Masters Programme in Computer and Information Engineering

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Contents

Chapter 1

Basic Course in Mathematics

1.1 Basic

1.1.1 Mängder

Naturliga tal: $\mathbb{N} = \{1, 2, 3...\}$
Heltal: $\mathbb{Z} = \{... - 2, 1, 0, 1, 2...\}$
Rationella tal: $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b$
Irrationella tal: $\mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \{x | x \in \mathbb{R}, x \notin \mathbb{Q}\}$
Reella tal: $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$

1.1.2 Potenslagar

$$(\frac{a}{b})^{-3} = (\frac{b}{a})^3$$
$$\sqrt{a} = a^{\frac{1}{2}}$$

1.1.3 Logaritmer

Logaritmlagar:

Va vaksam över när a<0 då är logarytmen odefinerad.

(1): $b = a^x \Leftrightarrow \log_a(b) = x$ för: $a > 0, b > 0, a \neq 1$

(2): $\log_a(\frac{b}{c}) = \log_a(b) - \log_a(c)$

(3): $\log_a(b * c) = \log_a(b) + \log_a(c)$

(4): $\log_a(b^d) = d \log_a(b)$

(5): $\log_a(b) = \frac{\log_f(b)}{\log_f(a)}$

(6): $\log_a(a) = 1$

(7): $\log_a(1) = 0$

(8): $a^{\log_a(x)} = x$

(9): $\log_{a^c}(b) = \frac{1}{c} \log_a(b)$

1.1.4 Tillvägagångs sätt

$$\log_3(x) + \log_x(\frac{1}{27}) = 2 \tag{1.1}$$

Lösning

$$\begin{aligned}
&\log_3(x) - 3\log_x(3) = 2\\
&\log_3(x) - 3\frac{\log_3(3)}{\log_3(x)} = 2\\
&\log_3(x) - 3\frac{1}{\log_3(x)} = 2\\
&\log_3(x)^2 - 3 = 2\log_3(x)\\
&y = \log_3(x)^2\\
&y^2 - 2y - 3 = 0\\
&\text{pq-formeln: } y = 1 \pm \sqrt{4} = 1 \pm 2\\
&\log_3(x) = 3 \Leftrightarrow x = 3^3 = 27\\
&\log_3(x) = -1 \Leftrightarrow x = 3^{-1} = \frac{1}{3}
\end{aligned}$$

1.1.5 Intervall

$$[\,a, b]\; = \{x | a \leq x \leq b\} \tag{1.2}$$

$$[\,a, \infty[\tag{1.3}$$

$$]-\infty, \infty[\tag{1.4}$$

1.1.6 Tillvägagångs sätt

$$\frac{2}{x-3} < \frac{5}{x} \tag{1.5}$$

Lösning

$$\begin{aligned}
&\frac{2}{x-3} - \frac{5}{x} < 0\\
&\frac{(x)2}{x(x-3)} - \frac{5(x-3)}{x(x-3)} < 0\\
&\frac{2x - 5x - 15}{x(x-3)} < 0\\
&\frac{-3x - 15}{x(x-3)} < 0\\
&\frac{-3(x-5)}{x(x-3)} < 0\\
&x \neq 0, x \neq 3
\end{aligned}$$

Värde tabell:

	x<0	0<x<3	3<x<5	5<x
x-5	-	-	-	+
-3	-	-	-	-
x	-	+	+	+
x-3	-	-	+	+
hela	+	-	+	-

1.2 Komplexa tal

$z = a + bi$
$\text{Re}(z) = a$
$\text{Im}(z) = b$
$\overline{\text{z}} = a - bi$
$|z| = \sqrt{a^2 + b^2}$ Där b(Im(z)) är för y-axeln och a(Re(z)) är för x-axeln

Samma räkneregler för reella tal gäller för komplexa tal

1.2.1 Polärform

$\arg(z) = \alpha + 2\pi * n$
arg(z) är vinkeln mellan a (x-axeln) linjen |z|. n är heltal
$\text{Arg}(z) \in] - \pi, \pi[$
$\text{Arg}(z) \in \arg(z)$
$a = |z| \cos \alpha$
$b = |z| \sin \alpha$
$z = |z| \cos \alpha + i * |z| \sin \alpha$

Polärform:

$$z = |z|(\cos \alpha + i * \sin \alpha) \tag{1.6}$$

Eulers formel:

$$z = |z|e^{i\alpha} \tag{1.7}$$

Sats:

$$|z * w| = |z| * |w|$$
$$\arg(z * w) = \arg(z) + \arg(w)$$
$$\arg(\frac{z}{w}) = \arg(z) - \arg(w)$$

De Movre's formel:

$$(\cos(\theta) + i * \sin(\theta))^n = \cos(n * \theta) + i * \sin(n * \theta)$$

Binomisk ekvation

$$(\cos(\theta) + i * \sin(\theta))^n = \cos(n * \theta) + i * \sin(n * \theta)$$

Tillvägagångs sätt

$$\text{Visa lösningarna i det komplexa tal planet } z^5 = \sqrt{3} + i \tag{1.8}$$

$$
\begin{aligned}
&|\sqrt{3} + i| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2 \\
&\begin{cases} \sqrt{3} = 2\cos(\alpha) \\ 1 = 2\sin(\alpha) \end{cases} \\
&\alpha = \frac{\pi}{6} \\
&|z|^5(\cos(5 * \theta) + i * \sin(5 * \theta)) = 2(\cos(\frac{\pi}{6}) + i * \sin(\frac{\pi}{6})) \\
&\text{Vilket ger:} \\
&|z|^5 = 2 \Leftrightarrow |z| = 2^{\frac{1}{5}} \\
&5 * \theta = \frac{\pi}{6} + 2\pi n \Leftrightarrow \theta = \frac{\pi}{30} + \frac{2\pi n}{5} n \in \mathbb{Z} \\
&\text{I polärform blir det då:} \\
&z = 2^{\frac{1}{5}}(\cos\frac{\pi}{30} + \frac{2\pi n}{5} + i * \sin\frac{\pi}{30} + \frac{2\pi n}{5}) \\
&\text{Eulers formel: } z = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{2\pi n}{5})} \\
&n = 0, 1, 2, 3, 4 \\
&z_1 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30})} \\
&z_2 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{2\pi}{5})} \\
&z_3 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{4\pi}{5})} \\
&z_4 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{6\pi}{5})} \\
&z_5 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{8\pi}{5})}
\end{aligned}
$$

1.3 Absolut Belopp

1.3.1 Tillvägagångs sätt

$$|2x-8|+|1-x|-2|x-3|=8+3x \tag{1.9}$$

lösning

$$\begin{pmatrix} 2x-8=0 \\ x=4 \end{pmatrix} \begin{pmatrix} 1-x=0 \\ x=1 \end{pmatrix} \begin{pmatrix} x-3=0 \\ x=3 \end{pmatrix}$$

$$\mathrm{I}\begin{cases} x\leq 1 \\ -(2x-8)+(1-x)+2(x-3)=8+3x \end{cases}$$

$$\mathrm{I}\begin{cases} x\leq 1 \\ -4x=5 \end{cases} \quad \begin{cases} x\leq 1 \\ x=-\frac{5}{4} \text{ lösning} \end{cases}$$

$$\mathrm{II}\begin{cases} 1<x\leq 1 \\ -(2x-8)-(1-x)+2(x-3)=8+3x \end{cases}$$

$$\mathrm{II}\begin{cases} 1<x\leq 3 \\ -2x=7 \end{cases} \quad \begin{cases} 1<x<3 \\ x=-\frac{7}{2} \text{ Ej lösning} \end{cases}$$

$$\mathrm{III}\begin{cases} 3\leq x<4 \\ -(2x-8)-(1-x)-2(x-3)=8+3x \end{cases}$$

$$\mathrm{III}\begin{cases} 3\leq x\leq 4 \\ -(2x-8)-(4x-3)=8+3x \end{cases} \quad \begin{cases} 3x<4 \\ x=\frac{5}{6} \text{ Ej lösning} \end{cases}$$

$$\mathrm{IV}\begin{cases} x\geq 4 \\ (2x-8)-(1-x)-2(x-3)=8+3x \end{cases}$$

$$\mathrm{IV}\begin{cases} x\geq 4 \\ x=-\frac{11}{2} \text{ Ej lösning} \end{cases}$$

1.4 Summor

1.4.1 Aritmetiska summor

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n\,(a_1 + a_n)}{2} \tag{1.10}$$

1.4.2 Geometriska summor

Börjar altid med exponenten 0 och gör om summan så att den passar i följade talföljd:

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\,(k^n - 1)}{k - 1} \tag{1.11}$$

1.4.3 Tillvägagångs sätt

$$\sum_{k=n}^{2n} (2^k - k) \tag{1.12}$$

Lösning

$$\begin{aligned}
&\text{sätter f = 0 = k -n} \\
&\sum_{f=0}^{n} (2^{f+n} - (f+n)) = 2^n * \sum_{f=0}^{n} (2^f) - \sum_{f=0}^{n} (f+n) \\
&\frac{2^n(2^{n+1} - 1)}{2 - 1} - \frac{3n(n+1)}{2} = 2^{2n+1} - 2^n - \frac{3n(n+1)}{2}
\end{aligned}$$

1.5 Kombinatorik

1.5.1 Multiplikations principen

permetationer p(a,b) då ordningen spelar roll.
Antal sätt: (exemplet: antal sätt av måltider 7 företer 5 varmrätter 4 efterätter (7*5*4)

$$n_1 \cdot n_2 \cdot \ldots \cdot n_m \tag{1.13}$$

1.5.2 Kombinationer

Kombination c(a,b) då ordningen inte spelar roll.
Antal sätt: (exemplet: antal del-mängder två element (n = element k = antal element som kan väljas))

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} \tag{1.14}$$

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n \cdot (n-1)(n-2) \cdot \ldots \cdot (n-1(k-1))}{k!} \tag{1.15}$$

Pascal triangel

n							
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1
	0	1	2	3	4	5	6
				k			

1.5.3 Binomial satsen

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^k b^{n-k} \tag{1.16}$$

1.6 Funktioner och kordinatsystem

Kom ihåg att när det stor (x-a) förflytas den i x-axeln a steg till höger –> medans (x+a) förflytas den a steg till vänster <–

1.6.1 Avståndsformeln

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.17}$$

1.6.2 Elipser

Förenkla till denna formeln:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.18}$$

Där a är avståndet på x-axeln och b är avståndet på y-axeln

Hyperbol

Ser väldigt anerlunda ut från elipser

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1.19}$$

Circkel

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \tag{1.20}$$

Där a = r

1.7 Polynom division

1.7.1 Tillvägagångs sätt

Man vet att ekvationen $z^4-2z^3-7z^2+26z-20=0$ har roten $z=2+i$. Lös ekvationen fullständigt. (1.21)

$z = 2+i$ Är en läsning är också konjugatet en lösning enligt faktorsatsen $\overline{z} = a - bi$

$z = 2 \pm i$

Vilket betyder att följande går att factorisera ut polynomet

$(z-(2+i))(z-(2-i)) = z^2 - 4z + 5$

Långdivition (liggande stolen):

$$
\begin{array}{r|l}
 & x^2 + 2x - 4 \\ \hline
x^2-4x+5 & x^4 - 2x^3 - 7x^2 + 26x - 20 \\
 & -x^4 + 4x^3 - 5x^2 \\ \hline
 & 2x^3 - 12x^2 + 26x \\
 & -2x^3 + 8x^2 - 10x \\ \hline
 & -4x^2 + 16x - 20 \\
 & 4x^2 - 16x + 20 \\ \hline
 & 0
\end{array}
$$

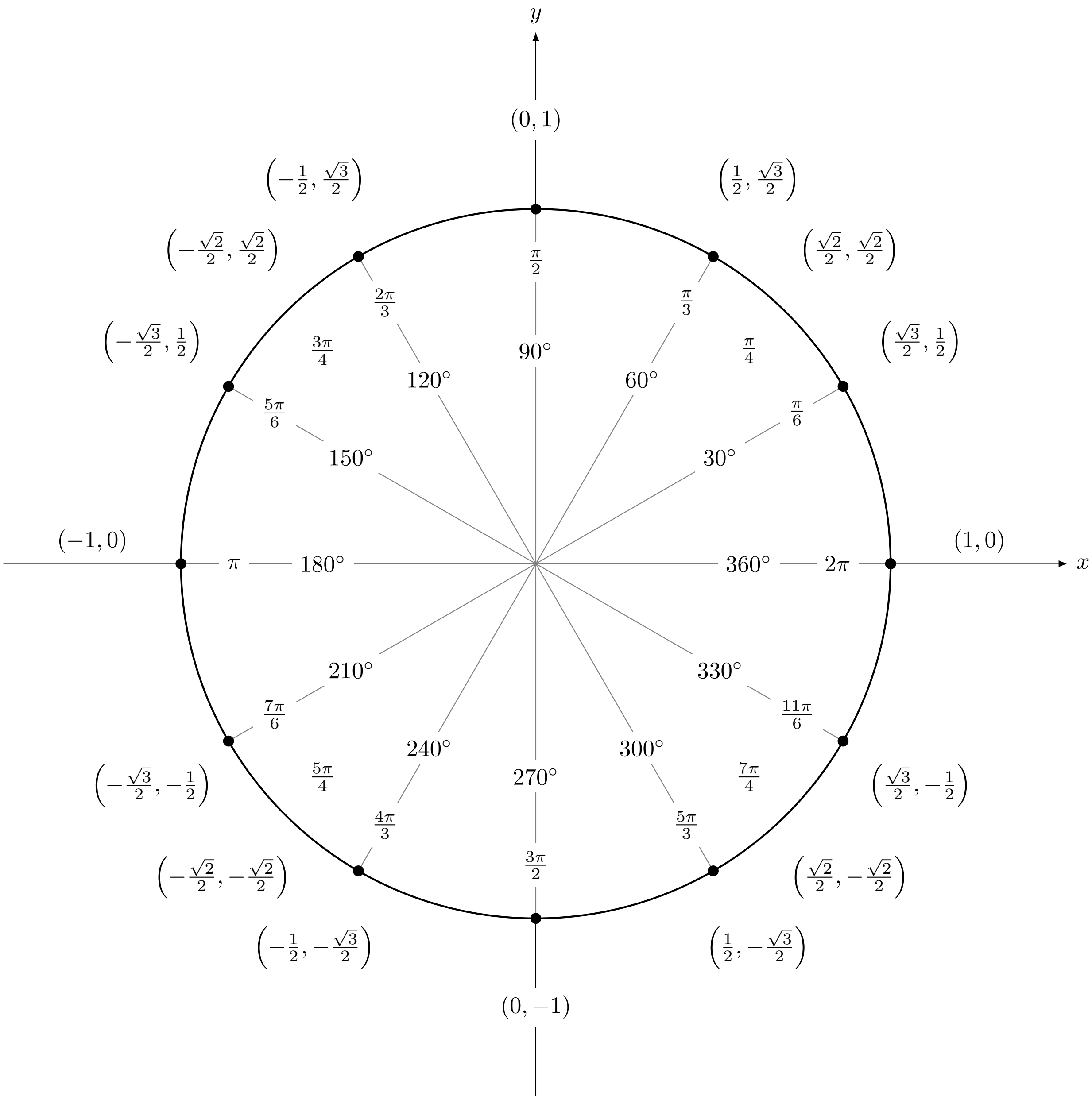
$z^2 + 2z - 4 = 0$ Är också en läsning som tilslut ger följande

$z = -1 \pm \sqrt{5}$

$z = 2 \pm i$

Varge n grads polynom har altid n stycken komplexa lösningar

1.8 Trigonometri



Sats:

$$
\begin{aligned}
&360° = 2\pi rad \\
&v_g = v_r * \frac{180°}{\pi} \\
&v_r = v_g * \frac{\pi}{180°}
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&-1 \leq \sin t \leq 1 \\
&-1 \leq \cos t \leq 1
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&\cos(-t) = \cos(t) \\
&\sin(-t) = -\sin(t) \\
&\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)}
\end{aligned}
$$

Additionsformlerna:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha)
\end{aligned}
$$

Trigonometriska ettan:

$$
\begin{aligned}
&(\sin t)^2 + (\cos t)^2 = 1 \\
&\sin^2 t + \cos^2 t = 1
\end{aligned}
$$

1.8.1 Tillvägagångs sätt

$$
\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

Chapter 2

Program Design and Data Structures

2.1 Coding convension

VARIANT
A (recursive) function terminates if it has a variant. The variant need to follow all of the flowing rules

- Needs to decrease every recursive call
- All ways positive

Side effects
All IO functions have side effects in order to separate pure Haskell function with impure functions that changes the state with is commonly the case with imperative and object oriented programming. Every IO function has a side effects.
INVARIANT
A data types invariant is what value are allowed for the data type to work. Similar to preconditions for a function. An example is integer data type that can only use positive integers therefor the invariant is positive integers.

2.2 Design approach

- top-down design (Cheating): Is to break down a complex system in to subsystems to solve the problem. Most often is to write everything by scratch.
- bottom-up design (Stacking): Is to piece existing system together to create a more complex system. little is programmed, most is copied.
- dodging: Get some code working more quickly, make progress with some part of the system and back-paddle to the dodged part later. The reason is to develop insight that will help solve the larger problem.

2.2.1 Process

1. Data Description
2. Data Examples
3. Function Description
4. Function Examples
5. Function Template
6. Code
7. Tests
8. Review and Refactor

Programming to an Interface More dynamic, can change models, less code to write and a layer of abstraction. ADT

2.3 Recursion

2.3.1 Recursion types

1. Simple recursion: There is at most one recursive call (in each branch) and the argument is decremented by one.

2. Complete recursion: Some argument becomes smaller in the recursive call, but not necessarily.

3. Multiple recursion: There are multiple recursive calls (in the same branch).

4. Mutual recursion: Two or more functions are defined in terms of each other.

5. Nested recursion: An argument to a recursive call is computed by a recursive call.

6. Recursion on a generalized problem: Sometimes, no suitable recursion scheme is obvious.

2.4 Complexity

2.4.1 Growth

1. Big θ Notation: estimate of growth in intervals determine by constants Definition For non-negative functions f and g, $f(n) = \theta(g(n))$ if and only if there exist $n_0 \geq 0$ and $c_1, c_2 > 0$ such that for all $n > n_0$ $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$. $\theta(g(n))$ is the set of all functions $f(n)$ that are bounded below and above

2. Big Ω Notation: estimate of growth Lower bound

3. Big O: Notation: estimate of growth upper bound

Relation

$O(g(n)) \cap \omega(g(n)) = \theta(g(n))$

2.5 Recurrences

Example:
sumList [] = 0
sumList (x:xs) = x + sumList xs

1. pattern matching [] takes t_0 time

2. pattern matching $(x : xs)$ takes t_1 time

3. Adding x with recursive call takes t_{add}

4. Then the recursive call takes $T(n-1)$

$$T(n) = \begin{cases} t_0 & \text{if } n = 0 \\ T(n-1) + t_{add} + t_1 & \text{if } n > 0 \end{cases}$$

2.5.1 Closed Form

1. Use the substitution method to obtain a closed form for the following recurrence:

$$
\begin{aligned}
&f(0) = 5 \\
&f(n) = f(n-1) + n + 2, n > 0
\end{aligned}
$$

Hint: $1 + 2 + 3 + \ + n = \frac{n(n+1)}{2}$ **— you do not need to prove this fact.**

Expansion Method

$$
\begin{aligned}
&f(0) = 5 \\
&f(1) = f(0) + n + 2 = n + 5 + 2 \\
&f(2) = f(1) + n + 2 = 2n + 5 + 2 + 2 \\
&f(3) = f(2) + n + 2 = 3n + 5 + 2 + 2 + 2 \\
&f(n) = +n^2 + 5 + n \cdot 2
\end{aligned}
$$

Induction proof

Step1: test with base case sense the base case is predefine for 0 we do $n = 1$
$f(1)_{VL} = 1^2 + 5 + 1 \cdot 2 = 8,$
$f(1)_{HL} = f(0) + 1 + 2 = 5 + 3 = 8$
$f(1)_{VL} = f(1)_{HL}$

Step2: assumption for p $f(p) = p^2 + 5 + p \cdot 2$
$f(p+1)_{VL} = f(p) + (p+1) + 2 = (p^2 + 5 + p \cdot 2) + (p+1) + 2 =$
$= (p^2 + (p+1)) + 5 + (p+1) \cdot 2 = (p^2 + 2p + 1) + 5 + (p+1) \cdot 2$
$f(p+1)_{HL} = (p+1)^2 + 5 + (p+1) \cdot 2 = (p^2 + 2p + 1) + 5 + (p+1) \cdot 2$
$f(p+1)_{VL} = f(p+1)_{HL}$

Conclusion: according to induction hypothesis the recurrence of the function is equal to
$2n + 5 + \dfrac{n(1+n)}{2}$

Substitution Method

$$\begin{aligned}
f(n) &= f(n-1) + n + 2 \\
&= (f(n-2) + (n-1) + 2) + 1n + 2 \\
&= f(n-2) + (n-1) + n + 2 \cdot 2 \\
&= (f(n-3) + (n-2) + 2)(n-1) + n + 2 \cdot 2 \\
&= f(n-3) + (n-2) + (n-1) + n + 3 \cdot 2 \\
&. \\
&= f(n-k) + (n-(k-1)) + (n-(k-2)) + (n-(k-3)) + \ldots + n + k \cdot 2 \\
&. \\
&= f(n-n) + (n-(n-1)) + (n-(n-2)) + (n-(n-3)) + \ldots + n + n \cdot 2 \\
&= f(0) + 1 + 2 + 3 + \ldots + n + n \cdot 2 = 5 + 1 + 2 + 3 + \ldots + n + n \cdot 2
\end{aligned}$$

We can see the following patterns

$$2n + 5 + \sum_{k=1}^{n}(k) = 2n + 5 + \frac{n(1+n)}{2}$$

Induction proof

Step1: test with base case since the base case is predefine for 0 we do $n = 1$

$f(1)_{VL} = 2 \cdot 1 + 5 + \dfrac{1(1+1)}{2} = 2 + 5 + 1 = 8,$

$f(1)_{HL} = f(0) + 1 + 2 = 5 + 3 = 8$

$f(1)_{VL} = f(1)_{HL}$

Step2: assumption for p $f(p) = 2p + 5 + \dfrac{p(1+p)}{2}$

$f(p+1)_{HL} = f(p) + (p+1) + 2 = (2p + 5 + \dfrac{p(1+p)}{2}) + (p+1) + 2 =$

$= 2(p+1) + 5 + \dfrac{p(1+p)}{2} + p = 2(p+1) + 5 + \dfrac{2(p+1) + p(1+p)}{2} =$

$= 2(p+1) + 5 + \dfrac{p^2 + 2p + p + 2}{2} =$

$f(p+1)_{HL} = 2(p+1) + 5 + \dfrac{(p+1)(p+2)}{2} = 2(p+1) + 5 + \dfrac{p^2 + 2p + p + 2}{2}$

$f(p+1)_{VL} = f(p+1)_{HL}$

Conclusion: according to induction hypothesis the recurrence of the function is equal to

$2n + 5 + \dfrac{n(1+n)}{2}$

2.6 Higher-Order Function

2.6.1 Higher-Order Functions on Lists

```
map :: (a -> b) -> [a] -> [b]
filter :: (a -> Bool) -> [a] -> [a]
foldl :: (a -> b -> a) -> a -> [b] -> a
foldr :: (a -> b -> b) -> b -> [a] -> b
```

map

maps a function to each element in list.

filter

filters elements with a condision

```
filter :: (a -> Bool) -> [a] -> [a]
filter (<6) [6,3,0,1,8,5,9,3] == [3,0,1,5,3]
```

foldl

foldl recurses over a list “from the left,” i.e., it initially applies the given operation to the first list element and the given start value. starts from the left (first element) and apply the function to with each element. Similar to an accumulator. No one uses it since it is some what useless.

```
foldl :: (a -> b -> a) -> a -> [b] -> a
foldl (*) 1 [1,2,3,4] == 24
```

foldr

foldr recurses over a list “from the right,” i.e., it initially applies the given operation to the last list element and the given start value. starts from the right (last element) and apply the function to with each element. Similar to an accumulator.

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr (*) 1 [1,2,3,4] == 24
```

2.7 Data types

2.7.1 Basic

```
String :: ['char'] --list of charecters
List :: []         --undefine element types and elements
Tuple :: ()        --Predefine elements
Char :: ''         --single carecter
Int :: 1           --Hole number with define size
Integear :: 1      --Hole number with undefine size
Float :: 1.1       --Real number with double-precision
Double :: 1.1      --Real number with single-precision
```

2.7.2 Maybe Type

If the return is maybe "nothing" then the "Maybe type" is used, since it dose not have to return the a specific value. It is not polymorphic since you have to specify the type, however "Just" is at of it self polymorphic function. If a operation that requires a specific type one needs to remove "Just", for instance by a let function.

2.7.3 New types and enumeration types

New types: One create more relevant names and format of existing enumeration types. Overload is a problem with the use of the same operations that can not be used for the same data types. One needs to create new operations if it is not from the same type class.

Enumeration types: Instead of new types type enumeration types uses the operation call data. The difference is that enumeration types is independent from existing types therefor one becomes more flexible and precise.

example

```
data newTypeOfDataDerection = North | South | East | West
    deriving (Show) -- inorder to print

:t North
North :: newTypeOfDataDerection

-- We can use new types in pattern matching
oposit :: newTypeOfDataDerection -> newTypeOfDataDerection
oposit North = South
```

Type classes

A type class is a set of types that support certain related operations.

No function is applied by default to the new type, therefor you can write "deriving" the following functions are good to have

type classes to deriving for new data types

```
  deriving (Show) -- in order to print in ghci and print normal
  deriving (Eq) -- To test equality
  deriving (Ord) -- the first one has the smallest value, order matter for comparison
```

New type classes

```
class Eq a where
    (==) :: a -> a -> Bool
```

New type classes Instances

```
instance Eq Colour where
  (==) = eqColour
```

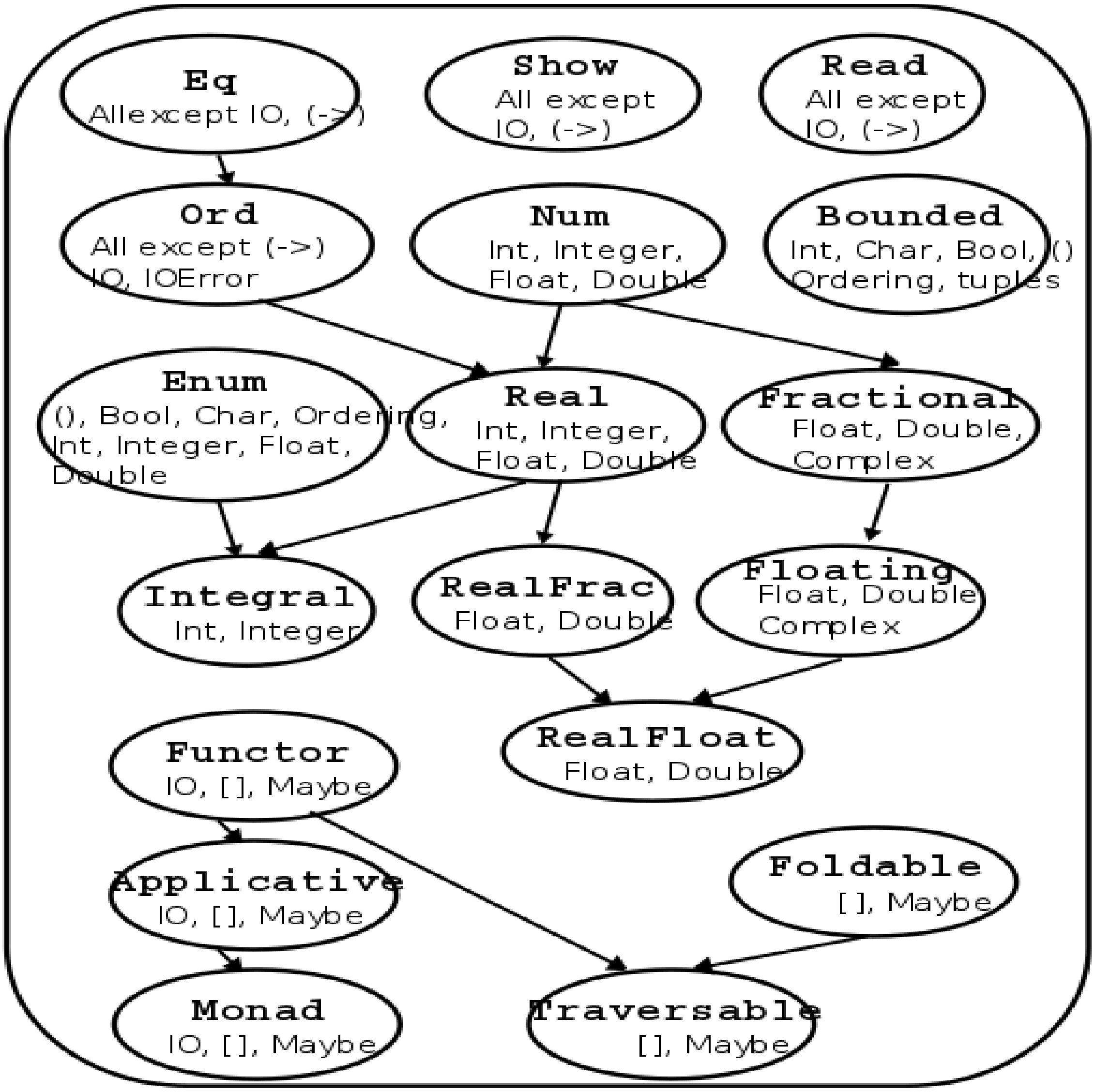


Figure 2.1: Type class (Wik)

2.7.4 Inductive Data Types

Uses a base case then the inductive step as cases of. Multiple arguments. Type constructure

```
data AExp = Atom Int
    | Plus AExp AExp
    | Times AExp AExp

eval (Atom i)      = i
eval (Plus a b)    = eval a + eval b
eval (Times a b)   = eval a * eval b
```

2.7.5 Trees

Terminology

1. Search tree: All nodes on the left side is less then the parent and opposite on the right side
2. Out-degree: is how many children it has. Binary trees has out-degree $0 - 2$
3. root Node: is a parent to any number of children at the top of the hierarchy
4. sub Node: is a parent to any number of children
5. Leaf: is a nod that has no children
6. Node: every element
7. Height: most steps from the root node to a leaf

Representation

1. Inorder: (Left, Root, Right)
2. Pre-order: (Root, Left, Right)
3. Post-order: (Left, Right, Root)

```haskell
data FBTree a = Leaf a
  | Node (FBTree a) a (FBTree a)
  deriving (Show)

Node (Leaf 1) 5 (Leaf 21)
rootValue :: FBTree a -> a
rootValue (Node _ a _) = a
rootValue (Node x) = x

height :: FBTree a -> Int
height (Leaf _) = 0
height (Node b a c) = 1 + max (height b) (height c)
```

Binary tree

Insertion: $O(1)$ ($O(h)$ if search tree)
Delition: $O(n)$ ($O(h)$ if search tree)
Search: $O(n)$ ($O(h)$ if search tree)
Height: $O(n)$ (n nodes)

1. Full binary tree: has node out-degree either 2 or 0 worst-case complexity of $O(\log n)$.
2. Binary tree: each node has up to an out-degree of 2

Binary search tree

Insertion: $O(h)$ ($O(n)$ if search tree)
Delition: $O(h)$
Search: $O(n)$
Height: $O(n)$ (n nodes)

Red and black trees

Insertion: $O(\log n)$
Deletion: $O(\log n)$
Search: $O(\log n)$
Element: $O(2^r)$ (rank r)
Height: $O(2 \cdot \log_2(n+1))$ (n nodes)

Better then a normal binary tree since it balance the tree, therefor it becomes smaller and more efficient to search in. One should use red and black tree when there is a large number of nodes, say 50.

Definition: A red-black tree is a binary search tree where every node is colored either red or black, with the following balancing invariants:

1. No red node has a red parent.
2. Every path from the root to an empty subtree contains the same.
3. A red-black tree with n nodes has height at most $2 \cdot \log 2(n+1)$.
4. there are 4 cases to rebalance (1;4) is similar so is (2;3).

Algorithm

1. Perform a standard binary-search-tree insertion.
2. Color the new node red.
3. Rebalance the tree, if there is a red node with a red parent.

Binomial Trees and heaps

Insertion: $O(\log n)$
Search: $O(\log n)$ (number of trees n)
Element: 2^r (rank r, n=1 then r=0)

A heap can be used to implement a priority queue, where elements are added to a pool and assigned a priority. In a min-priority queue, extraction of an element yields an element with minimum priority. The smallest node is the root and every child is equal or larger then its parent.

Binomial Trees is a data structure by linking trees of rank $r-1$ together. A binomial heap (Vuillemin, 1978) is a list of binomial trees such that each tree satisfies the min-heap property (hence the root of each tree contains its minimum key); and the trees have strictly increasing ranks.

Terminology

1. Link: putting together two trees.
2. Merge: putting together two heaps
3. Binomial Heap: a list (forest!) of Binomial Trees
4. Binomial Trees have the largest subtree to the left, while Binomial

Binomial heaps

1. Heaps, extracting minimum element in worst case $O(\log |h|)$
2. Binomial Trees, The height (here: number of edges on the longest branch) is r.
3. Binomial Trees, There are 2^r nodes in the tree.
4. Binomial Trees, There are $\begin{pmatrix} r \\ k \end{pmatrix}$ nodes at level k. (Hence its name!)
5. Binomial Trees, The root has r subtrees of ranks $r-1, r-2, \ldots, 1, 0$.
6. A binomial heap h has at most $[\lg |h|] + 1$ binomial trees.
7. Inserting a binomial tree into a binomial heap is like addition with base 2.
8. merging is made with too cases either (case 1) when one is smaller or (case 2) when they are equal

```
BinoTree = Node Int Int [BinoTree] -- Node rank key (subtrees with decreasing rank)
```

2.7.6 Other data types

Tables, Stacking and queuing

1. Table: a list of key-value pairs
2. Stacks: elements accessed in Last-In First-Out (LIFO) order
3. Queues: elements accessed in First-In First-Out (FIFO) order

Table operations

```
empty :: Table k v
insert :: Eq k => Table k v -> k -> v -> Table k v
exists :: Eq k => Table k v -> k -> Bool
lookup :: Eq k => Table k v -> k -> Maybe v -- value from key
delete :: Eq k => Table k v -> k -> Table k v
iterate :: Table k v -> (b -> (k, v) -> b) -> b -> b -- Foldr
keys :: Table k v -> (b -> k -> b) -> b -> b -- all keys
values :: Table k v -> (b -> v -> b) -> b -> b -- all values
```

Stack operations

```
-- interface
newtype Stack a = StackImpl [a] -- opaque!

empty :: Stack a
isEmpty :: Stack a -> Bool
push :: a -> Stack a -> Stack a -- insert
top :: Stack a -> a -- the first value
pop :: Stack a -> (a,Stack a) -- take out
```

Queue operations

```
-- interface
newtype Queue a = Q [a] -- opaque

empty :: Queue a
isEmpty :: Queue a -> Bool
head :: Queue a -> a
enqueue :: Queue a -> a -> Queue a  -- take out element
dequeue :: Queue a -> Queue a  -- insert element
toList :: Queue a -> [a]
```

Hastables

- Key value lookup (an index)
- Array is only define for small index, hash has no limit on avalible keys.
- Typically we have n possible keys from set U for a hashtable (which is an array) with m slots, where n $\geq$ m.
- since there is infinitely many elements and limited amount of key there will be element with the same key, therefor a coalition is created.
- Worst-Case Retrieval: time complexity of retrieving a element.
- Load Factor: How much data is in the table $\frac{elements}{slots}$
- Rehashing: make the hastable more balanced.

Collision Resolution by Chaining

- Most commonly used collision resolution
- Let each array slot (also called a bin) hold a list of elements (called a chain).
- In other words, When collision then add it to a list in that element

Collision Resolution by Open Addressing

- Start with a table with each element is $\perp$ previously used Δ.
- Probing: is a function to insert items in a hastable therefore resolves collations
- Types of probing:
  Linear probing: $f(i) = i$.
  Quadratic probing: $f(i) = c_2 \cdot i^2 + c_1 \cdot i,\ where\ c_2 \neq 0$.
  Double hashing: $f(i) = i \cdot h''(i)$, where h'' is another hash function.

- Inserting with Linear Probing: Insert it to the next avalible key
- Deleting with Linear Probing: Ignores $\perp$ and Δ idex will change.

2.7.7 Graphs

Types of graphs

- list
- tree
- forest
- Directed Acyclic Graph (DAG)

Treminolage

- Node, vertex (plural: vertices)
- Edge connects two nodes.
- Self-loop edge from node to itself
- Adjacent nodes connected by an edge
- Degree number of edges from or to a node
- In-degree number of edges to a node
- Out-degree number of edges from a node

Representation

- **Adjacency Matrix** — a 2-dimensional array A of 0/1 values, with $A[i, j]$ containing the number of edges between nodes i and j (undirected graph), or from node i to node j
  $+$ edge existence testing in $\theta(1)$ time
  $-$ finding next outgoing edge in $O|V|$ time
  $+$ compact representation for dense graphs (when $|E|$ is close to $|V|^2$)
- **Adjacency List** — a 1-dimensional array Adj of adjacency lists, with Adj[i] containing a list of the nodes adjacent to node i.
  $+$ finding next outgoing edge in $\theta(1)$ time
  $-$ edge existence testing in $O(|V|)$ time
  $+$ compact representation for sparse graphs (when $|E|$ is much smaller than $|V|^2$)
- **Edge List** — a list of tuples, (i, j), for each edge (i, j) (plus a list of the nodes).
  $-$ edge existence test in $\theta(|E|)$.
  $-$ finding next outgoing edge in $O(|E|)$ (unless appropriately sorted).
  $+$ compact representation for sparse graphs (when $|E|$ is much smaller than $|V|^2$)

topological sort

A topological sort is a linear ordering of all the nodes in a directed acyclic.

Algorithm

1. Select a node with in-degree 0.
2. Output it.
3. Remove it.
4. Repeat (from 1) until no nodes are left.

Total running time is $\theta(|V| + |E|)$.

Graph Traversals

- Breadth-first search (BFS). Uses a queue for eatch grey node.
  Time complexity: $\theta(|V| + |E|)$ — linear in the size of the graph.
- Depth-first search (DFS). Uses a stack for eatch (grey) node and has a rest of nodes (white).
  Time complexity: $\theta(|V| + |E|)$ — linear in the size of the graph.

Breadth-First Search: Algorithm
Input: Some node A.

1. Paint A gray. Paint other nodes white. Add A to an initially empty FIFO queue of gray nodes. All grey nodes is in the queue. It is a queue not a stack so first in first out.
2. Dequeue head node, X. Paint its undiscovered (white) adjacent nodes gray and enqueue them. Paint X black. Repeat until queue is empty. For every black node add it to BFS order (black)

Depth-First Search: Algorithm
Input: Some node A.

DFS(G)

1. Paint all nodes white.
2. For each node v in G: if v is (still) white, DFS-Visit(G,v). Each subsequent call to DFS-Visit in line 2 is called a restart.

DFS(G)

1. Colour v gray.
2. For each node u adjacent to v: if u is white, DFS-Visit(G,u).

Strongly-Connected Components

- Strongly connected component (SCC): maximal set of nodes where there is a path from each node to each other node.
- Many algorithms first divide a digraph into its SCCs, then process these SCCs separately, and finally combine the sub-solutions. (This is not divide & conquer, since a different algorithm is run on each SCC!)
- An undirected graph can be decomposed into its connected

1. Enumerate the nodes of G in DFS finish order, starting from any node
2. Compute the transpose G T (that is, reverse all edges)
3. Make a DFS in G T, considering nodes in reverse finish order from original DFS
4. Each tree in this depth-first forest is a strongly connected component

2.8 Important syntax

2.8.1 Let

```
let x = 1 in x * 2 == 2
case x of
1 -> "Hello"
2 -> "H"
3 -> "Hel"

input: 3 == "Hel"

# other examples
let f x y = x + 3 >= y + 3.1 in f 1 1 == False

f x = let g z = z+1 in g (g x)
f 1 == 3
```

```
:t div
div :: Integral a => a -> a -> a

:t (/)
(/) :: Fractional a => a -> a -> a
```

2.8.2 IO

monads Is used to return an IO and uses a operation (»=) that replaces do-notation

```
class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a
```

2.9 Sorting Algorithms

1. Insertion Sort: One at a time
2. Bubble Sort: sort in order two a time starting from left and work to the right side.
3. Merge Sort: Divide and Conquer, split into even pies and sort them and then merge/sort again.
4. Quicksort: Divide and Conquer, takes a pivot to spit smaller and larger.

Chapter 3

Algebra 1

3.1 logik

Utsaga: ett påstående som erhåller antigen värderna sann(S) eller falsk(F) vilket ge en stluten utsaga eller ej ett sannings värde vilket kallas för öpna utsagor
Kombinerade utsagor:
Kunjuktion utsagor: består av två utsagor som vi kallar A och B
där and: $\wedge\,(A \wedge B)$
Dissjuntion utsagor: består av två utsagor som vi kallar A eller B
or: $\vee\,(A \vee B)$
icke utsaga A (motsatsen): $\neg A$
Alla: $\forall$
Minst en: $\exists$
$\neg(\forall x : A) \Leftrightarrow \exists x : \neg A$
$\neg(\exists x : A) \Leftrightarrow \forall x : \neg A$

Exempel:

A: Alla reala tal x gäller att $(x+1)^2 = 0$
$\forall x : (x+1)^2 = 0$
$\neg(\forall x : (x+1)^2 = 0) = \exists x : (x+1)^2 \neq 0$
$\neg A$: Det finns reala tal x gäller att $(x+1)^2 \not 0$

3.1.1 värde tabeller

Kunjuktions värdetabel:

A	B	$A \wedge B$
S	S	S
S	F	F
F	S	F
F	F	F

Dissjuktions värdetabel:

A	B	$A \vee B$
S	S	S
S	F	S
F	S	S
F	F	F

3.1.2 Implikationer

A medför B: $A \Rightarrow B$ (implikaiton)
A och B medför varandra: $A \Leftrightarrow B$ (ekvivalens)

Implication värdetabell:

A	B	$A \Rightarrow B$
S	S	S
S	F	F
F	S	S
F	F	S

Ekvivalens värdetabel:

A	B	$A \Leftrightarrow B$
S	S	S
S	F	F
F	S	F
F	F	S

Exempel1:

$$x = \sqrt{6 - x} \tag{3.1}$$

$x = \sqrt{6-x} \Rightarrow x^2 = 6 - x \Leftrightarrow x^2 + x - 6 = 0$

pq-formeln: $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{24}{4}} \Leftrightarrow (x = -3) \vee (x = 2)$

Eftersom det är en implikation år höger och inte ekvivalens så behöver inte rötterna vara sanna

Testar för falska rötter:

$2 = \sqrt{6-2}$ Sann

$2 = \sqrt{6-(-3)}$ Falsk $2 \neq \sqrt{6-(-3)}$

Exempel2:

$$(x + 2)(x + 1) = 2x(x + 1) \tag{3.2}$$

$(x+2)(x+1) = 2x(x+1)$ is not $\Rightarrow (x+2) = 2x$ (x=0 is not allowed)

Insted do ass following:

$(x+2)(x+1) = 2x(x+1) \Leftrightarrow (x+2)(x+1) - 2x(x+1) = 0 \Leftrightarrow (x+1)(x+2-2x) = 0 \Leftrightarrow$

$\Leftrightarrow (x+1=0) \vee (2-x=0)$

svar ej: x=1 och x=2

svar: x=1 eller x=2

svar: $x = 1 \vee x = 2$

3.2 Mängder

$$\begin{aligned}
&\text{Naturliga tal: } \mathbb{N} = \{0, 1, 2, 3..\}\\
&\text{Heltal: } \mathbb{Z} = \{.. - 2, 1, 0, 1, 2..\}\\
&\text{Rationella tal: } \mathbb{Q} = \{\frac{a}{b}|a, b \in \mathbb{Z}, b\\
&\text{Irrationella tal: } \mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \{x|x \in \mathbb{R}, x \notin \mathbb{Q}\}\\
&\text{Reella tal: } \mathbb{R} = \mathbb{P} \cup \mathbb{Q}
\end{aligned}$$

$$\begin{aligned}
&A \cup B = \{x : (x \in A) \vee (x \in B)\}\\
&A \cap B = \{x : (x \in A) \wedge (x \in B)\}\\
&A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}\\
&A^{\#} = \{x : (x \in X) \wedge (x \notin A)\}
\end{aligned}$$

Exempel:

$$\text{Bevisa: } X \setminus (A \cup B) = (X \setminus A) \wedge (X \setminus B) \tag{3.3}$$

$$\begin{aligned}
&x \in (X \setminus (A \cup B)) \Rightarrow (x \in X) \wedge (x \notin (A \cup B)) \Rightarrow\\
&\Rightarrow (x \in X) \wedge (x \notin A) \wedge (x \notin B) \Rightarrow\\
&\Rightarrow (x \in X \setminus A) \wedge (x \in X \setminus B) \Rightarrow\\
&\Rightarrow x \in (X \setminus A) \wedge (X \setminus B) \Rightarrow\\
&\Rightarrow X \setminus (A \cup B) \subseteq (X \setminus A) \wedge (X \setminus B)
\end{aligned}$$

3.3 Bevis

3.3.1 Induktions bevis

- steg 1: bevisa att det gäller för basfallet.
- Steg 2: bevvisar att p => p.
  - a. Antar att det stämmer för pp.
  - b. Vissar att med att stopa in antagandet i pp+1 så blir hl = vl.

Tips: Förenkla hl först och sedan vl på klad papper fram och tillbacka.

Exempel Recursion:

$a_1 = 2, a_{n+1} = \frac{7a_n}{7-a_n}, n \in \mathbb{N}$

Vissa med induktion att $a_{n+1} = \frac{14}{7-2n}$

Bevis med induktions

steg 1: visar att påståendet som vi kallar p gäller för basfallet (n=1)

$VL_1 : a_{1+1} = a_2 = \frac{7 \cdot 2}{7-2} = \frac{14}{5}, HL_1 : a_{1+1} = a_2 = \frac{14}{7-2} = \frac{14}{5}$

steg 2: visar att $p_m \Rightarrow p_{m+1}$

steg 2a: antar att p_m gäller

$a_{m+1} = \frac{14}{7-2m}$

steg 2b: bevisar attt $p_m \Rightarrow p_{m+1}$ genom att andvända antagandet

$VL_{m+1} a_{m+2} = \frac{7a_{m+1}}{7-a_{m+1}} = \frac{7\frac{14}{7-2m}}{7-\frac{14}{7-2m}} = \frac{\frac{7 \cdot 14}{7-2m}}{\frac{7(7-2m)-14}{7-2m}} =$

$= \frac{7 \cdot 14}{7(7-2m+2)} = \frac{14}{7-2(m+1)}$

$HL_{m+1} : \frac{14}{7-2(m+1)}$

Enlight induktionsprincipen är p_m sann för alla $n = 1, 2, 3 \ldots$ VSB

3.3.2 Motsägelse bevis

- Steg 1: Formulera utsagan och icke utsagan
- Steg 2: Hitta en motsägelse med utsagan

Tips: Förenka båda led, tänk på teorin vi har, bättre att gå vaga moteveringar en inga alls

Bevis med motsägelse

Antar att motsatsen är sann

3.4 Delbarhet

$$\text{a är delbar med b, altså kvoten ger ingen rest. Vi följand: } a \mid b \tag{3.4}$$

Divitions algoritmen:

$$\begin{aligned}
&a, b \in \mathbb{Z}\\
&a \geq 0 \wedge b \geq 0\\
&a \mid b \Rightarrow (q \in \mathbb{Z} : q \geq 0) \wedge (r \in \mathbb{Z} : 0 \leq r \leq a) \text{ Sådant att}\\
&b = qa + r\\
&q = \text{ kvoten}, r = \text{ resten}
\end{aligned}$$

3.4.1 Största Gemensama Delaren (SGD)

$$\begin{aligned}
&SGD(a, b)\\
&a = bq + r\\
&0 \leq r \leq b
\end{aligned}$$

Euklides algoritm:

$$\begin{aligned}
&SGD(a, b)\\
&a = bq_1 + r_1\\
&b = r_1 q_2 + r_2\\
&r_1 = r_2 q_3 + r_3\\
&r_2 = r_3 q_4 + r_4\\
&.\\
&.\\
&.\\
&r_{k-3} = r_{k-2} q_{k-1} + r_k\\
&r_{k-2} = r_{k-1} q_k + 0\\
&SGD(a, b) = r_k\\
&\text{Om } r_k = 1 \Rightarrow a \in \text{primtal} \vee b \in \text{primtal}
\end{aligned}$$

Exempel:

$$
\begin{aligned}
&\text{Förenkla } \frac{114}{96} \\
&SGD(114, 96): \\
&114 = 1 * 96 + 18 \\
&96 = 5 * 18 + 6 \\
&18 = 3 * 6 + 0 \\
&\frac{114}{6} = 19 \\
&\frac{96}{6} = 16 \\
&\frac{19}{16}
\end{aligned}
$$

Lemma:

$$
\begin{aligned}
&a, b \in \mathbb{Z} \\
&x, y \in \mathbb{Z} \\
&SGD(a, b) = ax + by
\end{aligned}
$$

Aritmetiska fundamentalsatsen:

$$
\begin{aligned}
&a \in \mathbb{Z} \wedge a \geq 2 \Rightarrow \\
&\Rightarrow a \text{ kan endast primtalsfaktoriseras på ETT SÄTT}
\end{aligned}
$$

Lemma 2.7:

$$
\begin{aligned}
&a \geq 2 \wedge a \notin \text{ Primatal} \Rightarrow \\
&\Rightarrow q \in \text{ Primatal} \wedge q \mid a \wedge \text{ a går att primtals faktoriera}
\end{aligned}
$$

3.4.2 Primtal

Sats:

För att bestäma om tal a är ett primtal

$$\begin{aligned}
&a \geq 1 \wedge a \in \text{ Primatal} \\
&\text{om ett tall p finns som dellar a gäller följande} \\
&2 \leq p \leq \sqrt{a} \leq a
\end{aligned}$$

Exemple:

$$\text{Bestäm om 211 är ett primtal} \tag{3.5}$$

$$\begin{aligned}
&\text{Om 211 är ett primtal så finns det inte en äkta delare a} \\
&2 \leq a \leq \sqrt{211} \\
&\sqrt{211} \approx 16 \\
&a : \{\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{11}, \cancel{13}\} \\
&\text{a kan inte vara en äktadelare}
\end{aligned}$$

Euklides algoritm:

$$\text{Det finns oändligt många primtal}$$

Bevis:

Motsägelsebevis

$$\begin{aligned}
&\text{Antar att det finns ändligt många primtal} \\
&p_1, p_2, P_3, .., p_n \\
&M = \prod_{k=1} n + 1 \Rightarrow M > p_j, j = 1, 2, 3, .., n \\
&\text{Vissar att M är ett primtal} \\
&1 < b < M \wedge b \mid M \text{ Där b är minsta äkta delaren av M} \Rightarrow \\
&\Rightarrow M = p_1 * b \Rightarrow p_1 \mid 1 \wedge p \geq 2 \\
&\text{Detta är falskt eftersom båda utsågorna kan inte vara samtidigt} \\
&\text{Eftersom motsatsen inte fungerar resulterar det i att satsen är sann}
\end{aligned}$$

3.5 Diofantiska ekvationer

Sats:

$$
\begin{aligned}
&ax + by = c \wedge a, b, c \in \mathbb{Z} a \neq 0, b \neq b \\
&\Rightarrow \\
&ax + by = c \text{ Där } SGD(a,b) = 1 \\
&\text{Har den anmäla lösningen:} \\
&x = Cx_0 - nb \wedge y = Cy_0 + na
\end{aligned}
$$

Exempel:

$$\text{En lastbil lastas med 12kg packet och 20kg paket. Totalt väger lasten 296, hur många av varge packet?} \tag{3.6}$$

$$
\begin{aligned}
&ax + by = C \Leftrightarrow 12x + 20y = 296 \\
&\text{steg1: Testar om } SGD(a,b) \mid c \\
&SGD(20,12) = 4 \Rightarrow 4 \mid 269 \\
&\text{steg2: delar SGD med HL och VL} \\
&\frac{12x - 20y}{4} = \frac{269}{4} \Leftrightarrow 3x - 5y = 74 \\
&SGD(5,3): \\
&5 = 1 \cdot 3 + 2 \\
&3 = 1 \cdot 2 + 1 \\
&2 = 2 \cdot 1 + 0 \\
&\text{steg3: Hjälp ekvation för att hitta} x_0, y_0 \\
&3x_0 - 5y_0 = 1 \\
&1 = 3 - 1 * 2 \\
&1 = 3 - (5 - 3) \\
&1 = 2 \cdot 3 - 1 \cdot 5 \\
&x_0 = 2, y_0 = -1 \\
&\text{steg4: almäna lösningen } x = Cx_0 - bn, y = Cy_0 + an \\
&x = 74 \cdot 2 - 5n = 148 - 5n \\
&y = 74 \cdot (-1) + 3n = -74 + 3n \\
&\text{steg5: hittar godtyckliga lösningar} \\
&\text{Intervallet som n ligger i för x-termen:} \\
&n = \cancel{29}, 28, 27, \ldots, x = 148 - 5 \cdot 28 = 148 - 140 = 8 \\
&\text{Intervallet som n ligger i för y-termen:} \\
&n = \cancel{27}, 28, 29, \ldots, x = -74 + 3 \cdot 28 = 74 - 84 = 10 \\
&n = 28, x = 8, y = 10 \\
&12 \cdot 8 + 20 \cdot 10 = 296
\end{aligned}
$$

3.6 Talbaser

3.6.1 konvertera från decimal bass till annan bas

$$175_8 = 1 \cdot 8^2 + 7 \cdot 8^1 + 5 \cdot 8^0$$

3.6.2 konvertera från annan bas till decimal bas

$$1609_{10} = (3 \cdot 8^3 + 1 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0) = 3111_8$$

eller så kan man andvända euklides algoritm

$$\begin{aligned}
&\text{skriv 517 i talbas 3}\\
&517 = 172 \cdot 3 + 1\\
&172 = 57 \cdot 3 + 1\\
&57 = 20 \cdot 3 + 0\\
&20 = 6 \cdot 3 + 2\\
&6 = 2 \cdot 3 + 0\\
&2 = 0 \cdot 3 + 2\\
&\text{Svar: } 517_{tio} = 202011_{tre}
\end{aligned}$$

3.6.3 Andra exempel

Exempel: Skriv 137_{nio} i bass tre

$$\begin{aligned}
&137_{nio} = 1 \cdot 9^2 + 3 \cdot 9^1 + 7 \cdot 9^0 = 1 \cdot 3^4 + 3 \cdot 3^2 + 7 \cdot 3^0 = \\
&= 1 \cdot 3^4 + 1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 = 11021_{tre}
\end{aligned}$$

3.7 Functioner

Typer av funktioner:

Injektion: alla element x har olika värden y $f : A \to B, \{\forall x \in A : x_1 \neq x_2, f(x_1) \neq f(x_2)\}$
Surjektion: mängd D är definitions mängden $\{g : C \to D, g(x) = y, (\forall y \in D \wedge \exists x \in C)\}$
Bijektion: Injektion $\wedge$ Surjektion

Kareskaprodukten:

$A \times B = \{(a, b) : a \in A \wedge b \in B\}$
Låt $A = \{1, 2, 3\} \wedge B = \{x, y, z, w\}$
$A \times B :$
$\{(1, x), (1, y), (1, z), (1, w)$
$(2, x), (2, y), (2, z), (2, w)$
$(3, x), (3, y), (3, z), (3, w)\}$

3.7.1 Inversen

En funktions invers kan enda

$f : A \to B \wedge$ Bijektiv $\Rightarrow f^{-1}(x)$ Finns, där
(1) $x = f^{-1}(y) \Leftrightarrow y = f(x)$
(2) $D_{f^{-1}} = V_f \Leftrightarrow D_f = V_{f^{-1}}$
(3) $x = f^{-1}(f(x)), x \in D_f = V_{f^{-1}}$
(3) $y = f^{-1}(f(y)), y \in D_f = V_{f^{-1}}$

3.7.2 Relatioiner

Relation: xRy
Reflexiv: $\forall x \in X : xRx$
Symetrisk: $xRy \Rightarrow yRx, x \in X \wedge y \in X$
Transitiv: $(xRy) \wedge (yRz) \Rightarrow xRz, \forall x, y, z \in X$
Ekvivalnsrelation: Reflexiv och Symetrisk Transitiv

3.8 Summor

3.8.1 Aritmetiska summor

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n\,(a_1 + a_n)}{2} \tag{3.7}$$

3.8.2 Geometriska summor

Börjar altid med exponenten 0 och gör om summan så att den passar i följade talföljd:

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\,(k^n - 1)}{k - 1} \tag{3.8}$$

Exempel:

$$\sum_{k=n}^{2n} (2^k - k) \tag{3.9}$$

sätter f = 0 = k -n

$$\sum_{f=0}^{n} (2^{f+n} - (f+n)) = 2^n * \sum_{f=0}^{n} (2^f) - \sum_{f=0}^{n} (f+n)$$

$$\frac{2^n(2^{n+1}-1)}{2-1} - \frac{3n(n+1)}{2} = 2^{2n+1} - 2^n - \frac{3n(n+1)}{2}$$

3.9 Kongruensräkning

Räkneregler:

$$\begin{aligned}
&a+b \pmod n \equiv a \pmod n + b \pmod n\\
&a \cdot b \pmod n \equiv a \pmod n \cdot b (mod n)\\
&a^b \pmod n \equiv (a \pmod n)^b
\end{aligned}$$

Exempel: Vilket är det minsta positiva rest som kan erhållas vid division av 19^{18} med 17?

$19^{18} \equiv 2^{18} \pmod{17} \equiv 2^4 \cdot 2^4 \cdot 2^4 \cdot 2^4 \cdot 2^2 \pmod{17} \equiv (-1)^4 \cdot (-1)^4 \cdot (-1)^4 \cdot (-1)^4 \cdot 2^2 \pmod{17}$
svar: resten blir 4

3.10 Kardinalitet

Kardinalitet eller “mäktighet” är ett sett att räkna med mängders sorlek och alla oändliga mängder har samma kardinalitet fast det är en delmängd. Naturliga tall har samma kardinalitet som reala tal trots att naturliga tal är en del mängd av reala talen

Låt A och B vara mängder. Vi sägger att A och B har samma kardinalitet då det finns en bijektion
$\exists f : A \to B, A \sim B$
Vi säger att A står i relation med B omm A och B har samma kardinalitetARB

3.10.1 Uppräkneligamängder

En mängd X sägs vara upräknerlig omm X har samma kardinalitet som $\mathbb{N}$
$\exists g : \mathbb{N} \to X$ där g är bijektiv
Exempel på upräknerliga mängder är $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \{1, 5, 78\}$
Exempel på ej upräknerliga mängder $\mathbb{R}, (0, 1)$

3.11 Polynom

3.11.1 Polynom division

Triviala delare: ej heltals kvot med delaren, har en kostant sådant $\lambda \cdot f, \lambda \notin \mathbb{Z}$
Äkta delare: heltals kvot med delaren $f(x), \exists a \in polynom : a \mid f(x)$
Irreducible: om polynomet endast har triviala delare det finns lösningar heltaslösningar $f(x) = 0$
Reducible: om polynomet har äkta delare
Multiplisitet: vilken grad polynomet har

Exempel:

Man vet att ekvationen $z^4-2z^3-7z^2+26z-20 = 0$ har roten $z = 2+i$. Lös ekvationen fullständigt. (3.10)

$z = 2 + i$ Är en lösning är också konjugatet en lösning enligt faktorsatsen $\overline{z} = a - bi$
$z = 2 \pm i$
Vilket betyder att följande går att factorisera ut polynomet
$(z - (2 + i))(z - (2 - i)) = z^2 - 4z + 5$

Långdivition (liggande stolen):

$$
\begin{array}{rrrrr}
 & x^2 & +2x & -4 & \\
\hline
x^2-4x+5) & x^4-2x^3 & -7x^2 & +26x & -20 \\
 & -x^4+4x^3 & -5x^2 & & \\
\hline
 & 2x^3 & -12x^2 & +26x & \\
 & -2x^3 & +8x^2 & -10x & \\
\hline
 & & -4x^2 & +16x & -20 \\
 & & 4x^2 & -16x & +20 \\
\hline
 & & & & 0
\end{array}
$$

$z^2 + 2z - 4 = 0$ Är också en läsning som tilslut ger följande
$z = -1 \pm \sqrt{5}$
$z = 2 \pm i$
Varge n grads polynom har altid n stycken komplexa lösningar

3.11.2 Faktorsatsen

$$f(x) = (x - \alpha)p(x)$$

Exempel, Gemensam root hos två polynom:

$p(x) = x^4 - x^3 + x^2 + 2 = 0,\ g(x) = x^3 + 4x^2 + 4x + 3 = 0$ har en Gemensam root
Eftersom polynomen har en Gemensam root vet vi att det finns en gemensam$(x - \alpha)$

$f(x) = (x - \alpha)p_1(x)$
$g(x) = (x - \alpha)p_2(x)$
euklides algoritm:
$f(x) = (x - 5)g(x) + 17(x^2 + x + 1)$
$g(x) = (\frac{1}{17}(x + 3))(17(x^2 + x + 1)) + 17(x^2 + x + 1) + 0$

$h(x) = x^2 + x + 1$
$f(x) = (x - 5)(x + 3)h(x) + 17h(x) = h(x)((x - 5)(x + 3) + 17) = h(x)(x^2 - 2x + 2)$
$g(x) = (x + 3)h(x)$

$f(x) = 0 \Leftrightarrow (h(x) = 0 \vee x^2 - 2x + 2 = 0)$
$h(x) = 0 \Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
$x^2 + 2x + 2 = 0 \Rightarrow x = 1 \pm \sqrt{1 - 2} = 1 \pm i$

Exempel, Heltalslösning med okänd konstant:

$x^3 + bx^2 - 7x - 7 = 0$ har en heltals lösning
Eftersom ekvationen har en heltals lösning vet vi att $\alpha \mid 7$
Kandidaterna är $\{1, -1, 7, -7\}$
$I(x = 1)\ 1 + b \cdot 1 - 7 - 7 = 0 \Rightarrow b = 13 \in \mathbb{Z}$
$II(x = -1)\ -1 + b \cdot -1 + 7 - 7 = 0 \Rightarrow b = 1 \in \mathbb{Z}$
$III(x = 7)\ 7 + b \cdot 7 - 49 - 7 = 0 \Rightarrow b = \_ \notin \mathbb{Z}$
$IV(x = -7)\ 7 + b \cdot -7 + 49 - 7 = 0 \Rightarrow b = \_ \notin \mathbb{Z}$
svar: b=13, b=1

Exempel, Rationell root:

$g(x) = 2x^3 + 2x^2 + 2x + 3 = 0$ har en rationell root
Eftersom ploynomet har en rattionel root vet vi att:
$\frac{p}{q}\ p, q \in \mathbb{Z},\ SGD(p, q) = 1,\ p \mid 3 \wedge q \mid 2\ (a_0)$
Det möjliga delarna är $p = \pm 1, \pm 3,\ q = \pm 1, \pm 2$
$x = -\frac{3}{2}$ är den enda av kandidaterna som ger en sann root
Vilket vi löser genom polynom divition

Exempel, Renimaginär root:

$z^4 + 4z^3 + 8z^2 + 12z + 15 = 0$ har en renimachinär root
Eftersom ploynomet har en rentimaghinär root vet vi att $z = bi, b \in \mathbb{R}, b \neq 0$
$0 = b^4 i^4 + 4b^3 i^3 + 8b^2 i^2 + 12bi + 15 = b^4 - 4b^3 i - 8b^2 + 12bi + 15 = (b^4 - 8b^2 + 15) + (12b - 4b^3)i =$
$0 = b^4 - 8b^2 + 15$
$0 = 12b - 4b^3 = 4b(3 - b^2) \Rightarrow b = 0, b = \sqrt{3}, b = -\sqrt{3}$
$b = 0$ är ej en giltig lösning. Enlight faktorsatsen får vi följande
$p(z) = (z - \sqrt{3}i)(z + \sqrt{3}i)Q(x) = (z^2 + 3)Q(z)$
Vilket vi löser genom polynom divition

Exempel, Dubel root:

$p(t) = t^3 - 5t^2 + 3t + 9 = 0$ har en dubbel root
Eftersom ploynomet har en dubbel root vet vi att derivatan ger os en root$p'(t) = 0$
Att räkna ut derivatan gör man genm formeln $ax^b \Rightarrow b \cdot ax^{b-1}$
$p'(t) = 3t^2 - 10t + 3 = 0 \Rightarrow t_1 = 3, t_2 = \frac{1}{3}$ dock är sista en falsk root
$p(t) = x - 3^2 Q(t)$
Vilket vi löser genom polynom divition

Exempel, SGD polynom:

Förenkla $\dfrac{x^4+x^3+2x-4}{x^4-x^3-2x-4}$

$SGD(x^4+x^3+2x-4, x^4-x^3-2x-4):$

$x^4+x^3+2x-4 = 1\cdot x^4-x^3-2x-4+(2x^3+4x)$

$x^4-x^3-2x-4 = \dfrac{x}{2}-\dfrac{1}{2}\cdot(2x^3+4x)+(-2x^2-4)$ med polynom divition

$2x^3+4x = -x\cdot(-2x^2-4)+0$

Med SGD så kan vi ta ett polynom som är hjämt delbart ex: $-2(x^2+x)$

polynom divition: $\dfrac{x^4+x^3+2x-4}{x^2+x} = x^2+x-2$

polynom divition: $\dfrac{x^4-x^3-2x-4}{x^2+x} = x^2-x-2$

$\dfrac{x^4+x^3+2x-4}{x^4-x^3-2x-4} = \dfrac{x^2+x-2}{x^2-x-2}$

Chapter 4

Single Variable Calculus

4.1 Basic

4.1.1 Mängder

Naturliga tal: $\mathbb{N} = \{0, 1, 2, 3..\}$ or $\{1, 2, 3..\}$
Heltal: $\mathbb{Z} = \{.. -2, 1, 0, 1, 2..\}$
Rationella tal: $\mathbb{Q} = \left\{ \frac{a}{b} \middle| \, a, b \in \mathbb{Z} \right\}$
Irrationella tal: $\mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \, \{x | x \in \mathbb{R}, x \notin \mathbb{Q}\}$
Reella tal: $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$
$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$
$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}$$
$$A^{\#} = \{x : (x \in X) \wedge (x \notin A)\}$$

4.1.2 Intervall

$$\text{Open intervall} = (1,4)$$
$$\text{Closed intervall} = [\,1,4]$$

$$[\,a,b] \ = \{x|a \leq x \leq b\} \tag{4.1}$$

$$[\,a,\infty[\tag{4.2}$$

$$]-\infty,\infty[\tag{4.3}$$

Exempel: Olikheter och intervall

$$\frac{2}{x-3} < \frac{5}{x} \tag{4.4}$$

$$\frac{2}{x-3} - \frac{5}{x} < 0$$
$$\frac{(x)2}{x(x-3)} - \frac{5(x-3)}{x(x-3)} < 0$$
$$\frac{2x-5x-15}{x(x-3)} < 0$$
$$\frac{-3x-15}{x(x-3)} < 0$$
$$\frac{-3(x-5)}{x(x-3)} < 0$$
$$x \neq 0, x \neq 3$$

Värde tabell:

	x<0	0<x<3	3<x<5	5<x
x-5	-	-	-	+
-3	-	-	-	-
x	-	+	+	+
x-3	-	-	+	+
hela	+	-	+	-

4.1.3 Funktion

$f : A \to B$

A: domain/definitionsmängd, B: Målmängd/kodomängd/värdemängd

Injektion: alla element x har olika värden y $f : A \to B, \{\forall x \in A : x_1 \neq x_2, f(x_1) \neq f(x_2)\}$

Surjektion: mängd D är definitions mängden $\{g : C \to D, g(x) = y, (\forall y \in D \wedge \exists x \in C)\}$

Bijektion: Injektion $\wedge$ Surjektion

$f \circ g = f(g(x))$

Begränsad: functionen är inom ett intervall, altså nedåt och uppåt

Begränsat nedåt: functionenn är endast begrensad nedåt

Begränsat uppåt: functionenn är endast begrensad uppåt

Jämn function: altså är den symetrisk, ex: $x^2,\ f(-x) = f(x)$

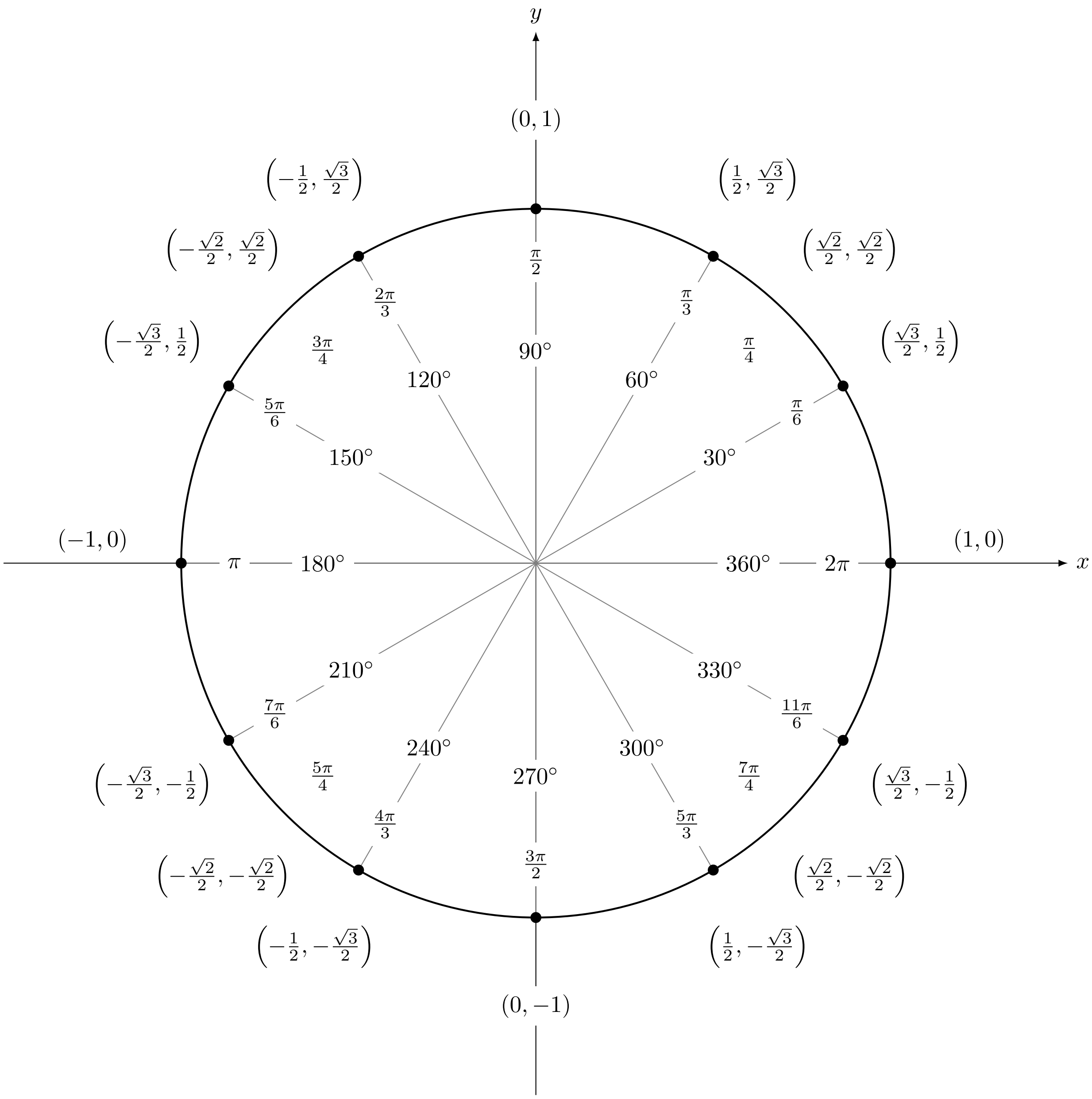
Udda function: altså är den spefelvänt symetrisk, ex: $x^3,\ f(-x) = -f(x)$

Polynom $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{n} (a_i x^i)$

Rationell funktion:$R(x) = \frac{P(x)}{Q(x)}$, ex: $x^{-1} = \frac{1}{x}$ ej polynom men rationell funktion

4.1.4 Trigonometri

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$



Sats:

$$
\begin{aligned}
360^\circ &= 2\pi rad \\
v_g &= v_r \cdot \frac{180^\circ}{\pi} \\
v_r &= v_g \cdot \frac{\pi}{180^\circ}
\end{aligned}
$$

Sats:

$$
\begin{aligned}
-1 \leq \sin t \leq 1 \\
-1 \leq \cos t \leq 1
\end{aligned}
$$

Sats:

$$
\begin{aligned}
\cos(-t) &= \cos(t) \\
\sin(-t) &= -\sin(t) \\
\tan(-t) &= \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)}
\end{aligned}
$$

Additionsformlerna:

$$
\begin{aligned}
\sin(\alpha + \beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
\sin(\alpha - \beta) &= \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \\
\cos(\alpha + \beta) &= \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \\
\cos(\alpha - \beta) &= \cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha)
\end{aligned}
$$

Trigonometriska ettan:

$$
\begin{aligned}
(\sin t)^2 + (\cos t)^2 = 1 \\
\sin^2 t + \cos^2 t = 1
\end{aligned}
$$

4.1.5 Exempel: Trigonometri

$$
\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

4.2 Gränsvärden

$\lim_{x\to x_0} f = L \wedge \lim_{x\to x_0} g = M \Rightarrow \lim_{x\to x_0}(f+g) = L+M$

Squeeze therum: $h(x) \leq f(x) \leq g(x)$, $\lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x_0) = L \Rightarrow \lim_{x\to x_0} f(x) = L$

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$\lim_{x\to\infty} \sin(x)$ är ej definerad

Example: Limit för sin

$$\text{Lös: } \lim_{x\to 0} \frac{\sin(2x^2)}{x^2}$$
$$\lim_{x\to 0} \frac{2\cdot\sin(2x^2)}{2x^2} = 2\cdot\lim_{x\to 0}\frac{\sin(2x^2)}{2x^2} = 2$$

Example: 0/0

$$\text{Lös: } \lim_{x\to 0} \frac{x\cdot\left(\cos(x)+\frac{\sin(x)}{x}\right)}{x+x^2}$$
$$\lim_{x\to 0} \frac{x\cdot\left(\cos(x)+\frac{\sin(x)}{x}\right)}{x(1+x)} = \frac{1+1}{1} = 2$$

Example: roten ur i nämnaren

$$\text{Lös: } \lim_{x\to 0} \frac{x}{\sqrt{x+9}-3}$$
$$\lim_{x\to 0} \frac{x(\sqrt{x+9}+3)}{(\sqrt{x+9}-3)(\sqrt{x+9}+3)} = \lim_{x\to 0}\frac{x(\sqrt{x+9}+3)}{x+9-9} = \lim_{x\to 0}\sqrt{x+9}+3 = 6$$

Example: roten ur i nämnaren

$$\text{Lös: } \lim_{x\to -\infty} \frac{2x-1}{\sqrt{3x^2+x+1}}$$
$$\lim_{x\to-\infty} \frac{x(2-\frac{1}{x})}{\sqrt{x^2(3+\frac{1}{x}+\frac{1}{x^2})}} = \lim_{x\to-\infty}\frac{x(2-\frac{1}{x})}{x\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \frac{-2}{\sqrt{3}}$$

Example: infinity sin and squeeze

$$\text{Lös: } \lim_{x\to\infty} \frac{\sin(x)}{x}$$
$$\left|\frac{\sin(x)}{x}\right| = \frac{|\sin(x)|}{|x|} \leq \frac{1}{|x|} \Rightarrow \text{ squeeze } \lim_{x\to\infty}\left|\frac{\sin(x)}{x}\right| = 0$$
$$\Rightarrow \lim_{x\to\infty}\frac{\sin(x)}{x} = 0$$

4.2.1 Kontinuitet

En funktion är kontnuelig i punkt x_0, $\forall \varepsilon \wedge \exists \delta : |x - x_0| < \delta \Leftrightarrow |f(x) - f(c_0)| < \varepsilon$.
Altså så finns det inga hopp i funktionen. Man kan dra penan på grafen utan att släppa om functionen bestor av flera utryck är funktionen kontenuerlig om alla utrycken är det

4.3 Derivator

Tangent: linjen som har samma lutining i en punkt som functionens derivata
$y - f(x_0) = f'(x_0)(x - x_0) \Rightarrow y = ax + b$
Samband: *Deriverbar* $\Rightarrow$ *Kontinuerlig*

Räkneregler

f	f'
c	0
x	1
x^n	nx^{n-1}
a^x	$a^x \ln a$
e^{kx}	ke^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\ln x$	$\frac{1}{x}$

$(f+g)' = f' + g'$
$(cf)' = cf'$
$(fg)' = f'g + fg'$ Produktregeln
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \text{ Kvotregeln, } g \neq 0$$

4.3.1 Kjedje regeln

Om functionen f är deriverbar i x och funktionen är deriverbar i $g(x)$
$\Rightarrow h(x) = f \circ g = f(g(x))$ deriverbar i x
$h'(x) = (f(g(x)))' = f'(g(x))g'(x)$

Example: Kjedje regeln

Lös: $(x^x)'$

$$(x^x)' = \left(e^{x \ln x}\right)' = e^{x \ln x}\left(\ln x + x\frac{1}{x}\right) = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x)$$

4.3.2 L'Hôpital's rule

$$f, g : \mathbb{R} \to \mathbb{R} \wedge \left(\lim_{x \to x_0} \frac{f}{g} = \frac{0}{0} \vee'' \lim_{x \to x_0} \frac{f}{g} = \frac{\pm\infty}{\pm\infty}'' \right) \Rightarrow \lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}$$

Example: L'Hôpital's rule

Lös: $\lim_{x \to 1} \frac{x^4 - 6x^3 + 5x^2}{x^3 - 8x^2 + 7x}$

$\lim_{x \to 1} \frac{x^4 - 6x^3 + 5x^2}{x^3 - 8x^2 + 7x} ='' \frac{0}{0}''$

L'Hôpital's rule $\Rightarrow \lim_{x \to 1} \frac{4x^3 - 18x^2 + 10x}{3x^2 - 16x + 7}$

$= \frac{-4}{-6} = \frac{2}{3}$

4.3.3 Medelvärdessatsen

Anta att f är kontinuerlig på $[a, b]$ och deriverbar på $(a, b) \Rightarrow \exists c \in (a, b)$

så att $\frac{f(b) - f(a)}{b - a} = f'(c)$

Example: Medelvärdessatsen

Bevisa att för varge $a > b$ gäller följande$|\cos a - \cos b| \leq |a - b|$

$f(x) = \cos x$ är kont och deriverbar i $\mathbb{R} \wedge [a, b]$

MVS $\Rightarrow \exists c \in (a, b) : f'(c) = \frac{f(b) - (a)}{b - a} = \frac{\cos(b) - \cos(a)}{b - a} = -\sin(c)$

$\Rightarrow |\frac{\cos(b) - \cos(a)}{b - a}| = |-\sin(c)| \Rightarrow \frac{|\cos(b) - \cos(a)|}{|b - a|} = |-\sin(c)| \leq 1$

$\frac{|\cos(b) - \cos(a)|}{|b - a|} \leq 1 \Rightarrow |\cos(b) - \cos(a)| \leq |a - b|$

4.3.4 Rolle

Anta att f är kontinuerlig på $[a, b]$ och deriverbar på (a, b). Om $f(a) = f(b) \Rightarrow \exists c \in (a, b) : f'(c) = 0$

4.3.5 växande funktioner

växande: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
strängt växande: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
avtagande: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
stänkt avtagande: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

4.3.6 Högreordnings devivator

andragrads derivator: $(f')' = f'' = \dfrac{d^2 f}{dx^2}$
tredjegrads derivator: $(f'')' = f''' = \dfrac{d^3 f}{dx^3}$
ntegrads derivator: $f^{(n)} = f^n = \dfrac{d^n f}{dx^n}$

4.3.7 Impericit derivering

deriverar båda sidorna om modifierat

Example: Impericit derivering

Bestäm tangenten till $x^3 + y^3 = 6xy\ i\ (3,3)$
$(3^3 + 3^3 = 6 \cdot 3 \cdot 3)$ -sant
$y - y_0 = f'(x_0)(x - x_0) \Rightarrow y - 3 = f'(3)(x - 3)$
Implicit derivering: $3x^2 + 3y^2y' = 6y + 6xy'$
$\Rightarrow 3x^2y' - 6xy' = 6y - 6x^2 \Rightarrow y'(3x^2 - 6x) = 6y - 3x^2 \Rightarrow y' = \dfrac{6y - 3x^2}{3x^2 - 6x}$
$\Rightarrow y'(3) = \dfrac{63 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = -1$
$\Rightarrow y - 3 = -(x - 3) \Rightarrow x + y = 6 \vee y = -x + 6$

4.3.8 invers funktioner

$f : A \to B \wedge Bijektiv \Rightarrow f^{-1}(x)$ Finns, där
(1) $x = f^{-1}(y) \Leftrightarrow y = f(x)$
(2) $D_{f^{-1}} = V_f \Leftrightarrow D_f = V_{f^{-1}}$
(3) $x = f^{-1}(f(x)), x \in D_f = V_{f^{-1}}$
(3) $y = f^{-1}(f(y)), y \in D_f = V_{f^{-1}}$

4.3.9 exponetial och logaritm

$$\frac{a}{b}^{-3} = \frac{b}{a}^{3}$$
$$\sqrt{a} = a^{\frac{1}{2}}$$
$$a^{x+y} = a^x a^y$$
$$(a^x)^y = a^{xy}$$
$$a^{-x} = \frac{1}{a^x}$$
$$a^{\frac{m}{n}} = \sqrt{a^m}^n$$

(1): $b = a^x \Leftrightarrow \log_a(b) = x$ för: $a > 0, b > 0, a \neq 1$

(2): $\log_a(\frac{b}{c}) = \log_a(b) - \log_a(c)$

(3): $\log_a(b \cdot c) = \log_a(b) + \log_a(c)$

(4): $\log_a(b^d) = d \log_a(b)$

(5): $\log_a(b) = \frac{\log_f(b)}{\log_f(a)}$

(6): $\log_a(a) = 1$

(7): $\log_a(1) = 0$

(8): $a^{\log_a(x)} = x$

(9): $\log_{a^c}(b) = \frac{1}{c} \log_a(b)$

4.3.10 odefinerad form

$$\frac{0}{0}, \infty \cdot 0, 1^{\infty}, \infty^0, 0^0$$

4.3.11 inversa trigometriska funktioner

arcsin

$$\begin{aligned}
&\cos : [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \\
&\arcsin 1 = \frac{\pi}{2} \\
&\arcsin 0 = 0 \\
&\arcsin \pi = \text{odefinerad} \\
&\sin(\arcsin(x)) = x \\
&\arcsin(\sin(x)) = x \\
&(\arcsin(x))' = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} = \\
&\frac{1}{\sqrt{\cos^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

arccos

$$\begin{aligned}
&\cos : [0, \pi] \to [-1, 1] \\
&\arccos 1 = 0 \\
&\arccos 0 = \frac{\pi}{2} \\
&\arccos \pi = \text{odefinerad} \\
&\cos(\arccos(x)) = x \\
&\arccos(\cos(x)) = x \\
&(\arccos(x))' = \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

arctan

$$\begin{aligned}
&\cos : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R} \\
&\tan(\arctan(x)) = x \\
&\arctan(\tan(x)) = x \\
&(\arctan(x))' = \frac{1}{1 + x^2}
\end{aligned}$$

exempel

$$
\begin{aligned}
\tan(\arccos x) &= \frac{\sin(\arccos x)}{\cos \arccos x} = \frac{\sqrt{1-x^2}}{x} \\
\cos(\arctan x) &= \cos \frac{\arcsin x}{\arccos x} = \frac{1}{1+x^2} \\
\sin(\arccos x) &= \sqrt{1-x^2} \\
\cos(\arcsin x) &= \sqrt{1-x^2}
\end{aligned}
$$

4.4 Grafritning

Ta reda på extrem punkterna och rita utifrån det

Extremvärden

global minimum punkt: $x_0,\ \forall x : f(x) \geq f(x_0)$
lokal minimum punkt: $x_0,\ \forall x$ nära $x_0 : f(x) \geq f(x_0)$
global maximum punkt: $x_0,\ \forall x : f(x) \leq f(x_0)$
lokal maximum punkt: $x_0,\ \forall x$ nära $x_0 : f(x) \leq f(x_0)$
Kritisk punkt: $f'(x) = 0$

Komplexity

konvex: Tangent liger altid under funktionen $y'' > 0$
konkav: Tangent liger altid över funktionen $y'' < 0$
Inflektionspunct: då funktionen byter från konvex till konkav eller tvärtom

Exemple: Datan som behövs beräknas vid grafritning

Rita functionen $y = (x^2 - 1)^3$

(1) Kollar om functionen är Konternuelig:

Eftersom funkrionen består av polynom är den konternuerlig

(2) Extrem punkter:

(I) Derivatan: functionen är deriverbar $y' = 3(x^2 - 1)^2 \cdot 2x = 0$

$\Rightarrow x = 0$

(II) Singulär punkter:

eftersom funktionen är deriverbar i alla punkter i definitions mängden

Så funns det inga singulär punkter

(III) End värden: Eftersom funktionen är ej definerad i ett intervall så finns det inga

(3) Komplexitet:

Andra derivatan avgör om funktionen är knvex eller konkav

$y'' = (6x(x^2 - 1)^2)' = (6x(x^4 - 2x^2 + 1))' = (6x^5 - 12x^3 + 6x)' = 30x^4 - 36x^2 + 6 = 0$

$\Rightarrow t^2 - \frac{6}{5}t + \frac{1}{5} = 0 \Rightarrow t = \frac{6}{10} \pm \sqrt{\frac{36}{100} - \frac{20}{100}} = \frac{6}{10} \pm \frac{4}{10}$

$t = 1 \vee t = \frac{1}{5} \Rightarrow x = \pm 1 \vee x = \pm\frac{1}{\sqrt{5}}$

(4) Asymptoter:

(I) lodräta asymptoter:

$\lim_{x->+\infty} y = +\infty$

$\lim_{x->-\infty} y = +\infty$

(II) vågräta asymptoter:

Eftersom funktionen inte är en kvot finns det inga sådana asymptoter

	$x < -1$	$x = -1$	$-1 < x < -\frac{1}{\sqrt{5}}$	$x = -\frac{1}{\sqrt{5}}$	...
f'	$-$	$-$	$-$	$-$	...
f''	$-$	0	$+$	0	...
f	avtagande	avtagande	avtagande	avtagande	...
	kokav	inflektions punkt	konvex	inflektions punk	...

4.5 Optemering

Sats: om funktionen $f(x)$ är knternuerlig på det slutna intervalet $[a, b]$
så antar det sitt största värde och minsta värde där $\exists x_1, x_2 \in [a, b]$
så att$f(x_1) \leq f(x) \leq f(x_2)$

Exempel: Max/Min värde

Låt $f(x) : [-1, 2] \to \mathbb{R}, f(x) = x^3 - 3|x|$
(1) Konternuelig:
f är konternuerlig i intervallet

(2) Extrem punkter:
(I) Derivatan: functionen är deriverbar i intervalet förutom då $x = 0$
låt f bestå av $f_1(x) = x^3 - 3x, x \geq 0 \wedge f_1(x) = x^3 + 3x, x < 0$
$\Rightarrow f_1'(x) = 3x^2 - 3, x \geq 0 \wedge f_2'(x) = 3x^2 + 3, x < 0$
$f_1'(x) = 0 \Rightarrow x = 1 \in [1, 2], f(1) = -2$
$f_2'(x) = 0 \Rightarrow x =$ Odefinerad
(II) Singulär punkter:
f är inte deriverbar då $x = 0, f(0) = 0$
(III) End punkterna: Eftersom funktionen är ej definerad i ett intervall så finns det inga
$f(-1) = -4$
$f(2) = 2$

(3) Största och minsta värdet:
$f(-1) < f(1) < f(0) < f(2)$
Svar: Max är 2, min är -4

4.6 Talfölder och serier

Def: Talföljder

En talföljd är en funktion $a : \mathbb{N} \to \mathbb{N}$
Vi skriver a_n istället för $a(n)$, $a_2 = a(2)$
Vi säger att $a_n \to a \in \mathbb{N}$ så är den **konvergent**
Vi säger att $a_n \to \infty$ så är den **divergent** eller ej existerande
Konvergent kan vara begränsad uppåt eller nedåt
Talföljed kan vara växande eller avtagande

$a_n \to a \vee b_n \to b$ Omm a och b existerar $a_n + b_n \to a + b$

Exempel: Derivatan av serier

$$\text{Ange } f'(2),\ f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 2^{2n}}$$

$$\text{Anger den första termen } \left(\frac{x-2}{4}\right)' = \frac{1}{4}$$

$$f'(x) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{n(x-2)^{n-1}}{n^2 2^{2n}} \text{ Inre och yttre derivatan}$$

$$f'(2) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{n(2-2)^{n-1}}{n^2 2^{2n}} = \frac{1}{4} + 0 = \frac{1}{4}$$

Sats:

$$\text{Om } (a_n)^{\infty} \text{ är Konvergent } \Rightarrow (a_n),\ n = 1 \text{ är begränsad}$$

Sats:

Låt $a > 0$
(I) $a^n \to 0$ om $a < 1$
(II) $a^n \to +\infty$ om $a > 1$

$$(a_n)_{n=1}{}^{\infty} = \sum_{k=0}^{n} a_n = a_1 + a_2 + a_3 + \ldots$$

Sats: Geometrisk serie

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\left(k^n - 1\right)}{k-1}$$

$$\sum_{n=0}^{\infty} r^n \Rightarrow$$

Konvergent om $|r| < 1$
divergent om $|r| < 1$

Sats:

$$\sum_{n=0}^{\infty} a_n \text{ konvergent } \Rightarrow a_n \to 0$$

Sats: p-serie

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \Rightarrow$$
$$\text{konvergent om } \Rightarrow p > 1$$
$$\text{divergent om } \Rightarrow p \leq 1$$

Sats:

Antag att $0 \leq a_n \leq b_n$ för varge $n \in \mathbf{N}$

$$\sum_{n=1}^{\infty} a_n \text{ konvergent } \Rightarrow a_n \to 0$$

$$(I) \sum_{n=1}^{\infty} b_n \text{ konvergerar} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ konvergerar}$$

$$(II) \sum_{n=1}^{\infty} a_n \text{ derigerar} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ derigerar}$$

Sats:

Antag att $a_n > 0,\ b_n > 0$ för varje $n \in \mathbf{N}$ och antag att $\dfrac{a_n}{b_n} \to L \neq 0 \ \wedge L < \pm\infty$

Då gäller att serien $\displaystyle\sum_{n=1}^{\infty} a_n$ och $\displaystyle\sum_{n=1}^{\infty} b_n$

är båda divergenta

Sats: kvotkriterium

Låt $a_n > 0 \wedge \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} \to L \Rightarrow$

$$(I)\ 0 \leq L \leq 1 \Rightarrow \text{ Konvergerar } \sum_{n=1}^{\infty} a_n (a_n \to 0)$$

$$(II)\ L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ derigerar } (a_n \to +\infty)$$

Exemple: Måste kunna

$$\sum_{n=1}^{\infty} \frac{5^n n!}{n^n}$$

$$\text{Låt } a_n = \frac{5^n n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{5^{n+1}(n+1)!/n^n}{5^n n!/n^n} = 5\frac{(n+1)n^n}{(n+1)^{n+1}} = 5(\frac{n}{n+1})^n =$$

$$= 5\frac{1}{(1+1/n)^n} \to \frac{5}{e} > 1 \Rightarrow \text{ derigerar}$$

Sats: Rotkriteriet (Ovanlig)

$$\text{Låt } a_n > 0 \wedge \lim_{n\to\infty} \sqrt[n]{a_n} \to L \Rightarrow$$

$$(I)\ 0 \le L \le 1 \Rightarrow \text{ Konvergerar } \sum_{n=1}^{\infty} a_n (a_n \to 0)$$

$$(II)\ L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ derigerar } (a_n \to +\infty)$$

4.6.1 Serier med varierande tecken

Sats: leibinz

$$\text{Antag att } \sum_{n=1}^{\infty} a_n \text{ är en alternerande serie}$$

$$\text{och att om } |a_n| \to 0 \wedge |a_n| \ge |a_{n+1}| \Rightarrow \textit{konvergent}$$

Def: absolut konvergent

$$\text{Endast om } \sum_{n=1}^{\infty} |a_n| \text{ konvagerar } \Rightarrow \text{ då är den absolutkonvergent}$$

om seriern är absolutkonvergent så är den också konvergent behöver inte vara tvärtom

4.6.2 Potensserier

Def: Potensserier

$$\text{En potensserier kring } x_0 \text{ är en serie på formen } \sum_{n=1}^{\infty} a_n (x - x_0)^n,\ a_n \in \mathbb{R} \text{ (koefficienten)}$$

x är en variabel

Def: Konvergens radie

$T(x) = \sum_{n=1}^{\infty} \frac{f(x_0)^{(n)}}{n!}(x - x_0)^n$
vid exempelvis grad 2 så blir det andra derivatan man räknar ut
$R = L^{-1} = \frac{1}{L}$ (L är där serien konvergerar, R är konvergens radie) $\sum_{n=1}^{\infty} a_n(x - x_0)^n,\ a_n \in \mathbb{R}$ (koefficienten)
x är en variabel
Om konvergens radien är noll så finns det inte några intervar för konvergens eller divergens

Sats: Potensserier

En potensserie kring x_0 är en serie på formen $\sum_{n=1}^{\infty} a_n(x - x_0)^n a_n \in \mathbb{R}\ a_n$ koficienten x är en variabel
När det står ex $(3x + 2)^n$ så måste det stå med i $a_n, 3^n$

Exempel: Potensserier

$\sum_{n=1}^{\infty} \frac{(x-7)^n}{n(n+1)}$ är en potensserie där $x_0 = 7, a_n = \frac{1}{n(n+1)}$

$|\frac{a_{n+1}}{a_n}| = |\frac{1/((n+1)(n+2))}{1/(n(n+1))}| = |\frac{n}{n+2}| = |\frac{n}{n(1+2/n)}|$

$\Rightarrow |\frac{1}{1}| = 1,$ då $n \to \infty$

$\Rightarrow R = \frac{1}{1} = 1 \Rightarrow$ vi får följande intervall av potenserien

Absolutkonverget för $|x - 7| < 1 \Leftrightarrow 6 < x < 8$
Divergent för $|x - 7| > 1 \Leftrightarrow 8 < x \vee x < 6$

Testar edge fallen $x = 6, x = 8$

$\sum_{n=1}^{\infty} \frac{(8-7)^n}{n(n+1)} \Rightarrow |\sum_{n=1}^{\infty} \frac{(1)^n}{n(n+1)}|,\ |\frac{(1)^n}{n(n+1)}| \to 0 \Rightarrow$ absolutkonvergerar

$\sum_{n=1}^{\infty} \frac{(6-7)^n}{n(n+1)} \Rightarrow |\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}|,\ |\frac{(-1)^n}{n(n+1)}| \to 0 \Rightarrow$ absolutkonvergerar

Svar:
Absolutkonverget för $6 \leq x \leq 8$
Divergent för $8 < x \vee x < 6$

4.6.3 Taylor serier

Def: Talorpolynom

Taylorpolynom av grad $n \in \mathbb{N}$, *kring* $x = x_0$

$$f(x) = P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k,\ 0! = 1, f^{(0)} = f$$

När $x \approx x_0 \Rightarrow f(x) \approx p_n(x)$

Sats: Talor sats

Antag att $f \in c^{n+1}$.

$f(x) = p_n(x) + R_{n+1}(x)$ Där felet är $R_{n+1}(x)$

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},\ x \vee x_0 \leq c \leq x_0 \vee x$$

exempel: felet

Upskata felet hos $f(x) = \sin x$, med grad 5

Sista termen är $\dfrac{x^5}{5!} \Rightarrow$ Felet $\dfrac{f^{(6)}(c)}{6!}x^6, (0 < c < 1)$

$$\left|\frac{f^{(6)}(c)}{6!}x^6\right| \leq \frac{|f^{(6)}(c)|}{6!} = \frac{|\sin c|}{6!} \leq \frac{1}{6!} = \frac{1}{720}$$

Exempel: grad n vid specificerad punkt

Lös: Hitta Taylorpolynomet, ording $2n - 1$, för $\sin 2x$ vid $x = \dfrac{\pi}{2}$

$f(x) = \sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f'(x) = -2\cos(2x)$ $\qquad f(\frac{\pi}{2}) = -2$

$f''(x) = -2^2\sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f^{(3)}(x) = -2\cos(2x)$ $\qquad f(\frac{\pi}{2}) = 2^3$

$f^{(4)}(x) = -2^2\sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f^{(5)}(x) = -2^4 f'(x)$ $\qquad f(\frac{\pi}{2}) = -2^5$

$$p_{2n-1}(x) = -2(x - \frac{\pi}{2}) + \frac{2^3}{3!}(x - \frac{\pi}{2})^3 - \frac{2^5}{5!}(x - \frac{\pi}{2})^5 + \ldots + \frac{(-1)^n \cdot 2^{2n-1}}{(2n-1)!}(x - \frac{\pi}{2})^{(2n-1)}$$

Def: maclaurin polynomial

Taylorpolynom av grad $n \in \mathbb{N},$ då $x_0 = 0$
Big-O notation: Vi säger att $g(x) = O(f(x))$ när $x \approx x_0$, värsta fall
Man kan också säga istället för Big-O notation en rest $R_{n+1}(x) = x^{n+1}H(x)$ där $H(x)$ begränsad i intervalet

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + O(x^{n+1})$$

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^{k-1}}{(2k-1)!} x^{2k-1} + O(x^{2n})$$

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^{k}}{(2k)!} x^{2k} + O(x^{2n+19})$$

$$\ln (1+x) = \sum_{k=0}^{n} \frac{x^k}{k} (x)^x + O(x^{n+1})$$

4.7 Integraler

Regler no.1

$$\int_a^a f(x)dx = 0$$

Regler no.2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Regler no.3 (Linjearitet)

$A, B \in \mathbb{R}$

$$\int_a^b (A \cdot f(x)dx + B \cdot g(x)) = A(\int_a^b f(x)dx) + B(\int_a^b g(x)dx)$$

Fungerar på subtraction men INTE multiplicatione eller divition!

Regler no.4

$c \in [a, b]$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Regler no.5

$f(-x) = -f(x)$ är udda

$$\int_{-a}^a f(x)dx = 0$$

Regler no.6

$f(-x) = f(x)$ är jämn

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Regler no.7

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

Regler no.8

$$f(x) \leq g(x) \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Sats: Medelvärdessatsen för integraler

$$f : [a, b] \to \mathbb{R} \text{ konternuelig } \wedge \exists c \in (a, b) \Rightarrow \int_a^b f(x)dx = f(c)(b - a)$$

Sats: Fanalysens huvudsats, Fundemetal theorem of calculus
Satsen är den vi andvänder för att lösa integraler utan geometrisk tolkning

$$f : [a, b] \to \mathbb{R} \text{ konternuelig } \Rightarrow F(x) = \int_a^x f(t)dt,\ a \leq x \leq b$$

Sats:

$$f : [a, b] \to \mathbb{R} \text{ konternuelig } \Rightarrow F \text{ är e primitiv funktion till } f(F'(x) = f(x))$$
$$\int_a^b f(x)dx = F(b) - F(a)$$
$$\text{Vi skriver } \int_a^b f(x)dx = [F(x)]_a{}^b = F(b) - F(a)$$

Exempel: arean mellan två grafer

$$\int_a^b (f(x) - g(x))dx \text{ där f är översta funnktionen och g är understa}$$
$$\int_0^1 (x - x^2)dx = [x^2/2 - x^3/3]_0{}^1 = 1/2 - 1/3 = 1/6$$

Primitiva funktioner (Obestämda integralen)

$\int f dx$	F
$\int 1dx$	$x + c$
$\int n^n dx$	$x^{n+1}/(n + 1) + c$
$\int 1/xdx$	$\ln \lvert x \rvert + c$
$\int \sin xdx$	$-\cos x + c$
$\int \cos xdx$	$\sin x + c$
$\int e^x dx$	$e^x + c$
$\int 1/\sqrt{1 - x^2}dx$	$\arcsin x + c$
$\int 1/(x^2 + 1)dx$	$\arctan x + c$
$\int 1/\cos^2 xdx$	$\tan x + c$

Exempel:

$$\int_a^b e^x dx = [e^x]_a{}^b = e^b - e^a$$
$$\int_e^x dx = e^x + c,\ c \in \mathbb{R}$$

Def: medelvärdet

$$Avg f = \frac{1}{b - a} \int_a^b f dx$$

4.7.1 variabelsubstition

$$\int f'(g(x))g'(x)dx = f(g(x)) + c$$
$$\int_e^x dx = e^x + c,\ c \in \mathbb{R}$$

Exempel:

$$\int e^{x2} 2x dx$$
$$\text{Låt } u = x^2 \Rightarrow du = 2xdx \Rightarrow \int e^{x2} 2xdx = \int e^u du =$$
$$= e^u + c = e^{x2} + c,\ c \in \mathbb{R}$$

Integraller som följer mönster

Integralles som inehåller $\sqrt{x^2 + a^2},\ a > 0$

$x = a \tan u \Rightarrow u = \arctan x,\ -\pi/2 < u < \pi/2$

$$\Rightarrow \sqrt{x^2 + a^2} = \sqrt{a^2 + a^2 \tan^2 u} = \sqrt{a^2(1 + \tan^2 u)} = a\sqrt{\frac{\sin^2 u + \cos^2 u}{\cos^2 u}} = a\frac{1}{\cos u}$$

Exempel: $\displaystyle\int \frac{1}{(1 + 25x^2)^{3/2}} dx$

$$\int \frac{dx}{\sqrt{1 + 25x^2}^3} = \int \frac{dx}{\sqrt{25(1/25 + x^2)}^3} = \frac{1}{5^3} \int \frac{dx}{\sqrt{1/205x^2}^3}$$

Låt $x = 1/5 \tan u \Leftrightarrow u = \arctan 5x$

$$\Rightarrow dx = \frac{1}{5\cos^2 u} du$$
$$\Rightarrow \frac{1}{5^3} \int \frac{dx}{\sqrt{1/205x^2}^3} = \frac{1}{5^3} \int \frac{\cos^3 u}{1/5^3} \frac{1}{5\cos^2 u} du = \frac{1}{5} \int \cos u du =$$
$$= \frac{1}{5} \sin u + c = \frac{1}{5} \sin \arctan 5x + c \text{ Kan nu förenkla med trigo regler}$$

och få: $\displaystyle\frac{x}{\sqrt{1 + 25x^2}} + c$

Integralles som inehåller $\sqrt{x^2 - a^2},\ a > 0$

$$x = \frac{a}{\cos u} \Rightarrow u = \arccos \frac{a}{x}$$

$$\Rightarrow \sqrt{x^2 - a^2} = \sqrt{\frac{a^2}{\cos^2 u} - a^2} = \sqrt{\frac{a^2 - a^2\cos^2 u}{\cos^2 u}} = \sqrt{\frac{a^2 \sin^2 u - a^2 \cos^2 u}{a^2 \cos^2 u}} = a|\tan u|$$

Integralles som inehåller $\sqrt{ax+b}$
$ax+b=u^2$

Exempel: $\displaystyle\int \frac{dx}{2+\sqrt{x}}$
Låt: $u^2 = x \Rightarrow 2udu = dx$
$$\Rightarrow \int \frac{2udu}{2+u} = 2\int \frac{u+2-2}{2+u}du = 2\left(\int du - 2\int \frac{du}{2+u}\right) = 2u - 4(\ln 2+u) + c$$

4.7.2 Integration av rationella funktioner

Om det står beräkna generaliserade integrallen så ska man beräkna med detta

Rationella funktioner: $R(x) = \dfrac{P(x)}{Q(x)}$, P, Q är polynom

Algoritm

1. Polynom divition om grad av täljare är stören än grad av nämnare
2. Faktorisera nämnaren i realla faktorer
3. Partiella bråk och sedan integrera dem, (dela up i mindre lösbara integraller)

Partiella bråk
När nämnaren har en förstagradsfaktor med potens $(x-\alpha)^n$ så ansätts termerna.
$$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_n}{(x-\alpha)^n}$$
Andragradsfaktor $(x^2+ax+b)^m$ get upphov till ansatsen
$$\frac{B_1x+C_1}{x^2+ax+b} + \frac{B_2x+C_2}{(x^2+ax+b)^2} + \ldots + \frac{B_mx+C_m}{(x^2+ax+b)^m}$$

$x-a$	$\frac{A}{x-a}$
$(x-a)^k$	$\frac{A_1}{x-a} + \frac{A_2}{x-a^2} + \ldots + \frac{A_k}{x-a^k}$
$x^2+bx+c,\ (b^2-4c<0)$	$\frac{Ax+B}{x^2+bx+c}$
$x^2+bx+c,\ (b^2-4c<0)$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \ldots + \frac{A_1x+B_1}{(x^2+bx+c)^k}$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$$
$$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln|x^2+a^2| + c$$
$$\int \frac{x}{x^2-a^2}dx = \frac{1}{2}\ln|x^2-a^2| + c$$
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^-1\frac{x}{a} + c$$
$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$$

$$\int \frac{x^3+2}{x^2-5x+4}dx$$

1. Polynomdivition

$P(x) = x^3 + 2,\ Q(x) = x^2 - 5x + 4$

Med polynom divition så får vi kvot $x + 5$ och rest $21x - 18$

$x^3 + 2 = (x + 5)(x^2 - 5x + 4) + 21x - 18$

2. Faktorisera nämnaren i realla faktorer

$$\Rightarrow \int \frac{x^3+2}{x^2-5x+4}dx = \int \frac{(x+5)(x^2-5x+4)+21x-18}{x^2-5x+4}dx \int (x+5)dx + \int \frac{21x-18}{x^2-5x+4}$$

3. Integrerar partiella bråk

$$\frac{21x-18}{x^2-5x+4} = \frac{21x-18}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} \Rightarrow \frac{21x-18}{(x-4)(x-1)} = \frac{Ax-A+Bx-4B}{(x-4)(x-1)}$$

$\Rightarrow A + B = 21 \wedge -A - 4B = -18 \Rightarrow A = 22 \wedge B = -1$

$$\int (x+5)dx + \int \frac{22}{x-4}dx - \int \frac{1}{x-1}dx$$

Svar: $x^2/2 + 5x + 22\ln|x-4| - \ln|x-1| + c,\ c \in \mathbb{R}$

4.7.3 Partiell integration

Om det är integraler med två funktioner så hjälper oftast partiell integration

Formel

$$\int_a^b f'g dx = [fg]_a^b - \int_a^b fg' dx$$
$$\int f'g dx = fg - \int fg' dx$$

Bevis

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \Rightarrow$$
$$\int_a^b (fg)' dx = \int_a^b f'g dx + \int_a^b fg' dx \Rightarrow \int_a^b f'g dx = \int_a^b (fg)' dx - \int_a^b fg' dx$$
$$\int_a^b f'g dx = [fg]_a^b - \int_a^b fg' dx$$

Exempel: polynom * trignomotrik

$$\int_0^{\pi/2} x \sin x dx$$
$$\int_0^{\pi/2} (-\cos x)' x dx = [-x \cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x (x)' dx =$$
$$[-x \cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2} = 1$$

Exempel: polynom * exponensiel

$$\int e^x x^2 dx$$
$$\int (e^x)' x^2 dx = [e^x x^2] - \int e^x 2x dx = e^x x^2 - 2 \int (e^x)' x dx = e^x x^2 - 2e^x x + 2 \int (e^x)' dx = e^x x^2 - 2e^x x + 2e^x + c$$

Exempel: exponensiel * trignomotrik

$$\int e^x \cos x dx$$
$$\text{Låt } I = \int e^x \cos x dx = \int (e^x)' \cos x = e^x \cos x - \int e^x \sin x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx \Rightarrow$$
$$I = e^x \cos x + e^x \sin x - I \Rightarrow I = \frac{e^x \cos x + e^x \sin x}{2} + c$$

Exempel: bonus ln

$$\int_1^2 \ln x dx$$
$$\int_1^2 x' \ln x dx = \ldots = 2 \ln 2 - 1$$

4.7.4 Generaliserande integraler

Def:

Antag att f är kontinuerlig i a, b och att $\lim_{x \to a^+} f(x) = \infty$

Vi definerar den generaliserade integralen $\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+c} f(x)dx, , , , konstingst$

Om gränsvärdet existerar och är ändlig säger vi att integralen är konvergentn, anart är den divergent

Betekning

$$\varepsilon \text{ Andvänds för små tal } \lim_{\varepsilon \to 0^+}$$
$$M \text{ Andvänds för stora tal } \lim_{M \to +\infty}$$
$$c \text{ Andvänds för konstanter } + c$$

Sats:

$$\int_a^{+\infty} \frac{dp}{x^p},\ a > 0$$
$$p > 1 \Rightarrow \text{ Konvergerar}$$
$$p \leq 1 \Rightarrow \text{ Divergerar}$$

Sats: Jämförelsesatsen

Anta att f och g är konternuerliga och $0 \leq f(x) \leq g(x), (a \in [-\infty, +\infty), b \in (-\infty, \infty])$

(I) om integralen är konvergent $\int_a^b g(x)dx \Rightarrow \int_a^b f(x)dx$ är också konvergent

(II) om integralen är divergent $\int_a^b f(x)dx \Rightarrow \int_a^b g(x)dx$ är också divergent

Sats:

Om $f(x)$ är positiv konternuerlig och avtagande i intervallet $x \geq \mathbb{N}$,

så är serien $\sum_{n=N}^{\infty} f(n)$ konvergent

precis när $\int_N^{\infty} f(x)dx$ är konvergent

Exempel:

Betämm om serien konvergerar eller divergerar $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$

$f(x) = \dfrac{1}{x \ln x (\ln(\ln x))^2}$

är positiv, konternuerlig och avtagande då nämnaren är stängt positivt ökande för $x \geq 10$

$\displaystyle\int_{10}^{+\infty} f(x)dx = \lim_{M\to+\infty} \int_{10}^{M} f(x)dx,\ (u = \ln\ln x \Rightarrow du = 1/\ln x \cdot 1/x dx) \Rightarrow \lim_{M\to+\infty} \int_{\ln\ln 10}^{\ln\ln M} \frac{1}{u^2}$

$\displaystyle\lim_{M\to+\infty} [-1/u]_{\ln\ln 10}^{\ln\ln M} = \lim_{M\to+\infty} (-1/\ln\ln M + 1/\ln\ln M) = 1/\ln\ln M$

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$ konverger

Sats:

Antag att $a_n > 0,\ b_n > 0$ för varje $n \in \mathbf{N}$ och antag att $\dfrac{a_n}{b_n} \to L \neq 0\ \wedge L < \pm\infty$

Då gäller att serien $\displaystyle\int_{n=1}^{\infty} a_n$ och $\displaystyle\int_{n=1}^{\infty} b_n$ är båda divergenta

4.7.5 Volymberäkningar

$$V = \int_a^b A(x)dx \text{ Rotationsvolymer runt x-axeln} \Leftrightarrow \pi \int_a^b ((g(x))^2 - (f(x))^2) \text{ g är övre f är undre}$$
$$V = 2\pi \int_a^b xf(x)dx \text{ Rotationsvolymer runt y-axeln} \Leftrightarrow 2\pi \int_a^b x(g(x) - f(x))dx \text{ g är övre f är undre}$$

exempel

Beräkna volymen av den kropp som uppstår när området som begränsas av kurnvan $y = 4x - x^2 - 3$ och x-axeln roteras kring y-axeln

$$y = 0 \Leftrightarrow 4x - 3 - x^2 = 0 \Leftrightarrow x = 1 \vee x = 3$$
$$\wedge\, y > 0,\ x \in [1, 3]$$
$$\Rightarrow V = 2n \int_1^3 x(4x - 3 - x^2)dx = 2n \int_1^3 (4x^2 - 3x - x3)dx$$
$$= 2n[4/3x^3 - 3/2x^2 - x^4/4]_1^3 = 16\pi/3$$

Kurvlängd

$$L = ||\vec{x_0} - \vec{x_0}|| = \sqrt{(a-c)^2 + (b-d)^2}$$
$$\Rightarrow L = \int_a^b \sqrt{1 + (f'(x))^2}$$

exempel

Find the lenght of curve $y = x^3/12 + 1/x$ from $x = 1$ to $x = 4$

$$f(x) = x^3/12 + 1/x$$
$$L = \int_1^4 \sqrt{1 + (f'(x))^2}dx = \int_1^4 \sqrt{1 + (x^2/4 - 1/x^2)^2}dx$$
$$= \int_1^4 \sqrt{1 + x^4/16 - 1/2 + 1/x^4}dx = \int_1^4 \sqrt{x^4/16 + 1/x^4 + 1/2}dx$$
$$= \int_1^4 \sqrt{(x^2/4 + 1/x^2)^2}dx = \int_1^4 (x^2/4 + 1/x^2)dx$$
$$= [x^2/4 + 1/x^2]_1^4 = 6$$

Rotationsarea

$$A = \int_a^b 2\pi|f(x)|\sqrt{1+(f'(x))^2}dx$$

exempel: Gabriel's Horn/Torricell's trumpet

Bestäm volymen och arean av den kropp som uppstår när området som begränsas av kurvan $y = 1/x, x \geq 1$ och x-axeln roteras kring x-axeln

$$V = \pi \lim_{M\to+\infty} \int_1^M \left(\frac{1}{x}\right)^2 dx = \pi \lim_{M\to+\infty} [-1/x]_1^M = \pi \cdot 1 = \pi$$

$$A = \lim_{M\to+\infty} \int_1^M 2\pi|1/x|\sqrt{1+(-1/x^2)^2}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{1+1/x^4}}{x}dx$$
$$= 2\pi \lim_{M\to+\infty} \int_1^M \frac{x^2\sqrt{1+1/x^4}}{x^3}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{x^4+1}}{x^3}dx$$
$$> 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{x^4}}{x^3}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{x^2}{x^3}dx$$
$$= 2\pi \lim_{M\to+\infty} \int_1^M \frac{1}{x}dx \text{ Diveigerar enlight p-satsen till } \infty$$

Eftersom integrallen har en undre begränsing som divergerar divegerar också integrallen till $+\infty$

4.8 Differential ekvationer

Tangent plan (Slope field): Andvänds för att se hur kurvan ser ut utefrån olika start värden.

grad: högsta graden på derivatan $ty'''(t) - 4y'(t) + 5t^2y(t) = e^t$ har grad 3

Linjär diffirential ekvation: diff funktionerna har ingen uphöjning
$y'' - 4t^2t' + e^ty = 0$ är linjär
$t^2y''' + 5ty' - 4y^2 = 5$ är inte linjär

Homogen: om $h(t) = 0 \Rightarrow$ ODE är homogen
Inhomogen: om $h(t) \neq 0 \Rightarrow$ ODE är inhomogen
$y''' - \sin^2(t)y' = 5y$ är homogen
$e^{t^2}y^{(5)} - y'' + 4ty(t) = e^t + t^3$ är inhomogen

Sats: superposition princple

Låt $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$
Om $y_1(x), y_2(x)$ är lösningar till differantials ekvanationen
$\Rightarrow Ay_1(x) + By_2(x),\ A, b \in \mathbb{R}$ är en lösning till differantials ekvanationen

4.8.1 superabla ekvationer

Def: superabla ekvationer

En differentialekvation är seperabel om den kan skrivas på formeln

$$\frac{dy}{dx} = f(x)g(y)$$

Exempel: Vilka är seperabel

(I) $y' = x + y$ är inte seperabel

(II) $\frac{dy}{dx} = 1 + e^y$ är separabel

Exempel: beräkning

Lös $y'(x) = (1 + e^{-x})(y^2 - 1)$

$$y = \pm 1$$

$$\frac{dy}{y^2 - 1} = (1 + e^-1)dx \Rightarrow \int \frac{dy}{y^2 - 1} = \int (1 + e^{-1})dx(*)$$

$$\int \frac{1}{(y+1)(y-1)}dy = \int \frac{((y+1) - (y-1))}{2(y+1)(y-1)}dy = \int \frac{1}{2(y-1)}dy - \int \frac{1}{2(y+1)}dy$$

$$(*) \Rightarrow \int \frac{1}{2(y-1)}dy - \int \frac{1}{2(y+1)}dy = \int (1 + e^{-1})dx$$

$$\Rightarrow 1/2 \ln|y-1| - 1/2 \ln|y+1| = x - e^{-x} + c \Rightarrow \ln|\frac{y-1}{y+1}| = 2x - 2e^{-x} + 2c$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}|\frac{y-1}{y+1}|} = e^{2x - 2e^{-x} + 2c}$$

$$\Rightarrow y = \frac{1 + e^{2x - 2e^{-x}} + 2c}{1 - e^{2x - 2e^{-x}} + 2c} \vee y = \frac{1 - e^{2x - 2e^{-x}} + 2c}{1 + e^{2x - 2e^{-x}} + 2c} \vee y = \pm 1$$

4.8.2 Linjära differentialekvationer av ordning 1

Methodology

$y' + p(x)y = q(x)$

Om $g(x) = 0 \Rightarrow$ homogen och därmed seperable

Om $g(x) \equiv 0$ Multiplisera med $e^{M(x)}$ För att kuna andvända produktregeln så vi kan slå ihop y, y'

$M(x) = \int p(x)dx$ är en primitiv dunktion till $p(x) \Rightarrow c = 0$

$\Rightarrow e^{M(x)}y' + e^{M(x)}p(x)y(x) = e^{M(x)}q(x)$ Antifunktionen slår ut $p(x)$

$\Rightarrow e^{M(x)}y' + (e^{M(x)})'y(x) = e^{M(x)}q(x)$ VL kan vi andvända produktregeln bakvänt

$\Rightarrow (e^{M(x)}y(x))' = e^{M(x)}q(x)$ Integrerar båda led

$\Rightarrow \int (e^{M(x)}y(x))'dx = \int (e^{M(x)}q(x))dx$ Antiderivatan slår ut derivatan

$\Rightarrow e^{M(x)}y(x) = \int (e^{M(x)}q(x))dx$ Får y ensamt

$\Rightarrow y(x) = e^{-M(x)} \int (e^{M(x)}q(x))dx$

Exempel

Lös $(1+t^2)y' + ty = \dfrac{1}{\sqrt{1+t^2}}$

$y' + \dfrac{t}{(1+t^2)y} = \dfrac{1}{(1+t^2)\sqrt{1+t^2}}$ Linjär ode, ordning 1

$p(t) = \dfrac{t}{(1+t^2)y},\ q(t) = \dfrac{1}{(1+t^2)\sqrt{1+t^2}}$

$M(t) = \int \dfrac{t}{(1+t^2)y}dt,$ Låt $u = 1+t^2 \Rightarrow dt = du/2t$

$\Rightarrow M(t) = \int \dfrac{1}{2u}du = \dfrac{1}{2}\ln|u| = \dfrac{1}{2}\ln t^2 + 1$

$e^{M(t)} = e^{\frac{1}{2}\ln t^2 + 1} = \sqrt{t^2+1}$

$\Rightarrow \sqrt{t^2+1}y' + \dfrac{t\sqrt{t^2+1}}{t^2+1}y = \dfrac{1}{1+t^2}$

$\Rightarrow \sqrt{t^2+1}y' + \dfrac{t}{\sqrt{t^2+1}}y = \dfrac{1}{1+t^2}$

$\Rightarrow (\sqrt{t^2+1}y)' = \dfrac{1}{1+t^2} \Rightarrow \sqrt{t^2+1}y = \int \dfrac{1}{1+t^2}$

$\Rightarrow \sqrt{t^2+1}y = \arctan(t) + c$

$\Rightarrow y = \dfrac{\arctan(t)}{\sqrt{t^2+1}} + \dfrac{c}{\sqrt{t^2+1}},\ c \in \mathbb{R}$

4.8.3 Linjära differentialekvationer av ordning 2

Methodology

Det finns 3 metoder för att lösa Linjära differentialekvationer av ordning 2
$ay'' + by' + cy = 0$ vilket är den generella formeln. Antag att lösningen är på formen
$y = e^{rx},\ y' = re^{rx},\ y'' = r^2e^{rx} \Rightarrow ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$
$(ar^2 + br + c)e^{rx} = 0 \Rightarrow ar^2 + br + c = 0$ andvänder pq-formeln och får följande
$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i)Om $b^2 - 4ac > 0 \Rightarrow r_1, r_2 \in \mathbb{R} \wedge r_1 \neq r_2 \Rightarrow y_1 = e^{r_1x},\ y_2 = e^{r_2x}$
är lösningen till $ay'' + by' + cy = 0 \Rightarrow y_k = c_1e^{r_1x} + c_2e^{r_2x},\ c_1, c_2 \in \mathbb{R}$ superposition
$(ii)b^2 - 4ac = 0 \Rightarrow r_1 = r_2 \in \mathbb{R}$ dubbelrot $y_1 = e^{r_1x},\ y_2 = xe^{r_1x}$

(iii)Om $b^2 - 4ac < 0 \Rightarrow r_1 \neq r_2 \in \mathbb{C}$
$r_1 = k + li,\ r_2 = k - li$ Koplexatals konjugat enlight euler formel, alla lösningar är
$y = c_1e^{r_1x} + c_2e^{r_2x}, c_1, c_2 \in \mathbb{C}$
$y = c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{(k+li)x} + c_2e^{(k-li)x} = c_1e^{kx}e^{lix} + c_2e^{kx}e^{-lix}$
$c_1e^{kx}(\cos lx + i\sin lx) + c_2e^{kx}(\cos -lx + i\sin -lx)$
$c_1e^{kx}(\cos lx + i\sin lx) + c_2e^{kx}(\cos lx - i\sin lx) = e^{kx}((c_1 + c_2)\cos lx + (c_1 - c_2)i\sin lx)$
$e^{kx}(\vec{c_1}\cos lx + \vec{c_2}\sin lx)$
$y = e^{kx}(c_1\cos lx + c_2\sin lx),\ c_1, c_2 \in \mathbb{R}$
Vi för ett svar som är realt, men vi andvänder genväg med komplexa tal

Exempel Positiv kvot

Lös $y''(x) - 4y'(x) + 3y(x) = 0$
$r^2(x) - 4r(x) + 3 = 0 \Rightarrow r_1 = 1,\ r_2 = 3 \Rightarrow y(x) = c_1e^x + c_2e^{3x},\ c_1, c_2 \in \mathbb{R}$

Exempel noll kvot

Lös $y''(t) - 4y'(t) + 4$
$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ dubbel rot
$y(t) = c_1e^{2t} + c_2te^{2t},\ c_1, c_2 \in \mathbb{R}$

Exempel koplex kvot

Lös $y''(t) + 25y(t) = 0$
$r^2 + 25 = 0 \Rightarrow r^2 = -25 \Rightarrow r = \pm 5i$
$y(t) = c_1\cos(5t) + c_2\sin 5t,\ c_1, c_2 \in \mathbb{R}$

Partikulär lösning

Methodology

Partikulär lösning är för att gissa lösningen till det ike homogena lösningarna
$ay'' + by' + cy = h(t),\ y(t) = y_h(t) + y_p(t)$

If $f(x) = P_n(x),$ try $y_p = x^m A_n(x)$
If $f(x) = P_n(x)e^{rx},$ try $y_p = x^m A_n(x)e^{rx}$
If $f(x) = P_n(x)e^{rx}\cos(kx),$ try $y_p = x^m e^{rx}(A_n(x)\cos(kx) + B_n(x)\sin(kx))$
If $f(x) = P_n(x)e^{rx}\sin(kx),$ try $y_p = x^m e^{rx}(A_n(x)\cos(kx) + B_n(x)\sin(kx))$

För att ta reda på t eller k så kan tänka att den homogena lösningen är ej en lösning
så börja med k=0 sen gå up tills det är sant

Exempel trigonometrisk

Lös $y'' + 9y = \sin 3x$
Sats: diff ekv kan skrivas på följande formel $y = y_h + y_p$

Homogena lösningen
$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow y_h = A\sin 3x + B\cos 3x$
Particulöra lösningen
Sats $y_p = x^m(A_1 \sin 3x + A_2 \cos 3x), m = 1$
är godtaglig då det är första ekv som inte kan skrivas på homogen förmel
$y_p = A_1 x \sin 3x + A_2 x \cos 3x$
$y_p' = A_1(\sin 3x + 3x\cos 3x) + A_2(\cos 3x - 3x\sin 3x)$
$y_p'' = A_1(3\cos 3x + 3\cos 3x - 9x\sin 3x) + A_2(-3\sin 3x - 3\sin 3x - 9x\cos 3x)$
$\Rightarrow y_p'' = A_1(6\cos 3x - 9x\sin 3x) + A_2(-6\sin 3x - 9x\cos 3x) + 9(A_1 x\sin 3x + A_2 x\cos 3x) = \sin 3x$
$6A_1\cos 3x - 6A_2\sin 3x = \sin 3x \Rightarrow A_1 = 0, A_2 = -1/6$
$\Rightarrow y_p = -x/6\cos 3x$
Den almäna lösningen blir på följande
$y(x) = A\sin 3x + B\cos 3x - x/6\cos 3x$

Exempel polynom och Exponensiel

Lös $y'' + y' - 2y = 4e^2x + x^2$

Sats: diff ekv kan skrivas på följande formel $y = y_h + y_p$

Homogena lösningen

$r^2 + r - 2 = 0 \Rightarrow r_1 = 1,\ r_2 = -2$

$y_h = c_1 e^x + c_2 e^{-2x},\ c_1, c_2 \in \mathbb{R}$

Particulöra lösningen

Sats $y_p = y_{p1} + y_{p2}$

$y'' + y' - 2y = 4e^{2x}$

$y_{p1} = x^m(Ae^{2x}), m = 0 \Rightarrow y'_{p1} = 2Ae^{2x} \Rightarrow y''_{p1} = 4Ae^{2x}$

$\Rightarrow 4Ae^2x + 2Ae^2x - 2Ae^2x = 4e^2x \Rightarrow A = 1 \Rightarrow y_{p1} = e^{2x}$

$y_{p2} = Ax^2 + Bx + C \Rightarrow y'_{p2} = 2Ax + B \Rightarrow y''_{p2}2A$

$\Rightarrow (2A) + (2Ax + B) - (2(Ax^2 + Bx + C)) = x^2$

$\Rightarrow -2A = 1,\ 2A - 2B = 0,\ 2A + B - 2C = 0 \Rightarrow A = -1/2,\ B = -1/2,\ C = -3/4$

$\Rightarrow y_{p2} = -1/2x^2 - 1/2x - 3/4$

$\Rightarrow y(x) = c_1 e^x + c_2 e^{-2x} + e^{2x} - 1/2x^2 - 1/2x - 3/4$

Resonans

Exempel Resonans

$$
\begin{aligned}
&\text{Lös} y'' + y = \sin 2t \\
&r^2 + 1 = 0 \Rightarrow r = \pm i \\
&y_h = c_1 \cos t + c_2 \sin t \\
&y_p = A \sin t + B \cos t \\
&y_p' = 2A \cos 2t - 2B \sin 2t \\
&y_p'' = -4A \sin 2t - 4B \cos 2t \\
&y_p'' + y_p = \sin 2t \Rightarrow -4A \sin 2t - 4B \cos 2t + A \sin t + B \cos t = \sin 2t \\
&\Rightarrow A = -1/3,\ B = 0 \Rightarrow y_p = -1/3 \sin 2t \Rightarrow y(t) = c_1 \cos t + c_2 \cos t - 1/2 \sin 2t
\end{aligned}
$$

Serielösningar

Antag att serien $\displaystyle\sum_{n=0}^{\infty} c_n (x - x_0)^n$ konvergerar för alla $n = 0$

$x \in x_0 - R, x_0 + R$ Då är funktionen $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x - x_0)^n$ deriverbar i

$(x_0 - R, x_0 + R) \wedge f'(x) \displaystyle\sum_{n=0}^{\infty} n c_n (x - x_0)^{n-1}$ Den sista serien konvergerar också i intervallet $(x_0 - R, x_0 + R)$

Chapter 5

Computer Architecture

5.1 ISA1

5.1.1 MIPS instructions

The instructions that a assembly languge use for the MIPS proses.
Terminology

- Register file: 32 registers from R0-R31, each is 32 bits.
- Memory: Large each adders stores a word (4 x Registers). Address is all ways increments of 4. Memory is segmented into 4 8-bits of data to create a word.
- PC (Project counter): Increase with 4 each time to go to the next memory address.
- Instruction Registers: Retrieve the current instructions.
- Control: Inserts data to the register file and add the operation to ALU (Compute).
- ALU (Compute): Calculates the value.
- Memory Address: Call the value from a specific address. From ex "lw".
- Data Register: Retries and directs value from memory to register file.

The operations most used

Function	Instruction	Effect
add	add R1, R2, R3	R1 = R2 + R3
sub	sub R1, R2, R3	R1 = R2 - R3
add immediate	addi R1, R2, 145	R1 = R2 + 145
multiply	mult R2, R3	hi, lo = R2 * R3
divide	div R2, R3	low = R2/R3, hi = remainder
and	and R1, R2, R3	R1 = R2 & R3
or	or R1, R2, R3	R1 = R2 | R3
and immediate	andi R1, R2, 145	R1 = R2 & 14
or immediate	ori R1, R2, 145	R1 = R2 | 145
shift left logical	sll R1, R2, 7	R1 = R2 << 7
shift right logical	srl R1, R2, 7	R1 = R2 >> 7
load word	lw R1, 145(R2)	R1 = memory[R2 + 145]
store word	sw R1, 145(R2)	memory[R2 + 145] = R1
load upper immediate	lui R1, 145	R1 = 145 << 16
branch on equal	beq R1, R2, 145	if (R1 == R2) go to PC + 4 + 145*4
branch on not equal	bne R1, R2, 145	if (R1 != R2) go to PC + 4 + 145*4
set on less than	slt R1, R2, R3	if (R2 < R3) R1 = 1, else R1 = 0
set less than immediate	slti R1, R2, 145	if (R2 < 145) R1 = 1, else R1 = 0
jump	j 145	go to 145
jump register	jr R31	go to R31
jump and link	jal 145	R31 = PC + 4; go to 145

Figure 5.1: MIPS oporation tabel. From (PH11).

note: memory operations like (sw R1, 0(R2)) stores the word in poison R2+0 where 0 is the increment since if it is a loop it is needed. The value at that position is R1.

5.1.2 Sequencing

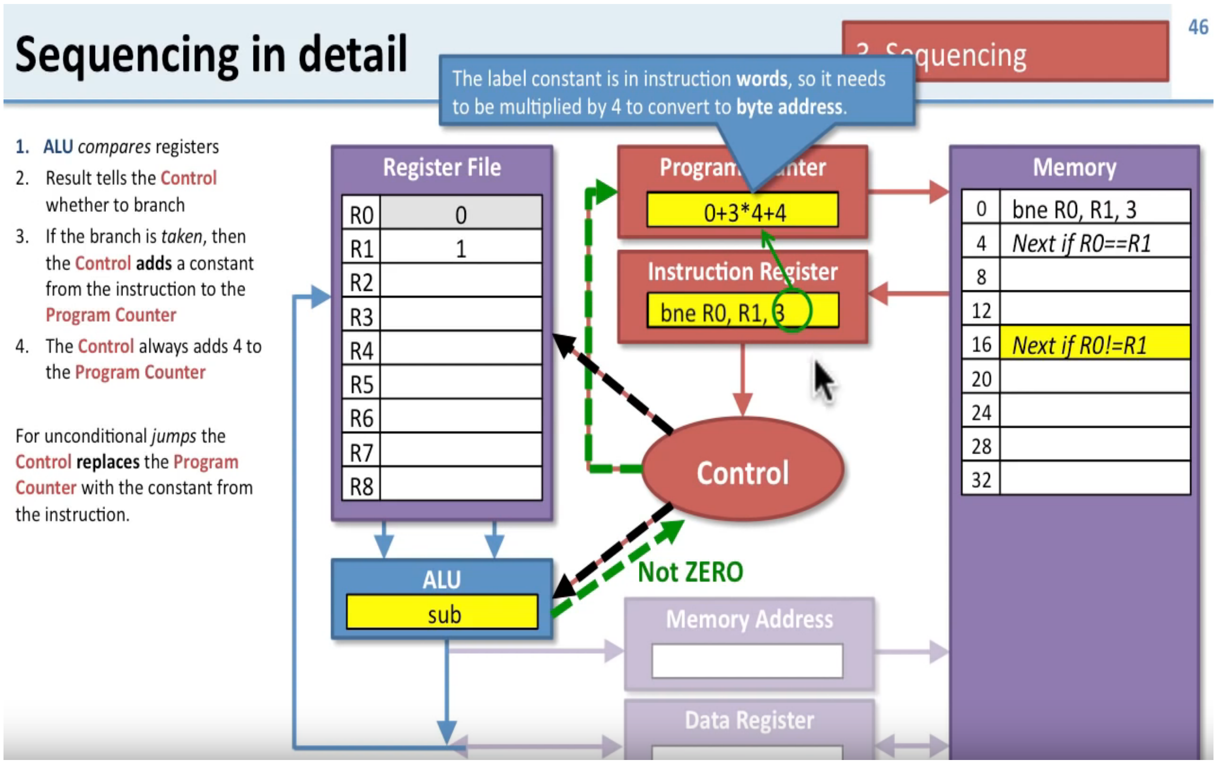


Figure 5.2: Sequencing. From (BS)

5.1.3 Converge to Assambly code

If Else example

