

# Problem Set Week 6

ETHZ Math Olympiad Club

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## 1 Problem (unknown)

We consider a game where two indistinguishable envelopes are presented to a player:

- One envelope contains an amount  $\alpha \in \mathbb{R}_{>0}$ .
- The other envelope contains  $2\alpha$ .

The game proceeds as follows:

1. The player randomly selects one envelope (with equal probability).
2. The player observes the content  $x$  of the selected envelope (without knowing  $\alpha$ ).
3. The player must decide whether to:
  - Keep the current envelope, or
  - Switch to the other envelope (this decision is irrevocable).

Although the game is played once, the player's objective is still to maximize their *expected gain*. Assuming access to *randomness*, how can they do better than always keeping the first envelope?

## 2 Problem A-3 (IMC 2018)

Determine all rational numbers  $a$  for which the matrix

$$A = \begin{bmatrix} a & a & 1 & 0 \\ -a & -a & 0 & 1 \\ -1 & 0 & a & a \\ 0 & -1 & -a & -a \end{bmatrix}$$

is the square of a matrix with all rational entries.

## 3 Problem A-4 (IMC 2005)

Find all polynomials

$$P(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \quad (a_n \neq 0)$$

satisfying the following two conditions:

1.  $(a_0, a_1, \dots, a_n)$  is a permutation of the numbers  $(0, 1, \dots, n)$ , and
2. all roots of  $P(X)$  are rational numbers.

## 4 Problem A-6 (IMC 2005)

Let  $m, n \in \mathbb{Z}$ . Given a group  $G$ , denote by  $G(m)$  the subgroup generated by the  $m$ -th powers of elements of  $G$ :

$$G(m) := \langle \{g^m \mid g \in G\} \rangle \leq G.$$

If  $G(m)$  and  $G(n)$  are commutative, prove that  $G(\gcd(m, n))$  is also commutative. Here,  $\gcd(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ .