Problem Set Week 9

ETHZ Math Olympiad Club

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1 Simon's Favorite Factoring Trick Problems

SFFT is often used in a Diophantine equation where factoring is needed. We let R be any unitary ring. If we have a multipolynomial in two formal variables X and Y, $P(X,Y) \in R[X,Y]$ of the form

$$P(X,Y) = aXY + bX + cY + d \in R[X,Y]$$

with $a \in R^{\times}$. According to *Simon's Favorite Factoring Trick*, this multipolynomial is:

$$P(X,Y) = a\left(X + a^{-1}c\right)\left(Y + a^{-1}b\right) + d - ca^{-1}b.$$

1.1 AMC 12 (2012)

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

1.2 AIME (1998)

An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

1.3 BMO (2005)

The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that N is a perfect square.

1.4 JBMO (2003)

Let n be a positive integer. A number A consists of $2n$ digits, each of which is 4; and a number B consists of n digits, each of which is 8. Prove that $A + 2B + 4$ is a perfect square.

1.5 AIME (2000)

The system of equations

$$\begin{aligned}
\log_{10}(2000xy) - \log_{10}(x)\log_{10}(y) &= 4, \\
\log_{10}(2yz) - \log_{10}(y)\log_{10}(z) &= 1, \\
\log_{10}(zx) - \log_{10}(z)\log_{10}(x) &= 0,
\end{aligned}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

2 Problem (Wu-Riddles)

Let $n \in \mathbb{N}_{>0}$ and $A, B \in \mathbb{R}^{n\times n}$ be real matrices. Now suppose $I_n - BA$ is invertible, where I_n is the identity matrix. Prove that $I_n - AB$ is also invertible.

3 Problem (unknown)

An enemy submarine is hidden somewhere along the infinite line $\mathbb{R}$. It travels silently, and you know that its path is described by a fixed rational polynomial rule $\sum_{i=0}^{n} c_i T^i \in \mathbb{Q}[X]$; for each time $t \in \mathbb{N}$, its position is given by

$$x(t) = \sum_{i=0}^{n} c_i t^i.$$

However, you do not know $n \in \mathbb{N}$, nor the rational coefficients $(c_i)_{i\in n} \in \mathbb{Q}^n$ that form this rational polynomial rule.

Each unit of time (starting from 0), you are allowed to launch a single torpedo at any chosen rational position. If the submarine is at that position at that time, it is struck and sinks. Assume you possess a potentially countably infinite arsenal of torpedoes and potentially countably infinite time.

Devise a strategy — a sequence of torpedo launches — such that, regardless of the submarine's unknown position, you will **eventually** hit it.

4 Problem B1 (Putnam 1960) & A1 (Putnam 1961)

Define

$$A := \{(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \,:\, x^y = y^x\}$$ [1]

and find:

$$A, \quad A \cap (\mathbb{Q} \times \mathbb{Q}) \quad \text{and} \quad A \cap (\mathbb{Z} \times \mathbb{Z})\,.$$

Bonus: Find $A \cap \left(\mathbb{Q}^{\mathrm{alg}} \times \mathbb{Q}^{\mathrm{alg}}\right)$ and $A \cap \left(\mathcal{O}_{\mathbb{Q}^{\mathrm{alg}}}(\mathbb{Z}) \times \mathcal{O}_{\mathbb{Q}^{\mathrm{alg}}}(\mathbb{Z})\right)$.

5 Problem (Sudakov & Milojević)

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ be non-constant sequences of rational numbers. Suppose that for all $i, j \in \mathbb{N}$,

$$(a_i - a_j)(b_i - b_j) \in \mathbb{Z}.$$

Prove that there exists a non-zero rational number $\gamma \in \mathbb{Q}^\times$ such that for all $i, j \in \mathbb{N}$,

$$\gamma (a_i - a_j), \quad \gamma^{-1} (b_i - b_j) \in \mathbb{Z}.$$

[1] Recall that the exponentiation $a^b := \exp(b \cdot \log(a))$ where $a > 0$ and $b \in \mathbb{R}$ can be extended to $a = 0$ and $b \geq 0$ by:

$$0^b = \begin{cases} 0 & (b > 0), \\ 1 & (b = 0). \end{cases}$$

and extended in the obvious way for $a \in \mathbb{R}^\times$ with $b \in \mathbb{Z}$.