

# Problem Set Week 5 Solutions

ETHZ Math Olympiad Club

10 March 2025

## 1 Problem (unknown)

Find all real solutions to the equation

$$9^x + 4^x + 2^x = 8^x + 6^x + 1.$$

**Answer:**

It is easy to see that  $x = 0$ ,  $x = 1$ , and  $x = 2$  are solutions. So the equation has at least 3 distinct real solutions. Let us introduce the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = 9^x + 4^x + 2^x - 8^x - 6^x + 1.$$

As stated above,  $f$  has at least 3 distinct zeros. We claim there are no other roots. By Rolle's theorem, if a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  has at least  $n \geq 2$  zeros  $x_1 < \dots < x_n$ , then the function  $g'(x)$  has at least  $n - 1$  zeros  $y_1 < \dots < y_{n-1}$ , where for each  $i = 1, \dots, n - 1$ , we have  $x_i < y_i < x_{i+1}$ . In particular, since for each  $a \in \mathbb{R}_{>0}$ ,  $a^{-x}g(x)$  has at least  $n$  zeros, we have that  $(a^{-x}g(x))'$  has at least  $n - 1$  zeros, and so does the function

$$h_a g(x) = a^x (a^{-x}g(x))' = g'(x) - \ln(a)g(x).$$

Suppose  $f$  has another zero not in  $\{0, 1, 2\}$ . Then  $f$  has at least 4 zeros, and thus

$$h_1 f(x) = f'(x) = \ln(9)9^x + \ln(4)4^x + \ln(2)2^x - \ln(8)8^x - \ln(6)6^x$$

has at least 3 zeros, which then implies that

$$\begin{aligned} h_6 h_1 f(x) &= f''(x) - \ln(6)f'(x) \\ &= \ln\left(\frac{9}{6}\right) \ln(9)9^x + \ln\left(\frac{4}{6}\right) \ln(4)4^x + \ln\left(\frac{2}{6}\right) \ln(2)2^x - \ln\left(\frac{8}{6}\right) \ln(8)8^x \end{aligned}$$

has at least 2 zeros, which again implies that

$$h_8 h_6 h_1 f(x) = \ln\left(\frac{9}{8}\right) \ln\left(\frac{9}{6}\right) \ln(9)9^x - \ln\left(\frac{4}{8}\right) \ln\left(\frac{4}{6}\right) \ln(4)4^x - \ln\left(\frac{2}{8}\right) \ln\left(\frac{2}{6}\right) \ln(2)2^x$$

has at least 1 zero.

The function  $h_8 h_6 h_1 f(x)$  is of the form  $k_2 2^x + k_4 4^x + k_9 9^x$ , for  $k_2, k_4, k_9 > 0$  and hence is always positive. Therefore,  $h_8 h_6 h_1 f(x)$  cannot have any real zero. This is a contradiction to the assumptions hence the solutions of the original equation are exactly  $\{0, 1, 2\}$ .

## 2 Problem 2 (Bernoulli Competition 2023)

Let  $e$  be Euler's number. Show that for any odd prime  $p$ , the integer

$$1! + 2! + 3! + \cdots + (p-1)! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by  $p$ .

**Answer:**

First note that:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!}.$$

Thus, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \left\lfloor \sum_{i=0}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \right\rfloor$$

Notice that:

$$\sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!} \in \mathbb{Z}$$

We argue that the tail  $\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \in ]0; 1[$ , indeed since  $p$  is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = \sum_{j=0}^{+\infty} \left( \frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right).$$

is certainly bigger than 0 since each term  $\left( \frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right) = \frac{(p-1)!}{(p-1+2j)!} \left( 1 - \frac{1}{p+2j} \right) > 0$ . Similarly since  $p$  is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = 1 - \sum_{j=0}^{\infty} \left( \frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right).$$

is certainly smaller than 1 since each term  $\left( \frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right) = \frac{(p-1)!}{(p+2j)!} \left( 1 - \frac{1}{p+2j+1} \right) > 0$ .

