Problem Set Week 11 Solutions

ETHZ Math Olympiad Club

19 May 2025

Problem 1 (Bernoulli Competition 20203)

1. Let $A = \{1, 2, \ldots, 100\}$ be the set of integers between 1 and 100.

(a) Let $B \subset A$ be a subset that doesn't contain two consecutive integers. What is the maximal cardinality of B?

(b) Let $C \subset A$ be a subset such that there is no n for which n and $2n$ are both in C. What is the maximal cardinality of C?

Solution:

Solution to part a: One can easily construct a legal set B with 50 elements by setting it equal to the set of odd numbers from 1 to 100, or the set of even numbers from 1 to 100. That leaves proving that one cannot do better. In order to do that, observe that A can be partitioned into 50 pairs of an odd number and the next even number, i.e. $A = \{1, 2\} \cup \{3, 4\} \cup \ldots \cup \{99, 100\}$. B can contain at most one element of each of these pairs, so it has at most 50 elements. Therefore, the maximum possible cardinality of B is $\boxed{50}$.

Alternate solution to part a: One can easily construct a legal set B with 50 elements by setting it equal to the set of odd numbers from 1 to 100, or the set of even numbers from 1 to 100. That leaves proving that one cannot do better. In order to do that, first let $x_1, \ldots, x_m$ be the elements of B in increasing order. The fact that no two elements of B are consecutive implies that $x_{i+1} \geq x_i + 2$ for all i, so $x_m \geq x_1 + 2(m-1) \geq 2m - 1$. However, $x_m \leq 100$ due to being in A, so it must be the case that $m \leq 50$. Therefore, the maximum possible cardinality of B is $\boxed{50}$.

Solution to part b: For each nonnegative integer m, let $A_m = A \cap \{(2m+1)2^i : i \in \mathbb{Z}\}$, and observe that these are disjoint and $A = \cup_{0 \leq i \leq 49} A_i$. The requirement on C is equivalent to saying that it never contains both $(2m+1)2^i$ and $(2m+1)2^{i+1}$, so $|C \cap A_i| \leq \lfloor |A_i|/2 \rfloor$ by an argument from the solution to the previous part. We can ensure that $|C \cap A_i| = \lfloor |A_i|/2 \rfloor$ for all m, such as by having C be the set of every element of A that can be expressed as an odd number times a power of 4. There are 50 odd numbers in A, 13 odd multiples of 4, 3 odd multiples of 16, and 1 odd multiple of 64. So, C has a cardinality of $50 + 13 + 3 + 1 = \boxed{67}$.

Alternate solution to part b: Observe that for any $51 \leq m \leq 100$, if we let $C' = C \cup \{m\} \backslash \{m/2\}$ then C' contains at least as many elements as C and does not contain both n and $2n$ for any n. So, we can assume that C contains n for all $51 \leq n \leq 100$. In this case, it cannot contain any $26 \leq n \leq 50$. Then by the same logic we can assume that C contains $\{13, \ldots, 25\}$, which forces it not to contain any $7 \leq n \leq 12$. Then we can assume that it contains $\{4, 5, 6\}$, which means it does not contain 2 or 3, at which point we can have it contain 1. So, the maximum possible cardinality of C is $50 + 13 + 3 + 1 = \boxed{67}$.

Problem (selected real analysis problem)

Determine wheter there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ (standard topology) such that $f \circ f = F$ where $F(x) = -x$, $F(x) = \exp(x)$, $F(x) = x^2 - 2$,

Solution:

(a) Prove that $f(0) = 0$ and investigate the sign of $f(x)$ for $x > 0$.

(b) Prove that f is strictly increasing and that $\inf f = a \in (-\infty, 0)$, $f(a) = 0$. Fix an arbitrary strictly increasing function $f_0 \in C([a, 0])$ satisfying the conditions $f_0(a) = 0$ and $f_0(0) = e^a$, and extend it to $\mathbb{R}$ by using the equation.

(c) Prove that f must be strictly increasing on $[0, +\infty)$ and strictly decreasing on $(-\infty, 0]$ and that $f(0) \geq 0$, which is impossible, because F, and hence f, takes negative values.

Bonus: If $F(x) = \cos(x)$?

