# Problem Set Week 7

ETHZ Math Olympiad Club

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## 1 Problem B-1 (IMC 2023)

Ivan writes the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

on the board. Then he performs the following operation on the matrix several times:

- He chooses a row or a column of the matrix, and
- He multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix

$$B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

after finitely many steps?

## 2 Vieta Jumping Problems

### 2.1 Problem 6 (IMO 1988)

Let $a$ and $b$ be positive integers such that $ab+1$ divides $a^2+b^2$. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

### 2.2 Problem (Kevin Buzzard & Edward Crane)

Let $a$ and $b$ be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

## 3 Problem A-3 (IMC 2015)

Let $F(0)=0$, $F(1)=\frac{3}{2}$, and

$$F(n) = \frac{5}{2}F(n-1) - F(n-2) \quad \text{for } n \geq 2.$$

Determine whether or not

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$$

is a rational number.

# 4 Hat Problems (Unknown)

> *A mathematician is a blind man in a dark room looking for a black hat which isn't there.*
> — Charles Darwin

In mathematical folklore, there is a long tradition of hat problems. Typically, a group of logicians or mathematicians (or, for some odd reason, gnomes) each wears a hat. They are able to see at least some of the other participants' hats, but not their own. Each player then guesses the colour of his own hat, and they all win or lose collectively based on the quality of their guesses. At first glance, it often seems impossible for the participants to win consistently: how should the colour of the other hats help me guess my own? Nonetheless, it is often possible to do surprisingly well. We refer to the book [1] for the history and many results on hat problems.

The following proposed problems deal with different kinds of hat problems, all considered in countable settings (both the number of mathematicians and the number of colours are countable). Of course, one can generalise these games to uncountable cardinals, increase the number of guesses per participant, allow simultaneous or recursive guessing, permit whispering or shouting, or vary how many other hats each mathematician can see.

The finite problems do not require any advanced mathematics, whereas the countably infinite versions require a bit more machinery, which should be manageable if you have studied naïve set theory up to the Axiom of Choice. For more general results, we advise those among you familiar with ordinal and cardinal arithmetic to consult the paper [2].

We leave the exact logistics of the realisation of these hat games to the imagination of the reader, but in the finite case, we assume the players can do anything finite (store any finite set, hear the guesses of all mathematicians, see everyone, ...). In the infinite case, we assume that the players have access to the Axiom of Choice (**AC**)[1] and that the players can use any choice function. We also assume that each player has a memory capable of storing sets of cardinality at most $2^{\aleph_0} = |\mathcal{P}(\mathbb{N})|$ and that each of them has perfect eyesight with infinite resolution (or simply that he knows the colour of each mathematician for each index).

The problems ask for the existence of a strategy verifying some winning property that the mathematicians can use to guess a colour. If we let $C$ be the set of colours and $N$ the set of mathematicians, with $N_i \subset N$ denoting the subset of mathematicians whose hat colours mathematician $i$ can see, then a general strategy for the mathematicians is an $N$-tuple $(f_i)_{i\in N}$ of functions $f_i\colon C^{N_i} \to C$. Here, each function $f_i$ is used by mathematician $i \in N$ to assign a colour in $C$ based on the hat colour configuration he observes, $\mathbf{c}|_{N_i} \in C^{N_i}$.

[1] That is, every collection of non-empty sets has a choice function:

$$\forall X \left[ (\forall Y \in X\, \exists z \in Y) \to \exists f : X \to \bigcup X \text{ such that } \forall Y \in X,\ f(Y) \in Y \right].$$

### 4.1

Consider $1 \leq N < \aleph_0$ mathematicians standing in a room. Each mathematician wears a hat that is either blue or red ($M = 2$). Each mathematician sees the colours of all the other hats but not his own. The game consists of the mathematician 0 shouting *only* "blue" or "red", which is the guess of his own hat colour (we assume everyone hears). Then **simultaneously** all other mathematicians $1 \leq i < N$ shout only "blue" or "red", which is the guess of their own hat colour. Before the game starts and the hats are placed on their heads, the mathematicians can devise a strategy. How can they ensure that they are all wrong or all correct?

