Problem Set Week 7

ETHZ Math Olympiad Club

31 March 2025

1 Problem B-1 (IMC 2023)

Ivan writes the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$$

on the board. Then he performs the following operation on the matrix several times:

- He chooses a row or a column of the matrix, and
- He multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

Can Ivan end up with the matrix

$$B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

after finitely many steps?

2 Vieta Jumping Problems

2.1 Problem 6 (IMO 1988)

Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

2.2 Problem (Kevin Buzzard and Edward Crane)

Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$.

3 Problem A-3 (IMC 2015)

Let $F(0)=0$, $F(1)=\frac{3}{2}$, and

$$F(n) = \frac{5}{2}F(n-1) - F(n-2) \quad \text{for } n \geq 2.$$

Determine whether or not

$$\sum_{n=0}^{\infty} \frac{1}{F(2^n)}$$

is a rational number.

4 Problems of Mathematicians (Unknown)

4.1

N mathematicians each wear a hat whose color is chosen from the set $\{1, 2, \ldots, N\}$ (with repetitions allowed). Each mathematician sees the colors of all the other hats but not their own. What strategy can they use to guarantee that at least one mathematician correctly guesses the color of their own hat?

4.2

N mathematicians each wear a hat that is either blue or red. Each mathematician sees the colors of all the other hats but not their own. What strategy can they adopt so that they are either all wrong or all correct simultaneously?

Bonus:[1] Solve the same problem when the set of mathematicians is countably infinite.

4.3

Consider a sequence of $N \geq 1$ mathematicians standing in a line. Each mathematician wears either a black or a white hat. Every mathematician can see the hat colors of all the mathematicians in front of them but not their own or those behind them. The game consists of the mathematicians recursively (starting from the first in line) shouting *only* "black" or "white", which will be interpreted as the guess of their own hat color. Before the game starts and the hats are placed on their heads, the mathematicians can devise a strategy. How can they ensure that all but at most one mathematician guess their hat color correctly?

Bonus:[2] If there are countably infinitely many mathematicians standing in line (say indexed by $\mathbb{N}$), how can they ensure that only finitely many of them guess incorrectly?

Paradox: Suppose the hats are assigned randomly and independently (for example, by tossing a fair coin for each mathematician). Then each mathematician has a probability of $\frac{1}{2}$ of guessing incorrectly. Let T_N denote the number of incorrect guesses among the first N mathematicians. By the strong law of large numbers, $\frac{T_N}{N} \overset{N\to+\infty}{\longrightarrow} \frac{1}{2}$ almost surely, so that $T_N \underset{N\to+\infty}{\sim} \frac{N}{2}$ almost surely, and thus T_N is unbounded almost surely. This means that with probability 1, infinitely many mathematicians guess incorrectly. How can we resolve this apparent paradox?

[1] We assume the axiom of countable choice ($\mathbf{AC}_\omega$), that is, every countable collection of non-empty sets has a choice function:

$$\forall X \left[(|X| = \aleph_0 \wedge \forall Y \in X \, \exists z \in Y) \to \exists f : X \to \bigcup X \text{ such that } \forall Y \in X,\ f(Y) \in Y\right].$$

We also assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $\aleph_0 = |\mathbb{N}|$.

[2] We assume the axiom of choice ($\mathbf{AC}$), that is, every collection of non-empty sets has a choice function:

$$\forall X \left[(\forall Y \in X \, \exists z \in Y) \to \exists f : X \to \bigcup X \text{ such that } \forall Y \in X,\ f(Y) \in Y\right].$$

We also assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $2^{\aleph_0} = |\mathscr{P}(\mathbb{N})|$.

4.4

We have a countably infinite sequence of mathematicians (indexed by $\mathbb{N}$), each wearing a hat of a unique color in $\mathbb{N}$ and each color is used (i.e., a permutation $\pi : \mathbb{N} \to \mathbb{N}$). Each mathematician can see the hat colors of all the mathematicians in front of them (an actual infinite sequence)[3] but not their own or those behind them.

Starting with the first mathematician, they recursively shout a color, which is interpreted as their guess for their own hat color. Devise a strategy such that every mathematician guesses their hat color correctly.

Now, suppose the mathematicians up to index $N \geq 0$ are muted (i.e., they do not guess), and the guessing begins with the mathematician of index $N + 1$. How can they ensure that at most N mathematicians guess incorrectly, and all remaining mathematicians guess correctly?

[3] We assume that each mathematician has a memory capable of storing sets of cardinality less than or equal to $\aleph_0$.