Solution:

To begin we do some preliminary work (some of the result are well known but for the sake of completeness we give the "proofs"):

-The function cos has a unique fixed point over $\mathbb{R}$ ($\exists! \alpha \in \mathbb{R}\,(cos(\alpha) = \alpha)$) which is moreover located between $]0, 1[$. We define in a classical manner (for this kind of fixed point problem) the function:

$$\begin{aligned} \varphi : \quad & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \longmapsto x - cos(x) = x - \textstyle\sum_{i \in \mathbb{N}} (-1)^i \cdot \frac{x^{2 \cdot i}}{(2 \cdot i)!} \end{aligned}$$

φ is clearly $C^{\infty}(\mathbb{R})$ and even analytic. We study φ and show that in fact it has only one zero: $|\varphi^{-1}[\{0\}]| = 1$ which will conclude the existence and unicity of the fixed point over $\mathbb{R}$. Notice that $\frac{d}{dx}(\varphi) = 1 - sin \geq \underline{0}$. The theorem relating the type of monotonicity and the sign of the derivative tell us therefore that φ is increasing. As $\varphi(0) = -1$ and $\varphi(1) = 1 - cos(1) \overset{1 \in]0; \frac{\pi}{2}[}{>} 0$, we have that:

$$\varphi[] - \infty; 0]] \subset] - \infty; -1] \subset] - \infty; 0[$$

and

$$\varphi[[1; +\infty[] \subset [\varphi(1); +\infty[\subset]0; +\infty[$$

Thus $\varphi^{-1}[\{0\}] \subset]0; 1[$ and moreover we have by the intermediate value theorem (or the more general topological version: image of a connected set through a continuous function is connected and knowing that only the intervals are precisely the connected set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$) that $0 \in [-1; \varphi(1)] \subset \varphi[[0; 1]]$. This shows that $|\varphi^{-1}[\{0\}]| \geq 1$. We know by the property of sin:

$$\forall x \in \mathbb{R}((\frac{d}{dx}(\varphi))(x) = 0 \leftrightarrow x \in \frac{\pi}{2} + 2 \cdot \pi\mathbb{Z})$$

Therefore we have the refinement that φ is strictly increasing over each connected component of $\mathbb{R} \setminus (\frac{\pi}{2} + 2 \cdot \pi\mathbb{Z})$. However $]\frac{\pi}{2} - 2 \cdot \pi; \frac{\pi}{2}[$ is one of the connected component. This means that φ is strictly increasing over $]0; 1[\subset]\frac{\pi}{2} - 2 \cdot \pi; \frac{\pi}{2}[$ in particular it is injective. We saw that

$\varphi^{-1}[\{0\}] \subset]0;1[$ so this means that $|\varphi^{-1}[\{0\}]| = 1$. This concludes that there is a unique fixed point of cos over $\mathbb{R}$. For the culture, this fixed point is called the *Dottie* number. The decimal expansion of the Dottie number is 0.739085133215160641655312087673873404... and one can show using some advanced techniques like the Lindemann–Weierstrass theorem that it is also a transcendental number. How can we attain such a number ? It is easy: cos is a contraction on $[0;1]$! Indeed its derivative $(-sin)$ is continuous therefore bounded over the compact $[0;1]$ and it appears that those bound are strictly less than 1 ! To be more precise, let $a, b \in [0;1]$, when $a \neq b$ we have by the mean value theorem ($cos \in C^\infty(\mathbb{R})$) that $\exists c \in]a;b[$ such that $-sin(c) = (\frac{d}{dx}cos)(c) = \frac{cos(a)-cos(b)}{a-b}$. Therefore we obtain:

$$|cos(a) - cos(b)| \leq max|-sin|[[0;1]] \cdot |a-b|$$

$$\overset{sin \text{ is strictly increasing over } [0;\frac{\pi}{2}]}{=} sin(1) \cdot |a-b|$$

This bounds (which works even for $a = b$) shows that cos is a contraction over $[0;1]$ **provided** $sin(1) < 1$ which is the case since sin is strictly increasing over $[0;\frac{\pi}{2}]$ ($1 < \frac{\pi}{2}$ so that $sin(1) < sin(\frac{\pi}{2}) = 1$). Therefore a common application of the Banach fixed point theorem for the complete metrix space $([0;1], |\cdot|)$ tell us not only that there is a unique fixed point of cos over $[0;1]$ (we already know this information) but also the way the fixed point is constructed. Take any $x \in [0;1]$, the fixed point is the limit ($[0;1]$ is complete) of the sequence $(cos^{\circ n}(x))_{n\in\mathbb{N}} \in [0;1]^{\mathbb{N}}$ where $cos^{\circ n}$ denotes the functional composition of cos with itself n times ($cos^{\circ 0} = Id$). This means that $lim_{n\to+\infty}(cos|_{[0;1]})^{\circ n} = cte_\alpha$ where α is the Dottie number. Apparently the generalized case $cos(z) = z$ where $z \in \mathbb{C}$ has infinitely many solutions (it uses Picard's theorem).

