Problem Set Week 8

ETHZ Math Olympiad Club

7 April 2025

1 Problem 2 (IMC 1999)

Does there exist a bijective map $\pi : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$\sum_{n=1}^{\infty} \frac{\pi(n)}{n^2} < \infty?$$

2 Problem 2 (IMC 1994)

Let $f \in C^1 \left( ]a, b[, \mathbb{R} \right)$ with $\lim_{x \to a^+} f(x) = +\infty$, $\lim_{x \to b^-} f(x) = -\infty$ and $f'(x) + f^2(x) \geq -1$ for $x \in]a, b[$. Prove that $b - a \geq \pi$ and give an example where $b - a = \pi$.

3 Problem B-3 (IMC 2005)

In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V, \quad \operatorname{tr}(XY) = 0.$$

(The trace of a matrix is the sum of the diagonal entries.)

4 Problem 4 (Bernoulli Competition 2024)

Let n and m be positive integers, with $m \geq 3$. Suppose a matrix $A \in \mathbb{Z}^{n \times n}$ of finite order[1] satisfies

$$A \equiv I_n \pmod{m}.$$

Prove that $A = I_n$.

5 problem (unknown)

For this problem, we strongly encourage you to try to recreate the scenario in the real world with a sufficiently long rope and some nails/pins fixed to a surface. We have $n \geq 1$ nails fixed to a wall and a sufficiently long rope wrapped around these nails in a non-trivial configuration (i.e., the rope must be physically engaged with the nails in such a way that it does not fall off initially). For any fixed $n \geq 1$, find **all** wrapping configuration around the n nails such that:

1. The rope remains securely wrapped (i.e., it does not fall off when all nails are present).

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[1]$\exists k \in \mathbb{N}^*$, $A^k = I_n$.

2. When **any single nail** is removed (regardless of which one), the entire rope falls from the wall (in practice, some friction might prevent it from falling, but we consider it as falling if it is no longer securely wrapped).

By "wrapping," we mean the rope can make multiple loops around the nails in non-trivial ways (e.g., making loops around single nails or multiple nails, or passing under/over certain nails). Here are examples where we've labeled the nails c_0, c_1, and c_2:

Figure 1: *Nontrivial rope wrappings for $n = 1$ **satisfying** the property. The first configuration has a single loop around c_0 (clockwise or counterclockwise), while the second has at least two loops around c_0 (clockwise or counterclockwise).*

Figure 2: *Nontrivial rope wrappings for $n = 2$ that **do not satisfy** the property. The first configuration has no loop around c_0 but at least two loops (clockwise or counterclockwise) around c_1. The second configuration has at least two loops (clockwise or counterclockwise) around both c_0 and c_1.*

Figure 3: *Nontrivial rope wrappings (performed from left to right) for $n = 3$ **not satisfying** the property. The first configuration has a single clockwise loop around c_0, no loop around c_1, and a single clockwise loop around c_2. The second configuration has at least two clockwise loops around c_0, no loop around c_1, and at least two counter-clockwise loops around c_2.*