```
if (i==j)
  h = i+j;

else
  h = i-j;
```

```
R1 = i; R2 = j; R3 = h

 bne R1, R2, DoElse
  add R3, R1, R2
  j SkipElse
DoElse:
  sub R3, R1, R2
SkipElse:
```

Figure 5.3: Assambly exampel. From ()

5.2 ISA2

5.2.1 MIPS type format

- R-format, Aretmetic and lodgical (add)
- I-format, Load/store, branch and intemidiats (addi, beq)
- J-format, Jump (j skipelse)

Name	Bit Fields						Notes
	6 bits	5 bits	5 bits	5 bits	5 bits	6 bits	(32 bits total)
R-Format	op	rs	rt	rd	shmt	funct	Arithmetic, logic
I-format	op	rs	rt	address/immediate (16)			Load/store, branch, immediate
J-format	op	target address (26)					Jump

Figure 5.4: MIPS Types. From ()

A instruction allwas ends with 00 it means that when a instruction is done we update the program couner to by 4 or more it changes the posistion of the 16bit intermidjet for i-format and can. (bne, beq)

5.2.2 Procedure

Procedure is how a function call is handeld. It is devided into two parts Caller and Callee, the caller call the procedure (callee). The oporation for contolling procedure is called jump-and-link (jal). It stores the return address (PC+4) in $ra (R31). Where (ra = register address (updates automatically)) (PC = project counter) (R31 = value). jr $ra returns the value.

Stacking

To avoid conflicts with writing over the callers register files, one uses staking in predefined ranges to allow store data independent from the callers. (R29 = store stack data $sp) When a register file is added the stock pointer is moved and when it is done it returns with (jr) and then reset the previously used register files. when it has used what it needs it removes them when leaded one value at a time

```
integer_array_sum:
        addi $sp, $sp, -4  # increase stack-pointer by one word
        sw $s0, 0($sp)     # save array index i to stack
        addi $sp, $sp, -4  # increase stack-pointer by one word
        sw $s1, 0($sp)     # save element n to stack

        addi $v0, $zero, 0     # Initialize Sum to zero.
        add $s0, $zero, $zero  # Initialize i to zero.

ia_loop:
        beq $s0, $a1, ia_done  # Done if i == N
        lw $s1, 0($a0)         # n = A[i]
        add $v0, $v0, $s1      # Sum = Sum + n
        addi $a0, $a0, 4       # address = ARRAY + 4*i
        addi $s0, $s0, 1       # i++
        j ia_loop              # next element



ia_done:
        lw $s1, 0($sp)     # loads i from stack-pointer
        lw $s0, 4($sp)     # loads n from stack-pointer
        addi $sp, $sp, 8   # restore stack-pointer

        jr   $ra           # return to caller
```

Figure 5.5: code ex

Notice that the register files for the callee save is $sx and for the caller $tx.

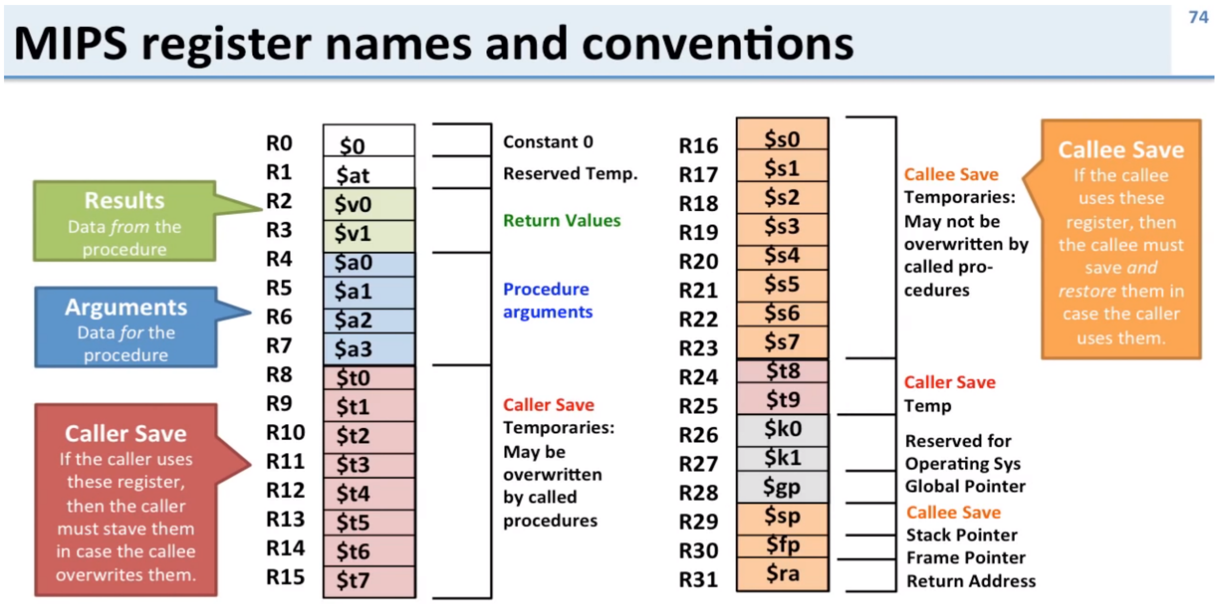


Figure 5.6: MIPS register. From ()

5.2.3 Other ISA

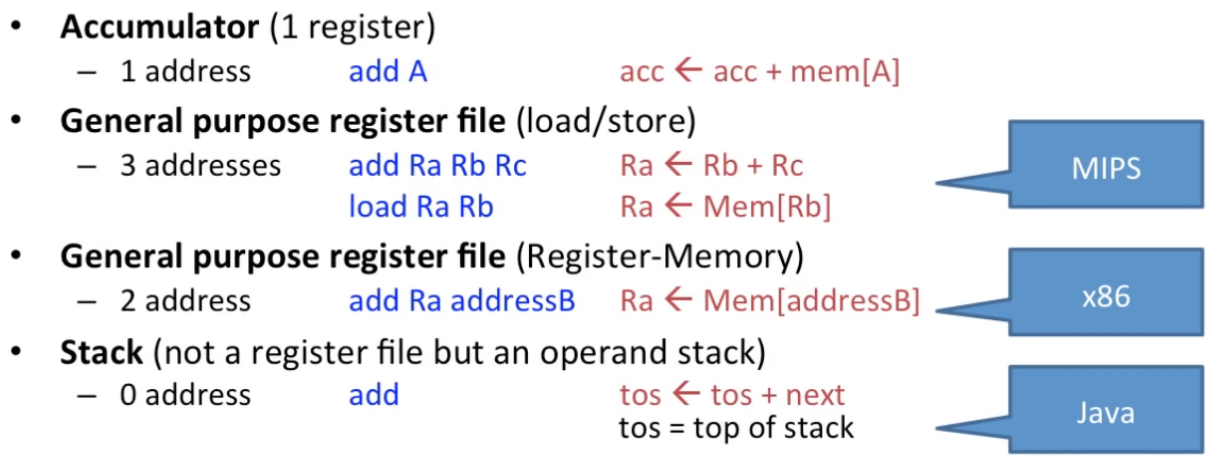


Figure 5.7: Other ISA. From ()

Stack	Accumulator	Register (Register-memory)	Register (Load-store)
Push A Push B Add Pop C	Load A Add B Store C	Load R1, A Add R1, B Store C, R1	Load R1, A Load R2, B Add R3, R2, R1 Store C, R3
JVM	PDP-8, 8008, 8051	x86	MIPS, PPC, ARM, SPARC

Figure 5.8: Other ISA instructions. From ()

5.3 Arithetic

5.3.1 Binary numbers

- Binary numbers reduses noice and disregard the exact voltitge to rusult in on off mode.
- In comuter sience octal and hexadecimals are also often used becose of it properties. every octo didgets represent 3 in decimal every hex didgets represent 4 in decimal
- msb=most significant bit
- lsb=lest significant bit
- Optemazation: $0010 * 0101$ 2 oporations $\rightarrow 0101 * 0010$ 1 oporations
- Multiplication: (fixed point need to shift desimal point) $0010*0101 = 0010+(0010 shifting[00]) = 1010$ $0011.010 * 0001.110 =$
- Addition: (Same for fixed points and integers) $1110+1000 = carry[1]0110$ Overflow $0101+0001 = 0110$ Not overflow

5.3.2 Negative integes

- Signed magnetude: msb is the sign $1011_2 = -(2+1)_{10} = -3_{10}$
- tow's complement: if msb is 1 then negative number every other digiet after is positiv $1011_2 = -8+2+1_{10} = -5_{10}$

5.3.3 Oporations

5.3.4 Non integer numbers (Floating and fixed point)

- Converting binary to decimal: $0.111 = 1/2 + 1/4 + 1/8 = 0.875$
- Fixed point: for defined ranges 1.001 and 0.100 one can chose large and small representations
- Floating point: IEEE standard, se following image
- Standard floting point is in binary so FFFF is at most 1111 and with two's complement 0111

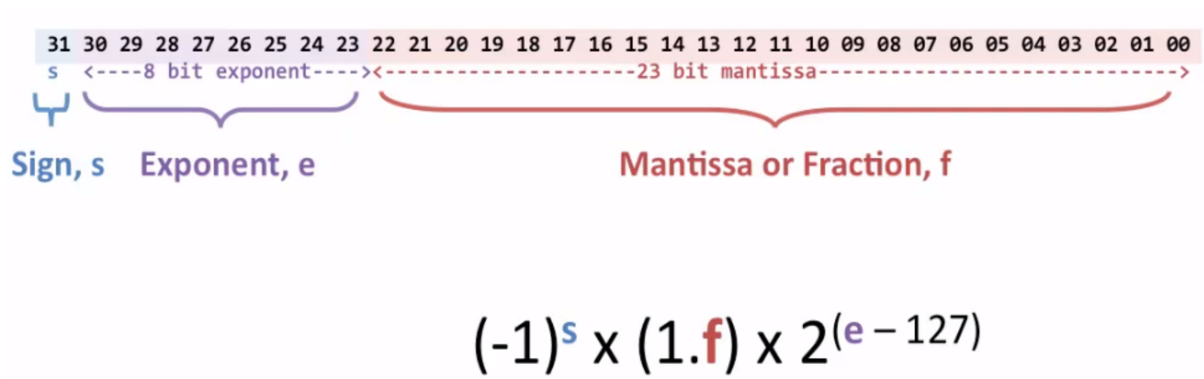
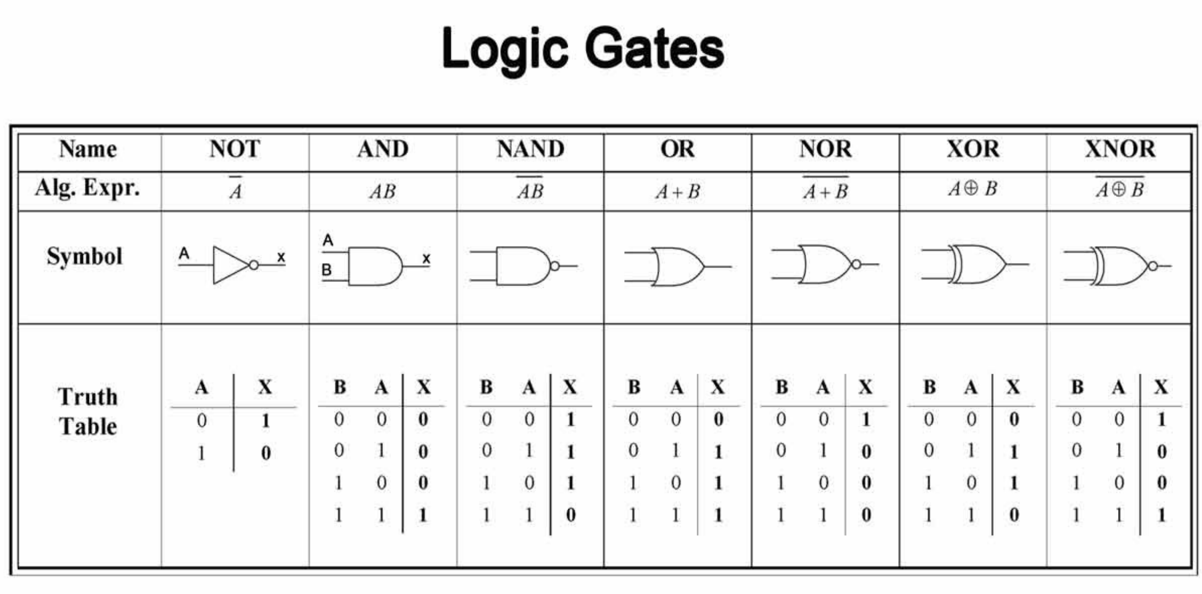


Figure 5.9: Floating Point. From ()

5.3.5 Overflow

- input: msb = 1 for boath numbers and output: msb = 0 (overflow)
- input: msb = 0 for boath numbers and output: msb = 1 (overflow)
- input: msb = 0 for boath numbers and output: msb = 0 (Not overflow)
- input: msb = 1 for boath numbers and output: msb = 1 (Not overflow)
- input: msb = 1 and msb = 0 (Maby overflow)

5.4 Logic



Logic Gates

Name	NOT	AND	NAND	OR	NOR	XOR	XNOR
Alg. Expr.	$\overline{A}$	AB	$\overline{AB}$	$A+B$	$\overline{A+B}$	$A\oplus B$	$\overline{A\oplus B}$
Symbol	A x	A B x					
Truth Table	A X 0 1 1 0	B A X 0 0 0 0 1 0 1 0 0 1 1 1	B A X 0 0 1 0 1 1 1 0 1 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 1	B A X 0 0 1 0 1 0 1 0 0 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 0	B A X 0 0 1 0 1 0 1 0 0 1 1 1

Figure 5.10: Floating Point. From ()

- truth table, is used to represent all possible inputs and then the corresponding output.
- To determined the possible schematics one can write for the input A and B “!A and B” not A and B. One often get a unnecessary complexity, one can often simplify the schematics. It is only done with the inputs that have an output of one.
- De Morgan’s Law $!(A+B) = !A \cdot !B$ + and $\cdot$ works the same way but are different.
- karno map is used to easily determine the schematics. It is a matrix. It is possible to use more inputs then two. In the matrix one locks at when output is one.
- A cabal with 4 wires can take 4 bits. Overflow needs one extra wire on output

Building blocks

- Building blocks are the first layer of abstraction since one can not se the logical gates constructed of. For example add is a building block.
- Two types of logic, combinational and sequential. combinational: output just depends on input sequential: output deepens on the state. State is stored in memory update on clock.
- MUXes (Mulitplexrs): choose input from a buss/ routing signal. 2-bit decision for input 4-bit selection. Starts from position 0.
- DEMUXes (Demulitplexrs): oposit, take on signal and decides where it want to be sent to Buss is a multi-bit signal “wire”
- Decoder: binary to hot, can choose position in array.
- Encoder: hot to binary.

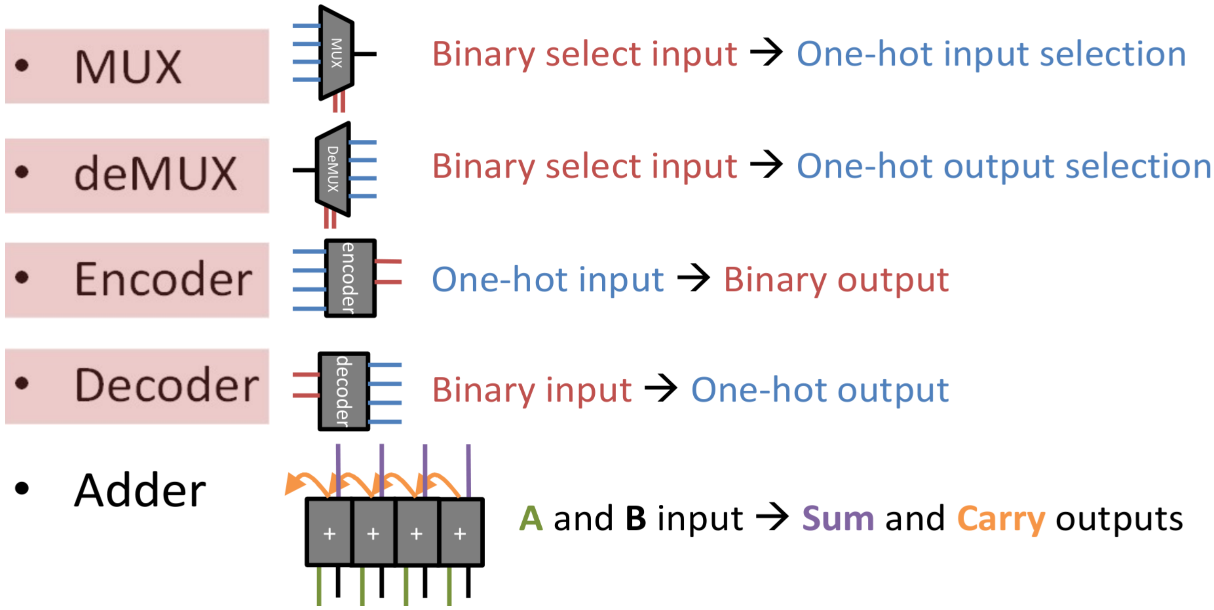


Figure 5.11: Floating Point. From ()

- Adders: adds carrys on extra bit to handel overflow

Latches

- One use is for counters with a clock statest trigers the count. So when input is one the output is 0 and when the clock clicks it is one then when the input is 0 the output is still 1.
- flip flop: is a edge trigerd latche has a clock trigger to open or close lathce and then can save it to SRAM cell

Memory

- SRAM (Static Random Access Memory) Big, fast and expensive. Loops thou value in order to store it. To write to memory one uses a switch to update value.
- DRAM (Dynamic Random Access Memory) Smaller, Slower and cheaper. Uses a transistor to store value and a capacitor to store the charge so the data dose not disappear. That is why it is slower become the capacitor needed to be recharge

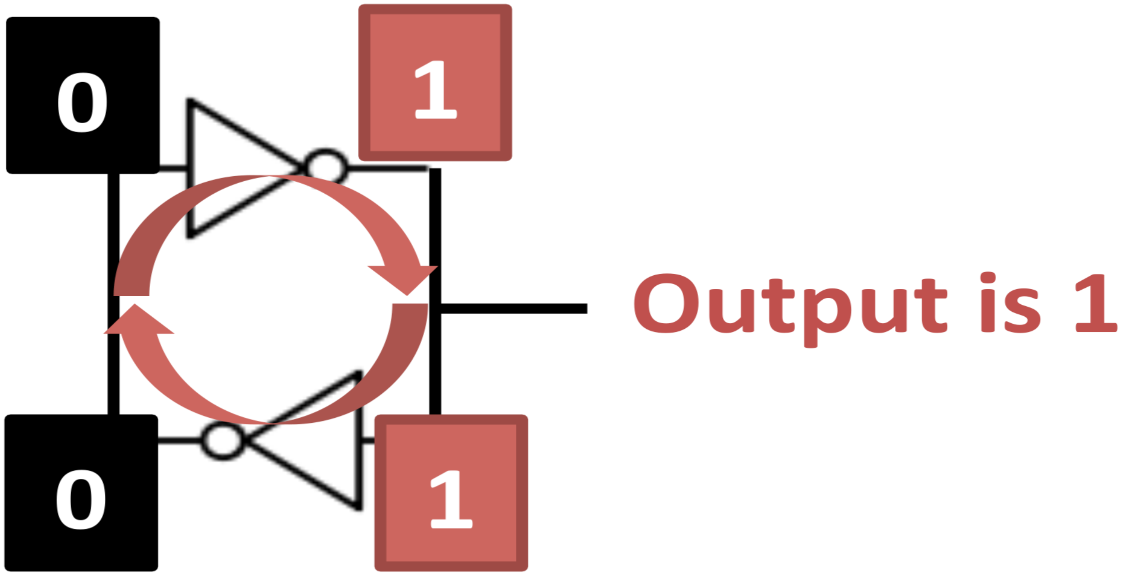


Figure 5.12: latche. From ()

5.5 processor control and datapath

There is 3 main part of the mips datapath as seen in the following image:

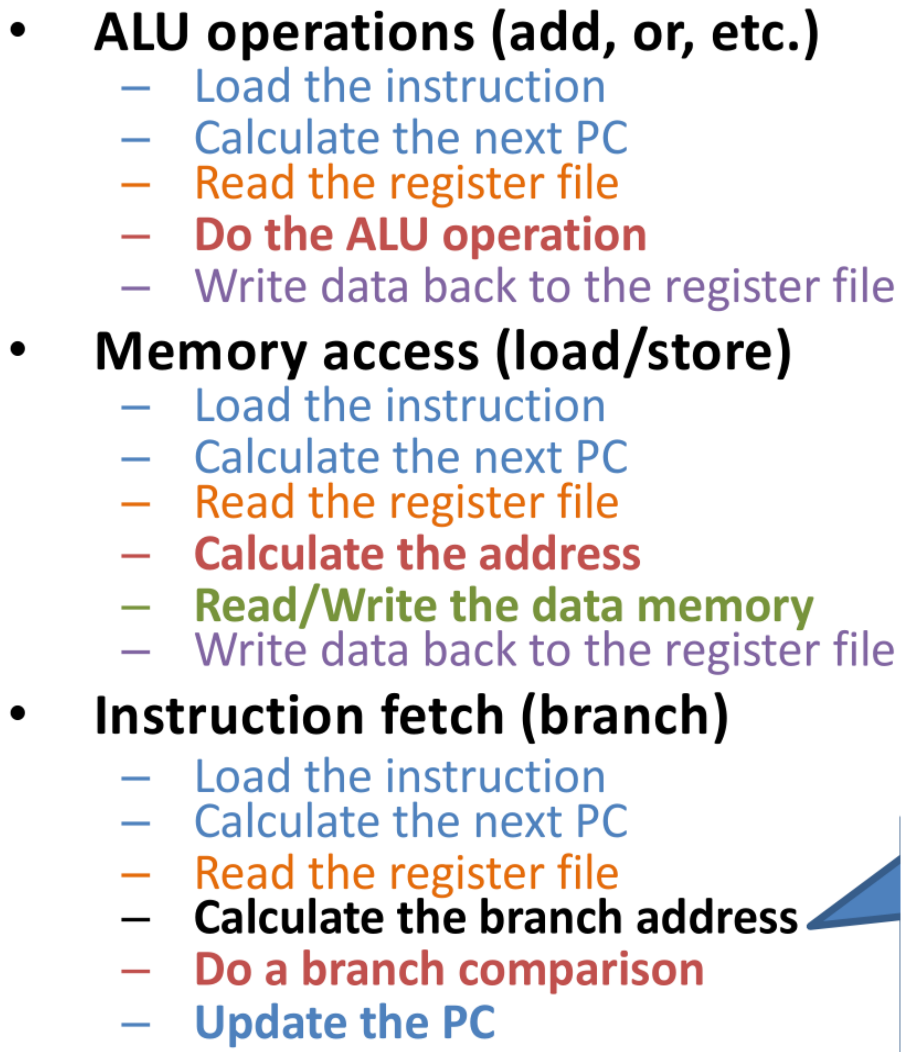


Figure 5.13: mips datapath 3 parts

An overview of the mips datapath with j-format instruction:
Think about the controller like a decoder that decodes the opt code from the different formats.

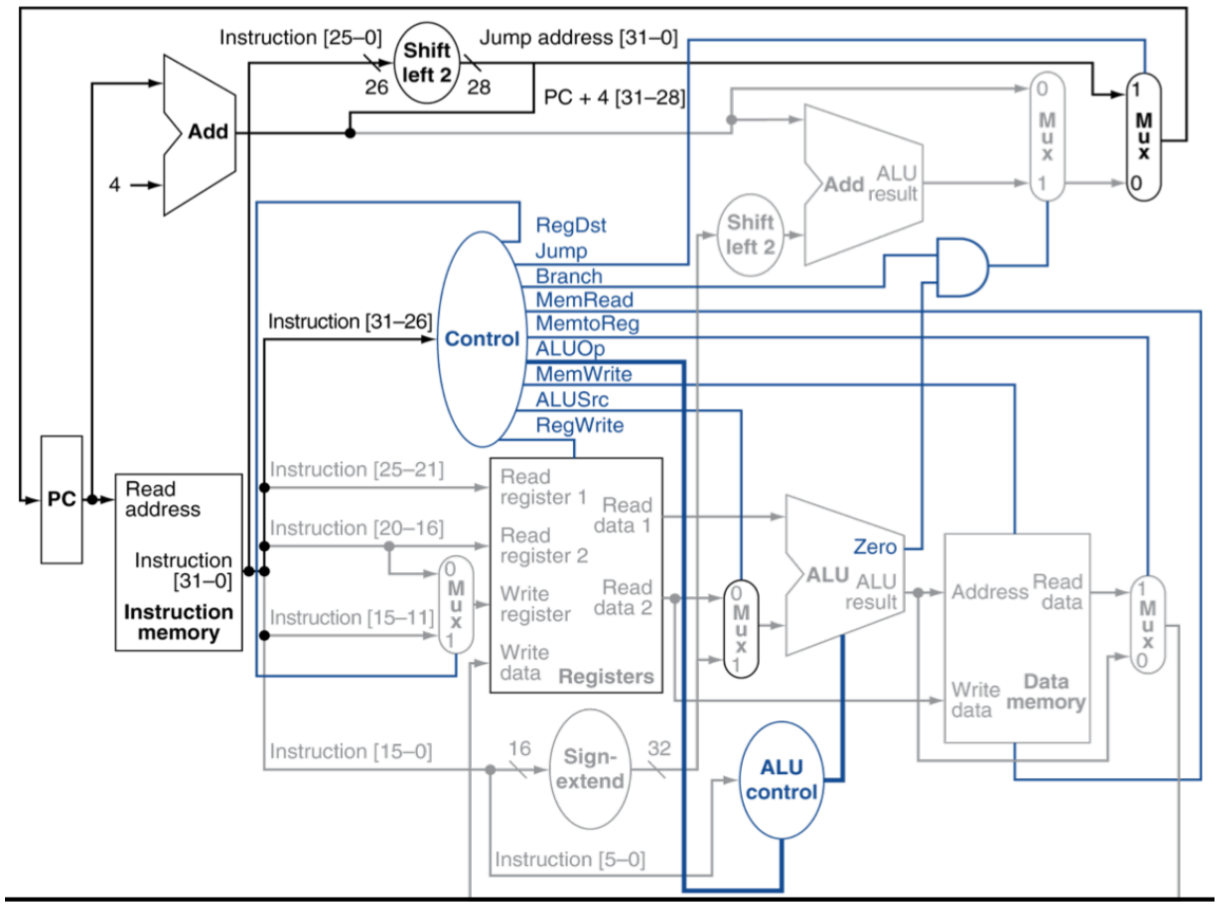


Figure 5.14: mips datapath. From ()

5.5.1 Clock

A clock is used to update the state and continue with the other instructions. Every state element uses a clock. In mips prossesor clock goes to memory, pc, rf and dm. The clock unit is in (MHz) converting from (ns) is simple. Ex 10ns: 1/10ns=0.1MHz.

5.5.2 Critical path

The longest path of the datapath is the critical path. Often PC is the fastest so one then calculates the amount of time it takes for the instruction has travel the data path and reternd to know what the critical path is. The longest part is data memory size it is big and slow. One often say read and write takes constant time regardless of the amount of different read and write.

5.6 pipline

The purpose is to split a instruction cycle into multiple stages to run instructions in peralel to improve preformans. Itch stage has its own pipline register file for storing the nessasary registers. Pipline registers has a preferment reduction since it takes time, therfore more pipeline stages dose not equal greater preformans over a certain amount. One issue of preformans is balance the stages inorder to get a good clock frequency. Not all stages are needed for eatch operation or instruction

Mips stages

- IF= Instruction fetch
- ID= Decode and RF Read
- Ex= ALU Execute
- MEM= Memory
- WB= RF Write back

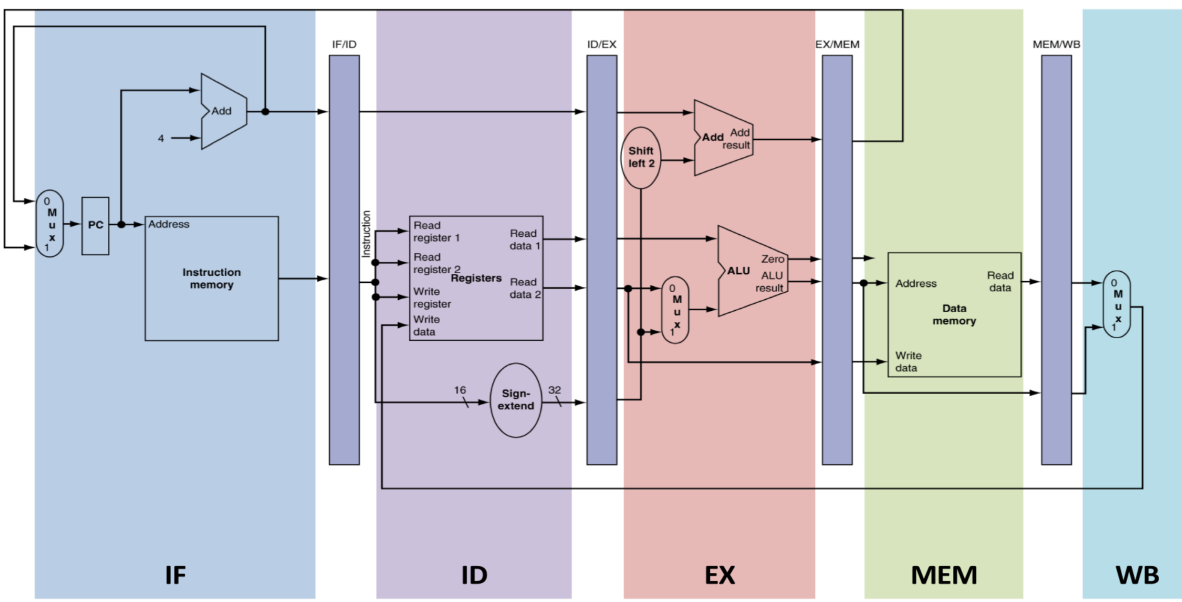


Figure 5.15: Pipline. From ()

Terminology

- Bubbles= Is detected by the hardware where there is no instuctions
- Nop= Is a sudo instruction for a stall type instruction (do not do anything)
- Delay slots= a stall type for branches. Try to fill in those slots with usful instructions.
- Interference= Can not read and write at the same time.
- Double pumping= Spiting write to first clock cycle and the second cycle is read.
- Forwarding= Getting data from a different pipeline stage from a register file.

Calculating time coplexity

- Latency: (stages*(penalty per stage))
  overhead: (instructions time (ex 100ns)) / (stages (ex 1000)) + (register overhead (1ns)) = 1.1ns Total time: 1.1*1000

5.7 Pipline hazards

5.7.1 Data hazards

The issue

- Data is not available where we need it (later in the pipeline; not written back yet).
- Data is not available when we need it (need to read memory first).

Fixing the issue

- Forward the data to where we need it.
- Stall if data is not ready yet (NOPs or Bubbles).

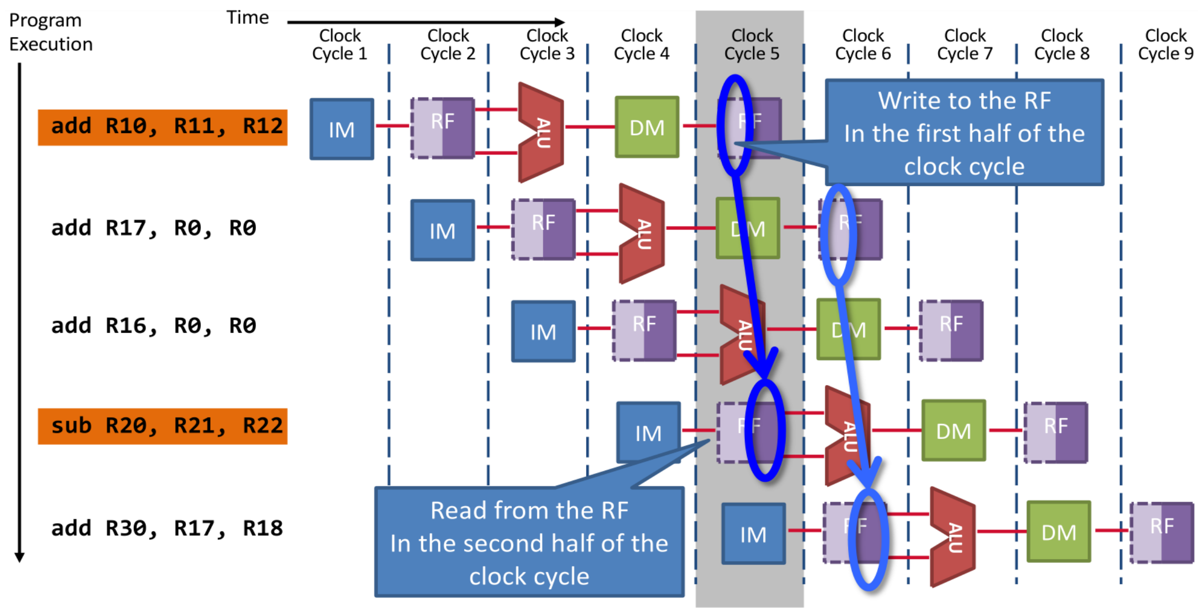


Figure 5.16: Bubble pump. From ()

5.7.2 Control hazards

The issue

- Don't know which instruction is next when we need to fetch it.

Fixing the issue

- Calculate branch as early as possible.
- Stall with a branch delay slot.

5.7.3 Structural hazards

The issue

- Can't do the instruction because the hardware is busy.

Fixing the issue

- Build more hardware (double-pumped register file). Double-pumped write at the first half cycle and read at the other half

Calculating time coplexity

- Branch delay
  (How many branches (ex 20%)) / (How many usful instuctions can be filled in (ex 50%)) = 20%/0.5 = 10%
- Instructions per cycle

5.8 Predicting Branches and Exceptions

When the prediction is wrong, clean up is needed. – "Kill" or "Squash" so they don't execute (they were wrong) – Prevent them from writing: disable RegWrite, MemWrite in the pipeline – Turn them into NOPs: change opcode to add R0, R0, R0 in the pipeline

5.8.1 Static pridictors

- Predict alwas not taken
- Predict alwas taken
- Backwards-Taken, Forward-Not-Taken (BTFNT)

5.8.2 Dynamic pridictors

The implementation of branch predictors look something like this:

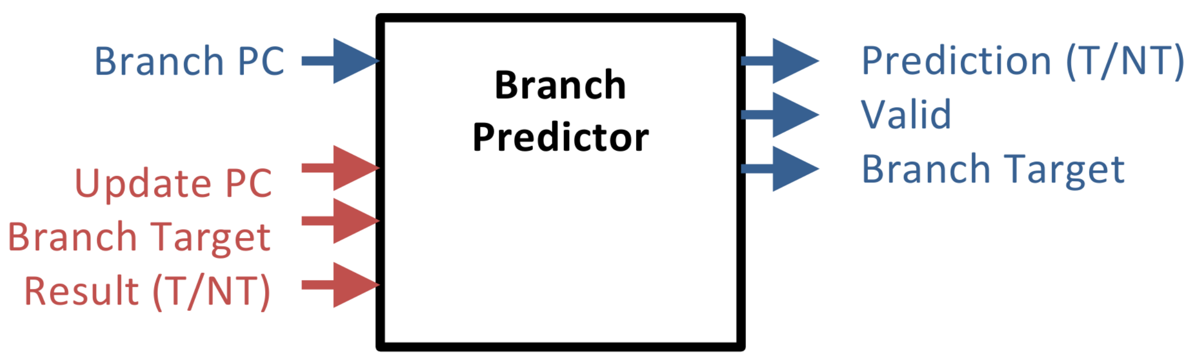


Figure 5.17: branch-predictor. From ()

BTB

- Branch Target Buffer (BTB)
- Save a table with PC in order to have history that we can predict on

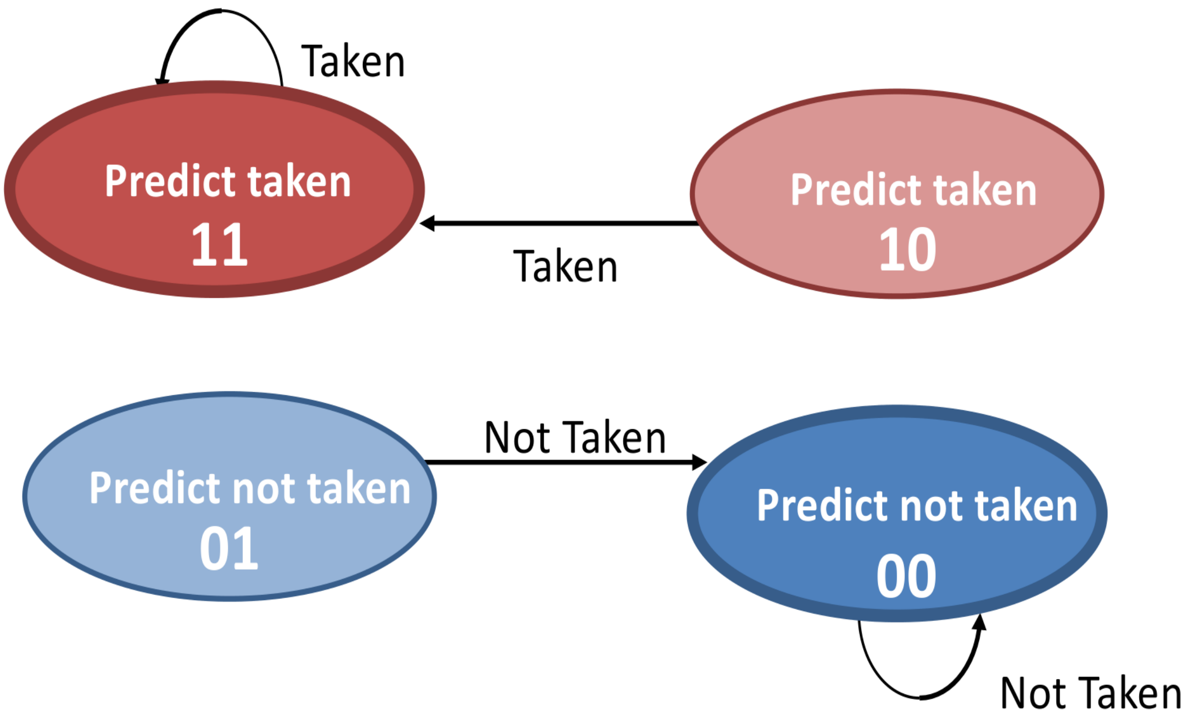


Figure 5.18: 2-bit predict. From ()

5.8.3 Exceptions

Exceptions are non-normal events that interrupt the normal flow of instructions

- Divide by zero
- Misaligned memory access
- Page fault
- Memory protection violation

Interrupts are external events that interrupt the normal flow of

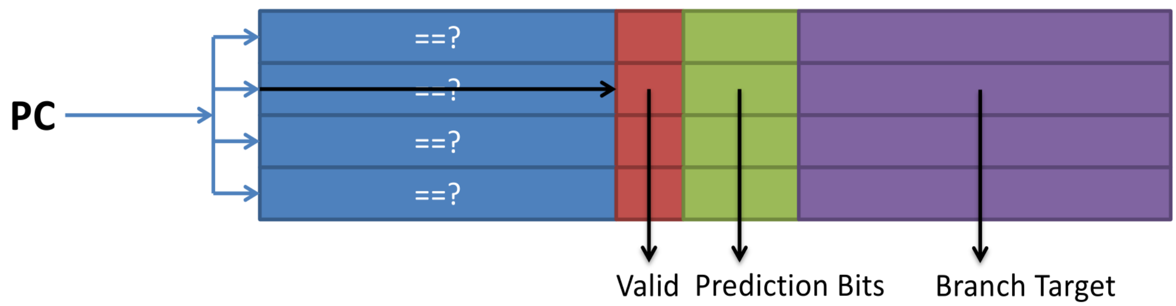


Figure 5.19: btb. From ()

5.9 Input/Output

Terminology

- nvram (flash): Similar to ram but it saves data when there is no power
- Busses: Parallel: many bits at once (e.g., 32 bits together in one clock) Used to be used everywhere Still used inside chips
- Serial links: Serial: one bit at a time (e.g., 32 clock cycles to send 32 bits) Used to be used only where distances were long (e.g., networks) Now used for most off-chip communications
- memory-mapped I/O: Manual handling of I/O devices Map portions of the address space to I/O devices Read and write to those addresses to access the
- direct memory access (DMA): Hardware who manage I/O devices (dynamically)
- Polling: The device puts its status somewhere The OS repeatedly checks for it to change
- Interrupt: When the device is ready, it gets the processor's attention by signaling an interrupt The OS then jumps to an interrupt handler to handle the event
- Throughput: x/s The read and write speed
- Latency: s/x Accessing time.
- Overhead: any combination of excess or indirect computation time, memory, bandwidth, or other resources that are required to perform a specific task

Data transfers: manual copy

```
add   R2, R0, R0        // Counter starts at 0
loop:
  lw R4, 0x12f0(R0)   // Read the I/O device
  sw R4, 0xfe00(R2)   // Store the results
  addi R2, R2, 4        // Next location
  bne R3, R2, loop
done:
```

Data transfers: DMA

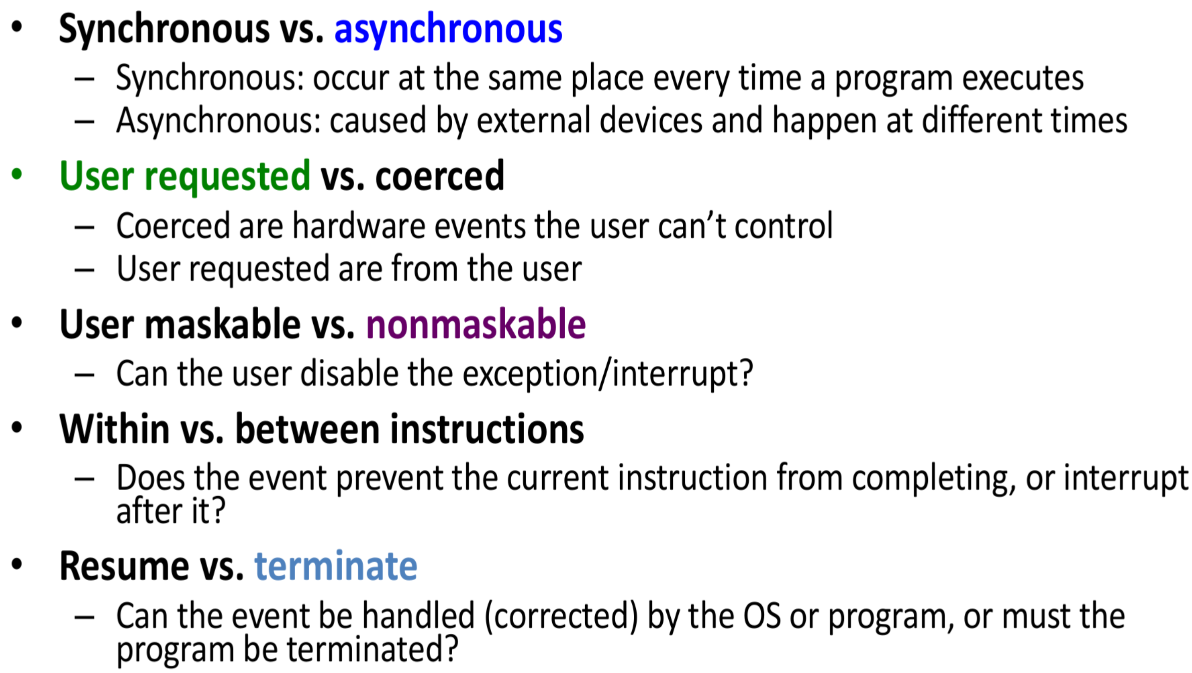


Figure 5.20: exceptions. From ()

