Problem Set Week 3 Solutions

ETHZ Math Olympiad Club

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1 Problem 1 (Pan African 2018)

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$(f(x+y))^2 = f(x^2) + f(y^2)$$

for all $x, y \in \mathbb{Z}$.

Answer:

Plug in $y = -x$, and let $c = f(0)$. We have $f(x^2) = \frac{c^2}{2}$. Plugging in $y = 0$, we have $f(x)^2 = \frac{c^2}{2} + c$. Plugging in $x = 0$ in this gives $c^2 = \frac{c^2}{2} + c$ $c = 0$ or $c = 2$. Now if $c = 0$, $f \equiv 0$ is the only solution. Else we have $f(x) = \pm 2$ for all x. This means $f(x^2) + f(y^2) = 4$, always, so for all x being perfect squares, $f(x) = 2$. So the solution in this case becomes $f(x) = 2$ for all perfect square x and ± 2 for all other x. It is easy to check that the solutions mentioned above work. Thus:

$$\left\{ f : \mathbb{Z} \to \mathbb{Z} \,\middle|\, \forall x, y \in \mathbb{Z}\ (f(x+y))^2 = f(x^2) + f(y^2) \right\}$$

$$= \{\underline{0}\} \cup \left\{ 2\mathbb{1}_A - 2\mathbb{1}_{\mathbb{Z}\setminus A} \,\middle|\, \exists A \subset \mathbb{Z}\ (\mathbb{N})^2 \subset A \right\}$$

2 Problem B-2 (IMC 2012)

Define the sequence $(a_n)_{n\geq 0}$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$, and

$$a_{n+1} = \frac{na_n^2}{1+(n+1)a_n} \quad \text{for } n \geq 1.$$

Show that the series

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$$

converges and determine its value.

Answer:

By induction, we establish that $a_i > 0$ for all $i \geq 0$, ensuring that the partial sums are well defined and form an increasing sequence. Furthermore, we observe that

$$ka_k = \frac{(1+(k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1} \quad \text{for all } k \geq 1.$$

Summing from $k = 0$ to n, we obtain

$$\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{n} (ka_k - (k+1)a_{k+1}) = \frac{1}{2} + 1 \cdot a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1}$$

for all $n \geq 1$. A quick verification confirms that this equality also holds for $n = 0$.

Since $(n+1)a_{n+1} > 0$, we deduce that for each $n \geq 0$,

$$\sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = 1 - (n+1)a_{n+1} < 1.$$

This implies that the partial sums are bounded, and thus, the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ is convergent (converging to the supremum of the partial sums). Consequently, the sequence $\frac{a_{k+1}}{a_k}$ must tend to zero. In particular, there exists an index $N \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} < \frac{1}{2}$ for all $n > N$.

For all $n > N$, we then have

$$a_n = \prod_{i=N}^{n-1} \frac{a_{i+1}}{a_i} < \frac{1}{a_N 2^{n-1-N}} = \frac{a_N^{-1} 2^N + 1}{2^n}.$$

In particular, for all $n > N$,

$$na_n < \left(a_N^{-1} 2^N + 1\right)\left(\frac{n}{2^n}\right).$$

Since $\frac{n}{2^n} \to 0$ as $n \to +\infty$, it follows that $na_n \to 0$ as $n \to +\infty$. Therefore,

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \lim_{n\to\infty} (1 - (n+1)a_{n+1}) = 1.$$

3 Problem 5 (Pan African 2018)

Let a, b, c and d be non-zero pairwise different real numbers such that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}=4 \text{ and } ac=bd.$$

Show that

$$\frac{a}{c}+\frac{b}{d}+\frac{c}{a}+\frac{d}{b}\leq -12$$

and that -12 is the maximum.

Answer:

First write $d=\frac{ac}{b}$ so we have $\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}=4$ and we want to show that $\frac{a}{c}+\frac{c}{a}+\frac{b^2}{ac}+\frac{ac}{b^2}$ or equivalently $(\frac{a}{b}+\frac{b}{a})(\frac{c}{b}+\frac{b}{c})\leqslant -12$.

Set $\frac{a}{b}+\frac{b}{a}=s$ and $\frac{c}{b}+\frac{b}{c}=t$. We have $s+t=4$ and we want $st\leqslant -12$.

We have $t=4-s$ and so we need $s(4-s)\leqslant -12 \Rightarrow s^2-4s-12\geqslant 0$. If $s\leqslant -2$, this holds (indeed, set $s=-2+k$ and we have $s^2-4s-12=(k-2)^2-4(k-2)-12=k^2-8k-16=(k-4)^2\geqslant 0$).

