Solutions exercises Queuing Theory and Simulation

Go to `https://github.com/Anton-4/queuing-exercises` **to get the most recent version of this document**

I found that the solutions written down in class often missed steps, important details and explanations, which is why I have created this document with detailed, annotated and complete solutions for most exercises.

If certain solutions are unclear or new exercises are given please create issues or contribute to the github repository so that we can maintain this document over the years.

This document was based on notes by ir. Mathias De Brouwer and ir. Anton Van Moere, LaTeXcreated by Anton Van Moere. Help in understanding and expanding solutions was provided by ir. Alexander Erreygers and dr. Sarah Dendievel. Course was given by Professor Dieter Fiems and Professor. ir. Joris Walraevens.

1 Probability refresher

1.

used formulas

mean discrete function $= E[g(X)] = \sum_{i=1}^{n} g(i) \cdot Pr[X = i]$

solution

N = number of students in avg class, with N a random variable
mean $= E[N] = 20$

for our example, a class has either 2 or 100 students $N - \mathcal{N} = \{2, 100\}$

$$\begin{aligned} E[N] &= 2 \cdot Pr(N = 2) + 100 \cdot Pr(N = 100) \\ &= 2 \cdot Pr(N = 2) + 100 \cdot (1 - Pr(N = 2)) \\ &= 100 - 98 \cdot Pr(N = 2) \end{aligned}$$

$\Rightarrow Pr(N = 2) = \frac{80}{98}, Pr(N = 100) = \frac{18}{98}$

$\widetilde{N}$ = nr students in class of randomly selected student

$Pr(\widetilde{N} = 100) = \frac{100 \cdot Pr[N=100]}{\sum_i i \cdot Pr[N=i]} = \frac{18 \cdot 100}{18 \cdot 100 + 80 \cdot 2} = \frac{1800}{1960} = 0.918$

$\Rightarrow$ Dean is right

2.

used formulas

$$
\begin{aligned}
E[a \cdot X] &= a \cdot E[X] \\
Var[X] &= E[(X - E[X])^2] = E[X^2] - E[X]^2 \\
E[X + Y] &= E[X] + E[Y] \\
V, W : Var[V + W] &= Var[V] + Var[W], \text{ if independent}
\end{aligned}
$$

solution

capital $c \in \mathbb{N}$ interest first strategy: $y_1 = c \cdot X$

$$
\begin{aligned}
E[y_1] &= E[c \cdot X] \\
&= c \cdot E[X]
\end{aligned}
$$

$$
\begin{aligned}
Var[y_1] &= Var[c \cdot X] \\
&= E[(cX - E[cX])^2] \\
&= E[(c \cdot X)^2] - (E[c \cdot X])^2 \\
&= c^2 E[X^2] - (c \cdot E[X])^2 \\
&= c^2 Var[X]
\end{aligned}
$$

interest second strategy: $y_2 = \sum_{i=1}^{c} X_i$

$$
\begin{aligned}
E[y_2] &= E[\sum_{i=1}^{c} X_i] \\
&= \sum_{i=1}^{c} E[X_i] \\
&= c \cdot E[X]
\end{aligned}
$$

$$
\begin{aligned}
Var[y_2] &= Var[\sum_{i=1}^{c} X_i] \\
&= \sum_{i=1}^{c} Var[X_i] \\
&= c \cdot Var[X]
\end{aligned}
$$

$\Rightarrow$ the second strategy is less risky because of lower variance

3.

used formulas

$$E[g(X)] = \sum_{i=1}^{n} g(i) \cdot Pr[X = i]$$
$$E[X \cdot Y] = E[X] \cdot E[Y]$$

solution

to prove or disprove: $E\left[\frac{A}{B}\right] = \frac{E[A]}{E[B]}$

$$\begin{aligned} A &: A \subset \mathbb{Z} \\ B &: B \subset \mathbb{Z}\backslash\{0\} \\ E\left[B\right] &\neq 0 \end{aligned}$$

$$\begin{aligned} E\left[\frac{A}{B}\right] &= \sum_{a\in A}\sum_{b\in B}\frac{a}{b}Pr\left[A=a, B=b\right] \\ &= \sum_{a\in A}\sum_{b\in B}\frac{a}{b}Pr\left[A=a\right]\cdot Pr\left[B=b\right] && \text{- joint of independent vars -} \end{aligned}$$

$$\frac{E\left[A\right]}{E\left[B\right]} = \frac{\sum_{a\in A} a\cdot Pr\left[A=a\right]}{\sum_{b\in B} b\cdot Pr\left[B=b\right]}$$

the above equations are clearly not equal

4.

used formulas

Properties exponential distribution:

$$\begin{aligned} \text{distribution function} &= F\left(x\right) = 1 - \exp\left(-\lambda\cdot x\right) \\ \text{density function} &= f\left(x\right) = \frac{d}{dx}F\left(x\right) = \lambda\cdot\exp\left(-\lambda\cdot x\right) \\ \text{mean} &= E\left[X\right] = \int_0^{+\infty} x\cdot f\left(x\right)dx = \int_0^{\infty} x\cdot\lambda\cdot\exp\left(-\lambda\cdot x\right) = \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} F\prime\left(t\right) &= \frac{F\left(t+\delta\right) - F\left(t\right)}{\delta} \\ Pr\left[A|B\right] &= \frac{Pr\left[B|A\right]\cdot Pr\left[A\right]}{Pr\left[B\right]} \\ &= \frac{Pr\left[A\cap B\right]}{Pr\left[B\right]} \\ F(t) &= Pr[X\leq t] \end{aligned}$$

property mean r.v.

$Y =$ positive continuous random var $\to \forall y < 0 | f_Y(y) = 0$

$$E[Y] = \int_0^\infty y f_Y(y)\, dy$$

- mean continuous r.v. -integration by parts: $\int u \cdot dv = u \cdot v - \int v \cdot du$

$$u = y, du = dy, dv = f_y(y)dy, v = F_y(y) - 1$$

$$= y(F_Y(y) - 1)\Big|_0^\infty - \int_0^\infty (F_y(y) - 1$$

- integral of density is cdf -

$$= \int_0^\infty (1 - F_y(y))\, dy$$

solution

given:

distribution F(t) of the hard drive lifetime X is a mixture of exponentials:

$$\begin{aligned} F(t) &= Pr[X \le t] = p \cdot (1 - \exp(-\lambda_1 \cdot t)) + (1-p) \cdot (1 - \exp(-\lambda_2 \cdot t)) \\ &= 1 - p \cdot \exp(-\lambda_1 \cdot t) - (1-p) \cdot \exp(-\lambda_2 \cdot t) \end{aligned}$$

The first exponential represents probability of failure during production, the second is prob of failure during normal use.

$$\begin{aligned} p &= 0.1 \\ \lambda_1 &= 10 \\ \lambda_2 &= 1 \end{aligned}$$

a)

failure rate $= \lambda(t) = \frac{F'(t)}{1-F(t)}$

$$\begin{aligned} \frac{F'(t)}{1-F(t)} &= \frac{1}{1-F(t)} \cdot F\prime(t) \\ &= \frac{1}{1-F(t)} \cdot \frac{F(t+\delta) - F(t)}{\delta} && \text{- formule afgeleide -} \\ &= \frac{1}{1 - Pr[T \le t]} \cdot \frac{F(t+\delta) - F(t)}{\delta} && \text{- } F(t) = Pr[T \le t] \text{ -} \\ &= \frac{1}{Pr[T > t]} \cdot \frac{Pr[T \le t+\delta] - Pr[T \le t]}{\delta} && \text{- } F(x) = Pr[T \le x] \text{ -} \\ &= \frac{Pr[t < T, T \le t+\delta]}{Pr[T > t] \cdot \delta} \\ &= \frac{Pr[T \le t+\delta | T > t]}{\delta} && \text{- bayes rule -} \end{aligned}$$

$\lambda(t) =$ probability it will fail at time $t + dt$ given it has not failed until time t

b)

remaining lifetime $= X_R = X - t$

$$\begin{aligned}
F_{X_R}(x|t) &= Pr\left[X_R \leq x | X > t\right] && \text{- property cdf -}\\
&= Pr\left[X - t \leq x | X > t\right] && \text{- } X_R = X - t \text{ -}\\
&= \frac{Pr\left[X \leq x + t, X > t\right]}{Pr\left[X > t\right]} && \text{- bayes rule -}\\
&= \frac{F(x+t) - F(t)}{1 - F(t)} && \text{- property cdf -}\\
&= \frac{F(x+t) + (1 - F(t)) - 1}{1 - F(t)}\\
&= 1 - \frac{1 - F(x+t)}{1 - F(t)}\\
&= 1 - \frac{p \cdot \exp\left(-\lambda_1 \cdot (t+x)\right) + (1-p) \cdot \exp\left(-\lambda_2 \cdot (t+x)\right)}{p \cdot \exp\left(-\lambda_1 \cdot t\right) + (1-p) \cdot \exp\left(-\lambda_2 \cdot t\right)} && \text{- fill in from given -}
\end{aligned}$$

c)

asked: mean of remaining life time =

$$\begin{aligned}
E\left[X_R | X > t\right] &= \int_0^\infty x \cdot dF_{X_R}(x|t)\\
&= \int_0^\infty \left(1 - F_{X_R}(x|t)\right) dx && \text{- see property mean r.v. -}\\
&= \dots\\
&= \frac{1}{\lambda_1 \cdot \lambda_2} \cdot \frac{p \cdot \lambda_2 \cdot \exp\left(-\lambda_1 \cdot t\right) + (1-p) \cdot \lambda_1 \cdot \exp\left(-\lambda_2 \cdot t\right)}{p \cdot \exp\left(-\lambda_1 \cdot t\right) + (1-p) \cdot \exp\left(-\lambda_2 \cdot t\right)}
\end{aligned}$$

lifetime increases when using it more (counterintuitive but true)

5.

used formulas

$$E\left[X\right] = E\left[E\left[X|Y\right]\right]$$

solution

S = file size

average file size = E[S] = 6K

a)

to prove: fewer than half of the files can have size >12K

$$\begin{aligned}
E\left[S\right] &= E\left[E\left[S|X\right]\right] && \text{- property conditional expectati}\\
&= E\left[S|S>12K\right]\cdot Pr\left[S>12K\right] + E\left[S|S\leq 12K\right]\cdot Pr\left[S\leq 12K\right] && \text{- def expectation -}\\
&= E\left[S|S>12K\right]\cdot Pr\left[S>12K\right] + E\left[S|S\leq 12K\right]\cdot\left(1-Pr\left[S>12K\right]\right)\\
6K &= \left(E\left[S\right]>12K\right)\cdot Pr\left[S>12K\right] + 0\cdot\left(1-Pr\left[S>12K\right]\right) && \text{- take lower bound -}
\end{aligned}$$

$\implies \frac{1}{2} > Pr\left[S>12K\right]$

b)

min file size is 3K, max amount of >12K files

$$\begin{aligned}
6 &> 12\cdot Pr\left[S>12K\right] + 3\cdot\left(1-Pr\left[S>12K\right]\right)\\
\frac{1}{3} &> Pr\left[S>12K\right]
\end{aligned}$$

6.

used formulas

$$Pr\left[X+Y\leq t\right] = \int_{-\infty}^{+\infty} Pr\left[X\leq t-y\right] dF_Y\left(y\right) \qquad \text{- see A.1.5 -}$$

solution

company pays fine if processing time exceeds 7 seconds

retrieving file takes time X exponentially distributed with mean 5

parsing file takes time Y uniformly distributed over [1, 3]

$T = X + Y > 7$

$$
\begin{aligned}
Pr\left[T > 7\right] &= 1 - Pr\left[T \leq 7\right] \\
&= 1 - F_T\left(7\right) && \text{- property cdf -}
\end{aligned}
$$

$$
\begin{aligned}
F_T\left(t\right) &= Pr\left[T \leq t\right] \\
&= Pr\left[X + Y \leq t\right] \\
&= \int_{-\infty}^{+\infty} Pr\left[X \leq t - y\right] dF_Y\left(y\right) && \text{- see formulas -}
\end{aligned}
$$

X is exponentially distributed, so based on table 1 at A.1.11:

$$
\begin{aligned}
F_X\left(x\right) &= 1 - \exp\left(-\lambda \cdot t\right) \\
E\left[X\right] &= 5 = \frac{1}{\lambda}
\end{aligned}
$$

Y is uniformly distributed, so based on table 1:

$$
F_y\left(y\right) = \begin{cases} 0 & y < 1 \\ \frac{1}{2}\left(y - 1\right) & 1 \leq y \leq 3 \\ 1 & y > 3 \end{cases}
$$

$$
f_Y\left(y\right) = \begin{cases} 0 & y < 1 \\ \frac{1}{2} & 1 \leq y \leq 3 \\ 0 & y > 3 \end{cases}
$$

We set boundaries to 1 and 3 because $f_Y(y) = 0$ everywhere else

$$
\begin{aligned}
F_T\left(t\right) &= \int_1^3 F_X\left(t - y\right) f_Y\left(y\right) dy \\
&= \int_1^3 (1 - e^{-\lambda \cdot (t-y)}) \frac{1}{2} dy \\
&= \frac{1}{2} \int_1^3 1 - e^{-\lambda \cdot (t-y)} dy \\
&= \frac{1}{2} \int_1^3 1 - \frac{1}{2} \int_1^3 e^{-\lambda \cdot (t-y)} dy \\
&= \frac{1}{2} \cdot 2 - \frac{1}{2} \int_1^3 e^{-\lambda t} \cdot e^{\lambda y} dy \\
&= 1 - \frac{1}{2} \int_1^3 e^{-\lambda t} \cdot e^{\lambda y} dy \\
&= 1 - \frac{1}{2} e^{-\lambda t} \int_1^3 e^{\lambda y} dy \\
&= 1 - \frac{1}{2} e^{-\lambda t} \left[-\frac{e^{\lambda y}}{\lambda}\right]_1^3 \\
&= 1 - \frac{e^{-\lambda t}}{2} \cdot \left(\frac{e^{-\lambda 3}}{\lambda} - \frac{e^{\lambda}}{\lambda}\right) \\
&= 1 - \frac{e^{-\lambda t}}{2 \cdot \lambda} \left(e^{\lambda 3} - e^{\lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
Pr\left[T>7\right] &= 1-Pr\left[T\leq 7\right]\\
&= 1-\frac{e^{-0.2\cdot 7}}{2\cdot 0.2}\left(e^{0.2\cdot 3}-e^{0.2}\right)\\
&= 0.3703
\end{aligned}
$$

7.

if $Pr\left[A|B\right]>Pr\left[A\right]$, prove that: $Pr\left[B|A\right]>Pr\left[B\right]$

$$
\begin{aligned}
&Pr\left[A\right]<Pr\left[A|B\right]\\
&Pr\left[A\right]<\frac{Pr\left[A\cap B\right]}{Pr\left[B\right]} && \text{- bayes theorem -}
\end{aligned}
$$

assume that $Pr\left[A\right]>0, Pr\left[B\right]>0$

$$
\begin{aligned}
&Pr\left[B\right]<\frac{Pr\left[A\cap B\right]}{Pr\left[A\right]}\\
&Pr\left[B\right]<Pr\left[B|A\right] && \text{- bayes theorem -}
\end{aligned}
$$

8.