```
addi R2, R0, 0xfe00 // destination
addi R3, R0, 230400  // number of words to copy
sw R1, 0x100b(R0) // set the device address
sw R2, 0x1008(R0) // set the destination address
sw R3, 0x1004(R0) // set the length and start
wait:
  lw R2, R0(0x1000) // Wait for it to be done
  bne R2, R0, wait
done:
```

5.10 Cache

5.10.1 Memory hierarchy

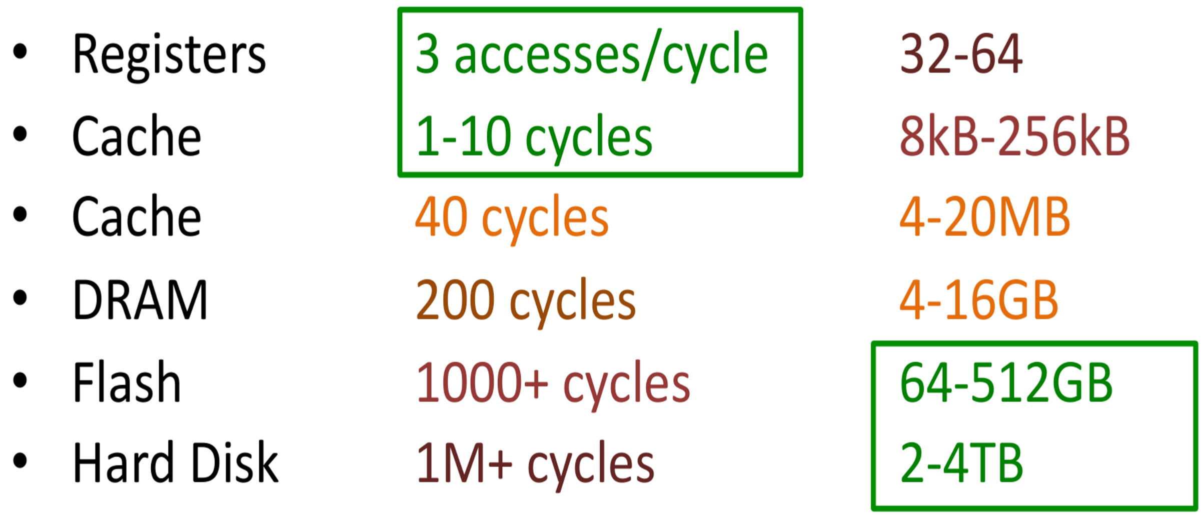


Figure 5.21: memory-hierarchy. From ()

3-types of cache types

- Fully-associative: Have to search all blocks, but very flexible
- Direct-mapped: Only one place for each block, no flexibility
- Set-associative: Only have to search one set for each block, flexible because each set can store multiple blocks in its ways

Write policies

- Write-through: slow, simple
- Write-back: fast (keeps the data just in the cache), more complex

Terminology

- Data: What is being stored
- Tag: What address the data has
- Index: What we use to determen where we want to put the data
- Cache line (block) size: How many tag's there are
- Valid bit: If the data is correct
- Dirty bit: The data has been change and we need to write it to memory
- Type of cache: Fully-associative (FA), Set-associative (SA), Direct-mapped (DM)
- Replacement policy: Write-through, Write-back

Cache blocks

- tag has 30bits and the 2 other bits is to determen what data 63 bits for eatch entry 32bit data 30bit tag 1bit valid-bit
- larger block of data -> fewer tags -> more efficient storage
- 1 word cache block we have 1 32bit data
- 2 word cache block we have 2 32bit data
- 4 word cache block we have 4 32bit data
- we load more data when we have space for it we will then have spatial locality
- Last 2: determine the byte within the word
- Next N: determine which word in the cache block
- Remaining 32-N-2: tag

Calculate overhead

- Data= Cache-Lines * Byte-Lines
- Overhead= Cache-Lines * (Tags-per-line + valid-bit)
- % data= Overhead / Data

Calculate Address (Tag Index Offset)

- Direct maped:
  Offest= log (number of data blocks) (+ 2-bit (4-bite cache block size if 64 then log 64))
  Index= log (number of cache lines)
  Tags= 32(bit processor)- (Index+Offset)
- x-set associative:
  Offest= log (number of data blocks) (+ 2-bit determens how the address looks like)
  Index= log ((number of cache lines)/x)
  Tags= 32(bit processor)- (Index+Offset)
- fully associative:
  Offest= log (number of data blocks) (+ 2-bit determens how the address looks like)
  Index= 0
  Tags= 32(bit processor)- (Offset)

Calculate Avrage memory access time

- Avrage-cycles= CacheL1*Cycles + CacheL2*Cycles + Dram*Cycles

5.10.2 Cache misses $3C's$

- cold/compulsory miss: When the program first begin it doesn't have anything in the cache
- conflicts miss: to small to store the necessary data
- capacity miss: map's at the same block an then overwrite unnecessarily

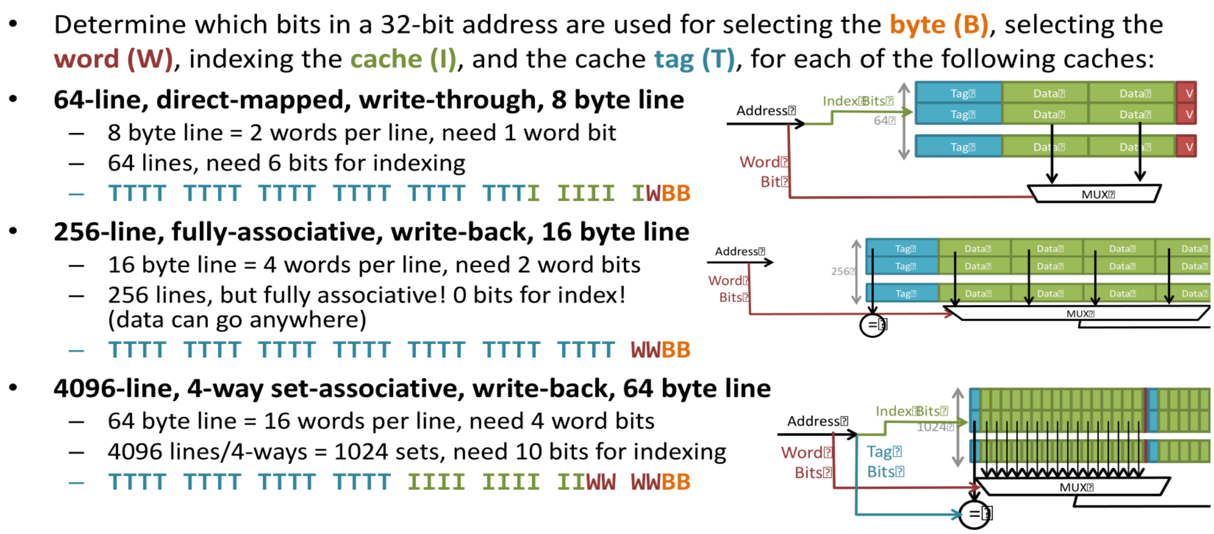


Figure 5.22: cache-format. From ()

5.11 Virtual Memory

Why use VM

- Map memory to disk ("unlimited" memory)
- Keep programs from accessing each other's memory (security)
- Fill holes in the RAM address space (efficiency)

Terminology

- Translation= map address
- Page tables= for each program keep track of all translations
- Fine grain= page table with spesific address
- Coarse grain= page table with address mapped ranges
- Page Table Entries (PTEs)= number of translation
- Translation Lookaside Buffer TLB= All of the pages, fast translation via hardware
- VA= Virtual program addresses
- PA= Physical RAM addresses
- Page offset= point to a range and then use the offset to determen where
- Translation Lookaside Buffer (TLB)= page table cache (Faster)
- Multilevel page table translation= page table point to other page tables (inseption)

Combining TLB and cache

- Physical caches: slow and needs the translation first and then save get the PA also known as Pysical-Index, Physical-Tagged (PIPT)
- Virtual caches: fast uses only virtual addresses, no translation, difficult for protection also known as Virtual-Index, Virtual-Tagged (VIVT):
- Virtual-Index, Phisically-Tagged (**VIPT**): VA for index PA for tags, fast dose translation and fetch from cache at the same time, most commonly used. We need a comperitor to se if the PA tags from cache matches the TLB PA only then we can say if we had a hit or a miss. The VA offset is what the PA tag is selected and the VA tag (number of virtual pages) is what the TLB uses. Mostly used for L1 cache not L2.

TLB

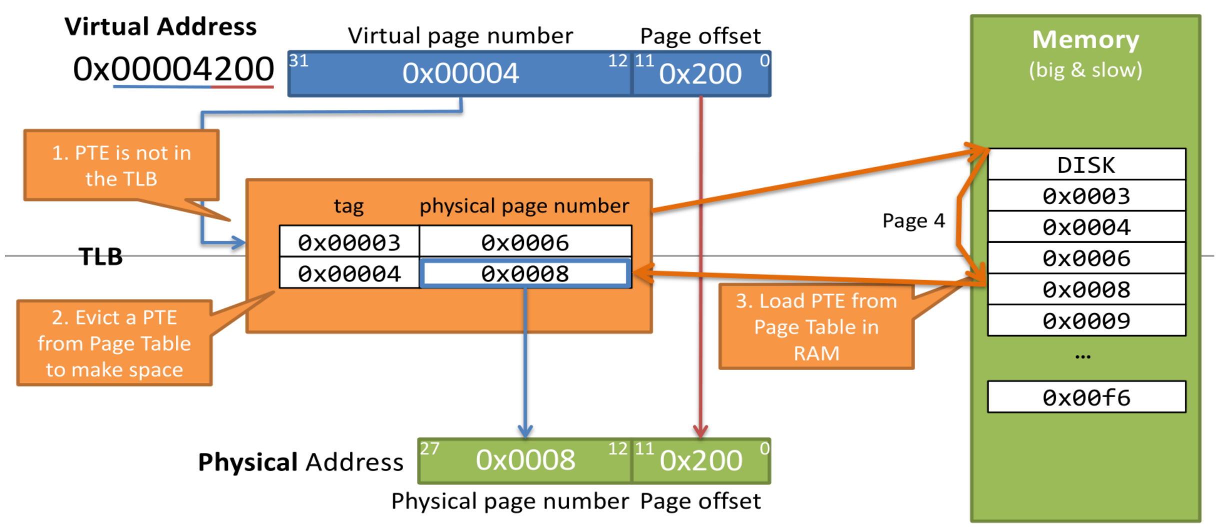


Figure 5.23: tlb

usful conversion: $2^n = xB$
Calculating Page sizing

- Number-of-Virtual-Pages= 2^{32}/pages
- Bits-used-for-Page-Offset= log(pages)
- Bits-used-for-VPN= 32(bit prossessor)- Bits-used-for-Page-Offset

Calculating TLB size

- TLB-size=Entries*Pages

0		10	1kB	20	1MB	30	1GB
1	2B	11	2kB	21	2MB	31	2GB
2	4B	12	4kB	22	4MB	32	4GB
3	8B	13	8kB	23	8MB		
4	16B	14	16kB	24	16MB		
5	32B	15	32kB	25	32MB		
6	64B	16	64kB	26	64MB		
7	128B	17	128kB	27	128MB		
8	256B	18	256kB	28	256MB		
9	512B	19	512kB	29	512MB		

Figure 5.24: conversion. From ()

5.12 Parallism

5.12.1 Multicore

powerwall

$$\begin{aligned}
&P = CV^2 f \\
&C = \text{ capasiter, smaller transistors smaller capasiters} \\
&V = \text{ voltage, decreasing makes it slower} \\
&f = \text{ frequency, reducing clock speed}
\end{aligned}$$

5.12.2 Paralel programming

Avrage processors

Calculate how many cores are used in avrage we need to know
how chunks (work) there is in total
how many time units (cycles think of a reverse pyramid)

there is 16 input data and we have 8-cores, we want to calculate the total sum
what is the avrage prossesor being used
$15 chunks,\ 4 timeunits \Rightarrow 15/4 = 3.75$

parallel issues

- Most programs can not utilize parallelism, need to devid the program to diffrent executions.
- We also need to share cache and therefore have performance issue since we cant use the entire cache.

How much faster

$75\% parallel,\ 25\% nonparallel$ we have hundred thousand cores
$\Rightarrow$ Parallaism takes $(0.75/100000 + 0.25 * 1)$
Singular takes $(0.75 * 1 + 0.25 * 1) \Rightarrow 4*$ faster

Speedup with Amdahl's law $Speedup = \dfrac{1}{(1-P)+P/S}$
$P =$ Parallel aktion
$S =$ Speed up of the parrallel part
$$\Rightarrow \frac{1}{(1-0.75)+0.75/100000} = \frac{1}{0.25+0} = 4$$

5.12.3 Synchronization

- Fix the issue with 2 processors accessing the same value at the same time.
- We can use atomic swap to first set lock to 1 then check the data.

locks

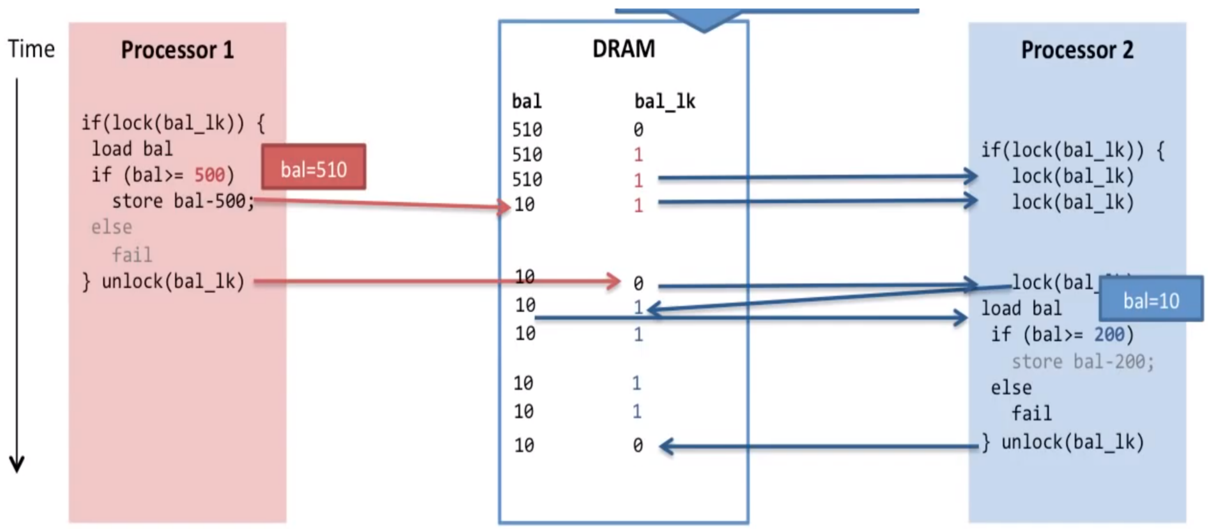


Figure 5.25: locks

5.12.4 Cache coherency

- How we use caches to save memory
- Locks: if the data has been acceded. (not a guaranty of protection)
- Snooping: Look what the other processors are storing in there private cache

snooping

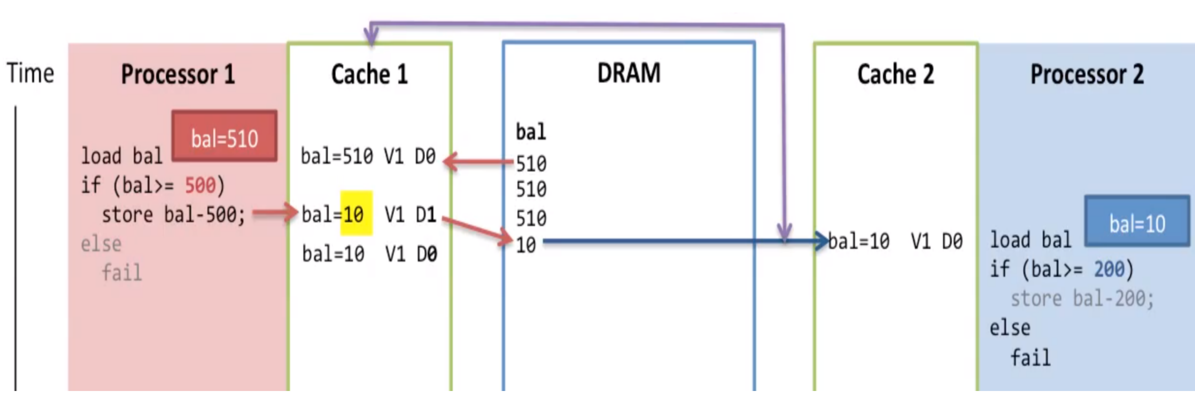


Figure 5.26: locks. From ()

5.12.5 ILP

- Instruction level parallelism (ILP)

- better to have out-of-oder execution since we can find independent instructions
- We use Dual-issue pipline so we can have one for memory instructions and one for non memory instruction
- It makes ISA promise with in order execution
- Issue with data hazards, more complexity

dual issue pipeline

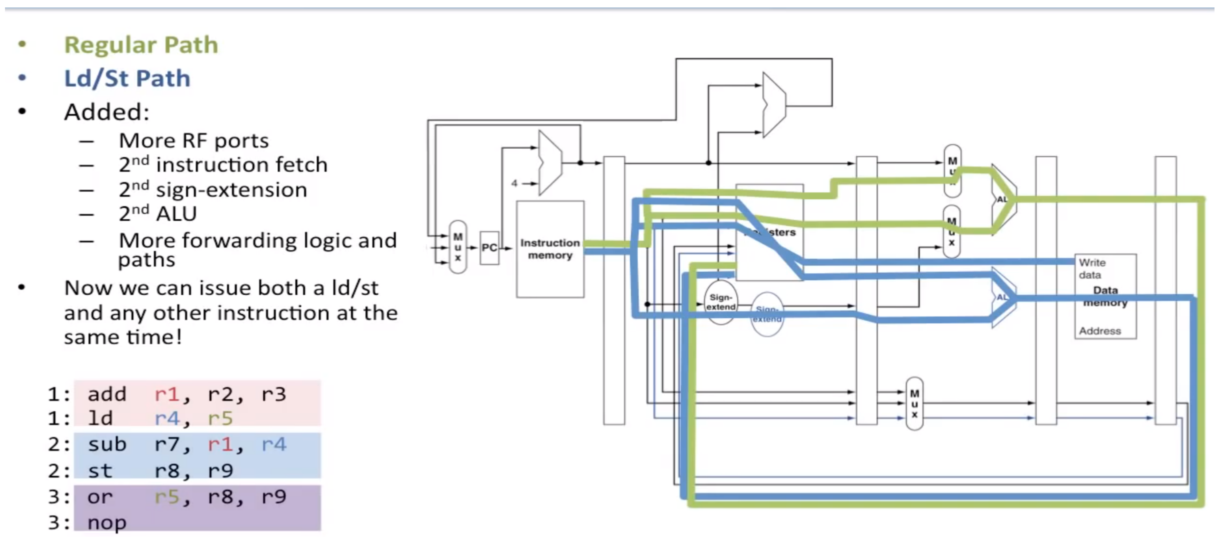


Figure 5.27: dual-issue-pipline. From ()

Chapter 6

Imperativ and Object-Oriented Programming Methodology

6.1 Introduction

6.1.1 Test

Innan:
Läs igenom standard instruktionerna.
Sätt upp mjukvaru miljö på skollans datorer.

Under:
Det är mycket tid.
Läs igenom instruktionerna på uppgiften nogrant.

6.2 Imperative programming i C

C is a strict programming language, we must therefore write every type of variables. Like any other imperative programming language it execute sequentially and every argument is evaluated in order. It also allow side effects like changing the state freely.
Vilkorsatser

```
if (expr) { expr; }
if (age > 100) { puts("Very old"); }

if (expr) { expr; } else { expr; }
if (age % 2 == 0) { puts("Even"); } else { puts("Odd"); }

expr ? expr : expr
a < b ? b : a;
```

Shortcuts loop operations

```
n_fakultet *= n--;
```

6.2.1 Data structurer

stack och heap

I kortids minnet som c kompilatorn hanterar skjälv så sappar datan i en stack. Stack funkar som talrikar som man staplar på varandra och tar den som du la senast ut.

Heap är för långtids mine och största data som arrayer. Det fungerar som ett rutat papper där varge ruta är en viss data size.
Shortcuts loop operations För att andvända heapen, (rita på det rutade papret) måste man följa 4 steg.
steg 1: räkna ut hur många rutor vi behöver
Använd plattformsoberoende hjälpmedel och vanlig aritmetik

```
sizeof(T) * antal
int size = sizeof(int) * 1024
```

steg 2: reservera motsvarande yta
Här kan det gå fel — det kanske inte finns plats på papperet

```
T *namn = malloc(antal bytes);
int *skonummer = malloc(size);
```

steg 3: använd ytan hejvilt
Men sudda först! (Beror på datastrukturen)
steg 4: lämna tillbaka platsen när du är klar
Annars kommer det gå dåligt i ett framtida steg 2

```
free(namn);
free(skonummer);
```

Structar

För att skapa egna data structurer som andvänder man sig av "struct" operatorn. Om man vill gömma fler data typer i en så kan man andvända en "union". För att göra en typ så andväder man "typedef" och avslutar med

`namnet_{t}`

Pekare

Pekare pekar var datan finns så man kan skicka information utan att sicka stora mängder data.