So assume that $s\geqslant -2$. Similarly, assume $t\geqslant -2$. We have then $\frac{a}{b}+\frac{b}{a}\geqslant -2\Rightarrow \frac{(a+b)^2}{ab}>0$, so $ab>0$ and $bc>0$, so a,b,c, are of the same sign, so $4=\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}\geqslant 2+2=4$, so $a=b=c$, contradiction, since $a\neq b\neq c\neq a$.

So, at least one of s,t is $\leqslant -2$ and the stated inequality holds.

The equality happens e.g. when $(a,b,c,d)=(2\sqrt{2}-3,3-2\sqrt{2},1,-1)$ (and this shows that -12 is the maximum).

4 Problem 3 (Silk Road 2019)

Find all pairs (a, n) of positive natural numbers such that $\varphi(a^n + n) = 2^n$.

($\varphi(n)$ is the Euler function, that is, the number of integers from 1 up to n, relatively prime to n.)

Answer:

Using the estimation $\varphi(b) > \sqrt{b}$ (classic) for all $b > 6$, we first check the (finitely many) cases when a solution (a, n) satisfies $a^n + n \leq 6$ to find that only:

$$(a, n) \in \{(2,1), (3,1), (5,1)\}$$

works.

Now for a solution (a, n) with $a^n + n > 6$ we get using the lower bound estimation:

$$\varphi(a^n + n) \overset{\text{hyp}}{=} 2^n > \sqrt{a^n + n}.$$

• If $a \geq 4$ the inequality cannot be true as then:

$$2^n > \sqrt{a^n + n} \geq \sqrt{4^n + n} \geq \sqrt{2^{2n}} = 2^n,$$

so we have to check the cases when $a = 1, 2, 3$.

• If $a = 1$ then using the trivial upper bound estimation $\varphi(b) \leq b - 1$ we get $n \geq \varphi(n+1) = 2^n$, which never holds for $n \geq 1$.

• If $a = 2$ or $a = 3$ this is more subtle. To have $3^n + n > 6$ we need for $a = 3$ that $n \geq 2$ and for $a = 2$ that $n \geq 3$. We see that $(3, 3)$ works: indeed $\varphi(30) = 30\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = 8 = 2^3$. Since $(3, 2)$ and $(2, 3)$ don't satisfy the property (easy to check) and $(3, 3)$ does, we can assume $n \geq 4$. Now we claim that there is no integer $n \geq 4$ satisfying for $a \in \{2, 3\}$:

$$\varphi(a^n + n) = 2^n.$$

We proceed as follows:

If an integer $m \geq 2$ satisfies

$$\varphi(m) = 2^k,$$

then every odd prime p dividing m must satisfy

$$p - 1 = 2^j.$$

Indeed, writing $m = 2^k \cdot p_1^{e_1} \cdots p_r^{e_r}$ for some distinct odd primes p_i, $k \geq 0$ and $e_1, \ldots, e_r \geq 1$, since $\varphi(m) = 2^{k-1} \cdot p_1^{e_1 - 1} \cdots p_r^{e_r - 1}(p_1 - 1) \cdots (p_r - 1)$, the only way for $\varphi(m)$ to be a power of 2 is that:

- For each $i \in \{1, \ldots, r\}$, we have $e_i = 1$.
- Every odd prime p_i satisfies $p_i - 1 = 2^{a_i}$ for some $a_i \geq 0$. In other words, every odd prime factor of m is a Fermat prime[1].

[1] A Fermat prime is a prime number of the form $2^j + 1$. One can easily see (mental exercise) that j is necessarily **again** a power of 2. Indeed, if j is not a power of 2, then there is an odd prime $p \mid j$ and $2^j + 1 = \left(2^{\frac{j}{p}}\right)^p - (-1)^p = \left(2^{\frac{j}{p}} + 1\right)\left(\sum_{s=0}^{p-1}(-1)^{p-1-s}\left(2^{\frac{j}{p}}\right)^s\right)$, which is a non-trivial factorization of $2^j + 1$ since $1 < 2^{\frac{j}{p}} + 1 < 2^j + 1$, contradicting the fact that $2^j + 1$ is a prime number. *Nota bene*: the only known Fermat primes are $3 = 2^{(2^0)} + 1$, $5 = 2^{(2^1)} + 1$, $17 = 2^{(2^2)} + 1$, $257 = 2^{(2^3)} + 1$, and $65537 = 2^{(2^4)} + 1$, and we don't know if there are infinitely many of them.

