Problem Set Week 5 Solutions

ETHZ Math Olympiad Club

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1 Problem (unknown)

Find all real solutions to the equation

$$9^x + 4^x + 2^x = 8^x + 6^x + 1.$$

Answer:

It is easy to see that $x = 0$, $x = 1$, and $x = 2$ are solutions. So the equation has at least 3 distinct real solutions. Let us introduce the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 9^x + 4^x + 2^x - 8^x + 6^x + 1.$$

As stated above, f has at least 3 distinct zeros. We claim there are no other roots. By Rolle's theorem, if a function $g : \mathbb{R} \to \mathbb{R}$ has at least $n \geq 2$ zeros $x_1 < \cdots < x_n$, then the function $g'(x)$ has at least $n-1$ zeros $y_1 < \cdots < y_{n-1}$, where for each $i = 1, \ldots, n-1$, we have $x_i < y_i < x_{i+1}$. In particular, since for each $a \in \mathbb{R}_{>0}$, $a^{-x}g(x)$ has at least n zeros, we have that $\left(a^{-x}g(x)\right)'$ has at least $n-1$ zeros, and so does the function

$$h_a g(x) = a^x \left(a^{-x}g(x)\right)' = g'(x) - \ln(a)g(x).$$

Suppose f has another zero not in $\{0, 1, 2\}$. Then f has at least 4 zeros, and thus

$$h_1 f(x) = f'(x) = \ln(9)9^x + \ln(4)4^x + \ln(2)2^x - \ln(8)8^x - \ln(6)6^x$$

has at least 3 zeros, which then implies that

$$h_6 h_1 f(x) = f''(x) - \ln(6)f'(x)$$

$$= \ln\left(\frac{9}{6}\right)\ln(9)9^x + \ln\left(\frac{4}{6}\right)\ln(4)4^x + \ln\left(\frac{2}{6}\right)\ln(2)2^x - \ln\left(\frac{8}{6}\right)\ln(8)8^x$$

has at least 2 zeros, which again implies that

$$h_8 h_6 h_1 f(x) = \ln\left(\frac{9}{8}\right)\ln\left(\frac{9}{6}\right)\ln(9)9^x - \ln\left(\frac{4}{8}\right)\ln\left(\frac{4}{6}\right)\ln(4)4^x - \ln\left(\frac{2}{8}\right)\ln\left(\frac{2}{6}\right)\ln(2)2^x$$

has at least 1 zero.

The function $h_8 h_6 h_1 f(x)$ is of the form $k_2 2^x + k_4 4^x + k_9 9^x$, for $k_2, k_4, k_9 > 0$ and hence is always positive. Therefore, $h_8 h_6 h_1 f(x)$ cannot have any real zero. This is a contradiction to the assumptions hence the solutions of the original equation are exactly $\{0, 1, 2\}$.

2 Problem 2 (Bernoulli Competition 2023)

Let e be Euler's number. Show that for any odd prime p, the integer

$$1! + 2! + 3! + \cdots + (p-1)! - \left\lfloor \frac{(p-1)!}{e} \right\rfloor$$

is divisible by p.

Answer:

First note that:

$$\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!}.$$

Thus, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \left\lfloor \sum_{i=0}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \right\rfloor$$

Notice that:

$$\sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!} \in \mathbb{Z}$$

We argue that the tail $\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} \in]0;1[$, indeed since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = \sum_{j=0}^{+\infty} \left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right).$$

is certainly bigger than 0 since each term $\left(\frac{(p-1)!}{(p-1+2j)!} - \frac{(p-1)!}{(p+2j)!} \right) = \frac{(p-1)!}{(p-1+2j)!} \left(1 - \frac{1}{p+2j} \right) > 0$. Similarly since p is odd:

$$\sum_{i=p-1}^{+\infty} \frac{(-1)^i (p-1)!}{i!} = 1 - \sum_{j=0}^{\infty} \left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right).$$

is certainly smaller than 1 since each term $\left(\frac{(p-1)!}{(p+2j)!} - \frac{(p-1)!}{(p+2j+1)!} \right) = \frac{(p-1)!}{(p+2j)!} (1 - \frac{1}{p+2j+1}) > 0$.