```
int  a
int *b
```

Syntaxen som andvänds är: pekartypen

`int *`

Skillnaden mellan arrayer och pointers är väldigt liten i c. Därför är samma oporation för string (

`char *`

som en pointer (

`int *`

Linked list

Varge element har data av någon typ samt en pekare till nästa element. Head and tail kan beskriva en sådan lista, då head är första ellementet och tail är rästen.

Tid komplexiteten är: On. Linjär

Hash table

Hash table är ett mer effectivt sätt att hantera data istället för en array. Då man inte behöver kålla varge element vad den finns utan andvänder en hash function för att skapa ett "nummer" som pekar till vad datan finns någonstans exempelvis kan det vara mod element i hash table då vet vi har vi ska ska söka efter. Ett problem som kommer upstå är att data mappas till samma plats, då får man andvända exempelvis linuer probing som säger att om det är fult på platsen gå till nästa tills det finns och om det inte finns så protesterar programet. Varge element inehåller inget värde eller deleted varde om det är tomma.

Tid komplexiteten är: O1. Konstant men inte i verkligheten

6.3 Object-Oriented programming i Java

kör den mest spesifica implementationen av metoden single dispage, vi bryr os ombjectet vi kör metoden på men inte argumentet

Chapter 7

Linear Algebra and Geometry I

7.1 Linjära ekvationssystem

3 gundläggande operationer för att lösa linjära ekvationer

(1) (Tvåpilar) Byt om på två ekvationer

(2) (Lambda) Multiplicera båda sidorna av en ekvation med $\lambda \neq 0$

(3) (Lambda pil) addera λ gånga en ekvation till en annan ekvation

Generall lösning

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

$$\text{Lambda pil upp} \begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

Ger nya ekvationssystemet

$$\text{Lambda pil upp} \begin{cases} (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y = c_1 + \lambda c_2 \\ a_2x + b_2y = c_2 \end{cases}$$

$$\text{-Lambda pil upp} \begin{cases} (a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y = c_1 + \lambda c_2 \\ a_2x + b_2y = c_2 \end{cases}$$

Exempel linjär ekvationssystem

$$\begin{cases} 2x + 3y = 4 \\ 3x - y = 6 \end{cases}$$

Andvänder rad operationer för att lösa ekv systemet

$$\left(-\frac{3}{2}\right) \text{pil ned} \begin{cases} 2x + 3y = 4 \\ 3x - y = 6 \end{cases}$$

$$(3x - y) - \frac{3}{2}(2x + 3y) = 6 - \frac{3}{2} \cdot 4 \Leftrightarrow$$

$$\left(3 - \frac{3}{2} \cdot 2\right) x + \left(-1 - \frac{9}{2}\right) y = 6 - 6 \Leftrightarrow -\frac{11}{2}y = 0$$

$$\left(-\frac{2}{11}\right) \text{ned} \begin{cases} 2x + 3y = 4 \\ -\frac{11}{2}y = 0 \end{cases}$$

$$(-3)\text{pil upp} \begin{cases} 2x + 3y = 4 \\ y = 0 \end{cases}$$

$$\left(\frac{1}{2}\right) \text{upp} \begin{cases} 2x = 4 \\ y = 0 \end{cases}$$

$$\begin{cases} x = 2 \\ y = 0 \end{cases}$$

Kontrol: stoppar in x och y i ekvationerna

$2 \cdot 2 + 3 \cdot 0 = 4$ (stämmer)

$3 \cdot 2 - \cdot 0 = 6$ (stämmer)

7.1.1 Total Matris

Termonologi

- Rader och Kolonner: Rader är vågräta delen av matrisen (ekvationen) Kolonner är lodräta delen (koificenterna)

- ledande ekvivalent:

- trappstegs matris: När ledande ekvivalent är i trapp form går ned max ett steg

- rad kanonisk: När alla av de ledande ekvivalent inte har någon i samma kolonn

Exempel matriser

$$\begin{cases} x + 2y + \quad z = -1 \\ 2x + (a+3)y + \quad 3z = -4 \\ x + (3-a)y + (a-2)z \quad = a-1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & a+3 & 3 & -4 \\ 1 & 3-a & a-2 & a-1 \end{array}\right)$$

$(-1 \text{ rad1 till rad2}), (-2 \text{ rad1 till rad3})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & a-1 & 1 & -2 \\ 0 & 1-a & a-3 & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a-1 & 1 & -2 \\ 0 & 0 & a-2 & a-2 \end{array}\right)$$

$a \neq 1 \wedge a \neq 2$

$(\frac{1}{a-2}\text{rad } 3)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a-1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(-1\text{rad } 3 \text{ till rad } 2), (-1\text{rad } 3 \text{ till rad } 1)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & a-1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(\frac{1}{a-1}\text{rad } 2)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & a-1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & \frac{3}{1-a} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(-2\text{rad } 2 \text{ till rad } 3)$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2-\frac{6}{1-a} \\ 0 & 1 & 0 & \frac{3}{1-a} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{cases} x & = -2 - \frac{6}{1-a} \\ y & = \frac{3}{1-a} \\ z & = 1 \end{cases}$$

Kontroll: stoppar in x,y,z i ekvationerna

$$\begin{cases} \left(\frac{2a-8}{1-a}\right) + \quad 2 \cdot \left(\frac{3}{1-a}\right) + \quad 1 = -1 \\ 2 \cdot \left(\frac{2a-8}{1-a}\right) + (a+3) \cdot \left(\frac{3}{1-a}\right) + \quad 3 \cdot 1 = -4 \\ \left(\frac{2a-8}{1-a}\right) + (3-a) \cdot \left(\frac{3}{1-a}\right) + (a-2) \cdot 1 = a-1 \end{cases}$$

7.2 Vektorer/koordinater i planet och rummet

Räkne regler vektorer

$$
\begin{aligned}
&\text{A1 } \vec{u}+\vec{v}=\vec{v}+\vec{u}\\
&\text{A2 } \vec{u}+(\vec{v}+\vec{w})=(\vec{v}+\vec{u})+\vec{w}\\
&\text{A3 } \vec{u}+\vec{0}=\vec{u}\\
&\text{A4 } \vec{u}+\vec{v}=\vec{0}\Leftrightarrow\vec{v}=\vec{-u}\\
&\text{M1 } 1\vec{u}=\vec{u}\\
&\text{M2 } k(l\vec{u})=(kl)\vec{u}\\
&\text{M3 } (k+l)\vec{u}=k\vec{u}+l\vec{u}\\
&\text{M4 } k(\vec{u}+\vec{v})=k\vec{u}+k\vec{v}
\end{aligned}
$$

$$\vec{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}\text{ för }\mathbb{R}^2$$

$$\vec{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\text{ för }\mathbb{R}^3$$

Definition: Parallela vektorer

$$\text{Parallela omm }\exists k:\vec{u}=k\vec{v}\vee k\vec{u}=\vec{v}$$

7.2.1 Bas

Standard bas är välbikant då i planet är x och y axeln medans i ett rum är x, y och z. Baser som inte är standard är vektorer som ej är parralella som då skappar axlar som ej behöver vra vinkelräta.

Definition: Bas

$$
\begin{aligned}
&\text{Bas i plan }\forall\vec{x},\exists k_1,k_2:\vec{x}=k_1\vec{u}\vee k_2\vec{v}\\
&\text{Bas generell }\vec{x}=k_1\vec{u_1}+k_2\vec{u_2}+\ldots+k_n\vec{u_n}\\
&\underline{u}=(\vec{u_1}\vec{u_2}\ldots\vec{u_n})\\
&\vec{e_1}=\begin{pmatrix}1\\0\\0\end{pmatrix}\vec{e_2}=\begin{pmatrix}0\\1\\0\end{pmatrix}\vec{e_3}=\begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}
$$

Exempel: kordenatar för vektor i bas

Låt: $\vec{u} = \begin{pmatrix}2\\1\end{pmatrix}, \vec{v} = \begin{pmatrix}-2\\1\end{pmatrix}$

Hitta kordenarterna för vektorn $\vec{x} = \begin{pmatrix}0\\6\end{pmatrix}$

Lösning: vi måste hittar realla tal k_1 och k_2 så att

$$\begin{pmatrix}0\\6\end{pmatrix} = k_1\begin{pmatrix}2\\1\end{pmatrix} + k_2\begin{pmatrix}-2\\1\end{pmatrix} = \begin{pmatrix}2k_1-2k_2\\k_1+k_2\end{pmatrix}$$

$$\begin{cases} 2k_1 - 2k_2 = 0 \\ k_1 + k_2 = 6 \end{cases} \Rightarrow \begin{pmatrix}k_1\\k_2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix}$$

Exempel: Om det är en bas

För vilken värde på a är vektorerna en bas för $\mathbb{R}^3$

$$\vec{u_1} = \begin{pmatrix}1\\2\\3\end{pmatrix}, \vec{u_2} = \begin{pmatrix}1\\2\\a\end{pmatrix}, \vec{u_3} = \begin{pmatrix}1\\a\\3\end{pmatrix}$$

Vi måste hitta ett a sådant att

$$k_1\begin{pmatrix}1\\2\\3\end{pmatrix}, k_2\begin{pmatrix}1\\2\\a\end{pmatrix}, k_3\begin{pmatrix}1\\a\\3\end{pmatrix} = \begin{pmatrix}?\\?\\?\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\2&2&a\\3&a&3\end{pmatrix}\begin{array}{l} -2\ \ -3 \\ + \\ + \end{array} \sim \begin{pmatrix}1&1&1\\0&0&(a-2)\\0&(a-3)&0\end{pmatrix}$$

Om a = 2: då är sista ekvationen 0 =? och vi ser att lösningarna får parametrar och därför: antingen ingen lösningar eller oändligt många lösningar.

Oavsett -ej bas

Om a = 3: då blir det samma problem som a=2

Svar: de tre vektorer är en basa om a inte är 2 eller 3

Definition: Punkter i planet

Från origo: alla $P = (x, y)$ och $\overrightarrow{OP} = \begin{pmatrix}x\\y\end{pmatrix}$

Om $A = (a_1, a_2)$ och $B = (b_1, b_2)$ då är $\overrightarrow{AB} = \begin{pmatrix}b_1-a_1\\b_2-a_2\end{pmatrix}$

Definition: Längd

Längd vektor i plan $|\vec{v}| = \sqrt{a^2+b^2}$

Längd vektor i rum $|\vec{v}| = \sqrt{a^2+b^2+c^2}$

7.3 Skalärprodukt och vektorprodukt

7.3.1 Skalärprodukt

$$\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$
$$\vec{u} \bullet \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_3 v_3$$

Räkneregler

$$\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$$
$$\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$$
$$\lambda(\vec{u} \bullet \vec{v}) = (\lambda \vec{u}) \bullet \vec{v} = \vec{u} \bullet (\lambda \vec{v})$$
$$\vec{u} \bullet \vec{u} = |\vec{v}|^2$$
$$\vec{u} \bullet \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$$

Exempel: parrallel och ortogonal

För vilka värden på a och b är vektorerna i $\mathbb{R}^3$ $\begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix}$ och $\begin{pmatrix} b \\ 8 \\ a \end{pmatrix}$

(a) parallel?, (b) ortognala?

(a)

$$\begin{cases} k = b \\ ak = 8 \\ 2k = a \end{cases} \Rightarrow$$

$$\begin{cases} \qquad\qquad\qquad k = b \\ (2k)k = 8 \Rightarrow k^2 = 4 \Rightarrow k = 2 \\ \qquad\qquad\qquad 2k = a \end{cases}$$

$k = b = 2, a = 4$

(b)

$$\begin{pmatrix} 1 \\ a \\ 2 \end{pmatrix} \bullet \begin{pmatrix} b \\ 8 \\ a \end{pmatrix} = 1b + a2 + 2a = b + 4a = 0$$

Exempel: Längd-formeln

$$|\vec{v}| = \sqrt{|\vec{v}|^2} = \sqrt{|\vec{v}| \bullet |\vec{v}|}$$

Beräkna längden av (ON-bas): $\vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 2^2 + 1^2 + 2^2 = 9 \Rightarrow$$

$|\vec{v}| = \sqrt{9} = 3$

Därmed är längden: 3

Exempel: Vinkel-formeln

Låt $\vec{u}$ och $\vec{v}$ vara vektorer och vinkel blir då:

$$\theta = \arccos \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|}$$

Beräkna vinkeln av (ON-bas): $\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ och $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot 2 + 2^2 + 2 \cdot 1 = 8$$

$$|\vec{u}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$$

$$|\vec{v}| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{9} = 3$$

$$\arccos \frac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|} = \arccos \frac{8}{3 \cdot 3} = \arccos \frac{8}{9} \approx 27.27°$$

Exempel: Längd av två vektorer

Låt u och v vara två vektorer sådana att

$|\vec{u}| = 4, |\vec{v}| = 2$ och vinkeln mellan $\vec{u}$ och $\vec{v}$ är $\frac{2\pi}{3}$

Bestäm längden av $3\vec{u} - 2\vec{v}$

$$\sqrt{|3\vec{u} - 2\vec{v}|^2} = \sqrt{9|\vec{u}|^2 + 4|\vec{v}|^2 - 2|\vec{u}||\vec{v}| \cos \frac{2\pi}{3}}$$

$$\sqrt{9 \cdot 16 + 4 \cdot 4 - 4 \cdot 3 \cdot 8 \cdot \frac{-1}{2}} = \sqrt{13 \cdot 16} = 4\sqrt{13}$$

Exempel: beräkna skalärprudukten

$$\underline{u} \begin{pmatrix} 9 \\ 9 \end{pmatrix} \bullet \underline{u} \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$\underline{u} = (\vec{u_1}, \vec{u_2}),\ \vec{u_1} \bullet \vec{u_1} = 9,\ \vec{u_1} \bullet \vec{u_2} = 6,\ \vec{u_2} \bullet \vec{u_2} = 8$$

$$\underline{u} \begin{pmatrix} 9 \\ 9 \end{pmatrix} \bullet \underline{u} \begin{pmatrix} -2 \\ -1 \end{pmatrix} = (9\vec{u_1} - 2\vec{u_2}) \bullet (9\vec{u_1} - 1\vec{u_2}) = -18|\vec{u_1}|^2 + (-9 - 18)\vec{u_1} \bullet \vec{u_1} - 9|\vec{u_2}|^2$$

$$= -18 \cdot 9 - 27 \cdot (6) - 9 \cdot 8 = -396$$

7.3.2 Ortogonal projektion

Hitta punkt på linjen som är närmast en punkt. Punkten är ortogonala (vinkelrätta)
Parralell koposant skrivs $\vec{v}_{||l}$ och ortogonal skrivs $\vec{v}_{\perp l}$.

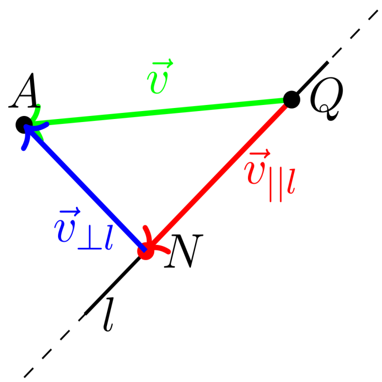


Figure 7.1: Ortogonal projektion

Ortogonal projektion Formeln

$$\vec{v}_{||l} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|^2}\vec{u}$$

Exempel: Hitta parralel och ortogonal komposanten

$$\vec{v} = \vec{v}_{\perp l} + \vec{v}_{||l}$$

$$\vec{v}_{||l} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|^2}\vec{u}$$

Beräkna ortogonal komposanten av:

$$\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \text{ och } \vec{u} = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}$$

Beräkna först parrallel komposanten:

$$\vec{v}_{||\vec{u}} = \frac{\vec{u} \bullet \vec{v}}{|\vec{u}|^2}\vec{u} = \frac{-18}{36}\vec{u} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Ortogonal komposanten blir då:

$$\vec{v}_{\perp\vec{u}} = \vec{v} - \vec{v}_{||\vec{u}} = \begin{pmatrix} 1-2 \\ -2+2 \\ 3-1 \end{pmatrix}$$

Svar: $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Exempel: Närmaste punkt och avståndet

Bestäm avståndet från punkten $P = (-2, 4, 3)$ till linjen

$$\begin{cases} x + 2y - z = 1 \\ x - y + 5z = 4 \end{cases}$$

Finn även den punkt på linjen l som ligger närmast punkten P

Skriver ekvationen på parameter form

$$\begin{pmatrix} 1 & 2 & -1 & | & 1 \\ 1 & -1 & 5 & | & 4 \end{pmatrix} \begin{matrix} {}^{-1} \\ {}^{+} \end{matrix} \quad | \cdot 1/3 \sim \begin{pmatrix} 1 & 2 & -1 & | & 1 \\ 0 & 1 & -2 & | & -1 \end{pmatrix} \begin{matrix} {}^{+} \\ {}^{-2} \end{matrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 3 & | & 3 \\ 0 & 1 & -2 & | & -1 \end{pmatrix} \begin{matrix} {}^{-1} \\ {}^{+} \end{matrix} \sim$$

$$l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \quad \text{dvs } (p_0 + t\vec{v})$$

Hittar en godtycklig punkt ex $t = 1 \Rightarrow Q = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

(1.) beräknar vektorn från godtykliga punkten till den givna

$$\overrightarrow{PQ} = \begin{pmatrix} 0 - (-2) \\ 1 - 4 \\ 1 - 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$$

(2.) Beräknar ortogonala projectionen

$$\overrightarrow{NQ} = \overrightarrow{PQ}_{||v} = \frac{\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}}{9 + 4 + 1} \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \frac{-14}{14} \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

(3.) Beräknar punkten från närmaste punkt till

$$\overrightarrow{ON} = \overrightarrow{OQ} + \overrightarrow{QN} = \overrightarrow{OQ} - \overrightarrow{NQ} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$$

(4.) Beräknar längden $\left| \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix} \right| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

Svar: avståndet är $\sqrt{3}$ och punten är $(-3, 3, 2)$

Exempel: Spegling

Bestäm speglingen av punkten $A : (1, 3, -3)$ i planet π
som går genom origo och innehåller punkterna
$(-1, 1, 1)$ och $(3, 3, 1)$

Beräknar planets ekvation

$$\vec{n} = \overrightarrow{OQ} \times \overrightarrow{OP} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$-x + 2y + 3z = 0$ Eftersom den går egenom origo

$$\overrightarrow{NA} = \overrightarrow{OA}_{||\vec{n}} = \frac{\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}}{1+4+9} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \frac{14}{14} \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$$

Beräkna speglingen rita då är det upenbart

$$\overrightarrow{OA'} = \overrightarrow{OA} - 2\overrightarrow{NA} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$$

Exempel Hitta punkt ortogonal projektion

Låt l vara linjen genom punkten $Q = (1, 2, 3)$ parallell med vektorn

$$\vec{u} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Hitta den punkt N på l som är närmast $A = (1, 7, 4)$

Lösning: Rita figur och tolka. Vi ortogonalt projeicerar

$$\vec{v} = \overrightarrow{QA} = \begin{pmatrix} 1-1 \\ 7-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \text{ på vektorn } \vec{u}$$

$$\vec{v}_{||l} = \vec{v}_{||\vec{u}} = \frac{\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = \frac{25}{50} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Detta är ju igen vektorn som pekar frpn Q till närmaste punkten som därför blir

$$N = \left(1 - \frac{1}{2}3, 2 - \frac{1}{2}4, 3 - \frac{1}{2}5\right) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)$$

Kontroll: kollar om $\overrightarrow{NQ} = k\vec{u}$, $k \in \mathbb{R}$

$$\begin{pmatrix} 1-(-1/2) \\ 2-0 \\ 3-1/2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 2 \\ 5/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} \text{ (stämmer)}$$

Räkneregler ortogonal

För vektorer $\vec{u}$, $\vec{v}$ och $\vec{w}$ och skalär $\lambda \in \mathbb{R}$
$(\lambda\vec{v})_{||\vec{u}} = \lambda(\vec{v}_{||\vec{u}})$
$(\vec{v}+\vec{w})_{||\vec{u}} = \vec{v}_{||\vec{u}} + \vec{v}_{||\vec{u}}$
$(\lambda\vec{v})_{\perp\vec{u}} = \lambda(\vec{v}_{\perp\vec{u}})$
$(\vec{v}+\vec{w})_{\perp\vec{u}} = \vec{v}_{\perp\vec{u}} + \vec{v}_{\perp\vec{u}}$

7.3.3 Enhetsvektorer och ON-baser

En enhetsvektor är en vektor med längd 1.
Om man har en vektor $\vec{u}$ kan man skala om den med ett positivt tal så den får längd 1.
En bas $\underline{\vec{u}} = (\vec{u_1}\vec{u_2}\ldots\vec{u_n})$kallas en ortonormal bas (ON-bas) omm:
(1) Alla vektorerna är enhetsvektorer $|\vec{u_1}|^2 = \vec{u_i} \bullet \vec{u_j} = 1$
(2) Varje par av vektorer $\vec{u_i}$ och $\vec{u_j}$ for $i \neq j$ är ortogonala:$\vec{u_i} \bullet \vec{u_j} = 0$
$\hat{\vec{u}} = \dfrac{1}{|\vec{u}|}\vec{u}$

Skalärproducten har samma räkneregler i ON-baser som i standard bas

7.3.4 Vektorprodukten

I högersystem kan representeras av en höger hand där tumen är vektorn 1 pekfingret är 2 och långfingret är 3. **Definition: högersystem**

En bas $\underline{u} = (\vec{u}_1\vec{u}_2\vec{u}_3)$ kallas ett högersystem omm:
set ifrån spetsen av $\vec{u}_3$ vridas $\vec{u}_1$ moturs till $\vec{u}_2$

Definition: Vektorprodukten

Givet två icke-parallella vektorer $\vec{u}$ och $\vec{u}$ med vinkel θ
mellan dem definieras $\vec{u} \times \vec{v}$ som den entydiga vektor som uppfyller
(a) $\vec{u} \times \vec{v}$ är ortogonal mot båda $\vec{u}$ och $\vec{v}$
(b) längden av $\vec{u} \times \vec{v}$ är $|\vec{u}\vec{v}\sin\theta|$
(c) $(\vec{u}\vec{v}\vec{u} \times \vec{v})$ är ett högersystem
I fallet där $\vec{u}$ och $\vec{v}$ är parallella är $\vec{u} \times \vec{v} = \vec{0}$

Definition: produkter

Sala om: tal $\cdot$ vektor = vektor
Skalär produkt: vektor $\bullet$ vektor = tal
Vektor produkt: vektor $\times$ vektor = vektor

Räkneregler: Vektorprodukten

För vektorer $\vec{u}, \vec{v}, \vec{w}$ och ett tal λ gäller

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$\lambda(\vec{u} \times \vec{v}) = \lambda(\vec{u}) \times \vec{v} = \vec{u} \times (\lambda\vec{v})$

Vektorprodukten metologi

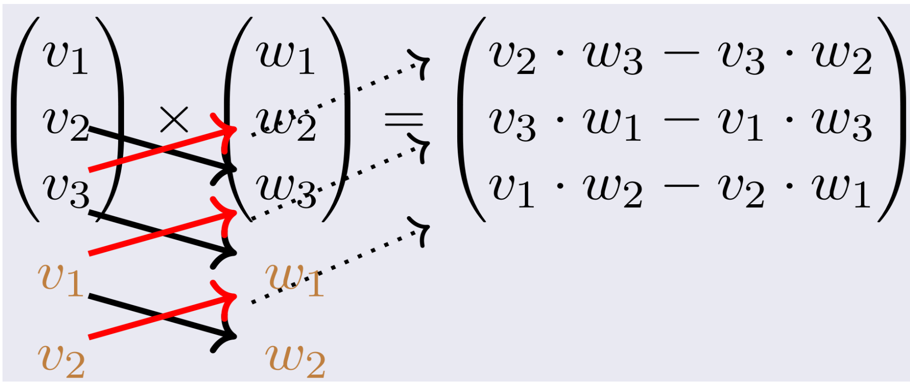


Figure 7.2: Vektorprodukten. From ()

Exempel: Hitta basen

Låt $\vec{u}_1 = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}$ och $\vec{u}_2 = \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix}$

Verifiera att dessa är ortogonala, och hitta en vektor $\vec{u}_3$

Så att $\underline{u} = (\hat{\vec{u}}_1 \hat{\vec{u}}_2 \hat{\vec{u}}_3)$ är en ON-bas (och skriv ut denna)

Lösning: Först kollar vi att skalärprodukten är 0 :

$\begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} = 16 - 8 - 8 = 0$

Där med ortogonala. Nu måste vi hitta en vektor som är ortogonal mot båda vektorprodukten är just det

$\vec{u}_3 = \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} \begin{pmatrix} -64+1 \\ 4+32 \\ -4-32 \end{pmatrix} = \begin{pmatrix} -63 \\ 36 \\ -36 \end{pmatrix}$

Längderna av dessa är (råkar vara lika)

$|\vec{u}_2|^2 = |\vec{u}_1|^2 = 4^2 + (\pm 8)^2 + (\pm 1)^2 = 81$

och vi kunna göra liknade beräkning för $\vec{u}_3$, men vi vet redan att:

$|\vec{u}_3| = |\vec{u}_1||\vec{u}_2| \sin\theta = 9 \cdot 9 \cdot 1 = 81$

Desras tre normering blir därför:

$\hat{\vec{u}}_1 = \frac{1}{9} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}, \hat{\vec{u}}_2 = \frac{1}{9} \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix} \hat{\vec{u}}_1 = \frac{1}{81} \begin{pmatrix} -63 \\ 36 \\ -36 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} -7 \\ 4 \\ -4 \end{pmatrix}$

Så basen är: $\underline{u} = \left(\frac{1}{9} \begin{pmatrix} 4 \\ 8 \\ 1 \end{pmatrix}, \frac{1}{9} \begin{pmatrix} 4 \\ -1 \\ -8 \end{pmatrix}, \frac{1}{81} \begin{pmatrix} -63 \\ 36 \\ -36 \end{pmatrix} \right)$

Exempel: Uttryck vektorer i varandra (2.5.8.b)

Uttryck w i u och v

$|u| = 6,\ |v| = 8, |w| = 7,\ u$ och v bildar vinkeln $\frac{\pi}{6}$,

u och w vinkeln $\frac{\pi}{2}$ och v och w vinkeln $\frac{2\pi}{3}$

Altså ska hitta $\vec{w} = k_1\vec{u} + k_2\vec{v},\ k_1, k_2 \in \mathbb{R}$

Vi ser att $k_2 < 0$ enlight bild, därmed får vi följande

$-k_2 \cdot \cos\frac{\pi}{3} = 7 \Rightarrow -k_2 \cdot \frac{1}{2} \cdot 8 = 7 \Rightarrow k_2 = \frac{7}{4}$

Beräknarn k_1

$6k_1 = \frac{7}{4} \cdot 8 \cdot \sin\frac{\pi}{3} \Rightarrow k_1 = \frac{7}{6}\sqrt{3}$

$\vec{w} = \frac{7\sqrt{3}}{6}\vec{u} - \frac{7}{4}\vec{v}$

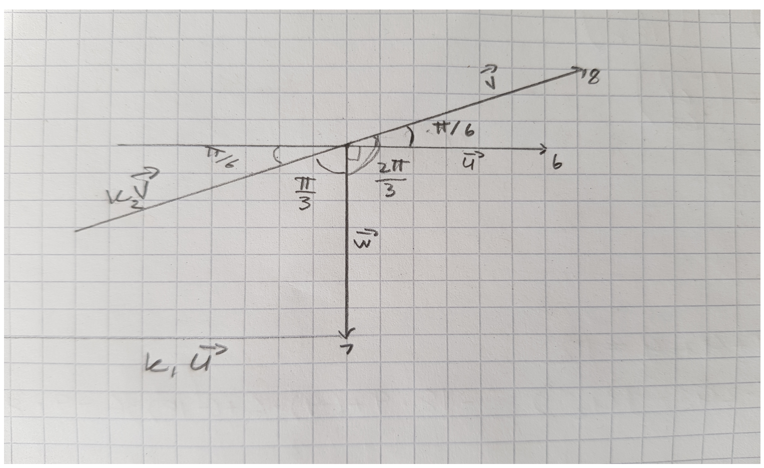


Figure 7.3: 2.5.8.b

7.3.5 Area och Volym

Area parralelogram: $|\vec{u}||\vec{v}|\sin\theta = |\vec{u}\times\vec{v}|$
Volym parralelogram: $|(\vec{u}\times\vec{v})\bullet\vec{w}|$

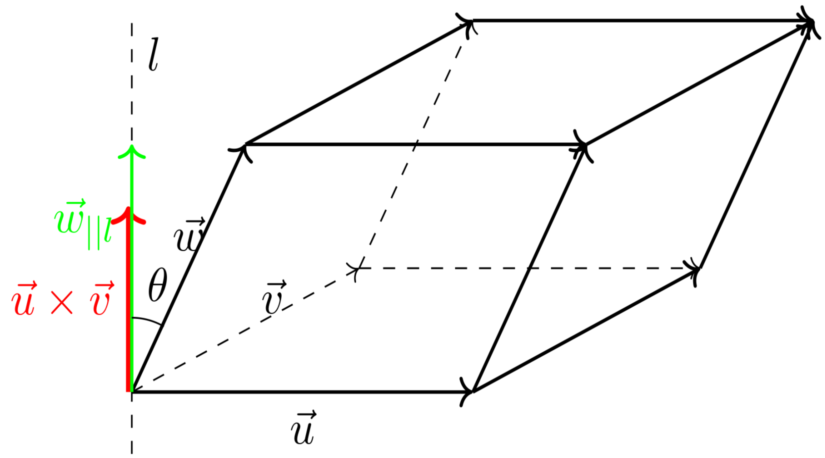


Figure 7.4: Ortogonal projektion

Exempel: Hitta volymen

Hitta volymen av parallellipipeden med sidorna

$\vec{u} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\vec{v} = \begin{pmatrix}2\\1\\1\end{pmatrix}$ $\vec{w} = \begin{pmatrix}1\\1\\2\end{pmatrix}$

och avgör om $(\vec{u}\vec{v}\vec{w})$ är ett högersystem,
ett vänstersystem, eller ej en bas.

Lösning: Vi beräknar

$\vec{u}\times\vec{v} = \begin{pmatrix}1\cdot 1 - 1\cdot 1\\1\cdot 2 - 1\cdot 1\\1\cdot 1 - 1\cdot 2\end{pmatrix} = \begin{pmatrix}0\\1\\-1\end{pmatrix}$

$(\vec{u}\times\vec{v})\bullet\vec{w} = 0 + 1 - 2 = -1$

Så det är ett vänstersystem och volymen är 1

7.4 Linjer och plan

$$\text{Parameterform: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, t \in \mathbb{R}$$

$$\text{Normalform: } ax + by = c \Rightarrow \vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\text{Plan: } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + t \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + s \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Exempel: Skriv i parameterform

Om vi lösar ekvationen (vars lösningar är en linje): $x + 3y = 4$;

$y = t \Rightarrow x = 4 - 3t \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Hitta parameterform för linje genom givna punkter

$A = \begin{pmatrix} -21 \\ 20 \end{pmatrix}, B = \begin{pmatrix} -24 \\ -22 \end{pmatrix}$

$\overline{AB} = \overline{OB} - \overline{0A} = \begin{pmatrix} -24 - (-21) \\ -22 - 20 \end{pmatrix} = \begin{pmatrix} -3 \\ -42 \end{pmatrix} \Rightarrow$

$L : \underline{e} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{e} \begin{pmatrix} -21 \\ 20 \end{pmatrix} + t\underline{e} \begin{pmatrix} -3 \\ -42 \end{pmatrix}$

Exempel: Hitta planets ekvation

Uppgift: Hitta en ekvation för planet som
går genom $(1, 2, 3)$ och är parallell med vektorna

$\vec{u} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \vec{v} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$

Lösnig: Vi måste hitta en vektorn $\vec{n}$ som är ortogonal mot planet
Inte så lätt i 3D som för linje i 2D. Men vi har en formel.

$\vec{n} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \times \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 45 - 48 \\ 42 - 36 \\ 32 - 35 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$

Det kan vi skriva på normal formen: $-3x + 6y - 3z = c$

För att hitta c så insätter vi det kända punkten $(1, 2, 3)$

$c = -3 \cdot 1 + 6 \cdot 2 - 3 \cdot 3 = 0$

Vi får formen: $-3x + 6y - 3z = 0$

Exempel: Beskriva linje på normalform

Uppgift: Hitta normalvektorn för linje $A = (16, 4), B = (9, 12)$

$$\overline{AB} = \begin{pmatrix} 9-16 \\ 12-4 \end{pmatrix} = \begin{pmatrix} -7 \\ 8 \end{pmatrix}$$

$$L : \overline{OA} + t\overline{AB} = \begin{pmatrix} 16 \\ 4 \end{pmatrix} + t\begin{pmatrix} -7 \\ 8 \end{pmatrix}$$

$$\begin{cases} x = 16 - 7t \\ y = 4 + 8t \end{cases}$$

$$t = \frac{16-x}{7} = \frac{y-3}{8} \Leftrightarrow 128 - 8x = 7y - 21$$

$$8x + 7y = 149$$

Exempel: Skärning mellan plan

$$\begin{cases} -3x + y + 4z = 4 \\ x - 4y - 5z = -5 \end{cases}$$

$$\begin{cases} 0 - 11y - 11z = -11 \\ x - 4y - 5z = -5 \end{cases}$$

$$\begin{cases} 0 + y + z = 1 \\ x - 4y - 5z = -5 \end{cases}$$

$$\begin{cases} x + 0 - z = -1 \\ 0 + y + z = 1 \end{cases}$$

$$\begin{pmatrix} t-1 \\ 1-t \\ t \end{pmatrix}$$

Exempel: Skärning mellan plan och linjen

$$3x + 3y + 4z = -7$$

$$\begin{cases} x = 2 - 3t \\ y = 1 - 3t \\ z = 3t \end{cases}$$

$$3(2-3t) + 3(1-3t) + 4(3t) = -7 \Rightarrow -6t = -16 \Rightarrow t = \frac{8}{3}$$

$$\begin{cases} x = 2 - 3t = -6 \\ y = 1 - 3t = -7 \\ z = 3t = 8 \end{cases}$$

7.4.1 ortogonal projektion på plan

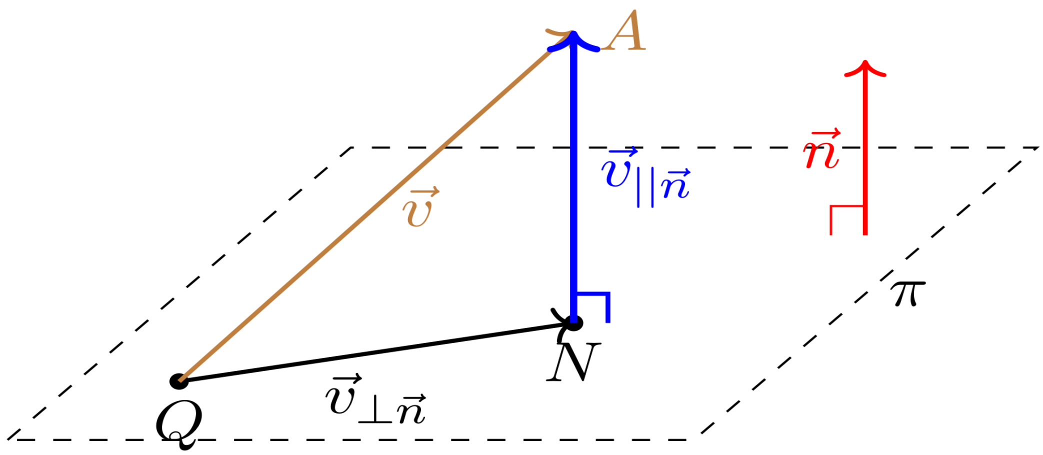


Figure 7.5: Ortogonal projektion på plan

Exempel: Hitta närmaste punkten i planet

Uppgift: hitta närmaste punkten N till punkten $A = (1, -5, 2)$
i planet $P : x + 2y - z = 1$ Hitta även avståndet mellan A och planet

Lösning: Först väljer vi godtycklig punkt $Q = (1, 0, 0)$ i planet, och beräknar

$\vec{v} = \overrightarrow{QA} = \begin{pmatrix} 1-1 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ 2 \end{pmatrix}$

Normalvektor till planet $x + 2y - z = 1 \vec{n} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

$\vec{v}_{||\vec{n}} = \dfrac{0 \cdot 1 + (-5) \cdot 2 + 2 \cdot (-1)}{1^2 + 2^2 + (-1)^2} \vec{n} = -2\vec{n}$

Så vi får koordinaterna:

$\overrightarrow{ON} = \overrightarrow{OA} - \overrightarrow{AN} = \overrightarrow{OA} - \vec{v}_{||\vec{n}} = \begin{pmatrix} 1 \\ -5 \\ 2 \end{pmatrix} - (-2)\vec{n} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$

så närmaste punkten är $N = (3, -1, 0)$ Avståndet är $|-2\vec{n}| = 2\sqrt{6}$

Exempel: Hitta närmaste punkten på linje till linje

Hitta den punkt på linjen

$$l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$$

som är närmast linjen

$$k : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \ s \in \mathbb{R}$$

$A = (-1+t, t, t)$, $B = (-1+s, 2s, -1+s)$

$$\overrightarrow{AB} = \begin{pmatrix} (-1+s)-(-1+t) \\ (2s)-(t) \\ (-1+s)-(t) \end{pmatrix} = \begin{pmatrix} s-t \\ 2s-t \\ -1+s-t \end{pmatrix}$$

Vektorn ska vara ortogonal mot: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{cases} 1(s-t)+1(2s-t)+1(-1+s-t)=0 \\ 1(s-t)+2(2s-t)+1(-1+s-t)=0 \end{cases} = \begin{cases} -1+4s+-3t=0 \\ -1+6s+-4t=0 \end{cases}$$

$$\Rightarrow s = \frac{3t+1}{4} \Rightarrow -1 + \frac{3}{2}(3t+1) - 4t = 0$$

$$\begin{cases} t=-1 \\ s=-1/2 \end{cases} \text{(Kontrollera)} -1-2+3=0,\ -1-3+4=0$$

$$A = (-2,-1,-1),\ B = (-3/2,-1,-3/2) \Rightarrow \overrightarrow{AB} = \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

Svar: punkten på linjen l är $A = (-2,-1,-1)$

7.5 Matrisräkning

Begräpp: matriser

diagonal
huvuddiagonalen
Kummuterar: $AB = BA$
$A = (a_{ij})_{r\times k}$
Rang: antalet ledande kofisent i trappsteksmatris

Räkneregler: matriser

Addition: endast i sama form
Multiplication: $(r \times k)(k \times m) \Rightarrow r \times x$
$(AB)C = A(BC)$
$\lambda(AB) = (\lambda A)B = A(\lambda B)$
$A(B + C) = AB + AC$
$(B + C)A = BA + CA$

$$
\begin{array}{cc}
\boxed{\begin{array}{c}\text{Villkor:}\\ \text{Kvadratisk}\end{array}} & \overbrace{\begin{pmatrix} 1 & 1 & 5 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}}^{B} \\
\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 5 \end{pmatrix}}_{A} & \underbrace{\begin{pmatrix} 3 & 1 & 9 \\ 12 & -2 & 28 \end{pmatrix}}_{AB}
\end{array}
$$

Figure 7.6: Ortogonal projektion på plan

Exempel: Multiplication av matriser

$$
\begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 2 \\ -5 & -2 & 1 \end{pmatrix}
\begin{pmatrix} a \\ b \\ c \end{pmatrix}
=
\begin{pmatrix} 2a & 3b & 4c \\ -a & 2b & 2c \\ -5a & -2b & c \end{pmatrix}
$$

Definition: Enhetsmatrisen

$$\begin{aligned}
&\text{Enhetsmatrisn } I_n \\
&A^0 I_n = I_n \\
&A : 2 \times 2 A I_2 = A \\
&I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

7.5.1 Transpornat

Definition: Transponaten

$$\begin{aligned}
&A^t \text{ betyder inte A upphöjt till t} \\
&A = (a_{ij})_{r \times k} \Rightarrow A^t = A = (\alpha_{ij})_{k \times r} \\
&A = (\alpha_{ij}) = (a_{ij})
\end{aligned}$$

Exempel: Transponaten

$$\begin{aligned}
&\text{Symetrisk } A = A^t \\
&\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}^t = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}
\end{aligned}$$

Räkneregler: Transponaten

$$\begin{aligned}
&(A + B)^t = A^t + B^t \\
&(\lambda A)^t = \lambda A^t \\
&(A^t)^t = A \\
&(AB)^t = B^t A^t
\end{aligned}$$

7.5.2 Matrisinvers

Räkneregler: Matrisinvers

$$\begin{aligned}
&AB = BA = I \\
&\text{Inversen finns endast om matrisen är kvadratisk} \\
&A \text{ är inventerba} \\
&AX = B \text{ har entydiga lösningar för alla B} \\
&AX = 0 \text{ har enbart lösningen} X = 0 \\
&A \text{ har rang} n \\
&A \sim I \text{ (är radekvivalent med) } I
\end{aligned}$$

Exempel: Matrisinvers 3x3

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right) \sim$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 3/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{array}\right) \sim$$

$$A^{-1} = \begin{pmatrix} -1/2 & 3/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 & 3 & -1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

Räkneregler: Matrisinvers

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

7.6 Determinanter

Ekvations system har unika lösningar då koefficient matrisen är inventerbar. Som är ekvivalent med att determinanten är nollskild.

Exempel: determinant många led genväg

$$\begin{pmatrix} 3 & 2 & -3 & 24 & 1005 \\ 0 & 1 & 23 & 14 & 15 \\ 0 & 0 & 3 & 7 & -5 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Vi det finns endast två producter som inte inehåller en nol faktor

$$\begin{pmatrix} (3) & 2 & -3 & 24 & 1005 \\ 0 & (1) & 23 & 14 & 15 \\ 0 & 0 & (3) & (7) & -5 \\ 0 & 0 & (2) & (4) & 3 \\ 0 & 0 & 0 & 0 & (5) \end{pmatrix} = 3 \cdot 1 \cdot 3 \cdot 4 \cdot 5 - 3 \cdot 1 \cdot 6 \cdot 2 \cdot 5 = 0$$

Exempel: determinant 3x3

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} =$$

$$= 1 \cdot 5 \cdot 9 - 1 \cdot 6 \cdot 8 - 2 \cdot 4 \cdot 9 +$$

$$+ 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 3 \cdot 5 \cdot 7 =$$

$$= 45 - 48 - 72 + 84 + 96 - 105 = 0$$

Sats:

Om B är matrisen A där man har bytt om på rad i och j är $detA = -detB$

(1) Bytt två rader om i A ändras determinaten sitt teken

(2) Skalas en rad om med λ skalas också determinaten om med λ

(3) Addera en rad gånger något på en annan rad i A ändrar inte dennas determinant

(4) $det(A) = det(A^t)$

$det(AB) = det(A)det(B)$

Exempel: determinant 3x3

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 8 & 7 \\ 1 & 3 & 6 & 5 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 3 & 8 & 7 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 3 \end{pmatrix} = - \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 3 & 8 & 7 \\ 0 & 0 & 2 & 3 \end{pmatrix} = - \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 2 & 3 \end{pmatrix}$$

$$- \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 11 \end{pmatrix} = -1 \cdot 1 \cdot (-1) \cdot 11 = 11$$

Exempel: okännt x (determinant)

$$\begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ x & x^2 & x^4 & x^5 \\ x^2 & x^3 & x^4 & x^6 \end{pmatrix}$$

$$x \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ x^2 & x^3 & x^4 & x^6 \end{pmatrix} = \text{rad 4 -rad 1} x^3 \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ 1 & x & x^2 & x^4 \end{pmatrix} = x^3 \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ 0 & 0 & 0 & x \end{pmatrix}^{R4} =$$

$$\text{rad 3 -rad 1} x^3(x^4 - x^3) \begin{pmatrix} 1 & x & x^2 \\ 1 & x^2 & x^3 \\ 1 & x & x^3 \end{pmatrix} = x^6(x-1)^2 \begin{pmatrix} 1 & x & x^2 \\ 1 & x^2 & x^3 \\ 0 & 0 & x \end{pmatrix} =$$

$$x^6(x-1)(x^3 - x^2) \begin{pmatrix} 1 & x \\ 1 & x^2 \end{pmatrix} = x^9(x-1)^3 = 0$$

7.6.1 Ko-faktorna

$det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}$ där a_{ij} är teknet o C_{ij} är kofisienten

$\tilde{A}^t$ är alla kofaktorerna frånAC_{ij}

$det(a^-1) = det(1/det(a))$

$$\begin{pmatrix} 2 & 1 & 1 & -1 \\ (1) & (0) & (2) & (0) \\ 1 & -2 & 1 & 2 \\ 3 & 1 & -1 & 5 \end{pmatrix}^{R2} = (-1)^{2+1} \cdot 1 \cdot \begin{pmatrix} 1 & 1 & -1 \\ -2 & 1 & 2 \\ 1 & -1 & 5 \end{pmatrix}$$

$$+ (-1)^{2+2} \cdot 0 \cdot \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 3 & -1 & 5 \end{pmatrix} + (-1)^{2+3} \cdot 2 \cdot \begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 2 \\ 3 & 1 & 5 \end{pmatrix} + (-1)^{2+4} \cdot 0 \cdot \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 1 & -1 \end{pmatrix}$$

$$-1(5+2-2-(-1-2-10)) + 0 - 2(-20+6-1-(6+4+5)) + 0 = -18 - 2 \cdot (-30) = 42$$

7.6.2 Geometri: parallellepiped

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \right)$$

7.7 Vektorer i $\mathbb{R}^n$

Skallär produkt och ärmed längd är samma regler som innan
Vektor mellan punkter är också samma regler
Vinkeln är samma som innan (ortogonal projection)
Vinkel formel: $\theta = \arccos \frac{\vec{v}\vec{w}}{|\vec{v}||\vec{w}|}$
(Cauchy-Schwarz olikheten) logist! $|\vec{v} \bullet \vec{w}| \leq |\vec{v}||\vec{w}|(|3-1| \leq |3||-1|)$

Linjärt oberoende: inga parametrar vid lösning av $c_1\vec{v_1} + c_2\vec{v_2} + \ldots + c_n\vec{v_n} = \vec{0}$
Linjära Höljet (spannet): mängden av alla möjliga linjärakombinationer av vektorerna
Det linjära höljet av vektorerna är hela $\mathbb{R}^n$ omm $RangV = n$

Bas definnerar lika dant i fler dimentioner, för att vara en bas $\mathbb{R}^n$
måste de vara linjärt oberoende och det linjära höljet är hela $\mathbb{R}^n$

7.8 Linjära avbildningar $\mathbb{R}^n \to \mathbb{R}^m$

7.8.1 Matristransformationer och linjära funktioner

Termenologi

Standardmatris: En matris som multipliseras med argumentet för att få svaret
Sammansätning: Låt $f : A \to B, g : B \to C$
$(g \circ f)(x) = g(f(x))$
Vektor produkt: $F : \mathbb{R}^3 \to \mathbb{R}^3$
$F(\vec{x}) = \vec{a} \times \vec{x}$

Definition: En funktion $T : \mathbb{R}^k \to \mathbb{R}^n$ kallas linjär om den uppfyller:

För alla $\vec{v}, \vec{w} \in \mathbb{R}^k$ och $\lambda \in \mathbb{R}$ gäller
$(i)\quad T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$
$(ii)\quad T(\lambda\vec{v}) = \lambda T(\vec{v})$

Formel standard matris

$$[T]\begin{pmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_k} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ T(\vec{v_1}) & T(\vec{v_2}) & \dots & T(\vec{v_k}) \\ | & | & & | \end{pmatrix}$$

Exempel: hitta standard matris

Hitta standardmatrisen $[T]$ för den linjära funktionen $T : \mathbb{R}^2 \to \mathbb{R}^3$ som uppfyller att

$$T\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ -8 \end{pmatrix} \wedge T\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -4 \end{pmatrix}$$

$$[T]\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 8 & -1 \\ 8 & 0 \\ -8 & -4 \end{pmatrix}$$

$$[T] = \begin{pmatrix} 8 & -1 \\ 8 & 0 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \\ 2 & 2 \end{pmatrix}$$

Kontrol: multiplisera standard matrisen med input och få output

Exempel: hitta standard matris ortogonal projection

Hitta standardmatrisen för $P : \mathbb{R}^3 \to \mathbb{R}^3$
där P är den ortogonala projektionen på planet$\pi : 2x + y + 3z = 0$

$$T(\vec{n}) = T \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hittar två godtykliga punkter på planet: $A = (1, 1, -1)$, $B = (-1, 2, 0)$

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$[T] \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \Rightarrow$$

$$[T] = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 5/7 & -1/7 & -3/7 \\ -1/7 & 13/14 & -3/14 \\ -3/7 & -3/14 & 5/14 \end{pmatrix}$$

Kontrol: multiplisera standard matrisen med input och få output

Exempel: hitta standard matris spegling

Hitta standardmatrisen för $P : \mathbb{R}^3 \to \mathbb{R}^3$
där P är speglingen i planet$\pi : -x + 2y - 2z = 0$

$$T(\vec{n}) = T \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

Hittar två godtykliga punkter på planet: $A = (0, 1, 1)$, $B = (2, 1, 0)$

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$[T] \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \Rightarrow$$

$$[T] = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \\ -2 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 7/9 & 4/9 & -4/9 \\ 4/9 & 1/9 & 8/9 \\ -4/9 & 8/9 & 1/9 \end{pmatrix}$$

Kontrol: multiplisera standard matrisen med input och få output

7.8.2 Injektiv/Surjektiv/Bijektiv

Termenologi

[T]	kolonnvektorerna i [T]	funktionen T
rang([T])=k	linjär oberoende	injektiv
rang([T])=n	spannet är hela $\mathbb{R}^n$	surjektiv
n=rang([T])=k	är en bas för $\mathbb{R}^n = \mathbb{R}^k$	bijektiv

Chapter 8

Linear Algebra II

8.1 Grudläggande teori

8.1.1 Ekvationssystem och matris räkning

$$\begin{cases} x+2y+ \quad z=-1 \\ 2x+(a+3)y+ \quad 3z=-4 \\ x+(3-a)y+(a-2)z \quad =a-1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & a+3 & 3 & -4 \\ 1 & 3-a & a-2 & a-1 \end{array}\right)$$

$(-1 \text{ rad1 till rad2}), (-2 \text{ rad1 till rad3})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 2 & a-1 & 1 & -2 \\ 0 & 1-a & a-3 & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a-1 & 1 & -2 \\ 0 & 0 & a-2 & a-2 \end{array}\right)$$

$a \neq 1 \wedge a \neq 2$

$(\frac{1}{a-2}\text{rad 3})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 0 & a-1 & 1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(-1\text{rad 3 till rad 2}), (-1\text{rad 3 till rad 1})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & a-1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(\frac{1}{a-1}\text{rad 2})$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & a-1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & 0 & \frac{3}{1-a} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$(-2\text{rad 2 till rad 3})$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2-\frac{6}{1-a} \\ 0 & 1 & 0 & \frac{3}{1-a} \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\begin{cases} x & = -2-\frac{6}{1-a} \\ y & = \frac{3}{1-a} \\ z & = 1 \end{cases}$$

Kontroll: stoppar in x,y,z i ekvationerna

$$\begin{cases} \left(\frac{2a-8}{1-a}\right) + \quad 2\cdot\left(\frac{3}{1-a}\right) + \quad 1 = -1 \\ 2\cdot\left(\frac{2a-8}{1-a}\right) + (a+3)\cdot\left(\frac{3}{1-a}\right) + \quad 3\cdot 1 = -4 \\ \left(\frac{2a-8}{1-a}\right) + (3-a)\cdot\left(\frac{3}{1-a}\right) + (a-2)\cdot 1 = a-1 \end{cases}$$

8.1.2 determinanter

$$\begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ x & x^2 & x^4 & x^5 \\ x^2 & x^3 & x^4 & x^6 \end{pmatrix}$$

$$x \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ x^2 & x^3 & x^4 & x^6 \end{pmatrix} = \text{rad 4 -rad 1} x^3 \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ 1 & x & x^2 & x^4 \end{pmatrix} = x^3 \begin{pmatrix} 1 & x & x^2 & x^3 \\ 1 & x^2 & x^3 & x^4 \\ 1 & x & x^3 & x^4 \\ 0 & 0 & 0 & x \end{pmatrix}^{R4} =$$

$$\text{rad 3 -rad 1} x^3(x^4 - x^3) \begin{pmatrix} 1 & x & x^2 \\ 1 & x^2 & x^3 \\ 1 & x & x^3 \end{pmatrix} = x^6(x-1)^2 \begin{pmatrix} 1 & x & x^2 \\ 1 & x^2 & x^3 \\ 0 & 0 & x \end{pmatrix} =$$

$$x^6(x-1)(x^3 - x^2) \begin{pmatrix} 1 & x \\ 1 & x^2 \end{pmatrix} = x^9(x-1)^3 = 0$$

8.1.3 flerdimisionel dvs R^n

räkneregler

8.1.4 Funktioner

polynom functioner vid en viss grad också

8.1.5 Linjer

$$l : \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Då är riktnings vektorn $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

Och går genom $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

8.2 Vektorrum

Definition: vektor rum

En mängd $\mathbb{V}$ kallas för en reellt vektorrum om
1. Det finns en operator på $\mathbb{V}$ som kallas addition och beräknas med $+$, sådant att om $\vec{u}, \vec{v} \in \mathbb{V}$ så gäller $\vec{u} + \vec{v} \in \mathbb{V}$
2. Det finns en poeration på $\mathbb{V}$ som kallas skalning eller multipliseras med reella tal, som beteknas med $\cdot$, sådan att om $\lambda \in \mathbb{R} \wedge \vec{v} \in \mathbb{V}$ så gäller $\lambda \cdot \vec{v} \in \mathbb{V}$

räknereglerna gäller som i la1

Axiomen: vektor rum

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Kommutativ lag)
2. $\vec{u}(\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (Associativ lag)
3. Det finns ett nollement $\vec{0}$ så att $\vec{v} + \vec{0} = \vec{v}$
4. Till varge $\vec{v} \in \mathbb{V}$ finns ett element $-\vec{v}$ så att $\vec{v} + (-\vec{v}) = 0$
5. $1 \cdot \vec{v} = \vec{v}$
6. $\lambda \cdot (\mu \cdot \vec{v}) = (\lambda \cdot \mu) \cdot \vec{v}$ (Associativ lag)
7. $(\lambda + \mu) \cdot \vec{v} = \lambda \cdot \vec{v} + \mu \cdot \vec{v}$ (Distributiv lag)
8. $\lambda \cdot (\vec{u} + \vec{v}) = \lambda \cdot \vec{u} + \lambda \cdot \vec{v}$ (Distributiv lag)

$\forall \lambda, \mu \in \mathbb{R} \wedge \vec{u}, \vec{v}, \vec{w} \in \mathbb{V}$

8.3 Underrum och linjära höljet

Ett underrum är en delmängd u ej 0 som är sluten under addition och skalning

Definition: Underrum

En delmängd $\mathbb{U}$ av ett vektorrum $\mathbb{V}$ kallas för
ett underrum eller delrum av $\mathbb{V}$ om $\mathbb{U}$ är ett vektorrum
med den additionoch den multiplikation med reella tal som definierats i $\mathbb{V}$.

Sats: Underrum

En icketom mängd $\mathbb{U}$ av ett vektorrum $\mathbb{V}$ är ett
underrum om och endast om följande gäller
1. Om $\vec{u} \in \mathbb{U}$ och $\vec{v} \in \mathbb{U}$, Så är $\vec{u} + \vec{v} \in \mathbb{U}$
2. Om $\vec{u} \in \mathbb{U}$ och $\lambda \in \mathbb{R}$, Så är $\lambda\vec{u} \in \mathbb{U}$

Regler: Underrum

1. Det snabbaste sättet att testa om det gäller är om nollvektorn finns
om den inte gör det så bryter det mot 2-lagen
2. Homogena ekvationer och linjer/plan genonom origo gäller det alltid för
3. Kontunerligt deriverbara, så gäller det att det är ett underrum

Exempel: Underrum (Ex 1)

$$\text{är } \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z = 5 \right\} \text{ ett underrum?}$$

Nej, då: $0 + 0 + 0 \neq 5$

Definition: linjära höljet

En delmängd $\mathbb{U}$ av ett vektorrum $\mathbb{V}$ kallas för
ett underrum eller delrum av $\mathbb{V}$ om $\mathbb{U}$ är ett vektorrum
med den addition och den multiplikation med reella tal som definierats i $\mathbb{V}$.

Exempel: Underrum (Ex 4)

Vilka vektrorer i $\mathbb{P}$ tillhör $u = [1, 1 + x, 1 + x + x^2]$

Svar: $u = P_2$ (eftersom) $1 \in u, x \in u,$ och $x^2 \in u$
Då alla $a \cdot 1 + b \cdot x + c \cdot x^2 \in u$
Inga plynom av grad > 2 tillhör u.
Vi för kombinera det olika elementen i u med addition och multiplication

8.4 Linjärt *o*beroende

Ide: linjärt beronde

Om vektorerna kan skrivas om en linjär kombination av det andra vektorerna
så är vektorn linjärt beroende då det inte behöver vektorn
vi får reda på beoende genom att ställa upp ekv
$x_1\vec{v_1} + x_2\vec{v_2} + \ldots + x_n\vec{v_n} = \vec{0}$
Om denna ekvation har icke triviala lösningar (ej noll) då är den beroende

Exempel: om linjärt oberonde

Är mängden $\left\{ \begin{pmatrix}1\\2\end{pmatrix} \begin{pmatrix}2\\1\end{pmatrix} \right\}$i $\mathbb{R}^2$ linjärt oberoende
$x_1\begin{pmatrix}1\\2\end{pmatrix} + x_2\begin{pmatrix}2\\1\end{pmatrix} = \vec{0} \Leftrightarrow \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 0 \end{array} \right)$
$x_1 = 0, x_2 = 0$Endast triviala lösningar

8.5 Bas

Ide: Bas och Dimention

Bas omm om vektorerna spanner upp hella spannet och det linjärt oberonde
Standard basen $\underline{e}$
Vi behöver tree vektorer för att utgöra en bas i $\mathbb{R}^3$ är $\vec{v_1}, \vec{v_2}, \vec{v_3} \in \mathbb{R}^3$

Dimentioner är antallet oberonde vektorer som spänner up "rummet"
"dimentioner blir då samma som antallet element som en bas av polynom"
$\mathbb{P}_3 = a + bx^1 + cx^2 + dx^3$ har 4 dimentioner
$\mathbb{R}^3$ har 3 dimentioner
Dimentioner för matriser är $n \times m(2 \times 2 - matris = dim4)$
En bas med tre vektorer där vektorerna $\mathbb{R}^4$ ger 3 dim

Exempel: Ta fram bas från plan

låt $2x - yz = 0$

Lösning: $x = y/2 - z/2$
Låt $y = s$, $z = t$ vi får då lösningen på parameter form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s/2 - t/2 \\ s \\ t \end{pmatrix} = s/2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t/2 \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Basen blir då $\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \right)$

8.6 Basomvandling

Vi kan ta inversen av matrisen av vektorerna som utger en bas och få matrisen som kan multipliseras med vektor för att få omvandlingen.

Sats: basbyte

$$\vec{v}_{\underline{e}} = Tv_f \text{ Där T är en matris}$$
$$T_{\underline{e}}^{f} = (T_{\underline{f}}^{e})^{-1}$$
$$\vec{v}_{\underline{f}} = T_{\underline{f}}^{e}\,\vec{v}_{\underline{e}}$$

Exempel: Från standar till annan bas

$$\text{Låt } \underline{f} = (f_1\, f_2) = \left(\begin{pmatrix}2\\1\\1\end{pmatrix} \begin{pmatrix}1\\2\\1\end{pmatrix}\right), \vec{v} = \begin{pmatrix}-2\\5\\1\end{pmatrix}$$

Utryck $\vec{v}$ i basen $\underline{f}$

$$\vec{v}_{\underline{e}} = \begin{pmatrix}-2\\5\\1\end{pmatrix} = x_1 \begin{pmatrix}2\\1\\1\end{pmatrix} + x_2 \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\begin{cases} 2x_1 + x_2 = -2 \\ x_1 + 2x_2 = 5 \\ x_1 + x_2 = 1 \end{cases} \Rightarrow \left(\begin{array}{cc|c} 2 & 1 & -2 \\ 1 & 2 & 5 \\ 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 2 & 1 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_2 = 4,\ x_1 = 1/2(-2-4) = -3 \Rightarrow v_{\underline{f}} = \begin{pmatrix}-3\\4\end{pmatrix}$$

Exempel: Finn matrisen för bas byte vektorer

$$\text{Låt } \underline{f} = \left(\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}2\\1\end{pmatrix}\right) \text{Bestäm bas omvandling matrisen}$$

$$\text{Lös } T_{\underline{f}}^{e} = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

Exempel: Finn matrisen för bas byte polynom

$$\text{Låt } \underline{e} = (1\, x) \text{ och } \underline{f} = (1\, 1-x)$$

$$\vec{f}_1 = 1 = \vec{e}_1 \Rightarrow \vec{f}_{1e} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\vec{f}_2 = 1 - x = \vec{e}_1 - \vec{e}_2 \Rightarrow \vec{f}_{2e} = \begin{pmatrix}1\\-1\end{pmatrix}$$

$$T_{\underline{e}}^{f} = \begin{pmatrix}1 & 1\\0 & -1\end{pmatrix} \Rightarrow (T_{\underline{e}}^{f})^{-1} T_{\underline{f}}^{e} =$$

$$\begin{pmatrix}1 & 1\\0 & -1\end{pmatrix}$$

Exempel: Finn matrisen för bas byte mellan olika matriser

Basbyte mellan $\underline{f} = \left(\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\right)$ och $\left(\begin{pmatrix}-1\\1\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}\right)$

Lös $T^{\underline{g}}_{\underline{f}}$: $\left(\begin{array}{cc|cc} 1 & 2 & -1 & 3 \\ 1 & 1 & 1 & 2 \end{array}\right) \sim$

$\left(\begin{array}{cc|cc} 1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right) \Rightarrow \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}$

Definition: Nollrum, Kolonrum, radrum.

låt A vara en $m \times n - matris$

1. dim(A:s kolonrum) = dim(A:s radrum) = rang A
2. dim(A:s kolonrum) + dim(A:s nollrum) = n

Exempel: Finn Nollrum, Kolonrum, radrum.

Hitta en bas för kolonrummet, radrummet och nollrumet till följande matris. Bestäm även rummens dimensioner

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 \\ 1 & 3 & 4 & 5 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$

Låt kolonernana vara vektorer, då får vi följande gäller

$$x_1v_1 + x_2v_2 + x_4v_3 + x_4v_4 = 0 \Leftrightarrow (v_1\, v_2\, v_3\, v_4)\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = 0 \Leftrightarrow Ax = 0$$

Gausselimination ger oss $A \sim \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-s+t\\-s-2t\\s\\t\end{pmatrix} = s\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} + t\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}$$

Basen för nollrumet blir då $\left(\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}\begin{pmatrix}1\\-2\\0\\1\end{pmatrix}\right)$

Basen för kolonrumet blir då det vektorer i A som har Pivot element dvs v_1, v_2

$$\left(\begin{pmatrix}1\\1\\-1\end{pmatrix}\begin{pmatrix}2\\3\\0\end{pmatrix}\right)$$

Basen för radrumet blir $\left(\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}\begin{pmatrix}0\\1\\1\\2\end{pmatrix}\right)$

8.7 Linjär avbildning

Definition: Linjär avbilding

låt $\mathbb{V}$ och $\mathbb{W}$ vara vektorrum. En funktion $F : \mathbb{V} \to \mathbb{W}$ kallas för linjär avbildning omm

1. $F(\vec{v} + \vec{w}) = F(\vec{v}) + F(\vec{w})$
2. $F(\lambda \cdot \vec{v}) = \lambda \cdot F(\vec{v})$

Exempel: Derivatan av polynom

Är följande en linjär avbildning

$F : C^1(a, b) \to C(a, b)$ definerad genom $F(f) = \dfrac{df}{dx}$

$\big(\dfrac{d}{dx}(af(x) + bg(x))\big) = a\dfrac{df}{dx} + b\dfrac{dg}{dx}$

$(\dfrac{d}{dx}(\lambda af(x))) = \lambda(\dfrac{d}{dx}(af(x)))$

Svar: ja F är en linjär avbilding

8.8 Matrisen av en linjär avbilding

$$\begin{tikzcd}
\mathbb{V} \arrow[r, "F"] & \mathbb{W} \arrow[d, "(?)_{\underline{\mathbf{w}}}"] \\
\mathbb{R}^n \arrow[u, "\underline{\mathbf{v}}\cdot ?"] \arrow[r, dashed, "(F)^{\underline{\mathbf{v}}}_{\underline{\mathbf{w}}}"] & \mathbb{R}^m
\end{tikzcd}$$

$$(F)^{\underline{\mathbf{v}}}_{\underline{\mathbf{w}}}(\mathbf{x})_{\underline{\mathbf{v}}} = (F(\underline{\mathbf{v}}(\mathbf{x})_{\underline{\mathbf{v}}})_{\underline{\mathbf{w}}} = (F(\mathbf{x}))_{\underline{\mathbf{w}}}$$

Figure 8.1: Matris av en linjär avbildning. From ()

Exempel: Rotations matrisen

$$\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}$$

Exempel: Hitta matrisen

Linjär avbildning$F : \mathbb{P}_3 \to \mathbb{P}_3,\ F = p + p' + p'' + p'''$Ange F:s matris i standarbasen.

Avbildar varge ellement i standardbasen $(1\, x\, x^2\, x^3)$
$F(1) = 1 + 1' + 1'' + 1''' = 1$
$F(x) = x + x' + x'' + x''' = x + 1$
$F(x^2) = x^2 + x^{2'} + x^{2''} + x^{2'''} = x^2 + 2x + 2 \cdot 1$
$F(x^3) = x^3 + x^{3'} + x^{3''} + x^{3'''} = x^3 + 3x^2 + 6x + 6 \cdot 1$
$$\begin{pmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

8.9 Basbyte av linjära avbildningar

Definition: Basbyte av linjära avblidningar

låt $F : \mathbb{V} \to \mathbb{W}$ vara linjär. Låt $\underline{e}$ och $\underline{v}$ vara baser i $\mathbb{V}$ och låt $\underline{f}$ och $\underline{w}$ vara baser av $\mathbb{W}$. Då gäller
$(F)^{\underline{e}}_{\underline{f}} = T^{\underline{w}}_{\underline{f}}(F)^{\underline{v}}_{\underline{w}}T^{\underline{e}}_{\underline{v}}$
Där vektorn kommer från höger (viktigt vid vilken ording transfomations matriserna står)

Ide: Vi omvandlar från en bas (ex standard bas) till en bas som är mer anpasat för uträkningen
Sedan omvandlar vi igen för att få svaret i den bas vi vill ha den i (ex standard bas)

Exempel: Basbyte av linjär avbildning vektor

Exempel: Basbyte av linjär avbildning polynom

Vad är matrisen av derivatan $\mathbb{P}_3 \to \mathbb{P}_2$ med avseende på basen $\underline{u} = (x^3\, x^2\, x\, 1) av \mathbb{P}_3$ och $\underline{v} = (x^2\, x\, 1)$ av $\mathbb{P}_2$?

$$(H)^{\underline{e}}_{\underline{f}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$T^{\underline{f}}_{\underline{v}}:$
$\vec{f}_1 = 1 = 0\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3$
$\vec{f}_2 = x = 0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3$
$\vec{f}_3 = x^2 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3$

$T^{\underline{u}}_{\underline{e}}:$
$\vec{u}_1 = x^3 = 0\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 + 1\vec{e}_4$
$\vec{u}_2 = x^2 = 0\vec{e}_1 + 0\vec{e}_2 + 1\vec{e}_3 + 0\vec{e}_4$
$\vec{u}_3 = x = 0\vec{e}_1 + 1\vec{e}_2 + 0\vec{e}_3 + 0\vec{e}_4$
$\vec{u}_4 = 1 = 1\vec{e}_1 + 0\vec{e}_2 + 0\vec{e}_3 + 0\vec{e}_4$

$$(H)^{\underline{u}}_{\underline{v}} = T^{\underline{f}}_{\underline{v}}(H)^{\underline{e}}_{\underline{f}}T^{\underline{u}}_{\underline{e}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

8.10 Kärna och bild av en linjär avbildning

Definition: Kärna och bild

Låt $F : \mathbb{V} \to \mathbb{W}$ vara en linjär avbilding

Kärnan eller **nollrummet** av den linjära avbildingen F
$\ker(F) = N(F) = \{\vec{v} \in \mathbb{V} | F(\vec{v} = \vec{0})\}$

Bilden eller värderummet av den linjära avbildingen F
$\mathrm{Im}(F) = V(F) = \{\vec{w} \in \mathbb{W} | \exists \vec{v} : F(\vec{v}) = \vec{w}\}$

Definition: Isomorfism, injektiv, surjektiv

$F : X \to y$ kallas
injektiv om $F(x) = F(x') \Rightarrow x = x'$
surjektiv om $\forall y \in Y \exists x \in X : f(x) = y$
bijektiv om det är både injektiv och surjektiv

Sats:

Låt $F : \mathbb{V} \to \mathbb{W}$ vara linjär
F är injektiv omm $\ker F = \{\vec{0}\}$
F är surjektiv omm $\mathrm{Im}\, F = \mathbb{W}$ omm $\mathrm{rang}((F)^{\underline{u}}_{\underline{w}}) = \dim \mathbb{W}$

Sats: Dimensionssatsen

För varje linjär avbilding $F : \mathbb{V} \to \mathbb{W}$ gäller
$\dim \mathbb{V} = \dim \ker(F) + \dim \mathrm{Im}(F)$

Sats: Isomorfism

Låt $F : \mathbb{V} \to \mathbb{W}$ vara en linjär avbilding. De följande är ekvivalenta
F är en isomorf
$\dim \mathbb{V} = \dim \mathbb{W}$ och F är injektiva
$\dim \mathbb{V} = \dim \mathbb{W}$ och F är surjektiva
$\det((F)^{\underline{v}}_{\underline{w}}) \neq 0$
$(F)^{\underline{v}}_{\underline{w}}$ är en kvadratisk matris av rang $\dim \mathbb{V} = \dim \mathbb{W}$

8.11 Egenvärden och egenvektorer

Definition: Egenvärde och egenvektor

Låt $F : \mathbb{V} \to \mathbb{W}$ vara en linjär avbilding. En vektor$\vec{v} \neq \vec{0}$
kallas en **egenvektor** till F om det finns $\lambda \in \mathbb{R}$ så att $F(\vec{v}) = \lambda\vec{v}$
I detta fall kallas λ en **egenvärde** till F
$\mathbb{V}_{F,\lambda} = \{\vec{v} \in \mathbb{V} | F(\vec{v}) = \lambda\vec{v}\}$kallas **egenrummet** av F till egenvärdet λ

Definition: Sekularpolynomet

För en matris A kallas polynomet $\chi_A(\lambda) = \det(A - \lambda I_n)$
sekularpolynomet till A
Om $A = (F)^{\underline{e}}_{\underline{e}}$, så kallas χ_A också sekularpolunomet till F

Exempel:

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \text{ matrisen i standardbasen är } \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$$

1. Hittar egenvärderna

$F(\vec{x}) = \lambda\vec{x} \Leftrightarrow A\vec{x} = \lambda\vec{x}$

$A\vec{x} - \lambda\vec{x} = \vec{0} \Leftrightarrow (A - \lambda I)\vec{x} = 0$

Där λ är egenvärdet, och $\vec{x}$ är egenvektorn

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{pmatrix}$$

$\det(A - \lambda I) = 4 - 2\lambda - (2-\lambda)(4 - 3\lambda + \lambda^2) = (2-\lambda)(-2 + 3\lambda - \lambda^2)$

$\det(A - \lambda I) = 0 \Leftrightarrow \lambda = 2 \vee \lambda = 1$

2. Hittar egenvektorerna

$\lambda = 2 :$

$$(A - 2I)\vec{x} \Leftrightarrow \left(\begin{array}{ccc|c} 1-2 & 2 & 2 & 0 \\ 0 & 2-2 & 1 & 0 \\ -1 & 2 & 2-2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \ t \in \mathbb{R}$$

Så egenrummet till $\lambda = 2$ är linjära höljet: $\left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right]$

$\lambda = 1 :$

$$(A - 1I)\vec{x} \Leftrightarrow \left(\begin{array}{ccc|c} 1-1 & 2 & 2 & 0 \\ 0 & 2-1 & 1 & 0 \\ -1 & 2 & 2-1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}$$

Så egenrummet till $\lambda = 1$ är linjära höljet: $\left[\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right]$

8.12 Diagonalisering

Definition: Diagonalisering

Låt $F : \mathbb{V} \to \mathbb{V}$ vara en linjär avbildning. Då kallas F **diagonaliserbar** om det finns en bas $\underline{v}$ som består av egenvektorer till F. (Det betyder att $(F)^{\underline{v}}_{\underline{v}}$ är en diagonalmatris)

Definition: Algebraisk och Geometrisk multiplicitet

Algebraisk multiplicitetet av en egenvärde λ_0 till en matris A är det maximala talet m, så att $\chi_A(\lambda) = (\lambda - \lambda_0)^m p(\lambda)$ för någon polunom p.

Algebraisk multiplicitetet av en egenvärde λ_0 till en matris A är $\dim \mathbb{V}_{A,\lambda_0}$

Exempel:

Avför om matrisen är diagoniserbar och isåfal vad är den matrisen

$$M = \begin{pmatrix} -2 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{pmatrix}$$

1. Hittar egenvärden: (se i kaptlet om enhetsvektorer)
$(-2 - \lambda)(2 - \lambda)(-2 - \lambda) = 0$
$\lambda = 2$ (alg mult 1) $\vee \lambda = -2$ (alg mult 2)

2. Hittar egenvekorer: (se i kaptlet om enhetsvektorer)

$$\lambda = 2 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; t \in \mathbb{R}$$

geo mult 1 därmed så är den hittils diagonaliserbar

$$\lambda = -2 \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ -t \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \; s, t \in \mathbb{R}$$

geo mult 2 därmed diagonaliserbar

3. Sätter upp diagonala matrisen
$F^{\underline{e}}_{\underline{e}} = T^{\underline{v}}_{\underline{e}} F^{\underline{v}}_{\underline{v}} T^{\underline{e}}_{\underline{v}} \Leftrightarrow$

$$\begin{pmatrix} -2 & 4 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

8.13 Inre produkrum

Definition: Skalärprodukt

Låt $\mathbb{V}$ vara ett vektorrum. En **skalärprodukt** på $\mathbb{V}$ är
en operation som tillordnar varge par av vketorer $\vec{u}, \vec{v} \in \mathbb{V}$ en skalär
$(\vec{u}|\vec{v}) \in \mathbb{R}$ så att följade vilkor gäller:
bilinjär
1. $(\vec{u} + \vec{v}|\vec{w}) = (\vec{u}|\vec{w}) + (\vec{v}|\vec{w})$
2. $(\lambda\vec{u}|\vec{v}) = \lambda(\vec{u}|\vec{v}) = (\vec{u}|\lambda\vec{v})$
Symetrisk
3. $(\vec{u}|\vec{v}) = (\vec{v}|\vec{u})$
Positiv definit
4. Om $\vec{u} \neq \vec{0}$ så $(\vec{u}|\vec{u}) > 0$

Definition: Avstånd och vinklar

Låt $\mathbb{V}$ vara ett vektorrum med enskalärprodukt och $\vec{u}, \vec{v} \in \mathbb{V}$
1. Längden (normen) av $\vec{u}$ betecknas $|\vec{u}|$ och defineras som
$|\vec{u}| = \sqrt{\vec{u}|\vec{u}}$
2. Avstånder mellan $\vec{u}$ och $\vec{v}$ defineras som $\vec{u} - \vec{v}$
3. Om $\vec{u} \neq 0$ och $\vec{v} \neq 0$ defineras vinkeln mellan $\vec{u}$ och $\vec{v}$ som
$\cos^{-1} \dfrac{(\vec{u}|\vec{v})}{|\vec{u}||\vec{v}|}$
4. Vi säger att $\vec{u}$ och $\vec{v}$ är ortogonala om
$(\vec{u}|\vec{v}) = 0$

Definition: Trianglar

Lår $\mathbb{V}$ vara ett vektorrum med en skalärprodukt och $\vec{u}, \vec{v} \in \mathbb{V}$.
Då bildar $\vec{u}, \vec{b}$ och $\vec{u} + \vec{v}$ en triangel
$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2(\vec{u}|\vec{v})$

Definition: Ortogonal projektion och komponent

1. $\vec{u}_{\|\vec{u}} := \dfrac{(\vec{v}|\vec{u})}{(\vec{u}|\vec{u})}\vec{u}$kallas den ortogonala projektionen av $\vec{v}$ på $\vec{u}$
2. $\vec{u}_{\perp\vec{u}} := \vec{v} - \vec{v}_{\|\vec{u}}$kallas den ortogonala kompnenten av $\vec{v}$ med avseende på $\vec{v}$

8.14 Symetriska och positiva definita matriser

Formel: skalärprodukt utrykt med matriser

$(\vec{u}|\vec{y}) = \vec{u}^t A\vec{y}$
där A är $(n \times n)$-matrisen som uppfyller följande krav
1. Symetrisk om $A^t = A$
2. Positivt definit om $\vec{x}^t A\vec{x} > 0$, $\forall \vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$

Exempel: skalärprodukt utrykt med matriser

På $\mathbb{R}^2$ definierar vi $(\vec{x}|\vec{y}) = x_1y_1 + 3x_1y_2 + ax_2y_1 + bx_2y_2$
För vilka värdenpå a och b är detta en skalärprodukt? $a, b \in \mathbb{R}$

Lösn:
1. Symetrisk: $(\vec{x}|\vec{y}) = (\vec{y}|\vec{x}) = y_1x_1 + 3y_1x_2 + ay_2x_1 + by_2x_2$
$(\vec{x}|\vec{y}) = \vec{x}^t \begin{pmatrix} 1 & 3 \\ a & b \end{pmatrix} \vec{y} \Rightarrow a = 3$pga summetri

2. Pos def: $(\vec{x}|\vec{x}) > 0$, $\forall \vec{x} \neq \vec{0}$
$(\vec{x}|\vec{x}) = (x_1x_2) \begin{pmatrix} 1 & 3 \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = {x_1}^2 + 3x_1x_2 + 3x_2 + x_1 + b{x_2}^2$
$= {x_1}^2 + 6x_1x_2 + b{x_2}^2 = (x_1 + 3x_2)^2 + (b-9){x_2}^2$
vilket endast är positivt då $b > 9$

8.15 On-baser och Gram-Schmidt-ortonormalisering

Definition: ON-bas

Låt $\mathbb{V}$ vara ett vektorrum med en skalärprodukt.
En mängd $\{\vec{u_1} \dots \vec{u_n}\} \subseteq \mathbb{V}$ kallas ortonormal (ON) om
$(\vec{u_i}|\vec{u_j}) = \begin{cases} 1\, i = j \\ 0\, i \neq i \end{cases}$
$|\vec{u_i}| = 1$

Dvs alla vektorer är ortogonala mot varandra och att längden ska vara 1

Definition: Koordinater i en ON-bas

Låt $\underline{u} = (\vec{u_1} \dots \vec{u_n})$ vara en ON-bas i $\mathbb{V}$ och $\vec{v}, \vec{w} \in \mathbb{V}$ Då är

1. $\vec{v}_{\underline{u}} = \begin{pmatrix} (\vec{v}|\vec{u}_1) \\ \cdot \\ \cdot \\ (\vec{v}|\vec{u}_n) \end{pmatrix}$
2. $(\vec{v}|\vec{w}) = (\vec{v}|\vec{u}_1)(\vec{w}|\vec{u}_1) + \dots + (\vec{v}|\vec{u}_n)(\vec{w}|\vec{u}_n) = \vec{v}_n \bullet \vec{w}_n$

Definition: Ortogonal komplement

Låt $\mathbb{V}$ vara ett vektorrum med en skalärprodukt och
$\mathbb{U} \subseteq \mathbb{V}$ ett underrum. Då är
$\mathbb{U}^{\perp} = \{\vec{v} \in \mathbb{V} | (\vec{u}|\vec{v}) = 0, \forall \vec{u} \in \mathbb{U}\}$
ett underrum som kallas det ortogonala koplementet till $\mathbb{U}$

Exempel: Hitta en ON-bas med Gram-Schidt ortonormalisering

$$U = [\vec{u}_1, \vec{u}_2, \vec{u}_3] \subseteq \mathbb{R}^4,\ \vec{u}_1 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}\ \vec{u}_2 = \begin{pmatrix}1\\0\\2\\1\end{pmatrix}\ \vec{u}_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

Hitta ON-bas

$[\vec{u}_1] \subseteq [\vec{u}_1, \vec{u}_2] \subseteq [\vec{u}_1, \vec{u}_2, \vec{u}_3]$
där $[\vec{u}_1] = U_1,\ [\vec{u}_1, \vec{u}_2] = U_2,\ [\vec{u}_1, \vec{u}_2, \vec{u}_3] = U_3$

ON-bas $U_1 : f_1 = \dfrac{1}{|\vec{u_1}|}\vec{u_1} = \dfrac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$

ON-bas $U_2 : f_1, f_2\ \vec{u}_{2\perp\vec{u_1}} = \vec{u_2} - (\vec{u_2}|f_1)f_1$

$$= \begin{pmatrix}1\\0\\2\\1\end{pmatrix} - \frac{1}{\sqrt{3}}(1+0+2+0)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\-1\\1\\1\end{pmatrix}$$

$$f_2 = \frac{1}{\vec{u}_{2\perp\vec{u_1}}}\vec{u}_{2\perp\vec{u_1}} = \frac{1}{\sqrt{3}}\begin{pmatrix}0\\-1\\1\\1\end{pmatrix}$$

ON-bas $U_3 : f_1, f_2, f_3\ \vec{u}_{3\perp\vec{u_2}} = \vec{u_3} - (\vec{u_3}|f_1)f_1 - (\vec{u_3}|f_2)f_2$

$$= \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{1}{\sqrt{3}}(0+0+1+0)\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{1}{\sqrt{3}}(0+0+1+1)\frac{1}{\sqrt{3}}\begin{pmatrix}0\\-1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}1\\1\\1\\0\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix} \Rightarrow f_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$

Kontrollera att f_1, f_2, f_3 är ortogonala mot varandra

8.16 Isometriska avbildningar och spektralsatsen

Definition: isometrier

Låt $\mathbb{V}$ och $\mathbb{W}$ vara vektorrum med skalärprodukter.
En linjär avbilding
$F : \mathbb{V} \to \mathbb{W}$
kallas en isomerik om $|F(\vec{v})| = |\vec{v}|$
En isometrisk linjär avbildling kallas också en isometri.

dvs. Så är längden oförendrad. F är en isometri omm
$(F(\vec{u})|F(\vec{v})) = (\vec{u}|\vec{u})$

Egenskaper: isometrier

Sats: Låt $F : \mathbb{V} \to \mathbb{W}$ vara en isometri. Då är F injektiv
Sats: Låt $F : \mathbb{V} \to \mathbb{W}$ vara en iventerbar isometri.
Då är $F^{-1} : \mathbb{W} \to \mathbb{V}$ en isometri
Sats: Låt $F : \mathbb{U} \to \mathbb{V}$ och $G : \mathbb{V} \to \mathbb{W}$ vara en isometrier.
Då är $G \circ F : \mathbb{U} \to \mathbb{W}$ en isometri

Sats: spektralsatsen

Låt $\mathbb{V}$ vara ett ändligt dimensionellt vektorrum med en skalär produkt och $F : \mathbb{V} \to \mathbb{V}$ en linjär avbildining.
Då är följande vilkor ekvivalenta
1. F är symmetrisk
2. $\mathbb{V}$ har en ON-bas av bestående av egenvektorer till F

Låt A vara en $(n \times n)$-matris. Då är följande vilkor ekvivalenta
1. A är symmetrisk
2. Det finns en ortonormal matris T så att $D = T^{-1}AT$ är en diagonalmatris

8.17 Andragradskurvor och andragradsytor

$$Q(x) = 1 \Leftrightarrow 1x^tAx = 1 \Leftrightarrow y^tDy = 1 \Leftrightarrow \lambda_1{y_1}^2 + \lambda_2{y_2}^2 = 1$$

Om $\lambda_1 = \lambda_2$ så berkriver $Q(x) = 1$ en cirkel
Om $\lambda_1 > 0$ och $\lambda_2 > 0$ så berkriver $Q(x) = 1$ en ellips
Om $\lambda_1 > 0$ och $\lambda_2 < 0$ så berkriver $Q(x) = 1$ en hyperbel

Exempel: Bestäm formen

$$Q(\vec{x}) = \frac{14}{5}{x_1}^2 + \frac{11}{5}{x_2}^2 + \frac{14}{5}x_1x_2$$

$$Q(\vec{x}) = (x_1\, x_2)\begin{pmatrix} 14/5 & 2/5 \\ 2/5 & 11/5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

1. Egenvärden $\det A = (\frac{14}{5} - \lambda)(\frac{11}{5} - \lambda) - \frac{4}{25} = 0$

$\lambda_1 = 2 > 0$

$\lambda_2 = 3 > 0$

Därmed så beskriver $Q(x) = 1$ en ellips

2. Egenvektor får vi en ON-bas

3. Ställer up ekvationen och ritar ut den

$$Q(\vec{y}) = y^-1\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\vec{y} = 2{y_1}^2 + 3{y_2}^2 = 1$$

8.18 System av linjära differentialekvationer

$$y' = Ay$$
$$T^{-1}AT = D$$
$$y = Tz \Rightarrow y' = Tz' \wedge y' = Ay \Leftrightarrow z' = dz$$
$$z = \begin{pmatrix} c_1e^{\lambda_1 t} \\ c_2e^{\lambda_2 t} \\ . \\ . \\ . \\ c_ne^{\lambda_n t} \end{pmatrix} \text{ där } c_i \in \mathbb{R}^n \text{ och } y = Tz$$
$$c = T^{-1}y_0 \text{ där } z(0) = c = \begin{pmatrix} c_1 \\ . \\ . \\ . \\ c_n \end{pmatrix}$$

Exempel: Bestäm formen

Lös följade system av differentialekvationer

$$\begin{cases} \frac{dx(t)}{dt} = 2x(t) + y(t) + z(t) \\ \frac{dy(t)}{dt} = x(t) + 2y(t) + z(t) & x(0) = 3,\ y(0) = 2,\ z(0) = 1 \\ \frac{dz(t)}{dt} = x(t) + y(t) + 2z(t) \end{cases}$$

1. Skriver up systemet

$$\begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \text{ Där matrisen är } A$$

2. Diagonaliserar

2.1 egenvärden

$\lambda_1 = 1$ (multiplicitet 2), $\lambda_2 = 4$ (multiplicitet 1)

2.1 egenvektor

$$\lambda_1 = 1 \Rightarrow A - I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \wedge \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 4 \Rightarrow A - 4I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Diagonal ekv blir $A = TDT^-1 \Rightarrow T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \wedge D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

3. Finner den almäna lösningen

$$u' = Du \Rightarrow \begin{cases} u_1 = c_1 e^t \\ u_2 = c_2 e^t \\ u_3 = c_3 e^{4t} \end{cases} \text{ Där } c_1, c_2, c_3 \in \mathbb{R}$$

$v' = Av$ Där $A^{-1}v' = Tu \Rightarrow v = Tu$

$$v = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^t \\ c_3 e^{4t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^t \\ c_2 e^t + c_3 e^{4t} \\ -c_1 e^t - c_2 e^t + c_3 e^{4t} \end{pmatrix}$$

4. Finner lösningen till begynelsevärdet

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right) \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\begin{cases} x(t) = e^t + 2e^{4t} \\ y(t) = 2e^{4t} \\ z(t) = -et + 2e^{4t} \end{cases}$$

Chapter 9

Computer System with Project Work

9.1 M1

9.1.1 CPU

The central processing unit (CPU) is the electronic circuitry within a computer that carries out the instructions of a computer program by performing the basic arithmetic, logical, control and input/output (I/O) operations specified by the instructions.

9.1.2 Register

A processor register is a quickly accessible location available to a computer's central processing unit (CPU). Registers usually consist of a small amount of fast storage. A CPU only has a small number of registers.

9.1.3 Memory

Memory refers to the computer hardware integrated circuits that store information for immediate use in a computer; it is synonymous with the term "primary storage". The memory is much slower than the CPU register but much larger in size. Sources

9.1.4 CPU context

At any point in time, the values of all the registers in the CPU defines the CPU context. Sometimes CPU state

9.1.5 Memory allocation

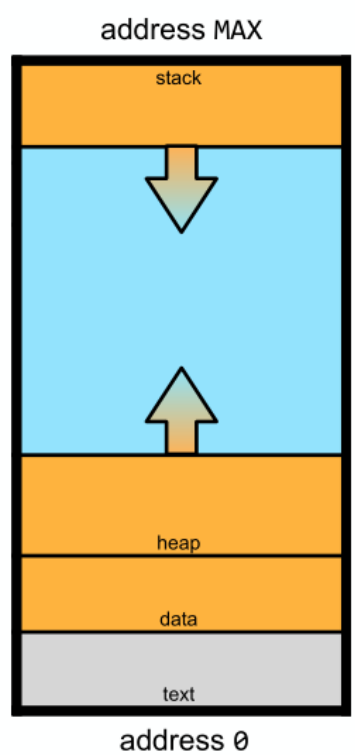


Figure 9.1: memory-allocation. From ()

9.1.6 Kernal

The kernel is a computer program that is the core of a computer's operating system, with complete control over everything in the system.

9.1.7 Multiprogramming

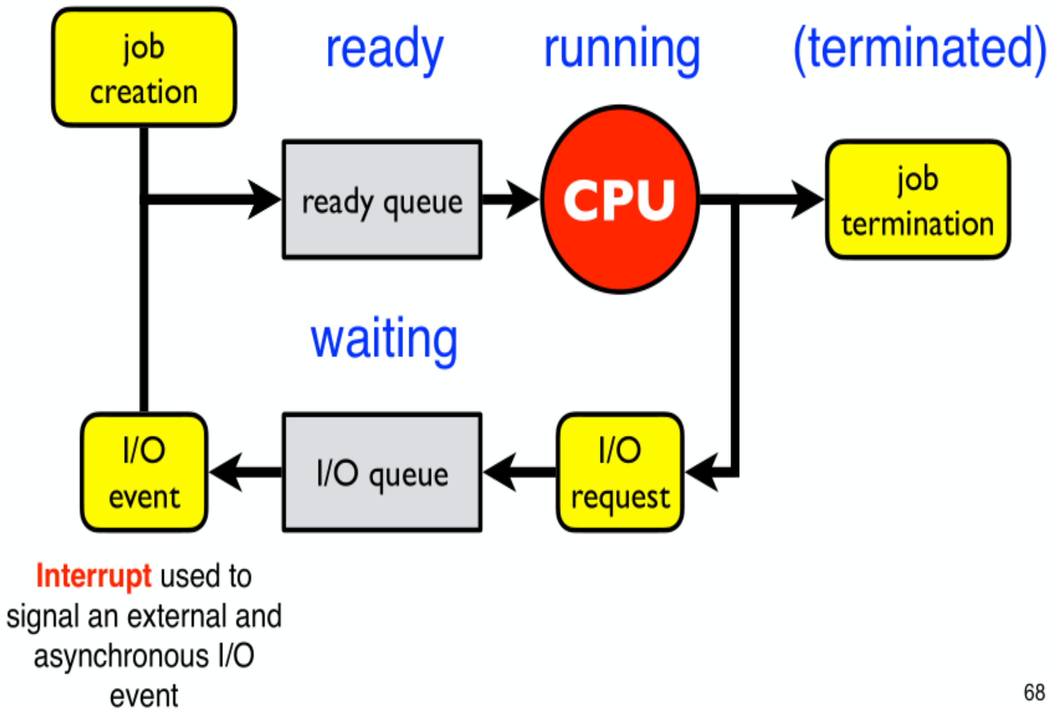


Figure 9.2: multiprogramming. From ()

9.2 M2

9.2.1 Network communication

- TCP for a reliable byte stream service
- UDP for an unreliable message forwarding service
- Four different delivery models: uni/broad/multi/any-cast
- The client/server model is common in application design
- Sockets API can be used for network programming
  - A socket is a communication handle abstraction
  - Properties of a socket can be set through socket options

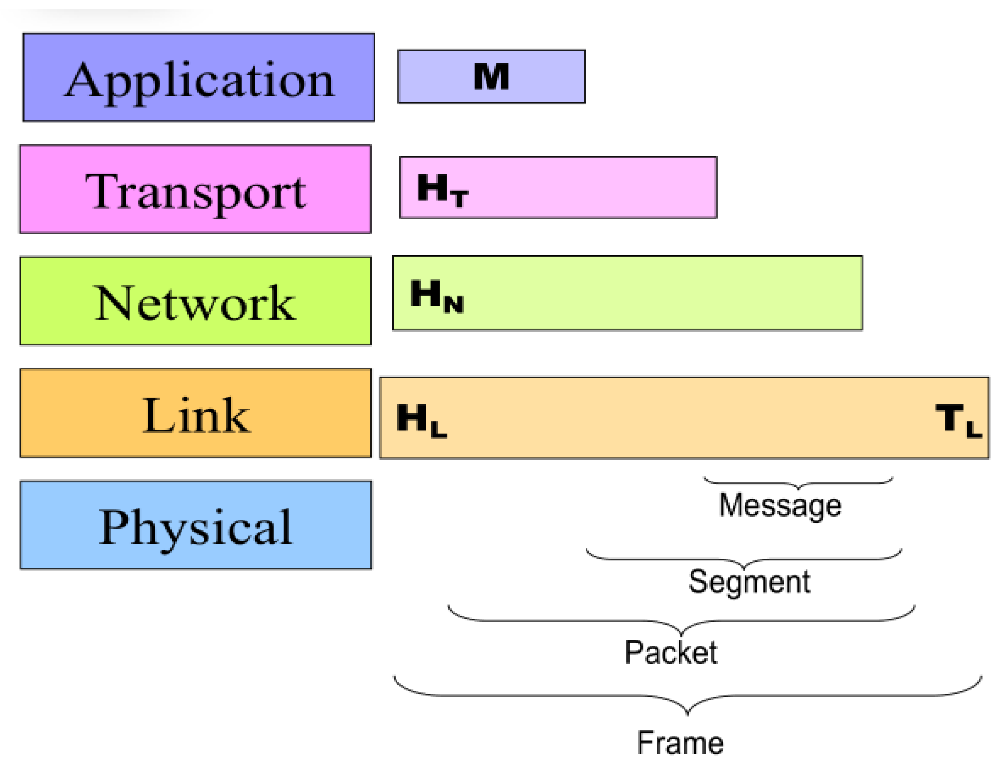


Figure 9.3: layering. From ()

9.2.2 Prossesses

- Zombie: A terminated process is said to be a zombie or defunct until the
- Orphan: An orphan process is a process whose parent process has terminated, though it remains running itself.
- Signals: are a limited form of inter-process communication used in
- File discriptor: The kernel keeps a table with information about a process's open file descriptors.
- Pipes: An (anonymous) pipe is a simplex FIFO communication channel that may be used for

9.3 M3

- DNS är ett distribuerat system av servrar
- ARP mappar IP-address till länklageraddress
- En IP-address har två delar för att identifiera nätve

9.3.1 Dispatcher

Another component that is involved in the CPU-scheduling function is the dispatcher, which is the module that gives control of the CPU to the process selected by the short- term scheduler. It receives control in

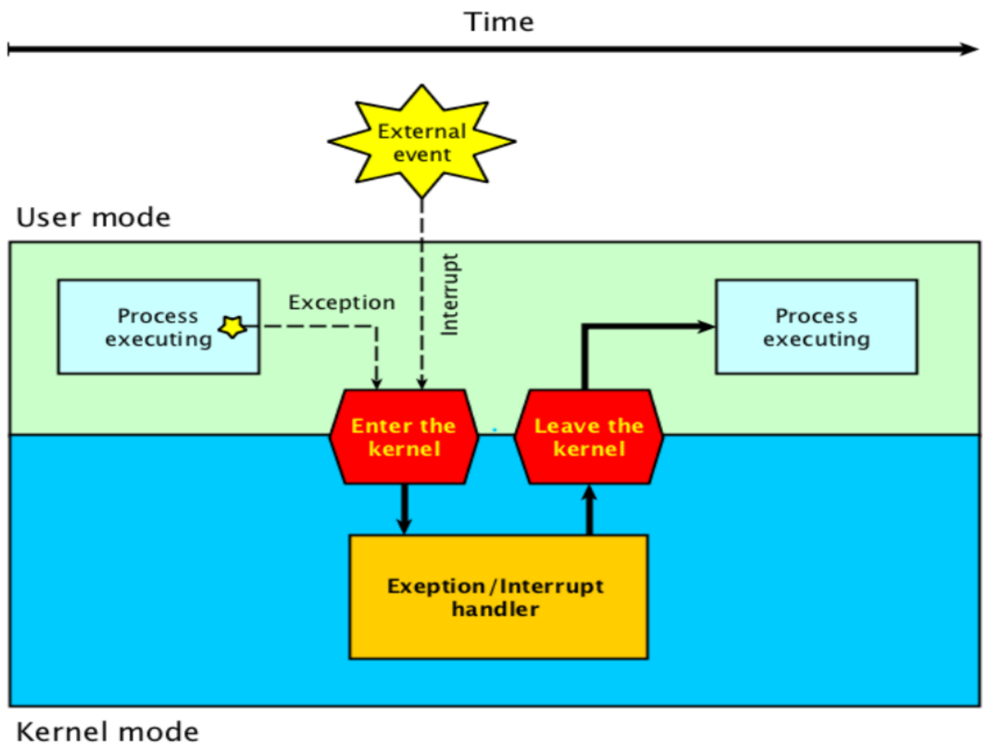


Figure 9.4: user-kernal-mode. From ()

kernel mode as the result of an interrupt or system call.

The CPU scheduler selects one process from among the processes in memory that are READY to execute. The scheduler dispatcher then gives the selected process control

9.3.2 PCB

The process control block (PCB) is a data structure in the operating system kernel containing the information needed to manage a particular process.

The Long-term scheduler (LTS) (aka job scheduler) decides whether a new process should be brought into the ready queue in main memory or delayed.

The medium-term scheduler (MTS) temporarily remove processes from main memory and places them in secondary storage and vice versa, which is commonly referred to as "swapping in" and "swapping out".

9.3.3 IO bound/CPU bound

An I/O-bound process spends more time doing I/O than computations and is characterised

An CPU-bound process spends more time doing computations and is characterised by few very long Interactive processes interact constantly with their human users.

Batch processes do not interact with human users.

Real-time processes have very strong scheduling requirements.

9.3.4 Schedular algorithems

The first come, first served (commonly called FIFO – first in, first out) process scheduling algorithm is the simplest process scheduling algorithm. Processes are executed on the CPU in the same order they arrive to the ready queue.

Shortest Job First (SJF) scheduling assigns the process estimated to complete fastest, i.e, the process with shortest CPU burst, to the CPU as soon as CPU time is avalible.

An extension of SJF where the currently running process is P preempted if the CPU burst of a process A arriving to the ready queue is shorted than the remaining CPU burst of the currently running process.

A priority number (integer) is associated with each process. The CPU is allocated to the process with the highest priority (smallest integer = highest priority).

Round Robin (RR) is a scheduling algorithm where time slices are assigned to each process in equal portions and in circular order.

A multi-level queue scheduling algorithm is used in scenarios where the processes can be classified into groups based on properties like process type, CPU time, IO access, memory size, etc.

In computer science, starvation is a problem encountered in multitasking where a process is perpetually denied necessary resources. Without those resources, the process can never finish its task.

Ageing is used to ensure that jobs with lower priority will eventually complete their execution.

9.4 M4

9.4.1 Concurrency

The ability of different parts or units of a program, algorithm, or problem to be executed out-of-order or in partial order, without affecting

9.4.2 Parallelism

In parallel systems, two tasks are actually performed simultaneously. Parallelism is when tasks

9.4.3 Client-Server modle

A distributed application structure that partitions tasks or workloads between the providers of a resource or service, called servers, and service requesters, called clients.

9.4.4 Threads

A thread of execution is the smallest sequence of instructions that can be managed independently by a scheduler, which is typically a part of the operating system.

Atomic

In concurrent programming, an operation (or set of operations) is atomic if it appears to the rest of the system to occur at once without being interrupted.

Race condition

A race condition or race hazard is the behaviour of an electronic, software or other system where the output is dependent on the sequence or timing of other uncontrollable events.

Data race

A data race occurs when two instructions from different threads access the same memory location

9.4.5 Locks

In general, any solution to the critical section problem requires a tool/abstraction - a lock.

9.4.6 Spinlock

A spinlock is a lock where a task simply waits in a loop ("spins") repeatedly checking until the lock becomes avalible.

9.4.7 TestAndSet

The TestAndSet instruction atomically first sets the value at the target address to True and return the old value stored at the target address.

9.4.8 Swap

The Swap instruction atomically swaps the content of two memory

9.4.9 Bounded waiting

A bound must exist on how many times a task has to wait in order to enter a critical section.

9.4.10 Semaphores

The hardware solutions to the critical section problem using Swap and TestAndSet are complicated for programmers to use directly.

9.4.11 Mutex lock

Many locking libraries provides a special mutex lock which can only be used to provide mutual exclusion to critical sections.

9.4.12 Deadlock

In concurrent computing, a deadlock is a state in which each member of a group is waiting for some other member to take action, such as sending a message or more commonly releasing a lock.

9.4.13 Mutual exclusion

A chopstick can only be held by one philosopher at the time. Updating the amount of rice must be atomic.

9.4.14 Deadlock prevention

Preventing deadlocks by constraining how requests for resources can be made in the system and how they are handled (system design).

9.4.15 Deadlock avoidance

The system dynamically considers every request and decides whether it is safe to grant it at this point,

9.4.16 Clock Synchronization

- External synchronization
- Internal synchronization
- Hard in an asynchronous system

9.4.17 Mutual exclusion

- Critical section
- Need for coordination:
- Evaluation criteria for different solutions

9.4.18 Desired properties of Transactions

- Atomicity
- Consistency
- Isolation
- Durability

9.4.19 WiFi

- Several standards, from 2Mbit/s to 100's of Mbit/s
  - Using multiple channels for higher speeds
- Range: in general 20-100m, depending on the environment
- Can be used in ad hoc or infrastructure mode
  - Ad Hoc: like a wireless Ethernet
  - Infrastructure: a coordinating base station
- Networks identified with SSID:s
  - Used to scramble signal somewhat
- Encryption with WEP (bad), WPA(better), WPA2 (best)

9.4.20 Bluetooth

- Several different versions
  - Different transmission power (and hence ranges)
- Different profiles for different applications
- Frequency hopping 1600 times/s in ISM band
- Peer to peer or piconet networks
- Max 7 simultaneous connections

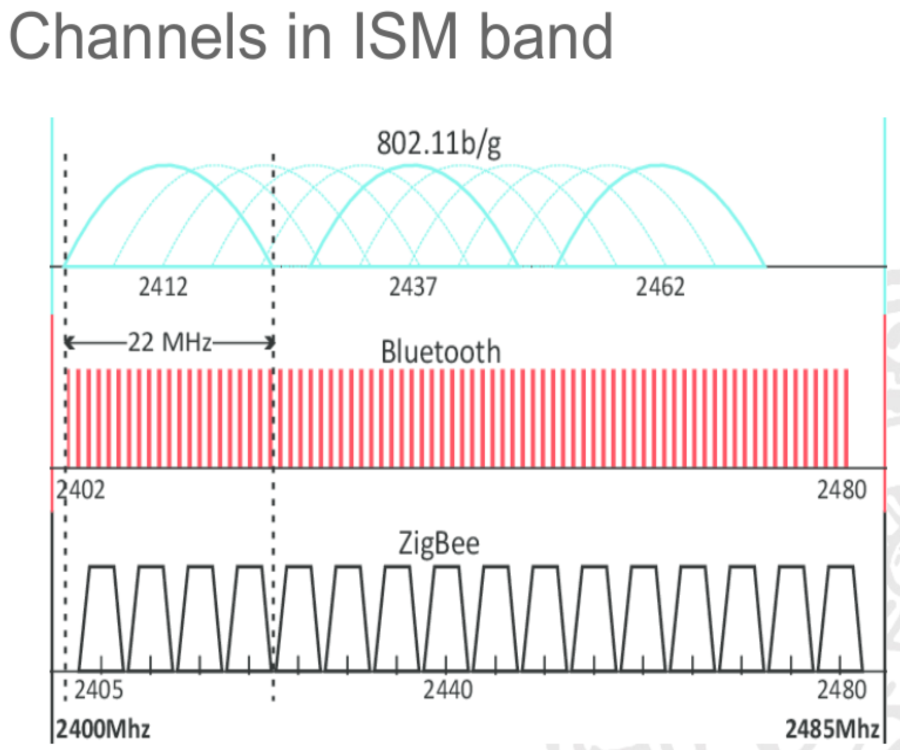


Figure 9.5: ISM-band. From ()

9.4.21 ISM-band

9.4.22 Properties of a medium

- Bandwidth B
  - Frequency range for signals transmitted in medium
- Signal/noise ratio S/N, SNR
  - Frequency range for signals transmitted in
  - Signal level / noise level
  - Measured at the receiver
  - Can be approximated with Maxwell's equations
- Capacity C=B log2 (1+S/N)

9.4.23 Sampling

Bandwidth B Hz is sampled at 2*B Hz

9.4.24 Multithreading models

9.4.25 Bounded buffer

A bounded buffer lets multiple producers and multiple consumers share a single buffer. Producers write data to

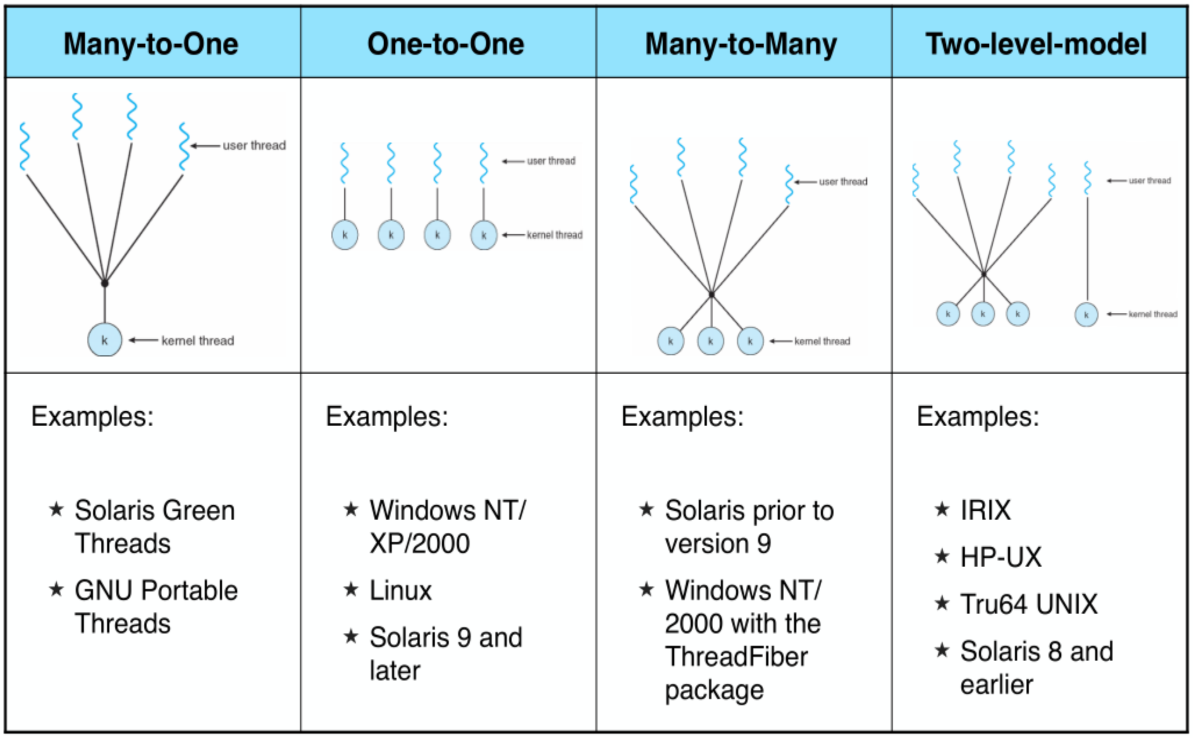


Figure 9.6: Multithreading-models. From ()

9.4.26 Priority inversion

A higher priority task is “preempted” by a lower priority one.

9.5 M5

9.5.1 Single contiguous

Single contiguous allocation is the simplest memory management technique. All the computer’s memory, usually with the exception of a small portion reserved for the operating system, is available to the single application.

9.5.2 Swapping

A process can be swapped temporarily out of memory to a backing store, and then brought back into memory for continued execution.

9.5.3 Memory resident

Certain programs, can be marked as being memory resident, which means that the operating system is not permitted to swap them out to a storage device; they will always remain in memory.

9.5.4 Partitioned allocation

Partitioned allocation divides primary memory into multiple memory partitions, usually contiguous areas memory.

9.5.5 Logical address space

A logical address is the address at which a memory cell appears to reside from the perspective of an executing application program.

9.5.6 Address binding

Address binding is the process of mapping the program's logical (or virtual addresses) to corresponding physical memory addresses.

9.5.7 Memory management unit (MMU)

The MMU is a computer hardware unit having all memory references passed through itself, primarily performing the translation of logical memory addresses to physical addresses.

9.5.8 Fragmentation

Fragmentation is a phenomenon in which storage space is used inefficiently, reducing capacity and often performance.

9.5.9 Compaction

Move all processes to one end of the address space producing one large hole at the other end of the address space.

9.5.10 Memory protection

Memory protection between processes implemented by associating a valid bit with each entry in the per process page table.

9.5.11 Shared pages

Paging makes it possible for processes to share parts of their memory space with each other.

9.5.12 Translation lookaside buffer

A translation lookaside buffer (TLB) is a memory cache that is used to reduce the time taken to access a user memory location. The TLB stores the recent translations of virtual memory to physical memory.

9.5.13 Operating System

Controls the hardware and coordinates its use among the various application programs for various users.

9.5.14 Access methods

Data can be accessed in different patterns. These patterns can be divided into two main categories: Sequential access, Random access (aka direct access)

9.5.15 RAM

The main memory aka primary memory is a form of Random Access Memory (RAM).

9.5.16 Volatile memory

Volatile memory, in contrast to non-volatile memory, is computer memory that requires power to maintain the stored information; it retains its contents while powered on but when the power is interrupted, the stored data is quickly lost.

9.5.17 Persistent data storage

In computer science, persistence refers to the characteristic of state that outlives the process that created it.

9.5.18 The file

A file is a named collection of related information that is recorded on secondary storage.

9.5.19 Block data storage

A file is made up of fixed length logical recordes (blocks)

9.5.20 File control block

A File Control Block (FCB) is a file system structure in which the state of an open file is maintained.

9.5.21 Directory

A directory is a file system cataloging structure which contains references to other computer files, and possibly other directories.

9.5.22 Contiguous allocation

Each file occupies a set of contiguous blocks on the disk.

9.5.23 Linked allocation

Each file is a linked list of disk blocks: blocks may be scattered anywhere on the disk.

9.5.24 Indexed allocation

Indexed allocation brings all pointers together into an index

9.5.25 FAT

File Allocation Table (FAT) is a computer file system architecture and a family of industry- standard file systems utilizing it. The FAT file system is a legacy file system which is simple and robust.

9.5.26 iNode

In Unix-style file systems the file control block is called iNode. The inode is a data structure that describes a filesystem object such as a file or a directory.

9.5.27 Directory

A directory is a file system cataloging structure which contains references to other computer files, and possibly other directories.

9.5.28 M6

9.5.29 Classic historical ciphers

Caesar cipher (monoalphabetic), Vigenere cipher (polyalphabetic).

9.5.30 Modern cryptography

If a lot of smart people for a long time have failed to solve a specific problem, it is unlikely that a solution will appear soon.

- Encryption method is well-known
- Secret guarded by a n-bit key
  - Encryption and decryption in $O(n)$ time
  - Key guessing in $O(2^n)$ time
- Key management is crucial
- CIA triad represent desirable properties
  - Confidentiality
  - Integrity
  - Availability

9.5.31 Public key exchange

$$\begin{array}{cc}
\text{Alice} & \text{Bob} \\
\text{Create secret key } S_A & \text{Create secret key } S_B \\
\text{Compute } T_A = g^{S_A} \, mod \, p & \text{Compute } T_B = g^{S_B} \, mod \, p \\
\text{Send } T_A \text{ to Bob} & \text{Send } T_B \text{ to Alice} \\
\text{Compute } K_1 = T_B^{S_A} \, mod \, p & \text{Compute } K_2 = T_A^{S_B} \, mod \, p
\end{array}$$

9.5.32 Certification Authority (CA)

Issues digital certificates: Digitally signed with the private key of the CA, Authorize a public key.

9.5.33 Chain of trust

9.5.34 Self-issued certificates

Some sites present self-issued certificates: A little lite designing your own drivers license.

9.5.35 Usage of modern cryptography

- Symmetric cryptography in encrypted sessions
  - public-key cryptography not fast enough
- Asymmetric cryptography in certain situations
  - To establish a symmetric key
  - Digital signatures and verification
- Cryptographic hashing for verification of authentic data

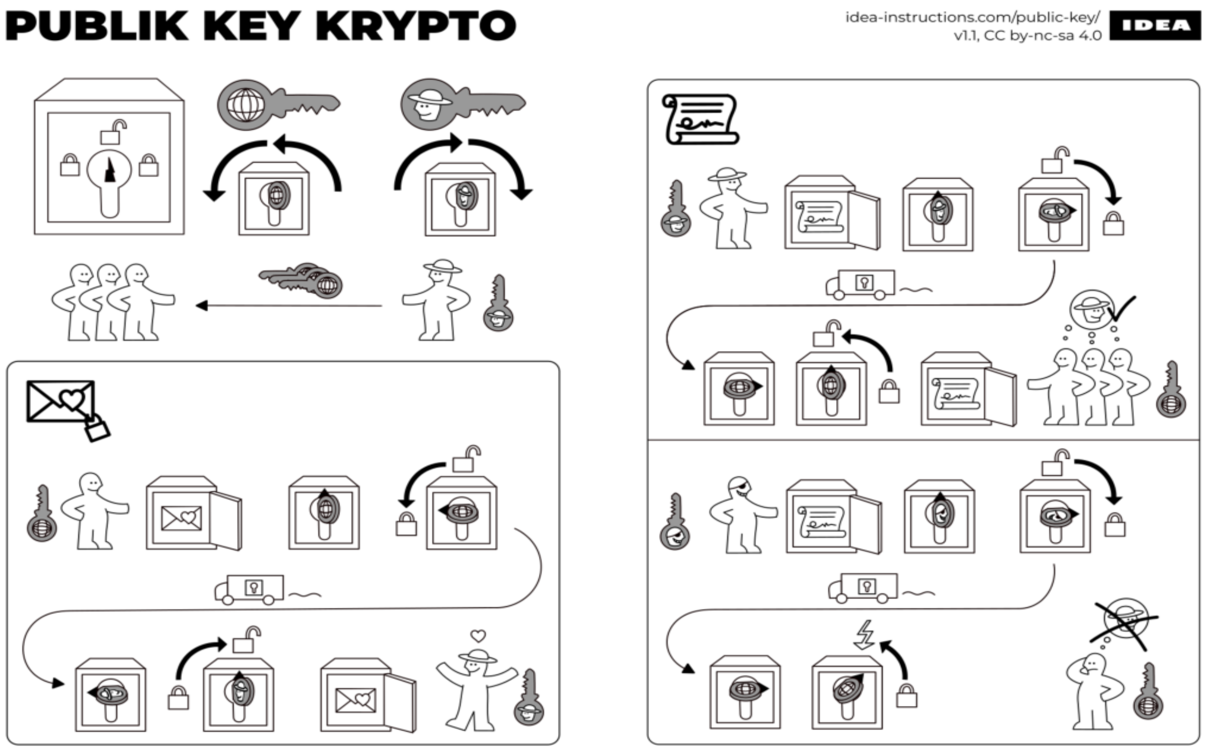


Figure 9.7: publik-key-krypto. From ()

9.5.36 Man-in-middle attack

the attacker secretly relays and possibly alters the communications between two parties who believe that they are directly communicating with each other

9.5.37 Chosen-plaintext attack

presumes that the attacker can obtain the ciphertexts for arbitrary plaintexts. The goal of the attack is to gain information that reduces the security of the encryption scheme

9.5.38 Side-channel attack

any attack based on information gained from the implementation of a computer system, rather than weaknesses in the implemented algorithm itself Timing information, power consumption, electromagnetic leaks or even sound can provide an extra source of information, which can be exploited.

9.5.39 Replay attack

valid data transmission is maliciously or fraudulently repeated or delayed. This is carried out either by the originator or by an adversary who intercepts the data and re-transmits it, possibly as part of a spoofing attack by IP packet substitution. This is one of the lower-tier versions of a man-in-the-middle attack. Replay attacks are usually passive in nature.

9.5.40 Security in the Internet stack

- Application layer: DNSSEC

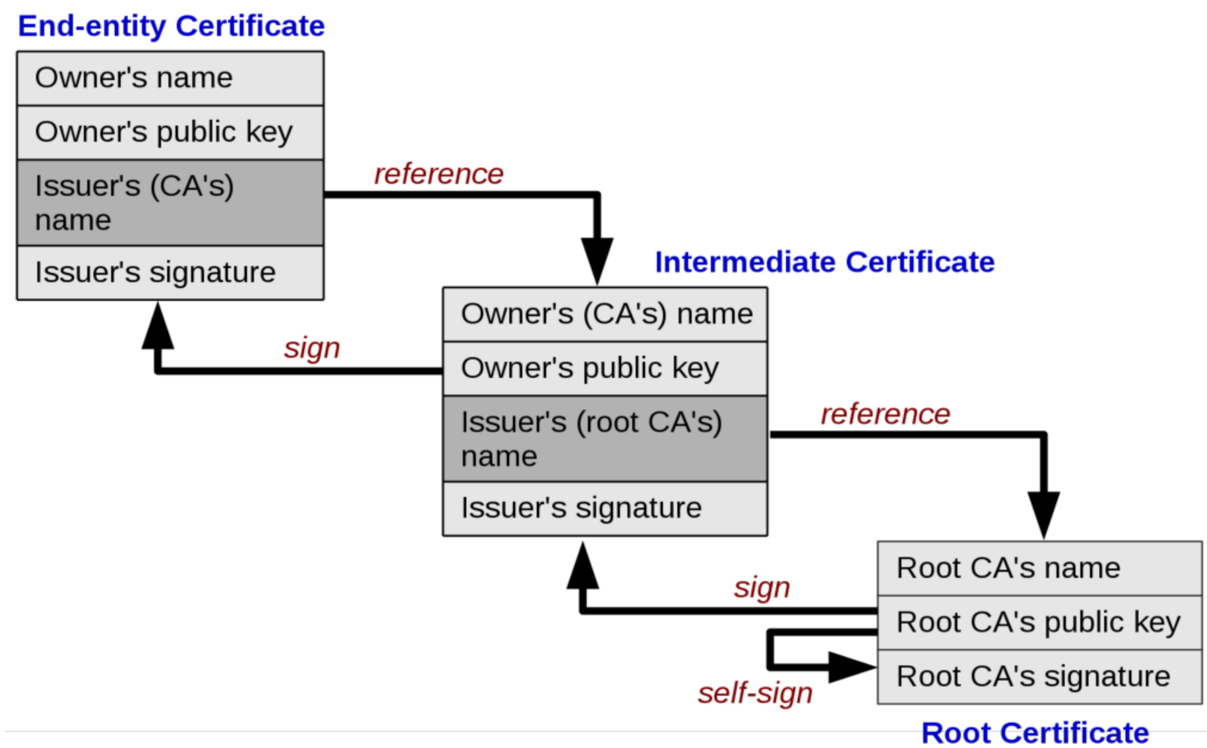


Figure 9.8: chain-of-trust. From ()

- Transport layer: SSL/TLS
- Network layer: IPsec
- Link layer: WEP/WPA, MAC filtering, ...

9.5.41 Firewall

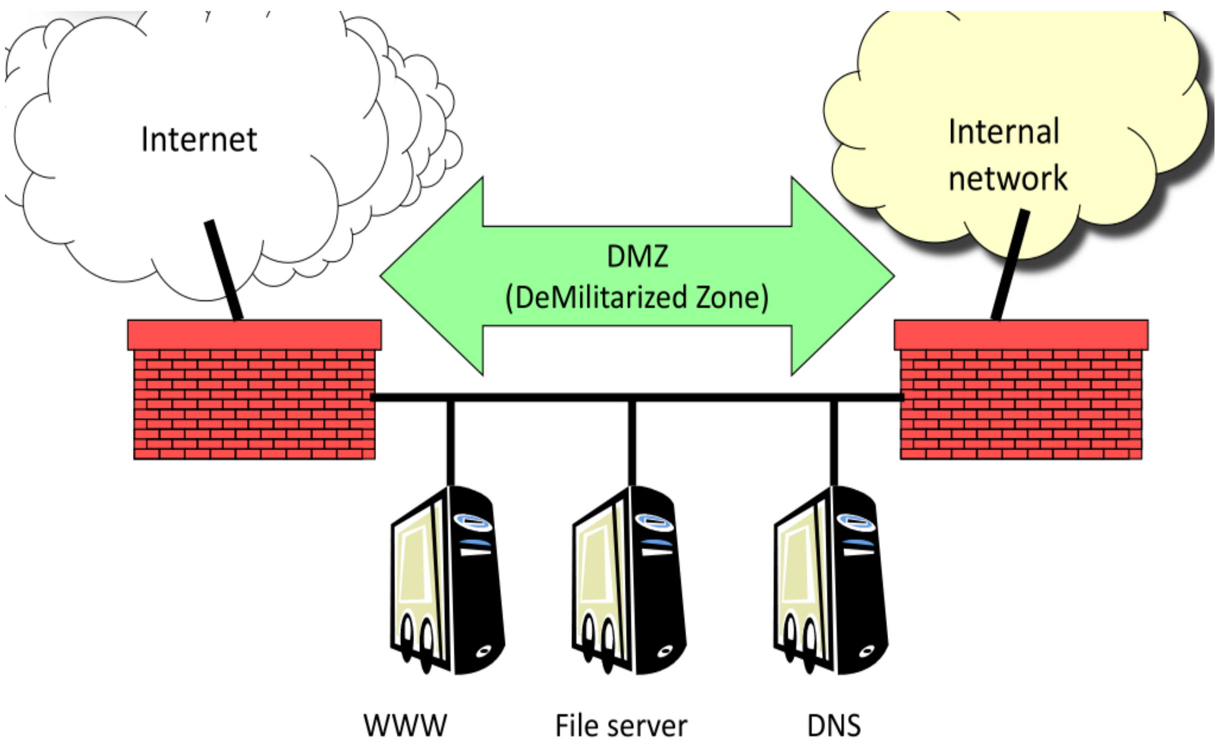


Figure 9.9: firewall. From ()

Chapter 10

System Design with User Perspective

10.1 Documentation

- One of the most important things in life is to understand...
- I want you to understand WHY you need to document a program!
- Then the HOW becomes much easier!
- The documentation is the user interface to the program code!
- It is possible to work without it, but who want to code in assembler?

10.1.1 Implicit documentation

- Names can tell a story (meaningful names)
- Names can also be classifying the variable (x,y positions. i,j,k index)
- Choose the right algorithms and data structures

10.1.2 Implicit documentation

- Several diffrent versions of documetation
  - Meta documentation (Each file: Discription, author, protocols, file dependencies)
  - External document (Manuals/Installation instruction, list of tree structures)
  - Inline documentation (Difficult code, TODO)

10.2 Human factors

10.2.1 Color combinations

There are some color combinations with humans have difficulty prossessing on of with is Red text on Blue backround and Blue text on red backround.

10.2.2 Personas

Who are the real people who will use this program. With that information mace sure it is well suted for them and there needs as well as for there difficulties.

10.2.3 Memory

Our memory is like a net wire eatch connection is from one peace of information to another there for memory needs to be assosiated with the nessesary areas. For instance when we design a web shop we should have the checkout on the right top since for most web shops have that and therfore people make that connection that this is a web shop there fore the checkout is on the top right.

10.2.4 What is needed to desing a system?

- Requirement specification
- Specify in detail the requirements on the software
  - What should it be able manage?
    - Number of customers, number of items in store, etc.
  - Which functions are needed?
- Order a beverage, check stored items, etc.

10.3 Model-View-Control

Initially a Object-Oriented (interface) Design model. Purpose: Making the View separated from the Mode.

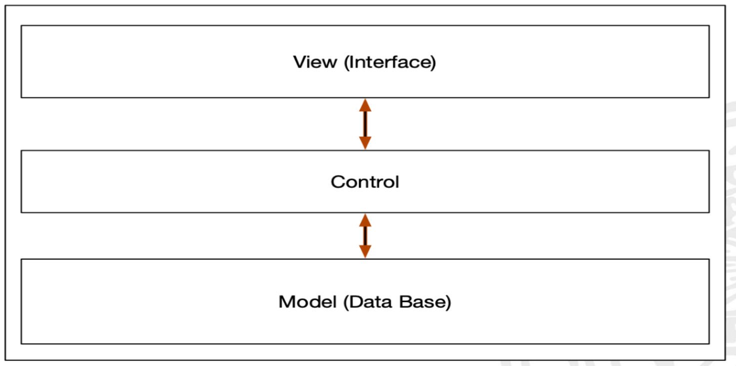


Figure 10.1: MVC

- View is only presentation of a Designed Interaction!
- Start by modelling the Model and Control (as a command Interface)
- Then implement the View (both control and presentation).
  - Create a good metaphor to the things that happen in

Chapter 11

Probability and Statistics DV

11.1 Statistisk mått och begreppet sannolikhet

11.1.1 Begrepp

- Deterministiska modeler: enkla modeler som inte tar hänsyn till fel
- Sannolikhetsteori: modelera slumpmäsiga fel
- s..k: a..u slumpmässig data
- Beskrivande statistik:
  - Population: Alla bilar i uppsala
  - Stickprov: 100 utvalda bilar i uppsala
  - Enhet: En av det 100 utvalda
  - Variabler: Motorstyrka, dragkrok, automat etc
    - Kvalitativa: (dragkrok, automati/manuel)
    - Kvantitativa: (Motorstyrka, vikt)
  - Tvärsnitts data: Befolkning i sverige städer 1 jan 2018
  - Longitudinella data: Befolking i uppsala under 1 jan 1960 till 2020
- Statistisk mått:
  - Lägesmått:
    - Aretmetisk modelera: $\overline{x} = \frac{\sum_{k=1}^{n} x_k}{n}$
    - median: $(\cancel{1}, \cancel{1}, \cancel{2}, 4, \cancel{4}, \cancel{4}, \cancel{7}), (\cancel{1}, \cancel{1}, 2, 4\ \cancel{4}, \cancel{4})$
  - Kvartiler: Tre punkter, fyra i quarters, som delar upp tal serie
    - 1:a kvartilen (Nedre kvartil) mittpunkten av den nedre halvan
    - 3:e kvartilen (Övre kvartil) mittpunken av den övre halvan
  - Spridningsmått:
    - Kvartal bred: 3:e kvartalen − 1:a kvartalen
    - Stickprovs standard divianse (standard diviaion): $s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2}$
    - Spridnings diagram: Positiv korelation, negativ korelation, ingen korelation, perfect korelation
- Diskret variable: räknerligt många värden
- Kontinuerlig variabel: tar oräknerligt många värden, värden i ett interval
- Dickreta visualiseras: med stolpdoagram
- Kontinuerlig variabel visualisering: med histogram
- Lådiagram:
- Sannolikhetsteori: Vi antar att ett försek genomförs n gånger oberonde av varandra. En händelse A inträfar f gånger
  - Frekvenskvoteten: $\frac{f}{n}$
  - Exprimentellt bestömt närmevärde: frekenskvoten på $p(A)$
    - Def: (Frekvensbaserad sannoliket): $p(A) \lim_{n\to\infty}$
    - Def: (Klassisk sannoliket) Anta att fersek kan utföras på m olika olika sätt varav g är gynnsama (innebär A). Då är $p(A) = g/m$

– Utfallsrum: Ω alla möjliga värden som slumpvariabeln kan ta
– Händelser: är delmängden av utfalsrummet
– Slumpvariabler: Stokastisk variabel

Kotmogorovs axiom

$$
\begin{aligned}
I\ & \text{För varje händelse } A \text{ gäller att } P(A) \geq 0 \\
II\ & \text{Sannolikhet för utfallsrumet är } 1 P(\Omega) = 1 \\
III\ & \text{Om } A \text{ och } B \text{ är händelser och } A \cap B = \not{o} \\
& \text{(oförsenliga hänelser) gäller } P(A \cup B) = P(A) + P(B) \\
& \Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)
\end{aligned}
$$

11.2 Sannolikheter och slumpvariabler

Användbara räkneregler

$$1\ P(A^*) = 1 - P(A)$$
$$2\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$3\ P(A|B) = \frac{A \cap B}{P(B)}$$

11.2.1 Betingade sannolikheter

Type	Dyglig	Defekt	Tot
Äldre	170	10	180
Ny	115	5	120

$A =$ "Slumpmässigt vald produkt är dyglig"
$B =$ "Slumpmässigt vald produkt är tillverkad vid äldre maskin"
$P(A) = \frac{285}{300} = 0.95$
$C =$ Slumpmässigt vald produkt är dyglig givet att den är tillvärkad vid äldre maskinger

$$P(C) = P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{170}{180} \approx 0.94$$

11.2.2 Kedjer av händelse

$$P(A \cap B \cap C) = P(C|A \cap B)P(A \cap B) = P(C|A \cap B)P(B|A)P(A)$$

Bayes sats

$$P(B|A) = \frac{P(B)P(A|B)}{P(A)}$$

Lagen om total sannolikhet

$$P(B) = P(B|A)P(A) + P(B|A^*)P(A^*)$$

11.2.3 Oberonde händelser

$P(A|B) = P(A)$

Betingad sannolikhet: $P(A \cap B) = P(A|B)P(B)$ om A och B är oberonder $\Rightarrow$
$\Rightarrow P(A \cap b) = P(A)P(B)$

Födelsedagsparadoksen

$$1-(\prod_{k=1}^{n}\frac{365-k}{365})$$

Slumpvariablel

Är en function från utfalsrumet Ω till någon mängd E $X:\Omega\to E$

$X\in\{0,1,2,3\}$
$P(X=0)+P(X=1)+P(X=2)+P(X=3)=1$ Kolmogorovis axiom

Sannolikhetsfaktor? är $P_X(x)$ $P_X(x)=P(X=x)$

11.3 Fördelningar

Diskreta fördelningar	Kontenuerliga fördelningar
$\Omega=\{1,2,3\}$ $\Omega=\{0,2,...\}$	$\Omega=[0,1]$ $\Omega=\mathbb{R}$
Sannolikhetsfunktion $p_X(x)=P(X=x)$	Täthetsfunktion $f_X(x)=\int_a^b f_X(x)dx=P(a\le x\le b)$ $P(X=x)=0$
$\sum_{x\in\Omega}p_X(x)=1$	$\int_\Omega f_X(x)dx=1$
$E[X]=\sum_{x\in\Omega}xp_X(x)$	$E[X]=\int_\Omega xf_X(x)dx$
$V[X]=E[X^2]-E[X]^2$	$V[X]=E[X^2]-E[X]^2$
$F_X(x)=\sum_{i\le x}p_X(i)$ $=p(X\le x)$	$F_X(x)=\int_{-\infty}^{x}f_X(x)dx=1$ $=P(X\le x)$ $=P(X<x)$

11.3.1 Binomial-fördelningar

Tillämpning: man utför något n anatal gånger med sannolheten p att det lyckas.

Om X är binomialfördelad med paramter n och p. Då gäller
$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x},\ x=0,1,2,..,n$
$X\sim Bin(n,p)$

$E[X]=np$
$V[X]=np(1-p)$

$dbinom(x,n,p)$
$pbinom(x,n,p,lower.tail=FALSE)$

11.3.2 Possion-fördelningar

Tillämpning: För att modelera sällsynta händelser.

Om X är possionsfördelad med paramter m. Då gäller

$P(X = x) = \frac{m^x}{x!} e^{-m}$, $x = 0, 1, 2, .., n$

$X \sim Po(\mu)$

$E[X] = \mu$

$V[X] = \mu$

$dpois(x, \mu)$

$ppois(x, \mu, lower.tail = FALSE)$

11.3.3 Likformig/rektangulär-fördelningar

Tillämpning: Lika fördelade inom ett intervall.

Om X är likformig på intervallet $[a, b]$. Då gäller

$f_X(x) = \frac{1}{b-a}$, $a \leq x \leq b$

$X \sim Re(a, b)$

$E[X] = (a + b)/2$

$V[X] = (b - a)^2/12$

$dunif(x, a, b)$

$punif(x, a, b)$

11.3.4 Exponential-fördelningar

Tillämpning: Livslängd/väntetid.

Om X är exponetialfördelad med paramter $a > 0$. Då gäller
$f_X(x) = \frac{1}{\lambda}e^{-x/\lambda}$, $x \leq lambda$
$X \sim Exp(\lambda)$

$E[X] = 1/\lambda$
$V[X] = 1/\lambda$

$dexp(x, \lambda)$
$pexp(x, \lambda)$

11.3.5 Normalfördelning-fördelningar

Tillämpning: Allt möjligt.

Om X är normalfördelad med paramter μ och σ^2. Då gäller
$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma}, -\infty < x < \infty$
$X \sim N(\mu, \sigma^2)$

$E[X] = \mu$
$V[X] = \sigma^2$

$dnorm(x, \mu, \sigma)$ täthetsfunktionen (behövs inte för kontinuerliga)
$pnorm(x, \mu, \sigma)$ fördelningsfunktion (behövs inte för kontinuerliga)

11.3.6 Läges och spridningsmått

Väntevärde

$$\text{(Diskret) } E[X] = \sum_{x \in \Omega} x P_X(x)$$
$$\text{(Kontinuerlig) } E[X] = \int_\Omega x f_X(x)$$

varians

Om X är slumpvariabler med väntevärde μ. Då gäller

(Diskret) $V[x] = \sum_{x \in \Omega} (x-\mu)^2$

(Kontinuerlig) $V[X] = \int_{\Omega} (x-\mu)^2 f_X(x) dx$

Alternativ form

(Diskret) $E[X^i] = \sum_{x \in \Omega} x^i P_X(x)$

(Kontinuerlig) $E[X^i] = \int_{\Omega} x^i f_X(x)$

$V[x] = E[x^2] - (E[x])^2$

11.4 Olikheter

11.4.1 Markovs olikhet

Om X är ickenegatic $(x \leq 0)$ och $a > 0$. Då gäller

$p(x \geq a) \leq \frac{E[x]}{a}$

11.4.2 Thebysjovs olikhet (chebyshev)

Om X är en slumpvariabel med $E[X] = \mu$ och $V[X] = \sigma^2$. Då gäller

$p(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

11.4.3 Fördelnings functioner

(Diskret) $p(X \leq x) = F_X(x) = \sum_{i \leq x} P(X = x)$

(Kontinuerlig) $p(X \leq x) = F_X(x) = \int_{-\infty}^{x} f_X(x) dx$

Normalfördelning

Vi kan inte integrera täthetsfunctionen -ingen stängd form för fördelings funktionen
special fall: $X \sim N(0,1)$(standard normalfördelning)
Vi betecknar fördelnings funktionen $\Phi(x) = P(X \leq x)$

kvartier

För $0 < x < 1$ defineras α-kvantilen x_α till
slupvariabel x som en lösning till
$F_X(x_\alpha) = 1 - \alpha$

11.4.4 Oberonede slupvariabel

Två slumpvariabler $x_1 \wedge x_2$ kallas oberonde om
$P(x_1 \in A \cap x_2 \in B) = P(x_1 \in A)P(x_2 \in B)$
för alla mängder $A \sqsubseteq \Omega_1$, $B \sqsubseteq \Omega_2$

$$E[Y] = a + bE[x]$$
$$V[Y] = b^2 V[x]$$

räkneregler

$Y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$ slupvaribler x_i konstanter a_i
$E[Y] = a_1 E[x_1] + a_2 E[x_2] + \ldots + a_n E[x_n]$

Exempel (505): räkneregler

Låt $X \sim N(10, 4), Y \sim N(3, 1)$ vara oberonde slumpvariabler.
Beräkna sannolikheten att $X > 3Y$

Låt $Z = X - 3Y \Rightarrow E[Z] = E[X] - 3E[Y] = 10 - 3 * 3 = 1$
$V[Z] = E[X] + (-3)^2 E[Y] = 10 = (-3)^2 * 3 = 13$
Enligt normalfördelingien så får vi $Z \sim N(1, 13)$
Därmed kan vi beräkna följande sannolikhet$P(X - 3Y > 0) = P(Z > 0) = 1 - P(z \leq 0)$
$1 - \Phi(-1/\sqrt{13}) = 1 - \Phi(1 - \Phi(1/\sqrt{13})) = \Phi(0.28) = 0.61$

väntevärde av produkter

Om $x_1, \ldots, X_n$ är oberonde då gäller
$E[x_1, \ldots, X_n] = E[x_1] \ldots E[x_n]$

Special fall

$$\overline{x} = \frac{1}{n}(x_1, \ldots, X_n)$$
där x är oberonde och diktördelade med
$$E[x_i] = \mu,\ V[x_i] = \sigma^2$$
$$E[x] = \mu,\ V[x] = \frac{\sigma^2}{n}$$

11.4.5 Fördelning av summor

Binomialfördelning, Possionfördelning, Linjärkombination av normalfördelade variabler.

11.4.6 Central gränsvärdessatsen (CGS)

Exakt fördelning för summor är svårt i allmänhet
därmed andvänder man CGS

$$Y = X_1 + X_2 + \ldots + X_n$$
$$Y \sim N(\mu_Y, \sigma_Y^2)$$

Example

$$X_i \sim Bin(1, 0.2)$$
$$Y = \sum_{i=1}^{30} X_i$$
$$P(Y \leq 8) = ?$$

CGS: $Y \sim N(\mu_Y, \sigma_Y^2)$
$$\mu_Y = E[Y] = 30 * E[X_i] = 30 * 0.2 = 6$$
$$\sigma_Y^2 = V[Y] = 30 * V[X_i] = 30(1 * 0.2 * 0.8) = 4.8$$
$$P(Y \leq 8) = pnorm(8, 6, sqrt(4.8)) \approx 0.82$$
$$P(\sum_{i=1}^{30} X_i \leq 8) \approx 0.87$$

11.5 Simulering av slumptal

11.5.1 äkta slummpmässiga tal

Hårdvaru genererade tal:

- Tärningar, rolethjul ..
- Radiaktivit sönderfall
- Atmosfäriskt brus (random.org)

11.5.2 Pseudoslumpmässiga tal

Dator generede slump tal. Kommer att upredasig.

- Tar in en seed som input för att generera sluptal
- Defoult seed är oftast tid
- det är diterministiska
- Har en persiod med nya tal sedan så uprepar det sig

Von Neumann

1. Väljer seed: $u_0 = 0.1111$
2. Skapar $y_0 = 1111$
3. Beräknar $y_0^2 = 1234321$
4. Fyller på från venster med noll för att få 8 siffror01234321
5. Skapar genom att ta det fyra mitersta siffrorna $y_1 = 2343$
6. $0.2343 \Rightarrow y_1^2 = 5489649 \Rightarrow y_2 = 4896 \Rightarrow u_2 = 0.4896$

Kongruens

$V_{n+1} = aV_n + b(\mod c)$
Vi är ett heltal mellan 0 och $c-1$ och $u_1 = \dfrac{V_i}{c}$
a,b,c måste väljas noggrant (talteori, c måste vara ett primtal)
vanliga val är $a = 7^7 = 16807$, $b = 0$, $c = 2^{31} - 1 = 2147483647$
hat svagheter, andvänds inte längre

XOR-generator

Extrem snabb, lätt att förstå
1. Väljer seed: m binära bits (heltal mellan 0 och $2^m - 1$)
2. Skifta alla bits l steg åt vänster och fyll i från höger med nollor
3. XOR med seed och det skiftade talet (seed update)
4. Skifta seed update $m - l = r$ steg åt höger och fyll i nollor från vänster
5. XOR med seed update och högerskiftet
6. Konvertera till ett tal mellan 0 och 1

Mersenne Twister

Exempel (505): räkneregler

Givet 5 psudoslumpmässiga tal från $Re[0,1]$
$u_1 = 0.8147$, $u_2 = 0.9058$, $u_3 = 0.1270$, $u_4 = 0.9134$, $u_5 = 0.634$
Simulera 5 ?dosernar? från slumpvariablen X med
$f_X(x) = \frac{x}{2}$, $0 \leq x \leq 2$
Beräna $E[x]$ och jämför med medvärder av de simmulerade dosetaner

Tar fram primitiva functionen $F_X(x) = \int_0^x \frac{t}{2} dt = [t^2/4]_0{}^x = \frac{x^2}{4}$
$(Tarframinversen) y = \frac{x^2}{4} \Rightarrow x = 2\sqrt{y}$
$x_1 = F_x^{-1}(u_1) = 2\sqrt{x} = 1.8052$
$x_2 = F_x^{-1}(u_2) = 2\sqrt{x} = 1.9035$
$x_3 = F_x^{-1}(u_3) = 2\sqrt{x} = 0.7127$
$x_4 = F_x^{-1}(u_4) = 2\sqrt{x} = 1.9114$
$x_5 = F_x^{-1}(u_5) = 2\sqrt{x} = 1.5905$
Beräknar väntevärdet $E[x] \int_0^2 x \frac{x}{2} dx = [\frac{x^3}{6}]_0^2 = \frac{8}{6} = \frac{4}{3} \approx 1.33$

11.6 Statistikens grunder

11.6.1 Allmänt

Skattning av en okänd parameter θ från en familj $F_X(\theta)$ är en funktion
$\hat{\theta} = t = g(x_1, x_2, ..., x_n)$

11.6.2 Medelfel

$\sigma^2 \wedge p$ kan vara okända
Vi kan definera medel felet genom att andvända skattningen
$s = \hat{\sigma} \wedge \hat{p}$
$V[\hat{\mu}] = \dfrac{\sigma^2}{n}$
$V[\hat{p}] = \dfrac{\hat{p}(1-\hat{p})}{n}$
$d[T_\mu] = \dfrac{s}{\sqrt{n}}$
$d[T_p] = \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$
Utifrån medelfelet definerar vi konsistens och effektivitet
Konsistans $V[T] \to 0$ då $n \to \infty$
Effektivitet Givet estimatorn T_1, T_2 så är T_1
effekten att T_2 om $V[T_1] < V[T_2]$ $(D[T_1] < D[T_2])$

Exempel

Vi beräknar variansen på T_μ och T_μ
$V[T_\mu] = V[1/n(x_1, x_2, ..., x_n)]$
$V[T_\mu] = 1/n^2(V[x_1] = V[x_2] + ... + V[x_n])$
$V[T_\mu] = \dfrac{1}{n^2} n\sigma^2 = \dfrac{\sigma^2}{n}$
$V[T_p] = V[x/n]$
$V[T_p] = \dfrac{1}{n^2} V[x]$
$V[T_p] = \dfrac{1}{n^2} np(1-p)$
$V[T_p] = \dfrac{p(1-p)}{n}$

11.6.3 Skattning av varians

$N(\mu, \sigma^2)$ -Vill skatta σ^2 det går att visa att
$S^2 = \hat{\sigma}^2 = \dfrac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2$
är en venteriktig skattning av variansen

Vi kan visa att
$S_p^2 = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2 + (n_3-1)S_3^2}{(n_1-1) + (n_2-1) + (n_3-1)}$
är en väntevärdsriktig skattning av σ^2

Def

Låt A och B vara funktioner av $x_1, x_2, ..., x_n$ så att
Då kallas $[A, B]$ ett $100(1-\alpha)$ -procent
konfidensintervall för θ (med konfidensgrad $1-\alpha$)

11.6.4 Väntevärdsriktig

Exempel

$X \sim N(m_1 - m_2, 4),\ Y \sim N(m_1 + m_2, 5)$
a. Visa att $\hat{m}_1 = (X+Y)/2$ är väntevärdesriktig skattning av m_1
$E(\hat{m}_1) = E[(X+Y)/2] = 1/2(E[X]+E[Y]) = 1/2(m_1 - m_2 + m_1 + m_2) = \frac{1}{2}2m_1 = m_1$
därmed så är den väntevärds riktig

b. Beräkna standardavvikelsen för $\hat{m}_1$
$D(\hat{m}_1) = \sqrt{V(\hat{m}_1)} = \sqrt{V(\frac{x+y}{2})} = \sqrt{\frac{1}{4}(V[X]+V[Y])} = \frac{\sqrt{9}}{2} = \frac{3}{2}$

11.6.5 Konfidensintervall för μ från $N(\mu, \sigma^2)$ med känt σ

Estimatorn $\hat{\mu} = \overline{x} = \frac{1}{n}(X_1 + X_2 + ... + X_n)$
$\overline{X} \sim N(\mu, \sigma^2/n)$
$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$A = \overline{X} - \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
$B = \overline{X} + \lambda_{\alpha/2}\frac{\sigma}{\sqrt{n}}$

11.6.6 Example

vid tillverkningsproncessen av axlar av rundstål kontrollers diametern.
Följande diametrar (mm) uppmättes:
30.02, 30.12, 30.07, 29.95, 30.05, 29.90, 30.01
Konstruera ett konfidensintervall, med konfidensgrad 0.95, för den förväntade diametern.

$n = 7$, $\alpha = 0.05$

$$\overline{X} = \frac{30.02 + 30.12 + 30.07 + 29.95 + 30.05 + 29.90 + 30.01}{n} = \frac{210.12}{7}$$

$$s = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{6}((30.02 - \overline{x})^2 + (30.02 - \overline{x})^2)$$

11.6.7 Konfidensintervall för p från binomialfördelad

Finns många alternativa metoder

$Bin(n,p) \sim N(np, np(1-p))$ om $np(1-p) \geq 10$
$\Rightarrow I_p = [\hat{p} \pm \lambda\sqrt{\hat{p}(1-\hat{p})/n}]$
konfidensintervall för θ (med konfidensgrad $1-\alpha$)

11.6.8 Konfidensintervall för skillnad i väntevärde

Vill ofta jämföra två grupper

$x_1, ..., x_n$ från $N(\mu_1, \sigma_1^2)$
$y_1, ..., y_n$ från $N(\mu_2, \sigma_2^2)$
Söker $\mu_1 - \mu_2$. Om detta intervall inhåller 0 då kan vi med konfidensgrad $1-\alpha$ säga att det är skillnad på väntevärdena

Okända varianser

$$s_p^2 = \frac{(n-1)s_1^p + (m-1)s_2^2}{(n-1)+(m-1)}$$

$$\frac{\overline{X} - \overline{Y} - (\mu_1 - \mu_2)}{s_p\sqrt{1/n + 1/m}}$$

$$I_{\mu_1-\mu_2} = [\overline{X} - \overline{Y} \pm t_{\alpha/2}(n+m-2)s_p\sqrt{1/n+1/m}]$$

11.6.9 Ensidiga intervall

Låt A och B vara funktioner av $x_1, ..., x_n$
För ett nedåt begänsat konfidensintervall gäller
$p(\theta \geq A) = 1 - \alpha$
För ett uppåt begrensat gäller
$p(\theta \leq B) = 1 - \alpha$

11.6.10 Stickprov i par

Fotgängare	A	B
1	43	32
2	81	90
3	11	7
4	49	31
5	22	26
6	143	168
7	24	31
8	56	39
9	31	29
10	53	57

Tiden det tar för A respektive B att upptäcka forgängaren baserat står i tabellen
Konstruera ett lämpligt konfidensintervall baserat på datan. Vilken algoritm borde andvändas

Vi väljet ett konfidensintervall på 95%, $Z = A - B$

$\overline{Z} = 3/10 = 0.3,\ s_Z^2 = \frac{1}{10-1}\sum_{i=1}^{10}(Z_i - \overline{Z}) = 13.08,\ t_{0.025}(t) = 2.26$

$I_\mu = [\overline{Z} \pm t_{0.025}(t)\frac{s}{\sqrt{n}}]$

$= [-9.05, 9.65]$

0 finns i intervallet, med konfidensgrad 0.95 kan ingen skillnad påvissas
mellan algoritmerna samla in mer data, allternativ andvänd vilken algoritm som helst

11.7 Regrission

11.7.1 Modell

Givet observationsparen $x_1, ..., x_n$ och $y_1, ..., y_n$ ansätter man följade modell
$Y_i = m + kx_i + \epsilon_i$ Där $\epsilon_i \sim N(0, \sigma^2)$

11.7.2 Modellens giltighet

$$\text{Konrelationskoeficent: } r = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}}$$

Förklaringsgrad

$$R^2 = \frac{S_{xy}}{\sqrt{S_{xx} \cdot S_{yy}}}$$

Residualer

$e_i = y_i - (\hat{m} + \hat{k}x_i)$

Residualer bör uppfylla visa krav
(1) konstat varians, oberonde av x
(2) Residualerna bör vara oberonde av varandra
(3) Residualerna bör vara normalfördelade

Vi bedömmer dessa visuelt
(i) Plottar residualerna i ett histogram, där vi kan se om det kan vara normal fördelat
(ii) Plottar residualerna i q-q plot
På x-axeln kvatiler från en s..fördelning
På y-axeln kvatiler från residualerna
(iii) Ritar ett spridnings diagram över x-värderna mot residualerna
ej ett mönster $\Rightarrow$ från X
(iiii) Ritar ett spridnings diagram över residualerna mot det förutspodda y-värdet
vill se ett jämt utpridning utan mönster

11.7.3 Användning av modellen

Konfidensintervallet för parametern k

$$V[\hat{m}] = \frac{\sigma^2}{n}\frac{1}{S_{xx}}\sum_{i=1}^{n} x_i^2$$

$$V[\hat{k}] = \frac{\sigma^2}{S_{xx}}$$

11.8 Stokastiska processer

11.8.1 Bornulli processer

Kommunikationskanal, överföring av slumpmäsiga data. Tiden är uppdelad i luckor (slots) $k = 1, 2, 3, ..., n$ varje lucka kan hantera ett paket

11.8.2 Poisson processer

Proposition:

För en poissonprocess $N(t), t \geq 0$ gäller att
(a) inkreten $N(t_1), N(t_2) - N(t_1), ..., N(t_k) - N(t_{k-1})$ är oberonde slumpvariabel för alla $0 \leq t_1 \leq ... \leq t_k$ och
(b) $N(t) - N(s) \sim Po(\lambda(t - s))$

Ex:

Man att aantal fel på en kommunikationskabel är 1.7. Total antal fel beskrivs av en poisson process med parameter $\lambda = 1.7$ Vad är sannolikheten att det finns mer än två fel på 0.5 kilometer?

$N(0.5) \sim Po(\lambda(0.5 - 0)) = Po(0.85) \Rightarrow P(N(0.5) > 2) = 1 - P(N(0.5) \leq 2)$
$= 1 - ppois(2, 0.85) \approx 0.0549$

Förtunning

Låt $N(t)$ vara en poisson process med parameter λ
Låt $J_n, n \in \mathbb{N}$ vara en följ av i.i.d. $Be(p)$
$P(J_i = 1) = p, P(J_i = 0) = 1 - p$

$$M(t) = \sum_{k=1}^{\infty} \mathbb{1}_{T_k \leq t} J_k$$

Vi kan tolka att vi skippar vissa händelser

Superposition

Låt $N(t)$ och $N_2(t)$ vara en poisson process med parameter λ_1 och λ_2
Vi vill visa att $M(t) = N_1(t) + N_2(t)$ är en possion process med parameter λ_1 och λ_2

Ex:

Antal inkomade samtal till en mobil kan beskrivas som en poissonprocess med paramter 0.5 per time och antal sms som en poissonprocess medparamter 2 två fel på 0.5 kilometer?

(a) sannoliket att ingen kommunikationskabel inkomer på en time
$N_1(t)$ med parameter $\lambda_1 = 2$, -antal sms
$N_2(t)$ med parameter $\lambda_2 = 0.5$, -antal samtal
$M(t) = N_1(t) + N_2(t)$ med parameter 2.5, -antal kommunicationer
$= 1 - ppois(2, 0.85) \approx 0.0549$

Spatial process

En samling punkter i en region $S \leq \mathbb{R}^2$
Om $N(A)$ räknar antal punkter i en mängd A (där A är mätbar)
då gäller att
*$N(A) \sim Po(\lambda|A|)$
* Om $A \cap B = \emptyset$ är $N(A)$ och $N(B)$ oberonde

M/M/∞-modellen

$N(t)$ -Poissonprocess, ankomster upp till tid t
ankomsterna tar tid att behandla
hur många ankomster pågår vid en viss tid?
X_t -antal pågående ankomster vid tid t
Vi antar att varje ankomst behandlas på $exp(\lambda)/tid$

11.9 Markovkedjor

En Stokastisk process X_n i diskret tid med diskret tillståndsrum E kallas en Markovkedja om den har Markovegenskapen
$P(X_n = x_n|X = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) = P(X_n = x_n|X_{n-1} = x_n - 1)$, $\forall x_1 \in E \wedge n \geq 1$
och övergångssanolikheten $p_{xy} = P(X_n = y|X_{n-1} = x)$ är oberonde från n
Vi fokuserar på $E = 0, 1, ..., r$ och $E = \mathbb{Z}_{\geq 0}$

Övergångsmatris:

$$\begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0r} \\ p_{10} & p_{11} & \cdots & p_{1r} \\ \vdots & \vdots & \cdots & \vdots \\ p_{r0} & p_{r1} & \cdots & p_{rr} \end{pmatrix}$$

11.9.1 Ehrenfestmodellen

Oavsett startpunkt tenderar kedjan mot ett ekvilibrium

Övergångsmatris:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 & 0 \\ 1/r & 0 & 1-1/2 & 0 & ... & 0 & 0 \\ 0 & 2/r & 0 & 1-2/r & ... & 0 & 0 \\ : & & & & & : & : \\ 0 & ... & ... & ... & ... & 1 & 0 \end{pmatrix}$$

Def: stationär

En fördelning π kallas stationär för en Markovkedja med övergångsmatris P om den löser ekvationssystemt

$\pi = \pi P$

- π är en egenvektor för P med egenvärde 1
- Om π är ursprungsfördelningen $p^0 = \pi \Rightarrow p^n = \pi$, $\forall n \geq 0$

Def: asymptotisk

En fördelning π är en asymptotisk fördelning för Markovkedjan X_n om $\lim\limits_{0 \to \infty} P(X_n = k) = \pi_k$, $\forall k \geq 0$ oberonde avursprungsfördelningen p^0

Asymptotiska fördelningar är alltid stationära, dock inte tvärt om

Def: Irreducibel, Aperiodisk

En markovkedja X_n kallas

-Irreducibel om $P(X_n = j|X_0 = i) > 0$ för något n och alla $i, j \in E$

-Aperiodisk om största gemensamma delaren av mängden $n : P(X_n = i|X_0 = 1) > 0$ är 1 för alla i

Om en kedja är irreducibel och något $p_{ii} > 0$ är kedjan aperiodisk

Def: Tillståndsrummet

Låt X_n vara aperiodisk och irreducibel

(1) Om tillsåndsrummet är ändligt finns en unik stationär fördelning som också är asymptotisk

(2) Om tillsåndsrummet är oändligt, då om en stationär fördelning existerar är den unik och asymptotisk

Exempel: Markovkedjor

$$P = \begin{pmatrix} 3/4 & 0 & 1/4 \\ * & 1/3 & 0 \\ 1/4 & 1/2 & ** \end{pmatrix}$$

(a) Ange $*$ och $**$

(b) Rita övergångsgrafen

(c) argumentera för att kedjan är aperiodisk och irreducibel

(d) Bestäm den stationära fördelningen

(a) $* = 1 - (1/3 + 0) = 2/3$, $** = 1 - (1/4 + 1/2) = 1/4$

(b) Rita övergångsgrafen

(c) Som man ser på övergångsgrafen att den man kan ta sig till alla posistioner därmed så är den irreducibel vilket också medföljer aperiodisk

(d) $(\pi_0\pi_1\pi_2) = (\pi_0\pi_1\pi_2) \cdot p \begin{pmatrix} 3/4 & 0 & 1/4 \\ 2/3 & 1/3 & 0 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$

$$\begin{cases} \pi_0 = 3/4\pi_0 + 2/3\pi_1 + 1/4\pi_2 \\ \pi_1 = 0\pi_0 + 1/3\pi_1 + 1/2\pi_2 \\ \pi_2 = 1/4\pi_0 + 0\pi_1 + 1/4\pi_2 \end{cases}$$

$\pi_2 = \pi_0/3$, $\pi_1 = \pi_0/4$

$\pi = (12\ 3\ 4)$ Normalisering: $\pi = (12/19\ 3/19\ 4/19)$

11.9.2 Google-kedjan

Google skapar en graph av alla sidor
Tillståndet efter n steg beskrivs av en Markovkedja
med denna övergångsmatrisen
Google letar efter den asymptotiska fördelningen på
kedjan och rankar sidorna i sökresultatet enligt sannolikheterna
i den asymptotiska fördelningen

11.9.3 Hashfunktioner

En funktion h som tar n visare och sparar som någon av m möjliga hasvärden $(m < n)$

11.9.4 Kollisionmodoll

Modeleras med Bernoulli-processen

11.9.5 Markov Buffer/Markovkö

X_n -anatal packet som anländer under slot n

Q_n -anatal packet i kö slutet av slot n

Chapter 12

Digital Technology and Electronics

Resorcess

- https://www.falstad.com/circuit/circuitjs.html
- http://www.32x8.com/var4kmapx.html
- https://www.tinkercad.com/dashboard?type=circuits&collection=designs
- https://www.symbolab.com/solver/system-of-equations-calculator

12.1 Circuit Theory

12.1.1 Basic electical quantities

Current

$i(A)$: amps

Positive repels Positiv. Negative repels Negative. Positive and Negative force of attraction. Copper is commonly used in wires since there is only one electron in last orbital. This makes it easy to move therefore it is higly conductive.

$I = q^{-}/sec$ Current is charge per second.

Current is writen from Positiv to Negative. Since back in history Ben Franklin who studied elektriesaty and came up with theories. However the were unaware that electrons exsisted and there fore made the false assumption that the current goese from positive to negative. With infact the electron is attracted to the posiive end and therefore flows that way. Ben Franklin 1747. JJ Thompson 1897 e^{-}. However positve chage dose exist like in your body then it would be true.

Electron current is from negative to possitive. Conventional current (the current we know as) is from positive to negative.

Voltage

$v(V)$: volt
Voltage is simmular to gravity. It is the potential difference in the two pols, in otherwords like potatial enegry like in the gravity analagy.

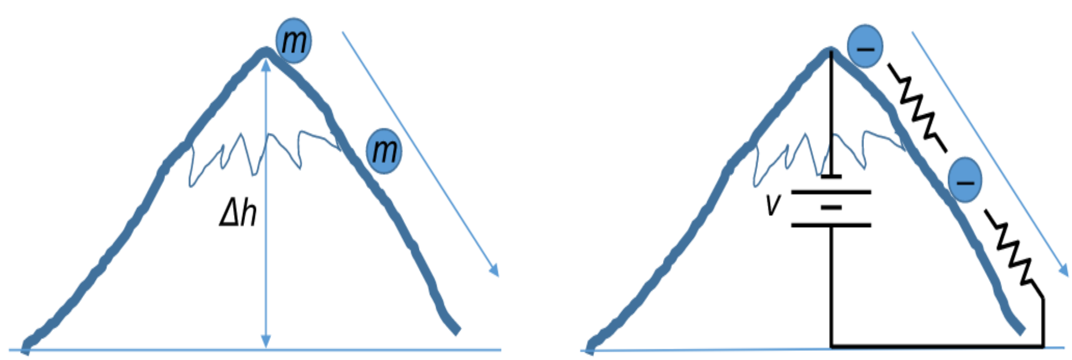


Figure 12.1: volt. From ()

$V = \frac{\Delta U}{q}$ Where ΔU is the potational energy diffrent and q is charge particle

Power

Power is defined as the rate energy (U) is transformed or transferred over time. We measure power in units of joules/second, also known as watts. 1 watt = 1 joules / second)

An electric circuit is capable of transferring power. Current is the rate of flow of charge, and voltage measures the energy transferred per unit of charge. We can insert these definitions into the equation for power: $p = \frac{dU}{dt} = \frac{dU}{dq} * \frac{dq}{dt}$. $(dU = \Delta U)$

$$p = v * i \tag{12.1}$$

12.1.2 Prefixes

Number	Prefix	Symbol	Note
10^{+10}	tera-	T	
10^{+9}	giga-	G	
10^{+6}	mega-	M	
10^{+3}	kilo-	k	the only > 1 prefix in lower case
10^{0}			
10^{-3}	milli-	m	
10^{-6}	micro-	μ	be careful μ mu doesn't turn into "m"
10^{-9}	nano-	n	
10^{-12}	pico-	p	

12.1.3 Unit grammar

Symbol name	example names	Symbol	example symbols	Named after
second	1 millisecond	s	$2ns$	
meter	300 kilometer	m	$35nm35$	
hertz	10 kilohertz	Hz	$100MHz$	Hertz
ohm	2 megohm	Ω	$47k\Omega$	Ohm
farad	10 picofarad	F	$220pF220$	Faraday
ampere	35 microamp	A	$65mA$	Ampère
volt	11 kilovolt	V	$5\mu V$	Volta

12.1.4 Ohm's law

- V (Voltage): V (volt)
- I (Current): A (amps)
- R (Resistance): Ω (omega)

$$V = IR \tag{12.2}$$

12.1.5 Circuit elements

12.1.6 Circut termenology

- *Closed circuit* – A circuit is closed if the circle is complete if all currents have a path back to where they came from.
- *Open circuit* – A circuit is open if the circle is not complete if there is a gap or opening in the path.

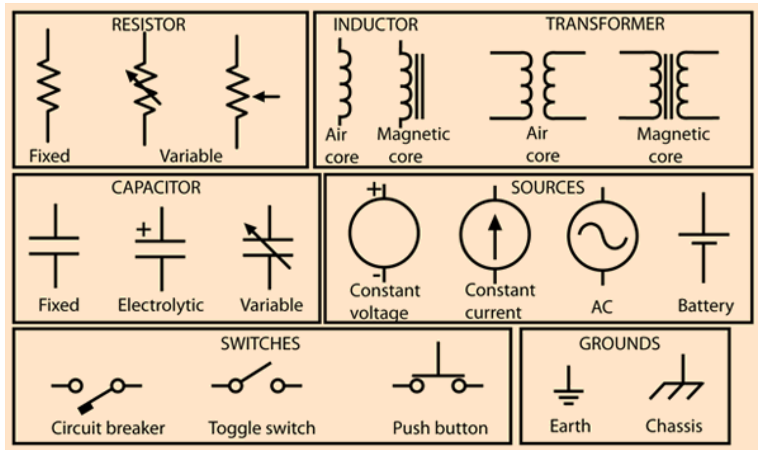


Figure 12.2: From () Circuit elements

- *Short circuit* – A short happens when a path of low resistance is connected (usually by mistake) to a component. The resistor shown below is the intended path for current, and the curved wire going around it is the short. Current is diverted away from its intended path, sometimes with damaging results. The wire shorts out the resistor by providing a low-resistance path for current (probably not what the designer intended).

- *Node* - Between elements

- *Branch* - path between two noodes.

- *Loop* - Close path in a circuit (Branch - Nodes + 1)

12.1.7 Series Resistor

$$R_{series} = R_1 + R_2 + \ldots + R_n \tag{12.3}$$

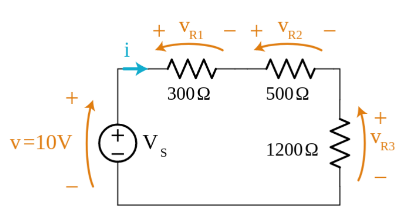


Figure 12.3: voltage distributes between resistors in series. From ()

$$R_{series} = 300\Omega + 500\Omega + 1200\Omega = 2000\Omega$$
$$i = \frac{v}{R_{series}} = \frac{10V}{2000\Omega} = 5mA$$

$$v_{R1} = i \cdot R1 = 5mA \cdot 300\Omega = 1.5V$$
$$v_{R2} = i \cdot R2 = 5mA \cdot 500\Omega = 2.5V$$
$$v_{R3} = i \cdot R3 = 5mA \cdot 1200\Omega = 6.0V$$
$$10V - 1.5V - 2.5V - 6.0V = 0V \text{ Therefore correct}$$

12.1.8 Parralel resistor

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} \tag{12.4}$$

$$R_{parallel} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} \tag{12.5}$$

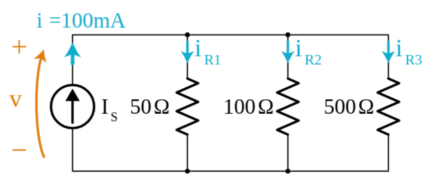


Figure 12.4: current distributes between resistors in parallel. From ()

$$R_{parallel} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$
$$R_{parallel} = \frac{1}{\frac{1}{0.02} + \frac{1}{0.01} + \frac{1}{0.002}} = \frac{1}{0.032} = 31.25\Omega$$
$$v = i \cdot R_{parallel} = 100mA \cdot 31.25\Omega = 3.125V$$

$$i_{R1} = \frac{v}{R1} = \frac{3.125V}{50\Omega} = 62.5mA$$
$$i_{R2} = \frac{v}{R2} = \frac{3.125V}{100\Omega} = 31.25mA$$
$$i_{R3} = \frac{v}{R3} = \frac{3.125V}{500\Omega} = 6.25mA$$
$$100mA - 62.5mA - 31.25mA - 6.25mA = 0mA \text{ Therefore correct}$$

12.1.9 Simplify resistor network

Start from the furthest point. Make parallel and serial resistors become one.

12.1.10 Voltage divider

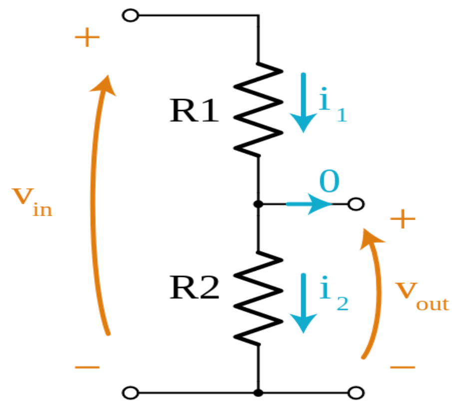


Figure 12.5: voltage-divider. From ()

$$
\begin{aligned}
v_{in} &= i(R_1 + R_2) \Rightarrow I = \frac{1}{R_1 + R_2} v_{in} \\
v_{out} &= iR_2 \Rightarrow v_{out} = \frac{R_2}{R_1 + R_2} v_{in} \\
&= \frac{1}{\frac{R_1}{R_2} + 1} v_{in}
\end{aligned}
$$

12.1.11 Kirchhoff's laws

Kirchhoff's current law

KCL

$$\sum i_{in} = \sum i_{out} \tag{12.6}$$

Kirchhoff's voltage law

KVL

$$\sum_n V_n = 0 \tag{12.7}$$

$$\sum V_{rise} = V_{drop} \tag{12.8}$$

12.1.12 Node voltage method

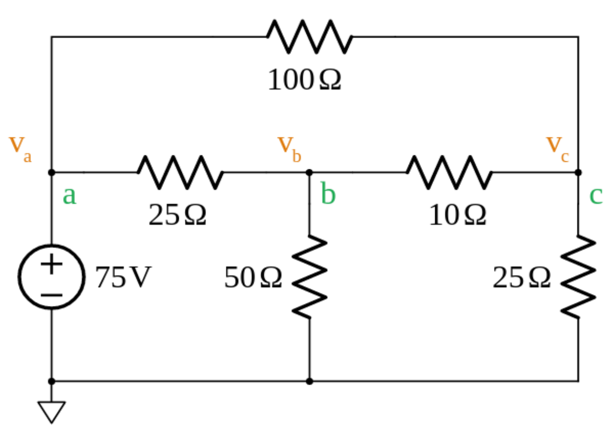


Figure 12.6: Example node voltage method. From ()

1. **Assign a reference node (ground).**
Assign it under Voltage sorce

2. **Assign node voltage names to the remaining nodes.**
See imgage for $v_a,\ v_B,\ v_c,\ a,\ b,\ c$

3. **Solve the easy nodes first, the ones with a voltage source connected to the reference node.**
V_a is the easyest since it is the same as the input $v_a = 75V$

4.**Write Kirchhoff's Current Law for each node. Do Ohm's Law in your head.**
Node b: $+\dfrac{v_a - v_b}{25} - \dfrac{v_b}{50} + \dfrac{v_c - v_b}{10} = 0$
Node c: $+\dfrac{v_a - v_c}{100} - \dfrac{v_b - v_c}{10} + \dfrac{v_c}{25} = 0$

5.**Solve the resulting system of equations for all node voltages.**
Node b: $-\dfrac{v_b}{25} - \dfrac{v_b}{50} - \dfrac{v_b}{10} + \dfrac{v_c}{10} = -\dfrac{v_a}{25}$
$\Rightarrow -\dfrac{8}{50}v_b + \dfrac{1}{10}v_c = -\dfrac{75}{25} = -3$
Node c: $+\dfrac{v_b}{10} - \dfrac{v_c}{100} - \dfrac{v_c}{10} + \dfrac{v_c}{25} = -\dfrac{v_a}{100}$
$\Rightarrow +\dfrac{1}{10}v_b - \dfrac{15}{100}v_c = -\dfrac{75}{100} = -\dfrac{3}{4}$

Solve by gauseelimination and then get $v_b = +37.5V,\ v_c = +30V$

6.**Solve for any currents you want to know using Ohm's Law.**
$i_{ab25\Omega} = \dfrac{75 - 37.5}{25} = 1.5A$ arrow pointing right
$i_{bg50\Omega} = \dfrac{37.5}{50} = 0.75A$ arrow pointing down
$i_{bc10\Omega} = \dfrac{37.5 - 30}{10} = 0.75A$ arrow pointing right
$i_{ac100\Omega} = \dfrac{75 - 30}{100} = 0.45A$ arrow pointing right
$i_{cg25\Omega} = \dfrac{30}{25} = 1.2A$ arrow pointing down

12.1.13 Mesh current method

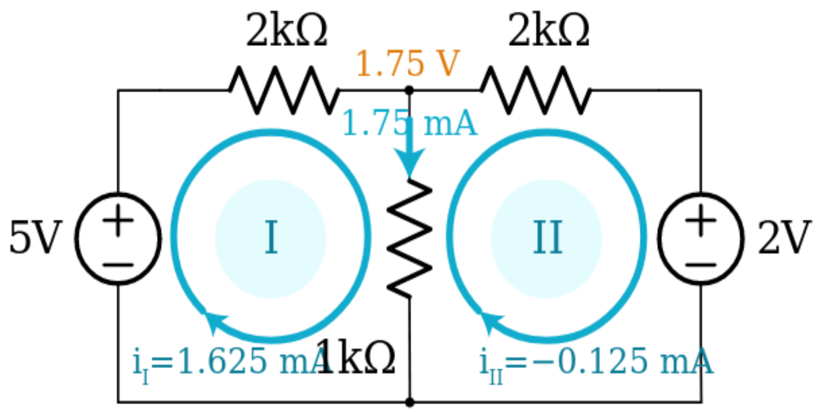


Figure 12.7: Mesh current method. From ()

Mesh I:
+ The voltage sorce − voltage drop over first resistor − voltage over second resitor = 0 (KVL)
$+5V - 2000i_I - 1000(i_I - i_{II}) = 0$

Mesh II:
+ Voltage over first resistor − Voltage over second resistor − Voltage sorce
$+1000(i_I - i_{II}) - 2000i_{II} - 2V = 0$

Gauseelimination gives us $i_{II} = -0.125mA,\ i_I = +1.625mA$

12.1.14 Capacitor i-v equation

$$i = C\frac{dv}{dt},\ v = \frac{1}{C}\int_0^T idt + v_0 \tag{12.9}$$

- C is the capacitance, a physical property of the capacitor (ex: $10\mu F$)
- $\frac{dv}{dt}$ reate of change for volt over time (ex: $100 volts/second$)

12.1.15 inductor i-v equation

$$v = L\frac{di}{dt},\ \frac{1}{L}\int_0^T vdt + i_0 \tag{12.10}$$

- L is the inductance, a physical property of the inductor (ex: $1mH$)
- $\frac{di}{dt}$ is the reate of change for the current over time (ex: $300 ampere/second$)

12.1.16 RC circuit

RC natural response

When initializes the circuit with a voltage and the capacitor will then be charged. Then we disconnect the voltage source, the capasitor will act as a battary and edvenshula discharge.

$$v(t) = v_0 e^{-t/RC} \tag{12.11}$$

- Time constant: $\tau = RC$. It is the time it takes to charge the capacitor, through the resistor.
- v_0 is the voltage at $t = 0$

Example:

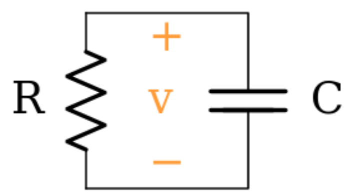


Figure 12.8: EX RC natural response. From ()

a. Write the expression for $v(t)$

$v(t) = V_0 e^{-t/RC} = 1.4e^{-t/(3k\Omega \cdot 1\mu F}1.4e^{-t/3ms}$

b. What is $v(t)$ when $t = RC$?

$\tau = RC = 3 \times 10^3 \cdot 1 \times 10^{-6} = 3ms$

$v(3ms) = 1.4e^{-3ms/3ms} = 1.4 \cdot 1.4 \cdot 0.3679 = 0.515V$

c. Plot $v(t)$

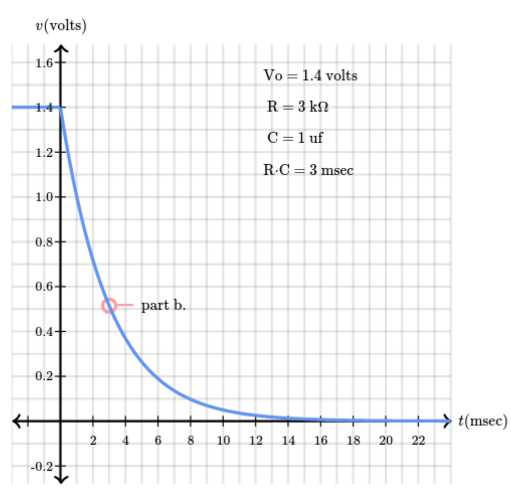


Figure 12.9: Plot RC natural response. From ()

RC step response

$$\text{Total} = \text{Forced response} + \text{Natural response} \tag{12.12}$$

- Foreced response: No start charge on capasitor and connected voltage supply
- Total responsee: what the circuit actoaly response.

$$v(t) = V_s + (V_0 - V_S)e^{-t/RC} \tag{12.13}$$

- V_S is the height of the voltage step
- V_0 is the inital voltage on the capacitor

LC natural response

$$s^2 + 1/LC = 0 \tag{12.14}$$

Euler's identities

$$e^{+jx} = \cos x + j \sin x$$
$$e^{-jx} = \cos x - j \sin x$$

where j is the name for $\sqrt{-1}$
Current function of time

$$i(t) = \sqrt{\frac{C}{L}} V_0 \sin \omega_o t \tag{12.15}$$

Natural frequency of LC circuit

$$\omega_0 \equiv \sqrt{\frac{1}{LC}} \tag{12.16}$$

RLC

The RLC end text circuit is the electronic equivalent of a swinging pendulum with friction.

$$s = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} \tag{12.17}$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} \tag{12.18}$$

where $\alpha = \frac{R}{2L}$ and $\omega_o = \frac{1}{\sqrt{LC}}$

12.2 Aplifier

Saturates mean that it will become a state line since it can not become more than the input nor can it be less than what is going in the opposite direction. $v_0 = A(v^+ - v^-)$

The current to v^- and v^+ will be close to zero and in an ideal circuit will be zero.
We can say $v^- = v^+$ when there is a feedback loop (negative terminal and noninverting amp).

12.2.1 Inside Op-Amp

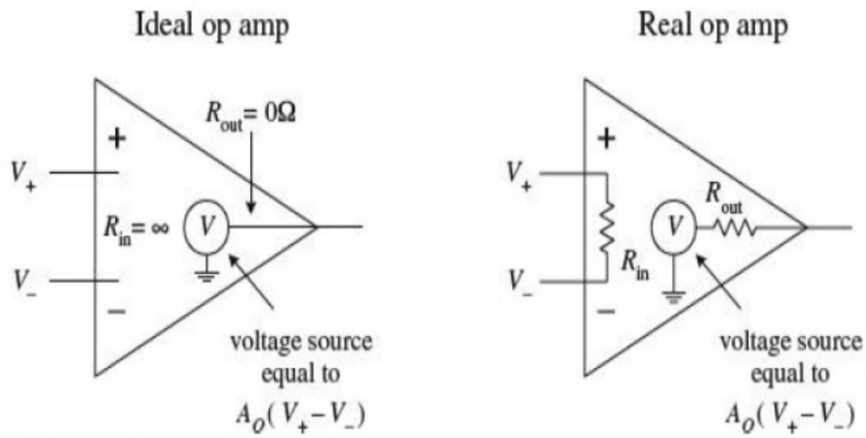


Figure 12.10: Inside opAmp. From ()

In ideal op amp the resistancce will be infinity. However the in realaty there is not infinit resistance therfore if one would have a voltage devider with larger resistanse then the internal resistanse the current will go threw V_+ and V_-. The output in 0Ω so we will not have any voltage drop, not in reality however.

12.2.2 Implementation of Op-Amp

Comparator

An op amp without negative feedback (a comparator)

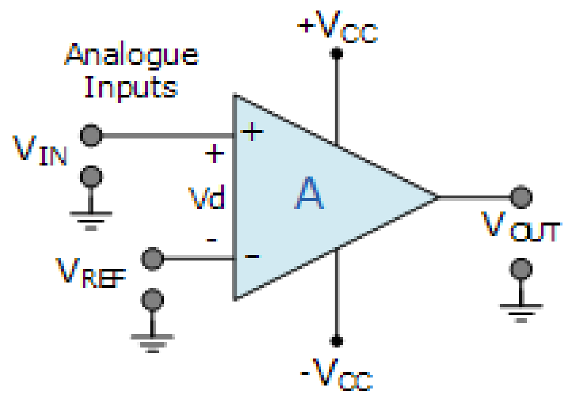


Figure 12.11: Op-Amp comparator. From ()

When $V_+ > V_-$ then $V_o = +V_{cc}$. When $V_+ < V_-$ then $V_o = -V_{cc}$

Non inverting (with feedback loop)

Amplefies the input but dose not change it sign (non inverting). An op amp with negative feedback (a non-inverting amplifier)

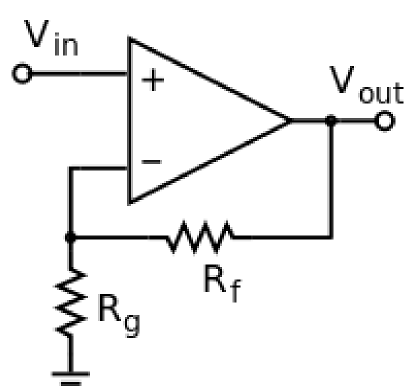


Figure 12.12: Op-Amp Feedback (Non inverting). From ()

Voltage follower

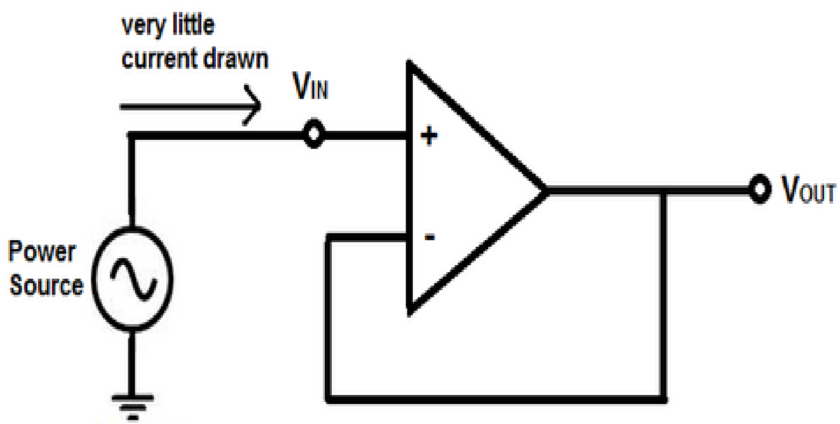


Figure 12.13: Op-Amp voltage follower. From ()

$v_+ = v_- = v_{in} = v_{out}$

Inverting

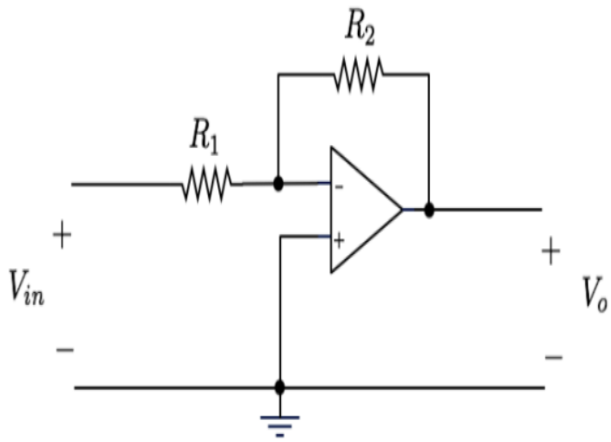


Figure 12.14: Op-Amp voltage follower. From ()

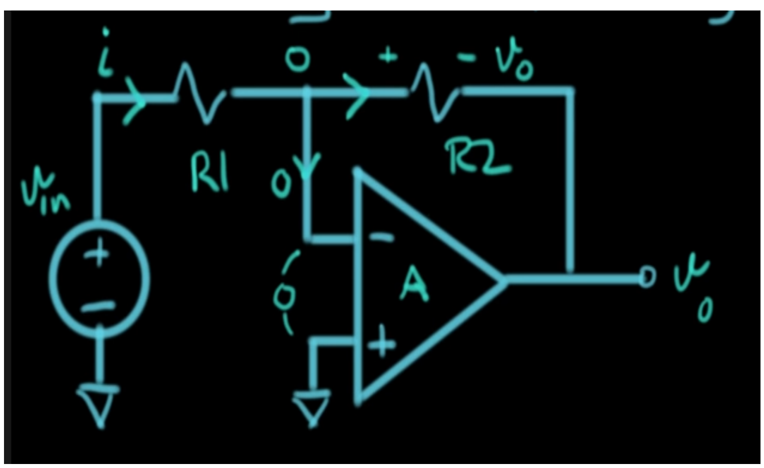


Figure 12.15: Op-Amp calculations for inverting. From ()

As seen from the image the current will be $i = v_{in}/R_1$ and $i = -v_0/R_2$. We combined them and get: $V_0 = \frac{-R_2}{R_1} V_{in}$

Summing

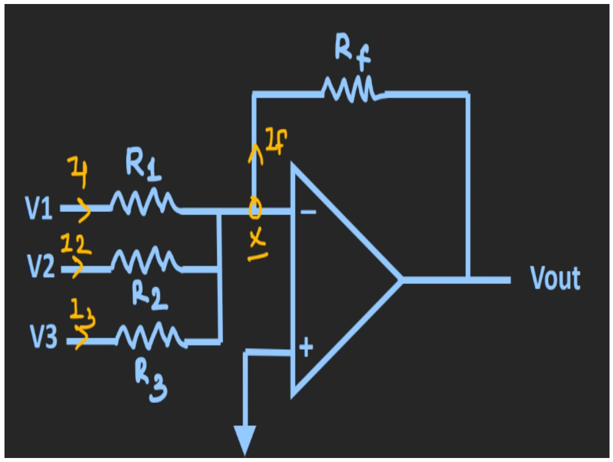


Figure 12.16: Op-Amp summing. From ()

Appling KCL we get: $I_f = I_1 + I_2 + I_3$ $\frac{v_1 - 0}{R_1} + \frac{v_2}{R_2} + \frac{V_3}{R_3} = \frac{0 - v_{out}}{R_f} \Rightarrow v_{out} = -(\frac{R_f}{R_1}V_1 + \frac{R_f}{R_2}V_2 + \frac{R_f}{R_3}V_3)$

12.2.3 Gerneral example

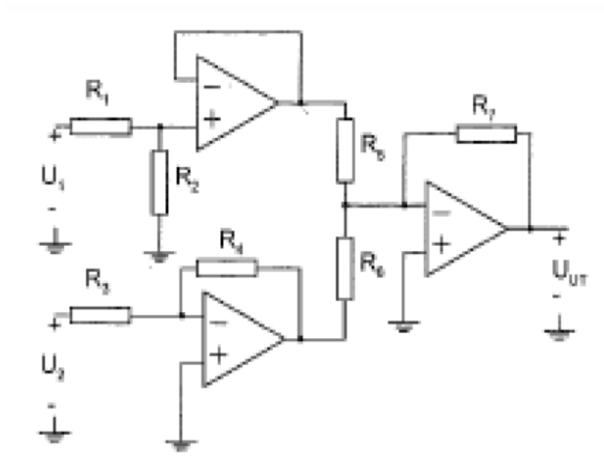


Figure 12.17: Op-Amp example. From ()

General opservations: We can divide the circuit into three pieces. The one on the top is a *Voltage follower*, we can call it circuit 1. The one on the bottom is a *Inverting op-amp* as is the middle one. We can name the bottom one circuit 2 and the middle one is circuit 3.

We start with circuit 3:
We name node **A** between R_5 and R_6. KCL: $I = I_5 + I_6$ and $I = I_7$ since there is no current in $-$ terminal.
$I = \frac{-U_{ut}}{R_7} = I_5 + I_6 = \frac{v_{O1}}{R_5} + \frac{v_{O2}}{R_6}$
Because of Ohm's law and $v_{overresistor} = v_{head} - v_{tail}$.

To solve v_{O1} we need to solve circuit 1:
Applying KCL between R_1 and R_2 we get $I_1 = I_2 + I_3$ were $I_3 = 0$ since there is no current going in the + terminal. $I_1 = I_2$ Ohm's law result in $\frac{U_1 - v_{O1}}{R_1} = \frac{V_{O1} - 0}{R_2}$ since in a voltage follow circuit
$v^- = v^+ = v_{out}$. We then get $v_{out} = \frac{R_2}{R_1 + R_2} U_1$
then $I_5 = \frac{R_2}{(R_1 + R_2) R_5} U_1$.

To solve v_{O2} we need to solve circuit 2:
KCL gives $I_3 = I_4$ since there is no current in $-$ terminal.
Ohm's law $\frac{U_2 - v^-}{R_3} = \frac{v^- - v_{O2}}{R_4}$.
$v^- = v^+ = 0$ with result in $\frac{U_2}{R_3} = \frac{-v_{O2}}{R_4}$.
$v_{O2} = \frac{-R_4}{R_3} U_2$
The current $I_6 = \frac{-R_4}{R_3 R_6} U_2$

Going back to circuit 3:
$I = I_5 + I_6 = \frac{R_2}{(R_1 + R_2) R_5} U_1 + \frac{-R_4}{R_3 R_6} U_2$
results in $\frac{-U_{ut}}{R_7} = \frac{R_2}{(R_1 + R_2) R_5} U_1 + \frac{-R_4}{R_3 R_6} U_2$.
Finnal result: $U_{out} = -R_7 (\frac{R_2}{R_5 (R_1 + R_2)} U_1 - \frac{R_4}{R_3 R_5} U_2)$

12.3 Semiconductors

Semiconductors electrical conductivity is between conductors and insulators. The material becomes more "conductivity" or less resistance when the temprature rises. This behavier is oposite to that of metals.

12.3.1 Diodes

Ideal diodes

Cundoct electrisity only one way. Current flowes threow Anode (A) to Cathode (K).

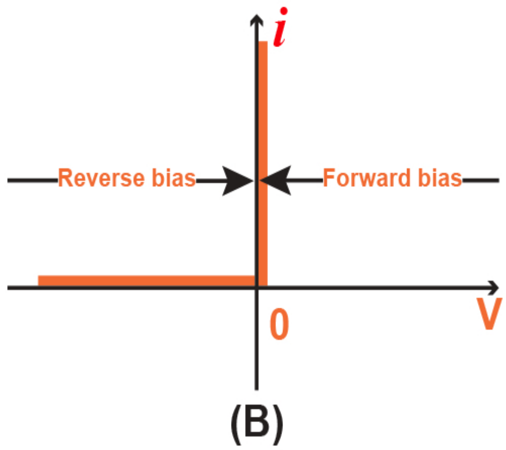


Figure 12.18: ideal diode iv-curv. From ()

Real diodes

reverse brakedown is when the diode lets negative current. On normal diodes it will perminantly fail, probably melts.

$$i = I_s(e^{qv_d/kT} - 1) \tag{12.19}$$

- k is boltsmans constant ($1.38 * 10^{-23} J/K$ Jouls/Kelvin) $0K = -273^oC$
- T is tempreture of device (kelvin)
- I_s is structual current (silicon $10^{-12}A$), the small reverse current
- q is the charge of an electron ($1.1609 * 10^{-19}C$)

Difficult to use formula unless numerical analasys.

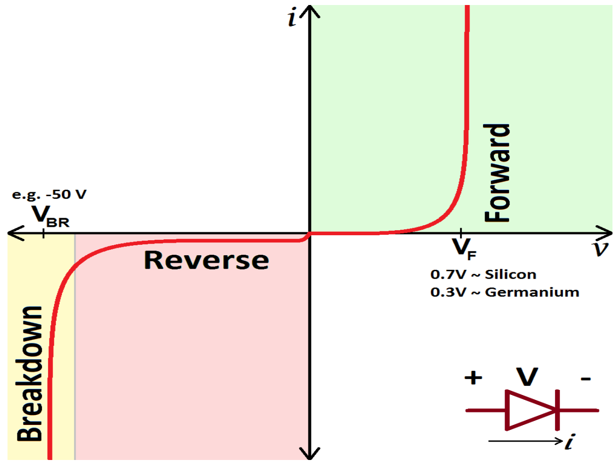


Figure 12.19: real diode iv-curv. From ()

The DC model of a diode

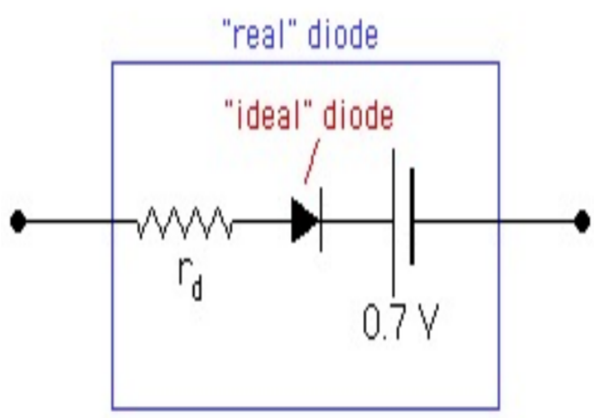


Figure 12.20: real diode dc model. From ()

Example: diode

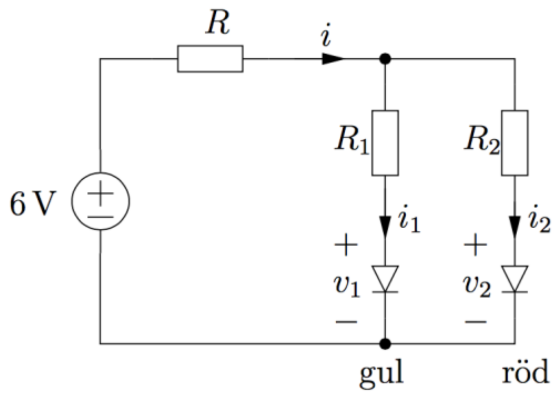


Figure 12.21: diodes example. From ()

$$i_1 = 40mA, v_1 = 2v$$
$$i_2 = 50mA, v_2 = 1.6v$$

$$\begin{aligned}
\text{Kirchhoffs strömlag: } & i = i_1 + i_2 = 40mA + 50mA = 90mA = 0.09A \\
\text{Kirchhoffs spänningslag: } & \text{(Loop 1)} \\
& +6 - iR - R_1 i_1 - 2 = 0 \\
\text{Kirchhoffs spänningslag: } & \text{(Loop 2)} \\
& +6 - Ri - R_2 i_2 - 1.6 = 0 \\
\text{Vi har 3 okända och 2 eqvationer.} & \text{Närmed så kan vi sätta ett värde på R} \\
\text{sådant att motstånden ärstöre än noll} & \\
& R = 20\Omega \Rightarrow R_1 = 55\Omega; R_2 = 52\Omega
\end{aligned}$$

Zener Diodes

Allows to enter and reenter the revese breakedown with out perminantly failing. symbol. Jusaly lower voltage for revese breakdown used for analog to digital conversion and vise versa and a voltage relulator.

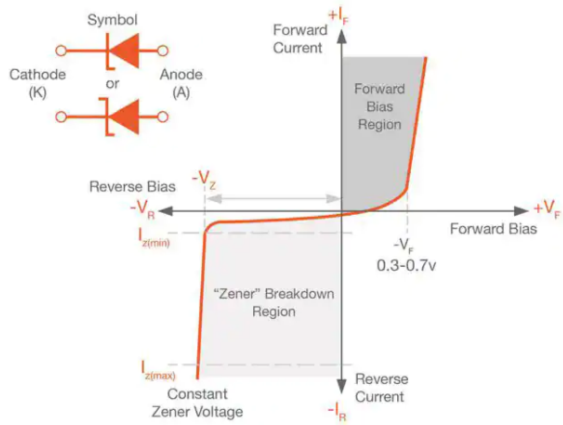


Figure 12.22: zener diode iv-curv. From ()

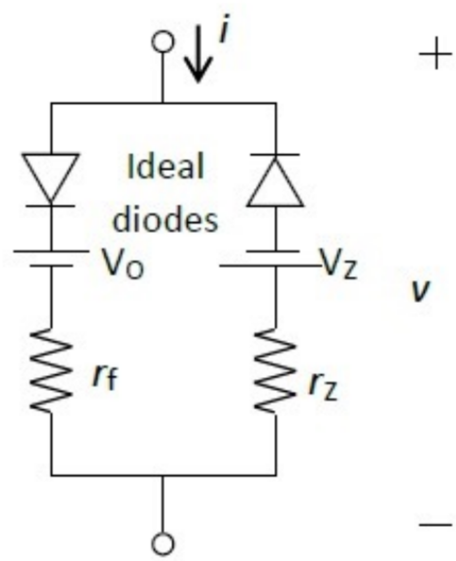


Figure 12.23: zener diode dc model. From ()

Example Let $V_{in} = 9V$ and $V_z = 5V$. Determent the minimum value of R if the maximum current is $20mA$ threw R, assume $R_L = \text{inf}$. Then determan the minimum resistance for R_L

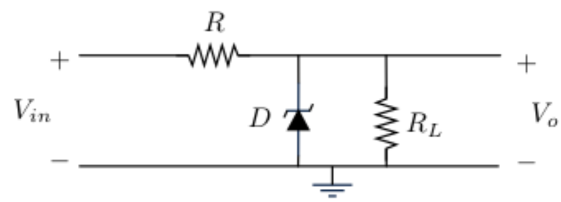


Figure 12.24: zener diode example. From ()

Let I be the current over R, I_1 the current over the zener diode and I_2 be the current over R_L.

$$\begin{aligned} I &= \frac{V_{in} - V_o}{R} \\ &\Rightarrow R > \frac{9-5}{20 \cdot 10^{-3}}\Omega = 200\Omega \end{aligned}$$

$$\begin{aligned} \text{KCL: } I &= I_1 + I_2 \\ I_1 &= 0 \;\text{ if } R_L \text{ is to low} \\ I_2 &= \frac{V_0 - 0}{R_L} \\ &\Rightarrow I = I_2 \Rightarrow 20mA = \frac{5v}{R_L} \\ &\Rightarrow R_L > \frac{5}{20 \cdot 10^{-3}}\Omega = 250\Omega \end{aligned}$$

12.3.2 Transistors

All Semiconductors are non linuer. But we can use transistors with linjer modules.
NPN and PNP symbols. Base (B), Collector (C) and Emitter (E)

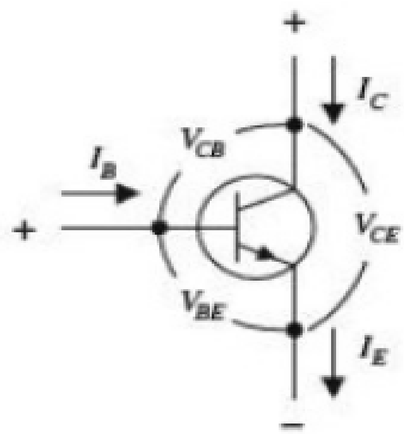


Figure 12.25: Transistor NPN. From ()

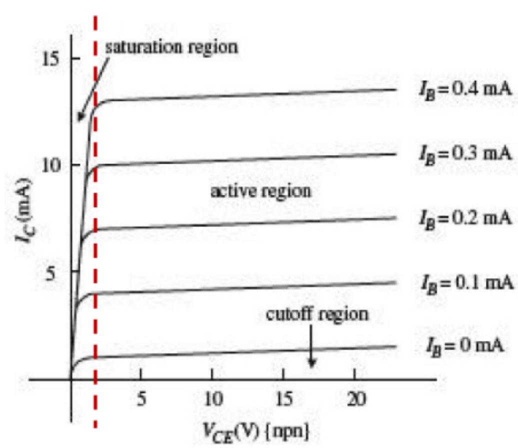


Figure 12.26: Transistor Operational mode. From ()

Transistor modes

Cutoff: Open switch

$$I_B = I_E = I_C = 0$$
$$V_{BE},\ V_{BC} < 0$$

Saturation: Closed switch

$$V_{CE} <= V_{CEsat}, I_B > 0$$

Active: Dependent current source

$$I_E = I_B + I_C,\ I_C = h_{EF} I_B = \beta I_B$$
$$V_{CE} > V_{CEsat}, I_B > 0, V_{BC} = V_B - V_C < 0$$

Example: Transistor What is the larges value tjat R_B can have so that the transistor behaves as a switch (saturation mode)? The transistor has a current gain h_{FE} between 100 and 300. Suppose that $V_{CE,sat} = 0.2V$ and $V_{BE,sat} = 0.7$.

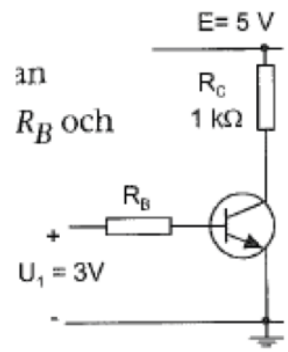


Figure 12.27: Transitor example. From ()

Om transistorn är bottnad (max current) gäller det att potatialen i C (mellan R_C och transistorn)

$$V_{CE} = V_C - V_E = V_C - 0 = V_{CE,sat}$$
$$V_C = V_{CE,sat} = 0.2V$$

$\Rightarrow$ Vi har då ett spänningsfall över R_C på $5 - 0.2 = 4.8V$.

$$R_C \cdot I_C = 4.8V \quad \text{Ohm's law} \Rightarrow I_C = \frac{4.8}{10^3} = 4.8mA$$

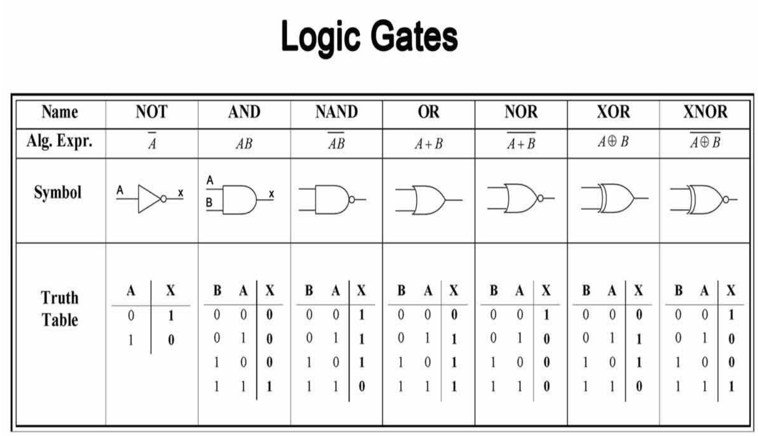
Vi vet att $I_C = \beta \cdot I_B$ men är ej säkra på β:s värde. Det kan vara så långt som 100.

Vi kan behöva $I_B = \frac{I_C}{100} = 48\mu A$ för att inte I_C ska bli för liten. Alltså: $I_B > 48\mu A$ för att garantera bottnad transistor.

Betrakta nu slingan där I_B går.
Potentialvandring: $U_1 - R_B I_B - V_{BE,sat} = 0$, $3 - R_B I_B - 0.7 = 0$
eller $2.3 = R_B I_B \Leftrightarrow I_B = \frac{2.3}{R_B}$

Använd olikhet $I_B > 48\mu A \Rightarrow \frac{2.3}{R_B} > 48\mu A \Rightarrow \frac{2.3}{48 \cdot 10^{-6}} > R_B$ eller $R_B < 47.9k\Omega$

12.4 Digital Circuits: Combinatorial and Sequential Networks



Logic Gates

Name	NOT	AND	NAND	OR	NOR	XOR	XNOR
Alg. Expr.	$\overline{A}$	AB	$\overline{AB}$	$A+B$	$\overline{A+B}$	$A\oplus B$	$\overline{A\oplus B}$
Symbol							
Truth Table	A X 0 1 1 0	B A X 0 0 0 0 1 0 1 0 0 1 1 1	B A X 0 0 1 0 1 1 1 0 1 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 1	B A X 0 0 1 0 1 0 1 0 0 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 0	B A X 0 0 1 0 1 0 1 0 0 1 1 1

Figure 12.28: Logic Gates. From ()

Bolean algebra (or: $+$), (and: $\cdot$)

$$(x')' = x$$

$$x + x = x \qquad x \cdot x = x$$

$$x + (y + x) = (x + y) + z$$

$$x(yz) = (xy)z$$

$$x + x' = 1 \qquad x \cdot x' = 0$$

$$x(y + z) = xy + xz$$

$$x + yz = (x + y)(x + z)$$

$$x + 1 = 1 \qquad x + xy = x$$

$$x \cdot 0 = 0$$

$$xy + \overline{x}z = xy + \overline{x}z + yz$$

$$x + 0 = x \qquad (x + y)(\overline{x} + z) = (x + y)(\overline{x} + z)(y + z)$$

$$x \cdot 1 = x \qquad \overline{(x + y)} = \overline{xy}$$

$$\overline{(xy)} = \overline{x} + \overline{y}$$

12.4.1 Combinational

$f(x_6, x_5, x_4, x_3, x_2, x_1, x_0) = x_6x_4(x_5 + x_3 + x_0) + x_2x_1$

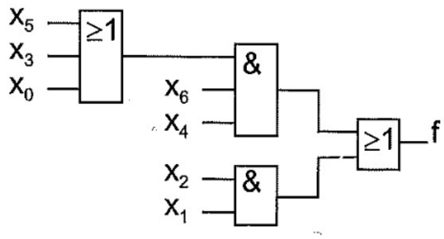


Figure 12.29: Bolean factorization. From ()

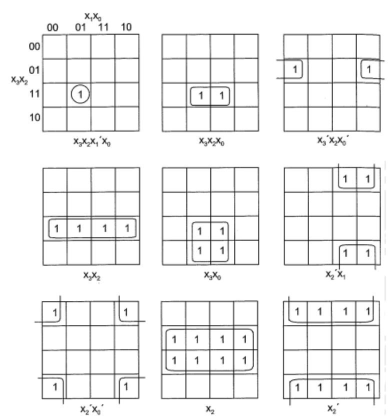


Figure 12.30: KMap patters. From ()

KMaps works from 4 variables.
Delays, there is a small delay for each gate.

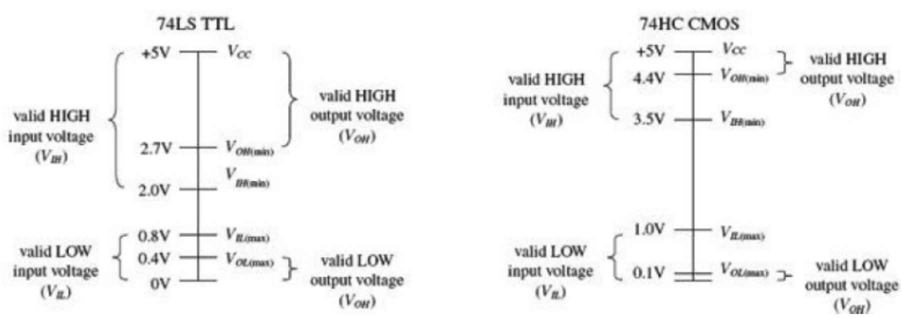


Figure 12.31: Logic families. From ()

Example: design circuit

$$f(x_3, x_2, x_1, x_0) = \sum(0, 2, 4, 5, 6, 7) + d(8, 10, 12, 14) \quad \text{- sum-of-products} \tag{12.20}$$

$0 = 0000$
$2 = 0010$
$4 = 0100$
$5 = 0101$
$6 = 0110$
$7 = 0111$

$8 = 1000$
$10 = 1010$
$12 = 1100$
$14 = 1110$

X_3X_2 \ X_1X_0	00	01	11	10
00	1	0	0	1
01	1	1	1	1
11	-	0	0	-
10	-	0	0	-

$$f = \overline{x_0} + \overline{x_3}x_2 \tag{12.21}$$

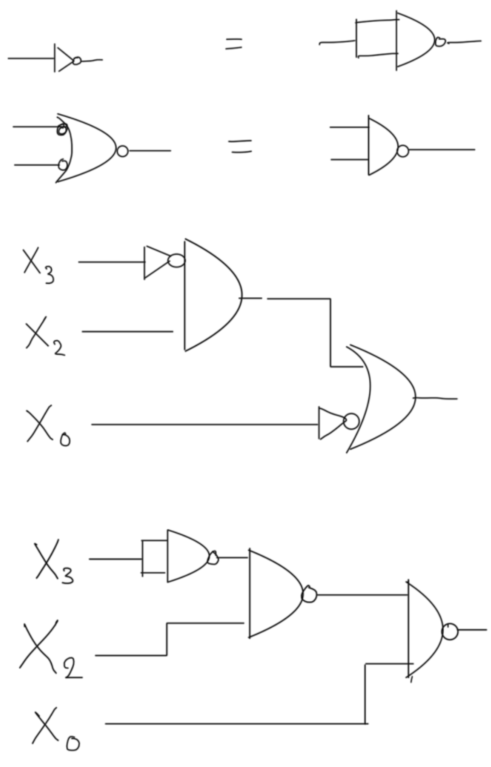


Figure 12.32: lodgic Combinational lodgic.

12.4.2 Sequential

Remembers the state what happend before.

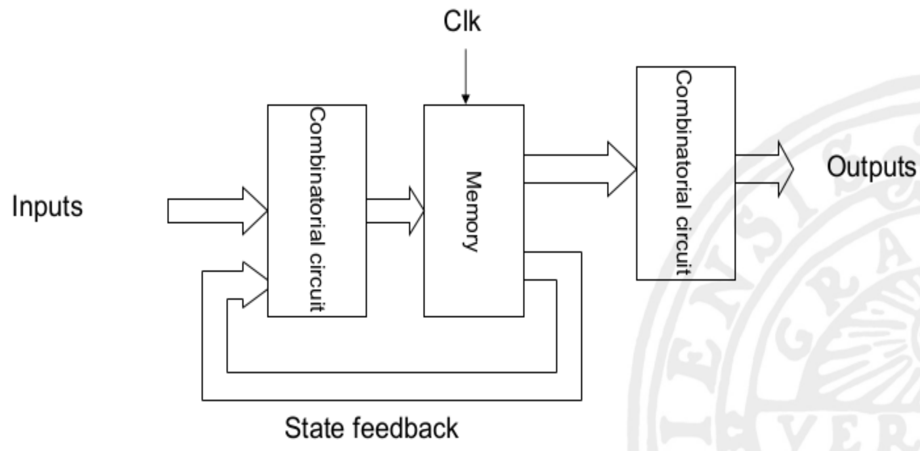


Figure 12.33: Sequential circuits moore type. From ()

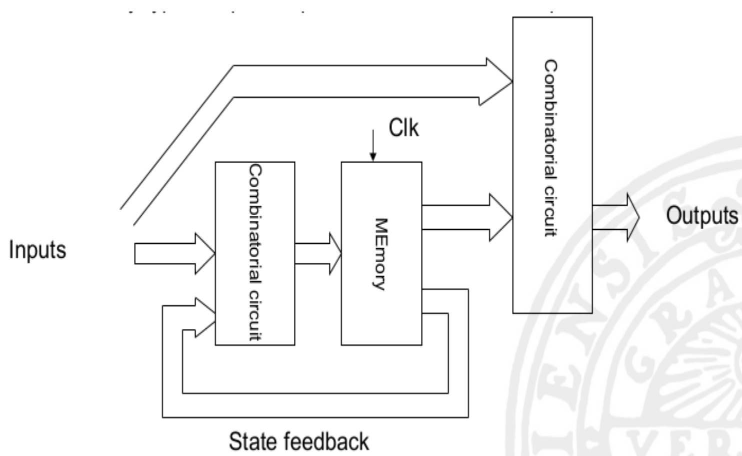


Figure 12.34: Sequential circuits mealy type. From ()

Memory

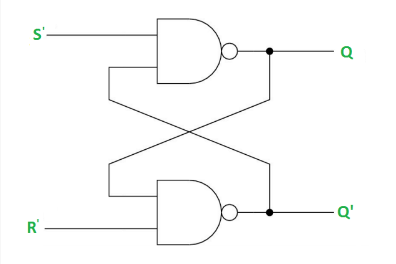
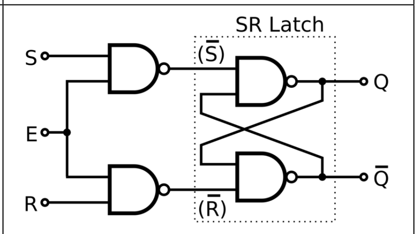
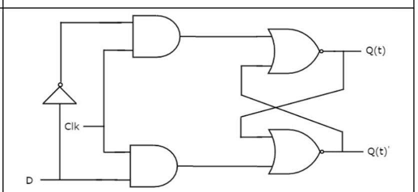
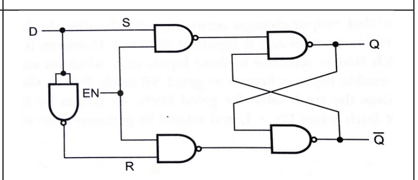
Description	Circuit
SR-latch (memory cell)	
Gated SR-latch	
Gated D-latch	
D Flip Flops	

Table 12.1: Memory. From ()

Timind diagram

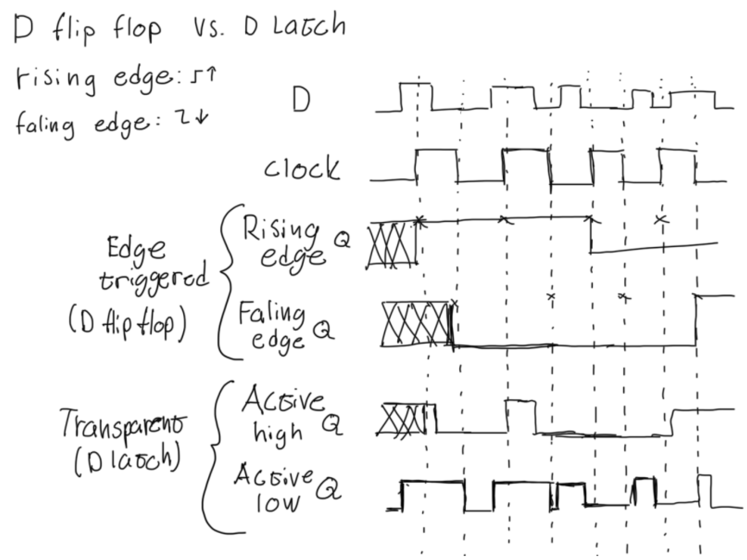


Figure 12.35: Timing diagram.

State diagram

State code

State name	q_1	q_0
S0	0	0
S1	0	1
S2	1	0
S3	1	1

State Table In sequential circuits we use *state table* instead of *truth table.* ex of state table format:

Previus state	Input	Next state	Output
0 0	0	0 0	0
0 0	1	0 1	0
0 1	0	0 1	1
0 1	1	1 0	1

Moore-circuit Note: in more machine we need a end state for output 1

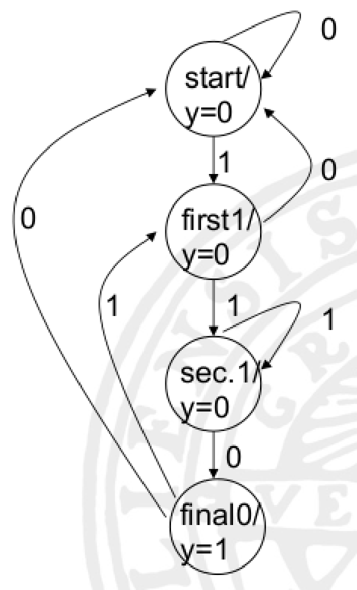


Figure 12.36: State diagram moore. From ()

Mealy-circuit Note: in mealy machine we dont need that.

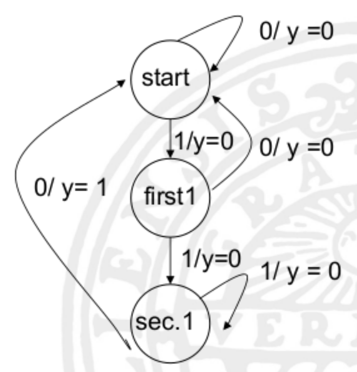


Figure 12.37: State diagram mealy. From ()

12.4.3 Shift registers

Each pit is prosest simultaniusly. Makes much simpler circuit.

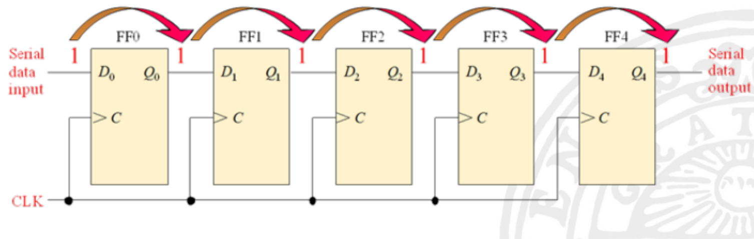


Figure 12.38: Shift register. From ()

12.4.4 Multiplixer

$f(A, B, C) = \sum(0, 1, 5, 6)$

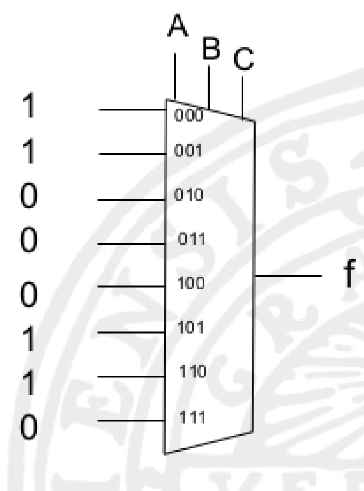


Figure 12.39: Mux. From ()

12.5 AC Circuit Analysis & Filtering

Note: $G(\omega) = H(\omega)$

- *sinusoidal steady state* same as AC (analysis)
- *phasor* a complex number representation of sinusoid
- *transfer function* relationsip with input and output voltage $G(jw) = \frac{v_{out}}{v_{in}}$
- *Impedance* representation of $\frac{v_{in}}{i}$ (z)
- *Angular frequency* $w = 2\pi f$ where f is frequency in Hz with is cycles per second
- *Operational frequency* $\frac{1}{\sqrt{2}}$

$$v(t) = v_p \cos wt + \phi \tag{12.22}$$

Phasor 3 representations

- Rectangular form $z = x + jy$
- polar form $z = r\angle\phi$
- exponential form $z = ze^{j\phi}$

Phasor conversion

$$\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
\phi &= \arctan \frac{x}{y} \\
x &= r\cos(\phi) \\
y &= r\sin(\phi) \\
e^{j\phi} &= \cos\phi + j\sin\phi \\
\cos\phi &= Re(e^{j\phi}) \\
\sin\phi &= Im(e^{j\phi})
\end{aligned}$$

Phasor operations

$$\begin{aligned}
\text{addition: } & z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \\
\text{subtraction: } & z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \\
\text{multiplication: } & z_1 z_2 = r_1 r_2 \angle(\phi_1 + \phi_2) \\
\text{division: } & \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\phi_1 - \phi_2) \\
\text{inverse: } & \frac{1}{z} = \frac{1}{r}\angle-\phi \\
\text{square root: } & \sqrt{z} = \sqrt{r}\angle(\phi/2) \\
\text{complex conjugate: } & z^* = x - jy
\end{aligned}$$

Transfer function

$$\begin{aligned}
H(j\omega) &= |H(j\omega)|\angle H(j\omega) \\
H(w) &= \frac{v_o(w)}{v_i(w}
\end{aligned}$$

Euler formula

$$\begin{aligned} e^{jx} &= \cos x + j \sin x \\ e^{-jx} &= \cos x - j \sin x \end{aligned}$$

They are used to represent sinus wave.

We instead calculate with complex exponsetioals with euler formula instead of sinus waves

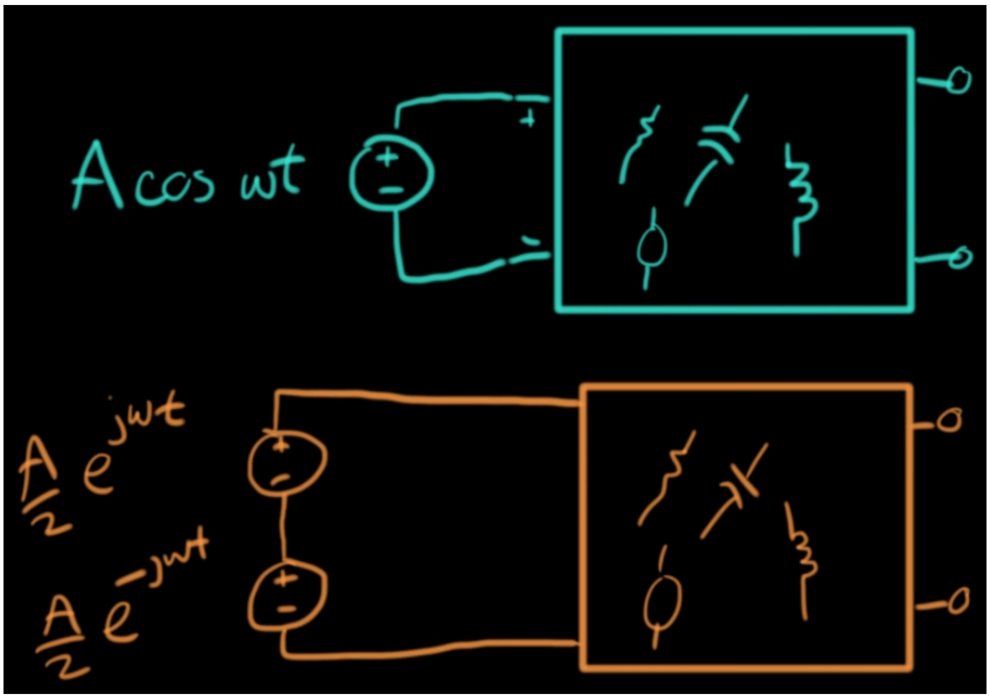


Figure 12.40: AC euler represent demostration. From ()

AC Resistor representation

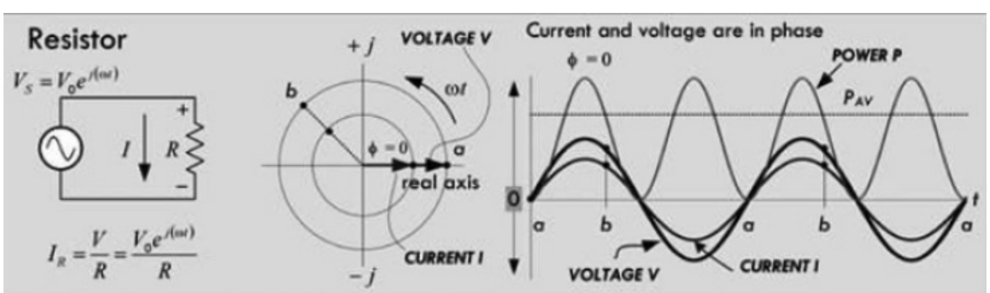


Figure 12.41: AC resistor represent. From ()

12.5.1 Impedance

Impedance $\frac{v_{in}}{i}$ (z)

$$\textbf{Resistor}: \quad \frac{v}{i} = R$$
$$\textbf{Inductor}: \quad \frac{v}{i} = jwl$$
$$\textbf{Capasitor}: \quad \frac{v}{i} = \frac{1}{jwc}$$

12.5.2 Filters

The filter can eather be active and passive filters:

- *Passive filters* only use active components like resistor, capasitor and inductor
- *Active filters* use opamp (can amplify output)

There are 4 common filtering types:

- *High-pass filter (HP)* lets higher frequencys threow
- *Low-pass filter (LP)* lets lower frequencys threow
- *Bandpass filter (BP)* within a frequency spand. Can be comind with HP and LP in series
- *Notch filter* oposit of BP

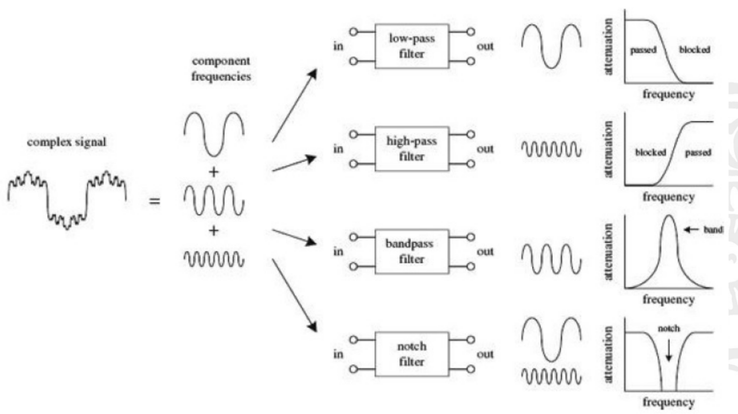


Figure 12.42: Signal filtering. From ()

Filters can be constructed as follows:
RC filter, *RL Filter*, *LC filter*, *L-filter*, π*-filter* and *T-filter*.

Some resources to construct filters

- RF tools
- Filterwizard

Example: HP filter

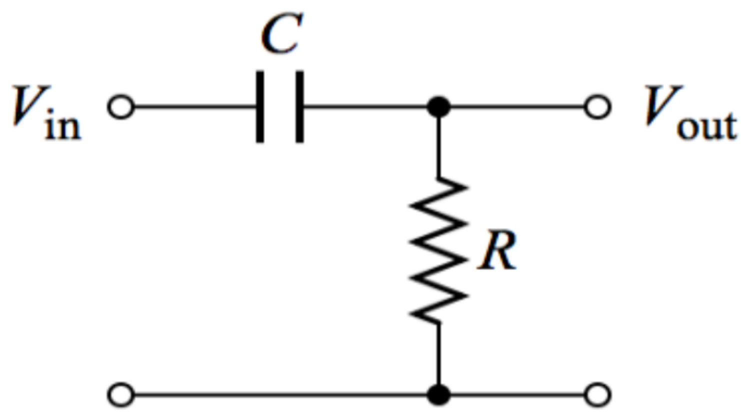


Figure 12.43: HP Filter Circuit.png. From ()

$$\begin{aligned} V_{out} &= V_{in}\frac{z_R}{z_C+z_R} && \text{-From voltage divider circuit} \\ &= V_{in}\frac{R}{\frac{1}{j\omega c}+R} && \text{-From impedance axiom} \\ &= V_{in}\frac{\frac{R}{1}}{\frac{R(j\omega c)+1}{j\omega c}} = V_{in}\frac{R(j\omega c)}{R(j\omega c)+1} \end{aligned}$$

Therefore we get $G_{HP} = \dfrac{R(j\omega c)}{R(j\omega c)+1}$

$$\begin{aligned} |G_{HP}| &= \frac{1}{\sqrt{2}} = \frac{R\omega c}{\sqrt{R^2\omega^2c^2+1}} \\ &\Rightarrow \frac{\sqrt{R^2\omega^2c^2+1}}{\sqrt{2}} = R\omega c \\ &\Rightarrow \frac{R^2\omega^2c^2+1}{2} = R^2\omega^2c^2 \\ &\Rightarrow R^2\omega^2c^2+1 = 2R^2\omega^2c^2 \\ &\Rightarrow R^2\omega^2c^2 = 1 \\ &\Rightarrow R\omega c = 1 \\ &\Rightarrow R = \frac{1}{\omega c} \end{aligned}$$

The following is given: $\omega = 2\pi f$, $1kHz$ for HP filter. The capacitor should not be larger than $2.2uF$. Therefore we chose $c = 1uF$, this gives us $R \approx 160\Omega$.

$$
\begin{aligned}
arg(G_{HP}) &= arg(\frac{160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j}{160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j + 1}) \\
&= arg(160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j) - arg(160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j + 1) \\
&= \frac{\pi}{2} - arctan(\frac{160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f}{1}) \\
&= \frac{\pi}{2} - arctan(\frac{\pi \cdot f}{3125})
\end{aligned}
$$

$$
\begin{aligned}
arg(G_{HP}) &= arg(\frac{160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j}{160 \cdot 2 \cdot \pi \cdot 10^-6 \cdot f \cdot j + 1}) \\
&\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2} \\
&\Rightarrow \frac{z_1}{z_2} = \frac{\sqrt{(0.32 \cdot 10^{-3} \cdot f\pi)^2}}{\sqrt{(0.32 \cdot 10^{-3} \cdot f\pi)^2 + 1}} \underline{/\arctan 0 - \arctan \frac{1}{0.32 \cdot 10^{-3} \cdot f\pi}} \\
&\Rightarrow arg(\frac{z_1}{z_2}) = -\arctan \frac{1}{0.32 \cdot 10^{-3} \cdot f\pi} \\
&= -\arctan \frac{1}{0.32 \cdot 10^{-3} \cdot f\pi} \\
&= \frac{\pi}{2} - \arctan \frac{f\pi}{3125}
\end{aligned}
$$

Example: exponential representation

$$
\begin{aligned}
H(j\omega) &= \frac{j\pi}{1 + j2\pi} = |H(j\omega)| e^{j arg(H(j\omega))} \\
\Rightarrow |H(j\omega)| &= \frac{\sqrt{\pi^2}}{\sqrt{1^2 + 2^2\pi^2}} = \frac{\pi}{\sqrt{1 + 4\pi^2}}
\end{aligned}
$$

$$
\begin{aligned}
arg(H(j\omega)) &= arg\Big(\frac{j\pi}{1 + j2\pi}\Big) = \arctan\Big(\frac{0}{\pi}\Big) - \arctan\Big(\frac{1}{2\pi}\Big) \\
&= \frac{\pi}{2} - \arctan\Big(2\pi\Big) \\
\Rightarrow H(j\omega) &= \frac{\pi}{\sqrt{1 + 4\pi^2}} e^{j \arctan \Big(2\pi\Big)}
\end{aligned}
$$

Series of filter

One can commbind filters to create more complex filters. The *transfer function* will be the multiplication of

$$G_{tot} = G_1 \cdot G_2 \cdot \ldots \cdot G_n \tag{12.23}$$

The *Gain* will be the gain for each individual transfer function multiplied

$$|G_{tot}| = |G_1| \cdot |G_2| \cdot \ldots \cdot |G_n| \tag{12.24}$$

The phase function is the sum of the phase functions

$$Arg(G_{tot}) = Arg(G_1) + Arg(G_2) + \ldots + Arg(\cdot G_n) \tag{12.25}$$

Chapter 13

Numerical Methods and Simulation

13.1 Matlab

https://www.egr.msu.edu/aesc210/systems/MatlabCheatSheet.pdf
Viktiga syntax/comandons