used formulas

$$
\begin{aligned}
E\left[X|Y=y\right] &= \sum_{i=1}^{\infty} Pr[X=i|Y=y]\cdot i\\
\sum_{n=1}^{\infty} n\cdot z^n &= \frac{z}{(1-z)^2}\\
\sum_{n=1}^{\infty} n^2 z^n &= \frac{z\left(1+z\right)}{(1-z)^3}
\end{aligned}
$$

solution

95% good chips, 5% bad chips
good chips will fail with probability 0.0001 each day
bad chips will fail with probability 0.01 each day
time until chip fails = T
compute $E\left[T\right]$ and $var\left[T\right]$

state of random chip = $S=\{g,b\}$

$$
\begin{aligned}
Pr\left[S=g\right] &= 0.95\\
Pr\left[S=b\right] &= 0.05\\
Pr\left[fail|S=g\right] &= 0.0001=p_g\\
Pr\left[fail|S=b\right] &= 0.01=p_b
\end{aligned}
$$

$$E\left[T\right] = E\left[E\left[T|S\right]\right] = Pr\left[S=g\right]\cdot E\left[T|S=g\right] + Pr\left[S=b\right]\cdot E\left[T|S=b\right]$$

$$\begin{aligned}
Pr\left[T=1|S=g\right] &= 0.0001\\
Pr\left[T=2|S=g\right] &= \left(1-Pr\left[fail|S=g\right]\right)\cdot Pr\left[fail|S=g\right]\\
Pr\left[T=n|S=g\right] &= \left(1-Pr\left[fail|S=g\right]\right)^{(n-1)}\cdot Pr\left[fail|S=g\right]\\
&= \left(1-p_g\right)^{(n-1)}\cdot p_g
\end{aligned}$$

$$E\left[T|S=g\right] = \sum_{n=1}^{\infty}\left(1-p_g\right)^{(n-1)}\cdot p_g\cdot n \quad \text{multiplied by n because we want to sum time, not probabilities}$$

$$\begin{aligned}
E\left[T|S=g\right] &= \sum_{n=1}^{\infty}\left(1-p_g\right)^{n-1}\cdot p_g\cdot n\\
&= \frac{p_g}{1-p_g}\cdot\sum_{n=1}^{\infty} n\cdot\left(1-p_g\right)^{n} \text{ - prop geom series -}\\
&= \frac{p_g}{1-p_g}\cdot\frac{1-p_g}{p_g^2}\\
&= \frac{1}{p_g}
\end{aligned}$$

$$E\left[T|S=b\right] = \frac{1}{p_b}$$

$=> E\left[T\right] = 0.95\cdot\frac{1}{p_g} + 0.05\cdot\frac{1}{p_b} = 9505$ days

$$\begin{aligned}
Var\left[T\right] &= E\left[\left(T-E\left(T\right)\right)^2\right]\\
&= E\left[T^2\right] - \left(E\left[T\right]\right)^2
\end{aligned}$$

$$E\left[T^2\right] = E\left[T^2|S=g\right]\cdot Pr\left[S=g\right] + E\left[T^2|S=b\right]\cdot Pr\left[S=b\right]$$

$$\begin{aligned}
Pr\left[T^2=1|S=g\right] &= Pr\left[T=1|S=g\right] = p_g\\
Pr\left[T^2=4|S=g\right] &= Pr\left[T=2|S=g\right] = \left(1-p_g\right)p_g\\
Pr\left[T^2=n^2|S=g\right] &= Pr\left[T=n|S=g\right] = \left(1-p_g\right)^{(n-1)}\cdot p_g
\end{aligned}$$

$$
\begin{aligned}
E\left[T^2|S=g\right] &= \sum_{n=1}^{\infty} n^2\left(1-p_g\right)^{(n-1)} p_g \\
&= \frac{p_g}{1-p_g} \cdot \sum_{n=1}^{\infty} n^2\left(1-p_g\right)^n \\
&= \frac{p_g}{1-p_g} \cdot \frac{\left(1-p_g\right)\left(2-p_g\right)}{p_g^3} && \text{- see formulas -} \\
&= \frac{2-p_g}{p_g^2}
\end{aligned}
$$

$$
E\left[T^2\right] = 0.95 \cdot \frac{2-p_g}{p_g^2} + 0.05 \frac{2-p_b}{p_b^2}
$$

$$
\begin{aligned}
Var\left[T\right] &= E\left[T^2\right] - \left(E\left[T\right]\right)^2 \\
&= 0.95 \cdot \frac{2-p_g}{p_g^2} + 0.05 \frac{2-p_b}{p_b^2} - 9505^2 \\
&= 99,646,470 \\
std &= 9982.3 \text{ days}
\end{aligned}
$$

properties geometric series:

$$
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \qquad \text{- if } |z|<1 \text{ -}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} n \cdot z^n &= z \cdot \sum_{n=1}^{\infty} n \cdot z^{n-1} = z \sum_{n=1}^{\infty} \frac{d}{dz} z^n = z \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \\
&= z \cdot \frac{d}{dz}\left(\frac{1}{1-z}\right) = z \cdot \frac{(-1)^2}{(1-z)^2} = \frac{z}{(1-z)^2}
\end{aligned}
$$

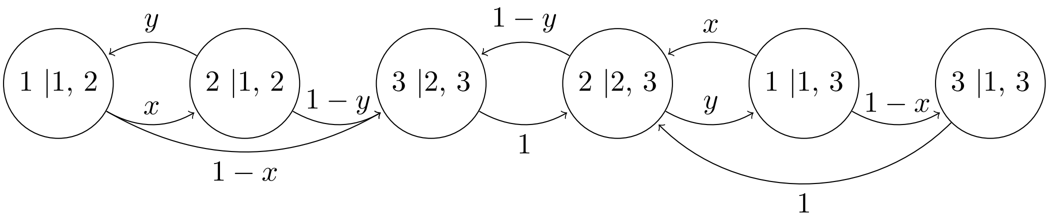
$$
\begin{aligned}
\sum_{n=1}^{\infty} n^2 z^n &= z \cdot \sum_{n=1}^{\infty} n^2 z^{(n-1)} \\
&= z \cdot \sum_{n=1}^{\infty} \frac{d}{dz}\left(n \cdot z^n\right) \\
&= z \cdot \frac{d}{dz}\left(\sum_{n=0}^{\infty} n \cdot z^n\right) \\
&= z \cdot \frac{(1-z)^2 + 2 \cdot (1-z)(-1)}{(1-z)^4} \\
&= \frac{z(1+z)}{(1-z)^3}
\end{aligned}
$$

2 Markov chains

1.

$p_{i,j} = Pr\left[W_{n+1} = j | W_n = i\right]$

webpages $= W = \{1, 2, 3\}$



proportion of time cache contains pages $k + l = t_{k,l}$

Ergodic?

- finite state space W? → yes
- gcd{2,3,4,5,...} = 1 for all sates → aperiodic
- irriducible? - yes
- >Markov chain is ergodic → limiting distribution = stationary distribution

transient state = time spent in state is 0 in the long run

$t_{1,2} = 0$, since it is a transient state

transient states can be removed, stationary equations:

$$\begin{aligned}
\pi_{(3,2,3)} &= \pi_{(2,2,3)} \cdot (1-y) \\
\pi_{(2,2,3)} &= \pi_{(3,2,3)} + \pi_{(3,1,3)} + x \cdot \pi_{1,1,3} \\
\pi_{(1,1,3)} &= y \cdot \pi_{(2,2,3)} \\
\pi_{(3,1,3)} &= (1-x) \cdot \pi_{(1,1,3)}
\end{aligned}$$

simplified:

$$\begin{aligned}
\pi_{(3,1,3)} &= (1-x) \cdot y \cdot \pi_{(2,2,3)} \\
\pi_{(3,2,3)} &= \pi_{(2,2,3)} \cdot (1-y) \\
\pi_{(1,1,3)} &= y \cdot \pi_{(2,2,3)} \\
1 &= \pi_{(3,1,3)} + \pi_{(3,2,3)} + \pi_{(1,1,3)} + \pi_{(2,2,3)} \qquad \text{- normalization condition -}
\end{aligned}$$

$$
\begin{aligned}
1 &= (1-x)\cdot y\cdot \pi_{(2,2,3)} + (1-y)\cdot \pi_{(2,2,3)} + y\cdot \pi_{(2,2,3)} + \pi_{(2,2,3)} \\
1 &= (y - xy + 1 - y + y + 1)\cdot \pi_{(2,2,3)} \\
\pi_{(2,2,3)} &= \frac{1}{y - xy + 2} \\
\pi_{(3,1,3)} &= \frac{(1-x)\cdot y}{y - xy + 2} \\
\pi_{(3,2,3)} &= \frac{1-y}{y - xy + 2} \\
\pi_{(1,1,3)} &= \frac{y}{y - xy + 2}
\end{aligned}
$$

$$
\begin{aligned}
t_{2,3} &= \pi_{(2,2,3)} + \pi_{(3,2,3)} \\
&= \frac{2-y}{y - xy + 2}
\end{aligned}
$$

$$
\begin{aligned}
t_{1,3} &= \pi_{(3,1,3)} + \pi_{(1,1,3)} \\
&= \frac{2y - xy}{y - xy + 2}
\end{aligned}
$$

b)

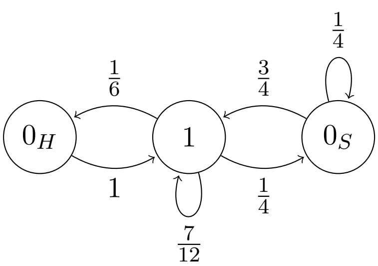
proportion of requests for cached pages
$p_{i,j}$ = prob that we will ask for j given that i is current page

$$
\pi_{(2,2,3)}\cdot p_{2,3} + \pi_{(3,1,3)}\cdot p_{3,1} + \pi_{(3,2,3)}\cdot p_{3,2} + \pi_{(1,1,3)}\cdot p_{1,3}
$$
$$
\pi_{(2,2,3)}\cdot (1-y) + \pi_{(3,1,3)}\cdot 0 + \pi_{(3,2,3)}\cdot 1 + \pi_{(1,1,3)}\cdot (1-x) = \frac{y\,(1-x)}{2 + y - xy}
$$

2.

a)

state space = $E = \{1, 0_H, 0_S\} H$ for hardware, S for software
we assume software and hardware problems cannot occur simultaneously



b)

ergodic?

- finite state space
- irreducible because single communicating class
- periodicity is same for all states in same class

$\rightarrow$ yes

c)

fraction working $= \pi_1$

$$\begin{aligned} \pi &= \pi \cdot P \\ \sum_{i=1}^{3} \pi_i &= 1 \\ P &= ((7/12, 1/6, 1/4), (1, 0, 0), (3/4, 0, 1/4)) \end{aligned}$$

$$\begin{aligned} \pi_1 \cdot \frac{1}{6} &= \pi_{0_H} \\ \pi_1 &= 6 \cdot \pi_{0_H} \\ \pi_1 \cdot \frac{1}{4} &= \frac{3}{4}\pi_{0_S} \\ \pi_1 &= 3 \cdot \pi_{0_S} \end{aligned}$$

$$\begin{aligned} 1 &= \pi_1 + \pi_{0_H} + \pi_{0_S} \\ 1 &= 6\pi_{0_H} + \pi_{0_H} + 2\pi_{0_H} \end{aligned}$$

$$\pi = (\pi_1, \pi_{0_H}, \pi_{0_S}) = \left(\frac{6}{9}, \frac{1}{9}, \frac{2}{9}\right)$$

The data center is up $\frac{2}{3}$ of the time

d)

mean time between backhoe failures = mean time between visits to π_{0_H}

$$E[T_{0_H}] = \frac{1}{\pi_{0_H}} = 9$$

3.

transition matrix P denotes the probabilities of following a link rank = r

$r_{n+1} = r_n \cdot P$
$r_0 \rightarrow$ solvable if P is ergodic, because then the limiting distribution exists
n_j = nr of links on page i to j
N_i = nr links on page i
$P_{(i,j)} = \frac{n_j}{N_i}$

a)

$$P = \left(\left(0, \frac{1}{2}, 0, \frac{1}{2}\right), \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), (1, 0, 0, 0), (0, 0, 0, 1)\right)$$

c)

$$\pi = \pi \cdot P$$
$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

$\rightarrow \pi = (0, 0, 0, 1)$ is stationary distribution, if ergodic (see b) it is also limiting distribution. That $\pi = (0, 0, 0, 1)$ is the stationary distribution makes intuitive sense, since once we arrive on webpage 4 we can never leave it.

b)

ergodic?:
irreducible? $\rightarrow$ no: there are 2 classes $\{4\}, \{1, 2, 3\}$
ergodicity is a sufficient but no necessary condition for the existence of a limiting distribution
eventually you will get stuck in state of page 4 so limiting distribution = stationary distribution

d)

with teleportation

$$P' = \left(\left(0, \frac{1}{2}, 0, \frac{1}{2}\right), \left(0, 0, \frac{1}{2}, \frac{1}{2}\right), (1, 0, 0, 0), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)\right)$$

e)

now chain is irreducible, also aperiodic and the state space is finite $\rightarrow$ ergodic

f)

solve system of equations $\pi = \pi \cdot P'$ with $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$
$\rightarrow \pi = \frac{1}{102}(30, 24, 21, 27)$

the limiting distribution defines final ranks for all pages

4.

based on the edges of the CTMC we fill in the transition rate matrix Q that defines the CTMC:

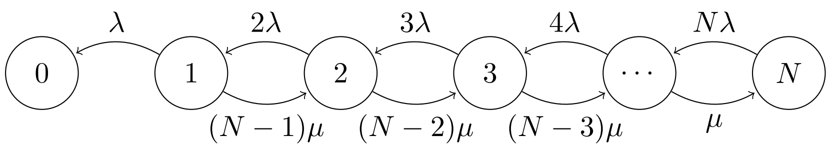
$$Q = ((x, 1, 0), (2, y, 2), (4, 3, z))$$
$$Q = ((-1, 1, 0), (2, -4, 2), (4, 3, -7)) \qquad \text{- rows have to sum to 0 -}$$

We fill in the transition matrix P for the corresponding DTMC, based on $p_{ij} = \frac{q_{ij}}{-q_{ii}}$

$$P = \left((0, 1, 0), \left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{4}{7}, \frac{3}{7}, 0\right)\right)$$

5.

a)



- chain is a birth-death chain: the state transitions are of only two types: "births", which increase the state variable by one and "deaths", which decrease the state by one
- $(N-1)\,\mu$ because data of $(N-1)$ drives has to be copied
- 2λ because each of the drives can fail with rate λ so we have to sum them
- the chain is not ergodic since it is reducible to state 0

b)

limiting probability mass function $\rightarrow (1, 0, 0, \ldots, 0)$, in the long run you will end up in state 0 and you will not be able to get out since there are no outgoing links.

3 Birth Death Queues

Preparatory Exercise

definition exponential series:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

1

$$\begin{aligned}
ze^z &= z \sum_{k=0}^{\infty} \frac{z^k}{k!} \\
&= z \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} \\
&= \sum_{k=1}^{\infty} \frac{z^k}{(k-1)!} \\
&= \sum_{k=1}^{\infty} k \frac{z^k}{k!} \\
&= \sum_{k=0}^{\infty} k \frac{z^k}{k!} && \text{- multiplication by zero adds nothing to sum -}
\end{aligned}$$

2

$$\begin{aligned}
(z+z^2)e^z &= z(ze^z + e^z) \\
&= z\left(\sum_{k=0}^{\infty} k \frac{z^k}{k!} + \sum_{k=0}^{\infty} \frac{z^k}{k!}\right) && \text{- use previous identities -} \\
&= z\left(\sum_{k=0}^{\infty} k \frac{z^k}{k!} + \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!}\right) \\
&= z\left(\sum_{k=1}^{\infty} (k-1) \frac{z^{k-1}}{(k-1)!} + \sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!}\right) \\
&= z \sum_{k=1}^{\infty} k \frac{z^{k-1}}{(k-1)!} \\
&= \sum_{k=0}^{\infty} k^2 \frac{z^k}{k!}
\end{aligned}$$

3

if $z = 1$ we have division by zero so we assume $z \neq 1$:

$$
\begin{aligned}
1 - z^{n+1} &= (1-z)\sum_{k=0}^{n} z^k \\
1 - z^{n+1} &= (1-z)\frac{1-z^{n+1}}{1-z} \qquad \text{-http://mathworld.wolfram.com/GeometricSeries.html-} \\
1 - z^{n+1} &= 1 - z^{n+1}
\end{aligned}
$$

4

$$
\begin{aligned}
\sum_{k=0}^{n} kz^k &= z\sum_{k=0}^{n} kz^{k-1} \\
&= z\sum_{k=0}^{n} \frac{dz^k}{dz} \\
&= \frac{z}{dz} d\left(\sum_{k=0}^{n} z^k\right) \\
&= \frac{z}{dz} d\left(\frac{1-z^{n+1}}{1-z}\right) \qquad \text{-http://mathworld.wolfram.com/GeometricSeries.html-} \\
&= z\frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2} \qquad \text{- derivative of fraction + simplify -}
\end{aligned}
$$

The infinite series have region of convergence $|z| \leq 1$ and limits:

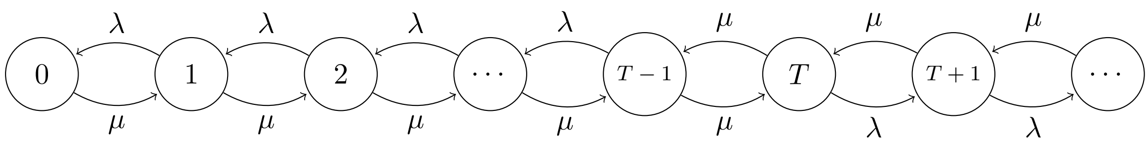
$$
\lim_{n\to\infty} \sum_{k=0}^{n} z^k = \lim_{n\to\infty} \frac{1-z^{n+1}}{1-z} = \frac{1}{1-z}
$$

$$
\lim_{n\to\infty} \sum_{k=0}^{n} kz^k = \lim_{n\to\infty} \frac{z(1-(n+1)z^n + nz^{n+1})}{(1-z)^2} = \frac{z}{1-z^2}
$$

1.

a)

state space = $\mathbb{N}$, S = system content.



b)

For which λ, μ, T is the Markov chain ergodic?

