E3.1

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Exercise session 24/2 | Birth-death queues

Preparatory exercise.
The analysis of many birth-death queueing systems relies on mathematical series. In particular, given the exponential series,

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

prove the following identities,

$$\sum_{k=0}^{\infty} k\frac{z^k}{k!} = ze^z, \qquad \sum_{k=0}^{\infty} k^2\frac{z^k}{k!} = (z+z^2)e^z.$$

Prove also the following identities,

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}, \qquad \sum_{k=1}^{n} kz^k = z\frac{1-(n+1)z^n+nz^{n+1}}{(1-z)^2},$$

and determine the region of convergence for the corresponding infinite series.

Exercise 1.
We define a threshold queue with parameter T. When the number of customers in the system is greater than or equal to $T \in \mathbb{N}$, customers arrive according to a Poisson process with rate $\lambda \in \mathbb{R}_{>0}$ and their service time is exponentially distributed with rate $\mu \in \mathbb{R}_{>0}$. However, when the number of customers is lower than T, customers arrive according to a Poisson process with rate μ and are served with exponential rate λ (i.e., the system is running in overload).

(a) Draw the state transition diagram of the Markov chain.

(b) For which λ, μ and T is the Markov chain ergodic?

(c) Calculate the mean number of customers in the system.

Exercise 2.
Consider a variant of the $M/M/2$ queue where the service rates of the two processors are not identical. Denote the service rate of the first processor by μ_1 and the service rate of the second processor by μ_2, where $\mu_1, \mu_2 \in \mathbb{R}_{>0}$ and $\mu_1 \geq \mu_2$. In the case of heterogeneous servers, the rule is that when both servers are idle, the faster server is scheduled for service before the slower one.

(a) Draw the state transition diagram of the Markov chain.

(b) For which λ, μ_1 and μ_2 is the Markov chain ergodic?

(c) Calculate the mean number of customers in the system.

(d) Would it sometimes make sense to get rid of the slow server altogether?

Exercise 3.
Assume an $M/M/1$ queue with feedback. Customers arrive according to a Poisson process with rate $\lambda \in \mathbb{R}_{>0}$, service times are exponentially distributed with parameter $\mu \in \mathbb{R}_{>0}$, and served customers are fed back to the back of the queue with probability $\alpha \in [0, 1)$ (this may happen multiple times for the same customer) or depart the system with probability $1 - \alpha$.

(a) Draw the state transition diagram of the Markov chain.

(b) For which λ, μ and α is the Markov chain ergodic?

(c) Calculate the mean number of customers in the system.

Exercise 4.

In the early days of computing, interaction with computer systems was by means of batch jobs which where executed one at a time. Consider a fixed number of users $N \in \mathbb{N}$ that are using the computer. The users repeatedly send requests to the computer which are served first-come-first-served, the computation time being exponentially distributed with rate $\mu \in \mathbb{R}_{>0}$. Upon completion of a job, the results are analysed by the user and a new job is prepared. Assume that this takes an exponentially distributed time with rate $\alpha \in \mathbb{R}_{>0}$.

(a) Draw the state transition diagram of the Markov chain.

(b) Calculate the probability mass function of the number of jobs in system. $= p(n)$

(c) Calculate the probability mass function of the number of users that are analysing a result or are preparing a new job.

Exercise 5. → buffer size

Assume the $M/M/1/5$ queue with load ρ.

(a) Assume $\rho = 0.4$. Which has a greater effect on lowering the loss probability: doubling the service rate or doubling the buffer capacity?

(b) Answer (a) for $\rho = 0.8$.

(c) Can you explain the answers intuitively?

Exercise 6.

Assume the $M/M/2/3$ queue with equal arrival and service rates.

(a) Draw the state transition diagram of the Markov chain.

(b) Suppose there are exactly two customers in system. What is the probability that a customer arrives before one departs?

(c) Calculate the probability mass function of the system content.

(d) Calculate the mean number of customers in system.

(e) Calculate the loss probability.

(f) Define the throughput as the mean number of served customers per time unit. Calculate this throughput.

(g) Calculate the distribution of the number of customers in system at moments that an arrival *effectively* enters the system.

(h) Consider the process of effective arrivals to the system. Is this a Poisson process? Why or why not?