Hence $2^n = \varphi(m) = 2^{k-1}\prod_{i=1}^r 2^{a_i}$, and so the following equality holds:

$$k-1+\sum_{i=1}^r a_i = n.$$

If we let $n \geq 4$, $a \in \{2,3\}$, and

$$m = a^n + n,$$

with $\varphi(m) = 2^n$, then with the same notation as before:

$$a^n + n = 2^k \cdot p_1 \cdots p_r = 2^k\left(2^{a_1}+1\right)\cdots\left(2^{a_r}+1\right),$$

for $k \geq 0$ and some distinct odd primes $p_i = 2^{a_i}+1$ with $a_i \geq 1$ (as p_i is odd) (the a_i are powers of 2, but this is not needed here). Moreover, the equality $k-1+\sum_{i=1}^r a_i = n$ holds. Without loss of generality, order the exponents:

$$1 \leq a_1 < \cdots < a_r.$$

Now let us attack each case ($a = 2$ or $a = 3$ with $n \geq 4$) separately:

-If $a = 2$, then if $6 < 2^n + n$ is a prime number, this means that $k = 0$ and $r = 1$, and $-1 + a_1 = n$, so that $2^n + n = 2^{a_1} + 1 = 2^{n+1} + 1$, and thus $n = 2^n + 1 > 2^n > n$, which is a contradiction. Thus, $2^n + n$ satisfying the property cannot be a prime number. Hence, it is composite. We have the following classical upper bound for the Euler totient for a composite natural number m:

$$\varphi(m) = m\prod_{p|m}\left(1-\frac{1}{p}\right) \leq m\left(1-\frac{1}{\min\{p\in\mathbb{P} \mid p \mid m\}}\right) = m - \frac{m}{\min\{p\in\mathbb{P} \mid p \mid m\}}.$$

And since m is composite (we use it here), $\min\{p\in\mathbb{P} \mid p \mid m\} \leq \sqrt{m}$, so that:

$$\varphi(m) \leq m - \sqrt{m}.$$

Using this upper bound estimation, we get:

$$2^n + n - \sqrt{2^n+n} \geq \varphi\left(2^n+n\right) = 2^n.$$

Noticing that the function f defined on $\mathbb{R}_+$ by $f(x) = x - \sqrt{2^x + x}$ is always negative (because for all $x \geq 0$ we have $x(x-1) < x^2 < 2^x$ since $\ln(x) < \frac{x}{2}$), we conclude that the above inequality cannot hold for $n \geq 1$. This shows that there is no integer $n \geq 4$ such that $\varphi(2^n + n) = 2^n$.

-If $a = 3$, then by induction, one has the following fact:

$$t \geq 1 \implies 2^t + 1 \leq 3^t \text{ with equality only at } t = 1,$$

$$t \geq 3 \implies 2^t < 2^t + 1 < 3^{t-1}.$$

With this in mind, we obtain easily:

If there exists a_i with $a_i \geq 3$ (in particular, this occurs if $r \geq 3$, as then for $r \geq i > 2$ we have $a_i \geq 3$), then $2^{a_i} + 1 < 3^{a_i - 1}$, which yields:

$$3^n < 3^n + n = 2^k\left(2^{a_1}+1\right)\cdots\left(2^{a_r}+1\right) < 2^k 3^{a_i-1}\prod_{\substack{t=1\\ t\neq i}}^r 3^{a_t} = 2^k 3^{n-k} \leq 3^n,$$

which is a contradiction. Thus, in particular, $1 \leq r \leq 2$, and all a_i belong to $\{1, 2\}$.

If $k \geq 3$, then:

$$3^n < 3^n + n = 2^k \left(2^{a_1} + 1\right) \cdots \left(2^{a_r} + 1\right) \leq 2^k 3^{a_1} \cdots 3^{a_r} \leq 2^k 3^{n+1-k} \leq 3^{k-1} 3^{n+1-k} = 3^n,$$

which is again a contradiction, so $0 \leq k \leq 2$.

Summarizing, we have that a solution $3^n + n$ must be in the following set:

$$\begin{aligned} &\left\{3^j + j \mid j \in \mathbb{N}, j \geq 4\right\} \cap \left\{2^k \left(2^a + 1\right)^{\epsilon_a} \left(2^b + 1\right)^{\epsilon_b} \mid k \in \{0,1,2\}, a, b \in \{1,2\}, \epsilon_a, \epsilon_b \in \{0,1\}, a \neq b\right\} \\ &= \left\{3^j + j \mid j \in \mathbb{N}, j \geq 4\right\} \cap \left\{2^k 3^{\epsilon} 5^{\epsilon'} \mid k \in \{0,1,2\}, \epsilon, \epsilon' \in \{0,1\}\right\} \\ &= \{85\} \cap \{1, 3, 5, 9, 15, 25, 2, 6, 10, 18, 30, 50, 4, 12, 20, 36, 60, 100\} = \varnothing. \end{aligned}$$

This shows that there is no integer $n \geq 4$ such that $\varphi(3^n + n) = 2^n$.

To summarize, the only pairs that work are $\{(2,1), (3,1), (3,3), (5,1)\}$, and this concludes.