- chain is irreducible
- is the chain positive recurrent ? $\rightarrow$ solve for stationary distribution:

detailed balance equations:

$$\begin{aligned} (1)\, \mu s(0) &= \lambda s(1) \\ (2)\, \mu s(n) &= \lambda s(n+1) && \text{- n = 0,1, ..., T-2 -} \\ (3)\, \mu s(T-1) &= \mu s(T) \\ (4)\, \lambda s(n) &= \mu s(n+1) && \text{- n = T+1, T+2... -} \end{aligned}$$

$\rho = \frac{\lambda}{\mu}$

$$\begin{aligned} (1) &\Rightarrow s(1) = \frac{\mu}{\lambda} s(0) = \frac{1}{\rho} s(0) \\ (2) &\Rightarrow s(n+1) = \frac{\mu}{\lambda} s(n) = \frac{\mu}{\lambda} \cdot \frac{\mu}{\lambda} \cdot s(n-1) \\ &= \ldots = \frac{1}{\rho^{(n+1)}} \cdot s(0) && \text{- n=0,1,...,T-2 -} \\ (3) &\Rightarrow s(T) = s(T-1) = \frac{1}{\rho^{(T-1)}} \cdot s(0) \\ (4) &\Rightarrow s(n) = \frac{\lambda}{\mu} s(n-1) = \rho \cdot s(n-1) && \text{- n = T+1, T+2... -} \\ &= \ldots = \rho^{n-T} s(T) = \rho^{(n+1-2T)} s(0) \end{aligned}$$

So:

$$\begin{aligned} s(n) &= \frac{1}{\rho^n} s(0) && \text{- n=0,1,...,T-2 -} \\ s(T) &= s(T-1) = \frac{1}{\rho^{(T-1)}} s(0) \\ s(n) &= \rho^{(n-2T+1)} s(0) && \text{- } n = T+1, \ldots \text{ -} \\ 1 &= \sum_{n=0}^{\infty} s(n) = s(0) \cdot \left(\sum_{n=0}^{T-1} \frac{1}{\rho^n} + \frac{1}{\rho^{T-1}} + \sum_{n=T+1}^{\infty} \rho^{n-2T+1} \right) \\ &= s(0) \cdot \left(\frac{1-\left(\rho^{-1}\right)^T}{1-\rho^{-1}} + \frac{1}{\rho^{T-1}} + \sum_{n=T+1}^{\infty} \rho^{n-2T+1} \right) && \text{- } \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z} \text{ -} \\ &= s(0) \cdot \left(\frac{1-\left(\rho^{-1}\right)^T}{1-\rho^{-1}} + \sum_{n=T}^{\infty} \rho^{n-2T+1} \right) && \text{- include } \frac{1}{\rho^{T-1}} \text{ in summation -} \\ &= s(0) \cdot \frac{1-\left(\rho^{-1}\right)^T}{1-\rho^{-1}} + \rho^{-T+1} \cdot \frac{1}{1-\rho} && \text{- } \sum_{n=T}^{\infty} \rho^n = \frac{\rho^T}{1-\rho} \text{ -} \end{aligned}$$

$\Rightarrow s(0) = \frac{\rho^{T-1}(1-\rho)}{1-\rho^T}$

Markov chain is ergodic when:

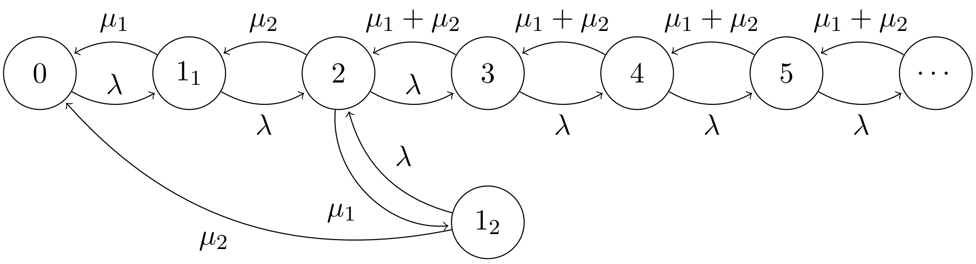
- To have s(0) positive, you need: $2 - \rho^T > 0 \Leftrightarrow \rho^T < 2$
- T is finite
- $0 < \rho < 1$

c)

$$\begin{aligned} E\left[N\right] &= \sum_{n=0}^{\infty} n \cdot s\left(n\right) \\ &= \dots \\ &= \frac{\left(1-\rho\right)\left(2T-1\right)+\rho^T}{\left(1-\rho\right)\left(2\cdot\rho^T\right)} \\ &= \frac{2T-1}{2\rho^T} + \frac{\rho^T}{\left(1-\rho\right)\left(2-\rho^T\right)} \end{aligned}$$

2.

a)



b)

irriducible $\rightarrow$ clear from state diagram
to prove: stationary probability mass function
local balance equations $\rightarrow$ combine states instead of writing down local ones state by state $\rightarrow$ is equivalent

$$\begin{aligned} (1)\, \lambda s\left(0\right) &= \mu_1 s\left(1,1\right) + \mu_2 s\left(1,2\right) \\ (2)\, \lambda \cdot s\left(1,1\right) &= \mu_2 s\left(2\right) + \mu_1 \cdot s\left(1,2\right) \\ (3)\, \left(\lambda+\mu_2\right) s\left(1,2\right) &= \mu_1 s\left(2\right) \\ (4)\, \lambda s\left(n\right) &= \left(\mu_1+\mu_2\right) s\left(n+1\right) \text{ - n=2,3,... -} \end{aligned}$$

$\rho = \frac{\lambda}{\mu_1+\mu_2}$
After some calculations:

$$
\begin{aligned}
s(1,1) &= \frac{\lambda(1+\rho)}{\mu_1(1+2\rho)} s(0) \\
s(1,2) &= \frac{\lambda \cdot \rho}{\mu_2 \cdot (1+2\rho)} s(0) \\
s(n) &= \frac{(\lambda+\mu_2)\cdot\lambda}{\mu_1 \cdot \mu_2} \cdot \frac{\rho^{n+1}}{1+2\rho} s(0) \text{ - } n = 2, 3 \ldots \text{ -}
\end{aligned}
$$

$$
s(1,1) + s(1,2) = s(0) \cdot \frac{\lambda(\lambda+\mu_2)}{\mu_1\mu_2(1+2\rho)}
$$

$$
1 = \sum_{n=0}^{\infty} s(n) = s(0)(\ldots)
$$

$$
s(0) = \left(1 + \left(\lambda \frac{\lambda+\mu_2}{\mu_1\mu_2(1+2\rho)(1-\rho)}\right)\right)^{-1}
$$

if $0 < \rho < 1$ the markov chain is ergodic

c)

$$
\begin{aligned}
E[N] &= \sum_{n=0}^{\infty} n \cdot s(n) \\
&= \ldots \\
&= \frac{1}{\frac{\mu_1\mu_2(1+2\rho)}{\lambda(\lambda+\mu_2)} + \frac{1}{1-\rho}} \\
&= \frac{1}{0 + \frac{1}{1-\rho}} \cdot \frac{1}{(1-\rho)^2} \\
&= \frac{1-\rho}{(1-\rho)^2} \\
&= \frac{1}{1-\rho}
\end{aligned}
$$

d)

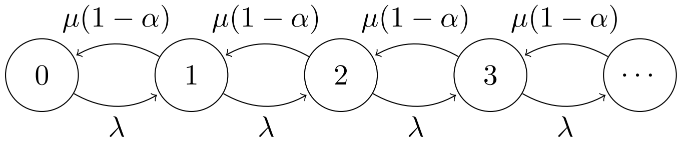
$\lambda < \mu_1 + \mu_2$

if $\mu_2 << \mu_1$, then $\rho \approx \frac{\lambda}{\mu_1}$, fill these params in in c) $\Rightarrow E[N] \approx \frac{1}{1-\rho}$
For a M/M/1 with params (λ, μ_1) -see syllabus- than we know $E[N] = \frac{\rho}{1-\rho}$ with $\rho = \frac{\lambda}{\mu_1} < 1$, this is smaller than before hence it is faster to use 1 server.

Conclusion : if server 2 is much slower than server 1 you are better off without server 2
better to let a customer wait a bit longer and then serve them quicker
the mean waiting time will increase a bit but the mean sojourn time will decrease much more

3.

a)



The feedback can be simplified by setting the service rate to $\mu(1-\alpha)$,
which results in a normal M/M/1 queue

b)

$s\left(0\right)$?

$$\lambda s\left(n-1\right) = \mu\left(1-\alpha\right)s\left(n\right) \qquad \text{- } n = 1, 2, \ldots \text{ -}$$

$$s\left(n\right) = \left(\frac{\lambda}{\mu\left(1-\alpha\right)}\right)^n s\left(0\right)$$

$$s(n) = \rho^n s\left(0\right) \qquad \text{- with } \rho = \frac{\lambda}{\mu\left(1-\alpha\right)} \text{ -}$$

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} s(n) \\ &= s\left(0\right)\sum_{n=0}^{\infty}\rho^n \\ &= s\left(0\right)\frac{1}{1-\rho} \qquad \text{- if } 0 < \rho < 1 \text{ then ergodic -} \\ s(0) &= 1-\rho \end{aligned}$$

c)

$$\begin{aligned} E\left[N\right] &= \sum_{n=0}^{\infty} n\cdot s\left(n\right) \\ &= \sum_{n=1}^{\infty} n\cdot\rho^n\cdot\left(1-\rho\right) \qquad \text{- leave out multiplication by zero -} \\ &= \frac{\rho}{1-\rho} \end{aligned}$$

4.

a)

$$
\begin{tikzcd}
0 \arrow[r, bend right, "N \cdot \alpha"'] & 1 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "(N-1)\alpha"'] & 2 \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "(N-2)\alpha"'] & \cdots \arrow[l, bend right, "\mu"'] \arrow[r, bend right, "\alpha"'] & N \arrow[l, bend right, "\mu"']
\end{tikzcd}
$$

b)

$s\left(n\right) = Prob\left(N-n\right)$

detailed balance equations:

$$
\begin{aligned}
(1)\, N \cdot \alpha \cdot s\left(0\right) &= \mu s\left(1\right) \\
(2)\, \left(N-1\right) \cdot \alpha \cdot s\left(1\right) &= \mu s\left(2\right) \\
(n)\, \left(N-n+1\right) \cdot \alpha \cdot s\left(n-1\right) &= \mu \cdot s\left(n\right) \qquad \text{- } n = 1, 2, \ldots, N \text{ -}
\end{aligned}
$$

$s\left(n\right) = c_n \cdot s\left(0\right)$

$$
\begin{aligned}
s\left(n\right) &= \frac{\left(N-\left(n-1\right)\right)}{\mu} \cdot \alpha \cdot s\left(n-1\right) \\
&= \frac{\left(N-\left(n-1\right)\right)}{\mu} \cdot \frac{\left(N-\left(n-2\right)\right)}{\mu} \cdot \alpha \cdot \left(s\left(n-2\right)\right) \\
&= \frac{\left(N-\left(n-1\right)\right) \cdot \alpha}{\mu} \cdot \ldots \cdot \frac{N\alpha}{\mu} \cdot s\left(0\right) \\
&= \frac{N!}{\left(N-n\right)!} \cdot \left(\frac{\alpha}{\mu}\right)^n \cdot s\left(o\right) \qquad \text{- } \forall n \text{ in } \mathbb{N}, \prod_{i=1}^{n} N-\left(n-i\right) = \frac{N!}{\left(N-n\right)!} \text{ -}
\end{aligned}
$$

$$
\begin{aligned}
1 &= \sum_{n=0}^{N} s\left(n\right) \\
&= \left(\sum_{n=0}^{N} \frac{N!}{\left(N-n\right)!} \cdot \left(\frac{\alpha}{\mu}\right)^n\right) s\left(0\right)
\end{aligned}
$$

$\rightarrow s\left(0\right) = \left(\sum_{n=0}^{N} \frac{N!}{\left(N-n\right)!} \cdot \left(\frac{\alpha}{\mu}\right)^n\right)^{-1}$

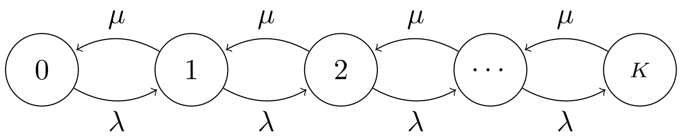
c)

$\left(Pr\left(\mu = n\right) = Pr\left(N-S=n\right)\right) = s\left(N-n\right) \forall n$ in $\{0, 1, \ldots, N\}$

$Pr\left(\mu = n\right) \rightarrow$ prob n users are analyzing completed job (preparing new job)

$Pr\left(N-S=n\right) \rightarrow$ prob N-n jobs are in the system

5.



$$\lambda s\left(n-1\right) = \mu s\left(n\right) \qquad \text{- } n = 1, 2 \ldots \text{ -}$$

$$s\left(1\right) = \rho s\left(0\right) \qquad \text{- } \rho = \frac{\lambda}{\mu} \text{ -}$$

$$s\left(2\right) = \rho^2 s\left(0\right)$$

$$s\left(K\right) = \rho^K s\left(0\right)$$

$$\begin{aligned} 1 &= \sum_{n=0}^{K} s\left(n\right) \\ &= s(0) \cdot \sum_{n=0}^{K} \rho^n \\ &= s(0) \frac{1-\rho^{K+1}}{1-\rho} \end{aligned}$$

$$s(K) = \rho^K \frac{1-\rho}{1-\rho^{K+1}}$$

a)

Assume $\rho = 0.4$. Doubling the service rate (using $\mu' = 2\mu$) results in a halving of the load:

$$\rho' = \frac{\lambda}{2\mu} = \frac{\rho}{2}.$$

The requested loss probabilities are summarized in the following table:

ρ	K	p_{loss}
0.4	5	$6 \cdot 10^{-3}$
0.2	5	$3 \cdot 10^{-4}$
0.4	10	$6 \cdot 10^{-5}$

Doubling the system capacity has a greater (positive) effect on the loss probability.

b)

Assume $\rho = 0.8$. The requested loss probabilities are summarised in the following table:

ρ	K	p_{loss}
0.8	5	$9 \cdot 10^{-2}$
0.4	5	$6 \cdot 10^{-3}$
0.8	10	$2 \cdot 10^{-2}$

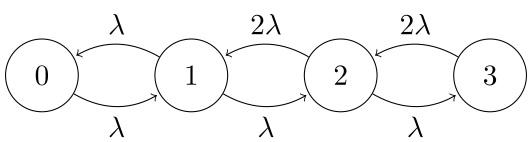
Doubling the service rate has a greater (positive) effect on the loss probability.

c)

The intuitive solution would be to increase the system capacity. If the load is high, then the system will be at full capacity for a large proportion of the time, even if the system has a high capacity.

