

# Problem Set Week 5 Solutions

ETHZ Math Olympiad Club

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## Problem (unknown)

We consider a game where two indistinguishable envelopes are presented to a player:

- One envelope contains an amount  $\alpha \in \mathbb{R}_{>0}$ .
- The other envelope contains  $2\alpha$ .

The game proceeds as follows:

1. The player randomly selects one envelope (with equal probability).
2. The player observes the content  $x$  of the selected envelope (without knowing  $\alpha$ ).
3. The player must decide whether to:
  - Keep the current envelope, or
  - Switch to the other envelope (this decision is irrevocable).

Although the game is played once, the player's objective is still to maximize their *expected gain*. Assuming access to *randomness*, how can they do better than always keeping the first envelope?

## Solution:

As outlined in the problem, we consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , where  $\Omega$  consists of two possible configurations of envelopes:  $\Omega = \{(\alpha, 2\alpha), (2\alpha, \alpha)\}$ . Since  $\Omega$  has only two elements, the only choice for the sigma-algebra (other than the trivial one) is its power set,  $\mathcal{F} = \mathcal{P}(\Omega)$ . The player selects an envelope at random with equal probability, implying that the probability measure  $\mathbb{P}$  follows Laplace's model. Thus, for any  $A \in \mathcal{F}$ :

$$\mathbb{P}(A) := \frac{|A|}{|\Omega|} = \frac{|A|}{2}.$$

In this framework, the "naive" strategies—always keeping the selected envelope or always switching—can be analyzed by computing the expected value of the first coordinate projection  $\pi_1 : \Omega \rightarrow \mathbb{R}$  (or equivalently, the second coordinate  $\pi_2$ ). These functions are obviously measurable and positive, and we compute:

$$\mathbb{E}(\pi_i) = \alpha \mathbb{P}_{\pi_i}(\{\alpha\}) + 2\alpha \mathbb{P}_{\pi_i}(\{2\alpha\}) = \alpha \frac{1}{2} + 2\alpha \frac{1}{2} = \frac{3}{2}\alpha.$$

However, since we assume access to randomness, we can improve upon these deterministic strategies by incorporating some randomization. The key observation is that while the player

does not know  $\alpha$ , i.e., they know only "partially"  $\Omega$  (having access to  $\Omega$  only through  $\pi_1$ ; opening one of the envelopes only gives them a value  $x$  and the other one may be  $2x$  or  $\frac{x}{2}$ ). They still know that one value is strictly greater than the other, i.e.,  $\alpha < 2\alpha$ .

To exploit this without knowing  $\Omega$ , the player, after having selected one envelope and having seen its content, introduces some randomization. Say, they will generate<sup>1</sup> a number between  $[0, 1]$  and *make a decision* with this number. Formally, we introduce an auxiliary probability space  $(\Sigma, \mathcal{A}, \mu)$  and let  $U : \Sigma \rightarrow [0, 1]$  be a uniform random variable, i.e.,  $U \sim \text{Unif}([0, 1])$ <sup>2</sup>, that is, for any  $a, b \in [0, 1]$  with  $a \leq b$  and any non-empty interval  $I \subset [0, 1]$  with  $\inf I = a$  and  $\sup I = b$  then  $\mu_U(I) = b - a$ . Adjoining this new probability space to the preceding one, we obtain a product space (which is a probability space):

$$(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{A}, \mathbb{P} \otimes \mu).$$

Choosing a  $(w, \xi) \in \Omega \times \Sigma$ , and looking at  $\pi_1(w)$ , we must now make a decision about switching or not with a deterministic choice involving the pair  $(\pi_1(w), U(\xi))$ . That is, we have a measurable function  $f : \mathbb{R}_{>0} \times [0, 1] \rightarrow \{0, 1\}$  that takes  $(\pi_1(w), U(\xi))$  and if  $f((\pi_1(w), U(\xi))) = 1$ , we keep, i.e., we take  $\pi_1(w)$ , and if  $f((\pi_1(w), U(\xi))) = 0$ , we switch, i.e., we take  $\pi_2(w)$ . Formally, the gain with respect to this choice of  $f$  is  $G_f : \Omega \times \Sigma \rightarrow \mathbb{R}$  where:

$$G_f((w, \xi)) = \pi_1(w) \cdot \mathbb{1}_{\{1\}}(f((\pi_1(w), U(\xi)))) + \pi_2(w) \cdot \mathbb{1}_{\{0\}}(f((\pi_1(w), U(\xi)))) ,$$

