

# Problem Set Week 3

ETHZ Math Olympiad Club

3 March 2025

## 1 Problem 1 (Pan African 2018)

Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that

$$(f(x+y))^2 = f(x^2) + f(y^2)$$

for all  $x, y \in \mathbb{Z}$ .

## 2 Problem B-2 (IMC 2012)

Define the sequence  $a_0, a_1, \dots$  inductively by  $a_0 = 1$ ,  $a_1 = \frac{1}{2}$  and

$$a_{n+1} = \frac{na_n^2}{1 + (n+1)a_n} \quad \text{for } n \geq 1.$$

Show that the series

$$\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$$

converges and determine its value.

## 3 Problem 5 (Pan African 2018)

Let  $a, b, c$  and  $d$  be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \quad \text{and} \quad ac = bd.$$

Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leq -12$$

and that  $-12$  is the maximum.

## 4 Problem 3 (Silk Road 2019)

Find all pairs  $(a, n)$  of positive natural numbers such that  $\varphi(a^n + n) = 2^n$ . ( $\varphi(n)$  is the Euler function, that is, the number of integers from 1 up to  $n$ , relatively prime to  $n$ )