6.

a)



2λ because once there are two tasks two servers can work on it.

b)

two costumers $\to s = 2$
$T_a \sim Exp(|\lambda), T_d \sim Exp(|2 \cdot \lambda)$

$$
\begin{aligned}
Pr\left(\text{arrival before departure}\right) &= Pr\left[T_a < T_d\right] \\
&= \int_0^{\infty} Pr\left(t < T_d\right) f_a\left(t\right) dt \\
&= \int_0^{\infty} (1 - F_{T_d}(t)) f_a\left(t\right) dt \\
&= \int_0^{\infty} e^{-2\lambda t} \lambda e^{-\lambda t} dt \qquad\qquad \text{- see table 1, A1.11 -} \\
&= \int_0^{\infty} \lambda \cdot e^{-3\lambda t} dt \\
&= \lambda \cdot \int_0^{\infty} e^{-3\lambda t} dt \\
&= \lambda \cdot \left[\frac{e^{-3\lambda t}}{-3\lambda}\right]_0^{\infty} \\
&= \lambda \cdot \left[0 - \frac{1}{-3 \cdot \lambda}\right] \\
&= \frac{1}{3}
\end{aligned}
$$

c)

$s(n) = \lim_{t\to\infty} Pr[x_t = n]$ $n = 0, 1, 2, 3, \ldots$

detailed balance equations:

$$\begin{aligned}
&(1)\ \lambda s(0) = \lambda s(1) \Rightarrow s(1) = s(0)\\
&(2)\ \lambda s(1) = 2\lambda s(2) \Rightarrow s(2) = \frac{1}{2}s(1) = \frac{1}{2}s(0)\\
&(3)\ \lambda s(2) = 2\lambda s(3) \Rightarrow s(3) = \frac{1}{2}s(2) = \frac{1}{2}\cdot\frac{1}{2}\cdot s(0)
\end{aligned}$$

$1 = s(0)\left(1 + 1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{11}{4}$

$$\begin{aligned}
s(0) &= \frac{4}{11}\\
s(1) &= \frac{4}{11}\\
s(2) &= \frac{2}{11}\\
s(3) &= \frac{1}{11}
\end{aligned}$$

d)

$E[S] = \sum_{n=0}^{3} n \cdot s(n) = 1$

$0 \cdot \frac{4}{11} + 1 \cdot \frac{4}{11} + 2 \cdot \frac{2}{11} + 3 \cdot \frac{1}{11}$

e)

probability of loss = prob arrival in state 3, there is no state 4, hence the customer will be lost: $p_{loss} = s(3) = \frac{1}{11}$ (PASTA)

f)

$$\begin{aligned}
\mu_1 &= \lambda\\
\mu_2 &= \mu_3 = 2\cdot\lambda
\end{aligned}$$

$$\begin{aligned}
\mu_{eff} &= \sum_{n=1}^{3} s(n)\,\mu_n\\
&= \frac{10}{11}\lambda \qquad\qquad \text{- nr of leaving customers per second -}
\end{aligned}$$

$\rightarrow \frac{10}{11}\lambda\left[s^{-1}\right] = (1 - p_{loss})\cdot\lambda = \frac{10}{11}\lambda$

$\rightarrow \mu_{eff} = (1 - p_{loss}) \cdot \lambda = \lambda_{eff}$
this should always be the case: states system is in equilibrium: at every time instant
nr incoming customers = nr outgoing customers

g)

An effective arrival occurs when a new customer arrives and the system is not full. The distribution of the number of customers in the system at an effective arrival is

$$Pr[S_{A,eff} = n] = Pr[S_A = n | S_A < 3]$$

where $S_{A,\text{eff}}$ is the system content at an effective arrival and S_A is the system content at an arrival. From the PASTA (Poisson Arrivals See Time Averages) property and the ergodicity of the CTMC, it follows that $S_A = S$. Therefore

$$\begin{aligned} Pr[S_{A,eff} = n] &= Pr[S = n | S < 3] \\ &= \frac{Pr[S = n, S < 3]}{Pr[S < 3]} \\ &= \frac{Pr[S = n, S < 3]}{\frac{10}{11}} \end{aligned}$$

$s(n) = Pr[S = n, S < 3] \rightarrow s(n)$ is already calculated with $S < 3$

which yields $Pr[S_{A,eff} = 0] = \dfrac{\frac{4}{11}}{\frac{10}{11}} = \frac{2}{5}$, $Pr[S_{A,eff} = 1] = \frac{2}{5}$ and $Pr[S_{A,eff]=2} = \frac{1}{5}$.

h)

The rate of a Poisson process is constant, and knowing when the previous Poisson event has happened is irrelevant for the occurrence of the next Poisson event. In this case, this means that the effective arrival rate λ_{eff} should be equal to the effective arrival rate λ_a right after an arrival. By definition,

$$\lambda_{eff} = (1 - s(3))\lambda = (1 - p_{\text{loss}})\lambda = \frac{10}{11}\lambda.$$

The effective arrival rate λ_a right after an arrival can be calculated similarly, but now the loss probability is different. The probability mass function of the system content S_a right after an effective arrival is computed from the relation

$$Pr[S_a = n] = Pr[S = n | S > 0] = \frac{s(n)}{1 - s(0)},$$

if n is equal to 1, 2 or 3. We find

$$\lambda_a = \frac{s(1) + s(2)}{1 - s(0)}\lambda = \frac{6}{7}\lambda.$$

As $\lambda_{eff} \neq \lambda_a$, the effective arrival process is not a Poisson process.

4 waiting times and delay

1.

a)

used formulas

stationary distribution M/M/1: $s_{\rho(n)} = (1-\rho)\cdot\rho^n$

solution

$$Pr\left[S_\rho \leq n\right] \text{ - } n \in \mathbb{N} \text{ - } = \sum_{i=0}^{n} Pr\left(S_\rho = i\right)$$

stationary distribution for M/M/1: $s_{\rho(n)} = (1-\rho)\cdot\rho^n$

$$\begin{aligned}
Pr\left(S_\rho \leq n\right) &= \sum_{k=0}^{n} Pr\left(S_\rho = k\right) \\
&= \sum_{k=0}^{n} s_\rho(k) \\
&= (1-\rho)\sum_{k=0}^{n} \rho^k \\
&= (1-\rho)\,\frac{1-\rho^{(n+1)}}{1-\rho} \\
&= 1-\rho^{n+1}
\end{aligned}$$

$\forall x \in \mathbb{R} : Pr\left(S_\rho \leq x\right) = 1 - \rho^{\lfloor x \rfloor + 1}$

b)

used formulas

Hopital:

$$\begin{aligned}
&\text{if } \lim_{x\to a} f\left(x\right) = \lim_{x\to a} g\left(x\right) = 0 \text{ or } \pm\infty \\
&\lim_{x\to a} \frac{f\left(x\right)}{g\left(x\right)} = \lim_{x\to a} \frac{f'\left(x\right)}{g'\left(x\right)}
\end{aligned}$$

solution

$$\begin{aligned}
Pr\left[(1-\rho)\,S_\rho \leq x\right] &= Pr\left[S_\rho \leq \frac{x}{1-\rho}\right] \\
&= 1 - \rho^{\frac{x}{\lfloor 1-\rho \rfloor}+1} && \text{- fill in from a) -}
\end{aligned}$$

$$1-\rho^{\frac{x}{1-\rho}} \leq 1-\rho^{\frac{x}{\lfloor 1-\rho \rfloor}+1} \leq 1-\rho^{\frac{x}{1-\rho}+1}$$

solve limit from both sides, side one:

$$\begin{aligned}
\lim_{\rho\to 1} \rho^{\frac{x}{1-\rho}} &= \lim_{\rho\to 1} \exp\left(\ln\left(\rho^{\frac{x}{1-\rho}}\right)\right) && f(x) = e^{ln(f(x))} \\
&= \lim_{\rho\to 1} \exp\left(\frac{x}{1-\rho}\ln(\rho)\right) && \text{- log power rule -} \\
&= \exp\left(\lim_{\rho\to 1}\frac{x}{1-\rho}\ln(\rho)\right) && \text{- pass limit through exponential -} \\
&= \exp\left(x\cdot\lim_{\rho\to 1}\frac{ln(\rho)}{1-\rho}\right) && \text{- factor out constant x -} \\
&= \exp\left(x\cdot\lim_{\rho\to 1}\left(\frac{\frac{1}{\rho}}{-1}\right)\right) && \text{- hopital -} \\
&= \exp\left(x\cdot\lim_{\rho\to 1}\left(\frac{\rho}{-1}\right)\right) \\
&= \exp(-x)
\end{aligned}$$

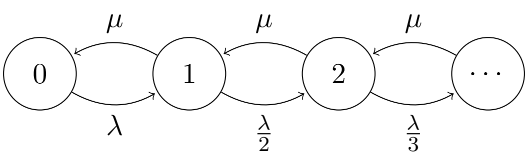
side two:

$$\begin{aligned}
&= \lim_{\rho\to 1} \rho^{\frac{x}{1-\rho}+1} \\
&= \exp(-x)
\end{aligned}$$

combine boths sides $\rightarrow \lim_{\rho\to 1}\left(1-\rho^{\frac{x}{\lfloor 1-\rho \rfloor}+1}\right) = 1-\exp(-x)$

$\rightarrow$ When ρ goes to 1, this distribution is converging to an exponential distribution with $\lambda =$ 1

2.



a)

λ_n =arrival rate when n customers in the system
μ_n =service rate when n customers in the system

balance equations:

$s(n)\cdot\lambda_n = s(n+1)\,\mu_{n+1}$

Normalization condition:

$\sum_{n=0}^{\infty} s(n) = 1$

$$\lambda_n = \frac{\lambda}{n+1} \qquad \text{- } \forall n \in \mathbb{N} \text{ -}$$

$$\mu_n = \mu \qquad \text{- } \forall n \in \mathbb{N} \text{ -}$$

Balance equations:

$$\begin{aligned} s(n+1) &= s(n) \cdot \frac{\lambda}{(n+1)\mu} \\ &= s(n-1) \cdot \frac{\lambda}{n\mu} \cdot \frac{\lambda}{(n+1)\mu} \\ &= s(0) \cdot \left(\frac{\lambda}{\mu}\right)^{n+1} \cdot \frac{1}{(n+1)!} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} s(n) &= 1 \\ s(0) \cdot \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{1}{n!} &= 1 \\ s(0) &= \exp\left(-\frac{\lambda}{\mu}\right) \qquad \text{- def exp series: } e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \text{ -} \end{aligned}$$

$$s(n) = e^{-\frac{\lambda}{\mu}} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \qquad \text{- } n \in \mathbb{N} \text{ -}$$

b)

used formulas

definition laplace transform =

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \qquad \text{- with s a complex var -} \\ &= E[e^{-sW}] \end{aligned}$$

$$\begin{aligned} \hat{s}(n) &: \text{distr after departure} \\ \widetilde{s}(n) &: \text{distr before departure} \\ \widetilde{s}(n+1) &= \hat{s}(n) \end{aligned}$$

solution

We use a different approach then usual, because λ is not fixed,
hence PASTA does not apply and so $S_A = S$ does not apply.
We use a more broadly applicable approach here

$$\begin{aligned} D^*(s) &= \text{laplace transform of the delay of a customer} \\ &= E\left[e^{-s\cdot W}\right] \\ &= \sum_n E\left[e^{-sW}|S^A = n\right] \cdot \hat{s}(n) \end{aligned}$$

$\hat{s}(n) = \widetilde{s}(n+1)$ - distr seen by someone just before departure -

$\widetilde{s}(n) = \dfrac{s(n)}{1 - s(0)}$ You can't have a departure when the system content is 0.

$$\begin{aligned} \hat{s}(n) &= \widetilde{s}(n+1) \\ &= \frac{s(n+1)}{1-s(0)} && \text{- see earlier result -} \\ &= \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^{n+1}\frac{1}{(n+1)!}}{1-e^{-\frac{\lambda}{\mu}}} && \text{- fill in from a) -} \end{aligned}$$

$E\left[e^{-sW}|S^A = n\right] = \left(\frac{\mu}{\mu+s}\right)^{n+1}$ see formula p 3.11

$$\begin{aligned} D^*(s) &= \sum_n E\left[e^{-sW}|S^A = n\right] \cdot \hat{s}(n) \\ &= \sum_{n=0}^{\infty}\left(\frac{\mu}{\mu+s}\right)^{n+1} \cdot \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^{n+1}\frac{1}{(n+1)!}}{1-e^{-\frac{\lambda}{\mu}}} \\ &= \frac{e^{-\frac{\lambda}{\mu}}}{1-e^{-\frac{\lambda}{\mu}}} \cdot \sum_{n=0}^{\infty}\left(\frac{\mu}{\mu+s}\right)^{n+1}\left(\frac{\lambda}{\mu}\right)^{n+1}\frac{1}{(n+1)!} \\ &= \frac{e^{-\frac{\lambda}{\mu}}}{1-e^{-\frac{\lambda}{\mu}}} \cdot \sum_{n=1}^{\infty}\left(\frac{\lambda}{\mu+s}\right)^{n}\frac{1}{n!} \\ &= \frac{e^{-\frac{\lambda}{\mu}}}{1-e^{-\frac{\lambda}{\mu}}} \cdot e^{\frac{\lambda}{\mu+s}} - 1 && \text{- } e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!} \text{ -} \\ &= \frac{e^{\left(\frac{\lambda}{\mu+s}\right)} - 1}{e^{\frac{\lambda}{\mu}} - 1} \end{aligned}$$

c)

TODO expand

Little's law: $E[S] = \lambda_{eff} \cdot E[D]$

$E[S] = \frac{\lambda}{\mu}$

$$\begin{aligned} \lambda_{eff} &= \sum_{n=0}^{\infty}\lambda_n \cdot s(n) && \text{- } \lambda_{eff} \text{ for any queue with infinite capacity -} \\ &= \mu \cdot \left(1 - e^{-\frac{\lambda}{\mu}}\right) && \text{- ? -} \end{aligned}$$

$$E\left[D\right] = -D^{*'}\left(0\right)$$
$$= \frac{\lambda}{\left(1 - e^{-\frac{\lambda}{\mu}}\right) \cdot \mu^2}$$

$$\frac{d\frac{\lambda}{\mu+s}}{ds} = -\frac{\lambda}{\left(\mu+s\right)^2}$$

$$D^{*'}\left(s\right) = \frac{1}{e^{\frac{\lambda}{\mu}} - 1} \cdot e^{\frac{\lambda}{\mu+s}} \cdot \frac{-\lambda}{\left(\mu+s\right)^2}$$
$$= \frac{\lambda \cdot e^{\frac{\lambda}{\mu+s}}}{\left(1 - e^{\frac{\lambda}{\mu}}\right)\left(\mu+s\right)^2}$$

$$-D^{*'}\left(0\right) = -\frac{\lambda \cdot e^{\frac{\lambda}{\mu}}}{\left(1 - e^{\frac{\lambda}{\mu}}\right) \cdot \mu^2}$$
$$= \frac{\mu \cdot e^{\frac{\lambda}{\mu}}}{e^{\frac{\lambda}{\mu}} \cdot \left(e^{-\frac{\lambda}{\mu}} + 1\right) \cdot \mu^2}$$

$$\lambda_{eff} \cdot E\left[D\right] = \mu \cdot \left(1 - e^{-\frac{\lambda}{\mu}}\right) \cdot \frac{\lambda}{1 - e^{-\frac{\lambda}{\mu}} \cdot \mu^2}$$
$$= \frac{\lambda}{\mu}$$
$$= E\left[S\right]$$

3.

$S\left(t\right) =$ nr of customers in the queue at time t

$$\bar{S} = \frac{1}{T}\int_0^T S\left(t\right)dt$$
$$= \frac{1}{T}\int_0^T \sum_{k=1}^{K} I_k\left(t\right)dt \qquad \text{- p3.7 -}$$
$$= \frac{1}{T}\sum_{k=1}^{K}\int_0^T I_k\left(t\right)dt \qquad \text{- switch sum and integral -}$$
$$= \frac{1}{T}\sum_{k=1}^{K} D_k \qquad \text{- p3.8 -}$$
$$= K \cdot \frac{1}{T} \cdot \frac{1}{K}\sum_{k=1}^{K} D_k \qquad \text{- split 1 into } K \cdot \frac{1}{K} \text{ -}$$
$$= \lambda \cdot \bar{D} \qquad \text{- see question -}$$

4.