-Let us denote the Dottie number by $\alpha \in]0;1[$. We claim that any function $g : \mathbb{R} \to \mathbb{R}$ satisfying $g \circ g = cos$ share the same fixed point of cos namely α and must be injective over $[0;\frac{\pi}{2}]$. Indeed suppose $g \circ g = cos$ then:

$$cos(g(\alpha)) = (g \circ g)(g(\alpha)) = g((g \circ g)(\alpha)) = g(cos(\alpha)) = g(\alpha)$$

Therefore $g(\alpha)$ is a fixed point of cos, by the unicity of the Dottie number we must have $g(\alpha) = \alpha$. The injectivity follows easily from the equality $g \circ g = cos$ and noting that cos is a bijection from $[0;\frac{\pi}{2}]$ to $[0;1]$ we must have (classic) from the equality $g \circ g = cos$ the injectivity of g over $[0;\frac{\pi}{2}]$ (and the surjectivity of g as well over [0;1]).

-Let I be a real interval. Let $h : I \to \mathbb{R}$ be an injective continuous real function. Then h is strictly monotone. Aiming for a contradiction, suppose h is not strictly monotone. If we let $\phi_{</>}(h) : \forall x \in dom(h) \forall y \in dom(h)\, (x < y \to h(x) < / > h(y))$ respectively. Not being strictly monotone therefore means:

$$\neg(\phi_<(h) \vee \phi_>(h))$$

That is equivalent to attention pas fini on utilise maintenant le fait que I soit un intervalle!!! voir video ou bien voir la serie 6.2 exo 3 de analyse I avec Hongler: there exist $x, y, z \in I$ with $x < y < z$ such that either:

$$(h(x) \leq h(y) \text{ and } h(y) \geq h(z)) \text{ or } (h(x) \geq h(y) \text{ and } h(y) \leq h(z))$$

Suppose $h(x) \leq h(y)$ and $h(y) \geq h(z)$. If $h(x) = h(y)$, or $h(y) = h(z)$, or $h(x) = h(z)$, h is not injective, which is a contradiction. Thus, $h(x) < h(y)$ and $h(y) > h(z)$. Suppose $h(x) < h(z)$. That is:

$$h(x) < h(z) < h(y)$$

As h is continuous on I, the Intermediate Value Theorem (same theorem as before) can be applied. Hence there exists $c \in]x, y[$ such that $h(c) = h(z)$. As $z \notin]x, y[$, we have $c \neq z$. So h is not injective, which is a contradiction. Suppose instead $h(x) > h(z)$. That is:

$$h(z) < h(x) < h(y)$$

Again, as h is continuous on I, the Intermediate Value Theorem can be applied so that there exists $c \in]y, z[$ such that $h(c) = h(x)$. But then h is again not injective, which is a contradiction. If we suppose $h(x) \geq h(y)$ and $h(y) \leq h(z)$, then by taking the function $\tilde{h} := -h$ which is also injective and continuous over I, we have that $\tilde{h}(x) \leq \tilde{h}(y)$ and $\tilde{h}(y) \geq \tilde{h}(z)$ and we obtain the same contradiction as above by performing the same proof for h. Therefore h is strictly monotone.