```
zeros(1,N);  % creates a vector of N lenght
zeros(N); % creates a matrix of N*N size
mean([1 2 2 3]); % avrage value with is 2 in this cases
vector(line,row);
hold on % holes the current plot and plots new over old
xlable("some x lable") % naming the plot x axis
ylable("some y lable") % naming the plot y axis
title("some title for the plot")
ode45(@(t,y) 2*t, tspan, y0);
[t,y] = ode15s(@(t,y) myode(t,y,A,B), tspan, y0);
linespace(StartPoint, EndPoint); % gemerates a row of vector
fzero(func, currentValueOfFunction, optimeset('Tolx', 1e-02));
[0; 1; 2]; % Vector not array
numel(x0); % the number of dimension of vector
repmat(x0(i), N, 1); % generate values
cumsum(rand) % the cumulative sum of Noise starting at the beginning of the first array dimension in N
```

13.2 Aritmatic

När beräkningar sker så andvänder den inte exaxta värdet utan konverterar till binärt och där med får ungefär väde. Räknar inte med bråk utan floting point.

(Mantissa m hur många index det finns för d_i), (Bas β), (exponent e). Man kan skriva ett decimaltal så här: $x = m\beta^e$. (Normaliserad form) är då första talet är en siffra ($1.23 * 10^{-1}$ ej 0.123) $m = d_0 * 2^0 + d_1 * 2^{-1} + d_2 * 2^{-2} + \ldots + d_n * 2^{-p}$. Normaliserad form ger att d_0 i altid 1 i binär form och därmed så behöver man inte sparra den. Den kallas hidden bit.

E andvänds för att konvertera till en normal form tex $0.d_1d_2..d_{52} \cdot 2^{-1022}$ i subnormal form blir $1.d_1d_2..d_{52} \cdot 2^{E-1022}$, där $E = 1$, i normal form.

13.2.1 IEEE

IEEE, ger en standard till olika representationer.

- Singleprecision (32 bit, enkel precision)
- Double precision (64 bit, dubbel precision)
- Extendedprecision (80 bit, utökad precision)

matlab andvänder double presission. Tal som är störe än realmax i matlab blir Inf och tal som är ogiltiga är NaN.

13.2.2 Maskinepsilon

Def: Gapet mellan 1 och nästa representerbara tal, minsta talet ϵ_M för vilket gäller $1 + \epsilon_M > 1$. Det är ett mått på tals systemetsprecision. $\epsilon_M \approx 10^{-16}$

```
if(x == y)
ifabs((x-y)/x)<tol
```

13.2.3 Diskretiseringsfel

Vid exempelet när man beräknar derivatan så är *avrundningsfelet* den dominanta. Medans vid Störe h så dominerar *Diskretiseringsfel*

13.3 ODE

Skillnaden mellan metod och modell är att moted är hur man gör, medans modell är strukturen av företelsen. För att man ska kunna lösa en ode så behöver den vara en entydlig lösning samt små skilnader på inparametrar ger littet skilnad i resultat.

Löser med ode45

```
%%%%%%%%% myODE.m
function yprim = myODE(t,y)
  % y'(t) = 3y(t)(1-y(t))
  yprim = 3*y*(1-y)
end
%%%%%%%%% script.m
% Tidsintervall mellan 0 till 10
% Begynnelse vilkor på 0.1
[t,y] = ode45(@myODE, [0 10], 0.1)
plot(t,y)
```

13.3.1 Numeriska metoder

När numrerisk fel är mindre än propotionelit ex h^2 då blir avrundningsfelet den dominanta

Implecita metoder Ovilkoliga stabilita.

Explicit metoder Kan vara effectivara, fämst för icke styva ode'r

Euler framåt (explicit metod)

Tar lutningen vid punkten man är i och beräknar värdet för ett givet steg.

Ex:

$\frac{y'(x)}{x} - y\sin(x) = 0,\ y(0) = 1$
Euler framåt: $y_{k+1} = y_k h_k f(t_k, y_k)$

$y'(x) = xy\sin(x),\ f(x,y) = xy\sin(x)$
$y_{i+1} = y_i + hf(x_i, y_i) = y_i + hx_i \cdot y_i \sin(x_i)$
Väljer $h = 0.1$ (h är steget mellan det approximerade linjerna)
$y_0 = 1$
$y_1 = y_0 + hx_0 y_0 \sin(x_0) = 1 + 0.1 \cdot 0 \cdot 1 \sin 0 = 1$
$y_2 = y_1 + hx_1 y_1 \sin(x_1) = 1 + 0.1 \cdot 0.1 \cdot 1 \sin 0.1 = 1.001$
$y_3 = y_2 + hx_2 y_2 \sin(x_2) = 1.001 + 0.1 \cdot 0.2 \cdot 1.001 \sin 0.2 = 1.005$
$*$
$y_{10} = 1.2895$

Aritmetiska lösnig (Räknat ut exakta värdet) $y(1) = e^{\sin(1) - 1 - \cos(1)} = 1.3514$
Fel $= 1.2895 - 1.3514 = -0.0619$
Väljer $h = 0.01,\ y_{100} = 1.3448$ Fel $= -0.00669$
Väljer $h = 0.01,\ y_{1000} = 1.35076$ Fel $= -6.74 \cdot 10^{-4}$
Vi ser att felet är propotionelit mot steg (h). Fel $\alpha\ h$

Euler bakåt (implicit metod)

Tar lutningen ett steg framåt.

Ex:

$$
\begin{aligned}
&\text{Euler bakåt: } y_{k+1} = y_k h_k f(t_{k+1}, y_{k+1})\\
\\
&y'(x) = xy\sin(x),\ y(0) = 1\\
&y_{i+1} = y_i + hx_{i+1}y_{i+1}\sin(x_{i+1})\\
&\Rightarrow y_{i+1} - hx_{i+1}\cdot y_{i+1}\sin(x_{i+1}) = y_i\\
&\Rightarrow y_{i+1} = \frac{y_i}{1 - hx_{i+1}\sin(x_{i+1})}\\
&\text{Väljer } h = 0.1\\
&y_0 = 1\\
&y_1 = \frac{y_0}{1 - hx_1\sin(x_1)} = \frac{1}{1 - 0.1\cdot 0.1\cdot\sin 0.1} = 1.000999\\
&y_2 = \frac{y_1}{1 - hx_1\sin(x_1)} = 1.00499\\
&*\\
&y_{10} = 1.42556\\
\\
&\text{Aritmetiska lösnig: } y(1) = e^{\sin(1) - 1 - \cos(1)} = 1.3514\\
&\text{Fel} = 1.42556 - 1.3514 = +0.07412\\
&\text{Väljer } h = 0.01,\ y_{100} = 1.35825 \text{ Fel} = +0.00681\\
&\text{Väljer } h = 0.001,\ y_{1000} = 1.35211 \text{ Fel} = +6.75\cdot 10^{-4}\\
&\text{Vi ser att felet är propotionelit mot steg } (h).\ \text{Fel } \alpha\ h
\end{aligned}
$$

Trapetsmetoden (implicit metod)

Trapetsmetoden uses the avrage value of the euler framåt and backåt.

Ex:

$$
\begin{aligned}
&\text{Trapetsmetoden } y_{i+1} = y_i + 0.5h_i(f(t_i, y_i) + f(t_{i+1}, y_{i+1})) && y' = xy\sin(x)\\
&y_{i+1} = y_i + h/2(x_i\cdot y_i\sin(x_i) + x_{i+1}y_{i+1}\sin(x_{i+1}))\\
&\Rightarrow y_{i+1} = \frac{y_i + h/2(x_i\cdot y_i\sin(x_i))}{1 - h/2(x_i) + \sin x_{i+1}} && y_0 = 1\ h = 0.1\\
&y_1 = 0.00499\\
&*\\
&y_{10} = 1.3343,\ \text{Fel} = 0.00286\\
&\text{Väljer } h = 0.01,\ y_{100} = 1.35147 \text{ Fel} = +2.86\cdot 10^{-5}\\
&\text{Väljer } h = 0.001,\ y_{1000} = 1.3514275 \text{ Fel} = +2.86\cdot 10^{-7}\\
&\text{Vi ser att felet är propotionelit mot steg } (h).\ \text{Fel } \alpha\ h^2
\end{aligned}
$$

Heuns metod (explicit metod)

Trapetsmetoden ger bra approximationer men är dock implicit vilket gör det svårt för icke linjära ODE. Det kan huens metod lösa genom att approximera implicita delen.

Metod:

Huens metod $y_{i+1} = y_n + h_n/2[f(t_n, y_n) + f(t_{n+1}, hf(t_n, y_n))]$

$k_1 = f(x_i, y_i)$
$k_2 = f(x_i, y_i)$
$y_{i+1} = y_i + h(k_1 + k_2)$
Vi ser att felet är propotionelit mot steg (h). Fel $\alpha\ h^2$

Ex:

$$y' = 10 \cdot y(1 - \frac{y}{1000}),\ y(0) = 1,\ h = 0.1$$

i. löser ut $y'(t)$
$y'(t) = 10 \cdot y(1 - y/1000)$

ii. Heuns metod, utryck för y_{i+1}
$y_{n+1} = y_n + h_n/2(k_1 + k_2)$

iii. Beräknar
$k_1 = f(t_0, y_0) = 10 \cdot 1(1 - \frac{1}{1000}) = 9.99$
$k_2 = f(t_1, y_0 + hk_1) = f(t_1, 1.999) = 10 \cdot 1.999(1 - \frac{1.999}{1000}) = 19.99(0.998001) \approx 19.95$
$\Rightarrow y_1 = y_0 + h_0/2(k_1 + k_2) = 1 + 0.05(k_1 + k_2) \approx 2.497$

Runge-Kutta (explicit metod)

Runge-Kutta är noggran numrerisk approximering, betydligt bättre än euler framåt/backåt. Man berärknar 4 lutningar och tar typ medelvärdet av dem.

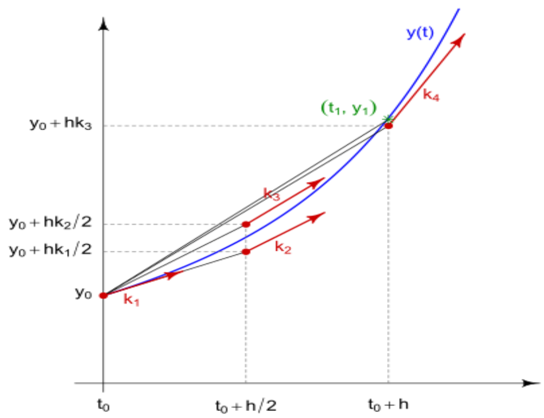


Figure 13.1: Runge-Kutta

Ex:

$$y'(t) - y(t) - t = 0,\ y(0) = 0,\ h = 0.1,\ \text{RK 1-steg}$$

i. löser ut $y'(t)$
$y'(t) = y(t) + t \Rightarrow f(t, y) = y + t$

ii. Runge-Kuttas, utryck för y_{i+1}
$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

iii. Beräknar
$k_1 = f(t_0, y_0) = y_0 + t_0 = 0$
$k_2 = f(t_0 + h/2, y_0 + hk_1/2) = f(0.05, 0) = 0.05$
$k_3 = f(t_0 + h/2, y_0 + hk_2/2) = f(0.05, 0.0025) = 0.0525$
$k_4 = f(t_0 + h, y_0 + hk_3) = f(0.1, 0.00525) = 0.10525$
$k = (k_1 + 2k_2 + 2k_3 + k_4)/6 = (0 + 2 * 0.05 + 2 * 0.0525 + 0.10525)/6 = 0.31025/6$
$y_1 = y_0 + h \cdot (0.31025/6) = 0.1 \cdot (0.31025/6) \approx 0.00517$

13.3.2 Högre årdningens ODE

$$x''(t) = -\frac{x(t)}{(\sqrt{x(t)^2 + y(t)^2})^3}$$
$$y''(t) = -\frac{y(t)}{(\sqrt{x(t)^2 + y(t)^2})^3}$$

$$\text{Låt } \overline{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$$
$$u_1 = x,\ u_1' = u_2$$
$$u_2 = x',\ u_2' = -\frac{u_1}{(\sqrt{u_1^2 + u_2^2})^3}$$
$$u_3 = y,\ u_3' = u_4$$
$$u_4 = y',\ u_4' = -\frac{u_3}{(\sqrt{u_1^2 + u_2^2})^3}$$
$$\overline{u}' = \begin{pmatrix} u_2 \\ -\frac{u_1}{(\sqrt{u_1^2+u_2^2})^3} \\ u_4 \\ -\frac{u_3}{(\sqrt{u_1^2+u_2^2})^3} \end{pmatrix}$$

Löser i matlab:
Ställer upp ODE i separat fil

```
function u_out = satellitODE(t,u)

u_out = [ u(2);
          -( u(1)/( sqrt( u(1)^2 + u(2)^2 )^3 ) );
          u(4);
          -( u(3)/( sqrt( u(1)^2 + u(2)^2 )^3 ) )];
```

Sedan kan man kalla med följande script

```
tidsintervall = [0, 100];

% satelitens position (x(0),y(0))
% satelitens hastighet (x'(0), y'(0))
u0 = [0; 22223; 17000; 100];

[t, u] = ode45(@satellitODE, tidsintervall, u0, odeset)

plot(t,u)
```

13.4 Analys

Styv mångra ädringar under kort tid. *Adaptivt steglängdsval* som ode15s och ode45. Anpassar steglängd efter styvhet.

13.4.1 Analys av metoder

$$\begin{aligned}&\text{ODE: } y' = f(t, y)\\&\text{Metod: } y_{i+1} = y_i + h \cdot \Psi_k(t_i, y_{i+1}, y_1)\end{aligned}$$

13.4.2 Konsistent

Metoden är konsistent med ode'n om:

$$\lim_{h \to 0} \Psi_h = \Psi_0(t_i, y_i) = f(t_i, y_i)$$

13.4.3 Rättstält

$$\begin{aligned}&||\overline{F}(x, \overline{u}) - \overline{F}(x, \overline{z}) \leq L \cdot ||\overline{u} - \overline{z}||\\&L = max||J(x, \overline{u})|| \quad \text{-jacobianen}\end{aligned}$$

13.4.4 Noggrannhetsordning

Låt $\phi(0)$ vara en funkion som uppfyller ode'n och är tillräklight deriverbar. Bilda lokala trunkteringsfelet

$$\tau = \phi(t_{i+1}) - (\phi(t_i) + h\Psi_h(\phi(t_{i+1}), \phi(t_i)))$$

och taylorutveckla kring t_i och låt $h \to 0$. Om $\Psi \to C \cdot h^{P+1}$ (dominerande term) Så har metoden noggranhetsordning P

$$\tau_h \to C \cdot h^{P+1} \Leftrightarrow \frac{\phi(t_{i+1}) - \phi(t_i)}{h} + \Psi_h(t_i, \phi(t_{i+1}), \phi(t_i)) \to Ch^P$$

Exemple: Visa nu att Euler bakåt (implicit Euler) har noggranhetsordning 1
Lokala trunkteringsfelet τ för implicit Euler metod är:

$$\tau = y(t_{k+1}) - y(t_k) - hf(t_{k+1}, y(t_{k+1}))$$

Vi kan då ersätta $f(t_{k+1}, y(t_{k+1}))$ med $y'(t_{k+1})$ (ode'n) vilket ger:

$$\tau = y(t_{k+1}) - y(t_k) - hy'(t_{k+1})||$$

Sedan analyserar vi det lokala trunkteringsfelet genom taylorutveckla den implecita delen av utryck vid punkt $t = t_k$. Där av så är $t_{k+1} = t_k + h$

$$\begin{aligned}\tau &= y(t_{k+1}) - y(t_k) - hy'(t_{k+1})\\&= y(t_k) + hy'(t_k) + \frac{h^2}{2}y''(t_k) + O(h^3)\\&- y(t_k) - h(y'(t_k) + y''(t_k) + O(h^2))\\&= (\frac{h^2}{2} - h^2)y''(t_k) + O(h^3) = O(h^2)\end{aligned}$$

Därmed så är det lockala trunkteringsfelet $O(h^2)$ och nogranhets ordningen p i $\tau_h = Ch^{p+1} \Rightarrow p = 1$. V:S:V

13.4.5 Stabilitet

Vi undersöker Stabilitet på testekvationen

$$y' = \lambda y \quad \text{där } re(\lambda) \leq 0$$

Euler framåt har stabilitetsområdet

$$|1 + h\lambda| \leq 1$$

Reella λ ger $-1 \leq 1 + \lambda h \Leftrightarrow h \leq \frac{-2}{\lambda}$

Euler bakåt har stabilitetsområdet

$$\frac{1}{|1 - \lambda h|} \leq 1 \text{ dvs ovilkorligt stabil}$$

Exemple: Visa nu att Euler bakåt (implicit Euler) är ovillkorligt stabil

Testekvationen ses som sådan:

$$\begin{aligned} y' &= \lambda \cdot y \\ y_{i+1} &= y_i + hf(t_{i+1}, y_{i+1}) \end{aligned}$$

Euler bakåt:

$$\begin{aligned} y_{i+1} &= y_i + h \cdot f(t_{i+1}, y_{i+1}) \\ &= y_i + h\lambda \cdot y_{i+1} \end{aligned}$$

Vi bryter ut y_{i+1}:

$$\begin{aligned} y_{i+1} - h\lambda \cdot y_{i+1} &= y_i \\ y_{i+1} \cdot (1 - h\lambda) &= y_i \\ y_{i+1} &= \frac{1}{1 - h\lambda} \cdot y_i \end{aligned}$$

Då ser vi att vårt stabilitetsvillkor är följande:

$$\left| \frac{1}{1 - h\lambda} \right| \leq 1$$

Vi antar (enligt uppgift) att λ är reellt och att $Re(\lambda) \leq 0$. Detta medför att följande alltid kommer vara mindre än 1. Nämnaren blir större än täljaren för alla värden på λ.

$$1 - h\lambda \geq 1 \Rightarrow \left| \frac{1}{1 - h\lambda} \right| \leq 1$$

för alla våra värden på λ. På grund av detta vet vi att oavsett värde på h så kommer den numeriska lösningen vara stabil, d.v.s ovillkorligt stabil.

13.4.6 Stabilitet generella ekvationer

Betrakta små rörelser av y (momentant)

$$y = \tilde{y} + z \quad \text{z-litet, } \tilde{y} - konstant$$
$$z' = f(t, \tilde{y}) \approx f(t, \tilde{y}) + z \cdot \frac{\delta f}{\delta y}(t, \tilde{y}) = C_1 + C_2 z$$

Sats om den numeriska metoden är stabil för alla

$$C_2 = \frac{\delta f}{\delta y}(t, \tilde{y})$$

i lösningsområdet på testekvationen ($\lambda = C_2$)

13.4.7 Stabilitet för system

$$\overline{u} = F(t, \overline{u}) \; \overline{u} \; F(t, \overline{u}) \begin{bmatrix} f_1(t, \overline{u}) \\ f_2(t, \overline{u}) \\ . \\ f_3(t, \overline{u}) \end{bmatrix}$$

Betrakta små rörelser (mementant)

låt $\overline{u} = \tilde{u} + \overline{z} \approx F(t, \tilde{u}) + F'(t, \tilde{u})\overline{z}$

$$\text{där } F'(t, \tilde{u}) = \begin{bmatrix} \frac{\delta f_1}{\delta u_1} & \frac{\delta f_1}{\delta u_2} & \dots & \frac{\delta f_1}{\delta u_n} \\ \frac{\delta f_1}{\delta u_1} & \frac{\delta f_1}{\delta u_2} & \dots & \frac{\delta f_1}{\delta u_n} \\ . & & & \\ \frac{\delta f_n}{\delta u_1} & \frac{\delta f_n}{\delta u_2} & \dots & \frac{\delta f_n}{\delta u_n} \end{bmatrix}$$

$F(t, \tilde{u})$ -konstantlägre odningens term påverar ej stabiliteten

Studera: $\overline{z}' = A\overline{z}$ där $A = F'(t, \tilde{u})$ (konstant matris) Låt $\overline{w} = s^{-1}\overline{z} \Leftrightarrow \overline{z} = S\overline{w} \Rightarrow S\overline{w}' = AS\overline{w} \Leftrightarrow \overline{w}' = S^{-1}AS\overline{w}$ där $D = S^{-1}AS$. Vi väljer S så att D blir diagonal med A's egenvärde i diagonalen, $\overline{w}' = Dw$

$$\Rightarrow \begin{cases} w_1' = \lambda_1 w_1 \\ w_2' = \lambda_2 w_2 \\ . \\ w_n' = \lambda_n w_n \end{cases} .$$

sats: Om den numeriska metoden är stabil för $w_i' = \lambda_i w_i$, alla $\lambda_i = \lambda(t, \tilde{u})$ i lösningsområdet så ör deb stabil för $\overline{u}' = F(t, \overline{u}) \Rightarrow$ Utanför stabilitetsanalysen på testekvation $y' = \lambda y$ men med $\lambda_i(t, \tilde{u})$ (egenvärdena till jacobianen)

Not: λ - kan vara komplext (äen om $\overline{u}$ och $F(t, \overline{u})$ är reela)

13.5 Stokastiska metoden

- Deterministisk metoder: Det vi tidigare kållat på. Får samma svar för sama input
- Stokastiska metoder: Varierar för same input, då slumpen spellar roll.

13.5.1 Monte Carlo