$\lambda_h =$ high priority Poisson arrival rate
$\lambda_l =$ low priority Poisson arrival rate
$\mu =$ service times rate

a)

Merging of Poisson arrivals: $\lambda = \lambda_h + \lambda_l$

M/M/1 so $s(n) = (1-\rho)\rho^n$ where $\rho = \frac{\lambda}{\mu} = \frac{\lambda_h+\lambda_l}{\mu}$

b)

because HP arrivals are either immediately served or enter HP queue we can basically ignore the LP customers for this distribution:
$s_h(n) = (1-\rho_h)\cdot\rho_h^n$ where $\rho_h = \frac{\lambda_h}{\mu}$

c)

$$E[S] = \frac{\rho}{1-\rho} \qquad \text{- always for M/M/1 -}$$
$$E[S_h] = \frac{\rho_h}{1-\rho_h}$$

$$\begin{aligned}
E[S_l] &= E[S] - E[S_h] \\
&= \frac{\rho}{1-\rho} - \frac{\rho_h}{1-\rho_h} \\
&= \frac{\rho - \rho_h + (\rho\rho_h - \rho\rho_h)}{(1-\rho)(1-\rho_h)} \\
&= \frac{\rho - \rho_h}{(1-\rho)(1-\rho_h)} \\
&= \frac{\frac{\lambda_h+\lambda_l}{\mu} - \frac{\lambda_h}{\mu}}{(1-\rho)(1-\rho_h)} \\
&= \frac{\frac{\lambda_l}{\mu}}{(1-\rho)(1-\rho_h)}
\end{aligned}$$

$$E[D] = \frac{E[S]}{\lambda} \qquad \text{Little's law}$$

$$\begin{aligned}
E[D_h] &= \frac{E[S_h]}{\lambda_h} \\
&= \frac{\frac{\lambda_h}{\mu}}{\lambda_h\left(1 - \frac{\lambda_h}{\mu}\right)} \\
&= \frac{1}{\mu - \lambda_h}
\end{aligned}$$

$$
\begin{aligned}
E\left[D_l\right] &= \frac{E\left[S_l\right]}{\lambda_h} \\
&= \frac{\lambda_l}{\mu\left(1-\rho\right)\left(1-\rho_h\right)\lambda_l} \\
&= \frac{1}{\left(1-\rho\right)\left(\mu-\mu\cdot\frac{\lambda_h}{\mu}\right)} \\
&= \frac{1}{\left(1-\rho\right)\left(\mu-\lambda_h\right)}
\end{aligned}
$$

5.

For M/M/1:

$$
\begin{aligned}
E\left[e^{-sW}\right] &= D^{*(s)} \\
&= \frac{\mu-\lambda}{\mu-\lambda+s}
\end{aligned}
$$

a)

LST of $\left(1-\rho\right)D_\rho$:

$$
\begin{aligned}
E\left[e^{-s\cdot(1-\rho)D_\rho}\right] &= E\left[e^{-(s\cdot(1-\rho))D_\rho}\right] \\
&= D^{*((1-\rho)s)} && \text{- } E\left[e^{-sW}\right] = D^{*(s)} \text{ -} \\
&= \frac{\mu-\lambda}{\mu-\lambda+\left(1-\rho\right)s} \\
&= \frac{\mu-\lambda}{\frac{(\mu-\lambda)(\mu+s)}{\mu}} \\
&= \frac{(\mu-\lambda)\mu}{\left(\mu-\lambda\right)\left(\mu+s\right)} \\
&= \frac{\mu}{\mu+s}
\end{aligned}
$$

b)

TODO expand

$\rho \to 1$

The limit is $\frac{\mu}{\mu+s}$(see a)) or $\frac{\lambda}{\lambda+s}$ since for $\rho \to 1, \lambda \to \mu$

discuss result

The delay $\left(1-\rho\right)D_\rho \sim \exp\left(\mu\right)$ (given), $D_\rho \sim \exp\left(\frac{\mu}{1-\rho}\right)$

$$\begin{aligned} Pr\left[(1-\rho)\,D_\rho < t\right] &= 1-e^{-\mu\cdot t} \\ Pr\left[\frac{(1-\rho)\,D_\rho}{1-\rho} < \frac{t}{1-\rho}\right] &= 1-e^{-\mu\cdot t} \\ Pr\left[D_\rho < \frac{t}{1-\rho}\right] &= 1-e^{-\mu\cdot t} \\ &= 1-e^{-(1-\rho)\mu\cdot\frac{t}{1-\rho}} \qquad \text{- split 1 -} \end{aligned}$$

$$\to Pr\left[D_\rho < y\right] = 1-e^{-(1-\rho)\mu\cdot y} \to D_\rho \sim \exp\left((1-\rho)\,\mu\right)$$

5 Mean Value Analysis

Theory

M/G/1 queue

- M: poisson arrivals, λ
- G $\rightarrow B(t)$ = distribution function of service times
- 1 server
- $b(t)$ = density function of service times
- load of the system: $\rho = \lambda \cdot E[B]$
- average service rate $\mu = \frac{1}{E[B]}$
- stability condition: $\rho < 1$
- remaining service time = $E[R] = \frac{E[B^2]}{2 \cdot E[B]}$ = inspection paradox
- $X \sim Exp(\mu) => E[X] = \frac{1}{\mu} => E[X^2] = \frac{2}{\mu^2}$
- sum of r exponential distributions $E_r \sim Erlang(r, r \cdot \mu) => E[X] = \frac{1}{\mu}$

1.

a)

M/D/1:

$$\begin{aligned} E[R] &= \frac{E[B^2]}{2 \cdot E[B]} \\ &= \frac{B}{2} && \text{deterministic, so } E[B] = B \end{aligned}$$

$$\begin{aligned} E[W] &= \frac{\rho}{1-\rho} \cdot E[R] \\ &= \frac{\rho}{1-\rho} \cdot \frac{B}{2} \\ &= \frac{\lambda \cdot B^2}{2(1-\rho)} && \text{- } \rho = \lambda \cdot B \text{ -} \\ &= \frac{\lambda \cdot E[B]^2}{2(1-\rho)} \end{aligned}$$

b)

M/M/1:

$$
\begin{aligned}
E\left[R\right] &= \frac{1}{\mu} && \text{- } \mu = \frac{1}{E[B]}\text{, memoryless so } E\left[R\right] = E\left[B\right] \text{ -}\\
E\left[W\right] &= \frac{\rho}{(1-\rho)} \cdot E[R] \\
&= \frac{\lambda}{(1-\rho)\cdot\mu^2} \\
&= \frac{\lambda \cdot E\left[B\right]^2}{1-\rho}
\end{aligned}
$$

c)

Erlang distribution = sum of r Exponential distributions
phase = one of the r successive exponential periods of time.
M/E_r/1:
$o\left(i\right)$ = phase i of the service is ongoing

$$
\begin{aligned}
E\left[R\right] &= \sum_{i=1}^{r} E\left[R|o\left(i\right)\right] \cdot Pr\left[o\left(i\right)\right] && \text{- sum over } R \text{ for all phases} \cdot \text{prob of being in phase -}\\
&= \sum_{i=1}^{r} (r-i+1) \cdot \frac{1}{\mu r} \cdot Pr\left[o\left(i\right)\right] && \text{- phases i till r, and } \frac{1}{\mu r} \text{ for each -}\\
&= \sum_{i=1}^{r} \frac{r-i+1}{\mu \cdot r} \cdot \frac{1}{r} && \text{- } Pr\left[o\left(i\right)\right] = \frac{1}{r}\text{, since it is equally distributed -}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{r^2 \cdot \mu} \sum_{i=1}^{r} (r-i+1) \\
&= \frac{1}{r^2 \cdot \mu} \left(\sum_{i=1}^{r} r + \sum_{i=1}^{r} 1 - \sum_{i=1}^{r} i \right) \\
&= \frac{1}{r^2 \cdot \mu} \cdot \left(r^2 + r - \sum_{i=1}^{r} i \right) \\
&= \frac{1}{r^2 \cdot \mu} \cdot \left(r^2 + r - \frac{r\left(r+1\right)}{2} \right) \\
&= \frac{1}{r^2 \cdot \mu} \cdot \frac{2r^2 + 2r - r^2 - r}{2} \\
&= \frac{1}{r^2 \mu} \cdot \frac{r(r+1)}{2} \\
&= \frac{r+1}{2 \cdot r \cdot \mu}
\end{aligned}
$$

$$
\begin{aligned}
E\left[W\right] &= \frac{\rho}{1-\rho}\cdot E\left[R\right] \\
&= \frac{r+1}{r}\cdot\frac{\rho}{2\cdot(1-\rho)\cdot\mu} \\
&= \frac{r+1}{r}\cdot\frac{\lambda\cdot E\left[B\right]^2}{2\cdot(1-\rho)} && \text{- } \rho=\lambda\cdot E\left[B\right],\ \frac{1}{\mu}=E\left[B\right] \text{ -}
\end{aligned}
$$

comparison of the three

- If $B=\frac{1}{\mu}$ for M/D/1: $E\left[R\right]=\frac{1}{2\cdot\mu}$
- For M/M/1: $E\left[R\right]=\frac{1}{\mu}$
- observe: $\frac{1}{2\cdot\mu}<\frac{1}{2\cdot\mu}+\frac{1}{2\cdot\mu\cdot r}\leq\frac{1}{\mu}$

2.

Theory

- $E[R]=\int_0^\infty E[R|\tilde{B}=t]\tilde{b}(t)dt$
- $\tilde{B}$ =random variable with $\tilde{b}$ its density
- $\tilde{B}=t\rightarrow$ condition on total length of the period that includes a randomly chosen point
- Relationship: $\tilde{b}(t)=\frac{t\cdot b(t)}{E[B]}$

a)

TODO expand

$$
\begin{aligned}
P\left[t<R<t+dt\right] &= \int_0^\infty P\left[t<R<t+dt|\widetilde{B}=u\right]\cdot\widetilde{b}\left(u\right)du && \text{- see theory -} \\
r(t)dt &= \int_0^\infty \frac{dt}{u}\cdot\widetilde{b}\left(u\right)du \\
r\left(t\right) &= \int_t^\infty \frac{\widetilde{b}(u)}{u}du && \text{- divide both sides by dt -} \\
&= \int_t^\infty \frac{b\left(u\right)}{E\left[B\right]}du && \text{- see theory -} \\
&= \frac{1}{E\left[B\right]}\cdot\int_t^\infty b\left(u\right)du
\end{aligned}
$$

$\int_t^\infty b\left(u\right)du=P\left[B\geq t\right]$

b)

$$
\begin{aligned}
R^*(s) &= \int_0^\infty r(t)\, e^{-st} dt && \text{- definition LST -}\\
&= \frac{1}{E[B]} \cdot \int_0^\infty \int_t^\infty b(u) \cdot du \cdot e^{-st} \cdot dt && \text{- substitute r(t) from a) -}\\
&= \frac{1}{E[B]} \cdot \int_0^\infty b(u) \int_0^u e^{-st} \cdot dt \cdot du && \text{- switching int limits: youtu.be/j6A44yQrGfU -}\\
&= \frac{1}{E[B]} \cdot \int_0^\infty b(u) \left[\frac{e^{-st}}{s}\right]_{t=0}^{t=u} du\\
&= \frac{1}{E[B]} \cdot \int_0^\infty b(u)\, \frac{1-e^{-su}}{s} du\\
&= \frac{1}{E[B]} \cdot \frac{1}{s} \int_0^\infty b(u) \left(1 - e^{-su}\right) du\\
&= \frac{1}{E[B]} \cdot \frac{1}{s} \left(\int_0^\infty b(u)\, du - \int_0^\infty b(u) \left(e^{-su}\right) du\right)\\
&= \frac{1}{s \cdot E[B]} \cdot (1 - B^*(s)) && \text{- def LST } B^*(s) \text{ -}\\
&= \frac{1 - B^*(s)}{s \cdot E[B]}
\end{aligned}
$$

Remark: $E[X^n] = (-1)^n \frac{d^n X^*(s)}{d \cdot s^n}\bigg|_{s=0}$

c)

TODO expand

$E\left[R^2\right] = \frac{E\left[B^3\right]}{3 \cdot E[B]} \to$ calculation similar to $E[R]$ below

Remark: $E[R^n] = \frac{E\left[B^{(n+1)}\right]}{(n+1) \cdot E[B]}$

$$
\begin{aligned}
E[R] &= -\frac{dR^*(s)}{ds}\bigg|_{s=0}\\
&= -\lim_{s \to 0} \frac{-\frac{dB^*(s)}{ds} \cdot s - (1 - B^*(s))}{s^2 \cdot E[B]} && \text{- } R^*(s) = \frac{1 - B^*(s)}{s \cdot E[B]}\text{, derivative of fraction -}\\
&= \frac{E\left[B^2\right]}{2 \cdot E[B]}
\end{aligned}
$$

3.

a)

$$
\begin{aligned}
E[B] &= 0 \cdot q + \frac{1}{1-q} \cdot (1-q)\\
&= 1
\end{aligned}
$$

b)

$\rho < 1 \Leftrightarrow \lambda \cdot E\left[B\right] < 1 \Leftrightarrow \lambda < 1$

c)

$$
\begin{aligned}
E\left[R\right] &= \frac{E\left[B^2\right]}{2 \cdot E\left[B\right]} \\
E\left[B^2\right] &= 0^2 \cdot q + \left(\frac{1}{1-q}\right)^2 (1-q) = \frac{1}{1-q} \\
\rightarrow E\left[R\right] &= \frac{1}{2 \cdot (1-q)}
\end{aligned}
$$

d)

$$
\begin{aligned}
E\left[W\right] &= \frac{\rho}{1-\rho} \cdot E\left[R\right] && \text{- syllabus p 5.3 -} \\
&= \frac{\lambda \cdot E\left[B\right]}{1 - \lambda \cdot E\left[B\right]} \cdot E\left[R\right] \\
&= \frac{1}{2} \cdot \frac{\lambda}{1-\lambda} \cdot \frac{1}{1-q} && \text{- fill in from a) and c) -}
\end{aligned}
$$

e)

$q \rightarrow 1$?
$E\left[W\right] \rightarrow \infty \rightarrow$ based on d)
$\rightarrow$ because some users will have an infinitely long service time

4.

a)

$$
\begin{aligned}
E\left[W\right] &= E\left[\sum_{j=1}^{Q_A} B_j^A\right] + Pr\left[S^A > 0\right] \cdot E\left[R\right] && \text{- definition p 5.2 -} \\
E\left[B_j^A\right] &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot E\left[B_1\right] + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot E\left[B_2\right] = E\left[B\right] \\
E\left[R\right] &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot E\left[R_1\right] + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot E\left[R_2\right] \\
\rho &= (\lambda_1 + \lambda_2) \cdot E\left[B\right] \\
&= \lambda_1 \cdot E\left[B_1\right] + \lambda_2 \cdot E\left[B_2\right] \\
Pr\left[S^A > 0\right] &= Pr\left[S > 0\right] = \rho
\end{aligned}
$$

5.

a)

- $\rho < 1 \Leftrightarrow \lambda \cdot E[c] \cdot E[B] < 1$
- $\lambda \cdot E[c] =$ effective arrival rate, λ_{eff}

b)

TODO explain ?? steps

$$E[W] = E\left[\sum_{j=1}^{Q^A+F} B_j^A\right] + Pr\left[S^A > 0\right] \cdot E[R]$$

