Problem Set 1

ETHZ Math Olympiad Club

23 Sep 2025

Autumn Math Competition Bulgaria problem 12.1

Given the sequence $\{x_n\}_{n=0}^{+\infty}$, where $x_0 = 1$ and $x_{n+1} = \sin(x_n) + \frac{\pi}{2} - 1$, prove that it converges, and find the limit.

Solution:

$\sin x - x$ is decreasing between 0 and $\frac{\pi}{2}$ by a simple derivative test, so $\sin x_n - x_n \geq 1 - \frac{\pi}{2}$ (value at intermediate point vs. value at endpoint). This immediately implies $x_{n+1} \geq x_n$.
$\frac{\pi}{2}$ is an easy upper bound for the sequence, so it converges by the Monotone Convergence theorem. The limit must be a fixed point, the unique solution between 0 and $\frac{\pi}{2}$ being $\frac{\pi}{2}$, which must be our limit.

Autumn Math Competition Bulgaria problem 12.3

Find all solutions in the natural numbers to the equation: $\sqrt[n]{m} + \sqrt[m]{n} = 2 + \frac{2}{mn(m+n)^{\frac{1}{m}+\frac{1}{n}}}$

Solution:

For all natural n and m, $\lfloor \sqrt[n]{m} \rfloor \geq 1$, we will prove that $\{ \sqrt[n]{m} \} \geq \frac{1}{nm}$. Let us call the floor value a, and the fractional value x:
$1 \leq m - a^n = (m^{\frac{1}{n}})^n - a^n = x \sum_{i=0}^{n-1} m^{\frac{i}{n}} a^{n-1-i} \leq xnm \;\; => \;\; x \geq \frac{1}{nm}$
We get $\sqrt[n]{m} + \sqrt[m]{n} \geq 2 + \frac{2}{nm} > 2 + \frac{2}{mn(m+n)^{\frac{1}{m}+\frac{1}{n}}}$, so there are no integer solutions.

Electrostatics problem

There is an equilateral triangle with equal charges kept on its vertices. Find the number and location of points with electric field $\vec{E}_{total} = 0$.
For a point charge q,

$$\vec{E}(\vec{r}) = \frac{q(\vec{r} - \vec{r}_0)}{4\pi\epsilon_0|\vec{r} - \vec{r}_0|^3}$$

where $\vec{r}_0$ is the location of the point charge. Take $q = 1$ and $4\pi\epsilon_0 = 1$. (The side length of the triangle can be chosen arbitrarily).

Radius of Curvature for $\mathbb{R} \to \mathbb{R}$ functions

Consider a particle moving along a curve $y = f(x)$. It is forced to move at a unit speed along the x direction. Find the expression of radius of curvature R, using the centripetal acceleration $\vec{a}_{\perp}$.

$$|\vec{a}_{\perp}| = \frac{|\vec{v}|^2}{R}$$

Problem 922. (American Monthly - Toyesh Prakash Sharma)

Let F_n be the nth Fibonacci number defined by $F_1 = 1$, $F_2 = 1$ and for all $n > 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{n\geq 1} \left(\frac{1}{9}\right)^{F_{n+2}}$$

is irrational number but not a transcendental number.