

Problem Set Week 13

ETHZ Math Olympiad Club

19 May 2025

1 Problem 1 (Bernoulli Competition 2023)

1. Let $A = \{1, 2, \dots, 100\}$ be the set of integers between 1 and 100.
 - (a) Let $B \subset A$ be a subset that doesn't contain two consecutive integers. What is the maximal cardinality of B ?
 - (b) Let $C \subset A$ be a subset such that there is no n for which n and $2n$ are both in C . What is the maximal cardinality of C ?

2 Problem (Selected Real Analysis Problem)

For each function $g \in \{-id_{\mathbb{R}}, \exp, x \mapsto x^2 - 2\}$, determine whether there exists a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f \circ f = g$.

Bonus: Solve the same problem with $g \in \{\cos, \sin\}$.

3 Problem B4 (Putnam 2001)

Let $S := \mathbb{Q} \setminus \{-1, 0, 1\}$. Define $f : S \rightarrow S$ by $f(x) = x - \frac{1}{x}$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times.

4 Problem (Hongler)

We have a linked list of elements, x_0, x_1, \dots, x_n , where n is unknown (but it is known that the list is finite). When at x_0 , we have a pointer to go to x_1 , which leads to x_2 , and so on, until we reach x_n , where we learn that it is the end. We have a bounded amount of memory (we cannot simply store the entire list in an array and choose an element from the array once we reach the end).

- (a) How can we select a random element from the list, uniformly, if we are allowed to traverse the list only once?
- (b) Why might solving this problem be useful in practice?

5 Problem (Hongler)

Let $U \subset \mathbb{C}$ be a domain containing the disc \mathbb{D} ; that is, $\mathbb{D} \subset U$, and let $f : U \rightarrow \mathbb{C}$ be a holomorphic function. Show that if $f(\partial\mathbb{D}) = \gamma$ is a simple loop (i.e., it can be parametrised by a continuous path which intersects only at the endpoints), and if $f|_{\partial\mathbb{D}} : \partial\mathbb{D} \rightarrow \gamma$ is injective, then $f|_{\mathbb{D}}$ is injective.