- $B_j^A =$ service time of the j-th customer
- $Q^A =$ queue content
- $F =$ nr of arrivals before the tagged customer in the same batch

$$\begin{aligned}
E[W] &= E\left[Q^A + F\right] \cdot E\left[B^A\right] + Pr\left[S^A > 0\right] \cdot E[R] \text{ - p5.3 -} \\
&= E\left[Q^A\right] \cdot E\left[B^A\right] + E[F] \cdot E\left[B^A\right] + \rho \cdot E[R] \\
&= \lambda \cdot E[c] \cdot E[W] \cdot E\left[B^A\right] + E[F] \cdot E\left[B^A\right] + \rho \cdot E[R] \text{ - little} \rightarrow E[Q] = \lambda \cdot E[c] \cdot E[W] \text{ -} \\
&= \rho \cdot E[W] + E[F] \cdot E\left[B^A\right] + \rho \cdot E[R] \text{ - } \rho = \lambda \cdot E[c] \cdot E[B] \text{ -} \\
&= \frac{E[F] \cdot E\left[B^A\right] + \rho \cdot E[R]}{1-\rho}
\end{aligned}$$

$$\begin{aligned}
E[R] &= \frac{E\left[B^2\right]}{2 \cdot E[B]} \\
E[F] &= \sum_{i=1}^{\infty} E[F|\widetilde{c} = i] \cdot Pr[\widetilde{c} = i] && \text{- ?? -} \\
Pr[\widetilde{c} = i] &= \frac{i \cdot Pr[c = i]}{E[c]} && \text{- ?? -} \\
E[F] &= \sum_{i=1}^{\infty} \frac{i-1}{2} \cdot i \cdot \frac{Pr[c=i]}{E[c]} && \text{- } \frac{i-1}{2} \rightarrow \text{on avg, half of others in batch are in front of you -} \\
&= \frac{1}{2 \cdot E[c]} \cdot \sum_{i=1}^{\infty} (i-1)\, i Pr[c=i] \\
&= \frac{E[(c-1)\,c]}{2 \cdot E[c]} && \text{- ?? -}
\end{aligned}$$

c)

- similar to b)
- or using little's law:

$$\begin{aligned} E[Q] &= \lambda_{eff} \cdot E[W] \\ &= \lambda \cdot E[c] \cdot E[W] \end{aligned}$$

6 Transform analysis M/G/1

Theory

$$\begin{aligned} \text{LST of X} : X^*(s)\, E\left[e^{-sX}\right] &= \int_0^\infty e^{-st} \cdot dF(t) & \text{- } F(t) \rightarrow \text{cdf -} \\ &= \int_0^\infty e^{-st} \cdot x(t)\, dt & \text{- } x(t) \rightarrow \text{density -} \end{aligned}$$

$$\text{LST of X} \sim Exp(\eta) : X^*(s) = \frac{\eta}{\eta + s}$$

$$\begin{aligned} \text{PGF of Y} : Y(\eta) &= \sum_{i=0}^{\infty} y(i) \cdot \eta^i \\ &= \sum_{i=0}^{\infty} Pr[y = i] \cdot \eta^i \\ &= E[\eta^y] \end{aligned}$$

load $\rho = \lambda \cdot E[B]$

PGF of system content in M/G/1:

$$S(z) = (1-\rho)\frac{B^*(\lambda(1-z))(1-z)}{B^*(\lambda(1-z)) - z} \qquad \text{- with } B^*(s) \text{ LST of service distribution -}$$

LST of sojourn time:

$$D^*(s) = (1-\rho)\frac{s \cdot B^*(s)}{s - \lambda + \lambda B^*(s)}$$

$$\begin{aligned} D &= W + B \\ D^*(s) &= W^*(s) \cdot B^*(s) \end{aligned}$$

$$\int_{-\infty}^{+\infty} f(x)\, d(x - x_0) \cdot dx = f(x_0) \qquad \text{- for deterministic distr -}$$

1.

a)

LST of $Exp\left(1\right) \rightarrow B_1^*\left(s\right) = \frac{1}{1+s}$, see theory
LST of $B_2 = 1 \rightarrow B_2^*\left(s\right) = e^{-s\cdot 1}$
$\Rightarrow$ LST of dirac function(LST= e^{-sc}), because we have a r.v. with a spike in one place (1)
$\rightarrow B^*\left(s\right) = B_1^*\left(s\right) \cdot B_2^*\left(s\right) = \frac{e^{-s}}{1+s}$

b)

remember

if X a r.v. :

$$E\left[X^n\right] = \left(-1\right)^n \cdot \left.\frac{d^n X^*\left(s\right)}{ds}\right|_{s=0}$$

$$\begin{aligned} S\left(z\right) &=? \\ \rho &= \lambda \cdot E\left[B\right] \\ &= -\lambda \cdot \left.\frac{dB^*\left(s\right)}{ds}\right|_{s=0} \\ &= -\lambda \cdot \left.\left(\frac{e^{-s}\cdot\left(-1\right)\left(1+s\right) - e^{-s}\cdot 1}{\left(1+s\right)^2}\right)\right|_{s=0} && \text{- derivative } \frac{e^{-s}}{1+s} \text{ -} \\ &= -\lambda \cdot \left.\left(\frac{-2\cdot e^{-s} - s\cdot e^{-s}}{\left(1+s\right)^2}\right)\right|_{s=0} \\ &= -\lambda \cdot \frac{-2-0}{1^2} \\ &= 2\cdot\lambda \end{aligned}$$

$$B^*\left(\lambda\left(1-z\right)\right) = \frac{e^{-\lambda\left(1-z\right)}}{1+\lambda\left(1-z\right)} \qquad \text{- see a), theory -}$$

Thus

$$S\left(z\right) = \left(1-2\cdot\lambda\right)\frac{e^{-\lambda\left(1-z\right)}\left(1-z\right)}{e^{-\lambda\left(1-z\right)} - z\left(1+\lambda-\lambda\cdot z\right)} \qquad \text{- see theory -}$$

c)

$$\begin{aligned} D^*\left(s\right) &= \left(1-2\cdot\lambda\right)\cdot\frac{s\cdot\frac{e^{-s}}{1+s}}{s-\lambda+\lambda\cdot\frac{e^{-s}}{1+s}} && \text{- see theory -} \\ &= \left(1-2\cdot\lambda\right)\cdot\frac{s\cdot e^{-s}}{\left(s-\lambda\right)\left(1+s\right)+\lambda\cdot e^{-s}} \end{aligned}$$

d)

$$
\begin{aligned}
W^*(s) &= \frac{D^*(s)}{B_1^*(s)} && \text{- see theory -}\\
&= (1+s)\cdot D^*(s) && \text{- see a) -}\\
&= (1-2\lambda)\,\frac{(1+s)\,s\cdot e^{-s}}{(s-\lambda)(1+s)+\lambda+\lambda\cdot e^{-s}} && \text{- fill in from c) -}
\end{aligned}
$$

2.

a)

$$
\begin{aligned}
S_{k+1}^D &= S_k^D - 1 + C_{k+1} && \text{- if } S_k^D \neq 0, C_{k+1} = \text{nr arrivals during service of customer k+1 -}\\
S_{k+1}^D &= C_{k+1} + F_{k+1} && \text{- if } S_k^D = 0, F = \text{nr arrivals during set-up -}
\end{aligned}
$$

b)

$$
\begin{aligned}
S_{k+1}^D(z) &= E\left[z^{S_{k+1}^D}\right] && \text{- definition -}\\
&= E\left[z^{S_k^D-1}\right]\cdot E\left[z^{C_{k+1}}\right]\cdot E\left[z^{F_{k+1}}\right] && \text{- see a) -}
\end{aligned}
$$

$$
\begin{aligned}
E\left[z^{F_{k+1}}\right] &= E\left[z^{F_{k+1}}|S_k^D=0\right]\cdot Pr\left[S_k^D=0\right] + E\left[z^{F_{k+1}}|S_k^D>0\right] Pr\left[S_k^D>0\right]\\
&= E\left[z^{F_{k+1}}|S_k^D=0\right]\cdot Pr\left[S_k^D=0\right] + E\left[z^0\right] Pr\left[S_k^D>0\right] && \text{- ?? -}\\
&= E\left[z^{F_{k+1}}|S_k^D=0\right]\cdot Pr\left[S_k^D=0\right] + 1\cdot Pr\left[S_k^D>0\right]
\end{aligned}
$$

$$
\begin{aligned}
E\left[z^{F_{k+1}}|S_k^D=0\right] &= \int_0^\infty E\left[z^{F_{k+1}}|S_k^D=0, I_{k+1}=t\right] i(t)\,dt && \text{- ?? -}\\
&= \int_0^\infty e^{\lambda t(z-1)} i(t)\cdot dt && \text{- ?? -}\\
&= I^*(\lambda(1-z))
\end{aligned}
$$

$$
\begin{aligned}
E\left[z^{F_{k+1}}\right] &= I^*(\lambda(1-z))\cdot S_k^D(0) + 1\cdot\left(1-S_k^D(0)\right)\\
&= 1 + S_k^D(0)\left(I^*(\lambda(1-z))-1\right)
\end{aligned}
$$

$$
S_{k+1}^D(z) = \frac{\left(1+S_k^D(0)\left(I^*(\lambda(1-z))-1\right)\right)B^*(\lambda(1-z))\left(S_k^D(z)+(z-1)\,S_k^D(0)\right)}{z}
$$

$$
S^D(z) = \frac{\left(1+S^D(0)\left(I^*(\lambda(1-z))-1\right)\right)B^*(\lambda(1-z))(1-z)\cdot S^D(0)}{\left(1+S_k^D(0)\left(I^*(\lambda(1-z))-1\right)\right)B^*(\lambda(1-z))-z}
$$

To find $S^D(0)$, say $S^D(1)=1$ and solve to $S^D(0)$ using Hopital, see p 5.11

c)

$$S(z) = S^D(z)$$ - see p 5.8-9 -

d)

substitute $I^*(s) = \frac{\alpha}{\alpha+s}$

3.

a)

condition on whether customers arrive during vacation or not

$$E\left[z^{S^D_{k+1}}\right] = E\left[z^{S^D_{k+1}}|S_{k=0}\right] Pr\left[S_k = 0\right] + E\left[z^{S^D_{k+1}}|S_{k\neq 0}\right] Pr\left[S_k \neq 0\right]$$

G = nr customers arriving during server vacation after serving customer

$$\begin{aligned}E\left[z^G\right] &= \int_0^\infty E\left[z^G|S^D_{k+1} = t\right] v(t)\, dt \\ G(z) &= V^*(\lambda(1-z))\end{aligned}$$

$$\begin{aligned}G(z) &= E\left[z^G|G>0\right](1 - P[G=0]) + E\left[z^G|G=0\right] P[G=0] \\ E\left[z^G|G>0\right] &= \frac{G(z) - E\left[z^0\right] P[G=0]}{1 - P[G=0]} \\ &= \frac{G(z) - 1\cdot G(0)}{1 - G(0)} \\ &= \frac{V^*(\lambda(1-z)) - V^*(\lambda)}{1 - V^*(\lambda)}\end{aligned}$$

5.

PGF of X:

$$\begin{aligned}X(z) &= \sum_{i=0}^{\infty} x(i)\cdot z^i \\ &= E\left[z^x\right]\end{aligned}$$

LST of X:

$$X(s) = E\left[e^{-sX}\right]$$

a)

$$
\begin{aligned}
S^{r^*}(s) &= E\left[e^{-s\cdot S^r}\right] \\
&= E\left[e^{-s\cdot(1-\rho)S}\right] && \text{- fill in from given -} \\
&= E\left[\left(e^{-s(1-\rho)}\right)^S\right] \\
&= S\left(e^{-s(1-\rho)}\right) && \text{- } E[z^S] = S(z) \text{ -} \\
&= (1-\rho)\cdot\frac{B^*\left(\lambda\cdot\left(1-e^{-s(1-\rho)}\right)\right)\left(1-e^{-s(1-\rho)}\right)}{B^*\left(\lambda\cdot\left(1-e^{-s(1-\rho)}\right)\right)-e^{-s(1-\rho)}} && \text{- see theory -}
\end{aligned}
$$

b)

$$
\begin{aligned}
\lim_{\rho\to 1} S^{r^*}(s) &= \lim_{\lambda\to\frac{1}{E[B]}} S^{r^*}(s) \\
&= \dots \\
&= \frac{2\cdot E[B]^2}{E[B^2]\cdot s + 2E[B]^2} \\
&= \frac{\frac{2\cdot E[B]^2}{E[B^2]}}{\frac{2\cdot E[B]^2}{E[B^2]}+s} && \text{- divide num and denom by } E[B^2]\text{-} \\
\lim_{\rho\to 1} S^r \sim Exp(\alpha), \alpha &= \frac{2\cdot E[B]^2}{E[B^2]} && \text{- LST of Exp -}
\end{aligned}
$$

c)

$$
\begin{aligned}
P[S^r > x] &\approx e^{-\alpha x} && \text{- } Pr[S^r \leq x] \approx 1-e^{-\alpha\cdot x} \text{ -} \\
P[(1-\rho)S > x] = P\left[S > \frac{x}{1-\rho}\right] &\approx e^{-\alpha\cdot x} && \text{- fill in from given a) -} \\
P[S > y] &\approx e^{-\alpha\cdot y(1-\rho)} && \text{- for } y = \frac{x}{1-\rho}, \rho\approx 1 \text{ -}
\end{aligned}
$$

$$
Pr\left[S > \frac{x}{1-\rho}\right] \approx e^{-\alpha\cdot\frac{x}{1-\rho}\cdot(1-\rho)} = e^{-\alpha\cdot x}
$$

7 Inversion of PGFs and LSTs

Theory

$M/E_r/1$, r phases, mean value $\frac{1}{\mu} = \frac{r}{r\cdot u}$

LST of service times:

$$B^*(s) = \left(\frac{r \cdot u}{s + r \cdot u}\right)^r$$

PGF of X

$$\begin{aligned} x(z) &= \sum_{i=0}^{n} x_i z^i \\ &= \sum_{i=0}^{n} Pr[x = i] z_i \end{aligned}$$

M/G/1, PGF of system content:

$$S(z) = (1 - \rho) \cdot \frac{B^*(\lambda(1 - z))(1 - z)}{B^*(\lambda(1 - z)) - z} \qquad \text{- } B^* = \text{LST of service time -}$$

1.

a)

$$\begin{aligned} S(z) &= \sum_{i=0}^{\infty} s(i) z^i \qquad \text{- definition PGF -} \\ s(i) &= ? \end{aligned}$$

example for M/Er/1

$$\begin{aligned} S(z) &= \frac{1 - \rho}{1 - \rho \cdot z} = (1 - \rho) \frac{1}{1 - \rho \cdot z} \\ &= (1 - \rho) \cdot \left(\sum_{i=0}^{\infty} \rho^i z^i\right) \\ &= \sum_{i=0}^{\infty} (1 - \rho) \rho^i z^i \end{aligned}$$

$$B^*(s) = \left(\frac{z\cdot u}{s+z\cdot u}\right)^z \qquad \text{- see theory -}$$

$$\begin{aligned}
S(z) &= (1-\rho)\cdot\frac{(2\mu)^2\cdot(1-z)}{(2\mu)^2 - z\cdot(\lambda\cdot(1-z)+2\mu)^2} \qquad \text{- see theory -}\\
&= (1-\rho)\cdot\frac{4\mu^2\cdot(1-z)}{4\mu^2 - z\cdot\left((\lambda\cdot(1-z))^2+4\mu^2\right)+2\cdot\lambda(1-z)\cdot 2\mu}\\
&= (1-\rho)\cdot\frac{\mu^2\cdot 4(1-z)}{\mu^2\left(4-\frac{z\cdot\lambda^2\cdot(1-z)^2}{\mu^2}\right)-4z-\frac{4z\cdot\lambda(1-z)}{\mu}}\\
&= (1-\rho)\cdot\frac{4(1-z)}{4-z\cdot\rho^2\cdot(1-z)^2-4z-4z\cdot\rho(1-z)}\\
&= \frac{4(1-\rho)}{4-z\cdot\rho^2(1-z)-4z\cdot\rho}\\
&= \frac{4(1-\rho)}{4-(\rho^2+4\rho)z+\rho^2z^2}
\end{aligned}$$

We want to split up in partial fractions → find zeros of denominator(poles) of $S(z)$:

$$\rho^2z^2-\left(\rho^2+4\rho\right)z+4=0$$

$$\begin{aligned}
D &= b^2-4\cdot a\cdot c\\
&= \left(\rho^2+4\rho\right)^2-4\rho^2\cdot 4\\
&= \rho^4+8\rho^3+16\rho^2-16\rho^2\\
&= \rho^4+8\rho^3
\end{aligned}$$

$$\begin{aligned}
z_{1,2} &= \frac{-b\pm\sqrt{D}}{2a}\\
&= \frac{\rho^2+4\rho\pm\sqrt{\rho^4+8\rho^3}}{2\rho^2}\\
&= \frac{\rho+4\pm\sqrt{\rho^2+8\rho}}{2\rho}
\end{aligned}$$