Therefore,

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!}.$$

Now note that for each  $0 \leq j \leq p-1$  we have  $j \equiv -(p-j) \pmod{p}$  and thus for fixed  $0 \leq i < p-1$ :

$$\begin{aligned} \frac{(-1)^i (p-1)!}{i!} &= (-1)^i (i+1)(i+2) \cdots (p-1) \\ &\equiv (-1)^i (p-(i+1))(p-(i+2)) \cdots 2 \cdot 1 \cdot (-1)^{p-(i+1)} \equiv (p-(i+1))! \pmod{p}, \end{aligned}$$

where we used that there is  $p-(i+1)$  factor in  $\frac{(p-1)!}{i!}$  and the fact that  $p$  is odd again. Hence, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor \equiv \sum_{i=0}^{p-2} (p-(i+1))! \equiv \sum_{i=1}^{p-1} i! \pmod{p},$$

since  $i \mapsto p-(i+1)$  is a bijection from  $\llbracket 0, p-2 \rrbracket$  to  $\llbracket 1, p-1 \rrbracket$ . This shows the problem's statement.

### 3 Problem in example page 140 (PUTNAM and BEYOND)

Find all real solutions to the equation

$$4^x + 6^{x^2} = 5^x + 5^{x^2}.$$

**Answer:**

Note that  $x = 0$  and  $x = 1$  satisfy the equation from the statement. Are there other solutions? The answer is no, but to prove it we use the amazing idea of treating the numbers 4, 5, 6 as variables and the presumably new solution  $x$  as a constant.

Thus let us consider the function  $f(t) = t^{x^2} + (10 - t)^x$ . The fact that  $x$  satisfies the equation from the statement translates to  $f(5) = f(6)$ . By Rolle's theorem there exists  $c \in (5, 6)$ , such that  $f'(c) = 0$ . This means that

$$x^2 c^{x^2-1} - x(10 - c)^{x-1} = 0,$$

or

$$xc^{x^2-1} = (10 - c)^{x-1}.$$

Because exponentials are positive, this implies that  $x$  is positive.

If  $x > 1$ , then  $x^2 - 1 > x - 1$  and as  $c > 5$

$$(10 - c)^{x-1} = xc^{x^2-1} > c^{x^2-1} > c^{x-1} > (10 - c)^{x-1},$$

which is a contradiction.

If  $0 < x < 1$ , then  $x^2 - 1 < x - 1$  and:

$$(10 - c)^{x-1} = xc^{x^2-1} < xc^{x-1}.$$

Let us prove that

$$xc^{x-1} < (10 - c)^{x-1}.$$

With the substitution  $y = x - 1$ , the inequality can be rewritten as

$$y + 1 < \left( \frac{10 - c}{c} \right)^y.$$

which must be proven for  $y \in ] - 1; 0[$ .

Lets make a simple analysis of the two functions defined over  $\mathbb{R}$ .

The exponential has base less than 1, so it is strictly decreasing, while the affine function on the left is strictly increasing. The two meet at  $y = 0$  so we must have that strictly before  $y = 0$  the exponential is strictly bigger than the affine. The inequality (on  $] - 1; 0[$ ) follows. Using it we conclude again that:  $(10 - c)^{x-1} = xc^{x^2-1} < (10 - c)^{x-1}$  which is a contradiction. This shows that a third solution to the equation from the statement does not exist. So the only solutions to the given equation are  $x = 0$  and  $x = 1$ .

## 4 Problem 3 (Bernoulli Competition 2023)

Let  $n \geq 1$  and  $A$  be a  $n \times n$  symmetric matrix over  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$  with  $1_{\mathbb{F}_2}$ 's on the main diagonal. Show that the vector composed uniquely of  $1_{\mathbb{F}_2}$ 's is in the image of  $A$ .