Therefore,

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor = \sum_{i=0}^{p-2} \frac{(-1)^i (p-1)!}{i!}.$$

Now note that for each $0 \leq j \leq p-1$ we have $j \equiv -(p-j) \pmod p$ and thus for fixed $0 \leq i < p-1$:

$$\frac{(-1)^i (p-1)!}{i!} = (-1)^i (i+1)(i+2)\cdots(p-1)$$

$$\equiv (-1)^i (p-(i+1))(p-(i+2))\cdots 2 \cdot 1 \cdot (-1)^{p-(i+1)} \equiv (p-(i+1))! \pmod p,$$

where we used that there is $p-(i+1)$ factor in $\frac{(p-1)!}{i!}$ and the fact that p is odd again. Hence, we have

$$\left\lfloor \frac{(p-1)!}{e} \right\rfloor \equiv \sum_{i=0}^{p-2} (p-(i+1))! \equiv \sum_{i=1}^{p-1} i! \pmod p,$$

since $i \mapsto p-(i+1)$ is a bijection from $[\![0, p-2]\!]$ to $[\![1, p-1]\!]$. This shows the problem's statement.

3 Problem in example page 140 (PUTNAM and BEYOND)

Find all real solutions to the equation

$$4^x + 6^{x^2} = 5^x + 5^{x^2}.$$

Answer:

Note that $x = 0$ and $x = 1$ satisfy the equation from the statement. Are there other solutions? The answer is no, but to prove it we use the amazing idea of treating the numbers 4, 5, 6 as variables and the presumably new solution x as a constant.

Thus let us consider the function $f(t) = t^{x^2} + (10 - t)^x$. The fact that x satisfies the equation from the statement translates to $f(5) = f(6)$. By Rolle's theorem there exists $c \in (5, 6)$, such that $f'(c) = 0$. This means that

$$x^2 c^{x^2-1} - x(10 - c)^{x-1} = 0,$$

or

$$xc^{x^2-1} = (10 - c)^{x-1}.$$

Because exponentials are positive, this implies that x is positive.
If $x > 1$, then $x^2 - 1 > x - 1$ and as $c > 5$

$$(10 - c)^{x-1} = xc^{x^2-1} > c^{x^2-1} > c^{x-1} > (10 - c)^{x-1},$$

which is a contradiction.

If $0 < x < 1$, then $x^2 - 1 < x - 1$ and:

$$(10 - c)^{x-1} = xc^{x^2-1} < xc^{x-1}.$$

Let us prove that

$$xc^{x-1} < (10 - c)^{x-1}.$$

With the substitution $y = x - 1$, the inequality can be rewritten as

$$y + 1 < \left(\frac{10 - c}{c}\right)^y.$$

which must be proven for $y \in]-1; 0[$.

Lets make a simple analysis of the two functions defined over $\mathbb{R}$.
The exponential has base less than 1, so it is strictly decreasing, while the affine function on the left is strictly increasing. The two meet at $y = 0$ so we must have that strictly before $y = 0$ the exponential is strictly bigger than the affine. The inequality (on $]-1; 0[$) follows.
Using it we conclude again that: $(10 - c)^{x-1} = xc^{x^2-1} < (10 - c)^{x-1}$ which is a contradiction.
This shows that a third solution to the equation from the statement does not exist. So the only solutions to the given equation are $x = 0$ and $x = 1$.

4 Problem 3 (Bernoulli Competition 2023)

Let $n \geq 1$ and A be a $n \times n$ symmetric matrix over $\mathbb{F}_2 = {}^{\mathbb{Z}}/_{2\mathbb{Z}}$ with $1_{\mathbb{F}_2}$'s on the main diagonal. Show that the vector composed uniquely of $1_{\mathbb{F}_2}$'s is in the image of A.

Answer:

We write $1 := 1_{\mathbb{F}_2}$ and $0 := 0_{\mathbb{F}_2}$, define the $\mathbb{F}_2-$vector space $V = \mathbb{F}_2^n$ and define the standard binary product on V, i.e.,

$$\langle v, w\rangle = \sum_{i=1}^{n} v_i w_i.$$

It is easy to see that $\langle \cdot, \cdot \rangle$ is $\mathbb{F}_2-$linear in the first coordinate, symmetric (so $\mathbb{F}_2-$linear in the second coordinate and thus $\mathbb{F}_2-$bilinear) and non-degenerate that is:

$$\forall v \in V\,((\forall w \in V \langle v, w\rangle = 0) \to v = 0_V)$$

(*just plug the canonical basis for w that is for each $i \in n$ take $w = e_i$ and use the fact that $\mathbb{F}_2$ is a field to conclude $v_i = 0$, and thus $v = \underline{0} = 0_V$*).