(p 5.16) assume:

$$\begin{aligned}
S(z) &= A_1\cdot\frac{1}{1-\frac{z}{z_1}}+A_2\cdot\frac{1}{1-\frac{z}{z_2}}\\
&= A_1\cdot\sum_{i=0}^{\infty}\left(\frac{z}{z_1}\right)^i+A_2\cdot\sum_{i=0}^{\infty}\left(\frac{z}{z_2}\right)^i\\
&= \sum_{i=0}^{\infty}A_1\cdot z^i\cdot z_1^{-i}+\sum_{i=0}^{\infty}A_2\cdot z^i\cdot z_2^{-i}\\
&= \sum_{i=0}^{\infty}\left(A_1\cdot z_1^{-i}+A_2\cdot z_2^{-i}\right)z^i
\end{aligned}$$

We need to find A_1, A_2

$$\begin{aligned}
S(z) &= A_1 \cdot \frac{z_1}{z_1 - z} + A_2 \cdot \frac{z_2}{z_2 - z} \\
&= \frac{A_1 (z_2 - z) z_1 + A_2 (z_1 - z) z_2}{(z_1 - z)(z_2 - z)} \\
&= \frac{A_1 (z_2 - z) z_1 + A_2 (z_1 - z) z_2}{(z - z_1)(z - z_2)}
\end{aligned}$$

we know we miss a factor in the denominator:

$$Denominator(z) = \rho^2 (z - z_1)(z - z_2)$$

Thus:

$$A_1 (z_2 - z) z_1 + A_2 (z_1 - z) z_2 = \frac{4(1-\rho)}{\rho^2}$$

$$\begin{aligned}
z_1 \cdot z_2 &= \frac{(\rho+4)^2 - \left(\sqrt{\rho^2 + 8\rho}\right)^2}{4\rho^2} \\
&= \frac{\rho^2 + 16 + 8\rho - \rho^2 - 8\rho}{4\rho^2} \\
&= \frac{16}{4\rho^2} \\
&= \frac{4}{\rho^2}
\end{aligned}$$

We intelligently choose values of z to make one of the terms 0 in:

$$A_1 (z_2 - z) z_1 + A_2 (z_1 - z) z_2 = (4(1-\rho))$$

$z = z_1$:

$$\begin{aligned}
A_1 (z_2 - z_1) z_1 &= \frac{4(1-\rho)}{\rho^2} \\
A_1 &= \frac{4(1-\rho)}{\rho^2 z_1 (z_2 - z_1)} \qquad \text{- } z_2 = \frac{4}{\rho^2 \cdot z_1} \text{ -}
\end{aligned}$$

$z = z_2$:

$$\begin{aligned}
A_2 (z_1 - z_2) z_2 &= \frac{4(1-\rho)}{\rho^2} \\
A_2 &= \frac{(1-\rho) z_1}{z_1 - z_2}
\end{aligned}$$

b)

$$\begin{aligned}
Pr\left[S > i\right] &= \sum_{n=i+1}^{\infty} s\left(n\right) \\
&= \sum_{n=i+1}^{\infty} \left(A_1 z_1^{-n} + A_2 z_2^{-n}\right) \\
&= \sum_{n=i+1}^{\infty} \left(1-\rho\right)\left(\frac{z_2 \cdot z_1^{-n} - z_1 \cdot z_2^{-n}}{z_2 - z_1}\right) \\
&= \frac{1-\rho}{z_2 - z_1}\left(\sum_{n=0}^{\infty} z_2 \cdot \left(\frac{1}{z_1}\right)^{n+i+1} - \sum_{n=0}^{\infty} z_1 \cdot \left(\frac{1}{z_2}\right)^{n+i+1}\right) \\
&= \frac{1-\rho}{z_2 - z_1}\left(z + 2z_1^{-i-1}\sum_{n=0}^{\infty}\left(\frac{1}{z_1}\right)^{n} - z_1 z_2^{-i-1}\sum_{n=0}^{\infty} z_1 \cdot \left(\frac{1}{z_2}\right)^{n}\right) \\
&= \frac{1-\rho}{z_2 - z_1}\left(z + 2z_1^{-i-1}\frac{1}{1-\frac{1}{z_1}} - z_1 z_2^{-i-1}\frac{1}{1-\frac{1}{z_2}}\right) \\
&= \frac{1-\rho}{z_2 - z_1}\frac{z + 2z_1^{-i-1}}{z_1 - 1} - \frac{z_1 z_2^{-i-1}}{z_2 - 1}
\end{aligned}$$

c)

$$\begin{aligned}
D^*\left(s\right) &= \left(1-\rho\right)\frac{s \cdot \left(\frac{2\mu}{2\mu+s}\right)^2}{s - \lambda + \lambda\left(\frac{2\mu}{2\mu+s}\right)^2} \qquad \text{- definition LST of sojourn time -} \\
&= \left(1-\rho\right)\frac{s \cdot 4\mu^2}{\left(s-\lambda\right)\left(2\mu+s\right)^2 + \lambda 4\mu^2} \\
&= \left(1-\rho\right)\frac{4\mu^2 s}{4\mu^2 s + 4\mu^2 s + s^3 - \lambda 4\mu^2 - \lambda 4\mu s - \lambda s^2 + \lambda 4\mu^2} \\
&= \left(1-\rho\right)\frac{4\mu^2}{s^2 + \left(4\mu - \lambda\right)s + 4\mu\left(\mu - \lambda\right)}
\end{aligned}$$

poles of denominator:

$$\begin{aligned}
D &= \left(4\mu - \lambda\right)^2 - 4 \cdot 1 \cdot 4\mu\left(\mu - \lambda\right) \\
&= 16\mu^2 + \lambda^2 - 8\mu \cdot \lambda - 16\mu^2 + 16\mu\lambda \\
&= \lambda^2 + 8\mu\lambda
\end{aligned}$$

$$s_{1,2} = \frac{-\left(4\mu - \lambda\right) \pm \sqrt{\lambda^2 + 8\mu \cdot \lambda}}{2}$$

we know: $1 \cdot \left(s - s_1\right)\left(s - s_2\right) =$ denominator of $D^*\left(s\right)$

$$\begin{aligned}
D^*\left(s\right) &= A_1 \cdot \frac{s_1}{s - s_1} + A_2 \cdot \frac{s_2}{s - s_2} \\
&= \frac{A_1 \cdot s_1\left(s - s_2\right) + A_2 \cdot s_2\left(s - s_1\right)}{\left(s - s_1\right)\left(s - s_2\right)}
\end{aligned}$$

$$A_1 s_1 (s - s_2) + A_2 s_2 (s - s_1) = \frac{(1-\rho)\,4\mu^2}{1}$$
$$A_1 s_1 (s_1 - s_2) = (1-\rho)\,4\mu^2$$
$$A_1 = \frac{(1-\rho)\,4\mu^2}{s_1 (s_1 - s_2)}$$
$$A_2 s_2 (s_2 - s_1) = (1-\rho)\,4\mu^2$$
$$A_2 = \frac{(1-\rho)\,4\mu^2}{s_2 (s_2 - s_1)}$$

$$D^*(s) = -A_1 \cdot \frac{-s_1}{-s_1 + s} + -A_2 \frac{-s_2}{-s_2 + s}$$

$$\begin{aligned} d(t) &= -A_1 (-s_1) e^{s_1 t} - A_2 (-s_2) e^{s_2 t} \\ &= A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t} \end{aligned}$$

hyperexponential distribution with 2 phases;

- A_i is the prob of phase i
- s_i is the rate if the i-th exponential
- see p 5.23-24

3.

a)

$$X_c(z) = \sum_{i=0}^{n} Pr[x > i]\, z^i$$
$$X(z) = \sum_{i=0}^{n} Pr[x = i]\, z^i$$
$$Pr[x > i] = \sum_{i=0}^{\infty} Pr[x = n]$$

$$\begin{aligned} X_c(z) &= \sum_{i=0}^{\infty} \left(\sum_{n=i+1}^{\infty} Pr[x = n]\, z^i \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=0}^{n-1} Pr[x = n]\, z^i \right) \\ &= \sum_{n=1}^{\infty} Pr[X = n] \sum_{i=0}^{n-1} z^i \\ &= \sum_{n=0}^{\infty} Pr[X = n] \frac{1 - z^n}{1 - z} && \text{- n=0 because product is 0 for n=0 -} \\ &= \frac{1}{1-z} \sum_{n=0}^{\infty} Pr[X = n] (1 - z^n) \\ &= \frac{1 - x(z)}{1 - z} \end{aligned}$$

b)

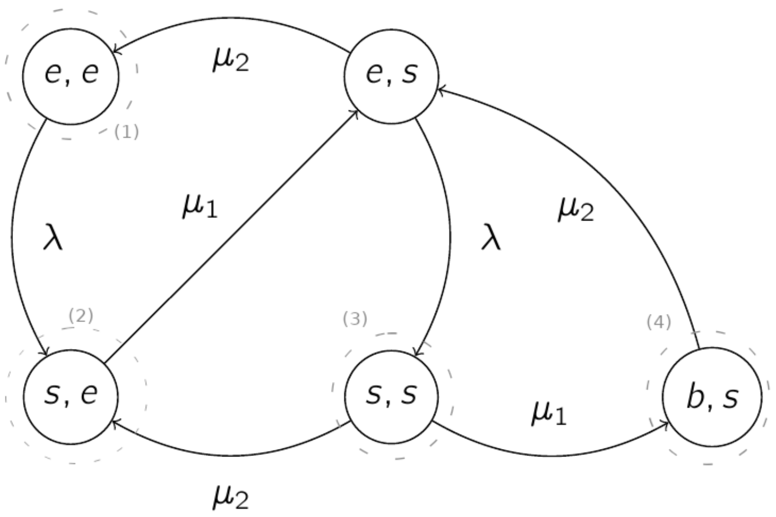
the singularities are the same

8 Queueing networks

1.

a)

e = empty, s = in service, b = blocked



b)

stationary distribution → global balance equations:

$$
\begin{aligned}
(1)\, \lambda \cdot p\,(e,e) &= \mu_2 \cdot p\,(e,s) \\
(2)\, \lambda \cdot p\,(e,e) + \mu_2 \cdot p\,(s,s) &= \mu_1 \cdot p\,(s,e) \\
(3)\, \lambda \cdot p\,(e,s) &= (\mu_2 + \mu_1) \cdot p\,(s,s) \\
(4)\, \mu_1 \cdot p\,(s,s) &= \mu_2 \cdot p\,(b,s)
\end{aligned}
$$

normalization condition: $1 = p\,(e,e) + p\,(s,e) + p\,(e,s) + p\,(s,s) + p\,(b,s)$

$$\ldots$$

$$p(e,e) = \frac{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$
$$p(e,s) = \frac{\lambda\mu_1\mu_2\left(\mu_1+\mu_2\right)}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$
$$p(s,s) = \frac{\lambda^2\mu_1\mu_2}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$
$$p(s,e) = \frac{\lambda\mu_2^2\left(\lambda+\mu_1+\mu_2\right)}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$
$$p(b,s) = \frac{\lambda^2\mu_1^2}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$

c)

$$E\left[S\right] = 1\cdot\left(p\left(s,e\right)+p\left(e,s\right)\right)+2\cdot\left(p\left(s,s\right)+p\left(b,s\right)\right)+0\cdot p\left(e,e\right)$$
$$E\left[S\right] = \cdots$$
$$E[S] = \frac{\lambda\left[\lambda\left(2\mu_1^2+2\mu_1\mu_2+\mu_2^2\right)+\mu_2\left(\mu_1+\mu_2\right)^2\right]}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$

d)

$p\left(e,e\right)+p\left(e,s\right) =$ proportion of time arrival can actually occur $=$ when 1st node is empty(e)

$$\lambda_{eff} = \lambda\cdot\left(p\left(e,e\right)+p\left(e,s\right)\right)$$
$$\lambda_{eff} = \cdots$$
$$\lambda_{eff} = \frac{\lambda\mu_1\mu_2\left(\lambda+\mu_2\right)\left(\mu_1+\mu_2\right)}{\mu_1\mu_2^2\left(\mu_1+\mu_2\right)+\lambda\mu_2\left(\mu_1+\mu_2\right)^2+\lambda^2\left(\mu_1^2+\mu_1\mu_2+\mu_2^2\right)}$$

e)

$$p_{block} = P[\text{admitted customer gets blocked}]$$

blocking requires two conditions:

- first server is empty on arrival and second server is serving
- service of 1st node finishes before 2nd node

$$\begin{aligned} p_{block} &= P[(e,s) \text{ on arrival x 1st service finishes before 2nd} \mid \text{customer is admitted }] \\ &= P[(e,s) \text{ on arrival } |\text{customer is admitted}] \cdot P[\text{1st service finishes before 2nd}] \\ &= \frac{p\,(e,s)}{p\,(e,e) + p\,(e,s)} \cdot \frac{\mu_1}{\mu_1 + \mu_2} \\ &= \frac{\lambda \mu_1}{(\lambda + \mu_2)\,(\mu_1 + \mu_2)} \end{aligned}$$

f)

$$\begin{aligned} E\,[D] &= \frac{E\,[S]}{\lambda_{eff}} && \text{- little's law -} \\ &= \dots \\ &= \frac{(\lambda + \mu_2)\,(\mu_1 + \mu_2)^2 + \lambda \cdot (\mu_1)^2}{\mu_1 \cdot \mu_2 \cdot (\lambda + \mu_2)\,(\mu_1 + \mu_2)} \\ &= \frac{1}{\mu_1} \cdot \mu_2 + p_{block} \cdot \frac{1}{\mu_2} \end{aligned}$$

2.

we solve without allowing use of reversibility property otherwise Burke's theorem would immediately lead to the solution.