**Answer:**

We write  $1 := 1_{\mathbb{F}_2}$  and  $0 := 0_{\mathbb{F}_2}$ , define the  $\mathbb{F}_2$ -vector space  $V = \mathbb{F}_2^n$  and define the standard binary product on  $V$ , i.e.,

$$\langle v, w \rangle = \sum_{i=1}^n v_i w_i.$$

It is easy to see that  $\langle \cdot, \cdot \rangle$  is  $\mathbb{F}_2$ -linear in the first coordinate, symmetric (so  $\mathbb{F}_2$ -linear in the second coordinate and thus  $\mathbb{F}_2$ -bilinear) and non-degenerate that is:

$$\forall v \in V ((\forall w \in V \langle v, w \rangle = 0) \rightarrow v = 0_V)$$

(just plug the canonical basis for  $w$  that is for each  $i \in n$  take  $w = e_i$  and use the fact that  $\mathbb{F}_2$  is a field to conclude  $v_i = 0$ , and thus  $v = \underline{0} = 0_V$ ).

With this being introduced, lets take an  $n \times n$ -matrix  $A$  with  $\text{diag}(A) = \underline{1}$ . Since  $\langle \cdot, \cdot \rangle$  is symmetric and non-degenerate, we have for any  $\mathbb{F}_2$ -subspace  $W \subset V$ ,

$$(W^\perp)^\perp = W.$$

where  $Z^\perp = \{v \in V \mid \langle v, z \rangle = 0 \forall z \in Z\}$  for any  $\mathbb{F}_2$ -subspace  $Z \subset V$ . For more information on this equality see the appendix 1. Now write  $A = (a_{ij})_{1 \leq i, j \leq n}$  with  $a_{ii} = 1$  and  $a_{ij} = a_{ji}$ . Then we have for any  $v \in V$ ,

$$\langle v, Av \rangle = \sum_{1 \leq i, j \leq n} v_i v_j a_{ij} = \sum_{i=1}^n v_i^2 + 2 \sum_{i < j} v_i v_j a_{ij} = \sum_{i=1}^n v_i = \langle v, \underline{1} \rangle,$$

because we are working over  $\mathbb{F}_2$ . In particular for any  $z \in \text{Im}(A)^\perp \subset V$ ,

$$\langle z, \underline{1} \rangle = \langle z, Az \rangle = 0,$$

since  $Az \in \text{Im}(A)$  and  $z \in \text{Im}(A)^\perp$ . As  $z \in \text{Im}(A)^\perp$  was arbitrary, we must have

$$\underline{1} \in (\text{Im}(A)^\perp)^\perp = \text{Im}(A).$$

## 5 Problem (unknown)

Find all differentiable functions  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  such that  $\forall x \in \mathbb{R}_{>0}$ :

$$1 \leq f'(x) = \frac{f(x)}{(f \circ f)(x)}.$$

**Answer:**

The identity function obviously works. We claim this is the only solution. Let  $f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  be differentiable with  $\forall x \in \mathbb{R}_{>0}$ :

$$1 \leq f'(x) = \frac{f(x)}{(f \circ f)(x)}.$$

The solution is done in two steps: we show that  $f$  admits a fixed point  $\beta > 0$ , then we use this fixed point to show that no other solution exists. Note that  $f$  takes positive values, so we must have that  $f' = \frac{f}{f \circ f}$  takes strictly positive values. Therefore,  $f$  must be strictly increasing and hence injective. From this, we derive the equivalence  $\forall \beta > 0$ :

$$f'(\beta) = 1 \Leftrightarrow f(\beta) = f(f(\beta)) \Leftrightarrow f(\beta) = \beta.$$

Thus, showing the existence of a fixed point  $\beta > 0$  is equivalent to showing that  $f'(\beta) = 1$ .