With this being introduced, lets take an $n \times n$-matrix A with $\operatorname{diag}(A) = \underline{1}$. Since $\langle \cdot, \cdot \rangle$ is symmetric and non-degenerate, we have for any $\mathbb{F}_2-$subspace $W \subset V$,

$$\left(W^{\perp}\right)^{\perp} = W.$$

where $Z^{\perp} = \{v \in V \mid \langle v, z\rangle = 0 \ \forall z \in Z\}$ for any $\mathbb{F}_2-$subspace $Z \subset V$. For more information on this equality see the appendix 1. Now write $A = (a_{ij})_{1 \leq i,j \leq n}$ with $a_{ii} = 1$ and $a_{ij} = a_{ji}$. Then we have for any $v \in V$,

$$\langle v, Av\rangle = \sum_{1 \leq i,j \leq n} v_i v_j a_{ij} = \sum_{i=1}^{n} v_i^2 + 2\sum_{i<j} v_i v_j a_{ij} = \sum_{i=1}^{n} v_i = \langle v, \underline{1}\rangle,$$

because we are working over $\mathbb{F}_2$. In particular for any $z \in \operatorname{Im}(A)^{\perp} \subset V$,

$$\langle z, \underline{1}\rangle = \langle z, Az\rangle = 0,$$

since $Az \in \operatorname{Im}(A)$ and $z \in Im(A)^{\perp}$. As $z \in Im(A)^{\perp}$ was arbitrary, we must have

$$\underline{1} \in \left(\operatorname{Im}(A)^{\perp}\right)^{\perp} = \operatorname{Im}(A).$$

5 Problem (unknown)

Find all differentiable functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $\forall x \in \mathbb{R}_{>0}$:

$$1 \leq f'(x) = \frac{f(x)}{(f \circ f)(x)}.$$

Answer:

The identity function obviously works. We claim this is the only solution. Let $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be differentiable with $\forall x \in \mathbb{R}_{>0}$:

$$1 \leq f'(x) = \frac{f(x)}{(f \circ f)(x)}.$$

The solution is done in two steps: we show that f admits a fixed point $\beta > 0$, then we use this fixed point to show that no other solution exists. Note that f takes positive values, so we must have that $f' = \frac{f}{f \circ f}$ takes strictly positive values. Therefore, f must be strictly increasing and hence injective. From this, we derive the equivalence $\forall \beta > 0$:

$$f'(\beta) = 1 \Leftrightarrow f(\beta) = f(f(\beta)) \Leftrightarrow f(\beta) = \beta.$$

Thus, showing the existence of a fixed point $\beta > 0$ is equivalent to showing that $f'(\beta) = 1$.

If $\exists \beta > 0$ with $f'(\beta) = 1$, then f has the fixed point β. Otherwise, by the condition on f, it must be that $\forall x > 0$, $f'(x) > 1$. In particular, $f'(1) > 1$, so $f(1) > f(f(1))$. Since f is strictly increasing, we must have $1 > f(1)$ (if $1 \leq f(1)$, then $f(1) \leq f(f(1))$, so $f(1) < f(1)$, a contradiction). Note that f is continuous (since f is differentiable), thus f' is continuous and hence integrable over any compact interval in $\mathbb{R}_{>0}$. Since $\forall t > 0$, $f'(t) > 1$, we have for any $0 < x < 1$:

$$f(1) - f(x) = \int_x^1 f'(t)\,\mathrm{d}t \geq \int_x^1 1\,\mathrm{d}t = 1 - x$$

($\int_x^1 \_\mathrm{d}t$ is increasing from the space of real valued integrable functions over $[x, 1]$).

Thus, $\forall x > 0$ with $x < 1$,

$$f(x) \leq (f(1) - 1) + x.$$

But then, for any $0 < y < 1 - f(1) < 1$, we have $f(y) < 0$, contradicting the positivity of f.

Hence, f admits a fixed point $\alpha > 0$.

Now, we use this fixed point to show that f must be the identity. Since f is continuous, we can integrate it and define for any $x > 0$:

$$F(x) := \int_\alpha^x f(t)\,\mathrm{d}t.$$

By the fundamental theorem of calculus, $F : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous. Moreover, since f is continuous, F is everywhere differentiable with $F' = f$. Because f is always positive, F is strictly increasing and thus injective.

Fix $x \in \mathbb{R}_{>0}$. From the given equation (valid for all $t > 0$),

$$f(t) = f(f(t))f'(t) = F'(f(t))f'(t) = (F \circ f)'(t),$$

we obtain:

$$F(x) = \int_\alpha^x f(t)\,\mathrm{d}t = \int_\alpha^x (F \circ f)'(t)\,\mathrm{d}t = (F \circ f)(x) - (F \circ f)(\alpha),$$

where we used the fundamental theorem of calculus again for the function $x \mapsto \int_\alpha^x (F \circ f)'(t)\,\mathrm{d}t$ (this holds for both $x \geq \alpha$ and $x < \alpha$).

Now, using the fact that α is a fixed point of f, we get $F(f(\alpha)) = F(\alpha) = 0$, so $F(x) = F(f(x))$. By the injectivity of F, we conclude that $f(x) = x$. Since $x > 0$ was arbitrary, the proof is complete.

