Problem Set Week 12

ETHZ Math Olympiad Club

12 May 2025

1 Problem B3 (Putnam 2001)

For each $n \in \mathbb{N}$, let $\langle n \rangle$ denote the closest integer to $\sqrt{n}$. Find $\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle}+2^{-\langle n \rangle}}{2^n}$.

2 Problem 3 (Bernoulli Competition 2024)

Suppose $x_0 \in \mathbb{R}$ and $x_{n+1} = \sum_{i=0}^{n}(-1)^i \sin(x_i)$ for $n \geq 1$.
1) What is the range for the x_0 such that $\lim_{n\to\infty} x_n$ exists? What is the value of the limit depending on x_0 in the range?
2) Suppose $x_0 = 1/4$. Find $\lim_{n\to\infty} \dfrac{\log\left(|\log(x_n)|\right)}{n}$.

3 Problem 4 (unknown)

Let $(S, \cdot)$ be a non-empty magma; that is S be a non-empty set, with an internal binary operation. Suppose that it satisfies the following:
-$(S, \cdot)$ forms a semi-group i.e the binary operation $\cdot$ is associative. -The binary operation $\cdot$ is injective in left and right coordinate $\forall a, b, c \in S$;

$$\cdot((a,b)) = \cdot((a,c)) \Rightarrow b = c \; i.e. \; a \cdot b = a \cdot c \Rightarrow b = c$$
$$\cdot((b,a)) = \cdot((c,a)) \Rightarrow b = c \; i.e. \; b \cdot a = c \cdot a \Rightarrow b = c$$

-$\forall a \in S, a^{\mathbb{Z}_{\geq 1}} := \{a^n | \; n \in \mathbb{Z}_{\geq 1}\}$ is finite, where $a^1 := a \; and \; for \; n > 1, \; a^n = a \cdot a^{n-1}$. (Note that this definition could be independent where we start to compute since $\cdot$ is associative.)

Show that S can be turned into a group where the group operation is $\cdot$.

4 Problem (unknown)

There is an odd number $(2 \cdot n + 1 > 0)$ of stones with real weights satisfying the following property: if we remove any stone from the $2 \cdot n + 1$, then there is a way to partition the rest of the stones into two sets of size n, such that the sum of the weights of the stones in both sets is equal. Show that all stones have the same weight.

5 Problem B-6 (IMC 2024)

Show that any function $f : \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfy the following propertie:

$$\exists a, b, c \in \mathbb{Q} \text{ with } a < b < c \text{ such that } f(a), f(c) \leq f(b)$$