E4.1

4

Exercise session 9/3 | Waiting times and delay

Exercise 1.

We investigate the $M/M/1$ queue with ρ close to 1. To this end, we investigate the limit of the system content distribution for $\rho \to 1$.

(a) Calculate the probability $\Pr[S_\rho < x]$, where S_ρ is the system content of an $M/M/1$ queue for load ρ.

(b) Calculate the limit

$$\lim_{\rho\to 1} \Pr[(1-\rho)S_\rho \leq x]$$

and discuss the result.

Exercise 2.

Consider a queueing system with exponential service times with rate μ and with arrival rate $\lambda_n = \lambda/(n+1)$ when there n customers in the system.

(a) Find the stationary distribution of the system content.

(b) Calculate the LST of the delay.

(c) Verify that Little's result holds.

Exercise 3.

Consider a queueing system with a finite (possibly random) number of arrivals, say K, and let time T be a time instant after the departure of the last customer. Moreover, let D_k be the delay of the kth customer and let $S(t)$ be the number of customers in the queue. Define the following variables:

$$\overline{D} = \frac{1}{K}\sum_{k=1}^{K} D_k\,, \qquad \lambda = \frac{K}{T}\,, \qquad \overline{S} = \frac{1}{T}\int_0^T S(t)dt\,.$$

Show that $\overline{S} = \lambda\overline{D}$.

Exercise 4.

Consider a preemptive priority queueing system with two priority classes and a single server. In such a system, the server always serves high-priority customers if such customers are present, and only serves low-priority customers if there are no high-priority customers. Whenever a high-priority customer arrives while a low priority customer is being served, the low-priority customer immediately leaves the server and rejoins its queue. Assume a high- and low-priority Poisson arrival process with rates λ_h and λ_l, and exponential service times with rate μ.

(a) Find the distribution of the total number of customers in the system.

(b) Find the distribution of the number of high-priority customers in the system.

(c) Determine the mean system content and delay for both classes.

Exercise 5.

As in exercise 1, we investigate the $M/M/1$ queue with ρ close to 1. To this end, we investigate the limit of the waiting time distribution for $\rho \to 1$.

(a) Find the LST of $(1-\rho)D_\rho$, where D_ρ is the delay of a customer in an $M/M/1$ queue with load ρ.

(b) Calculate the limit $\rho \to 1$ of this LST assuming fixed λ and assuming fixed μ and discuss the result.

Exercise 6.

We investigate rational incentives for queueing or for not queueing. Assume an $M/M/1$ queue with arrival rate λ and service rate μ. Each customer decides upon wether or not to join the queue whereby the profit of joining the queue is $\beta - w$. if the customers waiting time is w for some constant value $\beta > 0$. We consider two situations.

(a) The customer does not know the queue content upon arrival. Therefore, he or she joins with a probability p such that his/her expected profit is non-negative. Find the probability p and the corresponding mean queue and system content.

(b) The customer knows how many customers there are in the queue upon arrival and he joins if his/her expected profit is non-negative. Determine the rule by which a customer enters the queue, and find expressions for the corresponding mea n queue and system content.

Exercise 7.

Consider a birth-death queueing system. The arrival process is Poisson with rate λ. There are two identical servers, each with rate μ, but server 1 is deactivated if there are less than $T = 5$ customers in the system.

1. Calculate the system content distribution
2. Calculate the LST of the delay