Problem Set 4

Math Olympiad Club Zurich

Fall 2025

Problem 1: (Bulgarian National Math Olympiad 2025 prob. 1)

Find all sequences $a_1,\ a_2\ldots$ of real numbers, such that:
$\forall n,\ m \in \mathbb{N} : a_{m^2+m+n} = a_m^2 + a_m + a_n$.

Problem 2: (Bulgarian National Math Olympiad 2025 prob. 2)

For an integer n we look at a $n \times n$ square grid of white cells. For a given choice of n cells to color black, we define the cost pf the coloring as the minimal number of additional cells to color black needed, to make the black cells a connected component (between any two black cells you have a path traveling through cells sharing a side, such that all cells on the path are black). We define $f(n)$ to be the maximum among the costs for all colorings of the $n \times n$ grid. Find α such that $\forall n \geq 100,\ \frac{n^\alpha}{3} \leq 3n^\alpha$.

Problem 3: (Bulgarian National Math Olympiad 2025 prob. 5)

A number is squarefree, if it doesn't have a repeated prime factor. Prove that for every integer n there is an integer a such that $\{a+1,\ a+2\ldots a+n\}$ contains exactly $\frac{n}{4}$ squarefree numbers.

Problem 4: (Balkan Math Olympiad 2024 problem 2)

Let $n \geq k \geq 3$ be integers. Show that for every integer sequence

$$1 \leq a_1 < a_2 < \cdots < a_k \leq n$$

one can choose non-negative integers $b_1, b_2, \ldots, b_k$ satisfying:

[(i)]

1. $0 \leq b_i \leq n$ for each $1 \leq i \leq k$,
2. all the positive b_i are distinct,
3. the sums $a_i + b_i$, $1 \leq i \leq k$, form a permutation of the first k terms of a non-constant arithmetic progression.

Problem 5: (Balkan Math Olympiad 2024 problem 3)

Let a and b be distinct positive integers such that $3a + 2$ is divisible by $3b + 2$.
Prove that $a > b^2$.

Problem 6: (Balkan Math Olympiad 2024 problem 4)

Let $\mathbb{R}^+ = (0, 1)$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and polynomials $P(x)$ with non-negative real coefficients such that $P(0) = 0$ and which satisfy

$$f(f(x) + P(y)) = f(x - y) + 2y$$

for all real numbers $x > y > 0$.

Problem 7: (Bulgarian National Math Olympiad 2024 prob. 4)

Do there exist 2024 nonzero real numbers $a_1, a_2, \ldots, a_{2024}$, for which

$$\sum_{i=1}^{2024} \frac{a_i^2 + 1}{a_i^2} \; + \; 2\sum_{i=1}^{2024} \frac{a_i}{a_{i+1}} \; + \; 2024 \; = \; 2\sum_{i=1}^{2024} \frac{a_i + 1}{a_i} \; ?$$

(in the second sum $a_{2025} = a_1$.)

Problem 8: (Bulgarian National Math Olympiad 2024 prob. 5)

Fix an integer m and look at a set S of 5^m elements. We look at a family $\mathcal{F}$ of subsets of S of size 4, such that for any two subsets in the family, their intersection does not have size 2. What is the maximum value of $|\mathcal{F}|$, i.e. what is the maximum number of subsets with for elements we can choose, such that the property holds?