-Now we can prove the result: we argue by contradiction. Let us fix a real continuous function $f : (\mathbb{R}, \mathcal{T}_{\mathbb{R}}) \longrightarrow (\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ with the property that $f \circ f = cos$. Then we know by our second result that $f(\alpha) = \alpha$ and $f|_{[0;\frac{\pi}{2}]}$ is an injection. Since f is continuous, $f|_{[0;\frac{\pi}{2}]}$ must be a continuous injection. By our third result $f|_{[0;\frac{\pi}{2}]}$ is strictly monotone. Since $f(\alpha) = \alpha \in]0; 1[\subset [0; \frac{\pi}{2}]$, and $f|_{[0;\frac{\pi}{2}]}$ is continuous we have that $\alpha \in (f|_{[0;\frac{\pi}{2}]})^{-1}[]0; 1[] =: U \in \mathcal{T}^{\mathbb{R}}_{[0;\frac{\pi}{2}]}$. In this case, the composition $f \circ f$ must be strictly increasing over U; for if $x, y \in U \subset [0; \frac{\pi}{2}]$ with $x < y$ then by construction of U we have $f(x), f(y) \in [0; \frac{\pi}{2}]$ (important). Now by the strict monotonicity of $f|_{[0;\frac{\pi}{2}]}$, we have two cases. If $f|_{[0;\frac{\pi}{2}]}$ is strictly increasing then $f(x) < f(y)$ so that again we have $f(f(x)) < f(f(y))$. If $f|_{[0;\frac{\pi}{2}]}$ is strictly decreasing then $f(x) > f(y)$ and so $f(f(x)) < f(f(y))$. In all case $(f \circ f)(x) < (f \circ f)(y)$. Therefore $cos|_U = (f \circ f)|_U$ is strictly increasing. This is a contradiction with the fact that *cos* is strictly decreasing over $[0; \frac{\pi}{2}] \supset U$ **if** U contains at least 2 elements. The latter is true: by construction U is of the form $[0; \frac{\pi}{2}] \cap V$ with V an open set of $\mathbb{R}$, in particular (and by construction of $\mathcal{T}_{\mathbb{R}}$) V contains a basis element (which is an open interval $]c, d[$ of positive length) containing α. Therefore:

$$\alpha \in]c, d[\cap]0; 1[\subset]c, d[\cap[0; \frac{\pi}{2}] \subset U$$

so $\varnothing \subsetneq [\alpha; min\{d, 1\}[\subset U$, and U necessarily contains infinitely many points.

Remark: If f was differentiable at $\alpha = f(\alpha)$ then the third part would be much easier. Indeed :

$$0 \overset{\alpha\in]0;1[}{>} -sin(\alpha) = (\frac{d}{dx}cos)(\alpha) = (\frac{d}{dx}f \circ f)(\alpha) = (\frac{d}{dx}f)(f(\alpha)) \cdot (\frac{d}{dx}f)(\alpha) = (\frac{d}{dx}f)(\alpha)^2 \geq 0$$

a contradiction.

Problem B4 (Putnam 2001)

Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - \frac{1}{x}$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

Solution:

The intersection is empty. To see this, analyze the behavior of denominators under iteration of f. Let $x = \frac{m}{n} \in S$, where m, n are coprime integers. Applying f:

$$f\left(\frac{m}{n}\right) = \frac{m}{n} - \frac{n}{m} = \frac{m^2 - n^2}{mn}.$$

Since $\gcd(m^2 - n^2, mn) = 1$ (as m, n are coprime), the denominator becomes $|mn|$. For $m \neq 1$, $|mn| \geq 2|n|$. If $m = 1$, then $f\left(\frac{1}{n}\right) = \frac{1-n^2}{n}$, and since $n \neq \pm 1$, the numerator $|1 - n^2| \geq 3$.

Iterating f, the denominator grows at least exponentially. Specifically:

- For $x \in S$, if the denominator of $f^{(k)}(x)$ is d_k, then $d_{k+1} \geq 2d_k$.
- Thus, $d_k \geq 2^k d_0$, where d_0 is the initial denominator.

For any rational $x = \frac{a}{b}$ (in reduced form), choose k such that $2^k > b$. Then $d_k > b$, so $x \notin f^{(k)}(S)$. Hence, x cannot belong to $\bigcap_{n=1}^{\infty} f^{(n)}(S)$. Since x was arbitrary, the intersection is empty.

Problem (Hongler)

12. On a une liste chaînée d'éléments, $x_0, x_1, \dots, x_n$, où on ne connaît pas n (mais on sait que la liste est finie). Quand on est à x_0 on a un pointeur pour aller à x_1, qui amène à x_2, etc, jusqu'au moment où on arrive à x_n, où on apprend que c'est la fin. On a une quantité de mémoire bornée (on peut pas juste stocker toute la liste dans un tableau et choisir un élément dans la tableau quand on est arrivé à la fin).
(a) Comment prendre un élément aléatoire dans la liste, uniformément si on a le droit de parcourir la liste une seule fois?
(b) Pourquoi est-ce que résoudre ce problème peut être utile en pratique ?
13. On a 100 mathématiciens emprisonnés dans une salle. Ils ont été capturés par un sorcier qui va les soumettre à l'épreuve suivante.

Solution:

Problem (Hongler)

Let U be a domain such that $U \supseteq \mathbb{D}$ and f a holomorphic function $U \to \mathbb{C}$. Show that if $f(\partial\mathbb{D}) = \gamma$ is a simple loop and $f|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \gamma$ is injective, then $f|_{\mathbb{D}}$ is injective.

Solution: