Exercise session 21/10/2016 | Simulation

Exercise 1.
Use the inversion method to generate samples x from the random variable X with density

$$f(x) = \begin{cases} 3x^2 & \text{for } 0 \leq x \leq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.
The *von Mises distribution* has density

$$f(x) = \frac{e^{\kappa \cos(x)}}{2\pi I_0(\kappa)}$$

for $-\pi \leq x \leq \pi$ (and is zero otherwise), where $\kappa > 0$ and $I_0(\kappa)$ is the modified Bessel function of order 0.

(1) Show that $f(x)$ is maximal in $[-\pi, \pi]$ for $x = 0$.

(2) Use the acceptance-rejection method with uniform prior to generate von Mises distributed random variables.

(3) Suggest alternative priors, what does a good prior look like?

Exercise 3. *Exam 2015–2016*
The *Kumaraswamy distribution* and density function are given by,

$$F_K(x) = 1 - (1 - x^a)^b\,, \quad \text{and} \quad f_K(x) = abx^{a-1}(1 - x^a)^{b-1}\,,$$

respectively for $0 \leq x \leq 1$, where a and b are strictly positive real-valued parameters. Note the similarity with the density function of the beta distribution,

$$f_B(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1}\,,$$

defined for $0 \leq x \leq 1$, where α and β are strictly positive real-valued parameters and $B(\alpha, \beta)$ is a known constant.

(1) Show that Kumaraswamy random variables can be generated by the inversion method.

(2) Show that a Kumaraswamy random variable is the ath root of a beta distributed random variable with $\alpha = 1$ and $\beta = b$.

(3) Use the former observation to create a random number generator for beta distributed random variables with $\alpha = 1$.

Exercise 4.
Explain how antithetic variables can be used in obtaining an estimate of the quantity

$$J = \int_0^1 \int_0^1 e^{(x+y)^2} \mathrm{d}y \mathrm{d}x.$$

Is it clear in this case that using antithetic variables is more efficient than generating a new pair of random numbers?

Exercise 5.
If Z is a standard normal random variable, design a study using antithetic variables to estimate $J = \mathrm{E}[Z^3 e^Z]$.

Exercise 6.
Suppose that X is an exponential random variable with mean 1. Give another random variable that is negatively correlated with X and that is also exponential with mean 1.

Exercise 7.

Let X and Y be independent real-valued random variables with respective distributions F and G and with expected values $\mu_X \in \mathbb{R}$ and $\mu_Y \in \mathbb{R}$. For a given value $t \in \mathbb{R}$, we want to determine

$$J = \Pr[X + Y \leq t].$$

(1) Give the raw simulation approach to estimating J.

(2) Use conditioning to obtain an improved estimator.

(3) Give a control variable that can be used to further improve upon the estimator in (b).