- T: distribution of time between departures
- s_1: system content just before departure
- $s_1 - 1$: system content just after departure
- $d_0 = Pr\,[s_1 - 1 = 0]$
- $1 - d_0 = Pr\,[s_1 - 1 > 0]$
- (i) $s - 1 > 0 \rightarrow T \sim Exp\,(|\mu)$
- (ii) $s - 1 = 0 \rightarrow T \approx Exp\,(|\lambda) + Exp\,(|\mu)$

$$\begin{aligned} T^*\,(s) &= E\left[e^{-sT}\right] = E\left[E\left[e^{-sT}|s_1 - 1\right]\right] \\ &= E\left[e^{-sT}|s_1 - 1 = 0\right] \cdot d_0 + E\left[e^{-sT}|s_1 - 1 > 0\right] \cdot (1 - d_0) \\ &= d_0 \cdot \frac{\lambda}{\lambda + s} \cdot \frac{\mu}{\mu + s} + (1 - d_0) \cdot \frac{\mu}{\mu + s} \end{aligned}$$

We know from 3.14: system content after departure $\sim$ system content at arrival instants

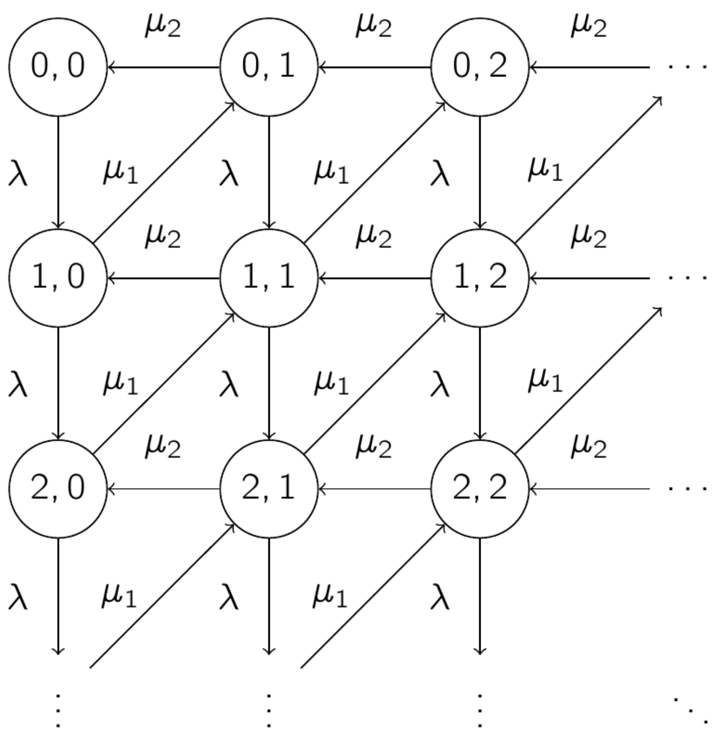
through PASTA we know: $d_0 = 1 - \rho, \rho = \frac{\lambda}{\mu}$

$$T^*(s) = \frac{\lambda}{\lambda + s} \implies T \sim Exp(\ldots|\lambda) \qquad \text{- } \rightarrow \text{ same as with Burke -}$$

3.

a)

Define S_i as nr of customers in node $i, i = 1, 2$. Then (S_1, S_2) is a CTMC with state space $\mathbb{N} \times \mathbb{N}$ and state transition diagram:



b)

$$\begin{aligned}
\lambda s(0,0) &= \mu_2 s(0,1) \\
(\lambda + \mu_1)\, s(n_1, 0) &= \lambda s(n_1 - 1, 0) + \mu_2 s(n_1, 1) \\
(\lambda + \mu_2)\, s(0, n_2) &= \mu_1 s(1, n_2 - 1) + \mu_2 s(0, n_2 + 1) \\
(\lambda + \mu_1 + \mu_2)\, s(n_1, n_2) &= \lambda s(n_1 - 1, n_2) + \mu_1 s(n_1 + 1, n_2 - 1) + \mu_2 s(n_1, n_2 + 1)
\end{aligned}$$

4.

used formulas

- M/M/1 with arrival rate λ, service rate μ
- $Pr\left[s=n\right]=(1-\rho)\,\rho^n$ with $\rho=\frac{\lambda}{\mu}$, $n=0,1,2\ldots$
- $E\left[S\right]=\frac{\lambda}{\mu-\lambda}=\frac{\rho}{\rho-1}$

solution

a)

stability condition for Jackson network: $\lambda\leq\mu \implies$ for our network: $2\cdot\lambda<\mu_1+\mu_2 \implies \lambda_1=\lambda_2=\lambda \implies$ Burke's theorem

b)

$E\left[D\right]=\frac{E[S]}{\lambda}$ little's law

we have a feedforward network:

$$\begin{aligned}
E\left[S\right] &= E\left[S_1+S_2\right]\\
&= \sum_{n_1,n_2}(n_1+n_2)\cdot Pr\left[S_1=n_1,S_2=n_2\right]\\
&= \sum_{n_1,n_2}(n_1+n_2)\cdot Pr\left[S_1=n_1\right]\cdot Pr\left[S_2=n_2\right]\\
&= E\left[S_1\right]+E\left[S_2\right] && \text{- because Jackson Network -}\\
&= \frac{\lambda}{\mu_1-\lambda}+\frac{\lambda}{\mu_2-\lambda}\\
&= \frac{\lambda}{\mu_1-\lambda}+\frac{\lambda}{(\mu-\mu_1)-\lambda}
\end{aligned}$$

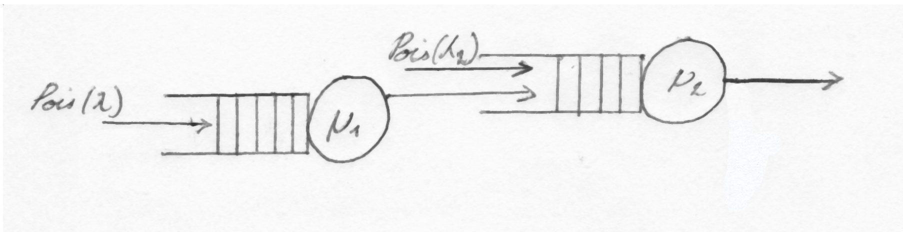
We minimize E[S], through Little's law this results in a minimal E[D]:

$$\begin{aligned}
\frac{d}{d\mu_1}E\left[S\right] &= \frac{d}{d\mu_1}\left[\frac{\lambda}{\lambda-\mu_1}+\frac{\lambda}{\mu-\mu_1-\lambda}\right]=0\\
\frac{d}{d\mu_1}E\left[D\right] &= \frac{d}{d\mu_1}\left[\frac{\lambda}{\lambda-\mu_1}+\frac{\lambda}{\mu-\mu_1-\lambda}\right]=0\\
\Leftrightarrow \mu_1 &= \frac{\mu}{2}
\end{aligned}$$

$$\begin{aligned}
\mu_{1opt} &= \frac{\mu}{2} && \mu_{1opt} \text{ in }]\lambda,\mu-\lambda[\\
\mu_{2opt} &= \mu-\frac{\mu}{2}=\frac{\mu}{2}
\end{aligned}$$

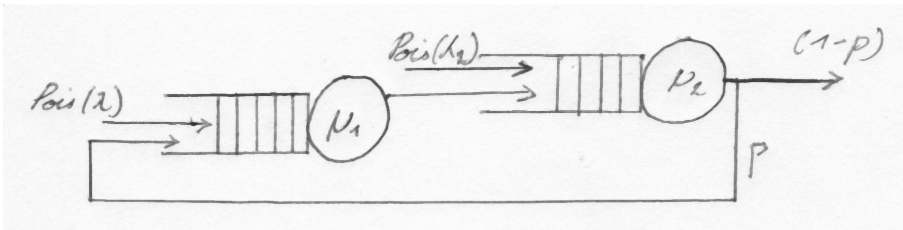
stability condition: $\lambda<\mu_1$

c)



$$\mu_1 = \frac{\mu-\gamma}{2} = \lambda + \frac{\mu-2\lambda-\gamma}{2} \qquad \text{and} \qquad \mu_2 = \frac{\mu+\gamma}{2} = \lambda+\gamma+\frac{\mu-2\lambda-\gamma}{2}$$

d)



traffic equations:

$$\lambda_1 = \lambda + p\cdot\lambda_2$$
$$\lambda_2 = \lambda_1 + \gamma$$

$$\lambda_1 = \frac{\lambda + p\cdot\gamma}{1-p}$$
$$\lambda_2 = \frac{\lambda+\gamma}{1-p}$$

as before we solve $\frac{dE[S]}{d\mu_1} = 0$ for $\mu_1 \ldots$

The constraint now becomes:

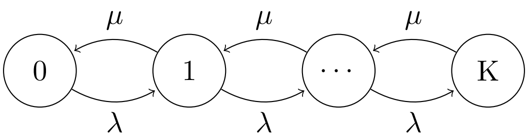
$$\mu > \frac{2\lambda + (1+p)\gamma}{1-p}$$

The optimal distribution of the total service rate μ in this case is:

$$\mu_1 = \frac{\lambda+p\gamma}{1-p} + \frac{(1-p)\mu - 2\lambda - (1+p)\gamma}{2(1-p)} \quad \text{and} \quad \mu_2 = \frac{\lambda+\gamma}{1-p} + \frac{(1-p)\mu - 2\lambda - (1+p)\gamma}{2(1-p)}$$

5.

a)



Local balance equation:

$$\lambda s(n) = \mu \cdot s(n+1) \qquad\qquad n = 0, \ldots, K-1$$

A (normalized) solution for these balance equations can be found. Therefore—see (CN: Section 2.2, p. 6.6)—the CTMC is reversible.

b)

Burke's theorem states that when a queue is reversible the arrival process must correspond to the departure process. The arrival process is not poisson because customers get discarded if K customers are in the queue $\Rightarrow$ finite capacity.

9 Simulation

1.

inversion method: calculate cdf by integrating density and invert this cdf.

Calculate the cumulative distribution function F(x) $\rightarrow$ integrate f(x):

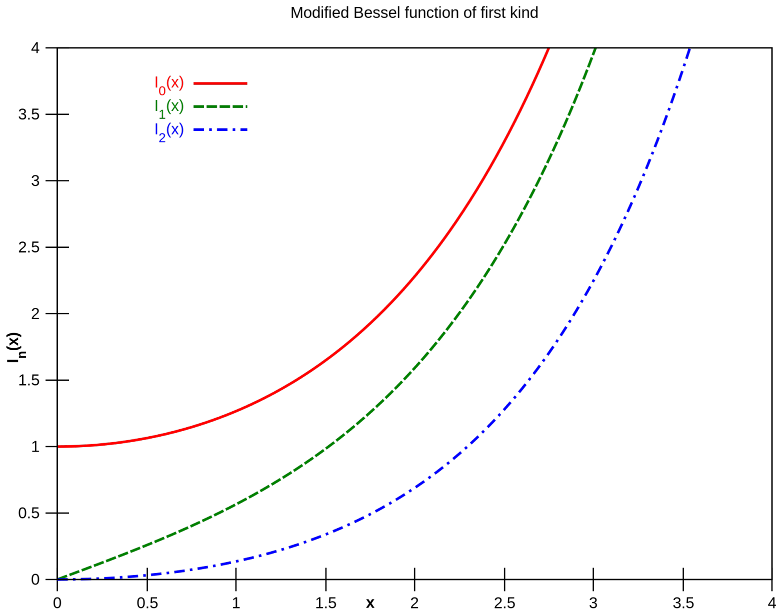
$$F(x) = \begin{cases} x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

inverse function: instead of returning y from x, the inverse function returns x given y: $F^{-1}(u) = \sqrt[3]{u}$ with $u =$ random sample $\in [0, 1]$

2.

definitions

- $I_0(\kappa) =$ modified Bessel function of order 0
- $\kappa > 0$
- density von Mises Distribution: $f(x) = \frac{e^{\kappa \cos(x)}}{2\pi I_0(\kappa)}$ for $-\pi \leq x \leq \pi$, density is 0 otherwise



(1)

based on plot: $I_0(\kappa) > 1$ finding extremum → derivative:

$$\frac{-K \sin(x) e^{\kappa \cos(x)}}{2\pi I_0(\kappa)}$$

fill in for K=1, since we just want to know the sign: slope is positive for $f'(x) < 0$, 0 for $f'(x) = 0$ and negative for $f'(x) > 0$, hence the maximum is at 0 when $-\pi \leq x \leq \pi$.

(2)

- find f_y such that $f_y \sim f_x$
- find $C \in \mathbb{R} > t$
- $C f_y(x) \geq f_x(x), \forall x \in \mathbb{R}$
- sample u and y → if $u(f_y(y) \leq f_x(y))$ return y else sample again

$$f_y(y) = \begin{cases} \frac{\pi}{2} & -\pi < y \leq \pi \\ 0 & otherwise \end{cases}$$

$$C = f(0)2\pi = \frac{e^{\kappa}}{I_0(\kappa)}$$

3

(1)

$F^{-1}(u) = (1 - (1-u)^{1/b})^{1/a}$

(2)

$B \sim Beta\left(|\alpha, \beta\right)$

$$\begin{aligned}
Pr\left[B^{\frac{1}{a}} \leq x\right] &= Pr\left[K \leq x\right] \\
&= Pr\left[B \leq x^{a}\right] \\
&= F_B\left(x^{a}\right) \\
&= \int_0^{x^a} f_B\left(t\right) dt \\
&= \int_0^{x} f_B\left(S^{a}\right) as^{a-1} ds && \text{- clean int limits} \rightarrow \text{ substitute } t \text{ with } S^a, a \in \mathbb{N} \text{ -} \\
&= \int_0^{x} \frac{1}{B\left(\alpha, \beta\right)} \left(1 - S^{a}\right)^{\beta - 1} as^{\alpha - 1} ds && \text{- solve } f_B(S^a),\ \left(1 - S^{a}\right)^{\beta - 1} as^{\alpha - 1} \approx f_K\left(s\right) \text{ -} \\
&= Pr\left[K \leq x\right] \\
&= F_k\left(x\right)
\end{aligned}$$

(3)

- uniform random variable: $u \in [0, 1]$
- $x_K = \left(1 - (1-u)^{\frac{1}{b}}\right)^{\frac{1}{a}}$
- $x_B = x_K^a = \left(1 - (1-K)^{\frac{1}{b}}\right) \rightarrow$ given: $\alpha = 1$

4.

$$\begin{aligned}
J &= \int_0^1 \int_0^1 e^{(x+y)^2} dy dx \\
&= E\left[e\left(x+y\right)^2\right] && \text{- double integral is like expectation, } x, y \sim \text{ uniform [0,1] -} \\
&= \int_0^1 \int_0^1 e^{(x+y)^2} f_x\left(x\right) f_y\left(y\right) dx dy
\end{aligned}$$

$$J_K = \frac{1}{K} \sum_{i=1}^{K} e^{(x_i + y_i)^2}$$

monte carlo $\rightarrow x_1, \ldots, x_K y_1, \ldots, y_K$ generate samples from uniform intervals correlation $(y_{i,} 1 - y_i) < 0$

$$\widehat{J}_K = \frac{1}{K} \sum_{i=1}^{K} \frac{1}{2} e^{(x_i + y_i)^2} + e^{(2 - x_i - y_i)^2}$$

negative correlation antithetic $\rightarrow$ see syllabus $\rightarrow$ smaller amount of samples for same conf variables

5.

samples $Z_1, \ldots, Z_K$

$$\widehat{J}_K = \frac{1}{K}\sum_i = 1^K Z_i^3 e^{Z_i}$$

- corr(Z, -Z) = E([Z-E(Z)][-Z-E(-Z)]) = -1 -

$$\widehat{J}_K^a = \frac{1}{K}\sum^K \frac{1}{2}\left(Z_i^3 e^{Z_i} - Z_i^3 e^{-Z_i}\right)$$

monotone, negative correlation $\rightarrow$ lower variance

6.

u is uniformly distributed, $\widetilde{u} = 1 - u$ is uniformly distributed. $F_x(x) = 1 - e^{-x}$
$x \rightarrow \widetilde{u} = e^{-x} \rightarrow u = 1 - e^{-x} \rightarrow -\ln\left(1 - e^{-x}\right)$
$\exp \rightarrow uniform \rightarrow uniform \rightarrow \exp$
$corr\left(\widetilde{u}, u\right) < 0 - \ln() \rightarrow corr\left(-\ln\left(\left(\widetilde{u}\right), -\ln\left(u\right)\right)\right) < 0 \rightarrow corr\left(x, -\ln\left(1 - e^{-x}\right)\right) < 0$

7.

(1)

$$\prod_{\{x+y\leq t\}}(x+y) = \begin{cases} 1 & x+y<t \\ 0 & x+y>t \end{cases}$$

$$Pr\left[x+y\leq t\right] = E\left[\prod_{\{\}}(x+y)\right]$$

$$\widehat{J}_K = \frac{1}{K}\sum_{i=1}^K \prod_{\{\}}(x_i + y_i)$$

(2)

definitions

- h is the indicator function
- we leave the out the subsript of $\prod_{\{\}}$ for clarity

solution

$$
\begin{aligned}
\widehat{J}_K^{cond} &= E\left[\prod^{*}(X+Y)\right] \\
&= E\left[E\left[\prod^{*}(X+Y)\,|X\right]\right] \\
&= E\left[\prod_{X+Y\leq t}(X+Y)\,|X\right] \\
&= E\left[\prod_{y\leq t-x}(Y)\,|X\right] \\
&= G(t-x) \\
&= Pr\left[y\leq t-X|X\right]
\end{aligned}
$$

$$
\begin{aligned}
J &= E\left[\prod^{*}(X+Y)\right] = E\left[G(t-x)\right] \\
\widehat{J}_K^{cond} &= \frac{1}{K}\sum_{i=1}^{K} G(t-x_i) & \text{- better because only one variable to sample -}
\end{aligned}
$$

(3)

definitions

- $u, E\left[h(u), V, g(V), E(g(V))\right]$

solution

$$
\begin{aligned}
E\left[h(u)\right] &= E\left[h(u)+\left[g(V)-E(g|V)\right]\right]K & \text{- } K\in\mathbb{R}\text{ -} \\
E\left[G(t-X)\right] &= E\left[G(t-X)+K\left[X-\mu_x\right]\right] & \text{- determine K online -}
\end{aligned}
$$

$$
\widehat{J}_K = \frac{1}{K}\sum_{i=1}^{K}\left(G(t-x_t)+K(x_i-\mu_x)\right)
$$