If  $\exists \beta > 0$  with  $f'(\beta) = 1$ , then  $f$  has the fixed point  $\beta$ . Otherwise, by the condition on  $f$ , it must be that  $\forall x > 0$ ,  $f'(x) > 1$ . In particular,  $f'(1) > 1$ , so  $f(1) > f(f(1))$ . Since  $f$  is strictly increasing, we must have  $1 > f(1)$  (if  $1 \leq f(1)$ , then  $f(1) \leq f(f(1))$ , so  $f(1) < f(1)$ , a contradiction). Note that  $f$  is continuous (since  $f$  is differentiable), thus  $f'$  is continuous and hence integrable over any compact interval in  $\mathbb{R}_{>0}$ . Since  $\forall t > 0$ ,  $f'(t) > 1$ , we have for any  $0 < x < 1$ :

$$f(1) - f(x) = \int_x^1 f'(t) dt \geq \int_x^1 1 dt = 1 - x$$

( $\int_x^1 \_ dt$  is increasing from the space of real valued integrable functions over  $[x, 1]$ ).

Thus,  $\forall x > 0$  with  $x < 1$ ,

$$f(x) \leq (f(1) - 1) + x.$$

But then, for any  $0 < y < 1 - f(1) < 1$ , we have  $f(y) < 0$ , contradicting the positivity of  $f$ .

Hence,  $f$  admits a fixed point  $\alpha > 0$ .

Now, we use this fixed point to show that  $f$  must be the identity. Since  $f$  is continuous, we can integrate it and define for any  $x > 0$ :

$$F(x) := \int_{\alpha}^x f(t) dt.$$

By the fundamental theorem of calculus,  $F : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  is continuous. Moreover, since  $f$  is continuous,  $F$  is everywhere differentiable with  $F' = f$ . Because  $f$  is always positive,  $F$  is strictly increasing and thus injective.

Fix  $x \in \mathbb{R}_{>0}$ . From the given equation (valid for all  $t > 0$ ),

$$f(t) = f(f(t))f'(t) = F'(f(t))f'(t) = (F \circ f)'(t),$$

we obtain:

$$F(x) = \int_{\alpha}^x f(t) \, dt = \int_{\alpha}^x (F \circ f)'(t) \, dt = (F \circ f)(x) - (F \circ f)(\alpha),$$

where we used the fundamental theorem of calculus again for the function  $x \mapsto \int_{\alpha}^x (F \circ f)'(t) \, dt$  (this holds for both  $x \geq \alpha$  and  $x < \alpha$ ).

Now, using the fact that  $\alpha$  is a fixed point of  $f$ , we get  $F(f(\alpha)) = F(\alpha) = 0$ , so  $F(x) = F(f(x))$ . By the injectivity of  $F$ , we conclude that  $f(x) = x$ . Since  $x > 0$  was arbitrary, the proof is complete.

*Remark.* If we drop the restriction that  $f'(x) \geq 1$  for all  $x > 0$ , the problem becomes significantly more challenging. The proof of the result proceeds in the same way, but we now have to handle the additional case where  $f'(x) < 1$  for all  $x > 0$ . If  $f(x) < x$ , this case can be handled trivially, but the situation where  $f(x) > x$  presents substantial difficulties. Since  $f'(x)$  is determined by the value of  $f$  at  $x$  and its composition  $f \circ f$  at  $x$ , and because  $f(f(x)) > f(x)$ , the derivative  $f'(x)$  depends on values of  $f$  at points beyond  $f(x)$ . This leads to a non-causal delay differential equation. For further discussion, see the related [MathStack Exchange thread](#).