```
Indata: N (antal försök)
for i=1:N
  Gör en stokastisk simulering
  resultat(i)=resultat av simulering ovan
end
slutresultat = mean(resultat)
```

Ensemble-prognoser är att man kör flertal gånger och kan ta medelvärdet för att få den troliga lösningen men sen också gemföra med och se hur stora fel/ osäkerhet är prognosen.

Initialtilståndet

Matlab har flera olika slumptals genererare

- r̈and:̈ slumptal mellan 0 och 1
- r̈andn:̈ normalfördelade slumptal

Nogrannhetsordning Monte Carlo är oberonde av dimensioner (vilket ex trapetsmetoden inte är).

$$\begin{aligned} \text{Trapetsmetoden - Fel } &\sim O(n^{-2/d}) \\ \text{MC - Fel } &\sim O(1/\sqrt{N}) \end{aligned}$$

13.5.2 Invers Transform Sampling (ITS)

ITS är en algorithm för att generera slumptal ut ifrån en given fördelning.

Cumulative distribution function (CDF): is that you take the primitive function of a distribution. Man kan då bestäma τ som är tiden för reaktion ska ske.

Exempel: ITS

$$f_\lambda(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

1. Hitta primitiva funktionen

$$F_{\lambda(x)} = \int_0^x \frac{1}{\lambda} e^{-t/\lambda} dt = [-e^{-t/\lambda}]^x{}_0 = 1 - e^{-x/\lambda}$$

2. Hitta inversen

F^{-1} : Löser $y = 1 - e^{-x/\lambda}$

$y - 1 = -e^{-x/\lambda} \Rightarrow ln(1-y) = -x/\lambda \Rightarrow x = -\lambda ln(1-y)$

Vilket är exponentialfördelad

Diskret slumptalsfördelning

Exempel: ITS Diskret slumptalsfördelning

$$\begin{aligned} 1\text{: Röd; } P_1 &= 0.45 \\ 2\text{: Gul; } P_2 &= 0.10 \\ 3\text{: Grön; } P_3 &= 0.45 \\ u &= 0.61 \end{aligned}$$

ITS innebär i den diskreta versionen att r,
numret på nästa utfall, ska väljas som det minsta tal x
så att $F(x) \leq u$, där $F(x)$ är den kumulativa fördelningsfunktionen.
Med de givna värdena kan $F(x)$ representeras av vektorn

$$F = \begin{bmatrix} 0.45 \\ 0.45 + 0.10 \\ 0.45 + 0.10 + 0.45 \end{bmatrix} = \begin{bmatrix} 0.45 \\ 0.55 \\ 1.00 \end{bmatrix}$$

Vi ser att $F(3)$ är den första som är störe än 0.61.
Slutsatsen är att det kommer då vara $r = 3$, färgen grön som slumpas fram.

13.5.3 Gillespie algorithm/Stochastic Simulation Algorithm (SSA)

Simulering av reaktioner oftast i kemiska kontext.

Exemple

$$\begin{aligned} S'(t) &= -\beta \frac{I(t)}{N} S(t) \\ I'(t) &= \beta \frac{I(t)}{N} S(t) - \gamma I(t) \\ R'(t) &= \gamma I(t) \end{aligned}$$

Reaktioner:

$$\begin{aligned} r_1: &\quad \beta \frac{I(t)}{N} S(t) \\ r_2: &\quad \gamma I(t) \end{aligned}$$

Reaktionsmatris:

$$\begin{array}{cc} & \begin{array}{ccc} S(t) & I(t) & R(t) \end{array} \\ \begin{array}{c} r_1 \\ r_2 \end{array} & \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{array} = R$$

Då har vi även reaktionshastighets-vektorn som följande.
Samt att $C = \{\beta, \gamma, N\}$.

$$\mathbf{P}(\tilde{\mathbf{y}}, \mathbf{C}) = \begin{bmatrix} C(1) \dfrac{y(1)}{C(3)} y(2) \\ C(2) y(1) \end{bmatrix}$$

I matlab:

13.6 Ordlista

- *Adaptiv steglängd*: Att den numeriska metoden i varje steg automatiskt ställer in steglängden så att det uppskattade felet blir lägre än toleransen.
- *Cumulative distrubution function*: du finner primitiva functionen av distributionen.
- *Deterministisk metoder*: En algoritm utan slumpmoment, så att utdata beror entydigt på indata.
- *Deterministisk modell*: En modell utan slumpmoment, så att utdata beror entydigt på indata.
- *Diskretiseringsfel*: Vid exempelet när man beräknar derivatan så är dominerar Diskretiseringsfel
- *Explicit metod*: En metod för numerisk lösning av ODE, där högerledet i metoden bara beror på kända värden.
- *Global trunkteringsfelet*: Det sammanlagda felet i från det tidigare stegen med den numeriska metoden utgående från exakta lösningen i senaste tidpunkten.
- *Implecit metod*: En numerisk metod för lösning av ODE, där högerledet i metoden innehåller y_{i+1} , så att man måste lösa en ickelineär ekvation i varje steg.
- *ITS*: en algorithm för att generera slumptal ut ifrån en given fördelning
- *Konsistent*: Den numeriska metoden går mot differentialekvationen då h går mot noll.

- *Kancellation*: Förlust av signifikanta siffror vid subtraktion mellan jämnstora flyttal.
- *Konvergens*: Den numeriska lösningen går mot den exakta lösningen då stegläng- den h går mot noll.
- *lockala trunkteringsfelet*: Felet i ett steg med den numeriska metoden utgående från exakta lösningen i senaste tidpunkten.
- *Mantissa*: bråkdelen i floting point
- *Maskinepsilon*: Det relativa fel man får då tal lagras i en dator (eller när beräkningar utförs) beror på hur talen lagras internt i datorn.
- *Monte Carlo-metod*: Upprepad stokastisk simulering samt statistik på de samlade re- sultaten.
- *Normalisering*: Konventionen att i flyttalsrepresentation ha precis en nollskild siffra före "decimalpunk- ten" (så att flyttalsrepresentationen blir entydig).
- *Noggrannhetsordning*: h-potensen i globala trunkeringsfelet hos en numerisk metod för lösning av ODE (där h är steglängden).
- *Overflow*: Motsatsen till underflow.
- *Stokastiska modell*: En modell med slumpmoment, så att utdata inte beror entydigt på indata.
- *Styv*: En ODE för vilken en explicit differensmetod behöver ta mycket kortare steg för stabilitet än vad som skulle krävas för tillräcklig noggrannhet.
- *Stabilitet*: Störningskänsligheten hos den numeriska metoden.
- *SSA/Gillespie algorithm*:
- *Underflow*: Det som uppstår när vi försöker representera ett tal vars abso- lutbelopp är mindre än det till beloppet minsta normaliserade flyttalet.
- *jacobianen*

Chapter 14

Signal and Transforms

14.1 Introduction

14.1.1 System Clasification

- *Continuos-time*: Defined for a all times in a time period $x(t)$
- *Discrete-time*: Defined for a some times in a time period $x[k]$
- *Analog*: Continous time signal time-varing
- *Digital*: Discreate time signal
- *Linear/ Non-linear*: if $\alpha x_1(t) + \beta x_2(t) \mapsto \alpha y_1(t) + \beta y_2(t)$ (supper position)
- *Periodic/ Aperiodic*: $x(t) = x(t + T)$
- *Casual*: time dependent on on previus not next values
- *time-varieng/ time-invariant* $x(t - T) \mapsto y(t - T)$
- *Deterministic* Same result each time no random.
- *stochastic* random -signals (won't look at)
- *Stable/ unstable* bounded $lim_{t\to\infty} x(t) < \infty$
- *Invertible* Can invert
- *SISO* (signal inlput signal output)
- *MIMO* (multiple input multiple output)
- *BIBO* (bounded input-bounded output) stable/undtable

Linear, time invariant (LTI)	Linear, time variant
non-linear, time invariant (LTI)	non-linear, time variant

We only look at LTI only.
Anny signal can be expressed by its even and odd componetnt

$$x(t) = x_e(t) + x_o(t) \tag{14.1}$$

Sampling: continuous time signal.

$$x[k] \triangleq x(kT_s) \tag{14.2}$$

T_s is the sampling time. 'The discreate signals at time k equals to the continuous time signal $x(t)$ at time $t = kT_s$'.

$$x(t) \mapsto y(t) \tag{14.3}$$
$$x[t] \mapsto y[t] \tag{14.4}$$

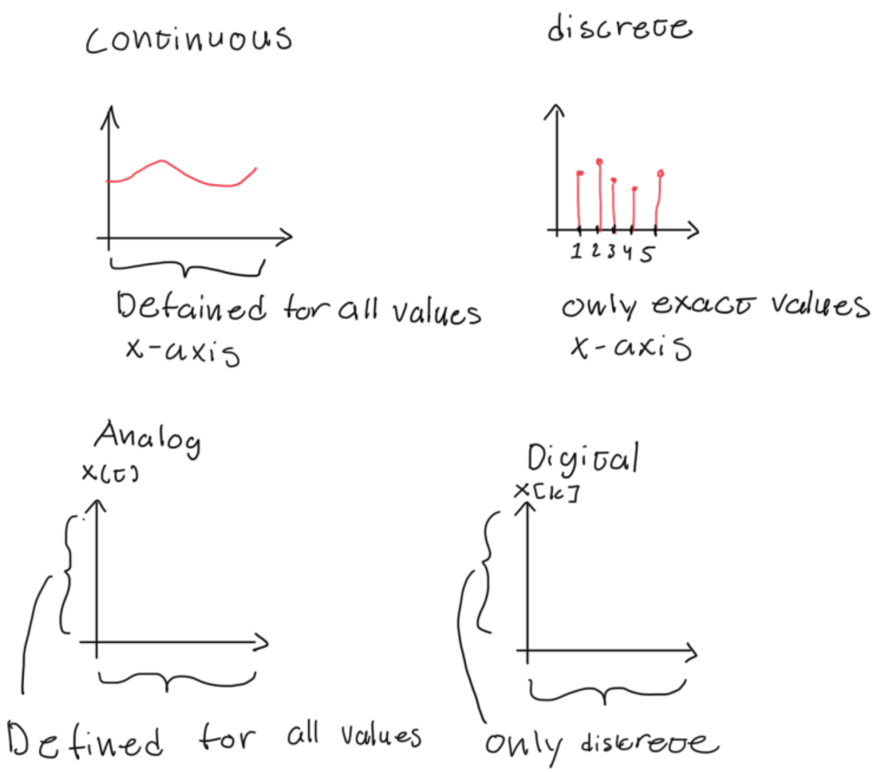


Figure 14.1: Signal plots

Transform methods

A transform is a operator $\mathrm{T}\{\dot{}\}$ from one domain to another.

$$X(\varsigma) = \mathrm{T}\{x(t)\}$$

$x(t)$ and $X(\varsigma)$ are a *transform pairs*

$$x(t) \circ\!\!-\!\!\bullet\; X(\varsigma)$$

The inverse transform $\mathrm{T}^{-1}\{\dot{}\}$ restores the original signal.

$$x(t) = \mathrm{T}^{-1}\{X(\varsigma)\}$$

Example: Linear

Is $\frac{dy(t)}{dt} + 3ty(t) = t^2x(t)$ linear? Assume zero initial conditions.
Solution:

$$\frac{dy(t)}{dt} + 3ty(t) = t^2x(t)$$

$$\frac{d(\alpha y_1(t) + \beta y_2(t))}{dt} + 3t(\alpha y_1(t) + \alpha y_2(t)) = t^2(\alpha x_1(t) + \beta x_2(t))$$

$$\alpha\frac{dy_1(t)}{dt} + \beta\frac{dy_2(t)}{dt} + \alpha 3ty_1(t)\beta 3ty_2(t) = \alpha t^2x_1(t) + \beta t^2x_2(t)$$

therefore is the system linear.

Example: Time invariant

Is $y(t) = tx(t-2)$ time invariant?
Solution:

$$y(t-\tau) = tx((t-\tau)-2 \neq (t-\tau)x((t-\tau)-2$$

therefore is the system time variant.

Example: Casual

Is $y(t) = x(-t)$ casual?
Solution: When $t < 0$ then $t < -t$ with is not allowed in a casual system since it then depends on upcoming values. If we were to restrict that $t \geq 0$ then it would be casual.

Euler's identities

$$\begin{aligned}
e^{+jx} &= \cos x + j \sin x \\
e^{-jx} &= \cos x - j \sin x \\
\sin(\alpha) &= \frac{e^{j\alpha} - e^{-j\alpha}}{2j} \\
\cos(\alpha) &= \frac{e^{j\alpha} + e^{-j\alpha}}{2}
\end{aligned}$$

$f_0 = \frac{1}{T_0}$, $w = 2\pi f_0$ $w_0 = \frac{2\pi}{T_0}$

14.2 Continuous-time signals and time-domain analysis

14.2.1 Continous Time Signals

- **Sine**: $A\sin(\omega_0 t + \varphi)$
  - ω is *angular frequency* $\omega_0 t = 2\pi f_0$ fpr *frequency domain*
  - t is time since it is in *time domaian*
  - φ is the *phase*
  - $\sin(\omega_0 t) = \dfrac{e^{j\omega_0 t} - e^{-t\omega_0 t}}{2j}$
- **Cosine**: $A\cos(\omega_0 t + \varphi)$
  - $\cos(\omega_0 t) = \dfrac{e^{j\omega_0 t} + e^{-t\omega_0 t}}{2}$
- **Dirac Delta function**: $\varsigma(t)$
  - $\delta(t) = 0$ for $t \neq 0$
  - $\int_{-\infty}^{\infty} \delta(t)dt = 1$
  - $\int_{-\infty}^{\infty} x(t)\delta(t)dt = x(0)$
  - $\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
- **Rectangular pulse**: $rect(\frac{t}{T_0})$
- **Unit step**: $u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$
- **Sinc function**: $sinc(\dfrac{t}{T_0}) = \dfrac{\sin(\frac{\pi t}{T_0})}{\frac{\pi t}{T_0}}$
- **Periodic Signals**: $x(t) = x(t + T_0)$
- **Energy Signals**: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$
- **Power Signals**: $P = \lim_{T\to\infty} \dfrac{1}{T} \int_{T/2}^{-T/2} |x(t)|^2 dt$
- **Orthogonal Signals**: $\langle x, y\rangle = x^{\mathrm{T}} y = \sum_i x_i y_i$

Signal Caractaristic

- **Casual**: $x(t) = 0$ for $t < 0$
- **Even**: (flip on y-axis) $x(t) = x(-t)$
- **Odd**: (flip on x-axis and y-axis) $x(t) = -x(-t)$

Signal opertations

- **Amplitude scaling** (Increase/decrease the amplitude value): $y(t) = \alpha x(t)$
- **Offsetting** (Moves signal in y-direction): $y(t) = x(t) + y_0$
- **Time shifting** (Move signal in x-direction): $y(t) = x(t - \Delta t)$
- **Time scaling and inversion** (Increase/decrease width): $y(t) = x(\alpha t)$

14.2.2 Time Domain Analysis of Contious Time Systems

Main properties

$$\delta(t) \text{ is even function}$$
$$\delta(at+b) = \frac{1}{|a|}\delta(t+\frac{b}{a})$$
$$\phi(t)\delta(t-\tau) = \phi(\tau)\delta(t-\tau)$$
$$\int_{-\infty}^{\infty} \phi(\tau)\delta(\tau-t)dt = \phi(t)$$
$$\text{Convolution: } (x*h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)dt \Rightarrow x(t)*\delta(t-T) = x(t-T)$$

Impulse responce

Deffinition

$$x(t) = \delta(t) \mapsto y(t) = h(t)$$

Properties

$$\textit{Impulse response: } h(t)$$
$$\delta(t) \mapsto h(t) \text{ since LTI system with continus time}$$
$$\alpha\delta(t-T) \mapsto \alpha h(t-T) \text{ since LTI}$$

Convolvution integral

Convolution integral. For any input $x(t)$, the output of an LTI system is (*convolution*):

$$y(t) = x(t) * h(t) = h(t)$$

Definition of convolution:

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \tag{14.5}$$

For causal systems ($h(t) = 0$ for $t < 0$):

$$\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau \tag{14.6}$$

Example: Convolution 1

$$\begin{aligned}
\sin(t)u(t) * u(t) &= \int_{-\infty}^{\infty} \sin(\tau)u(\tau)u(t-\tau)d\tau \\
&= \int_{0}^{t} \sin(\tau)d\tau = [-\cos(\tau)]_0^t \\
&= -\cos(t) + \cos(0) = 1 - \cos(t)
\end{aligned}$$

Example: Convolution 2

$$\begin{aligned}Y(\omega) &= \frac{1}{2\pi}X(\omega) * \pi[\delta(\omega-25)+\delta(\omega+25)] \\ &= \frac{1}{2}[X(\omega)*\delta(\omega-25)+X(\omega)*\delta(\omega+25)] \quad \text{-(Constants can be moved out)} \\ &= \frac{1}{2}[X(\omega-25)+X(\omega+25)] \quad \text{-(Convolution of direct delta is just time shift)}\end{aligned}$$

Sin In, Sine Out Principle

The frequency dose not change.

$$x(t) = A_x \sin(\omega_0 t - \varphi_x) \mapsto y(t) = A_y \sin(\omega_0 t - \varphi_y) \tag{14.7}$$

Amplitude and phase reletionships ($H(\omega_0)$: complex number):

$$A_y = |H(\omega_0)|A_x \quad \text{and} \quad \varphi_y = \varphi_x + \underline{/H(\omega_0)} \tag{14.8}$$

14.3 Continuous Time Fourier Series and Transform

14.3.1 Continuous Time Fourier Series

Trigonometric continuos time Fourier series Any periodic continius time signal can be expressed as

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

$$\begin{aligned}a_0 &= \frac{1}{T_0}\int_{(T_0)} x(t)dt \\ a_n &= \frac{2}{T_0}\int_{(T_0)} x(t)\cos(n\omega_0 t)dt \\ b_n &= \frac{2}{T_0}\int_{(T_0)} x(t)\sin(n\omega_0 t)dt\end{aligned}$$

Exponential continuos time Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$\begin{aligned}\omega_0 &= \frac{2\pi}{T_0} \\ c_n &= \frac{1}{T_0}\int_{(T_0)} x(t)e^{-jn\omega_0 t}dt\end{aligned}$$

Coefficients relationships of Exponential and trigonometric Fourier series

$$c_0 = a_0, \quad c_n = \frac{a_n - jb_n}{2}, \quad c_{-n} = \frac{a_n + jb_n}{2}, \quad \text{for } n \geq 1$$

$$a_0 = c_0, \quad a_n = 2Re\{c_n\}, \quad b_n = -2Im\{c_n\}, \quad \text{for } n \geq 1$$

Complex conjugate pair $c_n = c^*_{-n}$

- (Trigonometric) If $x(t)$ is an even signal, $b_n = 0$
- (Trigonometric) If $x(t)$ is an odd signal, $a_n = 0$
- (Exponential) If $x(t)$ is an even signal, c_n is purely real
- (Exponential) If $x(t)$ is an odd signal, c_n is purely imaginary

14.3.2 Continuos Time Fourier Transforms

For Aperiodic we use Fourier transform

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Spectral Analysis

Analysing figure of magnitude $|X(\omega)|$ and the phase $\underline{/X}(\omega)$. Phase plot is very important.

The magnitude plot shows us what frequencies is dominant and not. It is even. One can also determin the output from an input signal from the spectrum since from the magnitude you can do times that of the input signal. The phase plot determines the phase shift for a frequency of the input signal w_0.

Phasor operations

$$\begin{aligned}
\text{addition: } & z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \\
\text{subtraction: } & z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \\
\text{multiplication: } & z_1 z_2 = r_1 r_2 \underline{/(\phi_1 + \phi_2)} \\
\text{division: } & \frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/(\phi_1 - \phi_2)} \\
\text{inverse: } & \frac{1}{z} = \frac{1}{r} \underline{/-\phi} \\
\text{square root: } & \sqrt{z} = \sqrt{r} \underline{/(\phi/2)} \\
\text{complex conjugate: } & z^* = x - jy
\end{aligned}$$

Properties

- **Linjarity**: $x_1(t) \circ\!\!-\!\!\bullet X_1(\omega)$, and $x_1(t) \circ\!\!-\!\!\bullet X_1(\omega) \Rightarrow \alpha x_1(t) + \beta x_2(t) \circ\!\!-\!\!\bullet \alpha X_1(\omega) + \beta X_2(\omega)$
- **Duality**: $X(t) \circ\!\!-\!\!\bullet 2\pi x(-\omega)$
- **Time and frequency scaling**: $x(at) \circ\!\!-\!\!\bullet \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$ for $a \neq 0$
- **Time and frequency shifting**: $x(t - \Delta t) \circ\!\!-\!\!\bullet X(\omega) e^{-j\omega\Delta t} \Rightarrow x(t)e^{j\omega_0 t} \circ\!\!-\!\!\bullet X(\omega - \omega_0)$
- **Convolution and multiplication**: $x_1(t) * x_2(t) \circ\!\!-\!\!\bullet X_1(\omega)X_2(\omega) \Rightarrow x_1(t) * x_2(t) \circ\!\!-\!\!\bullet X_1(\omega)X_2(\omega)$
- **Differentiation and integration**: $\frac{d^n x(t)}{dt^n} \circ\!\!-\!\!\bullet (j\omega)^n X(\omega)$ where n is for derivatives $\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \circ\!\!-\!\!\bullet \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega)$
- **Symmetries**: $X^*(\omega) = X(-\omega)$
  - $|X(\omega)|$ is even
  - $\underline{/X}(\omega)$ is odd
  - $Re\{X(\omega)\}$ is even
  - $Im\{X(\omega)\}$ is odd
- **Periodicity**:
  - If $x(t)$ is periodic, $X(\omega)$ is discrete
  - If $x(t)$ is discrete, $X(\omega)$ is periodic
- **Parseval's theorem**:
  - $W = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$
  - $P = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)| dt = \sum_{n=-\infty}^{\infty} |c_n|^2$

Fourier Transforms of Elementary Signals

- **Dirac delta**:
  - $X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{j\omega t} dt = e^{-j\omega 0} = 1$
  - $x(t) = \delta(t) \circ\!\!-\!\!\bullet\ X(\omega) = 1$
- **Constant**: (Duality property)
  - $X(\omega) = 2\pi\delta(-\omega)$
  - $x(t) = 1 \circ\!\!-\!\!\bullet\ X(\omega) = 2\pi\delta(\omega)$
- **Rectangular pulse**:
  - $X(\omega) = \int_{-\infty}^{\infty} rect\Big(\frac{t}{T_0}\Big) dt = \frac{e^{-j\frac{\omega T_0}{2}} - e^{j\frac{\omega T_0}{2}}}{-j\omega}$
  - $X(\omega) = \frac{2\sin(\frac{\omega T_0}{2})}{\omega} = T_0 sinc\Big(\frac{\omega T_0}{2\pi}\Big)$
  - $x(t) = rect\frac{t}{T_0} \circ\!\!-\!\!\bullet\ X(\omega) = T_0 sinc\Big(\frac{\omega T_0}{2\pi}\Big)$
- **Sinc function**:
  - $sinc\Big(\frac{t}{T_0}\Big) = X_{rect}\Big(\frac{2\pi t}{T_0}\Big)$
  - $x(t) = sinc\Big(\frac{t}{T_0}\Big) \circ\!\!-\!\!\bullet\ X(\omega) = T_0 rect\Big(\frac{\omega T_0}{2\pi}\Big)$
- **Periodic signals**: It's fourier transform becomes discrete even if the input signal is continues
  - $X(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} e^{-j\omega t} dt = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$

14.3.3 Frequency Domain Analysis of Continuous Time Systems

Frequency domain input-output relationship

$Y(\omega) = H(\omega)X(\omega)$ -(Since in the time domain $y(t) = x(t) * h(t)$)

Where $Y(\omega)$ is the fourier transform LTI systems output signal,

$H(\omega)$ is the *frequency response* and $X(\omega)$ is the input signals Fourier transform

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

Polar form of Frequency domain input-output relationship

$$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$$
$$Y(\omega) = |X(\omega)||H(\omega)| e^{j(\angle X(\omega) + \angle H(\omega))}$$

Example: find output signal

What is the output signal $y(t)$ if the input signal is $x(t) = 2\cos\left(\frac{\pi}{2}t\right)$ and the impulse response is $h(t) = rect\left(\frac{t-1}{2}\right)$?

Solution:

$$x(t) = 2\cos\left(\frac{\pi}{2}t\right) \mapsto y(t) = 2|H\left(\frac{\pi}{2}\right)|\cos\left(\frac{\pi}{2} + \angle H\left(\frac{\pi}{2}\right)\right)$$

$$\begin{aligned}
H(\omega) &= \mathcal{F}\{h(t)\} = \mathcal{F}\{rect\left(\frac{t-1}{2}\right)\} \\
&= 2e^{-j\omega} sinc\left(\frac{2\omega}{2\pi}\right) \\
\Rightarrow H\left(\frac{\pi}{2}\right) &= 2e^{-j\frac{\pi}{2}} sinc\left(\frac{2\frac{\pi}{2}}{2\pi}\right) \\
&= 2e^{-j\frac{\pi}{2}} sinc\left(\frac{1}{2}\right) \approx 1.3e^{-j\frac{\pi}{2}} \\
&\Rightarrow |H\left(\frac{\pi}{2}\right)| = 1.3, \angle H\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}
\end{aligned}$$

Therefore is the output signal:

$$\begin{aligned}
y(t) &= 2 \cdot 1.3\cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right) \\
&= 2.6\cos\left(\frac{\pi}{2}t - \frac{\pi}{2}\right)
\end{aligned}$$

14.4 Laplace transform

General version of fourier transform. One issue with Fourier transform is that there could be undefined values. Therefor we introduce an exponential expression in order for it to converge quickly enuff. The definition *unitary laplace transform* is:

$$\begin{aligned}
X(s) &= \mathcal{L}\{x(t)\} = \int_0^\infty x(t)e^{st}dt \\
x(t) &= \mathcal{L}^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds
\end{aligned}$$

$X(s)$ is the unilateral laplace transform
$x(t)$ is a continuous time signal
$s = \sigma + j\omega$ is a complex variable
σ is dampening factor
ω is the frequency

Region of conversions (ROC) is needed for the laplace transform to exist. In other words, that the integral over $x(t)e^{st}$ is finite and a transform exists

Example: Convert transfer function to inpulse responce

Determine is the impulse response of:

$$H(s) = \frac{1}{(s+2)(s+3)}$$

Solution: To determine the impulse response from the transfer function do we need to to the inverse laplace transfer of the transfer function.

$$\begin{aligned} H(s) &= \frac{c_1}{s+2}\frac{c_2}{s+3} = \frac{c_1(s+3)}{s+2} + \frac{c_2(s+2)}{s+3} \\ &= \frac{c_1 s + 3c_1 + c_2 s + 2c_2}{(s+2)(s+3)} \end{aligned}$$

Therefore do we get the following system of equations

$$\begin{cases} c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \\ 3c_1 + 2c_2 = 1 \end{cases}$$

Hence, the constants are $c_1 = 1$ and $c_2 = -1$.

$$\begin{aligned} H(s) &= \frac{1}{s+2} - \frac{1}{s+3} \\ h(t) &= e^{-2t}u(t) - e^{-3t}u(t) \quad \text{-(transfer table)} \end{aligned}$$

Transform pairs

Time domain $x(t)$	Laplace transform $X(s)$	ROC
Dirac delta function		
$x(t) = \delta(t)$	$X(s) = 1$	All s
Unit step		
$x(t) = u(t)$	$X(s) = \frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
Exponential		
$x(t) = e^{-at}u(t)$	$X(s) = \frac{1}{a+s}$	$\mathrm{Re}\{s\} > -a$
Ramp		
$x(t) = tu(t)$	$X(s) = \frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
Higher order ramp		
$x(t) = t^n u(t)$	$X(s) = \frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
Cosine		
$x(t) = \cos(\omega_0 t)u(t)$	$X(s) = \frac{s}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Sine		
$x(t) = \sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Decaying cosine		
$x(t) = e^{-at}\cos(\omega_0 t)u(t)$	$X(s) = \frac{a+s}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$
Decaying sine		
$x(t) = e^{-at}\sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$

Figure 14.2: Unilateral Laplace transform pairs. From ()

14.4.1 Properties

- **Linjarity**: $\alpha x_1(t) + \beta x_2(t) \circ\!\!-\!\!\bullet\ \alpha X_1(s) + \beta X_2(s)$
- **Time and frequency scaling**: $x(at) \circ\!\!-\!\!\bullet\ \frac{1}{a}X(\frac{s}{a})$ for $a > 0$
- **Time and frequency shifting**:
  - $x(t - \Delta t) \circ\!\!-\!\!\bullet\ X(s)e^{-s\Delta t}$
  - $x(t)e^{at} \circ\!\!-\!\!\bullet\ X(s - a)$
- **Convolution and multiplication**:
  - $x_1(t) * x_2(t) \circ\!\!-\!\!\bullet\ X_1(s)X_2(s)$

  - $x_1(t)x_2(t) \circ\!\!-\!\!\bullet \frac{1}{2\pi j} X_1(s) * X_2(s)$

- **Differentiation and integration**: $\frac{d^n x(t)}{dt^n} \circ\!\!-\!\!\bullet sX(s) - x(t)$
- **Initial and final value theorems**:
  - $\lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)$
  - $\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

14.4.2 Transfer function

Solving differential equations

Can solve Differentiation equations. Use Laplace transforms then do the inverse transform. The analyzing system is useful to use transform. It is done the same way as for solving differential equations. This works for all dimensions.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^M b_M + s^{M-1} b_{M-1} + \ldots + b_0}{s^N b_N + s^{N-1} b_{N-1} + \ldots + a_0}$$
$$Y(s) = H(s)X(s)$$

Transfer functuion and impuslse response

$$h(t) \circ\!\!-\!\!\bullet H(s)$$
$$H(s) = \int_0^\infty h(t) e^{-st} dt$$

Transfer functuion and frequency response

The relationship between the transfer function $H(s)$ and the frequency response $H(j\omega)$ of a continues time LTI system is given by

$$H(j\omega) = H(s)|_{j\omega}$$

Poles and Zeros, and Stability

$$H(s) = K \frac{(s - z_1)(s - z_s) \ldots (s - z_M)}{(s - p_1)(s - p_2) \ldots (s - p_N)}$$
$$\text{where } z_M \text{ is zeros}$$
$$\text{and } p_N \text{ is poles}$$

The system is proper if $N \geq M$ and strictly proper if $N > M$. It is stable if all $p_i < 0$.

If $p_i = 0$ then the output will never convert just alternate to infinity (unstable).
If $p_i > 0$ then it will increase it's amplitude (unstable).

LTI system is stable if and only if the real part of all of its poles is negative.

Steady-state is when amplitude is $A = |H(s)|_{s=j\omega_0}$ and phase $\phi = H(s)_{s=j\omega_0}$

The poles tells us when the transfer function $H(s)$ becomes a division by zero. The zeros meanwhile tells us when then the function is zero. The Pole-Zero Map is a display from the top looking down where x is the marked when the expression goes to infinite and the zero is when the expression is zero.

Example: Poles and Zeros

Consider a linear, time-invariant system describe ed by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 13y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Assume that all initial conditions are zero. Determine wether or not the system is stable.
Solution: Determine the transfer function $H(s)$

$$\begin{aligned}
&s^2Y(s) + 6sY(s) + 13Y(s) = sX(s) + 2X(s)\\
&Y(s)(s^2 + 6s + 13) = X(s)(s + 2)\\
&\frac{Y(s)}{X(s)} = \frac{s+2}{s^2 + 6s + 13} = H(s)
\end{aligned}$$

we se that the system is proper since the degree of the enumerated is less the denominator
Zeros:

$$\begin{aligned}
&s + 2 = 0\\
&s = -2 \Rightarrow z_1 = -2
\end{aligned}$$

Poles:

$$\begin{aligned}
&s^2 + 6s + 13 = 0\\
&s = -\frac{6}{2} \pm \sqrt{3^2 - 13} = -3 \pm \sqrt{-4} = -3 \pm 2j\\
&\Rightarrow p_{1,2} = -3 \pm 2j
\end{aligned}$$

There of do we see that both poles are negative, there fore is the system stable.

qualitative impulse response.

14.4.3 Bod plot

Bode plots can relatively easily be constructed by hand, which was paramount before the computer era for analyzing complex dynamic systems.

Bode plots are a standardized way of plotting a system's frequency response function and a Bode plot actually consists of two plots: A Bode magnitude plot (or simply magnitude plot) and a Bode phase plot (or simply a phase plot).

Asymptote is the line where define the direction of the line.

$$|H(j\omega)|_{dB} = 20\log_{10}(|H(jw)|)$$

Bode Form of Transfer Function

$$\begin{aligned}
H(s) &= K\frac{(s-z_1)\dots(s-z_M)}{s^l(s-p_1)\dots(s-p_{N-l})}\\
H(s) &= K_0\frac{(\frac{s}{z_1}+1)\dots((\frac{s}{\omega_{n,i}})^2+2s\frac{\zeta_i}{\omega_{n,i}}+1)\dots}{s^l(\frac{s}{p_1}+1)\dots((\frac{s}{\omega_{n,j}})^2+2s\frac{\zeta_j}{\omega_{n,j}}+1)\dots}\\
K_0 &= K\frac{\prod_i(-z_i)}{\prod_i(-p_i)}
\end{aligned}$$

There are l poles in the origin (the term s^{-l}).

Sketching Bode Plots

Example: ODE solving with transfer functions

Sketch Bode plots for the following transfer functions.

$$H(s) = \frac{s(s+100)}{(s+2)(s+20)}$$

Solution: First we write the transfer function as in bode normal Form

$$H(s) = \frac{s(s+100)}{(s+2)(s+20)} = \frac{100s(\frac{s}{100}+1)}{2\cdot 20(\frac{s}{2}+1)(\frac{s}{20}+1)}$$
$$H(j\omega) = \frac{100j\omega(\frac{j\omega}{100}+1)}{2\cdot 20(\frac{j\omega}{2}+1)(\frac{j\omega}{20}+1)} = \frac{100j\omega(1+j\frac{\omega}{100})}{2\cdot 20(1+j\frac{\omega}{2})(1+j\frac{\omega}{20})}$$

$$\lim_{\omega\to 0} H(j\omega) = 0, (-\infty dB)$$
$$\lim_{\omega\to\infty} H(j\omega) = 1, (0dB)$$

We can write the magnitude by adding the factor together

$$|H(j\omega)|_{dB} = 20\log_{10}(\frac{100}{2\cdot 20}) + 20\log_{10}(|j\omega|) - 20\log_{10}(\left|1+j\frac{\omega}{2}\right|)$$
$$-20\log_{10}(\left|1+j\frac{\omega}{20}\right|) + 20\log_{10}(\left|1+j\frac{\omega}{100}\right|)$$

The phase shift can be done similarly.

$$\angle\frac{100}{2\cdot 20} = 0$$
$$\angle|j\omega| = \frac{\pi}{2}$$
$$\angle\frac{1}{1+j\frac{\omega}{2}} = -\angle 1+j\frac{\omega}{2} = \begin{cases} 0 & w=0 \\ -\frac{\pi}{4} & w=2 \\ -\frac{\pi}{2} & w\to\infty \end{cases}$$
$$\angle\frac{1}{1+j\frac{\omega}{20}} = -\angle 1+j\frac{\omega}{20} = \begin{cases} 0 & w=0 \\ -\frac{\pi}{4} & w=20 \\ -\frac{\pi}{2} & w\to\infty \end{cases}$$
$$\angle 1+j\frac{\omega}{100} = \begin{cases} 0 & w=0 \\ \frac{\pi}{4} & w=100 \\ \frac{\pi}{2} & w\to\infty \end{cases}$$

Now we can draw the bode plot see figure 14.3.

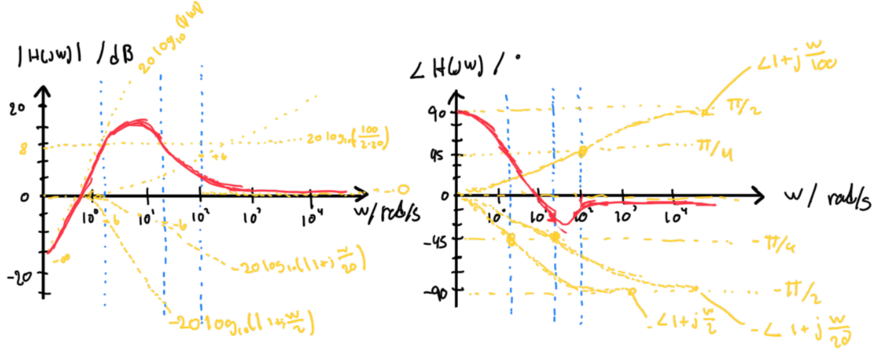


Figure 14.3: Bode plot example

14.5 Filtering theory

14.5.1 Filtering Theory and Ideal Filters

An ideal filter only contains a *passband* (the frequency range that the signal is not filterd out) and the *stopband* (the frequency witch is filtered out)

- **Lowpass filter**: $H_{LP}(\omega) = rect(\frac{\omega}{2\omega_c})$
- **Highpass filter**: $H_{HP}(\omega) = 1 - rect(\frac{\omega}{2\omega_c})$
- **Bandpass filter**: $H_{BP}(\omega) = rect(\frac{\omega}{2\omega_{c2}}) - rect(\frac{\omega}{2\omega_{c1}})$

Where ω_c is the cut-off frequency.

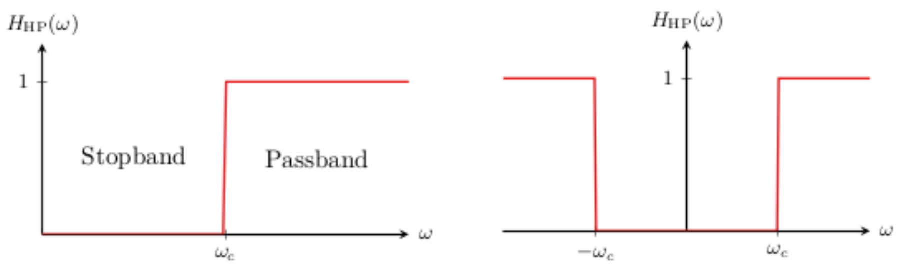


Figure 14.4: Lowpass and Highpass filter. From ()

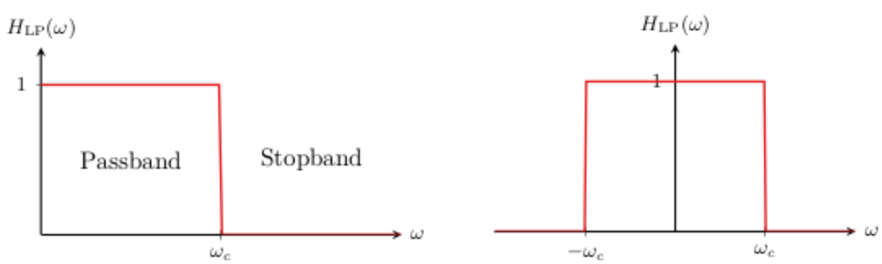


Figure 14.5: Bandpass and Bandstop filter. From ()

14.5.2 Filter Approximations

Ideal filters can not be implemented there for we use approximations.

- *passband ripple*: maximum amount of attenuation allowed in the passband.
- *passband edge frequency*: the frequency that markes the end of the requirment on the passband ripple
- *stopband attenuation*: the minimum attenuation in the stopband
- *stopband edge frequency*: the frequency that marks the start of the requirment on the stopband.

Cut-off frequency definition

$$|H(\omega_c)| = \frac{1}{\sqrt{2}} \approx 0.707 = -3dB$$

Roll-off rate is how quickly the real filter transition between the pass- and stopband.

$$R = \frac{d}{d\omega} \log_1 0(|H(\omega|^2)\Big|_{\omega=\omega_c} = 2\frac{d}{d\omega} \log_1 0(|H(\omega)|)\Big|_{\omega=\omega_c}$$

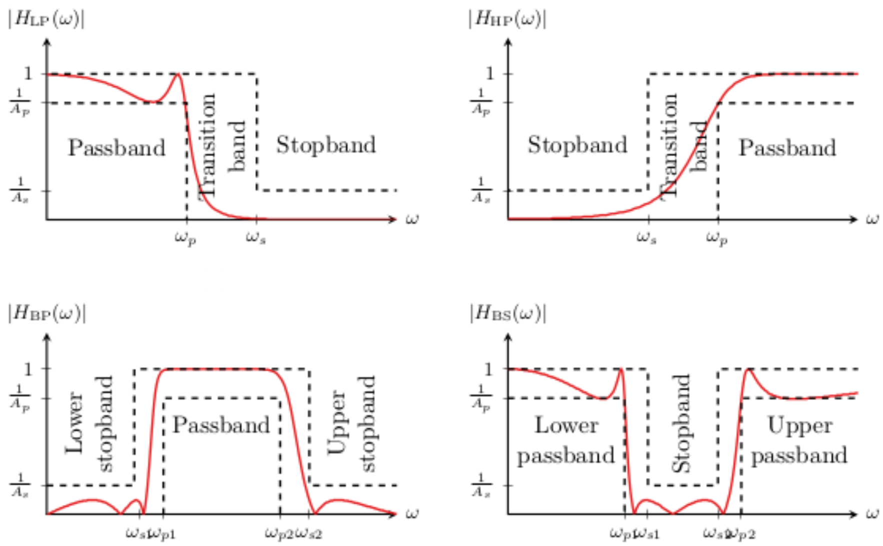


Figure 14.6: approximation-spectrum. From ()

Approximation Types

- **Butterworth filter**:
  - $|H(\omega)| = \dfrac{1}{\sqrt{1 + (\dfrac{\omega}{\omega_c})^{2N}}}$
  - $N \geq \dfrac{\log\left(\sqrt{\dfrac{A_s^2 - 1}{A_p^2 - 1}}\right)}{\log\left(\dfrac{2\pi f_s}{2\pi f_p}\right)}$
- **Chebyshev Type I filters**:
  - $|H(\omega)| = \dfrac{1}{\sqrt{1 + e^2 T_N^2(\dfrac{\omega}{\omega_p})}}$
  - $N \geq \dfrac{arccosh\left(\sqrt{\dfrac{A_s^2 - 1}{A_p^2 - 1}}\right)}{arccosh(\dfrac{2\pi f_s}{2\pi f_p})}$
  - $T_N = \begin{cases} \cos(Narccos(\omega)) \text{ for } |\omega| \leq 1 \\ cosh(Narccosh(\omega)) \text{ for } |\omega| > 1 \end{cases}$
- **Chebyshev Type II filters**:
  - $|H(\omega)| = \dfrac{1}{\sqrt{1 + \left[e^2 T_N^2(\dfrac{\omega_s}{\omega})\right]^{-1}}} = \sqrt{\dfrac{e^2 T_N^2(\dfrac{\omega_s}{\omega})}{1 + e^2 T_N^2(\dfrac{\omega_s}{\omega})}}$
  - $N =$
  - $A_p = 10^{(\text{passband ripple}/20)}$

  - $\epsilon = \sqrt{A_p^2 - 1}$
- **Elliptic filters**:
  - $|H(\omega)| =$
  - $N =$

Where A_p is passband attenuation, A_s stopband attenuation. ω_c is cutoff frequency in rad/s, f_c is cutoff frequency in Hz. ω_s is the stopband edge frequency. ϵ ripple control factor. N is the order.

- R_p, R_s (in dB).
- $A_p = 10^{R_p/20}$
- $A_s = 10^{R_s/20}$

Butterworth Chebyshev

Filter design

Example: Filter desing for Butterworth and Chebyshev type 1

TODO

Realed valued how to convert from dB to non dB what is sinc cosh arccosh

14.6 Sampling and Reconstruction

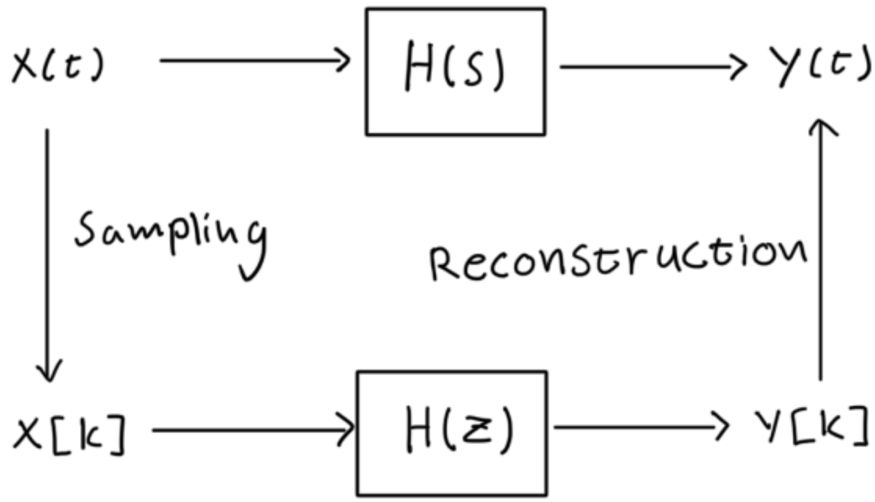


Figure 14.7: Sampling and Reconstruction

$$x[k] \triangleq x(kT_s)$$

Where T_s is the time delta for samplening and k is a integer.

We can do sampling in the time domain and in the frequncy domain. We us a sampling signal with is a diract delta function with is multiplied with the input signal in the time domain.

Bandlimiting is the limiting of signal's frequency domaian representation.

14.6.1 Sampling

The sampling frequncy ω_s is equal to the higest frerquency ω_b (b stands for bandwidth) time two.

$$\omega_s > 2\omega_b$$

Nyquist–Shannon Sampling Theorem

The creteria for sampling to work as inteded.

$$\begin{aligned}
&\omega_s = \frac{2\pi}{T_s} \\
&\omega_s > 2\omega_b \\
&f_s > 2f_c \text{ or } T_s < \frac{1}{2f_b} \\
&\omega_N = \frac{\omega_s}{2}
\end{aligned}$$

In order to get the sampled signal we need to do the following.

$$x[k] = x(kT_s) = x_s(t) = x(t)s(t)$$

The *impulse train* ($s(t)$) consisting of T_s-periodc Dirac delta functions.

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

Impulse train in the frequency domain yields:

$$S(\omega) = \mathcal{F}\{s(t)\} = \frac{2\pi}{T} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_s)$$

And the output then becomes:

$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * S(\omega)$$

Oversampling is the result of choosing a sampling frequency ω_s larger then the Nyquist frequncy ω_N, but still smaller then the maximum frequency ω_b.

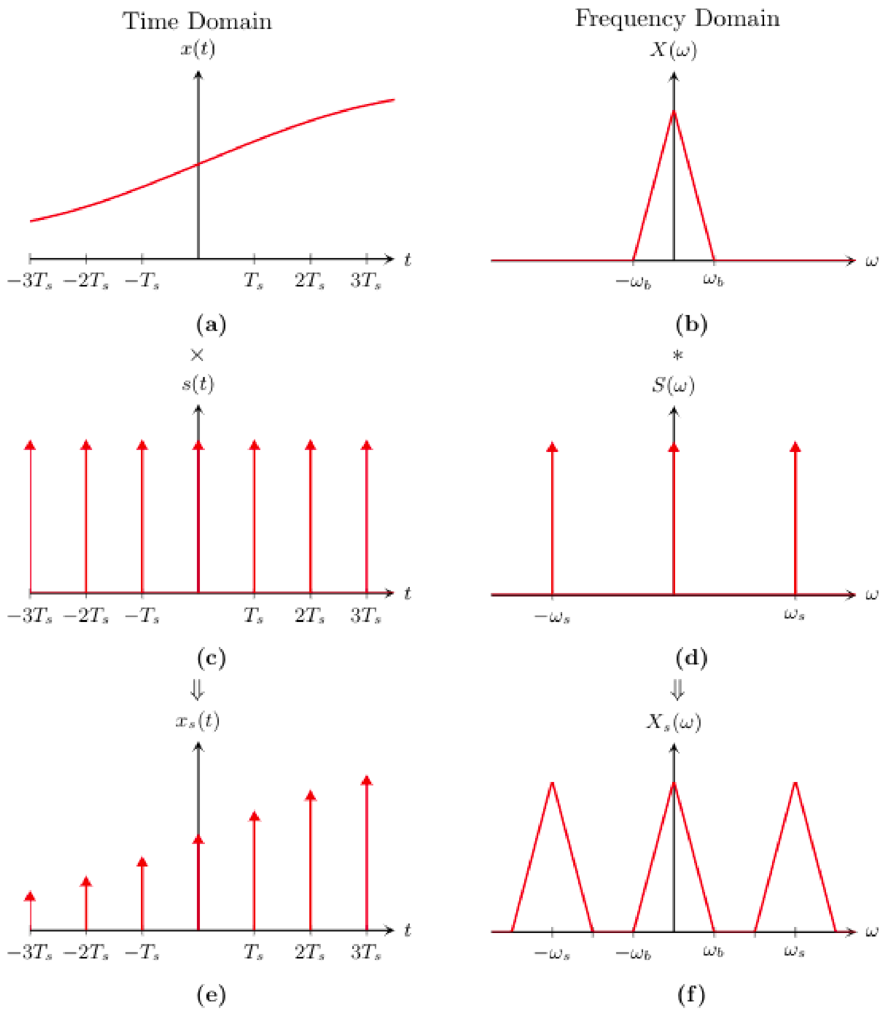


Figure 14.8: Sampling Graphs. From ()

Aliasing and Anti-Aliasing

Aliasing accures when

$$\omega_s < 2\omega_b$$

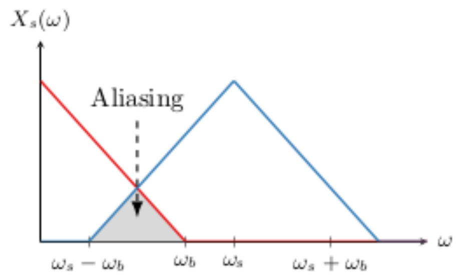


Figure 14.9: Aliasing. From ()

If there is overlap, one can not recover the signal and are therfore useless.

A anti-aliasing filter is plased before the annalog to digital conversion (ADC). Idealy would the filter block above the maximum signal frequncy ω_b but this it is not possible to implement such a filter witch garantes it and there for will there be a small amount of aliasing leaking through.

14.6.2 Reconstruction

A ideal low pass filter can be used for reconstruction, since one can get the signals frequencies. The filter is a rect in the frequency domain and there for a sinc in the time domain.

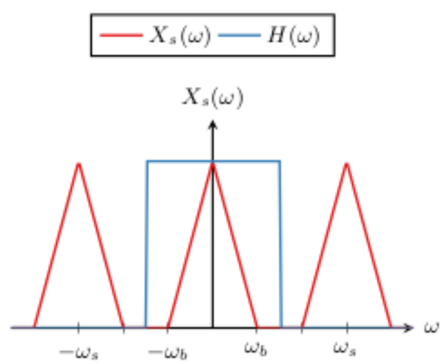


Figure 14.10: Ideal reconstruction. From ()

Zero-order hold (ZOH) is used to keep the output constant for the sampling period. After ZOH a smothing filter is used.

$$x_{zoh}(t) = x_s(t) * rect\Big(\frac{t - T_s/2}{T_s}\Big)$$
$$X_{zoh}(\omega) = X_s(\omega) T_s sinc\Big(\frac{\omega T}{2\pi}\Big) e^{j\omega \frac{T_s}{2}}$$

14.6.3 Quantization

The amount of lavels witch the digital signal can hold.

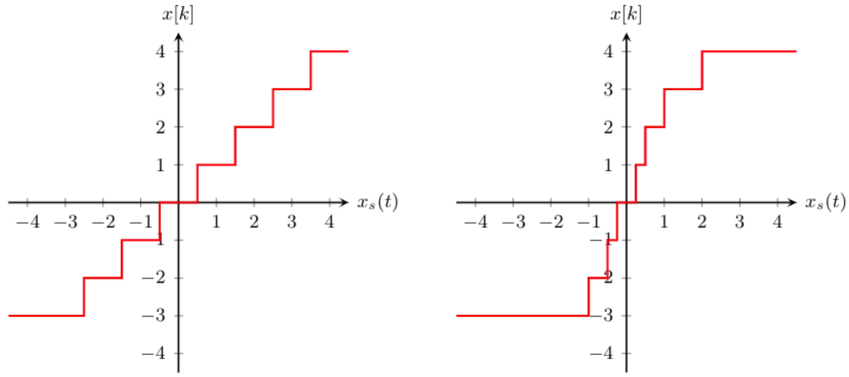


Figure 14.11: Two 3 bit quantizers ($L = 2^3 = 8$). (a) uniform quantizer (b) non-uniform quantizer. From ()

14.7 Discrete time signals and systems

Discrete time signals in only define on a valid integer k. Therefore can we not have a stem one 0.5 for instance.

K_0 is the descrete time period. Therefore is the following true for periodc dicrete time signals

$$x[k] = x[x + K_0]$$

also, the continus time period can be derived

$$K_0 T_s = T_0$$

Example: Determine how many samples is needed

If you were to calculate the DFT using K samples of the signal without zero padding, how many samples do you need when the signal is

$$x[k] = 0.5\cos\left(2\frac{2\pi}{10}k\right) + \cos\left(3\frac{3\pi}{10}k\right)?$$

We know $\frac{\omega_s}{K} \leq \Delta\omega$ where $x_1(t) = \cos(\omega_1 t)$ and $x_2(t) = \cos((\omega_1 + \Delta\omega)t)$

$$\begin{aligned} \Delta\Omega = \frac{2\pi}{K},\quad \Delta\Omega = 3\frac{2\pi}{10} - 2\frac{2\pi}{10} = \frac{2\pi}{10} \\ \frac{2\pi}{K} = \frac{2\pi}{10} \Rightarrow K = 10 \end{aligned}$$

Hence the we need 10 samples.

14.7.1 Discrete Time Signals

Elementary Discrete Time Signals

- **Sine and cosine signals**:
  - $x[k] = A\sin(\Omega_0 k + \varphi)$
  - $x[k] = A\cos(\Omega_0 k + \varphi)$

- **Unit impulse**:
  - $\delta[k] = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$
- **Rectangular pulse**:
  - $rect\left[\frac{k}{2N+1}\right] = \begin{cases} 1 & \text{for } |k| \neq N \\ 0 & \text{for } |k| > N \end{cases}$
- **Unit step**:
  - $u[k] = \begin{cases} 0 & \text{for } k < 0 \\ 1 & \text{for } k \geq 0 \end{cases}$
- **Sinc**:
  - $sinc\left[\frac{k}{K_0}\right] = \frac{\sin\left(\pi\frac{k}{K_0}\right)}{\pi\frac{k}{K_0}}$

Discrete Time Signal Properties

- **Periodic signals**: $x[k] = x[k + K_0]$
- **Causal signals**: $x[k] = 0$ for $k < 0$
- **Energy and power signals**: (can never be both)
  - $E = \sum_{k=-\infty}^{\infty} |x[k]|^2$ where $0 < E < \infty$
  - $P = \lim_{K\to\infty} \frac{1}{K+1} \sum_{k=-K/2}^{K/2} |x[k]|^2$ for aperiodic signals
  - $P = \frac{1}{K_0} \sum_{k=0}^{K_0-1} |x[k]|^2$ for periodic signals

$\Omega_0 = \frac{2\pi}{K_0}$

14.7.2 Time Domain Analysis of Discrete Time Systems

Impulse Response

$$\delta[k] \mapsto h[k]$$
$$\alpha\delta[k - k_0] \mapsto \alpha h[k - k_0]$$

Convolution Sum

input output relationship for descrite time (definition)

$$y[k] = \sum_{m=-\infty}^{\infty} h[k]x[k - m] = h[k] * x[k]$$
$$y[k] = \sum_{m=0}^{\infty} h[k]x[k - m] \text{ for casual systems}$$

Property	
Commutation	$x_1[k] * x_2[k] = x_2[k] * x_1[k]$
Distribution	$x_1[k] * (x_2[k] + x_3[k]) = x_1[k] * x_2[k] + x_1[k] * x_3[k]$
Association	$x_1[k] * (x_2[k] * x_3[k]) = x_1[k] * x_2[k] * x_3[k]$
Time shifting	$x_1[k - k_1] * x_2[k - k_2] = y[k - k_1 - k_2]$
Duration	$x_1[k] = 0$ for $k \geq K_1$, $x_2[k] = 0$ for $k \geq K_2 \Rightarrow x_1[k] * x_2[k] = 0$ for $k \geq K_1 + K_2$
Convolution with impulse function	$x[k] * \delta[k - K] = x[k - K]$
Convolution with step function	$x[k] * u[k] = \sum_{m=-\infty}^{k} x[m]$

Figure 14.12: Properties of the discrete time convolution sum. From ()

14.8 Discrete Time Fourier Analysis

14.8.1 Discrete Time Fourier Series and Transform

Definition and Properties

Definition of Discrete time Fourier transform (DTFT)

$$x[k] = \sum_{n=0}^{K_0-1}$$

$$c_n = \frac{1}{K_0} \sum_{k=0}^{K_0-1} x[k] e^{-jn\Omega_0 k}$$

Fourier transform for sampled continous tims signals:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) e^{-j\omega t} dt = \sum_{k=-\infty}^{\infty} x(kT_s) e^{-j\omega k T_s}$$

The frequency can be expression:

$$\Omega = \frac{2\pi}{K} \quad \text{witch is similar to } \omega = \frac{2\pi}{T}$$

Normalized frequency.

$$\Omega = \omega T_s = 2\pi \frac{f}{f_s}$$

Mapping frequencies:

$$\Omega = 2\pi \Leftrightarrow \omega = \omega_s$$

$$\Omega = \pi \Leftrightarrow \omega = \frac{\omega_s}{2} = \omega_N$$

Sampling feqeuency.

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\Omega k}$$

Definition of Discrete time Fourier transform

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\Omega k}$$
$$x[k] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{j\Omega k}d\Omega$$

Transfor pair and the important property that the fourier transform is alwase 2π for discrete time.

$$x[k] \circ\!\!-\!\!\bullet\ X(\Omega)$$
$$X(\Omega) = X(\Omega + 2\pi)$$

Property	Time Domain	Frequency Domain
Periodicity	-	$X(\Omega) = X(\Omega + 2\pi)$
Symmetry	Real-valued	$X(-\Omega) = X^*(\Omega)$
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(\Omega) + \beta X_2(\Omega)$
Time shifting (k_0: integer)	$x[k-k_0]$	$e^{-\mathrm{j}\Omega k_0}X(\Omega)$
Time differencing	$x[k] - x[k-1]$	$\left(1 - e^{-\mathrm{j}\Omega}\right)X(\Omega)$
Frequency domain differentiation	$-\mathrm{j}\,kx[k]$	$\frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}$
Time summation	$\sum_{n=-\infty}^{k} x[n]$	$\frac{X(\Omega)}{1-e^{-\mathrm{j}\Omega}} + \pi X(0)\sum_{m=-\infty}^{\infty}\delta(\Omega - 2\pi m)$
Convolution	$x_1[k] * x_2[k]$	$X_1(\Omega)X_2(\Omega)$
Multiplication†	$x_1[k]x_2[k]$	$\frac{1}{2\pi}X_1(\Omega) * X_2(\Omega)$
Parseval's theorem	$E = \sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2$	$E = \frac{1}{2\pi}\int_0^{2\pi} \lvert X(\Omega)\rvert^2\,\mathrm{d}\Omega$

Figure 14.13: Properties of the discrete time fourier transform. From ()

Hermitian symmetry: $X(-\omega) = X^*(\omega)$ for realed valued $x[k]$

Frequency Domain Analysis of Discrete Time Systems

$$y[k] = h[k] * x[k]$$
$$x_1[k] * x_2[k] \circ\!\!-\!\!\bullet\ X_1(\Omega)X_2(\Omega)$$
$$Y(\Omega) = H(\Omega)X(\Omega)$$
$$H(\Omega) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}$$

14.8.2 Discrete Fourier Transform

With DFT we can determine the spectrum without nowing the signal type of input signals $x[k]$.

Definition

$$X(l) = \sum_{k=0}^{K-1} x[k] e^{-jlk\frac{2\pi}{K}}$$
$$x[k] = \frac{1}{K} \sum_{l=0}^{K-1} X[l] e^{jlk\frac{2\pi}{K}}$$

Where L is the *Frequency bins*

Property	Time Domain	Frequency Domain
Periodicity	-	$X[l] = X[l+K]$
Symmetry	Real-valued	$X[l] = X^*[K-l]$
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1[l] + \beta X_2[l]$
Parseval's Theorem	$E = \sum_{k=0}^{K-1} \lvert x[k]\rvert^2$	$E = \frac{1}{K} \sum_{l=0}^{K-1} \lvert X[l]\rvert^2$

Figure 14.14: Properties of the discrete time Fourier transform. From ()

Graphical Interpretation of the DFT

Calculating the DFT of a sampled continous time signal consists fo the following three steps:

1. **Time domain sampling**. The convolution of $X(\omega)$ and $S(\omega)$ in the frequency domain, resulting in a spectrum for the sampled signal $x_s(t)$ with ω_s-periodic specrtum with frequency-shift copies of $X(\omega)$.
2. **Time limited**. Windowed signal $x_{sw}(t) = x_s(t)w(t)$, thus the Fourier transform is the convolution between the sampled signal's Fourier transform $X_s(\omega)$ and $W(\omega)$ (a sinc function). witch yields the spectrum $X_{sw}(\omega)$
3. **Frequency domain sampling**. Sampling the frequency domain with a sampling interval of $\frac{\omega_s}{L}$ This is a multiplication of the sampled signal in time domain and the window signal $x_{sw}(t)$ with an impulse train in the frequency domain.

Relationship to the Continuous and Discrete Time Fourier Transforms

Discrete Fourier Transform (DFT) and dicreate time Fourier Transform (DTFT) are not the same, however there are similar and there realation is

$$X(\Omega)|_{\Omega = l\frac{2\pi}{L}} = X[l]$$

Also DFT is non-zero.

For aperiodic CTFT can be approximated as

$$X(\omega)|_{\omega = l\frac{\omega_s}{L}} \approx T_s X[l]$$

if the continous time signal is limited to interval $0, \dots, (K-1)T_s$

In other words $\Omega_l = l\frac{2\pi}{L}$, $\omega_l = l\frac{\omega_s}{L}$, $f_l = l\frac{f_s}{L}$.

Zero-Padding, Graphical Resolution, and Spectral Resolution

- *Zero-Padding* is when calculating the DFT for $L > K$.
- *Graphical resolution* is the appearance of the spectrum plot.
- *Spectral resolution* is the resolution which determines how accurate the calculated Fourier transform is.

Windowing

- **Rectangular window**: Standard window. Most narrow mainlobe but the strongest sidelobes.
- **Bartlett and triangle windows**: smaller sidelobes then rectangular window but wither mainlobe.
- **Hamming window**: Smother transistions leades to smaller sidelobes and hamming window is the smothers and there for the smallest sidelobes.

Windowing causes *smering* (wide sidelobes) and *leakage* (new specral content with sidelobes).

Fast Fourier Transform (FFT)

One of the most important algorithems ever. It is used for deterimn the fourier transform of a discrete signal. FFT is an efficent way of implementing DFT when $K = 2^M$, from $O(K^2)$ to $O(K\log_2(K))$. Since the standard way of calculating the fourier transform giledes a matrix multiplication with $K \times K$, therefore the time complexity is $O(K^2)$. (the matrix is DFT). There exist symitry in the DFT and other redundent properties, therefore we can optimase it. FFT is a optimasiation with split odds and evense into sepreate colums and then multiple with a diaginal matrix times another matrix.

The *twiddle factor*

$$W_K = e^{-j\frac{2\pi}{K}}$$

$$\begin{aligned} X[l] &= \sum_{k=0}^{\frac{K}{2}-1} x_1[2k]W_K^{lk} + W_K^l \sum_{k=0}^{\frac{K}{2}-1} x[2k+1]W_K^{lk} \\ &= X_1[l] + W_k^l X_2[l] \end{aligned}$$

14.9 z-transform

14.9.1 Definition and Properties

Definition

The discrete time Fourier transform for causal signal can be expressed as

$$X(\Omega) = \sum_{k=0}^{\infty} x[k]r^{-k}e^{-j\Omega k}$$

where r^{-k} is a decaying exponential function with is used to scale $x[k]$. The z-transform can then be derived for $z = re^{j\Omega}$.

$$x[k] \circ\!\!-\!\!\bullet\ X(z)$$
$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k}$$
$$x[k] = \frac{1}{2\pi j} \oint X(z) z^{k-1} dz$$

Time domain $x[k]$	z-Transform $X(z)$	ROC
Unit impulse		
$x[k] = \delta[k]$	$X(z) = 1$	all z
Unit step		
$x[k] = u[k]$	$X(z) = \frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
Exponential		
$x[k] = a^k u[k]$	$X(z) = \frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Ramp		
$x[k] = ku[k]$	$x(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$	$\lvert z\rvert > 1$
Cosine		
$x[k] = \cos(\Omega_0 k) u[k]$	$X(z) = \frac{1 - z^{-1}\cos(\Omega_0)}{1 - 2z^{-1}\cos(\Omega_0) + z^{-2}}$	$\lvert z\rvert > 1$
Sine		
$x[k] = \sin(\Omega_0 k) u[k]$	$X(z) = \frac{z^{-1}\sin(\Omega_0)}{1 - 2z^{-1}\cos(\Omega_0) + z^{-2}}$	$\lvert z\rvert > 1$
Decaying cosine		
$x[k] = a^k \cos(\Omega_0 k) u[k]$	$X(z) = \frac{1 - az^{-1}\cos(\Omega_0)}{1 - 2az^{-1}\cos(\Omega_0) + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
Decaying sine		
$x[k] = a^k \sin(\Omega_0 k) u[k]$	$X(z) = \frac{az^{-1}\sin(\Omega_0)}{1 - 2az^{-1}\cos(\Omega_0) + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Figure 14.15: z-transform pairs. From ()

Properties

14.9.2 Transfer Function

Difference Equations and Transfer Functions

Input output relationship, is expressed as *difference equations* (recursive) Somtimes the difference equation are given as looking at N steps into the future, in that case the easies approtch is to shifting of diference equations.

Property	Time Domain	z-Domain
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(z) + \beta X_2(z)$
Time shifting	$x[k-m]$	$z^{-m}X(z)$
Convolution	$x_1[k] * x_2[k]$	$X_1(z)X_2(z)$
Scaling	$a^k x[k]$	$X\left(\frac{z}{a}\right)$
Time difference	$x[k] - x[k-1]$	$(1 - z^{-1})X(z)$
Accumulation	$\sum_{m=0}^{k} x[m]$	$\frac{1}{1-z^{-1}}X(z)$
Initial value theorem	-	$x[0] = \lim_{z\to\infty} X(z)$
Final value theorem	-	$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)X(z)$

Figure 14.16: properties z-transform. From ()

Definition: Transfer function of discrete time LTI systems

$$y[k] + a_1 y[k-1] + \ldots + a_N y[k-N] = b_0 x[k] + b_1 x[k-1] + \ldots + b_M x[k-M]$$
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$
$$Y(z) = H(z)X(z)$$

Transfer Function and Impulse Response

Definition: Relationship between the discrete time impulse response and transfrer function

$$h[k] \circ\!\!-\!\!\bullet\, H(z)$$
$$H(z) = \sum_{k=0}^{\infty} h[k] z^{-k}$$

Transfer Function and Frequency Response

Definition: Relationship between the discrete time transfer function and the frequncy reponse

$$H(\Omega) = H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}}$$

if the impulse responce is casual, $z = e^{j\omega}$ and the DTFT converges.

Poles and Zeros, and Stability

Similarly to the laplace transform we can derive the poles and zeros from the tranfer function. **Definition: Pole-zero form of the fiscrete time transfer function**

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}}$$
$$H(z) = b_0 z^{N-m} \frac{(z - z_1)(z - z_2)\ldots(z - z_M)}{(z - p_1)(z - p_2)\ldots(z - p_N)}$$

Spectrum behavior for diffrent pole and zero positions If the pole is inside the circle it will create a top wich dose not go to infiniy. It will however have a higher top the close it is to the circle.

Zeros will have the oposite affect, creating a dip. The closer the the circle the futher down the dip will reach.

Stabiltity

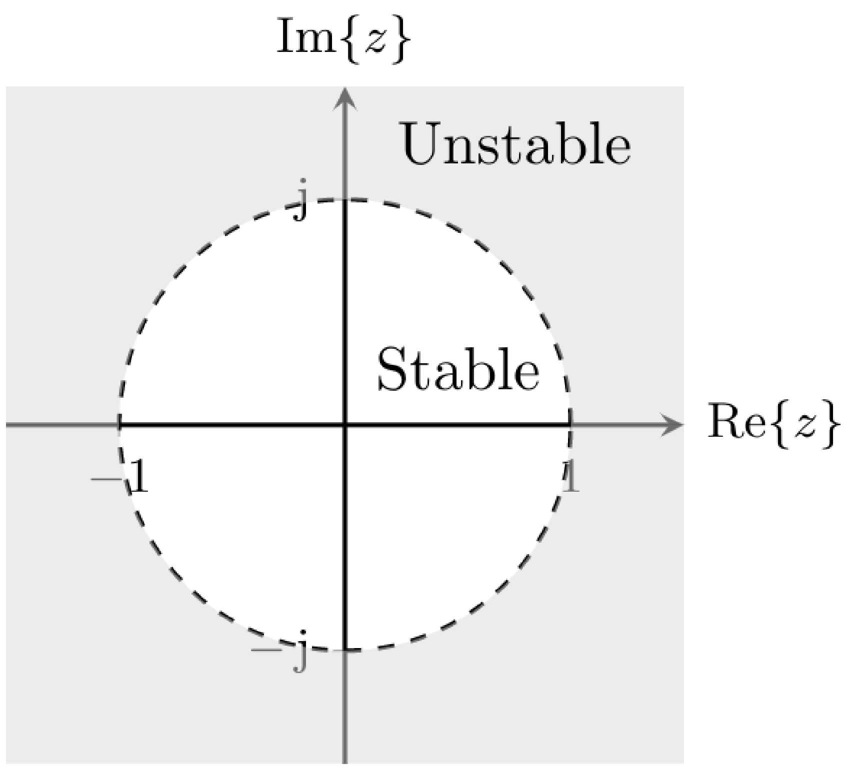


Figure 14.17: Stability for z-transforms. From ()

Example: stability casual and frequency responce

What is the systems frequency response and approximate it graphically? Also determin if the system is stable and casual.

$$y[k+2] + 5y[k+1] + 0.1y[k] = x[k+1]$$

We can see that the the output only depends on previus values there for it can be casual.
First we determin the systems transfer function

$$\begin{aligned}
&z^2Y(z) + 5zY(z) + 0.1 = zX(z) \quad \text{-(Time shifting property)}\\
&Y(z)(z^2 + 5z + 0.1) = zX(z)\\
&\frac{Y(z)}{X(z)} = \frac{z}{z^2 + 5z + 0.1} = H(z) \quad \text{-(Witch is the tranfer function)}
\end{aligned}$$

Next we determin the the poles to se if there are inside the complex unit circle, inorder to determin the stablitity.

$$\begin{aligned}
&z^2 + 5z + 0.1 = 0\\
&z = \frac{-5}{2} \pm \sqrt{\frac{5^2}{2^2} - 0.1} = \frac{-5}{2} \pm \frac{\sqrt{24.6}}{2} \quad \text{-(pq-formula)}\\
&p_1 \approx -0.0201,\ p_2 \approx -5
\end{aligned}$$

Since $|p_2| > 1$ is the system not stable.
Determine the frequency responce

$$H(\Omega) = H(z)|_{z=e^{j\Omega}} = \frac{e^{j\Omega}}{e^{2j\Omega} + 5e^{j\Omega} + 0.1}$$

Calculating some points to be able to approximate the graph:

$$\begin{aligned}
&|H(\Omega)|_{\Omega=2\pi}| = |H(\Omega)|_{\Omega=0}| = |\frac{e^0}{e^0 + 5e^0 + 0.1}| \approx 0.16 \quad \text{-(}2\pi\text{-periodic)}\\
&|H(\Omega)|_{\Omega=3\pi/2}| = |H(\Omega)|_{\Omega=\pi/2}| = |\frac{e^{\pi/2}}{e^{\pi} + 5e^{\pi/2} + 0.1}| = |\frac{-j}{-1 + 5(-j) + 0.1}| \approx 2\\
&|H(\Omega)|_{\Omega=\pi}| = |\frac{e^{\pi}}{e^{2\pi} + 5e^{\pi} + 0.1}| = |\frac{-1}{1 + 5(-1) + 0.1}| \approx 0.26
\end{aligned}$$

14.10 Digital Filters

14.10.1 Introduction

Background

Continous Time Requirements and Digital Filters

Digital Filters and LTI Systems

14.10.2 Finite Impulse Response Filters

Definition and Properties

Filter taps are the coefficients b_n. Thus, an Nth order filter has $N+1$ filter taps (starts from 0 and ends at N)

FIR Filters are allwase stable since it dose not have any poles only zeros.

FIR Filter Design

Ideal Frequency Response ($\pi \leq \Omega < \pi$)	Ideal Impulse Response
Lowpass	
$H(\Omega) = \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$	$h[k] = \frac{\Omega_c}{\pi}\,\text{sinc}\left[\frac{\Omega_c k}{\pi}\right]$
Highpass	
$H(\Omega) = 1 - \text{rect}\left(\frac{\Omega}{2\Omega_c}\right)$	$h[k] = \delta[k] - \frac{\Omega_c}{\pi}\,\text{sinc}\left[\frac{\Omega_c k}{\pi}\right]$
Bandpass	
$H(\Omega) = \text{rect}\left(\frac{\Omega}{2\Omega_{c2}}\right) - \text{rect}\left(\frac{\Omega}{2\Omega_{c1}}\right)$	$h[k] = \frac{\Omega_{c2}}{\pi}\,\text{sinc}\left[\frac{\Omega_{c2} k}{\pi}\right] - \frac{\Omega_{c1}}{\pi}\,\text{sinc}\left[\frac{\Omega_{c1} k}{\pi}\right]$
Bandstop	
$H(\Omega) = 1 - \text{rect}\left(\frac{\Omega}{2\Omega_{c2}}\right) + \text{rect}\left(\frac{\Omega}{2\Omega_{c1}}\right)$	$h[k] = \delta[k] - \frac{\Omega_{c2}}{\pi}\,\text{sinc}\left[\frac{\Omega_{c2} k}{\pi}\right] + \frac{\Omega_{c1}}{\pi}\,\text{sinc}\left[\frac{\Omega_{c1} k}{\pi}\right]$

Figure 14.18: Descrete time filter frequency responce and their impulse responses

Windowing
Parks-McClellan

14.10.3 Infinite Impulse Response Filters

Definition and Properties

IIR Filter Design Using the Bilinear Transform

IIR filter design methods can be divided into two types:

1. Indirect methods that are based on converting continuous time filters to discrete time
2. Direct methods that directly design digital IIR filters in the discrete time domain.

Requirements

1. Filter requirement specification (passband and stopband edge frequencies, passband ripple, stopband attenuation);
2. Design of the analog filter transfer function H(s) (see Chapter 5);
3. Approximation of the analog filter transfer function using the bilinear transform.

Filter Coefficient Quantization and Stability

There could stabile in theroy but unstable in practice. Since the computer may have to round the number in a way witch makes the system unstable.

14.11 How it is connected

Consept	Continous signals $x(t)$	Discreate signals $x[k]$	Description
Impulse reponse	$\delta(t) \mapsto h(t)$	$\delta[k] \mapsto h[k]$	Is used to determin the output of the system, since the output is $y[k] = h[k] * x[k]$
Frequency reponse	$H(\omega)$	$H(\Omega)$ with period 2π	Is the equvilant to the impulse responce in the frequency domain.
Signal's spectrum	$X(\omega)$	$X(\Omega)$	Is used to view the spectrum where one can se what frequencies dominates
Transfer function	$H(s)$ (Laplace)	$H(z)$ (z-transform)	Laplace and z-transform are use to analyse a system, with polse and zeros.
Discrete Fourier transform	-	$X[l]$	FFT is an algorith to compute the DFT, it turns the samples (wich become a vector) and convert it to a there frequency componetnts. We can not allwase use the DTFT since it requires knoladge of the full signal, not just finite set of samples.
Spectrum from DFT calculation	-	$X(f_l)$ with period f_s $(???X(f))$	We can not alwase determin the signals spectrum analyticly therfore we can approximate it by using the DFT

Note: The fourier tranforms are $X(\omega)$ for Continous tme, $X(\Omega)$ for discreate time and $X[l]$ for discreate (NO TIME).

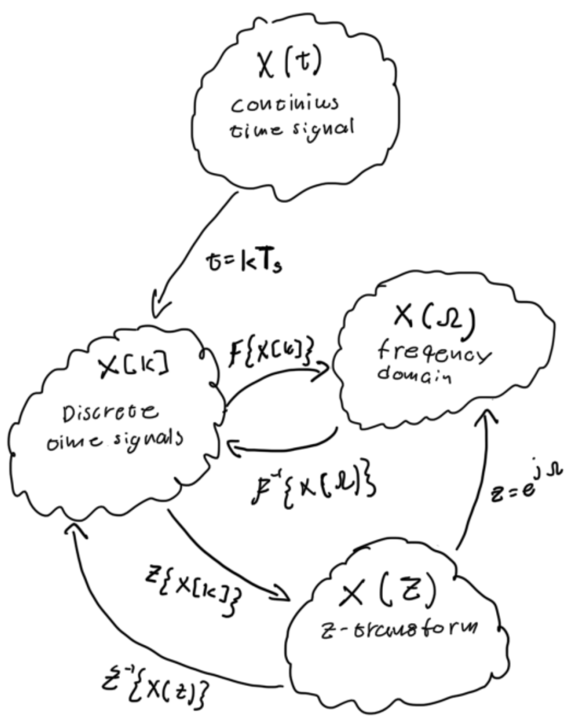


Figure 14.19: Connection mind map.

Example: sine in sine out for discrete time signal

The sampling time is $T_s = 0.05$

$$x(t) = 0.25\cos(2\pi/410t) \Rightarrow x[k] = 0.25\cos(2\pi/410kT_s) = 0.25\cos(\pi/4k)$$

$$H(\Omega) = H(z)|_{z=e^{j\Omega}} = \frac{1 - 0.8e^{-j\Omega} + 0.16e^{-2j\Omega}}{1 - e^{-j\Omega} + 0.5e^{-2j\Omega}} \quad \text{Where } \Omega_0 = \frac{\pi}{4}$$

$$|H(e^{j\pi/4})| = 1.7$$
$$\angle H(e^{j\pi/4}) = 0.14 \quad (7.8^\circ)$$

$$y[k] = 0.4\cos(\frac{\pi}{4}k + 0.14)$$

Appendices

Formula Tables
Signals and Transforms

October 19, 2021

1 Continuous Time Fourier Series

Trigonometric Continuous Time Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

with

$$a_0 = \frac{1}{T_0}\int_{(T_0)} x(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T_0}\int_{(T_0)} x(t)\cos(n\omega_0 t)\,\mathrm{d}t$$
$$b_n = \frac{2}{T_0}\int_{(T_0)} x(t)\sin(n\omega_0 t)\,\mathrm{d}t$$

Exponential Continuous Time Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{j}\,n\omega_0 t} \quad \text{with} \quad c_n = \frac{1}{T_0}\int_{(T_0)} x(t)e^{-\mathrm{j}\,n\omega_0 t}\,\mathrm{d}t$$

2 Continuous Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{j}\,\omega t}\,\mathrm{d}t \quad \text{and} \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{\mathrm{j}\,\omega t}\,\mathrm{d}\omega$$

Table 1. Properties of the continuous time Fourier transform.

Property	Time Domain	Frequency Domain
Symmetry	Real-valued	$X(-\omega) = X^*(\omega)$
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{\omega}{a}\right)$
Time shift	$x(t - \Delta t)$	$e^{-\mathrm{j}\,\omega\Delta t} X(\omega)$
Frequency shift	$x(t)e^{\mathrm{j}\,\omega_0 t}$	$X(\omega - \omega_0)$
Differentiation	$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$	$(\mathrm{j}\,\omega)^n X(\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{\mathrm{j}\,\omega} X(\omega) + \pi X(0)\delta(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega)X_2(\omega)$
Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,\mathrm{d}t$	$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,\mathrm{d}\omega$

Table 2. Continuous time Fourier transforms of elementary signals.

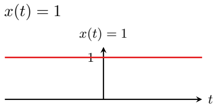
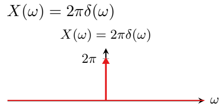
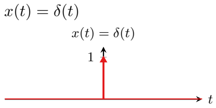
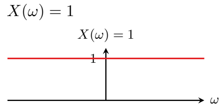
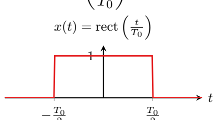
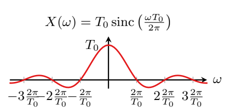
Time Domain $x(t)$	Fourier Transform $X(\omega)$
Periodic signals	
T_0-periodic ($\omega_0 = \frac{2\pi}{T_0}$)	$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$ with $c_n = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-\mathrm{j}\, n\omega_0 t}\,\mathrm{d}t$
Constant	
$x(t) = 1$	$X(\omega) = 2\pi\delta(\omega)$
	
Dirac delta function	
$x(t) = \delta(t)$	$X(\omega) = 1$
	
Rectangular pulse	
$x(t) = \mathrm{rect}\left(\frac{t}{T_0}\right)$	$X(\omega) = T_0 \,\mathrm{sinc}\left(\frac{\omega T_0}{2\pi}\right)$
	

Table 2. Continuous time Fourier transforms of elementary signals (cont'd.).

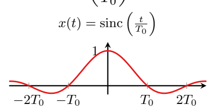
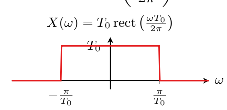
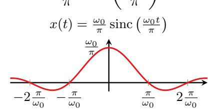
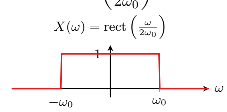
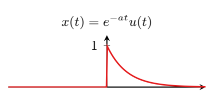
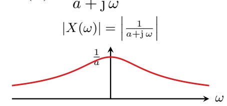
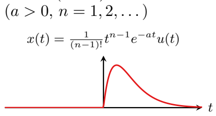
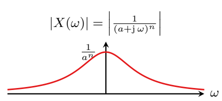
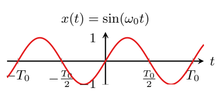
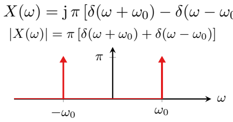
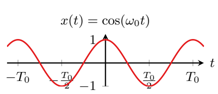
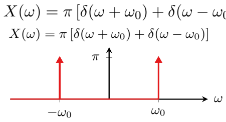
Time Domain $x(t)$	Fourier Transform $X(\omega)$
Sinc function	
$x(t) = \mathrm{sinc}\left(\frac{t}{T_0}\right)$	$X(\omega) = T_0 \,\mathrm{rect}\left(\frac{\omega T_0}{2\pi}\right)$
	
Sinc function (alternative parametrization)	
$x(t) = \frac{\omega_0}{\pi}\,\mathrm{sinc}\left(\frac{\omega_0 t}{\pi}\right)$	$X(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_0}\right)$
	
Exponential	
$x(t) = e^{-at}u(t)$ $(a > 0)$	$X(\omega) = \frac{1}{a + \mathrm{j}\,\omega}$
	

Table 2. Continuous time Fourier transforms of elementary signals (cont'd.).

Time Domain $x(t)$	Fourier Transform $X(\omega)$
Multiple real-valued poles	
$x(t) = \dfrac{1}{(n-1)!} t^{n-1} e^{-at} u(t)$ $(a > 0,\ n = 1, 2, \dots)$	$X(\omega) = \dfrac{1}{(a + \mathrm{j}\,\omega)^n}$
	
Sine $x(t) = \sin(\omega_0 t)$	$X(\omega) = \mathrm{j}\,\pi\,[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
	
Cosine $x(t) = \cos(\omega_0 t)$	$X(\omega) = \pi\,[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
	

3 Laplace Transform

$$X(s) = \int_0^\infty x(t) e^{-st}\,\mathrm{d}t \quad \text{and} \quad x(t) = \frac{1}{2\pi\,\mathrm{j}} \int_{\sigma - \mathrm{j}\,\omega}^{\sigma + \mathrm{j}\,\omega} X(s) e^{st}\,\mathrm{d}s$$

Table 3. Properties of the Laplace transform.

Property	Time Domain	Laplace Domain
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$
Scaling $(a > 0)$	$x(at)$	$\frac{1}{a} X\left(\frac{s}{a}\right)$
Time shift	$x(t - \Delta t)$	$e^{-s\Delta t} X(s)$
s-domain shift	$x(t)e^{at}$	$X(s - a)$
Differentiation (first order)	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s) - x(0)$
Differentiation (nth order)	$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$	$s^n X(s) - s^{n-1} x(0) - s^{n-2} x^{(1)}(0) - \dots - x^{(n-1)}(0)$
Integration	$\int_0^t x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s} X(s)$
Convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$
Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi\,\mathrm{j}} X_1(s) * X_2(s)$
Initial value theorem	$\lim_{t\to 0} x(t)$	$\lim_{s\to\infty} sX(s)$
Final value theorem	$\lim_{t\to\infty} x(t)$	$\lim_{s\to 0} sX(s)$

Table 4. Table of unilateral Laplace transform pairs for causal signals $x(t)$ ($x(t) = 0$ for $t < 0$).

Time domain $x(t)$	Laplace transform $X(s)$	ROC
Dirac delta function		
$x(t) = \delta(t)$	$X(s) = 1$	All s
Unit step		
$x(t) = u(t)$	$X(s) = \dfrac{1}{s}$	$\mathrm{Re}\{s\} > 0$
Exponential		
$x(t) = e^{-at}u(t)$	$X(s) = \dfrac{1}{a+s}$	$\mathrm{Re}\{s\} > -a$
Ramp		
$x(t) = tu(t)$	$X(s) = \dfrac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
Higher order ramp		
$x(t) = t^n u(t)$	$X(s) = \dfrac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
Cosine		
$x(t) = \cos(\omega_0 t)u(t)$	$X(s) = \dfrac{s}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Sine		
$x(t) = \sin(\omega_0 t)u(t)$	$X(s) = \dfrac{\omega_0}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Decaying cosine		
$x(t) = e^{-at}\cos(\omega_0 t)u(t)$	$X(s) = \dfrac{a+s}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$
Decaying sine		
$x(t) = e^{-at}\sin(\omega_0 t)u(t)$	$X(s) = \dfrac{\omega_0}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$

4 Discrete Time Fourier Series

$$x[k] = \sum_{n=0}^{K_0-1} c_n e^{\mathrm{j}\, n\Omega_0 k} \quad \text{with} \quad c_n = \frac{1}{K_0} \sum_{k=0}^{K_0-1} x[k] e^{-\mathrm{j}\, n\Omega_0 k}$$

5 Discrete Time Fourier Transform

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k] e^{-\mathrm{j}\,\Omega k} \quad \text{and} \quad x[k] = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{\mathrm{j}\,\Omega k}\, \mathrm{d}\Omega$$

Table 5. Properties of the discrete time Fourier transform.

Property	Time Domain	Frequency Domain
Periodicity	-	$X(\Omega) = X(\Omega + 2\pi)$
Symmetry	Real-valued	$X(-\Omega) = X^*(\Omega)$
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(\Omega) + \beta X_2(\Omega)$
Time shifting (k_0: integer)	$x[k - k_0]$	$e^{-\mathrm{j}\,\Omega k_0} X(\Omega)$
Time differencing	$x[k] - x[k-1]$	$\left(1 - e^{-\mathrm{j}\,\Omega}\right) X(\Omega)$
Frequency domain differentiation	$-\mathrm{j}\, k x[k]$	$\dfrac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}$
Time summation	$\sum_{n=-\infty}^{k} x[n]$	$\dfrac{X(\Omega)}{1 - e^{-\mathrm{j}\,\Omega}} + \pi X(0) \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$
Convolution	$x_1[k] * x_2[k]$	$X_1(\Omega) X_2(\Omega)$
Multiplication†	$x_1[k] x_2[k]$	$\dfrac{1}{2\pi} X_1(\Omega) * X_2(\Omega)$

Table 5. Properties of the discrete time Fourier transform (cont'd.).

Property	Time Domain	Frequency Domain
Parseval's theorem	$E = \sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2$	$E = \frac{1}{2\pi}\int_0^{2\pi} \lvert X(\Omega)\rvert^2 \,\mathrm{d}\Omega$

[†]The convolution is on the interval $0 \ldots 2\pi$.

Table 6. Discrete time Fourier transforms of elementary signals.

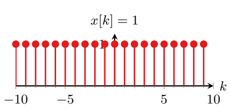
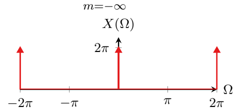
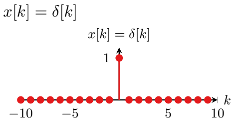
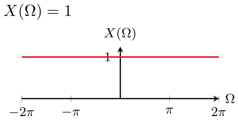
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Periodic signals	
K_0-periodic ($\Omega_0 = \frac{2\pi}{K_0}$)	$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\Omega - n\Omega_0)$ with $c_n = \frac{1}{K_0}\sum_{k=0}^{K_0-1} x[k] e^{-\mathrm{j}n\Omega_0 k}$
Constant	
$x[k] = 1$	$X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$
	
Unit impulse	
$x[k] = \delta[k]$	$X(\Omega) = 1$
	

Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

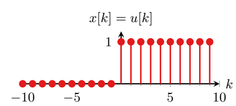
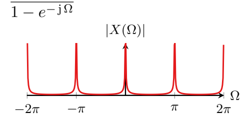
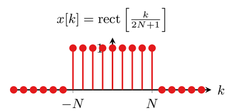
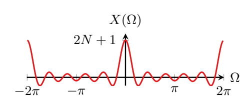
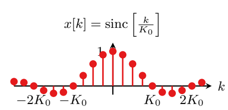
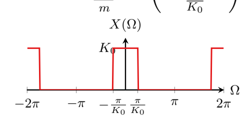
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Unit step	
$x[k] = u[k]$	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) + \frac{1}{1 - e^{-\mathrm{j}\Omega}}$
	
Rectangular pulse	
$x[k] = \mathrm{rect}\left[\frac{k}{2N+1}\right]$	$X(\Omega) = \frac{\sin\left(\frac{(2N+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$
	
Sinc function	
$x[k] = \mathrm{sinc}\left[\frac{k}{K_0}\right]$	$X(\Omega) = K_0 \sum_m \mathrm{rect}\left(\frac{\Omega - 2\pi m}{\frac{2\pi}{K_0}}\right)$
	

Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

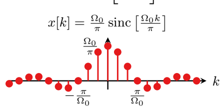
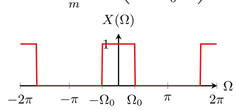
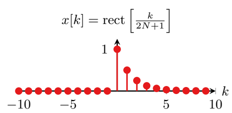
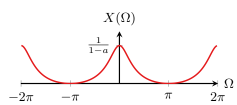
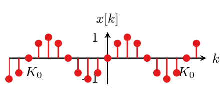
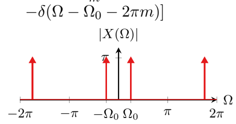
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Sinc function (alternative parametrization)	
$x[k] = \dfrac{\Omega_0}{\pi}\,\mathrm{sinc}\left[\dfrac{\Omega_0 k}{\pi}\right]$	$X(\Omega) = \sum_m \mathrm{rect}\left(\dfrac{\Omega - 2\pi m}{2\Omega_0}\right)$
	
Exponential	
$x[k] = a^k u[n]$ $(\lvert a\rvert < 1)$	$X(\Omega) = \dfrac{1}{1 - a e^{-\mathrm{j}\Omega}}$
	
Sine	
$x[k] = \sin(\Omega_0 k)$	$X(\Omega) = \mathrm{j}\,\pi \sum_m [\delta(\Omega + \Omega_0 - 2\pi m)$ $-\delta(\Omega - \Omega_0 - 2\pi m)]$
	
Cosine	
$x[k] = \cos(\Omega_0 k)$	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi m)$ $+\,\delta(\Omega - \Omega_0 - 2\pi m)]$

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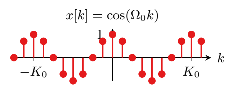
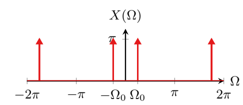
Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

Time Domain $x[k]$	Fourier Transform $X(\Omega)$
	

6 Discrete Fourier Transform

$$X[l] = \sum_{k=0}^{K-1} x[k] e^{-\mathrm{j}\,lk\frac{2\pi}{K}} \quad \text{and} \quad x[k] = \frac{1}{K}\sum_{l=0}^{K-1} X[l] e^{\mathrm{j}\,lk\frac{2\pi}{K}}$$

7 z-Transform

$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k} \quad \text{and} \quad x[k] = \frac{1}{2\pi\,\mathrm{j}} \oint X(z) z^{k-1}\,\mathrm{d}z$$

Table 7. Properties of the z-transform.

Property	Time Domain	z-Domain
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(z) + \beta X_2(z)$
Time shifting	$x[k-m]$	$z^{-m} X(z)$
Convolution	$x_1[k] * x_2[k]$	$X_1(z) X_2(z)$
Scaling	$a^k x[k]$	$X\left(\dfrac{z}{a}\right)$
Time difference	$x[k] - x[k-1]$	$(1 - z^{-1}) X(z)$
Accumulation	$\sum_{m=0}^{k} x[m]$	$\dfrac{1}{1 - z^{-1}} X(z)$
Initial value theorem	-	$x[0] = \lim_{z\to\infty} X(z)$

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Table 7. Properties of the z-transform (cont'd.).

Property	Time Domain	z-Domain
Final value theorem	-	$\lim\limits_{k\to\infty} x[k] = \lim\limits_{z\to 1}(z-1)X(z)$

Table 8. Table of unilateral z-transform pairs for causal signals $x[k]$ ($x[k] = 0$ for $k < 0$).

Time domain $x[k]$	z-Transform $X(z)$	ROC
Unit impulse		
$x[k] = \delta[k]$	$X(z) = 1$	all z
Unit step		
$x[k] = u[k]$	$X(z) = \dfrac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
Exponential		
$x[k] = a^k u[k]$	$X(z) = \dfrac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Ramp		
$x[k] = ku[k]$	$x(z) = \dfrac{z^{-1}}{(1 - z^{-1})^2}$	$\lvert z\rvert > 1$
Cosine		
$x[k] = \cos(\Omega_0 k)u[k]$	$X(z) = \dfrac{1 - z^{-1}\cos(\Omega_0)}{1 - 2z^{-1}\cos(\Omega_0) + z^{-2}}$	$\lvert z\rvert > 1$
Sine		
$x[k] = \sin(\Omega_0 k)u[k]$	$X(z) = \dfrac{z^{-1}\sin(\Omega_0)}{1 - 2z^{-1}\cos(\Omega_0) + z^{-2}}$	$\lvert z\rvert > 1$
Decaying cosine		
$x[k] = a^k\cos(\Omega_0 k)u[k]$	$X(z) = \dfrac{1 - az^{-1}\cos(\Omega_0)}{1 - 2az^{-1}\cos(\Omega_0) + a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
Decaying sine		
$x[k] = a^k\sin(\Omega_0 k)u[k]$	$X(z) = \dfrac{az^{-1}\sin(\Omega_0)}{1 - 2az^{-1}\cos(\Omega_0) + a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

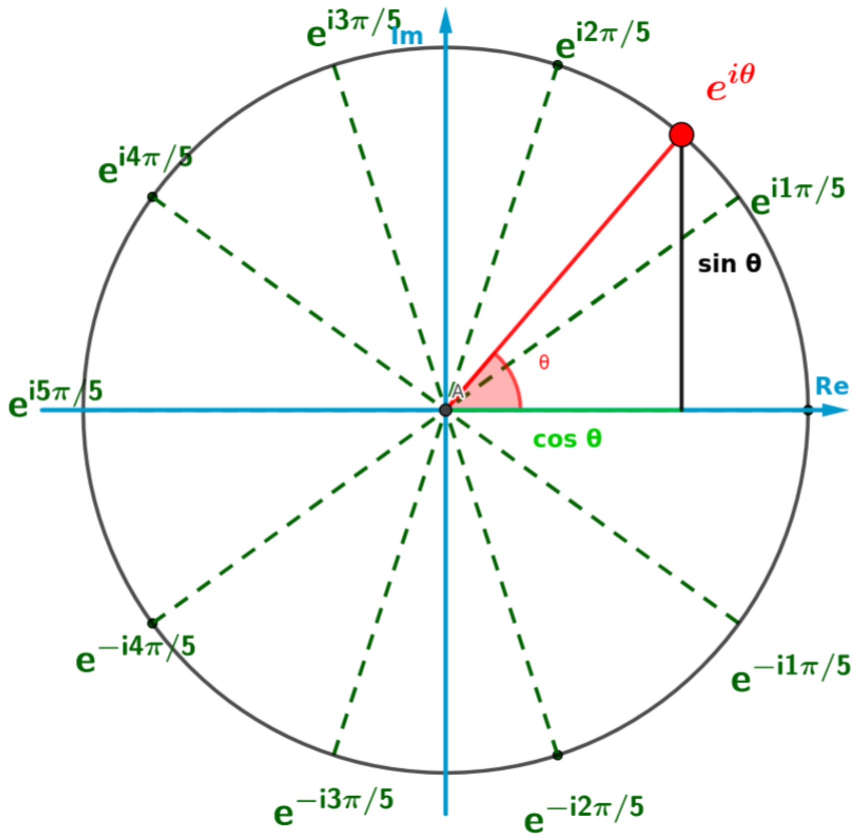


Figure 20: Complex unit circle. From ()

Chapter 15

Database Design I Samanfattning

Resorcess

- https://www.w3schools.com/sql/
- https://www.mysql.com/
- https://erdplus.com/

15.1 Ch1. Databases and Database Users

- *Database*: A collection of related data.
- *Data*: Known facts that can be recorded and have an implicit meaning.
- *Mini-world*: Some part of the real world about which data is stored in a database. For example, student grades and transcripts at a university.
- *Database Management System (DBMS)*: A software package/ system to facilitate the creation and maintenance of a computerized database.
- *Database System*: The DBMS software together with the data itself. Sometimes, the applications are also included.

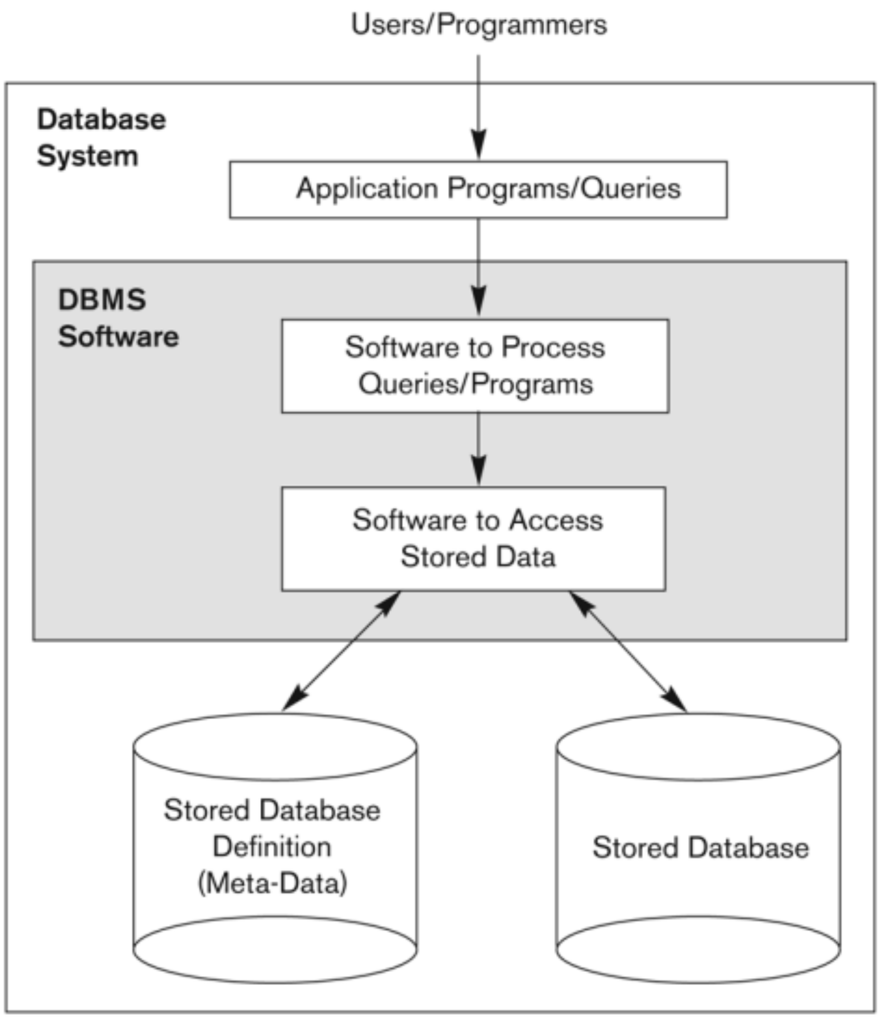


Figure 15.1: Database system. From ()

15.2 Ch3. Data Modeling Using the Entity-Relationship (ER) Model

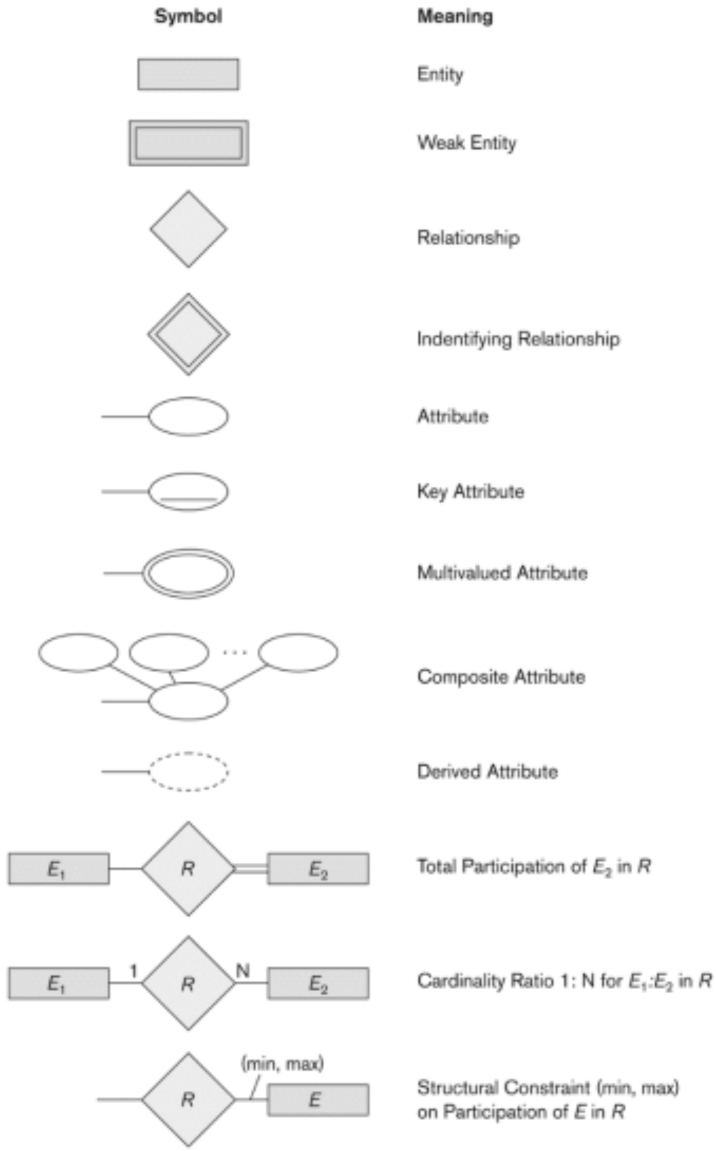


Figure 15.2: NOTATION ER Diagram. From ()

Definitions of ..

- *Entity*: Entities are specific things or objects in the mini-world that are represented in the database.
- *Entity set*: Each entity type will have a collection of entities stored in the database
- *Value sets*: Each simple attribute is associated with a value set
- *Weak Entity*: dependencies on other entity to exist
- *Relationship*: How entities are connected with one another
- *Relationship types*: Identifies the relationship name and the participating entity types and also identifies certain relationship constraints
- *Relationship Set*: The current set of relationship instances represented in the database
- *Recursive relationship*: Both participations are same entity type in different roles
- *Identifying Relationship*: Is the relationship for a entity to it's owner
- *Attribute*: Data with the entity stores
- *Simple Attribute*: Containing a single atomic value
- *Key Attribute*: Uniq attributes, can be used as keys
- *Multivalued Attributes*: Have multiple values like a list
- *Composite Attribute*: composed of several components
- *Derived Attributes*: Name(FirstName, MiddleName, LastName).

Relationship caractaristics

- Dubble line: Needs to have the relationship
- Single line: Can have the relationship
- 1: Only has one
- N: Many has relationship (Used for one to many 1:N and N:1)
- M: Many relationship (Only used for many to many N:M and M:N)
- (1:1-samband): one to one
- (1:N-samband): one to many
- (N:M-samband): many to many
- Doted key attribute: Is in combinations with the previus key

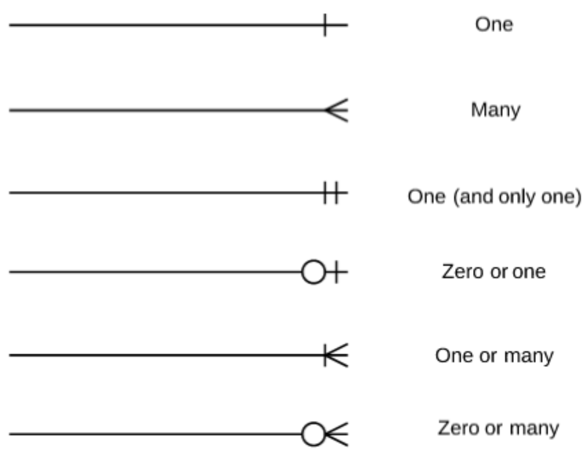


Figure 15.3: Crow foot notation. From ()

15.3 Ch4. Enhanced Entity-Relationship (EER) Modeling

EER includes all modeling consents of basic ER, but also have additional concepts witch are:

- *subclasses/superclasses*: additional meaningful subgroups of its entities, (ex: EMPLOYEE is futher grouped into SECRETERY, ENGINEER, MANAGER)
- *Specialization/generalization*: is the process of defining a set of subclasses of a superclass, (ex: SECRETERY, ENGINEER, MANAGER is spacialization of EMPLOYEE based upon job type)
  - *Disjointness Constraint*: an entity can be a member of at most one of the subclasses of the specialization
  - *Completeness Constraint*: Total specifies that every entity in the superclass must be a member of some subclass in the specialization/generalization

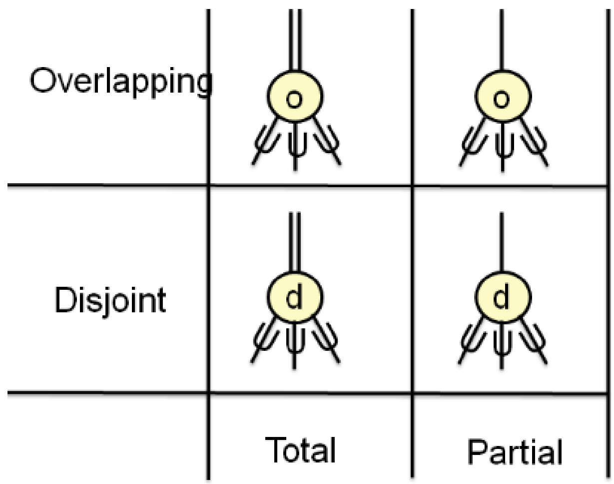


Figure 15.4: Overlapping, can be a member of multiple subclasses. Disjoint, can only be a member of at most one of the subclasses (Disjointness Constraint), Total, it must be a member of at least one (Completeness constraint). Partial, don't need to belong to any of the subclasses. From ()

15.4 Ch5. The Relational Data Model and Relational Database Constraints

Definitions

- *Schema*: Is the description of the relation
- *Tuple*: is an ordered set of values (row)
- *Domain*: each attribute has a domain or a set of valid values
- *Relations*: Is a set of data (table)

15.4.1 Constrains

Constraints determine which values are permissible and which are not in the database

- **Inherent or Implicit Constraints**: These are based on the data model itself. (E.g., relational model does not allow a list as a value for any attribute)
- **Schema-based or Explicit Constraints**: They are expressed in the schema by using the facilities provided by the model. (E.g., max. cardinality ratio constraint in the ER model)
- **Application based or semantic constraints**: These are beyond the expressive power of the model and must be specified and enforced by the application programs.

Relational Integrity Constraints

There are three main types of (explicit schema-based) constraints that can be expressed in the relational model:

- *Domain constraint*:
  if one of the attribute values provided for the new tuple is not of the specified attribute domain
- *Key constraints*:
  if the value of a key attribute in the new tuple already exists in another tuple in the relation
- *Referential integrity*:
  if a foreign key value in the new tuple references a primary key value that does not exist in the referenced relation
- *Entity integrity*:
  if the primary key value is null in the new tuple

15.5 Ch.13 Normalization

Youtube tutorial The process of decomposing unsatisfactory "bad" relations by breaking up their attributes into smaller relations

15.5.1 Anomalies

Having redudent information creates some

- *Insertion anomalies*: Can not Insertion unless there exist other data
- *Deletion anomalies*: By deleting an objects other objects will be deleted
- *Modification/Update anomalies*: An update result in updating many other objects attribute

EXAMPLE OF AN INSERT ANOMALY

- Consider the relation:
  - `EMP_PROJ(Emp#, Proj#, Ename, Pname, No_hours)`
- Insert Anomaly:
  - Cannot insert a project unless an employee is assigned to it.

EXAMPLE OF AN DELETE ANOMALY

- Consider the relation:
  - `EMP_PROJ(Emp#, Proj#, Ename, Pname, No_hours)`
- Delete Anomaly:
  - When a project is deleted, it will result in deleting all the employees who work on that project.
  - Alternately, if an employee is the sole employee on a project, deleting that employee would result in deleting the corresponding project.

EXAMPLE OF AN UPDATE ANOMALY

- Consider the relation:
  - `EMP_PROJ(Emp#, Proj#, Ename, Pname, No_hours)`
- Update Anomaly:
  - Changing the name of project number P1 from “Billing” to “Customer-Accounting” may cause this update to be made for all 100 employees working on project P1.

Note that it is not the relation *EMP_PROJ* who has the Anomaly, it is *PROJ*.

15.5.2 Normal forms

Database Normalization Basics Condition using keys and FDs of a relation to certify whether a relation schema is in a particular normal form

- 1NF: All attributes depend on the key. Explanation 1NF.
  - Requirements:
    1. Each table cell should contain a single value, no multi-value or composite attributes.
    2. Columns which contain sets of values or nested records are not allowed.
  - Steps:

    * Eliminate duplicative columns from the same table.
    * Create separate tables for each group of related data and identify each row with a unique column or set of columns (the primary key).

- 2NF: All attributes depend on the whole key. In other words there should not be any partial dependencies, meaning that a attribute depends on a key constructing a composite primary key. There for dose all attribute on the entire composite key. This is especially relevant for many to many reflations since a attribute can depend only on one foreign key but the primary key is the composition of the two foreign keys. Explanation 2NF.
  - Requirements:
    1. Meet the requirements of 1NF.
    2. Not have duplicates in any table, if present, normalize further that table. Create relations between these new tables that are created and use foreign keys between them in order to indicate a relation is present.
  - Steps
    * Meet all the requirements of the first normal form.
    * Remove subsets of data that apply to multiple rows of a table and place them in separate tables.
    * Create relationships between these new tables and their predecessors through the use of foreign keys.
- 3NF: All attributes depend on nothing but the key. In other words does there not exist any attribute with depend on a other attribute or key witch is not the primary key. Explanation 3NF.
  - Requirements:
    1. Meet the requirements of 2NF.
    2. Have no transitive functional dependencies.
  - Steps:
    * Meet all the requirements of the second normal form.
    * Remove columns that are not dependent upon the primary key.
- 4NF: (will not look at)
- BCNF: (will not look at)

primary key, full functional dependency Functional Dependencies FD
Transmit dependencies FD1 FD2 FD3 ..

15.6 Ch.9

ER-to-Relational Mapping Algorithm

- Step 1: Mapping of Regular Entity Types
- Step 2: Mapping of Weak Entity Types
- Step 3: Mapping of Binary 1:1 Relation Types
- Step 4: Mapping of Binary 1:N Relationship Types.
- Step 5: Mapping of Binary M:N Relationship Types.
- Step 6: Mapping of Multi-valued attributes.
- Step 7: Mapping of N-ary Relationship Types.

Mapping EER Model Constructs to Relations

- Step 8: Options for Mapping Specialization or Generalization.
- Step 9: Mapping of Union Types (Categories).

15.7 Ch.6 SQL

15.7.1 Terminology

- *Table*
- *Row*
- *Column*
- *SQL schema*: Identified by a schema name and includes an authorization identifier and descriptors for each element
- *Schema elements*: includes the document element (the top-level element) in a schema definitionTables, constraints, views, domains, and other constructs
- *Catalog*: Named collection of schemas in an SQL environment.
- *Base tables (base relations)*: Relation and its tuples are actually created and stored as a file by the DBMS
- *Virtual relations (views)*: Created through the CREATE VIEW statement. Do not correspond to any physical file.
- *Aliases*: a alternative name to the relation (table). Declared by **AS**
- *Tuple*: a row value

15.7.2 SQL types

- Numeric
  - Integer numbers: INTEGER, INT, SMALLINT
  - Floating-point: FLOAT or REAL, DOUBLE PRECISION
- Character-string
  - Fixed length: CHAR(n), CHARACTER(n)
  - Varing lenght: VARCHAR(n), CHAR VARYING(n), CHARACTER VARYING(n)
- Bit-string
  - Fixed lenght: BIT(n)
  - Varying length: BIT VARYING(n)
- Boolean
  - TRUE or FALSE or NULL
- DATE
  - YEAR, MONTH, DAY
- Additional data types
  - Timestamp: DATE, TIME
- INTERVAL

15.7.3 SQL examples

