Problem Set Week 4

ETHZ Math Olympiad Club

10 March 2025

1 Problem in example page 140 (PUTNAM and BEYOND)

Let $f: \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative. Prove that

$$f(x + f'(x)) \ge f(x),$$

for any real number x.

2 Problem A-2 (IMC 2011)

Does there exist a real 3×3 matrix A such that

$$\operatorname{tr}(A) = 0$$
 and $A^2 + A^T = I_3$,

where tr(A) denotes the trace of A, A^T is the transpose of A, and I_3 is the 3×3 identity matrix?

3 Problem B-2 (IMC 2014)

Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \le i < j \le n} a_{ii} a_{jj} \ge \sum_{1 \le i < j \le n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

4 Problem 414 (PUTNAM and BEYOND)

For any real number $\lambda \geq 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1+\ln x)=\lambda.$$

Prove that

$$\lim_{\lambda \to \infty} \frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = 1.$$

5 Problem A-4 (IMC 2014)

Let n>6 be a perfect number, and let $n=p_1^{e_1}\cdots p_k^{e_k}$ be its prime factorisation with

$$1 < p_1 < \ldots < p_k$$
.

Prove that e_1 is an even number.

A number n is perfect if s(n) = 2n, where $s(n) = \sum_{\mathbb{N} \ni d|_{\mathbb{Z}} n} d$ is the sum of the divisors of n.