ES.1

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Exercise session 11/3 | QBDs

Exercise 1.
The Erlang Distribution

Given: a random variable T_N that is Erlang distributed with N phases and parameter λ_N.

Questions:

(a) What should λ_N be so that $E[T_N]$ is equal to λ^{-1} (independently of N)?

(b) Assume the λ_N calculated in the first question. What is the variance of T_N for $N \to \infty$?

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(c) Assume the λ_N calculated in the first question. What is the squared coefficient of variation $C^2_{T_N} = \text{var}[T_N]/\,E[T_N]^2$?

Exercise 2.
The Degenerate Hyperexponential Distribution

Given: a random variable $T(p)$ that has a hyperexponential distribution with two phases, where one phase is indentically 0, i.e., $T(p)$ is exponentially distributed with parameter $\lambda(p)$ with probability p and is 0 with probability $1-p$.

Questions:

(a) What should $\lambda(p)$ be so that $E[T(p)] = \lambda^{-1}$ (independent of p)?

(b) Assume the $\lambda(p)$ calculated in the first question. What is the squared coefficient of variation $C^2_{T(p)} = \text{var}[T(p)]/\,E[T(p)]^2$?

(c) Assume the $\lambda(p)$ calculated in the first question. What values of the coefficient of variation can be obtained with this degenerate hyperexponential distribution?

Exercise 3.
Switching off the server

Given: Consider an $M/M/1$ queue with an arrival rate λ and service rate μ, and where the server immediately shuts off when it is idle (e.g., to save power). When a job arrives and finds the server off there is a setup time, I, required to get the server back on. In this problem we assume I to be exponentially distributed with parameter α.

Questions:

(a) Draw the state transition diagram of the QBD.

(b) For which λ, μ and α is the Markov chain ergodic?

(c) Calculate the R-matrix for this QBD.

(d) Calculate the distribution of the number of customers in the system.

(e) Compare the result with the system without setup times.

(f) Say that a server that is on costs C_{on} euros per time unit, a server that is turned off costs nothing and each time unit of delay per customer costs C_d euros. For which values of α is turning off the server when it is idle more optimal than leaving it on?

Exercise 4.

The PH/PH/1 queue

Given: Consider the PH/PH/1 queueing system. This queueing system has infinite capacity and a single server. The interarrival times have the phase-type distribution $PH(G^A, \boldsymbol{\alpha}^A)$ whereas the service times have the phase-type distribution $PH(G^S, \boldsymbol{\alpha}^S)$.

Questions:

(a) What is the state space of this Markov chain?

(b) Find the set of balance equations for the queueing system at hand

(c) Show the transition diagram when both service and interarrival times are Erlang distributed with two stages.

(d) Find the forward, local and backward matrices for the QBD of 3.

Exercise 5.

Two servers

Given: Consider a queueing system with an infinite capacity buffer Poisson arrivals and two non-identical servers. Service in the fast server is exponentially distributed with rate μ_f whereas service in the slow server is exponentially distributed with rate μ_s. Once a customer starts being served, it stays with the server till its service completes. If both servers are available upon arrival of a customer, the customer choses the fast server.

Questions:

(a) Define the state space of the queueing system and depict the transition diagram.

(b) What is the stability condition of this queueing system

(c) Show that this queueing system fits the QBD framework

(d) Calculate the stationary distribution