```
-- Creating tables

CREATE TABLE T1 (C1 VARCHAR(20), C2 VARCHAR(10), C3 INT);
INSERT INTO T1 VALUES ("a", "x", 1);
INSERT INTO T1 VALUES ("a", "y", 5);
INSERT INTO T1 VALUES ("b", "z", 2);
INSERT INTO T1 VALUES ("c", "z", 2);
INSERT INTO T1 VALUES ("d", "u", 3);

CREATE TABLE T2 (C1 VARCHAR(20), C2 VARCHAR(10), C3 INT);
INSERT INTO T2 VALUES ("a", "z", 4);
INSERT INTO T2 VALUES ("a", "x", 3);
INSERT INTO T2 VALUES ("b", "z", 2);
INSERT INTO T2 VALUES ("b", "x", 1);
INSERT INTO T2 VALUES ("c", "u", 2);

-- Select statements

SELECT COUNT(*)
FROM T1, T2;
-- T1 and T2 have 5 rows each there for is the result 5*5=25

SELECT Count(*)
FROM T1, T2
WHERE T1.C1=T2.C1;
-- 2+2+2+1=7

SELECT DISTINCT T1.C1
FROM T1, T2
WHERE T1.C2=T2.C2;
-- [a,b,c,d] (if the first is distinct) c,b,a,d (is the output)

SELECT T1.C2, AVG(T1.C3*T2.C3)
FROM T1, T2
WHERE T1.C2 = T2.C2
GROUP BY T1.C2;
-- [[u, 6],[x, 2],[z, 6]]

SELECT T1.C2, SUM(T1.C3*T2.C3)
FROM T1, T2
WHERE T1.C2 = T2.C2
GROUP BY T1.C2
HAVING SUM(T1.C3*T2.C3)>10
-- [z, 24] since u=6<10 and x=4<10 and z=2*4+2*2+2*4+2*2=24

SELECT count(*)
FROM T1 INNER T2 ON (T1.C1 = T2.C1);
-- Is an incorrect sql statement INNER JOIN is probably the right way

SELECT T1.C1
FROM T1 INNER JOIN T2 ON (T1.C1 = T2.C1)
WHERE (T1.C3>2);
```

```
-- [a, a] since there is only one a in T1 where T1.C3>2 and there is 2 a in T2

