Problem Set Week 9

ETHZ Math Olympiad Club

14 April 2025

1 Simon's Favorite Factoring Trick Problems

1.1 AMC 12 (2012)

How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

1.2 AIME (1998)

An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where m, n, and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

1.3 BMO (2005)

The integer N is positive. There are exactly 2005 ordered pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}.$$

Prove that N is a perfect square.

1.4 JBMO (2003)

Let n be a positive integer. A number A consists of $2n$ digits, each of which is 4; and a number B consists of n digits, each of which is 8. Prove that $A + 2B + 4$ is a perfect square.

1.5 AIME (2000)

The system of equations

$$\log_{10}(2000xy) - \log_{10}(x)\log_{10}(y) = 4, \tag{1}$$
$$\log_{10}(2yz) - \log_{10}(y)\log_{10}(z) = 1, \tag{2}$$
$$\log_{10}(zx) - \log_{10}(z)\log_{10}(x) = 0, \tag{3}$$

has two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2). Find $y_1 + y_2$.

2 Problem (Wu-Riddles)

Let $n \geq 1$ and $A, B \in \mathbb{R}^{n \times n}$ be real matrices. Now suppose $I_n - BA$ is invertible, where I_n is the identity matrix. Prove that this implies $I_n - AB$ is also invertible.

3 Problem (unknown)

An enemy submarine is hidden somewhere along the infinite number line $\mathbb{Z}$. It travels silently, its path described by a linear rule: at minute $t \in \mathbb{N}$, its position is given by

$$x(t) = x_0 + vt,$$

where both $x_0 \in \mathbb{Z}$ (the initial position), and $v \in \mathbb{Z}$ (the constant velocity) are unknown.

Each minute, you are allowed to launch a single torpedo at any chosen integer position on the number line. If the submarine is at that position at that minute, it is struck and sinks. You possess a potentially infinite arsenal of torpedoes and potentially infinite time. Devise a strategy — a sequence of torpedo launches — such that, regardless of the submarine's unknown position and velocity, you will eventually hit it.

4 problem (unknown)

Define $A := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \,:\, x^y = y^x\}$[1], and find:

$$A, A \cap \mathbb{Q} \times \mathbb{Q} \text{ and } A \cap \mathbb{N} \times \mathbb{N}.$$

5 Problem (Sudakov & Milojevic)

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ be non-constant sequences of rational numbers. Suppose that $\forall i, j \in \mathbb{N}$:

$$(a_i - a_j)(b_i - b_j) \in \mathbb{Z}$$

Prove that there exists a non-zero rational number $\gamma \in \mathbb{Q}^\times$ such that $\forall i, j \in \mathbb{N}$:

$$\gamma(a_i - a_j), \gamma^{-1}(b_i - b_j) \in \mathbb{Z}.$$

[1] Knowing the construction of $\mathbb{R}$ using equivalence classes of rational Cauchy sequences, recall that one can unambiguously define the power a^b (for $a > 0$ and $b \in \mathbb{R}$) in two equivalent ways.

First Method: After having developed the theory of power series and their convergence, using the exponential and logarithmic functions, we define $a^b := \exp(b \cdot \log(a))$.

Second Method: A constructive step-by-step approach involves defining a^b where b is an integer, then extending to the rationals, and finally to the real numbers.

Since exp is a group morphism with $\exp(1) > 0$, we can show for all $x \in \mathbb{R}$ the equality $\exp(x) = \exp(1)^x$, where the left-hand side uses power series and the right-hand side uses the constructive approach. Thus, we can easily show that the two definitions of exponentiation give the same result. The exponentiation a^b where $a > 0$ and $b \in \mathbb{R}$ can be extended to $a = 0$ and $b \geq 0$ by:

$$0^b = \begin{cases} 0 & (b > 0), \\ 1 & (b = 0). \end{cases}$$

and extended in the obvious way for $a \in \mathbb{R}^\times$ with $b \in \mathbb{Z}$. For more information, see the article on Exponentiation. You can find the proofs in the book of Terence Tao's *Analysis I*.