## A

Let  $E$  be a finite-dimensional vector space over a field  $K$ , and  $B : E \times E \rightarrow K$  a symmetric bilinear form. For any subspace  $Q \subseteq E$ , the orthogonal complement is defined as  $Q^\perp := \{v \in E \mid \forall q \in Q, B(q, v) = 0\}$ . It is clearly a  $K$ -subspace of  $E$ . The form  $B$  is non-degenerate if  $E^\perp = \{0\}$ . For a non-degenerate symmetric bilinear form  $B$ , the map

$$\varphi : E \rightarrow E^\vee, \quad v \mapsto B(\cdot, v)$$

is a vector space isomorphism. Indeed,  $B$  is bilinear, so  $\varphi$  is linear. The injectivity follows from

$$\varphi(v) = 0_{E^\vee} \implies \forall w \in E, B(w, v) = 0 \implies v \in E^\perp = \{0\} \quad (\text{by non-degeneracy}).$$

As  $\dim(E) = \dim(E^\vee)$  in finite dimensions (classical), injectivity implies surjectivity. Thus,  $\varphi$  is an isomorphism.

**Theorem 1** (Double Orthogonal Complement). *In this setting, if  $B$  is non-degenerate and  $Q \subseteq E$  is a  $K$ -subspace, then  $(Q^\perp)^\perp = Q$ .*

*Proof. Step 1.*  $Q \subseteq (Q^\perp)^\perp$ : For any  $q \in Q$ , by definition,

$$\forall v \in Q^\perp, B(q, v) = 0 \implies q \in (Q^\perp)^\perp.$$

**Step 2.** Using the isomorphism  $\varphi$ , we have that  $\varphi|_{Q^\perp}$  is an isomorphism onto its image:

$$\text{Im}(\varphi|_{Q^\perp}) = \{B(-, v) \mid v \in Q^\perp\} = \{f \in E^\vee \mid \forall q \in Q, f(q) = 0\} =: Q^\circ.$$

The middle equality's  $\supseteq$  inclusion follows from the surjectivity of  $\varphi$  and the definition of  $Q^\perp$ . Thus,  $\dim(Q^\perp) = \dim(Q^\circ)$ .

**Step 3.** Given an ordered basis  $(w_i)_{i \in \dim(Q)}$  of  $Q$ , complete it into an ordered basis of  $E$ :

$$(w_i)_{i \in \dim(Q)} \cup (v_j)_{j \in \dim(E) - \dim(Q)}.$$

There is a classical associated ordered basis of  $E^\vee$ :

$$(w_i^*)_{i \in \dim(Q)} \cup (v_j^*)_{j \in \dim(E) - \dim(Q)},$$

where each functional sends a vector  $v = \sum_{j \in \dim(Q)} \lambda_j w_j + \sum_{i \in \dim(E) - \dim(Q)} \gamma_i v_i$  of  $E$  to  $\lambda_j$  or  $\gamma_i$ , respectively. This basis satisfies the following property: if  $f \in E^\vee$ , then we can write

$$f = \sum_{i \in \dim(Q)} f(w_i) w_i^* + \sum_{j \in \dim(E) - \dim(Q)} f(v_j) v_j^*.$$

In particular, if  $f \in Q^\circ$ , then  $f = \sum_{j \in \dim(E) - \dim(Q)} f(v_j) v_j^*$ , because  $\{w_i \mid i \in \dim(Q)\} \subset Q$ . Thus,  $(v_j^*)_{j \in \dim(E) - \dim(Q)}$  generates  $Q^\circ$ . Since these vectors are  $K$ -linearly independent, we obtain

$$\dim(E) - \dim(Q) = \dim(Q^\circ).$$

**Step 4.** In total, we obtain that for any  $K$ -subspace  $Q$  of  $E$ ,

$$\dim(Q^\perp) = \dim(Q^\circ) = \dim(E) - \dim(Q).$$

Since  $Q^\perp$  is a  $K$ -subspace, we must have:

$$\dim((Q^\perp)^\perp) = \dim(E) - \dim(Q^\perp) = \dim(E) - (\dim(E) - \dim(Q)) = \dim(Q).$$

We conclude  $Q = (Q^\perp)^\perp$  since they have the same (finite) dimension and  $Q \subset (Q^\perp)^\perp$ .  $\square$