Problem Set Week 3

March 5, 2025

1 Problem 1 (Pan African 2018)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$(f(x+y))^2 = f(x^2) + f(y^2)$$

for all $x, y \in \mathbb{Z}$.

2 Problem B-2 (IMC 2012)

Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$ and

$$a_{n+1} = \frac{n a_n^2}{1 + (n+1) a_n} \quad \text{for } n \geq 1.$$

Show that the series

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$$

converges and determine its value.

3 Problem 5 (Pan African 2018)

Let a, b, c and d be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ and } ac = bd.$$

Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leq -12$$

and that -12 is the maximum.

4 Problem 3 (Silk Road 2019)

Find all pairs (a, n) of positive natural numbers such that $\varphi(a^n + n) = 2^n$. ($\varphi(n)$ is the Euler function, that is, the number of integers from 1 up to n, relatively prime to n)