} \left[{c_0}^1\right]_\sim = \left[\left\langle {c_1}^{-1}, {c_0}^{-1}, {c_1}^1, {c_0}^1 \right\rangle\right]_\sim.$$

  i.e., the physical wrapping corresponding to these generator are (for $j \in \{0, 1\}$):

  - 1 counterclockwise loop at c_j,
  - 1 counterclockwise loop at c_{1-j},
  - 1 clockwise loop at c_j,
  - 1 clockwise loop at c_{1-j}.

  So that:

  $$\mathcal{S}_2 = \left\langle \left\{ \left[\left\langle {c_0}^{-1}, {c_1}^{-1}, {c_0}^1, {c_1}^1 \right\rangle\right]_\sim, \left[\left\langle {c_1}^{-1}, {c_0}^{-1}, {c_1}^1, {c_0}^1 \right\rangle\right]_\sim \right\} \right\rangle \setminus \{[\varepsilon]_\sim\},$$

  i.e., the set of physical wrappings composed of the above physical wrappings that do not make a trivial physical wrapping **is** the set of all physical wrappings that have the required property.

- Similarly, for $n = 3$, $\mathcal{S}_3$ is the set of *wrapping words* generated by all combinations of the following generators that do not yield the trivial *wrapping word*:

  $$\left[\left\langle {c_0}^{-1}, {c_1}^{-1}, {c_0}^1, {c_1}^1, {c_2}^{-1}, {c_1}^{-1}, {c_0}^{-1}, {c_1}^1, {c_0}^1, {c_2}^1 \right\rangle\right]_\sim,$$

  $$\left[\left\langle {c_0}^{-1}, {c_2}^{-1}, {c_0}^1, {c_2}^1, {c_1}^{-1}, {c_2}^{-1}, {c_0}^{-1}, {c_2}^1, {c_0}^1, {c_1}^1 \right\rangle\right]_\sim,$$

  $$\left[\left\langle {c_1}^{-1}, {c_2}^{-1}, {c_1}^1, {c_2}^1, {c_0}^{-1}, {c_2}^{-1}, {c_1}^{-1}, {c_2}^1, {c_1}^1, {c_0}^1 \right\rangle\right]_\sim,$$

  $$\left[\left\langle {c_1}^{-1}, {c_0}^{-1}, {c_1}^1, {c_0}^1, {c_2}^{-1}, {c_0}^{-1}, {c_1}^{-1}, {c_0}^1, {c_1}^1, {c_2}^1 \right\rangle\right]_\sim,$$

  $$\left[\left\langle {c_2}^{-1}, {c_0}^{-1}, {c_2}^1, {c_0}^1, {c_1}^{-1}, {c_0}^{-1}, {c_2}^{-1}, {c_0}^1, {c_2}^1, {c_1}^1 \right\rangle\right]_\sim,$$

  $$\left[\left\langle {c_2}^{-1}, {c_1}^{-1}, {c_2}^1, {c_1}^1, {c_0}^{-1}, {c_1}^{-1}, {c_2}^{-1}, {c_1}^1, {c_2}^1, {c_0}^1 \right\rangle\right]_\sim.$$

This concludes.

A

We define formally in Set theory (**ZF**) the free group generated on $\{c_i \mid i < n\}$, $F(\{c_i \mid i < n\})$ as follows. First, we introduce an orientation by defining $c_i^1 := (c_i, 1)$ and $c_i^{-1} := (c_i, -1)$. Then, we take the set of all possible words over the alphabet $C_n := \{c_i^1 \mid i \in n\} \cup \left\{c_i^{-1} \mid i \in n\right\}$, that is, the set of all sequences w with a domain being an integer $\mathrm{dom}(w) \in \mathbb{N}$ and range being in the alphabet C_n, $\mathrm{ran}(w) \subset C_n$:

$$C_n^{<\omega} = \bigcup_{i \in \mathbb{N}} C_n^i.$$

We let the binary operation of concatenation $\frown : C_n^{<\omega} \times C_n^{<\omega} \to C_n^{<\omega}$ of two words $w, v \in C_n^{<\omega}$ defined by $w \frown v : \mathrm{dom}(w) + \mathrm{dom}(v) \to C_n$ defined by:

$$w \frown v = \begin{cases} w_i & \text{if } i < \mathrm{dom}(w), \\ v_j & \text{if } i = \mathrm{dom}(w) + j. \end{cases}$$

The reader may check that concatenation is internal, associative, admits a neutral element (namely the empty sequence $\varepsilon = \varnothing$), and that it is non-commutative as long as $n \geq 1$. Then, we let the smallest equivalence relation $\sim \subset C_n^{<\omega} \times C_n^{<\omega}$ containing the set:

$$C(n) := \left\{\left(\left\langle c_i^1, c_i^{-1}\right\rangle, \varepsilon\right) \mid i < n\right\} \cup \left\{\left(\left\langle c_i^{-1}, c_i^1\right\rangle, \varepsilon\right) \mid i < n\right\},$$

such that the operation of concatenation passes to the quotient:

$$\sim := \bigcap \left\{ R \in \mathscr{P}\left(C_n^{<\omega} \times C_n^{<\omega}\right) \;\middle|\; \begin{array}{l} \text{“}R \text{ is an equivalence relation”} \wedge C(n) \subset R \\ \wedge \quad \forall v, w, v', w' \in C_n^{<\omega} \\ \quad (v, v') \in R \wedge (w, w') \in R \to (v \frown w, v' \frown w') \in R \end{array} \right\}.$$

This set of such equivalence relations is non-empty, as it contains the trivial equivalence relation where everything is equivalent to everything, so the intersection is not over the empty set and is thus well-defined. It is easy to see that an arbitrary intersection of equivalence relations is an equivalence relation. Then $\forall v, w \in C_n^{<\omega}$: We can define a 2-ary relation $\frown \subset \left(C_n^{<\omega}/_\sim \times C_n^{<\omega}/_\sim\right) \times C_n^{<\omega}/_\sim$ by:

$$\frown := \left\{\left(\left([v]_\sim, [w]_\sim\right), [v \frown w]_\sim\right) \in \left(C_n^{<\omega}/_\sim \times C_n^{<\omega}/_\sim\right) \times C_n^{<\omega}/_\sim \mid \exists v, w \in C_n^{<\omega}\right\}.$$

The construction of $\sim$ gives us that in fact $\frown$ is a functional relation with domain $C_n^{<\omega}/_\sim \times C_n^{<\omega}/_\sim$:

$$\frown : C_n^{<\omega}/_\sim \times C_n^{<\omega}/_\sim \to C_n^{<\omega}/_\sim$$

defined by:

$$[v]_\sim \frown [w]_\sim = [v \frown w]_\sim .$$

(That is, $\frown$ commutes with $\frown$.) Now, the reader can verify that this new binary function $\frown$ makes $F(\{c_j \mid j < n\}) := C_n^{<\omega}/_\sim$ into a group (this construction inherits the properties of closure, associativity, and the existence of a neutral element, which is $[\varepsilon]_\sim$). The inverse element of the equivalence class of a word $w \in C_n^{<\omega}$ is the equivalence class of the word obtained by reversing the order of w and inverting each exponent (changing 1 to -1 and *vice versa*). More formally, if we denote by $p_2 : \{c_i \mid i < n\} \times \{1, -1\} \to \{1, -1\}$ the projection onto the second coordinate, this is:

$$[w]_\sim^{-1} = \left[\frown_{i \in \mathrm{dom}(w)} \left\langle w_{\mathrm{dom}(w)-1-i}^{-p_2\left(w_{\mathrm{dom}(w)-1-i}\right)} \right\rangle\right]_\sim$$

For example, since $\langle c_0^1 \rangle \frown \left\langle c_0^{-1} \right\rangle \sim \varepsilon$, we must have:

$$\left[\left\langle c_0^1 \right\rangle\right]_\sim^{-1} = \left[\left\langle c_0^{-1} \right\rangle\right]_\sim .$$

One can check that $\frown$ is again non-commutative as long as $n \geq 2$ (since $c_0 \neq c_1$). This group has a "free" property (which we will not develop here), so that we call $F(\{c_j \mid j < n\})$ the free group generated by $\{c_j \mid j < n\}$.

For the sake of simplicity, we define for any $w \in C_n^{<\omega}$ and $l \geq 0$ the abbreviation:

$$w^l := \frown_{s<l}^{l} w$$

(the order of concatenation doesn't matter here). For example, we have $w^0 = \varepsilon$, since we perform a concatenation with an index running over the empty set, which is then the empty set. Here is another example:

$$\left\langle c_0^1 \right\rangle^3 \frown \left\langle c_0^{-1} \right\rangle^4 = \left\langle c_0^1, c_0^1, c_0^1, c_0^{-1}, c_0^{-1}, c_0^{-1}, c_0^{-1} \right\rangle \sim \left\langle c_0^{-1} \right\rangle .$$

Every equivalent word will behave the same under the "new" concatenation by construction of $\sim$. In particular, we have for any words $w, v \in C_n^{<\omega}$, any $i < n$, and any $r, s > 0$ that:

$$\text{if } r \geq s: \quad \left(w \frown \left\langle c_0^1 \right\rangle^r \frown \left\langle c_0^{-1} \right\rangle^s \frown v\right) \sim w \frown \left\langle c_0^1 \right\rangle^{r-s} \frown v,$$

$$\text{and } \left(w \frown \left\langle c_0^{-1} \right\rangle^r \frown \left\langle c_0^1 \right\rangle^s \frown v\right) \sim w \frown \left\langle c_0^{-1} \right\rangle^{r-s} \frown v,$$

while:

$$\text{if } r < s: \quad \left(w \frown \left\langle c_0^1 \right\rangle^r \frown \left\langle c_0^{-1} \right\rangle^s \frown v\right) \sim w \frown \left\langle c_0^{-1} \right\rangle^{s-r} \frown v,$$

$$\text{and } \left(w \frown \left\langle c_0^{-1} \right\rangle^r \frown \left\langle c_0^1 \right\rangle^s \frown v\right) \sim w \frown \left\langle c_0^1 \right\rangle^{s-r} \frown v.$$

Remark: In fact, if we replace $\{c_i \mid i < n\}$ with an arbitrary set J, we obtain—*mutatis mutandis*—the construction of the free group $F(J)$ generated by J (even $J = \varnothing$). Note, however, that we have not yet discussed what makes these groups "free".