**Bonus:** Solve the same problem when there are $N = \aleph_0$ mathematicians.

### 4.2

Consider a sequence of $1 \leq N \in \mathbb{N}$ mathematicians standing in a line. Each mathematician wears a black or a white hat ($M = 2$). Every mathematician can see the hat colours of all the mathematicians in front of him, but not his own or those behind him. The game consists of mathematicians **recursively** (starting from the first in line) shouting *only* "black" or "white", which is the guess of their own hat colour (we assume that everyone hears, i.e. information propagates recursively to the rest of the queue). Before the game starts and the hats are placed on their heads, the mathematicians can devise a strategy. How can they ensure that all but at most one mathematician guesses his hat colour correctly?

**Bonus:** If there are $N = \aleph_0$ mathematicians standing in line, but this time they whisper their guesses **recursively** (no one hears what any other says) or equivalently they shout **simultaneously**, how can they ensure that only finitely many of them guess incorrectly?

**Paradox:** Suppose the hats are assigned randomly and independently (for example, by tossing a fair coin for each mathematician). Then each mathematician has a probability of $\frac{1}{2}$ of guessing incorrectly. Let $T_N$ denote the number of incorrect guesses among the first $N$ mathematicians. By the strong law of large numbers, $\frac{T_N}{N} \overset{N \to +\infty}{\longrightarrow} \frac{1}{2}$ almost surely, so that $T_N \underset{N \to +\infty}{\sim} \frac{N}{2}$ almost surely, and thus $T_N$ is unbounded almost surely. This means that with probability 1, infinitely many mathematicians guess incorrectly. How can we resolve this apparent paradox?

### 4.3

Consider $1 \leq N < \aleph_0$ mathematicians standing in a circle and $1 \leq M < \aleph_0$ colours. Each mathematician $0 \leq i \leq N - 1$ wears a hat whose colour is chosen from the set $M$ (with repetitions allowed). Each mathematician sees the colours of all the other hats but not their own. The game consists of the mathematicians **simultaneously** shouting a colour (a number) in $M$. Before the game starts and the hats are placed on their heads, the mathematicians can devise a strategy. For which number of colours $1 \leq M < \aleph_0$ can they **always** ensure that at least one mathematician correctly guesses the colour of his hat?

**Bonus:** What can you say at best when the number of mathematician $N$ is such that $1 \leq N \leq \aleph_0$ and the number of color $M$ is such that $1 \leq M \leq \aleph_0$ ?

### 4.4

In an infinite sequence, $N = \aleph_0$ many mathematicians stand one behind the other. Each mathematician has a natural number on their back, where each number appears exactly once but is assigned arbitrarily to a mathematician (i.e., a permutation $\pi : \mathbb{N} \to \mathbb{N}$). Each mathematician can see all the numbers on the backs of those standing in front of them (forming an actual infinite sequence) but not their own number or the numbers of those standing behind them.

(a) The game consists of the mathematicians recursively (starting from the first in line) shouting (we assume that everyone hears, i.e., information propagates recursively to the rest of the queue) *only* a number, which will be interpreted as the guess of their own number on their back. No mathematician is allowed to state a number they see on any back in front of them, and all of them are aware of this rule.

    How can they ensure that everyone guesses their number correctly?

(b) As in (a), but the first $0 \leq N' < \aleph_0$ mathematicians are silent (say nothing), meaning the mathematician $N'$ (in the $N' + 1$ position) starts the guessing process.

    How can they ensure that everyone except at most $N'$ guesses their number correctly?

## References


[1] Clyde Hardin and Allen Taylor. *The Mathematics of Coordinated Inference: A Study of Generalized Hat Problems*. Springer, 2013.

[2] Andreas Lietz and Jeroen Winkel. *Infinite Hat Problems and Large Cardinals*. https://arxiv.org/abs/2411.06002. arXiv:2411.06002 [math.LO]. Nov. 2024. DOI: 10.48550/arXiv.2411.06002.