Remark. If we drop the restriction that $f'(x) \geq 1$ for all $x > 0$, the problem becomes significantly more challenging. The proof of the result proceeds in the same way, but we now have to handle the additional case where $f'(x) < 1$ for all $x > 0$. If $f(x) < x$, this case can be handled trivially, but the situation where $f(x) > x$ presents substantial difficulties. Since $f'(x)$ is determined by the value of f at x and its composition $f \circ f$ at x, and because $f(f(x)) > f(x)$, the derivative $f'(x)$ depends on values of f at points beyond $f(x)$. This leads to a non-causal delay differential equation. For further discussion, see the related MathStack Exchange thread.

A

Let E be a finite-dimensional vector space over a field K, and $B : E \times E \to K$ a symmetric bilinear form. For any subspace $Q \subseteq E$, the orthogonal complement is defined as $Q^\perp := \{v \in E \mid \forall q \in Q B(q,v) = 0\}$. It is clearly a $K-$subspace of E. The form B is non-degenerate if $E^\perp = \{0\}$. For a non-degenerate symmetric bilinear form B, the map

$$\varphi : E \to E^\vee, \quad v \mapsto B(\cdot, v)$$

is a vector space isomorphism. Indeed B is bilinear so φ is linear, the injectivity follows from

$$\varphi(v) = 0_{E^\vee} \implies \forall w \in E\, B(w,v) = 0 \implies v \in E^\perp = \{0\} \quad \text{(by non-degeneracy)}$$

As $\dim(E) = \dim(E^\vee)$ in finite dimensions (classic), injectivity implies surjectivity. Thus φ is an isomorphism.

Theorem 1 (Double Orthogonal Complement)**.** *In this settings, if B is non-degenerate and $Q \subseteq E$ is a $K-$subspace, then $\left(Q^\perp\right)^\perp = Q$*

Proof. **Step 1.** $\left(Q \subseteq \left(Q^\perp\right)^\perp\right)$: For any $q \in Q$, by definition,

$$\forall v \in Q^\perp,\ B(q,v) = 0 \implies q \in \left(Q^\perp\right)^\perp.$$

Step 2. Using the isomorphism φ, we have that $\varphi|_{Q^\perp}$ is an isomorphism onto its image:

$$\text{Im}\left(\varphi|_{Q^\perp}\right) = \left\{B(-,v) \mid v \in Q^\perp\right\} = \{f \in E^\vee \mid \forall q \in Q,\ f(q) = 0\} =: Q.$$

The middle equality's $\supset$ inclusion follows from the surjectivity of φ and the definition of $Q^\perp$. Thus, $\dim(Q^\perp) = \dim(Q)$.

Step 3. Given an ordered basis $(w_i)_{i \in \dim(Q)}$ of Q, complete it into an ordered basis of E:

$$(w_i)_{i\in\dim(Q)} \frown (v_j)_{j\in\dim(E)-\dim(Q)}.$$

There is a classical associated ordered basis of $E^\vee$:

$$(w_i^*)_{i\in\dim(Q)} \frown (v_j^*)_{j\in\dim(E)-\dim(Q)},$$

where each functional sends a vector $v = \sum_{j\in\dim(Q)} \lambda_j w_j + \sum_{i\in\dim(E)-\dim(Q)} \gamma_i v_i$ of E to λ_j or γ_i, respectively. This basis satisfies the following property: if $f \in E^\vee$, then we can write

$$f = \sum_{i\in\dim(Q)} f(w_i)w_i^* + \sum_{j\in\dim(E)-\dim(Q)} f(v_j)v_j^*.$$

In particular, if $f \in Q$, then $f = \sum_{j\in\dim(E)-\dim(Q)} f(v_j)v_j^*$, because $\{w_i \mid i \in \dim(Q)\} \subset Q$. Thus, $(v_j^*)_{j\in\dim(E)-\dim(Q)}$ generates Q. Since these vectors are K-linearly independent, we obtain

$$\dim(E) - \dim(Q) = \dim\left(Q\right).$$

Step 4. In total, we obtain that for any K-subspace Q of E,

$$\dim(Q^\perp) = \dim\left(Q\right) = \dim(E) - \dim(Q).$$

Since $Q^\perp$ is a K-subspace, we must have:

$$\begin{aligned}\dim\left(\left(Q^\perp\right)^\perp\right) &= \dim\left(\left(Q^\perp\right)\right) = \dim(E) - \dim(Q^\perp) \\ &= \dim(E) - (\dim(E) - \dim(Q)) = \dim(Q).\end{aligned}$$

We conclude $Q = (Q^\perp)^\perp$ since they have the same (finite) dimension and $Q \subset \left(Q^\perp\right)^\perp$. $\square$