which is measurable as long as  $f$  is measurable. Then, with probability  $\frac{1}{2}$ , the first envelope has amount  $\alpha$ , and we decide to keep it with a probability  $\mu_{f(\alpha, U)}(\{1\})$  and we switch with probability  $\mu_{f(\alpha, U)}(\{0\})$ . With probability  $\frac{1}{2}$ , the first envelope has amount  $2\alpha$ , and we decide to keep it with probability  $\mu_{f(2\alpha, U)}(\{1\})$  and we switch with probability  $\mu_{f(2\alpha, U)}(\{0\})$ . The expected gain in this new probability space is given by:

$$\begin{aligned} \mathbb{E}(G_f) = & \alpha \left( \mathbb{P}_{\pi_1}(\{\alpha\}) \mu_{f(\alpha, U)}(\{1\}) + \mathbb{P}_{\pi_1}(\{2\alpha\}) \mu_{f(2\alpha, U)}(\{0\}) \right) \\ & + \\ & 2\alpha \left( \mathbb{P}_{\pi_1}(\{2\alpha\}) \mu_{f(2\alpha, U)}(\{1\}) + \mathbb{P}_{\pi_1}(\{\alpha\}) \mu_{f(\alpha, U)}(\{0\}) \right) \\ = & \frac{1}{2} \alpha \left( \mu_{f(\alpha, U)}(\{1\}) + \mu_{f(2\alpha, U)}(\{0\}) \right) + \alpha \left( \mu_{f(2\alpha, U)}(\{1\}) + \mu_{f(\alpha, U)}(\{0\}) \right). \end{aligned}$$

This formulation may appear abstract, and the machinery might seem overdeveloped, but the goal is to introduce the object naturally and provide a philosophical motivation for what follows.

We observe an asymmetry in this linear combination of  $\alpha$  and  $2\alpha$ , which suggests an opportunity to exploit it in order to increase the expected gain beyond  $\frac{3}{2}\alpha$ . A natural way to deterministically exploit randomness in our decision-making process (that is, a natural choice of  $f$ ) is to define a threshold: if  $U(\xi)$  is below a certain threshold, we keep the first envelope; otherwise, we switch. However, since we have access to the amount in the first envelope, we can dynamically adjust this threshold to take advantage of the inequality  $\alpha < 2\alpha$  and the

<sup>1</sup>Practically, this could involve sampling from a physical source (e.g., thermal noise) or a deterministic pseudorandom number generator (PRNG) seeded by an unpredictable value (e.g., system clock nanoseconds); see [Hardware random number generator](#) for more information.

<sup>2</sup>We can construct  $(\Sigma, \mathcal{A}, \mu)$  as  $([0, 1], \mathcal{B}([0, 1]), \lambda|_{\mathcal{B}([0, 1])})$ , where  $\mathcal{B}([0, 1])$  is the Borel sigma-algebra and  $\lambda$  is the Lebesgue measure on  $\mathbb{R}$ .  $U$  can then be defined as the identity function, i.e.,  $U(x) = x$  for  $x \in [0, 1]$  and it is obviously a random variable with the uniform law on  $[0, 1]$ .

asymmetry in the expected gain formula.

To formalize this idea, we introduce a measurable function  $g : \mathbb{R}_{>0} \rightarrow [0, 1]$  and define a decision function  $f_g : \mathbb{R}_{>0} \times [0, 1] \rightarrow \{0, 1\}$  by

$$f_g(x, y) = \mathbb{1}_{[0, g(x)]}(y),$$

which means we keep the first envelope if the generated number  $U(\xi)$  satisfies  $0 \leq U(\xi) \leq g(\pi_1(\omega))$ , and we switch otherwise. For example, if  $g = \text{cte}_0$  then we always switch, while if  $g = \text{cte}_1$  then we always keep.

For this specific choice of  $f_g$ , the probabilities of keeping and switching the envelope are as follows: with probability  $\frac{1}{2}$  the first envelope has amount  $\alpha$ , and we keep it if the randomly generated number in  $[0, 1]$  is below  $g(\alpha)$  that is we keep it with probability:

$$\mu_{f_g(\alpha, U)}(\{1\}) = \mu(\{\xi \in \Sigma \mid U(\xi) \in [0, g(\alpha)]\}) = g(\alpha)$$

and we switch with probability:

$$\mu_{f_g(\alpha, U)}(\{0\}) = \mu(\{\xi \in \Sigma \mid U(\xi) \in ]g(\alpha), 1]\}) = 1 - g(\alpha).$$

Similarly, with probability  $\frac{1}{2}$  the first envelope has amount  $2\alpha$ , and we keep it with probability:

$$\mu_{f_g(2\alpha, U)}(\{1\}) = \mu(\{\xi \in \Sigma \mid U(\xi) \in [0, g(2\alpha)]\}) = g(2\alpha)$$

and we switch with probability:

$$\mu_{f_g(2\alpha, U)}(\{0\}) = \mu(\{\xi \in \Sigma \mid U(\xi) \in ]g(2\alpha), 1]\}) = 1 - g(2\alpha).$$

Thus, the expected gain under this strategy is:

$$\begin{aligned} \mathbb{E}(G_{f_g}) &= \frac{1}{2}\alpha \left( \mu_{f_g(\alpha, U)}(\{1\}) + \mu_{f_g(2\alpha, U)}(\{0\}) \right) + \alpha \left( \mu_{f_g(2\alpha, U)}(\{1\}) + \mu_{f_g(\alpha, U)}(\{0\}) \right) \\ &= \frac{1}{2}\alpha (g(\alpha) + 1 - g(2\alpha)) + \alpha (g(2\alpha) + 1 - g(\alpha)) \\ &= \frac{3}{2}\alpha + \frac{\alpha}{2} (g(2\alpha) - g(\alpha)). \end{aligned}$$

To ensure  $\mathbb{E}(G_{f_g}) > \frac{3}{2}\alpha$ , it suffices to choose a measurable function  $g$  satisfying  $g(2\alpha) > g(\alpha)$ . Since the player does not know the exact value of  $\alpha$  but knows that  $\alpha < 2\alpha$ , they can select any strictly increasing (and hence measurable) function  $g : \mathbb{R}_{>0} \rightarrow [0, 1]$ . Examples of such functions include:

$$g(x) = \frac{x}{x+1} = 1 - \frac{1}{x+1}, \quad g(x) = 1 - e^{-x}.$$

With these choices, we obtain  $g(2\alpha) > g(\alpha)$ , effectively improving the expected gain over the naive strategy by:

$$\frac{\alpha}{2} (g(2\alpha) - g(\alpha)).$$

For the above examples of  $g$ , this improvement is given by:

$$\frac{\alpha}{2} \left( \frac{1}{1+\alpha} - \frac{1}{1+2\alpha} \right), \quad \frac{\alpha}{2} (e^{-\alpha} - e^{-2\alpha}) \text{ respectively.}$$

Ideally, one would aim to find an increasing function

$$g \in \mathcal{G} := \{g' \mid g' : \mathbb{R}_{>0} \rightarrow [0, 1] \text{ increasing}\}$$

that maximizes  $\inf \{g(2x) - g(x) \mid x \in \mathbb{R}_{>0}\}$  however:

$$\sup \{\inf \{g'(2x) - g'(x) \mid x \in \mathbb{R}_{>0}\} \mid g' \in \mathcal{G}\} = 0$$

because if we assume for contradiction that there exists  $c > 0$  and  $g' \in \mathcal{G}$  such that

$$g'(2x) - g'(x) \geq c \quad \text{for all } x > 0.$$

Then by simply iterating this inequality we obtain by induction on  $\mathbb{N}$ :

$$g'(2^n x) - g'(x) \geq nc \quad \text{for all } n \in \mathbb{N}.$$

Since  $g'$  is bounded above by 1 and bounded below by 0, this implies  $nc < 1$  for every  $n$ , which is impossible since  $c$  is not an **infinitesimal element** w.r.t to 1. However if we had more information on  $\alpha$  (say some lower or/and upper bound) we may tune hyperparameter  $t, c \in \mathbb{R}_{>0}$  by doing first order optimization on for examples:

*Power-Law Functions:*

$$g(x) = \frac{x^t}{x^t + c}.$$

For a given  $t$ , larger  $c$  makes it easier to switch for less and less big  $x$  emphasizing to keep on only the very big  $x$ , smaller  $c$  makes it harder to switch for bigger and bigger  $x$  emphasizing to keep most of the  $x$ . A fast transition occurs to the constant 1 when  $c \rightarrow 0$ . For a given  $c$ , larger  $t$  makes it harder to switch for less and less big  $x$  emphasizing to keep most of the  $x$ , smaller  $t$  makes it easier to switch for less and less big  $x$  emphasizing to keep only the very big  $x$ . A fast transitions to the constant  $\frac{1}{1+c}$  occurs when  $t \rightarrow 0$ .

*Logarithmic Functions:*

$$g(x) = \frac{\ln(1 + x^t)}{\ln(1 + x^t) + c}.$$

This is exactly the same as above but things are more evenly spread out and smoother.

*Exponential Functions:*

$$g(x) = 1 - e^{-tx}.$$

Larger  $t$  makes it very difficult to switch for all  $x$  emphasizing to change only on the very small  $x$ , smaller  $t$  makes it easier to switch emphasizing to keep only on the very big  $x$ .

*Remark.* We can generalize the contents of the envelopes to be  $\alpha$  and  $k\alpha$  for any real number  $k > 1$  and the previous analysis applies *mutatis mutandis*, where:

- The naive strategy yields an expected gain of  $\frac{k+1}{2}\alpha$ .
- The improved strategy, using any increasing function  $g : \mathbb{R}_{>0} \rightarrow [0, 1]$ , gives an expected gain of:

$$\frac{k+1}{2}\alpha + \frac{k-1}{2}\alpha(g(k\alpha) - g(\alpha)).$$

This setting is equivalent to considering  $\alpha$  and  $k\alpha$  where  $0 < k \neq 1$  since we can simply swap the roles of the envelopes:

- If  $k > 1$ , the situation remains unchanged.

- If  $0 < k < 1$ , we reparameterize by setting  $\alpha \leftarrow k\alpha$  and  $k \leftarrow \frac{1}{k}$ , reducing to the previous case.

In general, this framework is equivalent to considering any two distinct positive amounts  $\alpha$  and  $\beta$  with  $\alpha < \beta$ , since we can express  $\beta = k\alpha$  where  $k = \frac{\beta}{\alpha} > 1$ .

Here you can find a code to simulate:

```
import numpy as np import math
def g_power(x,t = 1.0,c = 1.0) : return(x * t)/(x * t + c)
def g_exponential(x,t = 1.0) : return1 - np.exp(-t * x)
def g_logarithmic(x,t = 1.0,c = 1.0) : returnnp.log(1 + x * t)/(np.log(1 + x * t) + c)
def simulate_game(g,alpha = 1.0,k = 2.0,n_trials = 10 * 5,random_seed = None) :
if random_seed is not None : np.random.seed(random_seed)
choices = np.random.choice([0, 1], size=n_trials)x_observed = np.where(choices == 0, alpha, k*
alpha)U = np.random.uniform(0, 1, size = n_trials)decision_keep = U <= np.vectorize(g)(x_observed)gain =
np.where(decision_keep, x_observed, np.where(choices == 0, k*alpha, alpha))returnnp.mean(gain)
def simulate_naive(alpha = 1.0,k = 2.0,n_trials = 10*5,random_seed = None) : if random_seed is not None :
np.random.seed(random_seed)
choices = np.random.choice([0, 1], size=n_trials)x_observed = np.where(choices == 0, alpha, k*
alpha)returnnp.mean(x_observed)
def theoretical_gain(g,alpha = 1.0,k = 2.0) : return((k + 1)/2) * alpha + ((k - 1)/2) *
alpha * (g(k * alpha) - g(alpha))
alpha = 10 k = 30 n_trials = 10 * 5
g_functions = "Power - law(t = 1, c = 1)" : lambda x : g_power(x, t = 1.0, c = 1.0), "Exponential(t = 1)" : la
print("Two-Envelopes Game Simulation") print(f"Parameters: alpha = alpha, k = k, trials
= n_trials")
naive_gain = simulate_naive(alpha, k, n_trials, random_seed = 42)print(f"Naive Strategy (Always Keep) Expe
naive_gain : .6f")
for label, g_func in g_functions.items() : theo = theoretical_gain(g_func, alpha, k)exp_gain =
simulate_game(g_func, alpha, k, n_trials, random_seed = 42)print(f"Strategy : {label}")print(f"Theoretical Expect
theo : .6f")print(f"Experimental Average Gain : exp_gain : .6f")print(f"Improvement over naive strategy :
theo - naive_gain : .6f") The code produces the following output:
Two-Envelopes Game Simulation
Parameters: alpha = 10, k = 30, trials = 100000
Naive Strategy (Always Keep) Expected Gain: 155.182700
Strategy: Power-law (t=1, c=1) Theoretical Expected Gain: 167.700091 Experimental
Average Gain: 167.554100 Improvement over naive strategy: 12.517391
Strategy: Exponential (t=1) Theoretical Expected Gain: 155.000000 Experimental Average
Gain: 155.182700 Improvement over naive strategy: -0.182700
Strategy: Logarithmic (t=1, c=1) Theoretical Expected Gain: 176.054627 Experimental
Average Gain: 175.888700 Improvement over naive strategy: 20.871927
```

## Problem A-3 (IMC 2018)

Determine all rational numbers  $a$  for which the matrix

$$A = \begin{bmatrix} a & a & 1 & 0 \\ -a & -a & 0 & 1 \\ -1 & 0 & a & a \\ 0 & -1 & -a & -a \end{bmatrix}$$

is the square of a matrix with all rational entries.

### Solution:

We will show that the only such number is  $a = 0$ .

Let  $A$  be as given above, and suppose that  $A = B^2$  for some matrix  $B$  with rational entries. It is easy to compute the characteristic polynomial of  $A$ , which is

$$P_{\text{char},A}(X) := \det_{\mathbb{Q}(X)}(XI_4 - A) = (X^2 + 1)^2.$$

By the Cayley-Hamilton theorem, we have  $P_{\text{char},A}(B^2) = P_{\text{char},A}(A) = 0$ . Since  $P_{\text{char},A}(X^2) \in \mathbb{Q}[X]$  annihilates  $B$ , we can take the unique (non-zero) monic minimal polynomial that annihilates  $B$ ,  $P_{\text{min},B}(X) \in \mathbb{Q}[X]$ . The minimal polynomial is irreducible over  $\mathbb{Q}[X]$  and divide all polynomials with rational coefficient that vanish at  $B$ ; in particular,  $P_{\text{min},B}(X)$  must be a divisor of the polynomial  $P_{\text{char},A}(X^2) = (X^4 + 1)^2$ . So  $P_{\text{min},B}(X)$  must be a divisor of the polynomial  $X^4 + 1$ . However  $X^4 + 1$  is the 8-th cyclotomic polynomial since:

$$\Phi_8(X) = \Phi_{2^3}(X) = \Phi_2(X^{2^{3-1}}) = X^4 + 1,$$

and therefore is irreducible over  $\mathbb{Q}[X]$ . Hence  $P_{\text{min},B}(X) = X^4 + 1$ . Therefore,

$$A^2 + I = P_{\text{min},B}(B) = 0.$$

Since we have

$$A^2 + I = \begin{bmatrix} 0 & 0 & 2a & 2a \\ 0 & 0 & -2a & -2a \\ -2a & -2a & 0 & 0 \\ 2a & 2a & 0 & 0 \end{bmatrix},$$

the equation  $A^2 + I = 0$  forces  $a = 0$ .

In case  $a = 0$ , we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}^2,$$

hence  $a = 0$  satisfies the condition.

## Problem A-4 (IMC 2005)

Find all polynomials of degree  $n \geq 1$

$$P(X) = \sum_{i=0}^n a_i X^i = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \quad (a_n \neq 0)$$

satisfying the following two conditions:

1.  $\{a_i \mid i \in \llbracket 0, n \rrbracket\} = \llbracket 0, n \rrbracket$  and
2. all roots of  $P(X)$  are rational numbers.

### Solution:

Let  $n \geq 1$ . If  $n = 1$ , then trivially the only such polynomial is  $X$ . Else,  $n \geq 2$ , and let  $P(X)$  be a polynomial of degree  $n$  satisfying the conditions. Note that  $P(X)$  does not have any positive root because  $P(x) > 0$  for every  $x > 0$ . Thus, we can represent the roots as  $-\alpha_i$  for  $i = 0, 1, \dots, n-1$ , where  $\{\alpha_i \mid i \in n\} \subset \mathbb{Q}_+$ .

If  $a_0 \neq 0$ , then the roots are strictly positive  $\{\alpha_i \mid i \in n\} \subset \mathbb{Q}_{>0}$ , and condition 1 gives that there exists some  $k \in \mathbb{N}$  with  $1 \leq k \leq n-1$  such that  $a_k = 0$ . Using Vieta's formulas for this coefficient  $a_k$ , we obtain:

$$\sum_{\substack{S \subset n \\ |S|=n-k}} \left( \prod_{i \in S} \alpha_i \right) = \frac{a_k}{a_n} = 0,$$

which is impossible since the left-hand side is strictly positive (as  $k < n$ ). Therefore,  $a_0 = 0$ , and one of the roots of  $P(X)$ , say (without loss of generality)  $\alpha_{n-1}$ , must be zero.

Consider the polynomial

$$Q(X) = \sum_{i=1}^n a_i X^{i-1} = a_n X^{n-1} + a_{n-1} X^{n-2} + \cdots + a_1.$$

One has  $P(X) = XQ(X)$ , so it has the zeros  $-\alpha_i$  for  $0 \leq i \leq n-2$ . Again, using Vieta's formulas for the coefficients  $\frac{a_1}{a_n}$ ,  $\frac{a_2}{a_n}$  and  $\frac{a_{n-1}}{a_n}$  (recall  $n \geq 2$ ), we get:

$$\prod_{0 \leq i < n-1} \alpha_i = \frac{a_1}{a_n}, \quad \sum_{0 \leq i < n-1} \left( \prod_{\substack{0 \leq j < n-1 \\ j \neq i}} \alpha_j \right) = \frac{a_2}{a_n}, \quad \sum_{0 \leq i < n-1} \alpha_i = \frac{a_{n-1}}{a_n}.$$

Dividing the second equation by the first, we obtain

$$\frac{\sum_{0 \leq i < n-1} \left( \prod_{\substack{0 \leq j < n-1 \\ j \neq i}} \alpha_j \right)}{\prod_{0 \leq i < n-1} \alpha_i} = \sum_{0 \leq i < n-1} \frac{1}{\alpha_i} = \frac{a_2}{a_1}.$$

Applying the AM-HM inequality (which is a corollary of the very useful-to-know **Generalized Mean Inequality**):

$$\frac{a_{n-1}}{(n-1)a_n} = \frac{\sum_{0 \leq i < n-1} \alpha_i}{n-1} \geq \frac{n-1}{\sum_{0 \leq i < n-1} \frac{1}{\alpha_i}} = \frac{(n-1)a_1}{a_2}.$$

Rearranging, we find

$$\frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2,$$

and since  $1 \leq a_2, a_{n-1} \leq n$  we have  $a_2 a_{n-1} \leq n^2$  (equality happens only when  $a_2 = a_{n-1} = n$ , which can furthermore only happen when  $n = 3$  since the  $a_i$  are all distinct). Since  $n \geq 2$ , we have  $1 \leq a_1 \neq a_n \leq n$ , so either  $1 \leq a_1 < a_n \leq n$  or  $1 \leq a_n < a_1 \leq n$ , and we have  $a_1 a_n \geq 2$  (equality happens only when  $\{a_1, a_n\} = \{1, 2\}$ ). In total:

$$\frac{n^2}{2} \geq \frac{a_2 a_{n-1}}{a_1 a_n} \geq (n-1)^2 \implies n^2 + 2 \leq 4n \implies \left(n - (2 + \sqrt{2})\right) \left(n + (-2 + \sqrt{2})\right) \leq 0,$$

which holds only for integers  $1 \leq n \leq 3$ . Since  $n \geq 2$ , the only possibility is  $n \in \{2, 3\}$ .

Summarizing, the only possible polynomials  $Q(X) \in \mathbb{Z}[X]$  satisfying the given conditions are  $X$ , and otherwise they have degree 2 or 3 with a constant term 0 and negative roots.

These polynomials can then be explicitly found by brute force (finitely many possibilities), and they form exactly the set:

$$\left\{ X, \quad X^2 + 2X, \quad 2X^2 + X, \quad X^3 + 3X^2 + 2X, \quad 2X^3 + 3X^2 + X \right\}.$$



## Problem A-6 (IMC 2005)

Let  $m, n \in \mathbb{Z}$ . Given a group  $G$ , denote by  $G(m)$  the subgroup generated by the  $m$ -th powers of elements of  $G$ :

$$G(m) := \langle \{g^m \mid g \in G\} \rangle \leq G.$$

If  $G(m)$  and  $G(n)$  are commutative, prove that  $G(\gcd(m, n))$  is also commutative. Here,  $\gcd(m, n)$  denotes the greatest common divisor of  $m$  and  $n$ .

### Solution:

If  $m = 0$  or  $n = 0$ , this is trivial. Suppose now  $|m|, |n| \geq 1$ .

Recall that if  $H$  is a group and  $H' \subset H$  is a subset, then the subgroup of  $H$  generated by  $H'$  is the smallest subgroup of  $H$  containing  $H'$ , that is,  $\langle H' \rangle = \bigcap_{H' \subset F \leq H} F$ . It is easy to see that:

$$\langle H' \rangle = \left\{ \prod_{i \in k} g_i^{\epsilon_i} \mid \exists k \in \mathbb{N} \exists g \in (H')^k \exists \epsilon \in \{-1, 1\}^k \right\}, \quad (1)$$

where the product is taken in the order of the integer  $k = \{i \in \mathbb{N} \mid i < k\}$  and the empty product is  $e_H$ .

Write  $d = \gcd(m, n)$ . Notice that:

$$G(d) = \langle G(m) \cup G(n) \rangle.$$

Indeed, this follows from the monotonicity of  $\langle \_ \rangle$  and the fact that every subgroup is a fixed point of  $\langle \_ \rangle$ . If  $z \in \{g^d \mid g \in G\}$ , then  $z = g^d$  for some  $g \in G$ . By Bézout's lemma, there exist two integers  $l, r \in \mathbb{Z}$  with  $lm + rn = d$ , and we have:

$$z = g^d = g^{lm+rn} = \left( (g^m)^{\text{sign}(l)} \right)^{|l|} \left( (g^n)^{\text{sign}(r)} \right)^{|r|}.$$

Because  $g^m \in G(m)$  and  $g^n \in G(n)$ , we have  $z \in \langle G(m) \cup G(n) \rangle$ . Since  $z$  was arbitrary, we get  $\{g^d \mid g \in G\} \subset \langle G(m) \cup G(n) \rangle$ , and thus:

$$G(d) = \langle \{g^d \mid g \in G\} \rangle \subset \langle G(m) \cup G(n) \rangle.$$

Similarly, let  $z \in \{g^m \mid g \in G\}$ . Then  $z = g^m$  for some  $g \in G$ , and thus:

$$z = \left( g^d \right)^{\frac{m}{d}} \in G(d),$$

since  $g^d \in G(d)$ . As  $z$  was arbitrary, we conclude  $\{g^m \mid g \in G\} \subset G(d)$ . Thus,  $G(m) \subset G(d)$ . In the exact same manner,  $G(n) \subset G(d)$ , and therefore:

$$\langle G(m) \cup G(n) \rangle \subset G(d).$$

We also have:

$$\langle G(m) \cup G(n) \rangle = \langle \{g^m \mid g \in G\} \cup \{h^n \mid h \in G\} \rangle^3.$$

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<sup>3</sup>If  $H$  is a group,  $I$  a set, and  $\{H_i \mid i \in I\}$  are subgroups of  $H$  each generated by  $\{S_i \mid i \in I\} \subset \mathcal{P}(H)$  respectively ( $\forall i \in I, H_i = \langle S_i \rangle$ ), then  $\langle \bigcup_{i \in I} H_i \rangle = \langle \bigcup_{i \in I} S_i \rangle$ . Indeed, we clearly have  $\bigcup_{i \in I} S_i \subset \bigcup_{i \in I} H_i$  (since  $S_i \subset \langle S_i \rangle$ ) and thus  $\langle \bigcup_{i \in I} S_i \rangle \subset \langle \bigcup_{i \in I} H_i \rangle$ . If we take an element  $z \in \bigcup_{i \in I} H_i$ , then there exists  $j \in I$  with  $z \in H_j = \langle S_j \rangle \subset \langle \bigcup_{i \in I} S_i \rangle$ , and so we have  $\bigcup_{i \in I} H_i \subset \langle \bigcup_{i \in I} S_i \rangle$ , which means that we have the other inclusion  $\langle \bigcup_{i \in I} H_i \rangle \subset \langle \bigcup_{i \in I} S_i \rangle$ .

It is also clear regarding equation (1) that if  $S \subset G$  is constituted of elements that commute with one another, then  $\langle S \rangle$  is a commutative subgroup (this follows from the fact that if  $a, b \in S$  commute, then any two elements in  $\{a^{-1}, b^{-1}, a, b\}$  commute). The converse of this statement is trivial. Therefore, by the two equalities above, showing commutativity of  $G(d)$  is equivalent to showing commutativity of any two elements in  $\{g^m \mid g \in G\} \cup \{h^n \mid h \in G\}$ . Because we know that any two elements in  $\{g^m \mid g \in G\}$  or  $\{h^n \mid h \in G\}$  commute (since  $G(m)$  and  $G(n)$  are commutative), we only need to show that any element in  $\{g^m \mid g \in G\}$  commutes with any other element in  $\{h^n \mid h \in G\}$ . So, without further ado, let any two generators  $a^m$  and  $b^n$  ( $a, b \in G$ ). Showing commutativity of these two elements is equivalent to showing neutrality of their commutator:

$$z := [a^m, b^n] = a^{-m}b^{-n}a^mb^n.$$

Then the relations

$$z = (a^{-m}ba^m)^{-n}b^n = a^{-m}(b^{-n}ab^n)^m$$

show that  $z \in G(m) \cap G(n)$ . But then  $z$  is in the center of  $G(d)$ . Indeed to show  $z \in Z(G(d))$ , we let  $x \in G(d) = \langle \{g^m \mid g \in G\} \cup \{h^n \mid h \in G\} \rangle$ , then there exists a natural number  $k$  and a sequence of length  $2k$  of element of the group  $\mathbf{g} \in G^{2k}$  with:

$$x = g_0^m g_1^n \cdots g_{2k-2}^m g_{2k-1}^n,$$

and as  $z \in G(m) \cap G(n)$ ,  $z$  commutes with any element in  $\{g^m \mid g \in G\} \cup \{h^n \mid h \in G\}$  so we have by induction  $zx = xz$ . Now, from the relation  $a^mb^n = b^na^mz$ , it easily follows by induction that for any integer  $l \geq 0$ , we have:

$$a^{ml}b^{nl} = b^{nl}a^{ml}z^{l^2}.$$

Indeed, for  $l = 0$ , we have:

$$a^{ml}b^{nl} = a^0b^0 = e_G = b^0a^0z^0 = b^{nl}a^{ml}z^{l^2},$$

and for  $l = 1$ , we have:

$$a^{ml}b^{nl} = a^mb^n = b^na^mz = b^{nl}a^{ml}z^{l^2},$$

where we used the above relation. Suppose this holds for  $l \geq 1$ . We show this holds for  $l + 1$ . We have:

$$a^{m(l+1)}b^{n(l+1)} = a^m(a^{ml}b^{nl})b^n = a^mb^{nl}a^{ml}z^{l^2}b^n = a^mb^{nl}a^{ml}b^n z^{l^2},$$

where we use in the second equality the induction hypothesis and in the third the fact that  $z$  is in the center and hence  $z^{l^2}$  as well. Now:

$$a^mb^{nl} = b^na^mz b^{n(l-1)} = b^n(a^mb^{n(l-1)})z = b^n(a^mb^{n(l-1)})z^{l-1 \text{ times}} = b^{nl}a^mz^l,$$

where we do an induction on  $l$  by iteratively using the known relation and the fact that  $z$  is in the center. Thus:

$$a^{m(l+1)}b^{n(l+1)} = a^mb^{nl}a^{ml}b^n z^{l^2} = b^{nl}a^{m(l+1)}b^n z^{l^2+l},$$

where we used the two last results and the fact that  $z^l$  is in the center. Similarly:

$$a^{m(l+1)}b^n = (a^{ml}b^n)a^mz = a^{ml}b^n a^m z^{l \text{ times}} = b^n a^{m(l+1)}z^{l+1},$$

where we do an induction on  $l + 1$  by iteratively using the known relation and the fact that  $z$  is in the center. Thus:

$$a^{m(l+1)}b^{n(l+1)} = b^{nl}a^{m(l+1)}b^n z^{l^2+l} = b^{n(l+1)}a^{m(l+1)}z^{l^2+2l+1} = b^{n(l+1)}a^{m(l+1)}z^{(l+1)^2},$$

where we used the two last results and  $l^2 + 2l + 1 = (l + 1)^2$ . This concludes the induction and proves the statement.

In particular, for any integer  $l \geq 0$ , we have:

$$z^{l^2} = a^{-ml} b^{-nl} a^{ml} b^{nl} = [a^{ml}, b^{nl}].$$

Setting the two integers  $k := \frac{m}{d}$  and  $k' := \frac{n}{d}$ , since  $nk = mk'$ , we obtain that  $a^{mk} = (a^k)^m$  and  $b^{nk} = (b^{k'})^m$  i.e.  $a^{mk}, b^{nk} \in G(m)$ , and thus they commute by hypothesis, which means:

$$z^{k^2} = [a^{mk}, b^{nk}] = e_G.$$

Similarly,  $a^{mk'} = (a^k)^n$  and  $b^{nk'} = (b^{k'})^n$  i.e.  $a^{mk'}, b^{nk'} \in G(n)$ , and thus they commute by hypothesis, which means:

$$z^{k'^2} = [a^{mk'}, b^{nk'}] = e_G.$$

So  $z^{(\frac{m}{d})^2} = e_G = z^{(\frac{n}{d})^2}$ . Clearly,  $\gcd(k, k') = 1$ , and thus  $\gcd(k^2, k'^2) = 1$ , so by Bézout's lemma, there exist two integers  $s, t \in \mathbb{Z}$  with  $sk^2 + tk'^2 = 1$ . Hence:

$$e_G = (e_G)^s (e_G)^t = \left(z^{k^2}\right)^s \left(z^{k'^2}\right)^t = z^{sk^2 + tk'^2} = z^1 = [a^m, b^n].$$

Since  $a, b \in G$  were arbitrary, we conclude that  $G(d)$  is commutative, as required.