Problem Set Week 7

ETHZ Math Olympiad Club

31 March 2025

1 Problem B-1 (IMC 2023)

Ivan writes the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

on the board. Then he performs the following operation on the matrix several times:

- He chooses a row or a column of the matrix, and
- He multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix

$$B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

after finitely many steps?

2 Vieta Jumping Problems

2.1 Problem 6 (IMO 1988)

Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is the square of an integer.

2.2 Problem (Kevin Buzzard and Edward Crane)

Let a and b be positive integers. Show that if $4ab - 1$ divides $(4a^2 - 1)^2$, then $a = b$.

3 Problem A-3 (IMC 2015)

Let $F(0) = 0$, $F(1) = \frac{3}{2}$, and

$$F(n) = \frac{5}{2}F(n-1) - F(n-2) \quad \text{for } n \geq 2.$$

Determine whether or not

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$$

is a rational number.

4 Problems of Mathematicians (unknown)

4.1

N mathematicians each wear a hat whose color is chosen from the set $[\![1, N]\!]$ (with repetitions allowed). Each mathematician sees the colors of all the other hats but not their own. What strategy can they use to guarantee that at least one mathematician correctly guesses the color of their own hat?

4.2

N mathematicians each wear a hat that is either blue or red. Each mathematician sees the colors of all the other hats but not their own. What strategy can they adopt so that they are either all wrong or all correct simultaneously?

Bonus:[1] Solve the same problem when the set of mathematicians is countably infinite.

4.3

Consider a sequence of $N \geq 1$ mathematicians standing in a line. Each mathematician wears either a black or a white hat. Every mathematician can see the hat colors of all the mathematicians in front of them but not their own or those behind them. The game consists of the mathematicians recursively (starting from the first in line) shouting (we assume that everyone hears, i.e., information propagates recursively to the rest of the queue) *only* "black" or "white," which will be interpreted as the guess of their own hat color. Before the game starts and the hats are placed on their heads, the mathematicians can devise a strategy. How can they ensure that all but at most one mathematician guesses their hat color correctly?

Bonus:[2] If there are countably infinitely many mathematicians standing in line (say indexed by $\mathbb{N}$), how can they ensure that only finitely many of them guess incorrectly?

Paradox: Suppose the hats are assigned randomly and independently (for example, by tossing a fair coin for each mathematician). Then each mathematician has a probability of $\frac{1}{2}$ of guessing incorrectly. Let T_N denote the number of incorrect guesses among the first N mathematicians. By the strong law of large numbers, $\frac{T_N}{N} \xrightarrow{N \to +\infty} \frac{1}{2}$ almost surely, so that $T_N \underset{N \to +\infty}{\sim} \frac{N}{2}$ almost surely, and thus T_N is unbounded almost surely. This means that with probability 1, infinitely many mathematicians guess incorrectly. How can we resolve this apparent paradox?

[1] We assume the axiom of countable choice ($\mathbf{AC}_\omega$), that is, every countable collection of non-empty sets has a choice function:

$$\forall X \left[(|X| = \aleph_0 \land \forall Y \in X \, \exists z \in Y) \to \exists f : X \to \bigcup X \text{ such that } \forall Y \in X,\ f(Y) \in Y \right].$$

We also assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $\aleph_0 = |\mathbb{N}|$.

[2] We assume the axiom of choice ($\mathbf{AC}$), that is, every collection of non-empty sets has a choice function:

$$\forall X \left[(\forall Y \in X \, \exists z \in Y) \to \exists f : X \to \bigcup X \text{ such that } \forall Y \in X,\ f(Y) \in Y \right].$$

We also assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $2^{\aleph_0} = |\mathscr{P}(\mathbb{N})|$.

4.4

In an infinite sequence, countably infinitely many mathematicians stand one behind the other. Each mathematician has a natural number on their back, where each number appears exactly once but is assigned arbitrarily to a mathematician (i.e., a permutation $\pi : \mathbb{N} \to \mathbb{N}$). Each mathematician can see all the numbers on the backs of those standing in front of them (forming an actual infinite sequence)[3] but not their own number or the numbers of those standing behind them.

(a) The game consists of the mathematicians recursively (starting from the first in line) shouting (we assume that everyone hears, i.e., information propagates recursively to the rest of the queue) *only* a number, which will be interpreted as the guess of their own number on their back. No mathematician is allowed to state a number they see on any back in front of them, and all of them are aware of this rule.

    How can they ensure that everyone guesses their number correctly?

(b) As in (a), but the first $N \geq 0$ mathematicians are silent (say nothing), meaning the mathematician in the $N+1$ position starts the guessing process.

    How can they ensure that everyone except at most N guesses their number correctly?

[3] We assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $\aleph_0$.