Problem Set Week 4

ETHZ Math Olympiad Club

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1 Problem in example page 140 (PUTNAM and BEYOND)

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, with positive second derivative. Prove that

$$f(x + f'(x)) \geq f(x),$$

for any real number x.

2 Problem A-2 (IMC 2011)

Does there exist a real 3×3 matrix A such that

$$\operatorname{tr}(A) = 0 \quad \text{and} \quad A^2 + A^T = I_3,$$

where $\operatorname{tr}(A)$ denotes the trace of A, A^T is the transpose of A, and I_3 is the 3×3 identity matrix?

3 Problem B-2 (IMC 2014)

Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric $n \times n$ matrix with real entries, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote its eigenvalues. Show that

$$\sum_{1 \leq i < j \leq n} a_{ii} a_{jj} \geq \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j,$$

and determine all matrices for which equality holds.

4 Problem 414 (PUTNAM and BEYOND)

For any real number $\lambda \geq 1$, denote by $f(\lambda)$ the real solution to the equation

$$x(1 + \ln x) = \lambda.$$

Prove that

$$\lim_{\lambda \to \infty} \frac{f(\lambda)}{\frac{\lambda}{\ln \lambda}} = 1.$$

5 Problem A-4 (IMC 2014)

Let $n > 6$ be a perfect number, and let $n = p_1^{e_1} \cdots p_k^{e_k}$ be its prime factorisation with

$$1 < p_1 < \ldots < p_k.$$

Prove that e_1 is an even number.
A number n is *perfect* if $s(n) = 2n$, where $s(n) = \sum_{\mathbb{N} \ni d |_{\mathbb{Z}} n} d$ is the sum of the divisors of n.