SELECT T2.C2
FROM T1 LEFT OUTER JOIN T2 ON (T1.C2 = T2.C2);
-- [x,x,NULL,z,z,z,z,u] the NULL comes from T1.C2="y"
```

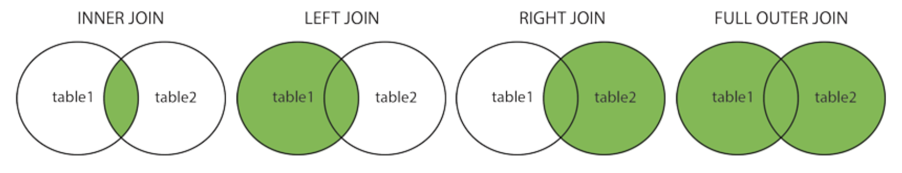


Figure 15.5: SQL JOIN statements. From ()

15.7.4 Constraints

- **Key** constraint: A primary key value cannot be duplicated.
- **Entity Integrity** Constraint: A primary key value cannot be null.
- **Referential integrity** constraints : The "foreign key " must have a value that is already present as a primary key, or may be null.

15.8 Ch.20 Transaction

Terminology

- *Transaction*: Describes local unit of database processing
- *Transaction processing systems*: Systems with large databases and hundreds of concurrent users. Require high availability and fast response time
- *Single-user DBMS*: At most one user at a time can use the system
- *Multi-user DBMS*: Many users can access the system (database) concurrently
- *Multiprogramming*: Executes commands from one process, then suspends that process and executes commands from another process, etc.
- *Interleave processing*: Interleaving is a process or methodology to make a system more efficient, fast and reliable by arranging data in a noncontiguous manner.
- *Parallel processing*: Parallel computing is a type of computation in which many calculations or processes are carried out simultaneously.
- *Read-only transaction*: (read_item(X))
- *Read-write transaction*: (read_item(X) and write_item(X))
- *Read set of transaction*: Set of all items read
- *Write set of a transaction*: Set of all items written
- *Concurrency control*: Transactions submitted by various users may execute concurrently
- *The lost update problem*: Occurs when two transactions that access the same database items have operations interleaved `https://www.youtube.com/watch?v=C_J6K8DodS8`
- *The temporary update problem*: Reading and updating from one transaction then read and write from another transaction. Then when reading another item we go back to the previous state. The problem is the dirty read of write then read from one transaction to another 6. `https://www.youtube.com/watch?v=d4Ziyuri0L0`
- *The incorrect summary problem*: T_3 reads X after N is subtracted and reads Y before N is added; a wrong summary is the result (off by N) 7. `https://www.youtube.com/watch?v=IIcR3G5w4ZI`
- *The unrepeatable read problem*: Transaction T reads the same item twice. Value is changed by another transaction T' between the two reads. T receives different values for the two reads of the same item.
- *Recovery*:
  - *Committed transaction*: Effect recorded permanently in the database
  - *Aborted transaction*: Does not affect the database
  - *Transaction faliures*:
    - *Computer failure*
    - *Transaction or system error*
    - *Local error or exception conditions detected by the transaction*
- *Transaction and System Concepts*
  - BEGIN_TRANSACTION
  - READ or WR

  - END_TRANSACTION
  - COMMIT_TRANSACTION
  - ROLLBACK (or ABORT)

- *system log*: System log keeps track of transaction operations
- *ACID properties*
  - *Atomicity*: Transaction performed in its entirety or not at all
  - *Consistency preservation*: Takes database from one consistent state to another
  - *Isolation*: Not interfered with by other transactions
  - *Durability or permanency*: Changes must persist in the database
- *Schedule*: (or history) Order of execution of operations from all transactions
- *Serializable schedules*: Places simultaneous transactions in series
- *serial schedules*: Everything in series results limit of concurrency
- *Conflict equivalence*: Relative order of any two conflicting operations is the same in both schedules.
- *DBMS enforces protocols*: Set of rules to ensure serializability.

Other SQL syntax

Set operations

- UNION: The UNION operator is used to combine the result-set of two or more SELECT statements.
- EXCEPT: The SQL EXCEPT clause/operator is used to combine two SELECT statements and returns rows from the first SELECT statement that are not returned by the second SELECT statement
- INTERSECT: The INTERSECT clause in SQL is used to combine two SELECT statements but the dataset returned by the INTERSECT statement will be the intersection of the data-sets of the two SELECT statements

Substring Pattern Matching

- LIKE: The LIKE operator is used in a WHERE clause to search for a specified pattern in a column.
- BETWEEN: The BETWEEN operator selects values within a given range.

Aliasing

15.8.1 Transaction Support in SQL

- No explicit Begin_Transaction statement
- Every transaction must have an explicit end statement
  - COMMIT
  - ROLLBACK
- Access mode is READ ONLY or READ WRITE
- Diagnostic area size option: Integer value indicating number of conditions held simultaneously in the diagnostic area
- Isolation level option
  - Dirty read
  - Nonrepeatable read
  - Phantoms

Appendices

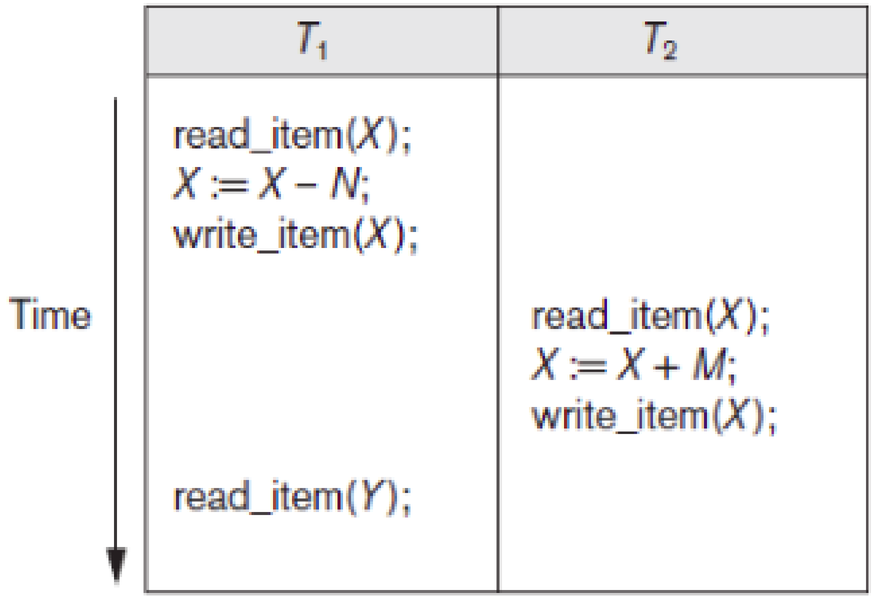


Figure 6: Temporary update problem. From ()

T_1	T_3
	sum := 0; read_item(A); sum := sum + A; ⋮

read_item(X);
X := X – N;
write_item(X);	

	read_item(X);
	sum := sum + X;
	read_item(Y);
	sum := sum + Y;

read_item(Y);
Y := Y + N;
write_item(Y);	

Figure 7: Incorrect summary problem. From ()

Chapter 16

Industrial Management

Resorcess

- https://hakan-kullven.webnode.se/
- https://www.youtube.com/watch?v=Q0o9S0q0Rr4
- https://www.youtube.com/watch?v=Kw-1nopchnA

16.1 Verksamheter

Företags former

	Enskild firma	Handels-bolag	Kommandit-bolag	Ekonomisk förening	Aktiebolag
Juridisk person?	Nej	Ja	Ja	Ja	Ja
Antal ägare	1	Minst 2	Minst en komplemen-tär, minst en kommandit-delägare	Minst 3	Minst 1
Ägarens betal-nings-ansvar	Personligt	Solidariskt och personligt	Komple-mentär: obegränsat; kommandit-delägare: insats	Insats	Insats

Figure 16.1: Control without feedback. From ()

Note: om du med i ett handelsbolag eller kommanditbolag se till att du inte har ett kompanions avtal med juridiskt ombud? Så att du inte råkar illa ut.

16.1.1 Entrepenörskap

- Entrepenörskap
  - Entrepenörskap är en process, där individer identifirerar möjligheter och baserar på deta går till handling.
  - Entreprenörsproccessen involverar en sökprocess
- Entreprenörskap bygger i mycket på att det finns riskvilligt kapital
  - På en aktiemarknad
  - Lån och bidrag, crowd funding, affärsänglar, venture capital, industriella partners, finnansinstitut, bootsrapping...

16.1.2 Inovation

- Produktutveckling
  - Alla aktiviteter som bidrar till utveckling och förbetrring av bolagets erbjudande till sina kunder
  - Ofta bedrivs detta i projekt
- Utvecklingsprocessen
  - Sekventiell
  - Parallell
  - Interativ
  - agil

    * Scrum
    * Pulse
  - Andra metoder
    * Prototyping
    * CAD, Computer-aided design
    * Set-based design
    * QFD, Quality function development
    * DFMA, Design for manufacturing and assembly
    * FMEA, Failure mode and effect analysis

16.1.3 Verksamhetstyp

- **Råvarufångst**:
  Karaktäristika: Stor personalrationalisering, mekaniserat, kapitalintensivt, världsmarknads- eller reglerade priser, kvalitetsortering, informations- och logistisystem viktiga konkurrensmedel
  Styrtal: Volym per tidsenhet, volym per arbetstimme, utbyte, kapitaloms'ttning, maksinutnyttjande, marginal, kostnad per ton - liter etc
- **Tillverkning**:
  Karkatäristika: Ofta öga personalkostnader och kapitalkostnader, produktiviteten viktig, ställ- och ledtider är viktiga
  Styrtal: Kapacitetsutnyttjande, produktivitet, ställtider, ledtider, materialutnyttjande, energif;rbrukning, leveranssäkerhet
  - **Legotillverkning**:
  - **Personalintensiv processtilverkling**:
  - **Kapitalalintensiv processtillverkning**:
  - **Sammansättning/motering**:
- **Varudistribution**:
  Karkatäristika: Mer logistik och system, kapacitetsutnyttjandet är viktigt, viktigt med stor kundkrets
  Styrtal: Intäkt per kapacitetsenhet, kostnad per kapacitetsenhet, antal kunder per dag, antal leveranser per dag, bruttomarginal, kundnöjdhet
  - **Godstransport**:
  - **Omlastning**:
  - **Detaljhandel**:
- **Kollektiva bastjänster**:
  Karkatäristika: Styrs ofta av lagar eller andra regler, budgeten är viktig, kräver ofta stora investeringar, kräver ofta stora volymer
  Styrtal: Kostnad mot budget, antal kunder, antal klagomål
  - **Myndighetsutövning**:
  - **Institutionella tjänster**:
  - **Abonnentrelaterad förvaltning**:
- **Servicesektorn**:
  Karkatäristika: Kompetens och speialisering är viktigt, arbetsintensiva verksamheter, hög och jämn beläggning är viktigt, kundservice är en konkurrensfaktor
  Styrtal: Debiteringgrad, Kapacitetsutnyttjande, kundnöjdhet, klagomål per kund.
  - **Lokala manuella tjänster**:

  - **Kunskapsintensiva uppdragsverksamhet**:
  - **Platsbunden konsumentservice**:
  - **Uthyrning**:
  - **Utbildning**:
  - **Distanssupport**:
  - **Artisteri**:

- **Spindleri**:
  Karkatäristika: Långvariga affärsrelationer, standardiserat affärskoncept, bruttomarginaler viktiga, mer och mer IT
  Styrtal: Marknadsandel, antal återkommande kunder täckningsbidrag per produkt
  - **Förläggande**:
  - **Kedjeorganiserande verksamhet**:
  - **Mäkleri**:

16.2 Planering för verksamheten

16.2.1 Företagets strategiska mål

- **Mission**:
  Vad företaget vill åstakommare
- **Vision**:
  En ide om ett möjligt önskat framtida tillstånd
- **Affärside**:
  En ide om hur ett företag ska kunna tjäna pengar på en affärsverksamhet
- **Strategier**:
  Handlingsvägar på olika nivåer i företaget
  Lågkostnadsstrategi, differentieringsstrategi, fokuseringsstrategi
- Differentation: Compete on a large market e.g. Apple
- Cost leadership: Compete on price e.g. Rianair
- Focus: Have a specific portion of the market e.g. Rolls Royce

16.2.2 Stratige modeler

SWOT-modellen

Figure 16.2: SWOT. From ()

Bostonmatrisen (BCG matrix)

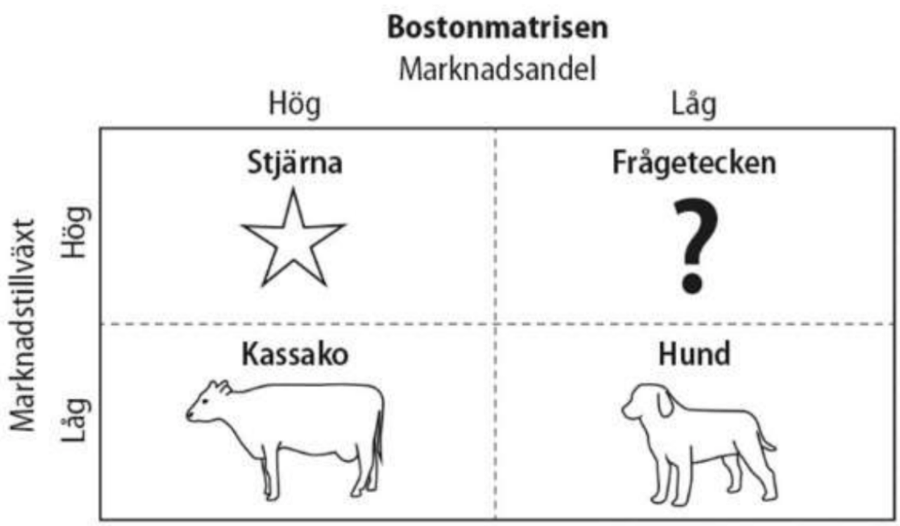


Figure 16.3: Bostonmatrisen. From ()

Produktlivscykeln (Product life cyckle)

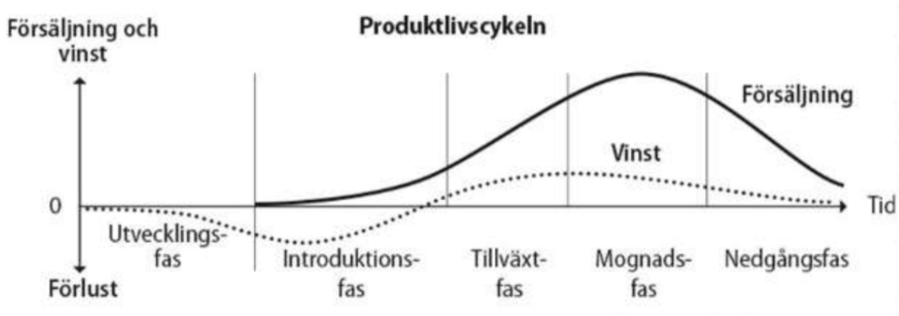


Figure 16.4: Produktlivscykeln. From ()

Intressentmodellen (Stakeholder analysis)

Figure 16.5: Intressentmodellen. From ()

16.2.3 Ekonomisktyrning (Managment control)

- Styrning av personer
- Styrning av kultur
- Styrning av resultat
- Styrning av agerandet

16.2.4 Ekonomiskstyrning modeller

Det balanserade styrkortet (Balance scorecard)

Prognoser (Forecasting)

- Försök att se in i framtiden och vad som ska hända då

Figure 16.6: balanserade styrkort. From ()

- System för planering på olika nivåer
  - Aktivetetsplaner
  - Målstättning
  - Förväntade värden
- Följa trender
  - Kritiska frmagångsfaktorer
  - Nyckelindikatorer
  - Viktigaste drivarna

Budgetering (Budgeting)

Budgetering är en **Handlingsplan** för

- Samordning
- Styrning
- Kommunikation
- Uppföljning och kontroll

Faser (uppbyggnad eller nedbrytning)

1. Budgetdirektive ges
2. Budget ställs samman
3. Budgetar faställs
4. Genomförande
5. Uppföljning

Fördelarna med budgetering

- Kan påverka
- För delegering

- Skapar kostnadsmedvetande
- informerar de anställda

Nackdelar med budgetering

- Tar mycket tid
- Slår aldrig in
- Begränsar möjligheter

16.3 Att agera på marknaden

16.3.1 Typer av marknader

- **Fullständig (perfekt) konkurrens**:
  Många små företag med samma produkter som därmed konkurrerar med priset
  - Den osynliga handen
- **Monopolistisk konkurrens**:
  Väldigt vanlig. Företagen han liknande produkter där vissa kunder föredrar produkter från enna företaget av diverse skäl och därmed konkurerar inte helt och hållet på pris.
- **Oligopol**:
  Ett fåtal företag som kontrollerar marknaden. Kan komma överens om pris
  - Varning för karteller
- **Monopol**:
  Endast ett företag som kontrollerar marknaden och kan sätta priset själv.
- **Marknadsekonomi**:
  Det ovanför är typer av marknadsekonomier.
- **Planekonomi**:
  Förekomer oftas i delar av samhälle, som försvaret.
- **Blandekonomi**:
  Kombination av marknadsekonomi och planekonomi.

16.3.2 Marknadsföring

Nätverssynsättet

- **Bygga upp förtroenden**:
- **Kundlojalitet**:
- **Eftermarknad**:
  Marknad för service och underhåll, reservdelar, tilläggsförsäljning, utbildning med mera till följd av tidigare försäljning av kapitalvaror och industriella produkter
- **Strategiskt partnerskap**:
  Konceptet bygger på att båda parter önskar förstärka en ömsesidig samverkan.
- **Relationsmarknadsföring**:
  innefattar att företag samlar in stora mängder data i olika relationsmarknadsföringsprogram
- **Marknadsföringsplan**:
  Den plan för att planera det följande konsepten

Konkurrensmedel, 4P

- **Produkt**: Varumärken
- **Pris**
- **Plats** (Distribution)
  - Råvaruindustri > Tillverkningsindustri > Grossist > Detaljist > konsument
- **Promotion** (reklam)

16.3.3 Strategisk prissättning

- **Värdebaserad**: (kundbaserad)
  Vad kunden är beräd att betala för den
- **Marknadsbaserad**: (konkurrentbaserad)
  Hur mycket den ska kostnad om man jämför med marknaden
- **Kostnadsbaserad**:
  Hur mycket företaget behöber att den kostar så det går runt.

Oftast är det en kombination av de tre.
Producentmarknad (B2B, Business to Business)

16.3.4 Tjänsteorientering

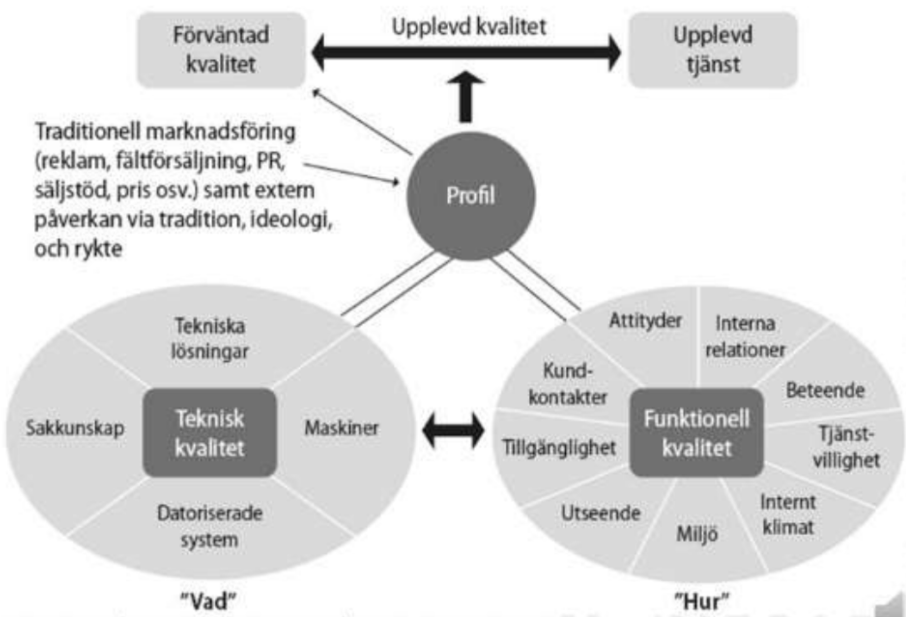


Figure 16.7: Tjänsteorientering. From ()

16.4 Organisera verksamhetern

16.4.1 Typer av hierarkiska organisationer

- **Linjeorganisation**: (U-form)
  Har den mest tydliga hirarkiska organisationenr med chefer och under chefer sedan arbetare

- **Linje-staborganisation**:
  Är som Linjeorganisation men det har en eller flera så kalande *stab* som är en avdelning som inte ingår i beslutsledet utan mer finns som stöd för övriga enheter, och benämns då en linje/stabs-organisation.

Operativa enheterna
  Det finns också mer produkt fokuserare strukturer som (M-form) där Divitionerna är olika produkter. En kombination av linje och produkt fokuserad struktur är MATRIX.

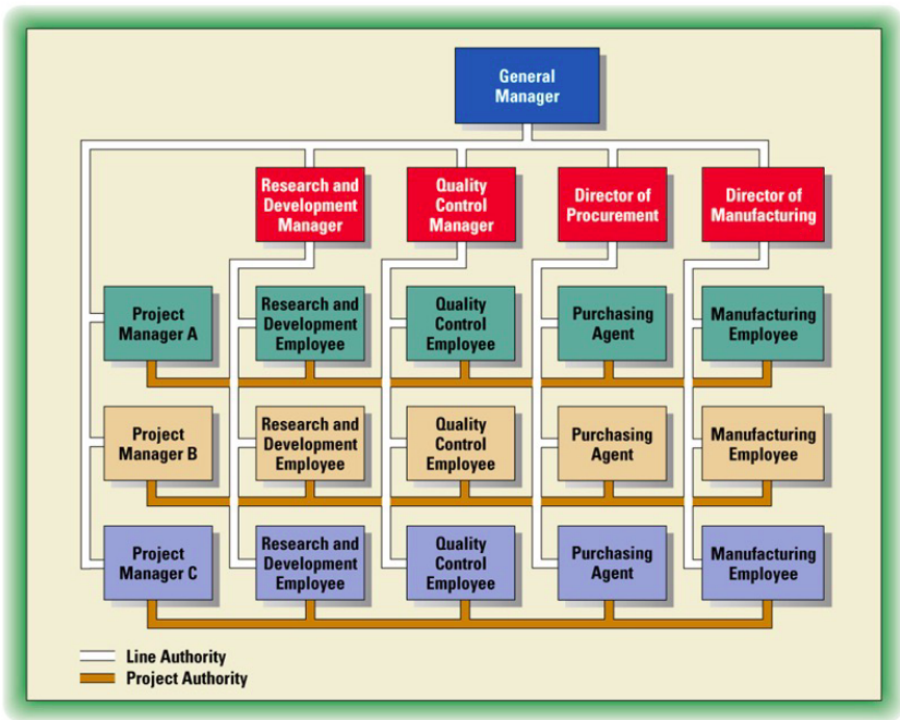


Figure 16.8: MATRIX. From ()

16.4.2 Projektorganisation

- För en uppgift
  - Balansera resultat, slutdatum och resurser
- Involverar
  - Projektledare
  - Projektägare
  - Projektgrupp
  - Projektråd
  - Styrgrupp
  - Delprojekt
- Planering
  - Indelat iarbetspaket med milsolpar.
  - Planneras med *PERT* och *CPM*

16.4.3 Rerulstatvärdesmetoden

$$\text{Bedömd kostnad vid färdigt} = \frac{\text{Faktisk kostnad}}{\text{Resultatvärde}} \cdot \text{Budgeterad total kostnad}$$

$$\text{Bedömd tid till färdigt} = \frac{\text{Planerad kostnad}}{\text{Resultatvärde}} \cdot \text{Budgeterad total tid}$$

16.4.4 Produktion

Automationsnivå

- Manuell produktion
- Semi-automatisk produktion
- automatisk produktion

Volym och variation

- Enstycksprocess
- Intermittent (batch) process
- Kontinuerlig process

Layout i lokalen

- Fast position
- Funktionell layout: Placering utifrån aktivitet i produktionen
- Födesgrupper: celler
- Linjebaserad layout: produktorienterad placering

16.4.5 Lager Optimering

$$\text{Optimal inköpskvantitet} = \sqrt{\frac{2 \cdot \text{Efertågan per år} \cdot \text{ordersärkostnad per order}}{\text{Lagerhållningskostnad per styck}}}$$

Exempel WrapUp AB

TODO

16.4.6 Vad gör en ledare?

- **Mintzberg**
  - Interpersonella rollen:
    Representant, vägledare och nätverkare
  - Informationsrollen:
    Övervakare, informationsspridare och talesperson
  - Beslutsrollen:
    Entreprenör, problemhanterare, resursfördelare och förhandlare
- **Hamel**
  - Formulera och planera målförverkligande

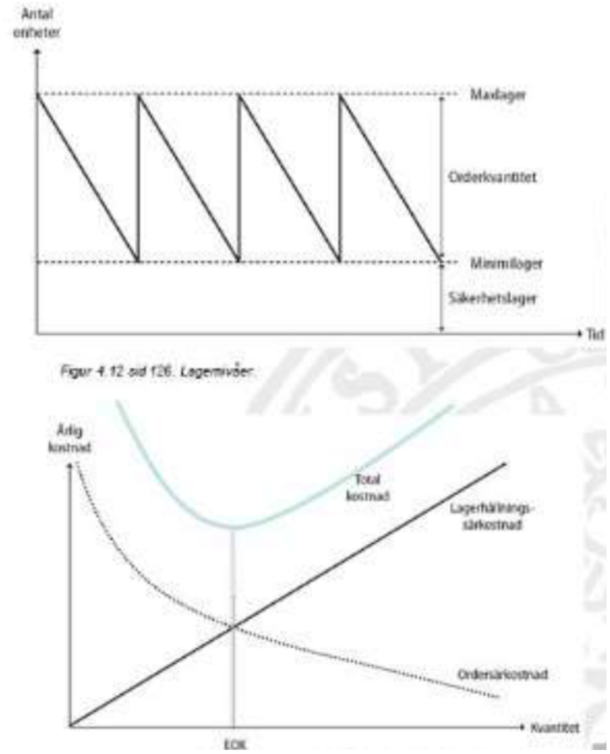


Figure 16.9: lager optimering grapher. From ()

- Motivera och samordna insatser
- Koordinera och styra aktiviteter
- Utveckla och tillsätta talanger
- Ackumulera och applicera kunskap
- Anskaffa och fördela resurser
- Skapa och vårda relationer
- Balansera och tillmötesgå krav från investerare

Bolagets ledning

- Bolagsstämma
  - Aktieägarna
  - Högsta beslutande organ
- Högsta ledningen
  - Styrelsen:
    Ansvarar för organisation och förvaltning
  - Företagsledningen:
    VD: löpande förvaltning
- Mellanchefer
- Operativa chefer
  - Första linjens chefer
  - Arbetsledare

Ansvar

- Räntabillitetsansvar (Investment center): (intäkt - kostnad)/ Kapital
  Exempelvis: CEO
- Resultatansvar (Profit center): Intäkt - Kostnad
  Exempelvis:
- Intäktsansvar (Revenue center): Intäkter och kostnad
  Exempelvis: Key accountant manager
- Kostnadsansvar (Cost center): Kostnad
  Exempelvis: Chef ekonomi ansvarig
- Timansvar (Hour center): Timmar
  Exempelvis: Lathe manager

16.5 Kalkyler för produkter och order

16.5.1 Begrepp i flödet

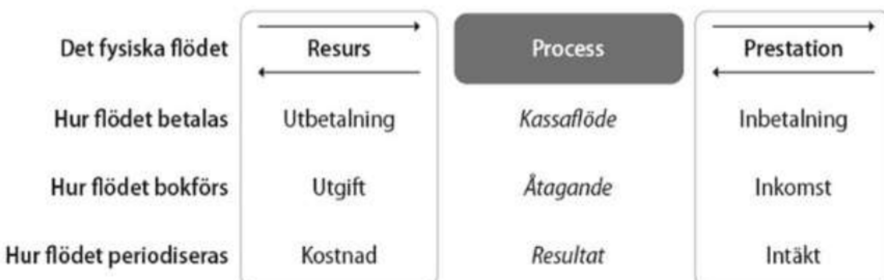


Figure 16.10: Begrepp i flödet. From ()

Inbetalning är när vi får pengarna på banken Inkomst är hur mycket våra kunder att betala oss Intäkt?

Periodisering

För att mäta produkts resultat så räknar vi resursenrna som krävdes för produkten (kostnad) och gemför dessa med hur produkten presterade (intäkt). Periodisera betyder att det är gjämförbara.

16.5.2 Kalkylbegrepp

Figure 16.11: Kalkylbegrepp. From ()

- Direkt kostnad: Kostnad spesific för produkten

- Indirekt kostnad: Kostnad som är delvis för den produkten (lathe)
- Särintäkter: Sär är relevant kostnad
- Samintäkter: Sam är irelevant kostnad.
- Rörlig kostnad: Förändrar sig med antal enhetet
- Fast kostnad: Är alltid samma

16.5.3 Kalkylmodeller

Det finns kalkyler som görs före och efter beslut.

Figure 16.12: Kalkylmodeller. From ()

16.5.4 Påläggskalkyl

Figure 16.13: Påläggskalkyl. From ()

Exempel Savox AB

Savox AB utför diverse lackeringsuppdrag för andra företag. För år 2018 har de budgeterat en direkt materialkostnad på 1 400 000 kronor, materialomkostnader på 280 000 kronor, direkta löner i produktionen på 400 000 kronor, indirekta produktionskostnader på 880 000 kronor för den planerade maskintiden 8 800 timmar, andra direkta produktionskostnader på 40 000 kronor, direkta försäljningskostnader på 100 000 kronor, och försäljning och administration på 1 200 000 kronor. Nu har företaget fått en order för lackering av en plastdetalj. För detta behövs färg för 5 000 kronor, 5 direkta arbetstimmar i produktionen vilka vardera kostar 1 200 kronor, 10 maskintimmar och en direkt försäljningskostnad på 1 000 kronor.
a. Till vilket belopp uppgår företagets självkostnad enligt budget? b. Vilka pålägg och timkostnader bör företaget använda under året? c. Vad är kostnaden för den order de nu fått?
...

16.5.5 Aktivitetsbaserad kalkyl, ABC

Figure 16.14: ABC. From ()

Exempel Savox AB-C

Savox AB har bestämt sig för att använda en del ABC i sitt kalkylsystem. Jämför denna övning med den föregående, där de använde en påläggskalkyl. För år 2018 har de budgeterat en direkt materialkostnad på 1 400 000 kronor, materialhantering för 280 000 kronor med liter färg som drivare, produktion som kostar 400 000 kronor, maskinbearbetning som kostar 880 000 kronor för den planerade maskintiden 8 800 timmar, andra direkta produktionskostnader på 40 000 kronor, direkta försäljningskostnader på 100 000 kronor, försäljningsaktiviteter för 400 000 kronor med antal order som drivare, vilket beräknas till 400 för året, samt försäljning och administration för 800 000 kronor för vilket de inte finner en lämplig drivare och vilket de inte anser orsakas av order som sådana. Nu har företaget fått en order för lackering av en plastdetalj. För detta behövs 140 liter färg för 5 000 kronor, 5 direkta arbetstimmar i produktionen vilka vardera kostar 1 200 kronor, 10 maskintimmar och en direkt försäljningskostnad på 1 000 kronor.
a. Till vilket belopp uppgår företagets självkostnad enligt budget? b. Vilka kostnader per drivare bör företaget använda under året? c. Vad är kostnaden för den order de nu fått?
...

16.5.6 Bidragskalkyl (Contribution Margin Costing)

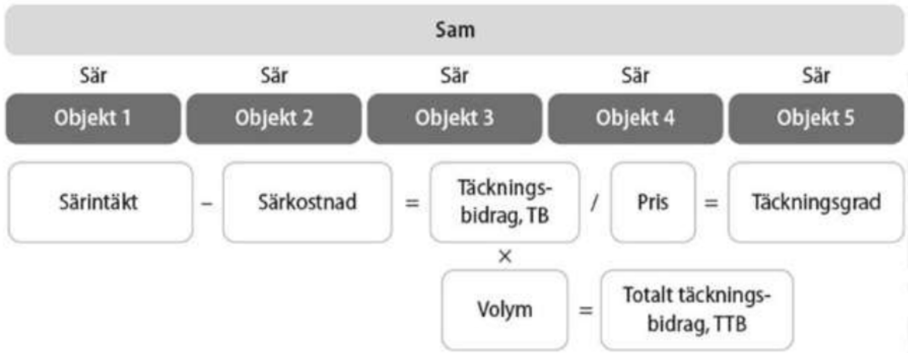


Figure 16.15: Bidragskalkyl. From ()

Särintäkt pris som betalt för produkten och särkostnaderna är vad det kostar att producera. räkningsgrad beteknas som bruttomarginal.

Beslut

Med bidragskalkylen så kan man andvända den till att fatta vissa beslut som:

- Ska ordern accepteras?
  Tar endast hänsyn till beslutrelevanta, orderspecifika effekter på sikt
- Produktval vid en begränsnig (en trång sektion)
  Den produkt som ger högst TTB, d.v.s högst TB per trång sektion, bör väljas
- Produktval vid flera begränsningar (flera tånga sektioner)
  Den kombination av produkter som ger högst TTB, enligt t.ex. linjär programmering, bör väljas
- Allmäna beslut om förändringar
  Tar endast hänsyn till beslutrelevanta, förändringsspecifika effekter på kort sikt

Exempel ChoicesAB

todo
Endast särkostnaden är relevant för beslutet. Tex så tar man inte in ens egna hyran i beräkningen för pris på en biobiljet är värt det eller inte.

16.5.7 Stegkalkyl

16.5.8 Andra självkostnadskalkyler

Exempel

16.6 Investeringsbeslut

Grundprincip: Insättning på bank

- Ränta på ränta, slutvärde: Kapital $\times(1+r)^n$
- Nuvärde: Kapital $\times\frac{1}{(1+r)^n}$
- Nuvärdesumma: Kapital $\times\frac{1-(1+r)^{-n}}{r}$

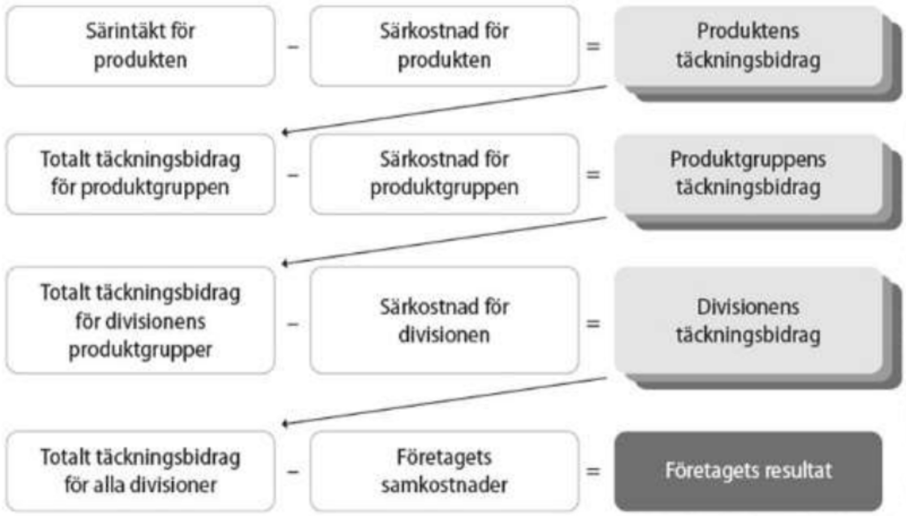


Figure 16.16: Stegkalkyl. From ()

- Annuitet: Kapital $\times \frac{r}{1-(1+r)^{-n}}$

Figure 16.17: Typer av investeringar. From ()

Metoder för investering

- Nuvärdemetoden
  - Positivt = lönsamt, ju högre desto bättre
  - Kan endast jämföra investeringar med samma ekonomiska livslängd
- Annuitetsmetoden
  - Positivt = lönsamt, ju högre desto bättre
- Internräntemetoden

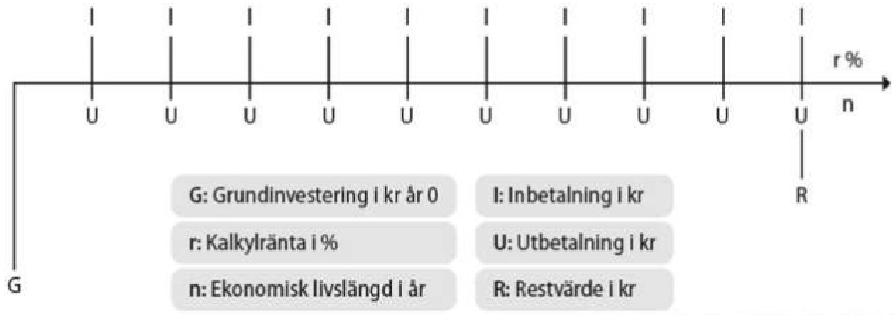


Figure 16.18: Kalkylbegrepp. From ()

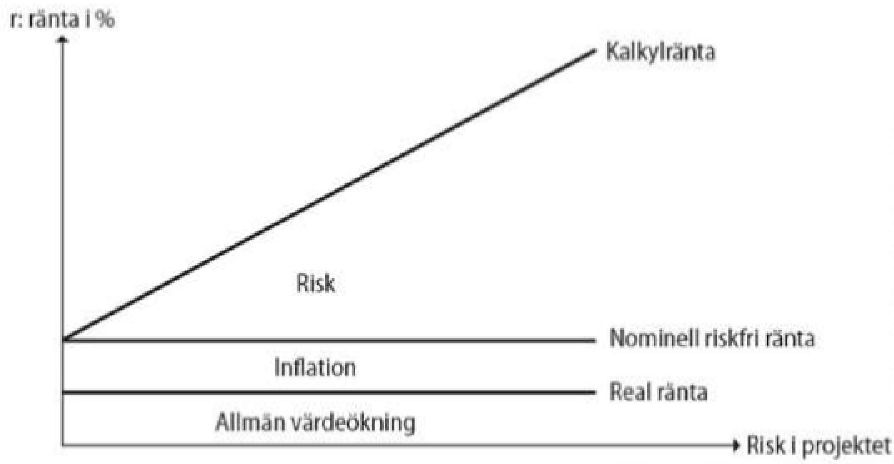


Figure 16.19: Kalkylräntan. From ()

  - Ju högre internränta desto bättre
  - Testa, eller dela grundinvesteringen med det årliga betalningsöverskottet
  - Kan bara använda om det också finns inbetalningar

- Återbetalningtid (pay-back)
  - $\sum$ inbetalningsöverskott = Grundinvesteringen, eller Grundinvestering/årligt inbetalningsöverskott
  - Ju kortare tid desto bättre
  - Problem; kortsiktigt, ingen ränta (i grundversionen)

- Investeringens räntabilitet
  - Genomsnittligt årligt inbetalningsöverskott divideras med genomsnittlig investering
  - Ju högre procent räntabilitet desto bättre
  - Problem: Tar inte hänsyn till ränta

shabloner, prusent (enkla) aktivitetbaserad kalkyl

—–

Balanse tekning inventarie, sånt som inte ändrar sig
nuvärde annueitet återbetalningid

Teknisk livslängd är så länge som en anläggning (eller del av anläggning) går att tekniskt/funktionellt att använda. Ekonomisk livslängd är den tid som det är ekonomiskt rationellt att använda anläggningen

kalkylränta är riks, real ränta och inflation real ränta hur det faktiskt förändras

återbetalningen metoden är inte riktigt rätt

räntan är inte bankräntan

16.7 Uppföljning av verksamheten

Bokföring med T-konton notera att kredit och debit för balansräkningen respektive resultaträkningen

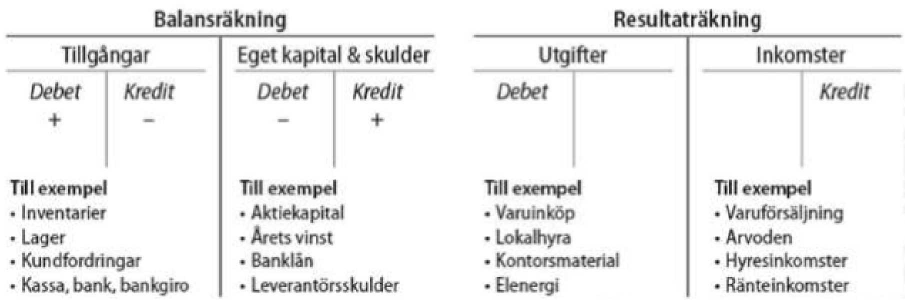


Figure 16.20: Bokforing med t-konton. From ()

ska vara samma. Debet betyder enbart vändstra sidan av räkningarna och Kredit är högra. Var man lägger tillgångar samt eget kapital & skulder respektive Utgifter och incomseter är upptill företaget. Men skenerält så är det rällativt licka sätt man lägger up det.

- SCOA2
- Kiosk är en ..
- FAST rapport
- årsredovisning
- årsmöte

Årsredovisningens innehåll

Figure 16.21: Årsredovisningens innehåll. From ()

Resultaträkning och balansräkning

Figure 16.22: Resultaträkning och balansräkning. From ()

16.7.1 Redovisningens regler

- Lagar och regelverk
  - Bokföringslagen, Aktiebolagslagen, Årsredovisningslagen, Inkomstskattelagen. Mervärdeskattelagen, Skattebetalningslagen m.m.
  - Svensk kod för bolagsstyrning, Sarbanes-Oxley Act SOX
- Redovisningens principer
  - Rättvisande bild, "true and fair view"
  - Periodisering, fortlevand, jämförbarhets, försiktighet, realisering, osv
- Normgivare: God redovisningssed
  - Bokföringsnämnden, Rådet för finansiell rapportering, FAR SRS
  - IASB (Eng) International Accounting Standards Board med IFRS (International Financial Reporting Standards)
  - FASB (USA) Financial Accounting Standards Board

Revisorn jobbar för aktieägarna. Det måste se till att det siffror som aktie ägarna får(redovisas) är rätt.
Namn och begrepp måste vara med och för årsredovisning
Det går att ändra räkenskapsår, men man måste anmäla att göra en sån ändring

16.7.2 Måt

- Omsättingstillgångar: Lägsta värdets princip (försiktighetsprincip). Är tillgångar som förväntas säljas om ett år.
- Anläggningstillgångar: Långtids investeringar, mer än 1 år
- Värdering och bokslutsdisponitioner:

EBITDA är ett annat finnansielt mått.

- Soliditet: överlevnadsförmåga
  $\frac{\text{Eget kapital}}{\text{Totalt kapital}}$
- Balanslikviditet:
  $\frac{\text{Omsättningstillgångar}}{\text{Kortfristiga skulder}}$
- Kassalikviditet:
  $\frac{\text{Omsättningstillgångar minus lager}}{\text{Kortfristiga skulder}}$

Lösamhet (räntabilitet)

- Räntabilitet på egen kapital, R_E: ägarnas avkastning
  $\frac{\text{Årets resultat}}{\text{Genomsnittligt eget kapital}}$
- Räntabilitet på totalt kapitel, R_t: total avkastning
  $\frac{\text{Resultat före skatt plus finansiella kostnader}}{\text{Genomsnittligt totalt kapital}}$

DuPoint-formeln

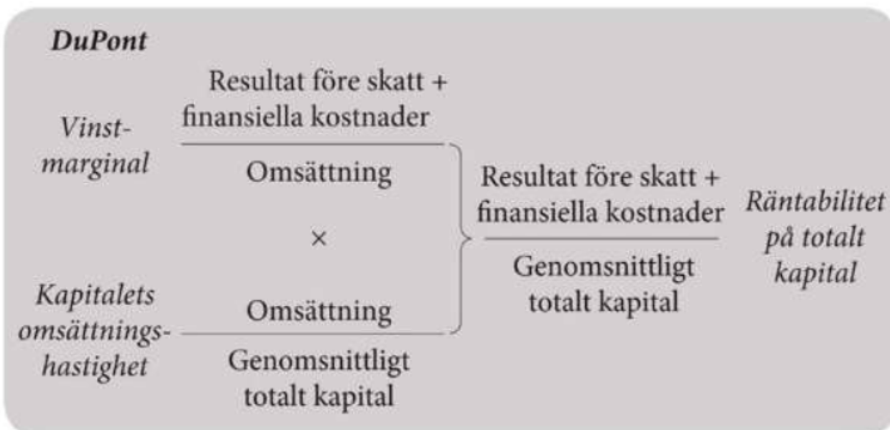


Figure 16.23: DuPoint-formeln. From ()

exampels
Vi har stabilt näringsiv i sverige. Vi har regler som skattar mycket när det går bra och lite när det går dårligt
Värde på tillgångar är upp till företaget att välja innom vissa sätta rammar som är samma för alla
Om man inte anmäler om konkurs när soliditet är för lågt så kommer man vara skildig till att betala ur egen plånbock.
Man har förökt redovisa hållbarhet, dock så har det inte kommigt så långt med det.

16.8 Analysera förändring

Rörliga kostnader

- Rörlig kostnad: rack linje (Kronor totalt x volym)
- Avtagande rörliga kostnader: avtagande ökande kurva (kronor totalt x volym)
- Ökande rörliga kostnader: exponentiel kurva (kronor totalt x volym)

Fasta kostnader

- Fast kostnad: horizontelt sträck (kronor, totalt x Volym)

- Halvfast kostnad: ökar stegvis beronde på volym
- Halvfast kostnad: blir biligare per styck ju störe volym men ökar vid en viss mängd, blir som en såg kurva

Volymanalys

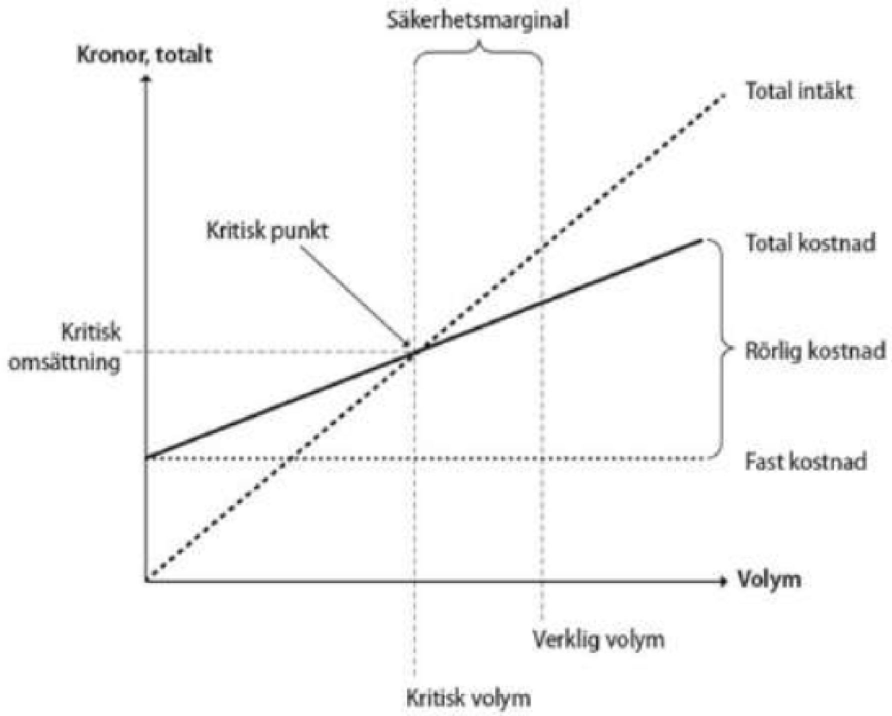


Figure 16.24: Volymanalys. From ()

Exempel: Fasta Rör AB

Företaget Fasta Rör AB säljer en produkt för 20 kronor per styck. Man hade i januari en total kostnad på 1 630 000 kronor, och i februari en total kostnad på 1 930 000 kronor. Inga andra skillnader noterades mellan perioderna än att volymen uppgick till 82 000 enheter i januari och till 102 000 enheter i februari.

- Direct costs: Costs that can directly ried to the production of specific goods or services. Ex: Direct labour and material.
- Inderect costs: Costs that are neccessary, but dificult to trace to the cot object. Ex: Overhead (rents, utulities, administation)
- Relevant costs: Costs that are affected by managerial choice in a cerain business situation. Ex: Variable or marginal costs, increamental costs, specific costs, avoidable ficed costs, oppertunity costs.
- Irrelevant costs: costs that won't be affected by a managerial decision. Ex: Sunk costs and costs which are same for different alternatives.
- Fixed costs: Costs that change in total when the volume changes. Ex: Rents, insurance, employee salaries, etc.
- Variable costs: Costs that don't change in total when the volume is changing. Ex: Sunk costs and costs which are same for different alternatives.

$$\text{Total Cost} = \text{Fixed Costs} + \text{Variable Costs}$$

Fundamental equaion

$$\text{Operating Profit} = \text{Revenues - Total VC - Total FC}$$

16.9 Verksamhetens finansiering

Kapitalbehov

Figure 16.25: Kapitalbehov. From ()

Olika typer av Risker

- Verksamhetsrisker
  - Risker på grund av verksamheten
  - till exempel marknadsrisk, tillverkningsrisk, leverantörsrisker, och så vidare
- Finansiella risker
  - Risker till följd av hur investeringarna finansieras
  - till exempel risken för förändringar i valutakurserna, risken för förändringar i det funanaiella nettot, och risken för nya skatteregler.

Kapitalrationalisering går ut på att förbättra ett företags räntabilitet genom en förbättrad kapitalomsättning

16.9.1 Anläggningskapital (Fixed capital)

describes a company's investment in long-term assets - relatively permanent - e.g., for property, plant and equipment on the balance sheet

16.9.2 Rörelsekapital (Working capital)

is the amount of available capital that a company can readily use for day-to-day operations (measures a company's liquidity, operational effeciency, short-term finnancial health) - e.g., for inventory, accounts receivables, etc. *Rörelsekapital*

- Procentmetoden
  - Procentuell ändring i posterna p g a t.ex. omsättningsökning
- Genomsnittsmetoden
  - Kundfordringar = (kredittid i dagar/360 dagar) × försäljning
  - Leverantörsskulder = (kredittid i dagar/360 dagar) × inköp

  - Förråd = (minimilager + maximilager)/2 (eventuellt sen × kronor)
  - Produkter i arbete i kronor = Antal produkter × halva produktkostnaden
  - Färdiglager i kronor = Antalet produkter × produktkostnaden

- Resursmetoden och balansräkningsmetoden

16.9.3 Safety capital

Amount of capital to cover for variations in sales and the like.

16.9.4 Other terminology

Resursmetoden

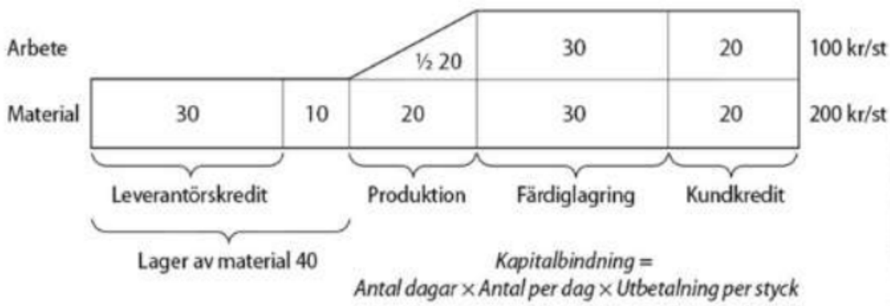


Figure 16.26: Resursmetoden. From ()

Kreditmarknaden
Främmande kapital kreditmarknad:

- Obligationslån
- Inteckningslån
- Reverslån
- Kontokredit
- Factoring
- Leasing
- Konvertibla skuldebrev

Aktiemarknaden
Eget kapital Aktiemarknad

- Sparade vinster
- Nyemission: fler aktie, för att göra det biligare att handla med aktien

16.9.5 Analyser

kassaflödes analys
Finansiell analys
Bank will get the money before the shareholder in case of bankrupsy

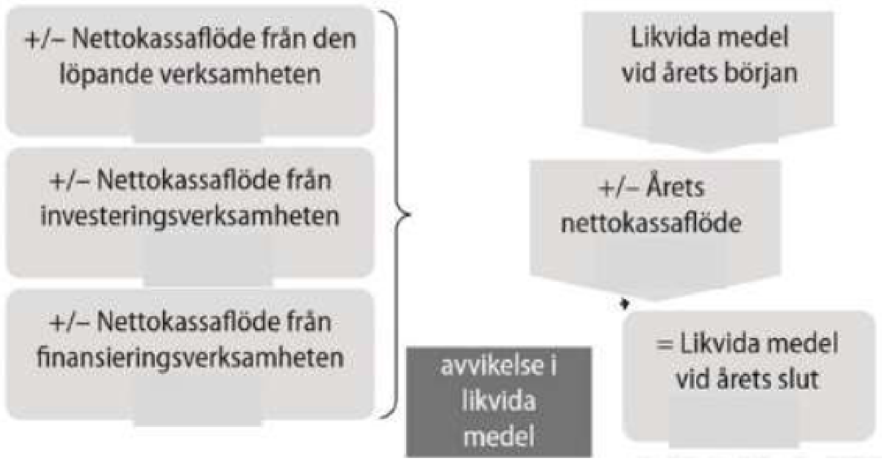


Figure 16.27: Kassaflödesanalysens komponenter. From ()

Figure 16.28: Kassaflodesanalys. From ()

Chapter 17

Introduction to Computer Control Systems

Resources

-

17.1 Intro

17.1.1 Control with and without feedback

Control without feedback is not as common in requires much more precise models and more sensors in order for it to work. In the case with a control for a segway without feedback requires a tone of sensors to measure every posable input with may change the angle of the handle.

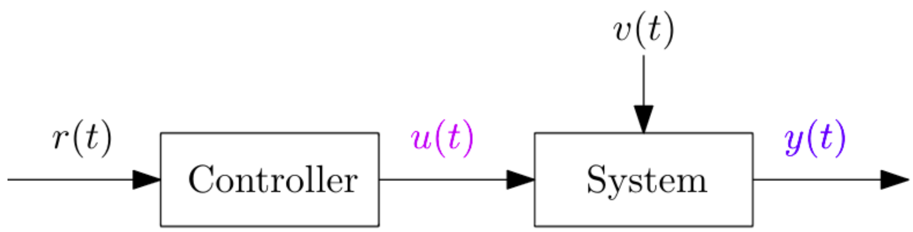
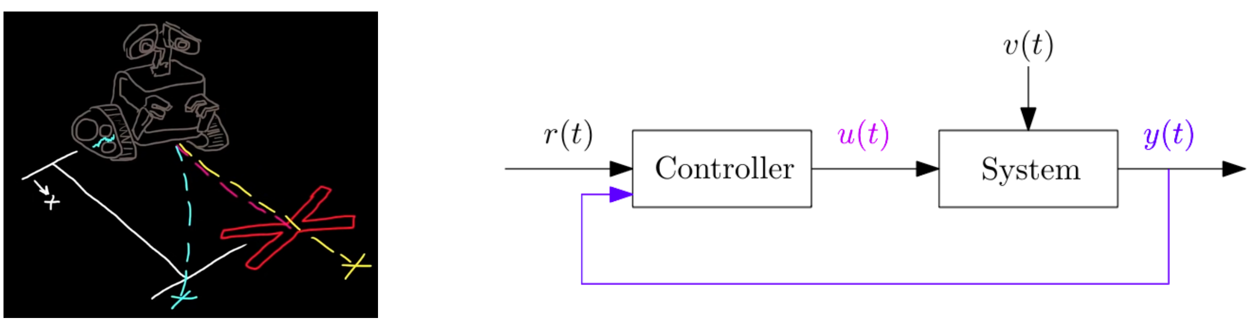


Figure 17.1: Control without feedback. From ()

Control with feedback is the most common. It is important to have feedback since the system rarely accts exactly like the model, for instance if you tell a robot to drive to a marking and without feedback you have to calculate the voltage needed to apply to the motors. Meanwhile with feedback you can addjust the stering and voltage in reference with the current position.



(a) Walle ilustration. From ()

(b) Block diagram. From ()

Figure 17.2: Control with feedback. From ()

The controller part including the reference $r(t)$ is Computer based and the Systems is generally physical.

17.1.2 Typical design requirements

1. **Stability**: Output $y(t)$ will be *bounded* when reference $r(t)$ is bounded
2. **Tracking**: $y(t)$ tracks $r(t)$ *quickly* with minimal *oscillations*
3. **Feasibility**: Controller should decide on feasible *input* $u(t)$
4. **Robustness**: If a disturbance occurs, output $y(t)$ should quickly *return* to the reference signal $r(t)$.

17.1.3 Laplace repetition

Transforms Solving ODE

17.1.4 Transfer function of a controller

$$Y(s) = G(s)U(s)$$

see image 17.2

The **Impulse response** is defined as follows

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

and

$$y(t) = \int_0^t g(\tau)\delta(t-\tau) \equiv g(t)$$

17.1.5 Poles and zeros

The **zeros** tells us how much the $G(s)$ oscillates and the **Poles** tells us how fast it converges. It is *stable* if the poles are negative.

Expressing $G(s)$ via the poles

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+\sigma_1} + \dots \frac{B_j(s)}{(s+\sigma_j)^2 + \omega_j^2} + \dots$$

Impulse response then becomes

$$g(\tau) = \sum_j b_j e^{Repolet} = b_1 e^{\sigma_1 \tau} + \dots + b_j \sin(\omega_j \tau + \phi_j) e^{\sigma_j \tau} + \dots$$

17.1.6 Static gain of system

Assume stable system $Y(s) = G(s)U(s)$ with input $u(t)$ as a step function. $U(s) = \frac{u_0}{s}$

Using the Final value theorem we get the static gain $G(0)$

$$y_f = \lim_{t\to\infty} y(t) = \lim_{s\to 0} Y(s) = lim_{s\to 0} sG(s)\frac{u_0}{s} = G(0)u_0$$

the static gain tells us what value dose the it converges

Static gain

System

$$G(s) = \frac{2}{s^2+s+1}$$

where the input to the system is a step response. What is the steady state value dose the output have?

From figure 17.2 we determine the output as $y(t) = g(t)u(t)$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)U(s) = \lim_{s\to 0} s\frac{2}{s^2+s+1}\frac{1}{s} = \frac{2}{1} = 2$$

17.2 PID control

https://www.youtube.com/watch?v=UR0hOmjaHp0
https://www.youtube.com/watch?v=IB1Ir4oCP5k

$$u(t) = \underbrace{K_p e(t)}_{P} + \underbrace{K_i \int_{\tau=0}^{t} e(\tau) d\tau}_{I} + \underbrace{K_d \dot{e}(t)}_{D}$$

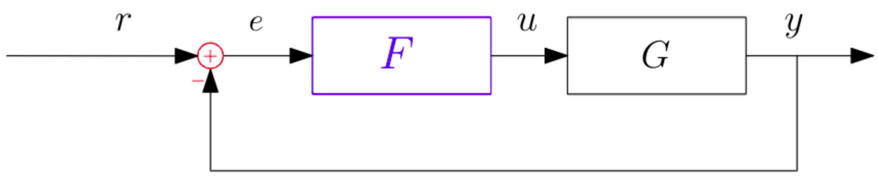


Figure 17.3: PID-controller. From ()

17.2.1 Feedback control based on error

$$e(t) \triangleq r(t) - y(t)$$

- current error: $\propto e(t)$ (**P**roportional)
  The value is the error veted with a constant. How fast the system response.
- past error: $\propto \int_{\tau=0}^{t} e(\tau) d\tau$ (**I**ntegral)
  The integral under the error from all past values. How fast the steady state error is removed
- change in error: $\propto \dot{e}(t)$ (**D**erivative)
  The rate of change of the error. Faster response and better preforming.

using laplace we get:

$$\begin{aligned} U(s) &= K_p E(s) + K_i \frac{1}{s} E(s) + K_d s E(s) \\ &= \Big(K_p + \frac{K_i}{s} + K_d s\Big) E(s) \\ &= F(s)E(s) \end{aligned}$$

- System from u to y: $G(s)$
- Closed-loop system from r to y: $G_c(s)$
- Open-loop system from e to y: $G_o(s) = G(s)F(s)$

The closed-loop is defined as:

$$G_c(s) = \frac{G(s)F(s)}{1 + G(s)F(s)}$$

We can define $Y(s)$ as:

$$Y(s) = G_c(s)R(s)$$

The open-loop system is defined as:

$$G_o(s) = F(s)G(s)$$

Derive close loop

Derive G_c TODO

Stationary error

$$\begin{aligned} e_f &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\frac{1}{1+G_0(s)}\frac{r_0}{s} = \lim_{s\to 0}\frac{r_0}{1+G(s)F(s)} \end{aligned}$$

e_f approaches 0 if $G(0)F(0) = \infty$. (E.g. when $F(s)$ contains $\frac{1}{s}$, i.e. integration)

17.2.2 Classical closed-loop design methods

- **Solve roots** of $G_c(s)$ as a function of parameters
- **Routh's algorithm** for Stability check. You only get if the poles are on the left side or not.
- **Root-locus** plot for single parameter

Solve roots

Solve roots

The output is expressed as:

$$Y(s) = \frac{4s(s+2)}{s^3 + (3+8K_d)^2 + (2+8K_p)s + 8K_i}V(s)$$

what is the poles of the system, if a P-controller is used?

$$\begin{aligned} Y(s) &= \frac{4s(s+2)}{s^3 + (3+8K_d)^2 + (2+8K_p)s + 8K_i}V(s) \\ &\text{set } K_i = K_d = 0 \\ &= \frac{4(s+2)}{s^2+3s+2+8K_p}V(s) \\ &\Rightarrow s^2+3s+2+8K_p = 0 \\ &\Rightarrow s = -1.5 \pm \sqrt{0.25-8K_p} \quad \text{-using pq-formula} \\ &\Rightarrow 8K_p > 0.25 \Rightarrow K_p > 0.03125 \end{aligned}$$

Routh's algorithm

The denominator of the system can be constructed to find the pols, i.e. when the denominator is zero.

$$a_0 s^n + b_0 s^{n-1} + a_1 s^{n-2} + b_1 s^{n-3} + \ldots = 0$$

Step 1: construct a table of the coefficient

$$\begin{array}{ccccc} a_0 & a_1 & a_2 & a_3 & \ldots \\ b_0 & b_1 & b_2 & b_3 & \ldots \\ c_0 & c_1 & c_2 & \ldots & \end{array}$$

Step 2:

$$c_k = \frac{b_0 a_{k+1} - a_0 bk + 1}{b_0}$$

Routh's algorithm

The output is expressed as:

$$Y(s) = \frac{4s(s+2)}{s^3 + (3+8K_d)^2 + (2+8K_p)s + 8K_i} V(s)$$

is the system stable, if a P-controller is used?

$$G_o(s) = \frac{4s(s+2)}{s^3 + (3)^2 + (2+8K_p)s} V(s)$$

$$\begin{array}{ccc}
1 & 2+8K_p & 0 \\
1 & 2+8K_p & 0 \\
(24K_p + 6 - 8K_i)/3 & 0 & 0 \\
(8K_i) & 0 &
\end{array}$$

$$\Leftrightarrow 24K_p + 6 - 8K_i > 0 \wedge K_i > 0$$
$$\Rightarrow 0 < K_i < \frac{3}{4} + 3K_p$$

Rootl-locus

Root-locus algorithm

$$G_0(s) = \frac{K}{s(s^2+2s+2)}$$
$$G_c(s) = \frac{K}{s(s^2+2s+2)+K.\underbrace{1}_{Q(s)}}$$

Step 1: Solve the poles $P(s) = s(s^2+2s+2)$ Starting point (K=0)

$$z_{p1} = 0$$
$$z_{p2} = -1+j$$
$$z_{p3} = -1-j$$

Step 2: $Q(s) = 1$ Ending points $Q(z_q) = 0 \Rightarrow$ no ending points $(K \to \infty)$
Step 3: Asymptotes number of asymptotes = $\underbrace{\text{number starting}}_{3} - \underbrace{\text{number ending}}_{0}$ the assymptot φ

$$\varphi_A = \frac{2q+1}{\text{number of asymptotes}}$$

$$\varphi_A(q=0) = \frac{2\cdot 0+1}{3}\cdot\pi = \frac{\pi}{3}$$
$$\varphi_A(q=1) = \frac{2\cdot 1+1}{3}\cdot\pi = \pi$$
$$\varphi_A(q=2) = \frac{2\cdot 2+1}{3}\cdot\pi = \frac{5\pi}{3} = \frac{-\pi}{3}$$

Intersection of the asymptotes

$$c = \frac{\sum_{p=1} z_p - \sum z_q}{\text{number of asymptotes}} = \frac{0-1+j-1-j}{3}$$

see image 17.4
Step 4: Intersection with imaginary axis

$$\begin{aligned}
&s = 6 + j\omega \\
&6 = 0 \Rightarrow s = j\omega \\
&s(s+1)(s+3)+k = 0 \\
&s^3+4s^2+3s+k = 0 \\
\Leftrightarrow& -j\omega^3 - 4\omega^2 + 3j\omega + K = 0 \\
\Leftrightarrow& \begin{cases} K - 4\omega^2 = 0 \\ w(w^2-3) = 0 \end{cases} \quad \Rightarrow \begin{cases} w = 0 \Rightarrow K = 0 \\ w = \pm\sqrt{3} \Rightarrow K = 12 \end{cases}
\end{aligned}$$

Step 5: Intersection with real axis

$$k = -s(s+1)(s+3)$$
$$\frac{\partial K}{\partial S} = 0 \Leftrightarrow 3s^2+8s+3 = 0 \qquad \Rightarrow \textbf{No real solutions}$$

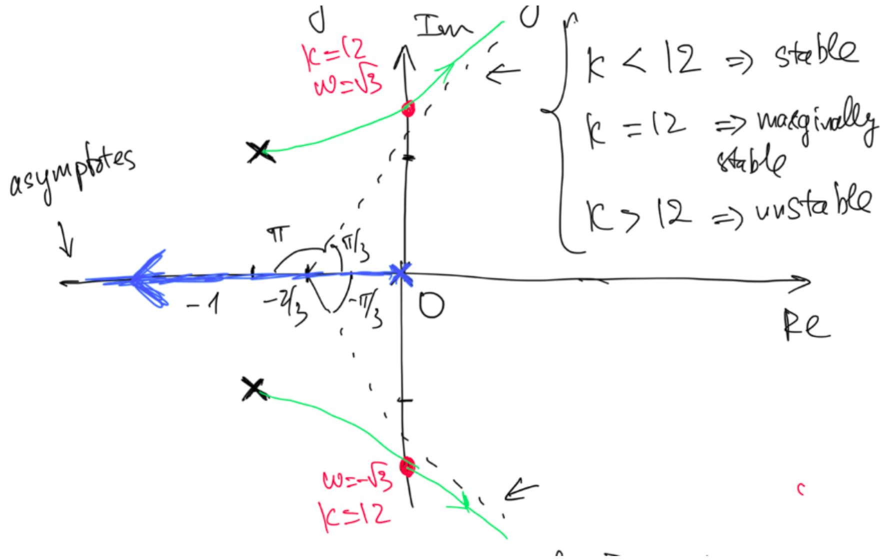


Figure 17.4: Root locus example

17.3 Good Controller?

17.3.1 Method 1: Frequency description of system

Sine in/sine out principle tels us that the frequency of a signal dose not change when applying it to a system (the frequency response dose not change the frequency of input signal).

System amplitude- and phase plot tells us how much the system amplifies the input signal, mean while the phase plot tells us how much how much it has been moved.

17.3.2 Tracking properties of closed-loop system

Any signal can be represented as a weighted sum of cosine- and sine signals

$$r(t) = \frac{1}{2\pi}\int R(i\omega)e^{i\omega t}d\omega$$

recall that $G_c(s)|_{s=i\omega} = G_c(i\omega)$ and that

$$Y(i\omega) = G_c(i\omega)R(i\omega) = |G_c(i\omega)|e^{i\arg\{G_c(i\omega)\}}R(i\omega)$$

where ideally the magnitude $|G_c(i\omega)| \approx 1$ and the phase $\arg\{G_c(i\omega)\} = 0$

Performance metric in time domain: quickness: rise time $T_r = t_{90\%} - t_{10\%}$ Damping: overshoot $M = (y_{max} - y_f)/y_f$ Accuracy: static control error $e_f = r_0 - y_f$

Performance metric in frequency domain: quickness: bandwidth ω_B where $|G_c(i\omega_B)| = |G_c(0)|/\sqrt{2}$ Damping: resonance peak level $M_p = max(|G_c(i\omega)|)$ Accuracy: static gain $G_c(0)$

Bode plot

Example bode plot

$$G(s) = \frac{100(s+1)}{s(s^2+6s+100)}$$

Step 1: determining the poles and zeros

$$\text{Zeros}: z_1 = -1$$
$$\text{Poles}: p_0 = 0 \quad \wedge \quad p_1, p_2 = -3 \pm i\sqrt{91} = 10$$

Step 2: sort poles and zeros

	$\lvert P_0 \rvert$		$\lvert Z_1 \rvert$		P_1 and P_2	
ω	0		1		10	
slope		-1		$+1$		-2

Step 3: draw $\log_{10}(G(i\omega))$ starting at the first pole or zero at the origin see figure 17.5

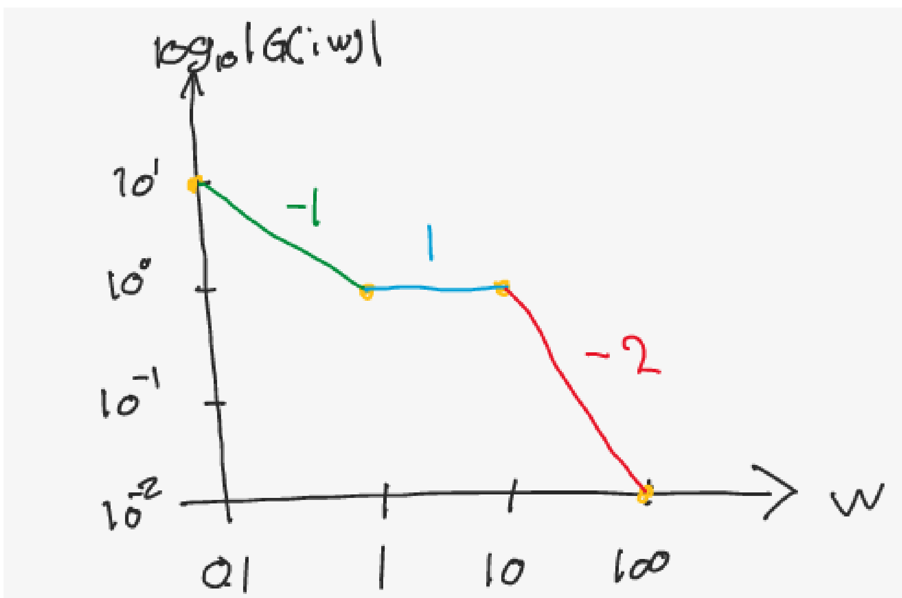


Figure 17.5: example bode plot

17.3.3 Method 2: Design via open-loop system

It is easier to design using Open-loop system instead of closed loop with we did in Method 1.

$$\text{Open-loop: } G_o(s) = F(s)G(s)$$
$$\text{Closed-loop: } G_c(s) = \frac{G_o(s)}{1+G_o(s)}$$
$$G_c(s) \text{ stable } \Leftrightarrow 1{+}G_o(s) \text{ no roots in right half-plane}$$

Nyquist curve

$$G_o(i\omega) = G(i\omega)F(i\omega), \text{ over } 0 \leq \omega < \infty$$

$$|G_c(i\omega)| = \frac{|G_o(i\omega)|}{|1 + G_o(i\omega)|} = \underbrace{\frac{|G_o(i\omega)|}{|G_o(i\omega) - (-1)|}}_{\text{distance to } -1}$$

Nyquist curve $G_o(i\omega)$

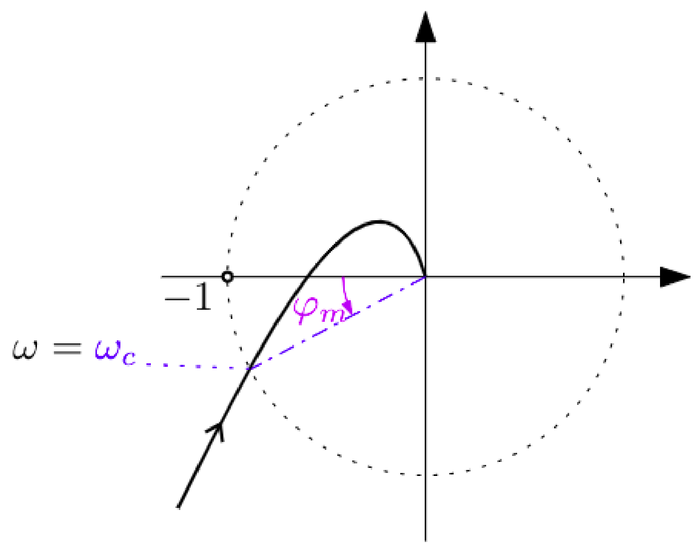


Figure 17.6: Nyquist curve. From ()

Note if the l̈ine is under -1 ẗhen it is stable.

Crossover frequency and **phase margin**

Find $\omega = \omega_c$ where $|G_o(i\omega_c)| = 1$ and then define
$\varphi_m = arg\{G_o(i\omega_c) + 180°$

The Nyquist curve also tells us quickness and dampining

- Crossover frequency $\omega_c \Rightarrow$ bandwidth ω_B of G_c (quickness)
- Phase margin $\varphi_m \Rightarrow$ resonance peak M_p of G_c (damping)

Controller $U(s) = F(s)(R(s) - Y(s))$ with three blocks:

$$F(s) = K \cdot F_{lead}(s)F_{lag}(s)$$

Design recipe:

1. Initialize with $F_{lead}(s)F_{lag}(s) \equiv 1$ (P-control)
2. Find a K that yields stable closed-loop system (e.g. via Nyquist criterion)

3. Adjust $F_{lead}(s)$ and $F_{lag}(s)$ to improve frequency characteristics of $G_o(s)$

Adjust frequency characteristics of open-loop system

$$\begin{aligned}\log|G_o(i\omega)| &= \log|G(i\omega)| + \log K + \log|F_{lead}(i\omega)| + \log|F_{lag}(i\omega)| \\ \arg G_o(i\omega) &= \arg G(i\omega) + 0 + \arg|F_{lead}(i\omega)| + \arg F_{lag}(i\omega)\end{aligned}$$

Nyquist curve

$$G_o(s) = \frac{K(s+10)}{s(s^2+3s+50)}$$

Denominator: $s(s^2+3s+50)$ How many roots not LHP
Step 1

$$s(s^2+3s+50) = 0 \Rightarrow \begin{cases} s = 0 \\ s = -\frac{3}{2} + j\frac{\sqrt{|9|}}{2} \\ s = -\frac{3}{2} - j\frac{\sqrt{|9|}}{2} \end{cases}$$

see image 17.7

A	$\rightarrow$	B	$\rightarrow$	C	$\rightarrow$	D
$s = j\omega$		$s = Re^{j\theta}$		$s = j\omega$		$s = \epsilon e^{j\theta}$
$\omega = 0^+ \rightarrow \infty$		$R \rightarrow +\infty$		$\omega : -\infty \rightarrow 0^-$		$\epsilon \rightarrow 0$

Step 2: Investigate each past in open-loop translate function
Part A: $s = j\omega$ $(\omega : 0^+ \rightarrow +\infty)$

$$\begin{aligned} G_A(s=j\omega) &= \frac{K(j\omega+10)}{j\omega(-w^2+3j\omega+50)} = \frac{K(j\omega+10)}{\omega(-3j\omega+j(50-w^2))} \\ &= \frac{K(j\omega+10)(-3\omega-j(50-\omega^2))}{\omega(-9j\omega^2+j(50-w^2)^2)} = \frac{K(20-\omega^2)}{9\omega^+(50-\omega^2)^2} - j\frac{K(500-7\omega^2)}{\omega(9\omega^2+(50-\omega^2)^2)} \end{aligned}$$

$$\begin{aligned} \omega \rightarrow 0^+ : &\quad G_A(j0^+) = \frac{K20}{50^2} - j\infty \\ \omega \rightarrow +\infty : &\quad G_A(j\infty) = 0 - j0 \end{aligned}$$

Intersection of of $G_A(j\omega)$ with real axis

$$\begin{aligned} Im[G_A(j\omega)] &= 0 && \Rightarrow w^2 - 3 = 0 \\ & && \Rightarrow w = \sqrt{3} \\ Re[G_A(j\sqrt{3})] &= \frac{-K}{18} \end{aligned}$$

Part B: $s = Re^{j\theta}$ $R \rightarrow +\infty$

$$\begin{aligned} G_A(s = Re^{j\theta}) = \frac{K(s+1)}{s^2(s+3)^2} = 0 \\ (R \rightarrow +\infty) \end{aligned}$$

Part C: and past A are asymmetric around real axis
Past D: $s = \epsilon e^{j\theta}$ $\epsilon \rightarrow +\infty$

$$\begin{aligned} G_A(s = \epsilon e^{j\theta}) = \frac{K(s+1)}{s^2(s+3)^2} = \infty \\ (\epsilon \rightarrow 0) \end{aligned}$$

Step 3: see image 17.8
Stable $\Leftrightarrow$ $[-1+j0]$ is not enclosed in Nyquist plot

$$\begin{aligned} &\Leftrightarrow -1 < \frac{-K}{18} \\ &\Rightarrow K < 18 \end{aligned}$$

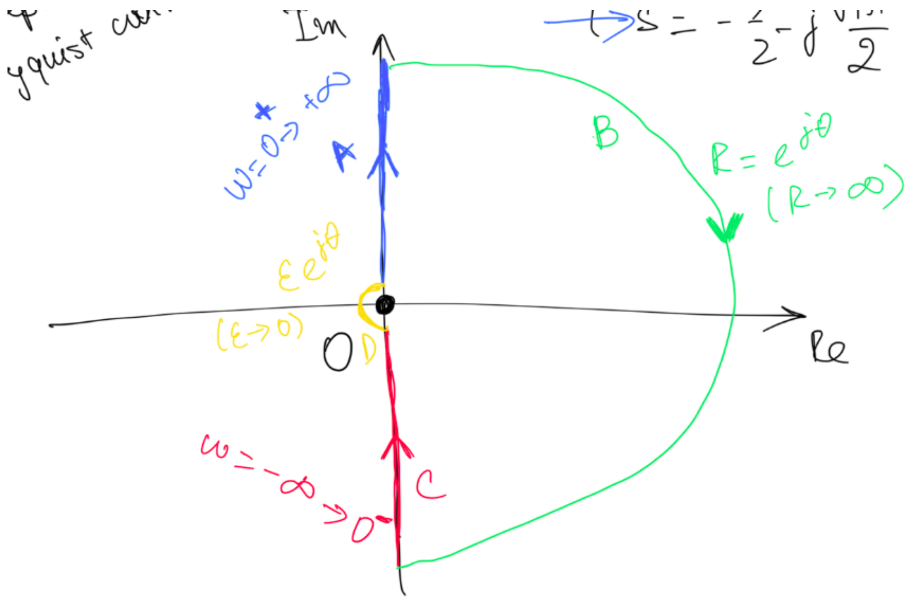


Figure 17.7: Nyquist curve example

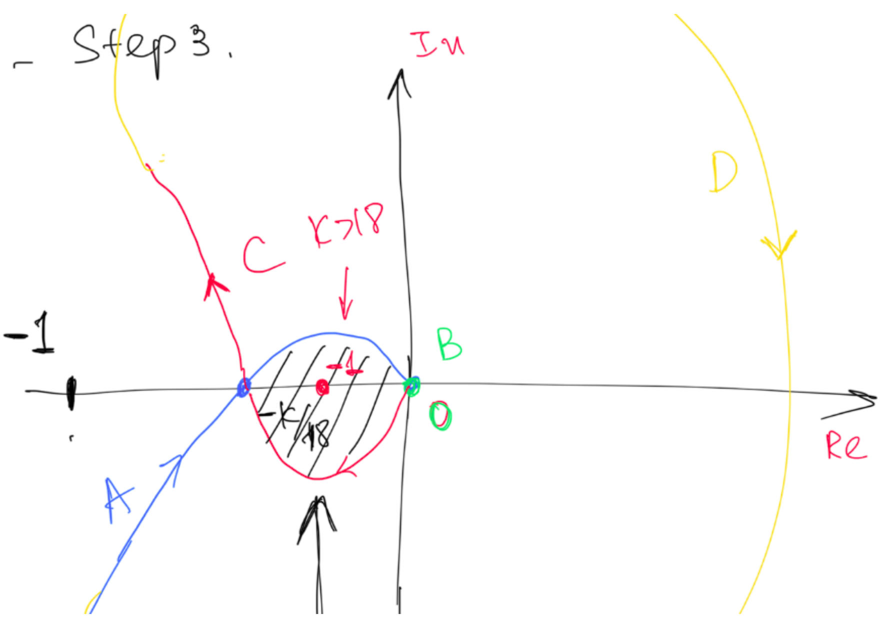


Figure 17.8: Nyquist curve example

The values for K kan be calculated by (-Negative Intersection * K $<$ 1)

Minimum phase systems

Definition: Minimum phase system

Among all systems $G(s)$ with the same magnitude curve, the system with the least negative phase shift is a **Minimum phase system**

Note: A system is minimum phase $\Leftrightarrow$ it has no poles nor zeros in the right half-plane, and contains no time delays.

17.3.4 General linear feedback control

$$U(s) = F_r(s)R(s) - F_y(s)Y(s).$$

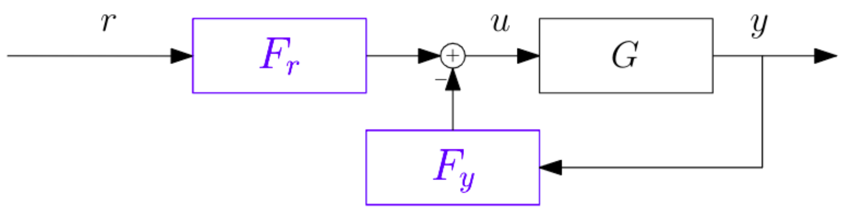


Figure 17.9: General linear feedback control. From ()

$$Y(s) = \frac{G(s)F_r(s)}{1+G(s)F_y(s)}R(s)$$

Note: $F_y(s) = F_r(s) \equiv F(s) \Rightarrow$ simple linear feedback

Feedback with measureable disturbances and intermediate signals

Feedback with measureable disturbances What if we added more sensors?

$$U(s) = F_r(s)R(s) - F_y(s)Y(s) + F_f(s)V(s).$$

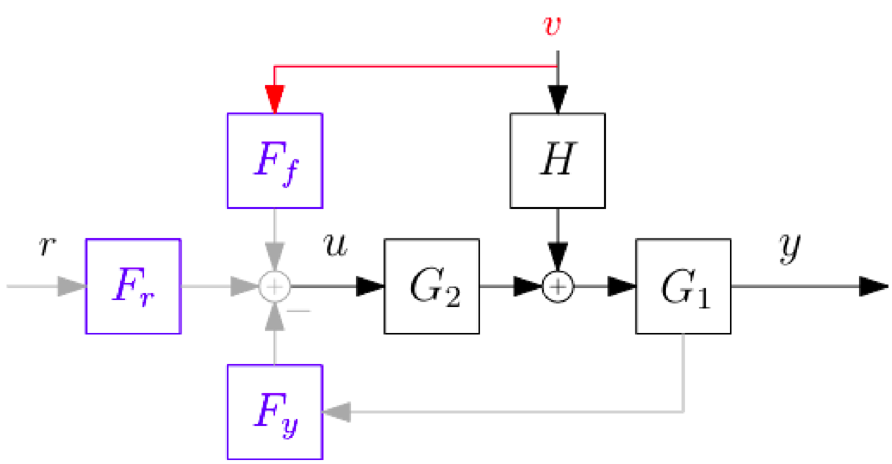


Figure 17.10: General feedback with measurable signals including disturbance. From ()

Feedback with intermediate signals TODO
Combining them TODO

17.4 Design general linear feedback

General feedback design methods include:

1. Pole placement (using observer)
2. ~~Linear quadratic regulation~~
3. ~~Model predictive control~~

17.4.1 System states

Definition: system states

System states at t = summary of its past from which we can determine the output $y(t)$
Transfer function yields

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\} = \int_{\tau=0}^{t} g(t)\,\underbrace{u(t-\tau)}_{\text{past input}}\,d\tau$$

$$\begin{aligned} y(t) &= c_1 x_t(t) + c_2 x_2(t) + \ldots + c_n x_n(t) + Du(t) \\ &= Cx(t) + Du(t) \end{aligned}$$

where

$$C = \begin{bmatrix} c_1 & \ldots & c_n \end{bmatrix} \text{ and } x(t) \triangleq \begin{bmatrix} x_1(t) \\ \vdots \\ x_n \end{bmatrix}$$

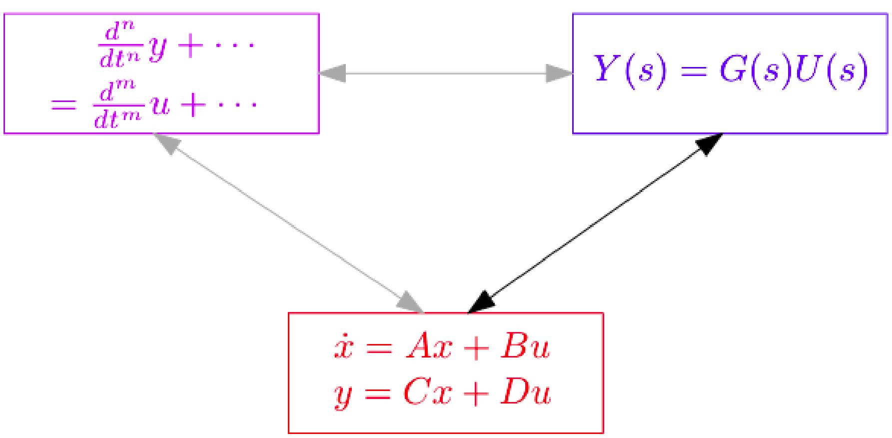


Figure 17.11: Translate LTI system models. From ()

$$G(s) = \underbrace{C}_{1\times n}\underbrace{(sI-A)^{-1}}_{n\times n}\underbrace{B}_{n\times n} + \underbrace{D}_{1\times 1} = \frac{b(s)}{a(s)}$$

Important property

- System matrix A:s eigenvalues $\{\lambda\}$ given by solution to $\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \ldots a_n = 0$
- $a(s) = \det(sI - A)$ is a polynomial of order n
- $G(s)$:s poles $p_i \subseteq A$:s eigenvalues λ_j

Given transfer function

$$G(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

one can choose e.g. controllable canonical form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 - a_1 b_0 & b_2 - a_2 b_0 & \cdots & b_n - a_n b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Figure 17.12: State-space form ← transfer function. From ()

Solution to state-space equation

$$x(t) = e^{At} x_0 + \int_0^t e^{A\tau} B u(t-\tau) d\tau$$

matrix exponential e^{At} defined as

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$$

- States $x(t)$ at t summarize all past inputs $u(t-\tau)$
- State space formulation easily incorporates initial condition
- Determine output $y(t) = Cx(t) + Du(t)$

Asymptotically stable if

$$u(t) \equiv 0 \Rightarrow \lim_{t\to\infty} x(t) = 0$$

stability

- A:s eigenvalues are strictly in left half-plane ⇔ system is asymptotically stable.
- A:s eigenvalues are strictly in left half-plane ⇒ system is input-output stable.

Example: State space

TODO Find transfer function F8 s14

17.5 Nonlinear time-invariant system

Feedback control using states

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \Leftrightarrow \quad Y(s) = G(s)U(s)$$

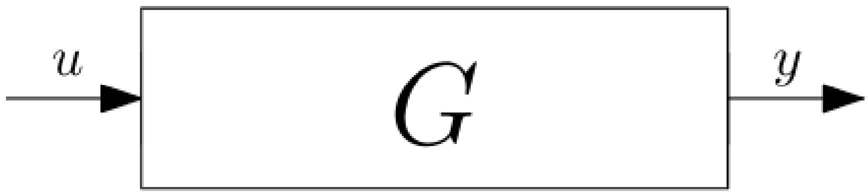


Figure 17.13: State-feedback control. From ()

State space description:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \qquad \Leftrightarrow \quad G(s) = C(sI - A)^{-1}B$$

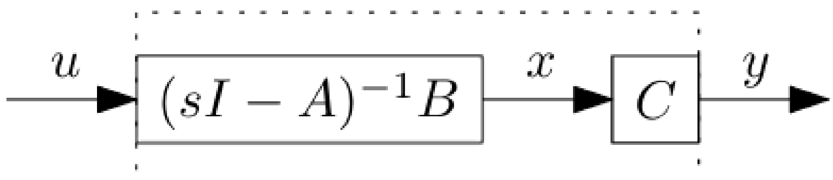


Figure 17.14: State-feedback control. From ()

$$u = -Lx + l_0 r$$

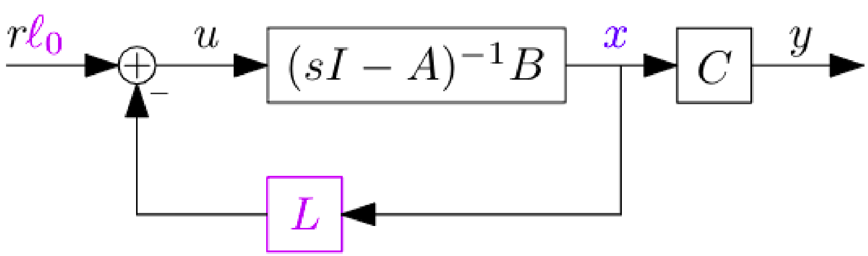


Figure 17.15: State-feedback control. From ()

state-space equation:

$$\dot{x} = Ax + B(-Lx + l_0 r) = (A - BL)x + Bl_0 r$$

The resulting open loop system

$$\begin{aligned}\dot{x} &= (A - BL)x + Bl_0 r \\ y &= Cx\end{aligned}$$

Controllabel?

A sought state x^* is *controllable* if we can move the system from $x(0) = 0$ to $x(T) = x^*$

$$\begin{aligned}x(T) &= e^{At}x_0 + \int_0^T e^{A\tau}Bu(T-\tau)d\tau \\ &= B\gamma_0 + AB\gamma_1 + \ldots + A^{n-1}B\gamma_{n-1}\end{aligned}$$

$$S \triangleq [B\ AB\ \ldots\ A^{n-1}B]$$

All states x^* are controllable S:s columns are linearly independent. $rank(S) = n$ or $\det(S) \neq 0$

$$\begin{aligned}y(t) &= Cx(t) \\ &= Ce^{At}x^* + 0\end{aligned}$$

$$Cx^* = 0,\ CAx^*, \ldots, CA^{n-1} = 0$$

$$O \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observable?

All states x^* are *observable* $\Leftarrow$ O:s columns are linearly independent

Example: controllable

Is the following system controllable?

$$\begin{aligned}x(t) &= \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)\end{aligned}$$

$$\begin{aligned}\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} &= [n = 2] = \begin{bmatrix} C \\ CA \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \end{bmatrix}}_{2\times 2} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\ \Rightarrow \det(\mathcal{O}) &= \det\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right) = (1)(-1) - (1)(-1) = 0\end{aligned}$$

Therefore the system is not controllable

17.5.1 Minimal realization of state-space form

State-space form of $G(s)$ is a *minimal realization* if vector x has the smallest possible dimension.

A state-space form is *minimal realization* $\Leftrightarrow$ controllable and observable $\Leftrightarrow$ A:s eigenvalues $=$ $G(s)$:s poles

17.5.2 Method: Design of state-feedback control

State-space model with controller $u = -Lx + l_0 r$ where

$$L = \begin{bmatrix} l_1 & l_2 & \dots & l_n \end{bmatrix}$$

The closed loop system is expressed as

$$\begin{aligned} \dot{x} &= (A - BL)x + Bl_0 r \\ y &= Cx \end{aligned}$$

The closed loop system as a transfer function

$$G_c(s) = C(sI - A + BL)^{-1} Bl_0$$

Calculations of eigenvalues/poles

$$\det(sI - A + BL) = 0$$

$$l_0 = \frac{1}{C(-A + BL)^{-1} B}$$

State-space form is controllable $\Leftrightarrow$ L can be designed to yield arbitrarily placed poles (real and complex-conjugated) of the closed-loop system

17.5.3 Estimation via simulation

Controller

$$u = -L\hat{x} + l_0 r$$

where $\hat{x}$ is an estimate of the actual state x that is unknown

we estimate via simulating the states

$$\dot{\hat{x}} = A\hat{x} + Bu, \quad \hat{x}(0) = \hat{x}_0$$

where $\hat{x}_0$ is an initial guess.

Estimation vi observer

Instead of simply guessing what $\hat{x}$ is. We use a so called observer

$$\dot{\hat{x}} = A\hat{x} + Bu + \underbrace{K(y - C\hat{x})}_{correction}, \quad \hat{x}(0) = \hat{x}_0$$

where K is a matrix

$$K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$$

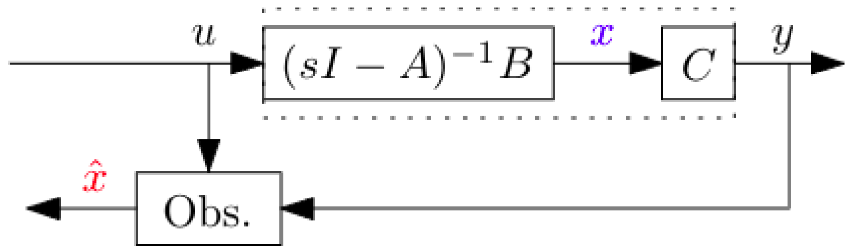


Figure 17.16: Estimating the states with observer. From ()

Estimation error

Definition: Estimation error

Estimation error:

$$\tilde{x} \triangleq x - \hat{x}$$

Errors of observer described as system

$$\tilde{x}(t) = e^{(A-KC)t}\tilde{x}(0)$$

and therefore $||\tilde{x}(t)||$ decays at a rate given by maximum

$$Re\{\tilde{s}_i\}$$

where $\tilde{s}_i$ are observer poles/eigenvalues of $(A-KC)$

State-space form is observable ($\det \mathcal{O} \neq 0$) $\Rightarrow$ matrix K can be chosen such that $\tilde{x}$ vanish arbitrarily quick.

K is solved by polynomial $\det(sI - A + KC) = 0$ with desired roots in left halfplane

Controller using estimated states

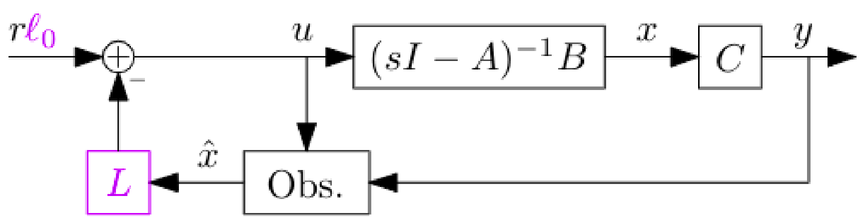


Figure 17.17: Estimating the states with observer. From ()

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \begin{cases} u = -L\hat{x} + l_0 r \\ \dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}) \end{cases}$$

$$L : \begin{cases} U(s) = -L\hat{X}(s) + l_0 R(s) \\ s\hat{X}(s) = A\hat{X}(s) + BU(s) + K(Y(s) - C\hat{X}(s)) \end{cases}$$

Closed-loop system with estimated states

Close loop system

$$\begin{aligned} \dot{x} &= (A - BL)x + \overbrace{BL\tilde{x}}^{\text{effect of estimation error}} + Bl_0 r \\ y &= Cx \end{aligned}$$

The closed-loop system with estimation error can be written as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} &= \underbrace{\begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} l_0 \, r \\ y &= \underbrace{\begin{bmatrix} C 0 \end{bmatrix}}_{\tilde{C}} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \end{aligned}$$

17.6 Sensitivity and robustness

Sensitivity function:

$$S(s) \triangleq \frac{1}{1 + G_o(s)}$$

Complementary sensitivity function:

$$T(s) \triangleq 1 - S(s) = \frac{G_o(s)}{1 + G_o(s)}$$

$$S(s) + T(s) \equiv 1 \quad \forall 1 \tag{17.1}$$

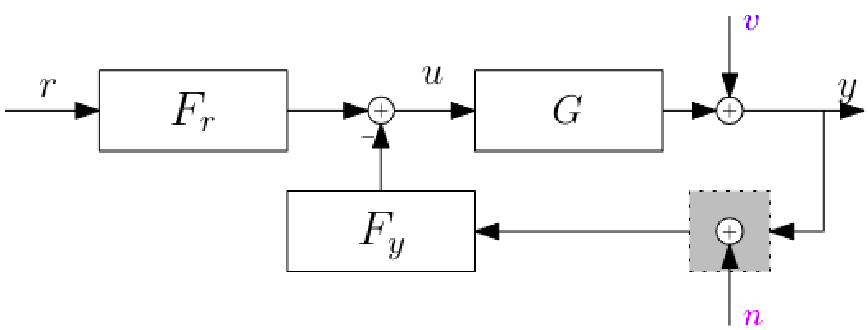


Figure 17.18: Control system with disturbances and noise. From ()

$$Y(s) = G_c(s)R(s) + S(s)V(s) + T(s)N(s)$$

where $V(s)$ is the disturbance from external factors on the system. For example a wind or other forces. $N(s)$ is the noise witch can be the for example from an inaccurate sensor or the sensor not coping with the quick changes.

We can either minimize the disturbance or the noise since $S(s)$ and $T(s)$ are connected see equation 17.1. Often the noise is high frequency and disturbance is lower frequency for example if somone is pusing the system the that is a log

Ideally both $|S(i\omega)|$ and $T(i\omega) \gg 1$

$$|S(i\omega)| + |T(i\omega)| \geq |S(i\omega) + T(i\omega)| \equiv 1$$

In addition we want Nyquist contour

$$G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}$$

17.6.1 Robustness: model error and stability

Assume

1. Controller $F(s)$ stability the assumed system $G(s)$
2. $G(s)$ and $G^0(s)$ have same number of poles in right half-plane.
3. Open-loop: $F(s)G(s) \to 0$ and $F(s)G^0(s) \to 0$ where $|s| \to \infty$

Robustness criterium If assumptions are valid and $T(i\omega)$ fulfills

$$|T(i\omega)| < \frac{1}{|\delta G(i\omega)|}, \quad -\infty \leq \omega \leq \infty$$

$\Rightarrow$ the real closed-loop system $G_c^0(s)$ is also stable!

17.7 Digital Controllers

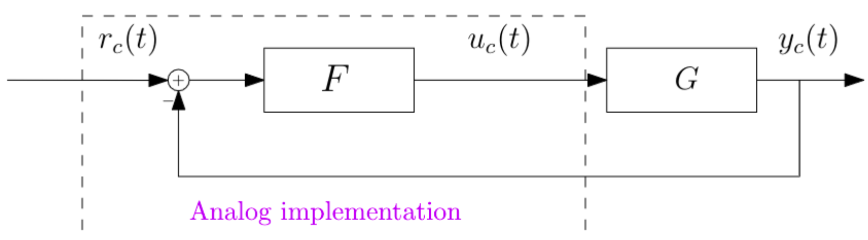


Figure 17.19: Digital systems and discrete-time clocks. From ()

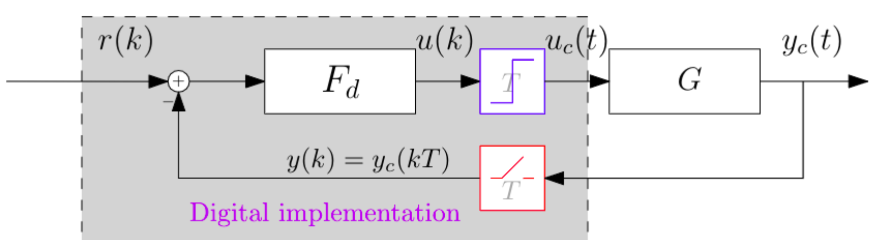


Figure 17.20: Digital systems and discrete-time clocks. From ()

17.7.1 Sampling continuos-time output

A/D: Sampling feedback signal $y(t) = y_c(kT)$

Nyquist sampling theorem: If $y_c(t)$ is bandlimited to $\pm\omega_B = \pm 2\pi f_b \Rightarrow$ it can be recovered exactly from $y_c(kT)$ when sampling frequency

$$f_s \equiv \frac{1}{T} > 2f_B$$

17.7.2 Generating continuous-time input

Zero-order hold: continuos-time input is piecewise constant for $k = 0, 1, 2, \ldots$

$$u_c(t) = u_c(kT), kT \le t < (k+1)T$$

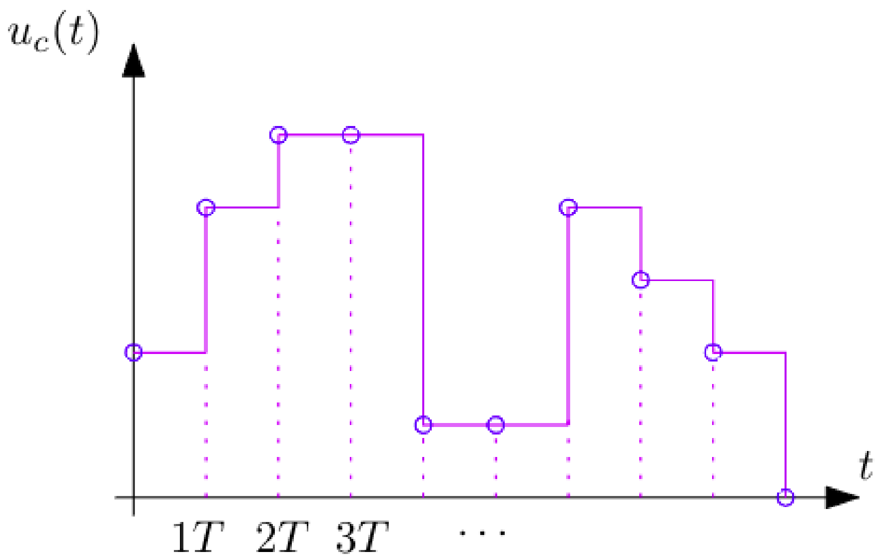


Figure 17.21: zero-order hold. From ()

State evolution when using zero-order hold input
State $x_c(t)$ at time $(k+1)T$ is determined at time kT as:

$$x(k+1) = Fx(k) + Gu(k)$$
$$y(k) = Hx(k)$$

which are discrete-time states using matrices

$$F = e^{AT}$$
$$G = \int_{\tau=0}^{T} e^{AT} d\tau B = [\text{if } A^{-1} \text{ exists}] = A^{-1}(e^{AT} - I)B$$
$$H = C$$

If eigenvalues of A: $Re(s) < 0 \Rightarrow$ stable
If eigenvalues of F: $|p| < 1 \Rightarrow$ stable

17.7.3 Discrete-time controllers

PID controller

$$u(k) = K_p e(k) + \underbrace{K_i T \sum_{n=0}^{k} e(n)}_{\simeq \int_{\tau=0}^{t} e_c(\tau) d\tau} + \underbrace{K_d \frac{e(k) - e(k-1)}{T}}_{\frac{de_c(t)}{dt}}$$

17.8 Final notes before exam

Remember:

- $G_0 = FG$ (useful when calculating the nyquist marginaly stable)
- poles of system in state space for is $\det sI - A$
- $Y(s) = G(s)U(s)$ logicaly
- $Y(s) = G_c(s)R(s)$ logicaly
- $G(s) = C(sI - A)^{-1}B$

Thing that were missing from the exam:

- Forward- och cascade control (yes we have)
- Very little final value theorem
- What the diffrent PID parts does
- Roths algorithm
- Is the preformance spesifications met look at step response
- witch is the minimum phase
- general feedback control. is logical
- what is F_r and F_y. use the formulas writen on exams. L9,p18

Other info:

- (L3, P:) The system static gain is therefore $G(0)$
- (L3, P17) Stationary error e_f approaches 0 if $G(0)F(0) = \infty$. (E.g. when F(s) contains 1 s, i.e. integration.)
- (L5, P19) $G(s) = K_0 \frac{\dots}{\dots}$
- (L6, P9) lead block lag block, formula is given in formula sheet
- (L6, P13) Among all systems G(s) with the same magnitude curve, the system with the least negative phase shift is a minimum phase system.
- (L8, P8) the close loop representation in state space form
- (L8, P21) G_c and l_0
- (L10, P6) Ideally S and T should be less then 1 but it is imposible since $|S(i\omega)| + |T(i\omega)| \geq 1$
- (L10, P9) Nyquist contour $G_o(i\omega) = F_y(i\omega)G(i\omega) = \frac{T(i\omega)}{S(i\omega)}$
- (L10, P11) Relative model error
- (L10, P14) $|T(i\omega)| < \frac{1}{|\Delta G(i\omega)|}$ image

Chapter 18

Introduction to Machine Learning

Resorcess

- `http://user.it.uu.se/~justin/Hugo/courses/machinelearning/`
- `https://scikit-learn.org/stable/`
- `https://numpy.org/`
- `https://pandas.pydata.org/`
- `https://matplotlib.org/`

18.1 Introduction

Overfitting is the property of a model such that the model predicts very well labels of the examples used during training but frequently makes errors when applied to examples that weren't seen by the learning algorithm during training. **Over fitting**: To high degree of model with to few data. **Bias**: The data that has been selected may not be a true representation of the true data set.

How many samples should we use? We know how many unknown parameters we have, then can we should have more data points then parameters at a minimum.

18.1.1 Types of Learning

- **Supervised Learning**:
  - Desc: You are given labelled data. For each data-point you know what the correct prediction should be.
  - Math Model: Labeled examples $\{(x_i, y_i)\}_{i=1}^{N}$ were each element x_i among N is called a **feature vector**.
  - Goal: The goal of a **supervised learning algorithm** is to use the dataset to produce a model that takes a feature vector xas input and outputs information that allows deducing the label for this feature vector.
- **Unsupervised Learning**:
  - Desc: You just have data which is not labelled. This is given to algorithm. The most common algorithms do some sort of clustering, data-points that are similar are grouped together.
  - Math Model: unlabeled examples $\{x_i\}_{i=1}^{N}$ where x is the feature vector
  - Goal: The goal of an **unsupervised learning algorithm** is to create a model that takes a feature vector x as input and either transforms it into another vector or into a value that can be used to solve a practical problem
- **Semi-Supervised Learning**:
  - Desc: both labeled and unlabeled examples
  - Goal: The goal of a **semi-supervised learning algorithm** is the same as the goal of the supervised learning algorithm
- **Reinforcement Learning**
  - Desc: Different actions bring different rewards and could also move the machine to another state of the environment
  - Goal: The goal of a **reinforcement learning algorithm** is to learn a **policy**

18.1.2 Notation and Definitions

- **Unbiased Estimators**:
  $\mathbb{E}[\hat{\theta}(S_X)] = \theta$
  $S_X = {x_i}_i^N = 1$, $\hat{\theta}$ is a sample static and θ is the real statistic.
- **Bayes' Rule**:
  $Pr(X = x|Y = y) = \frac{Pr(Y=y|X=x)Pr(X=x)}{Pr(Y=y)}$
- **Parameter Estimation**:
  Is a algorithm that can predict the parameter given there probability's?.

- **Parameters vs. Hyperparameters**:
A Hyperparameter, set of numerical valued parameters with is decided by the data analysis. Parameters are variables that define the model learned by the learning algorithm. Examples of hyperparameter is C logistic regression witch is the penalty strength.

- **Classification vs. Regression**:
  - **Classification**: Each data point should be put into one of a finite number of classes. For example email should be classified as Spam or Ham. Pictures should be classified into pictures of cats, dogs, or sleeping students.
  - **Regression**: Given the data the required prediction is some value. For example predicting house prices from the location and the size of the house

- **Model-Based vs. Instance-Based Learning**:
Model-Based create a model after training with data witch then can be discarded unlike Instance-Based Learning witch takes the data closes the the given inputs and predicts after that.

- **Shallow vs. Deep Learning** Shallow algorithm learns creates the parameters learned only on the training examples.

- **Categorical**
A finite set of categoric with the data is a subset of. For example sex (Man or Woman) is a categorical value and wight is not.

- **Scaling**
if one feature have a more dominant number (larger) then you can scale (multiple with some number) the smaller number so they become similar

- **Probabilistic Classification**
Try to predicate the probability that an put sample x belongs to a class

- **Odds ratio**
Given an event with probability p we can take the odds ratio of p happening and p not happening

$$\frac{p}{1-p}$$

- **Multiclass classification**
  - one vs all or one vs rest
Is when you set the one class to 1 and the rest to 0 for each of the classes. This will however result in uneven amount of positive and negative examples, thus preform badly
  - one vs one
Is when you compare two classes at a time

Hypothesises

$$h_{\theta_0,\theta_1(x)} = \theta_0 + \theta_1 x$$

where θ_0 and θ_1 are two weights/parameters.

Measuring Error - RMS

A way to measure the error.

$$J(\theta_0, \theta_1, x, y) = \frac{1}{2m}\sum_{i=1}^{m}(h_{\theta_0,\theta_1}(x^i) - y^i)^2$$

Also knowns ass loss error function.

Minimising regret, Minimising root mean square error, minimising classification error.

Parameters of algorithm are the things that learned from the data. In neural networks you would call this the weight space.

Training and Test Sets

The data should be dived into two parts

Training Sets This is the data you use to find the best parameters of the model or hypothesis. Machine learning can be seen as an optimisation problem find the parameters that best explain the data under some error/cost or loss function.

Test Sets The test set is what you use to validate your model. You are interested in the error/cost on this set.

Confusion matrices

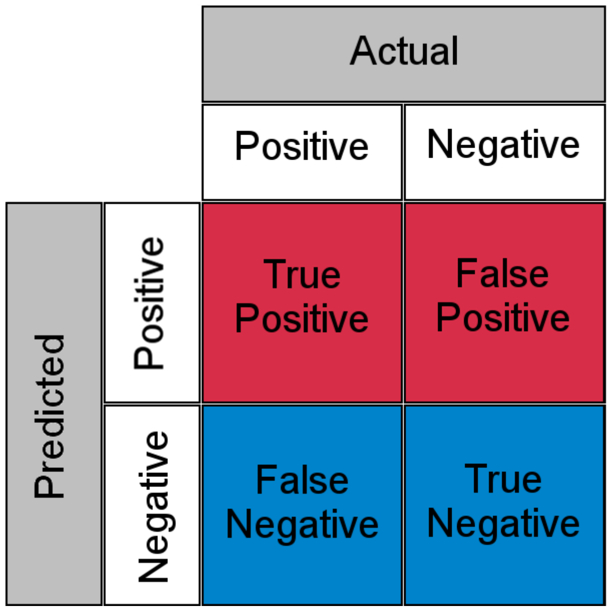


Figure 18.1: Actual prediction matrix. From ()

Accuracy

$$\frac{TP + TN}{TP + TN + TF + FN}$$

Precision

$$\frac{TP}{TP + FP}$$

18.2 Linear Regression as Machine Learning

Properties

- Supervised learning algorithm
- Numerical variables
- Regression problem

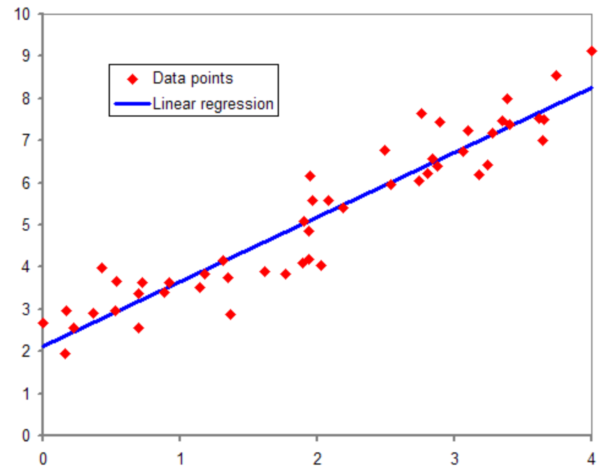


Figure 18.2: Linear regressionFrom ()

Explenation

With the linear hypothesis

$$h_{\theta_0,\theta_1(x)} = \theta_0 + \theta_1 x$$

we want the parameters θ_0 and θ_1 to give us as small of a error as possible

the error is calculated with **Cost function**

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} (h_{\theta_0,\theta_1}(x^i) - y^i)^2$$

Where the goal is to **minimize** $J(\theta_0, \theta_1)$.

18.2.1 gradient descent

Gradient descent can be used for many thing, not so often use for linear regression. Essentialy what is dose it take a 360 degree view and look at what is the fastest way to get down the slope. It will repete one step at a time untill it is at a local minimum.

$$\frac{\partial}{\partial \theta_j} J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)} - y^{(i)}) x_j^{(i)} \right)$$

Note that $x_0 = 1$

Linear Regression with Gradient Descent

$$\theta_0 \leftarrow \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^{m} (\theta_0 + \theta_1(x^{(i)}) - y^{(i)})$$

$$\theta_1 \leftarrow \theta_1 - \alpha \frac{1}{m} \sum_{i=1}^{m} (\theta_0 + \theta_1(x^{(i)}) - y^{(i)})x^{(i)}$$

where α is how aggressive the change should be and $\frac{\partial}{\partial \theta_j}$ is the angle at the current point, this tells us were we should go and how big of a step.

$\leftarrow$ and $:=$ is the same thing

If J is minimized then linear regression *always* converges to a global minimum not just a local minimum. The value of the global minimum dose not need to be 0 (a line docent need to have a derivative with is 0).

Example Discriptive example

18.3 Probability and Naive Bayes Classification

Properties

- Supervised learning algorithm
- Categorical variables
- Classification problem

By calculating the probability we can decide witch class it belongs to by seeing witch has the biggest probability.

Bayes theorem

$$P(H|E) = \frac{P(H)P(E|H)}{P(E)} = \frac{P(H)P(E|H)}{P(H)P(E|H) + P(\neg H)P(E|\neg H)}$$

where H is the hypothesis and E is the evidence. $P(H)$ is called the prior, $P(E|H)$ is called the likelihood and $P(H|E)$ is the posterior.

for several cases one can express as followed:

$$P(A|b_1, b_2, ..., b_n) = \frac{P(b_1|A)P(b_2|A)...P(b_n|A)P(A)}{P(b_1)P(b_2)...P(b_n)}$$

- Experiment: is an outcome of several possible outcomes
- Sample space: the set of all possible outcomes
- Event: A subset of a sample space

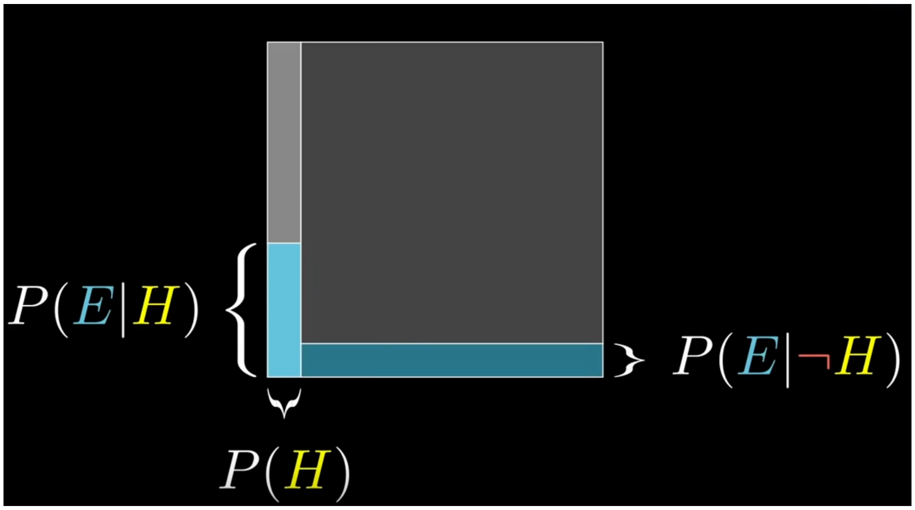


Figure 18.3: Bayes Theorem. From ()

18.4 Logistic Regression

Properties

- Supervised learning algorithm
- Numerical and categorical variables
- Classification algorithm
- binary classification problems

Linear regression is not good at classification problems, since not all classes can be linear separable. Logistic regression is used for classification better than linear regression. Logistic regression explanation

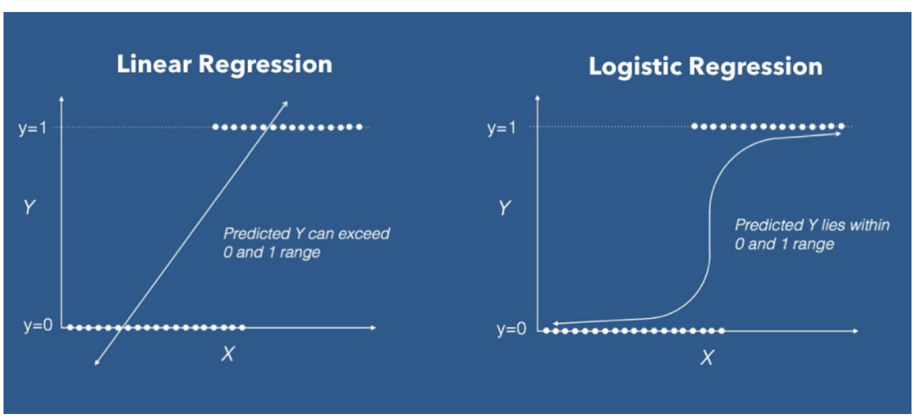


Figure 18.4: Linear vs logistic regression. From ()

The prediction will be for value between 0 and 1 so it is best for binary classifications. Sigmoid function is the same as logistic function.

$$0 \leq h_\theta(x) \leq 1$$

$$
\begin{aligned}
h_\theta(x) &= g(\theta^T x) \\
g(z) &= \frac{1}{1+e^{-z}} \\
&\Rightarrow h_\theta(x) = \frac{1}{1+e^{-\theta^T x}}
\end{aligned}
$$

$$h_\theta = g(\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2 + \ldots)$$

18.4.1 Cost function

$$
\begin{aligned}
J(\theta) &= \frac{1}{m}\sum_{i=1}^{m} Cost(h_\theta x^{(i)}, y^{(i)}) \\
&= -\frac{1}{m}\left[\sum_{i=1}^{m} y^{(i)} \log h_\theta(x^{(i)}) + (1-y^{(i)}) \log(1-h_\theta(x^{(i)}))\right]
\end{aligned}
$$

18.4.2 Gradient dissent

$$\theta_j := \theta_j - \alpha \sum_{i=1}^{m} (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)})$$

$$J(\theta, x, y) = \frac{1}{2m}\sum_{i=1}^{m}\left(\sigma\left(\theta^T x^{(i)} - y^{(i)}\right)\right)^2$$

Multiclass classification problem. We run classification problem for each class we do logistic regression for the class and not class. So if we have tree classes we would have to do tree logistic regression hypothesis. Then we can combine them to create a single hypothesis

Cost function with regularized

Underfitting is when we use to low of degree of polynomial for the classification. Overfitting is when the model is to fitted and specific training and therefore preforms badly on real data, i.e. to large degree.

We can manually look at the data set and select certain feature. We can also use a model to automatically do this with regularization.

If we set λ to a large number the parameter will automatically be small and therefore higher degrees will not effect the output as much. The legalization parameter λ can automatically be determine `https://www.youtube.com/watch?v=IXPgm1e0IOo`

$$J(\theta) := \frac{1}{m}\left[\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^{n} \theta_j^2\right]$$

Gradient descent with regularization

$$\theta_j := \theta_j \left(1 - \alpha\frac{\lambda}{m}\right) - \alpha\frac{1}{m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}\right)$$

18.5 Support Vector Machines

Properties

- Supervised learning algorithm
- Numerical variables
- classification and regression problems

SVM explained

Support Vector machines are models where tho goal is to find a hyperplane (the line for the two) with a margin either side that maximizes the space between the two clusters. The labels defining what is in the cluster and not is changed to -1 and 1.

We want to find weights $\bar{w} \in \mathbb{R}^d$ and a constant b such that

$$\begin{cases} \bar{w} \cdot \bar{x} - b \geq 1 & \text{if } y = 1 \\ \bar{w} \cdot \bar{x} - b \leq 1 & \text{if } y = -1 \end{cases}$$

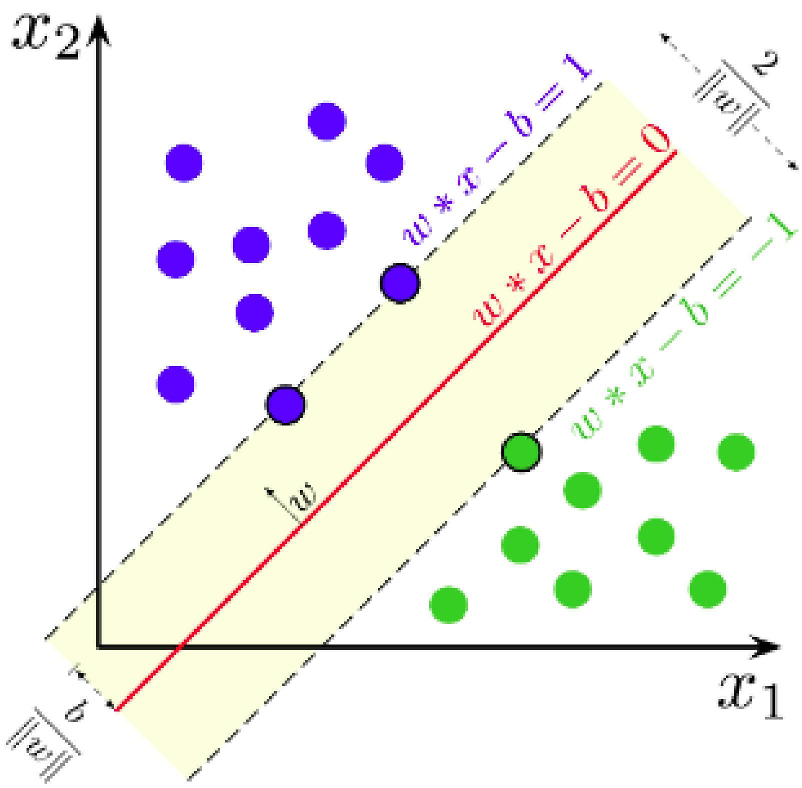


Figure 18.5: Bayes Theorem. From ()

Since the vectors are octagonal to each other the product is 0

$$\bar{w} \cdot \bar{x} = 0$$

If we want the vector $\bar{w}$ to be normal to the hyperplane

$$\bar{w} \cdot \bar{x} = b$$
$$\Rightarrow \bar{w} \cdot \bar{x} - b = 0$$

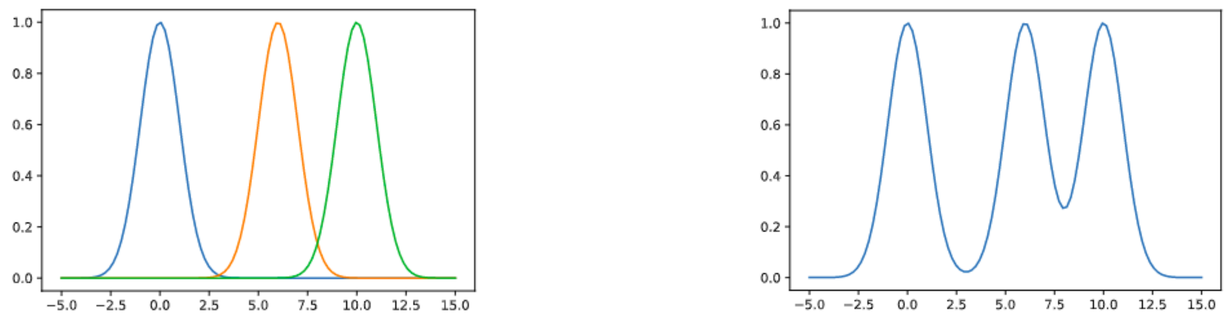


Figure 18.6: Gaussian Kernels. From ()

Hence the two points are $\bar{x}_1$ where $\bar{w} \cdot \bar{x}_1 - b = -1$ and $\bar{x}_2$ where $\bar{w} \cdot \bar{x}_2 - b = 1$

Maximisning the distance between the two hyperplanes $\bar{w}x - b = 1$ and $\bar{w}x - b = -1$ we want to maximize $t = \frac{2}{||\bar{w}||}$ so we minimize $\frac{1}{2}||\bar{w}||$

18.5.1 Quadratic programming

Graidient descent will not work for all cases if there are a lot of quadratic terms but the problem is convex then we can use quadratic programming.

18.5.2 Slack Variables

when there is overlap Quadratic programming will not work

18.5.3 kernel trick

We can compute the inner product in the high-dimensional space by using a function on the lower dimensional vectors, i.e. The kernel trick is a way to lower the demission to avoid componential expenses

18.5.4 Gaussian Kernels

18.6 Some feature engineering and Cross validation

- Model Bias
  A model is baiased if the error/loss/cost function is high on the training data. It makes bad perdition on the training data.

- Model Variance
  A model has high variance if the error/loss/cost function is high on the test data compared with the training data.

Bias variance trade off
Overfitting vs Bias
Two Goals

- Model Selection:
  Estimating the performance of different models in order to chose the best one

- Model assessment:
  Having chosen a final model, estimate its prediction error on new data.

- Training
  This is what we use to train our different algorithms. Typical split 50%

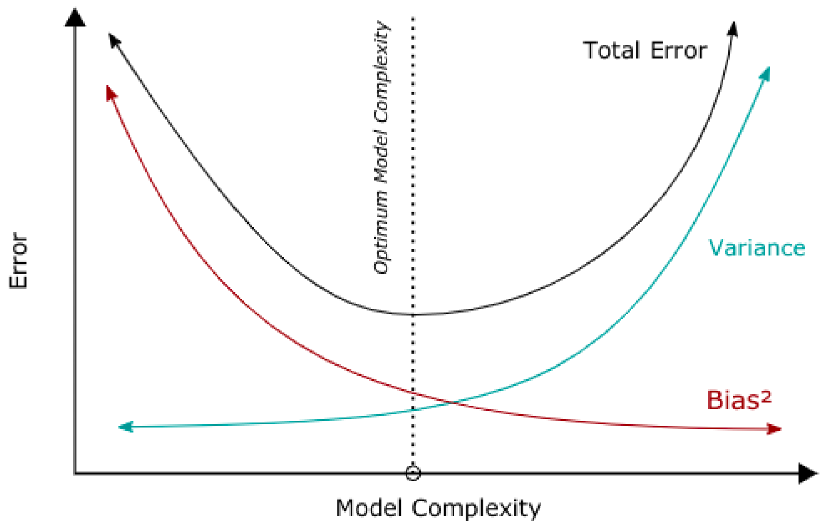


Figure 18.7: Bias variance trade off. From ()

- Validation
  This is what we use to choose our model we pick the model with the best validation score. Typical split 25%
- Test
  This is the data that you keep back until you have picked a model. You use this predict how well you model will do one real data. Typical split 25%

18.6.1 k-fold cross validation

It is used to evaluate with model is best.

If $k = 5$, then you have 5 parts $T_1, \ldots, T_5$ you would run 5 training runs

- Train on T_1, T_2, T_3, T_4 evaluate on T_5.
- Train on T_1, T_2, T_3, T_5 evaluate on T_4.
- Train on T_1, T_2, T_4, T_5 evaluate on T_3.
- Train on T_1, T_3, T_4, T_5 evaluate on T_2.
- Train on T_2, T_3, T_4, T_5 evaluate on T_1.

Good values of k are 5 or 10. Obviously the larger k is the more time it take to run the experiments.

Hyper-parameters

These are parameters to the learning algorithm that do not depend on the data. They are often continuos values such as the regularization parameter, but not allowance.

18.6.2 Estimating Hyper parameters - Grid search

We make divide our search space of the hyper parameters into a grid (evenly distributed possible values). Then we go through them all and find the parameters with minimize the error. Explanation

18.6.3 One-Hot encoding

Instead of classifying with natural number like 0, 1, 2 we only use 0 and 1. We use 1 for a class and 0 for the other classes.

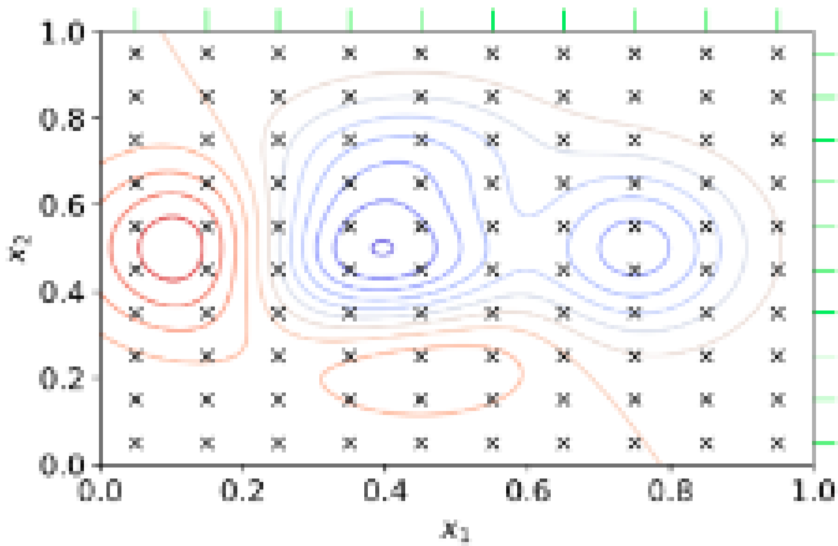


Figure 18.8: Grid search. From ()

18.6.4 Boosting for feature selection of linear models

Boosting is a general framework, and it can also be combined with cross-validation in a technique called bagging (bootstrap aggregating). The idea is very simple, learn you model one feature at time, at each stage pick the next feature that gives you optimal performance. You order the features in order of importance and this gives you models that are easier to interpret for humans.

Don't forget to scale your data, so that all dimensions have roughly the same range

Boosting for linear models advantages

- You order the variables in terms of importance
- There is the possibility to stop early when the model does not improve. This is a way of selecting a subset of the features

18.6.5 Co-variance matrix

Theremins the covariance between the features in a matrix. The larges eigen-value tell us the direction to project in order to maximize the variance in that dimension. `https://www.youtube.com/watch?v=152tSYtiQbw` `https://en.wikipedia.org/wiki/Covariance_matrix`

18.6.6 Correlation matrix

How two features are moving with one another. Note that the correlation matrix is diagonally symmetric.

18.6.7 Heatmap

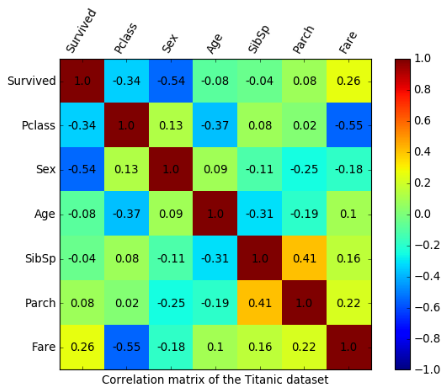
This can be done with hierarchical clustering.

18.7 Clustering and classifiers

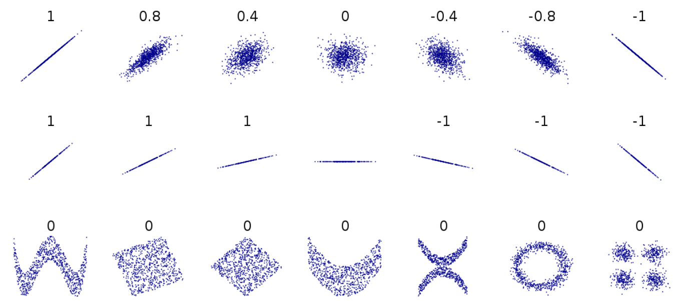
18.7.1 k-nearest neighbor classifier

Properties

- Un-supervised learning algorithm



(a) Correlation matrix. From ()



(b) Correlation. From ()

Figure 18.9: Correlation matrix, the two images are un related

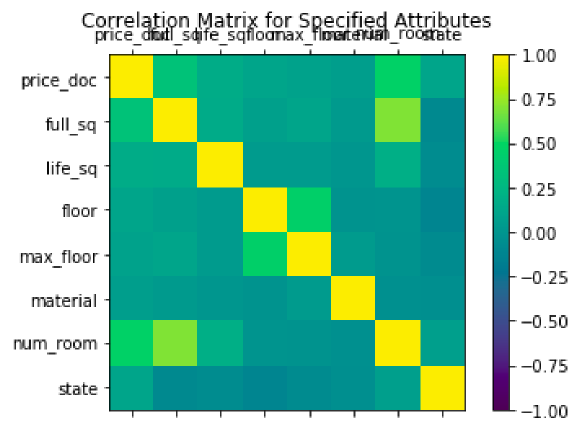


Figure 18.10: Heatmap. From ()

- Numerical variables

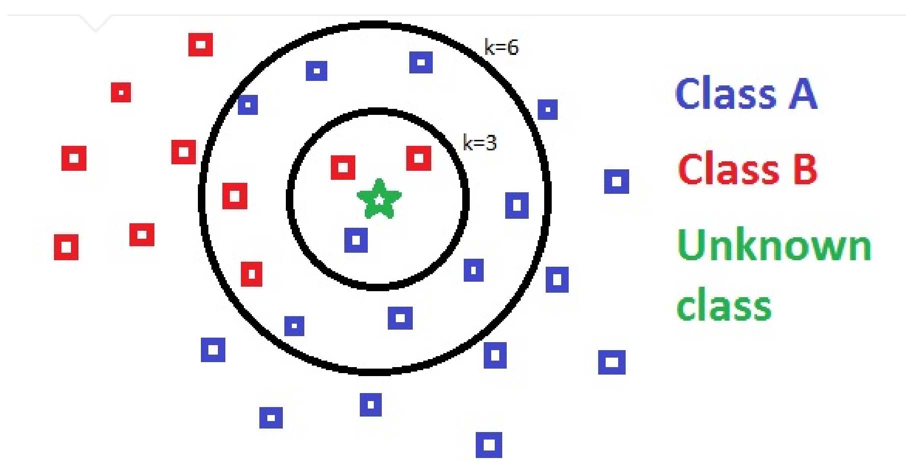


Figure 18.11: K-nn. From ()

Otherwise known as *k-NN*

- Very simple classifier
- Memory based, no model is learned you just have to remember the training data
- To classify a point, look at the k -closest points look at their classes and take a vote
- No need to do One-vs-Rest or One-vs-One.

Problems with k-NN

- As the size of the data set grows, and the more dimensions of the input data the computational complexity explodes. This is sometimes referred to as the curse of dimensionality
- With reasonable clever data structures and algorithms you can speed things up

18.7.2 Hierarchical clustering

Properties

- Un-supervised learning algorithm
- Numerical and categorical variables

Hierarchical clustering
Basic idea:

- Start with the same number of clusters as you have data points
- At each stage cluster together close together cluster
- Stop when you have enough clusters or everything is clustered together.

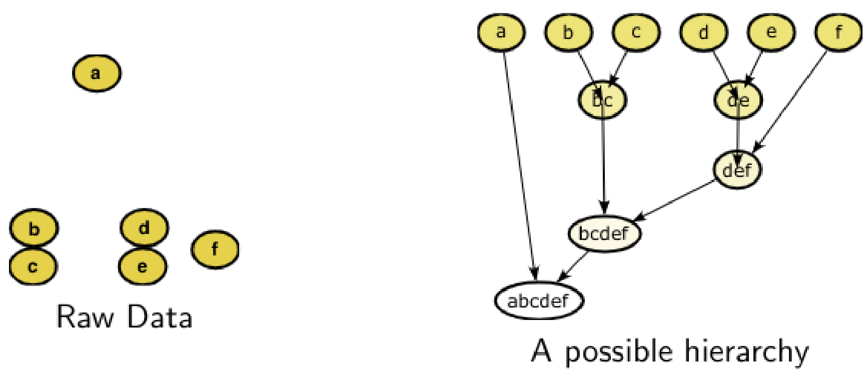


Figure 18.12: hierarchical clustering. From ()

Linkage Criteria

The act where a point is associated with a specific class.

- Maximum or complete linkage clustering: $max\{d(a,b) : a \in A, b \in B\}$
- Minimum of single-linkage clustering: $min\{d(a,b) : a \in A, b \in B\}$

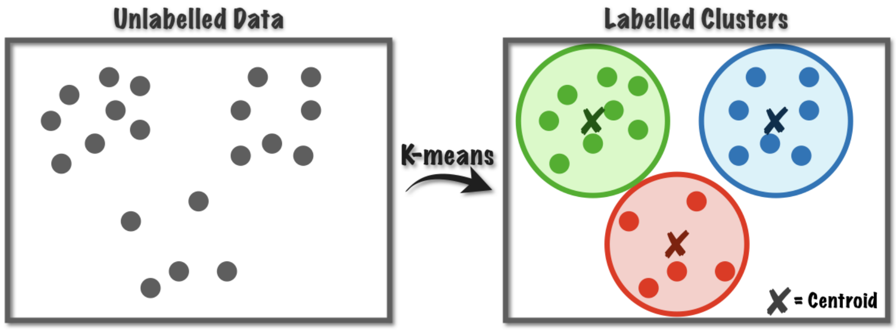


Figure 18.13: K-means. From ()

18.7.3 K-Means

Properties

- Un-supervised learning algorithm
- Numerical variables

$$\mu = \frac{1}{N}\sum_{i=1}^{n} x_i$$

We want to find k centres $\mu_1, \ldots, \mu_k$ that minimizes the spread or max distance in each cluster.

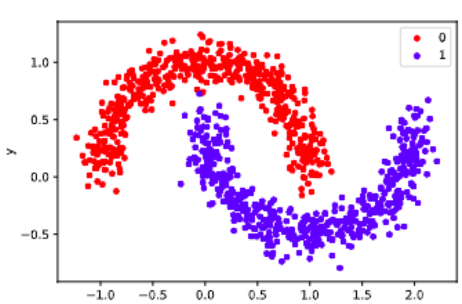
First we randomly select the clusters centroids. Then we associate each point to a cluster with we then replace the centroid so it minimizes it. Then we repeat this step.

H-Means explained

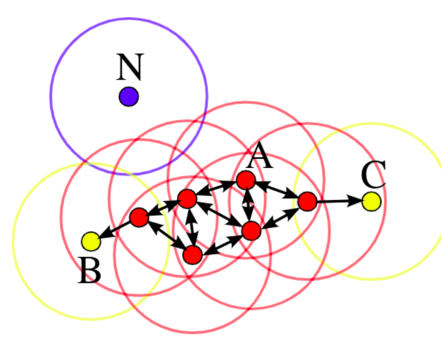
18.7.4 DBSCAN

Properties

- Un-supervised learning algorithm
- Numerical variables



(a) DBSCAN example. From ()



(b) DBSCAN algorithm. From ()

Figure 18.14: DBSCAN

18.8 Information theory and Decision Theory

18.8.1 Decision Trees

Properties

- Supervised learning algorithm
- Categorical variables
- Classification and regression

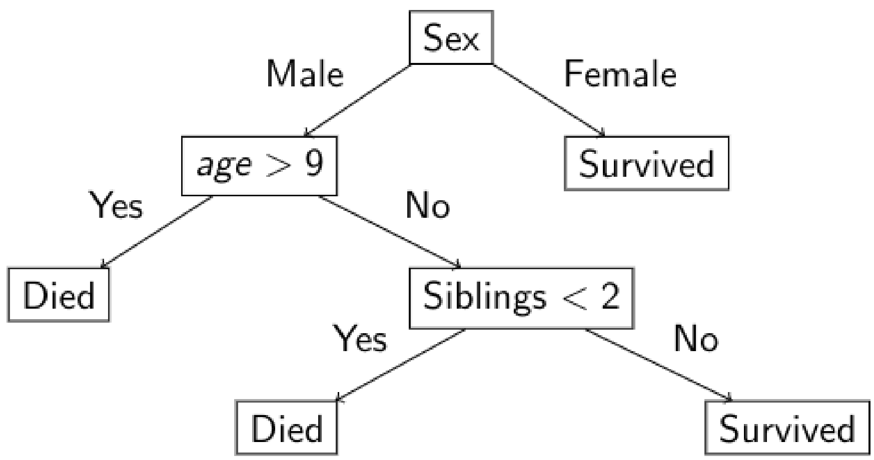


Figure 18.15: Decision Trees. From ()

It is NP-hard to find an ordering that gives the smallest tree. We can get exponential blow up in the size for different orders. To find the smallest tree we can use information theory.

18.8.2 ID3 algorithm

1. Take a feature (with the most entropy) and place it as the root node where the branches true or false or similar (dependent on entropy). Then remove the feature from the feature list.
2. Then take the new most entropy feature and place it as a node under the parent. Then remove that.
3. Then lastly we come to the conclusion (ex diabetes or not diabetes).

ID3 explained

18.8.3 Measuring Information

Information theory is the scientific study of the quantification, storage, and communication of digital information.

Entropy: a measurement of how spread out the data points are. High entropy means that the datapoints ave evenly spread out and 0 means there are not.

$$H(X) = -\sum_{i=1}^{n} p_i \log_2(p_i) = \sum_{i=1}^{n} p_i \log_2\left(\frac{1}{p_i}\right)$$

Properies of H and I Given an event that occurs with probablility p we want a $I(p)$ that measures the information of that event. We want I to have certain properties

- $I(p)$ is monotonically decreasing in p. The higher the probability of the event, the less information.
- $I(p) \geq 0$
- $I(1) = 0$. An even with probability 1 has no information.
- if p and q are independent events then we want $I(pq) = I(p) + I(p)$

Given these properties the only mathematically sensible choice is

$$I(p) = \log\left(\frac{1}{p}\right) = -\log(p)$$

Example: Unfair coin with probability p

$$H(p, 1-p) = -p\log p - (1-p)\log(1-p)$$

If you plot the graph it will look something like figure

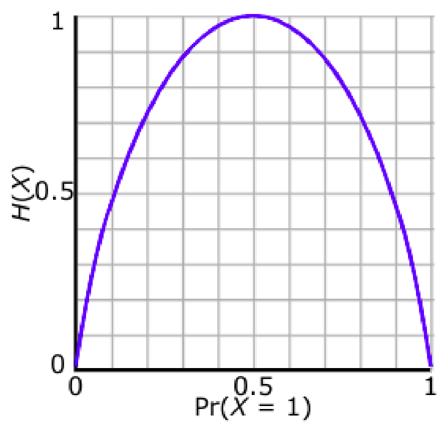


Figure 18.16: Decision Trees. From ()

Conditional entropy

$$H(X|Y = a) = -\sum_{x \in X} P(x|Y = a)\log_2 P(x|Y = a)$$

where $P(X|Y) = \frac{P(X,Y)}{P(X)}$

Information Gain

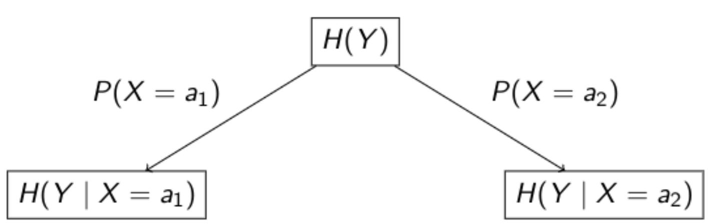


Figure 18.17: Decision Trees. From ()

$$H(Y) - \left(\underbrace{P(X = a_1)H(Y|X = a_1) + P(X = a_2)H(Y|X = a_2)}_{\text{average information when split}}\right)$$

ID3 algorithm the algorithm to construct a tree using information theory.

- ID3 is heuristic, i.e. practical solution but not optimal result, since we use information there to make it easier to solve.
- ID3 will not construct the smallest tree.

	Cloudy (c)	Not Cloudy ($\bar{c}$)
Raining (r)	24/100	1/100
Not Raining ($\bar{r}$)	25/100	50/100

Figure 18.18: Example Conditional Entropy. From ()

Example: Conditional entropy

Se figure 18.18 The amount of days it is raining $(24 + 1 = 25)$

- $P(c|r) = 24/25$
- $P(\bar{c}|r) = 1/25$

$$H(Y|X = r) = -\underbrace{\frac{24}{25}\log_2\frac{24}{25}}_{\text{Cloudy}} - \overbrace{\frac{1}{25}\log_2\frac{1}{25}}^{\text{NotCloudy}} \approx 0.24$$

Advantages of Decision trees

- Easy to understand what the algorithm has learned
- Can learn very non-linear boundaries
- Does not require that much processing
- Not many parameters to tune

Disadvantages of Decision trees

- Prone to overfilling
- Small changes in the data can mean that you learn very different trees.
- Computationally more expressive than other methods

18.9 Principle component analysis (PCA)

Is dimensionality reduction method, i.e. reducing the number of dimension of a training set. We try to compress the data, i.e. remove the redundant data.

covariance matrix is

$$B = X - \bar{X}$$
$$C = B^T B$$

where X is the data matrix.

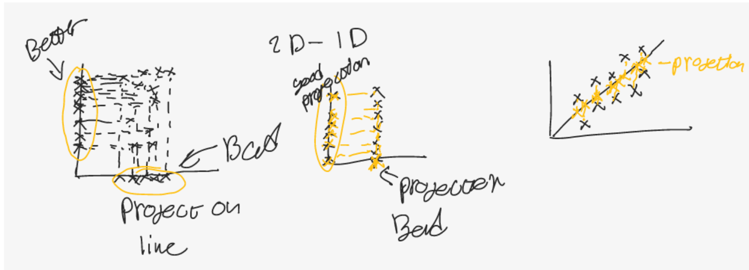


Figure 18.19: Example Conditional Entropy

$$C\omega = \lambda\omega$$

Where C is Covariance matrix, λ is the eigenvalue and ω is the eigenvector PCA discription

The principle components

$$T = B\omega$$

We then **reduse the dimension** by desigding on how many principle component we want this is done by taking the standard diviation to including $x\%$ of the data.

18.9.1 Covariance Matrix

Is a diagonaly symetric matrix of the covariance between two elements in the random vector.

The eigenvectors of the covariance matrix describes the direction of the line where the data points are maped to. `https://wiki.pathmind.com/eigenvector`

`https://www.youtube.com/watch?v=pmG4K79DUoI` `https://www.youtube.com/watch?v=rng04VJxUt4`

When the eigenvalues are relativly small we can remove those vectors since the do not contrebute much of the eigenvalue is in significant.

Chapter 19

Introduction to Machine Learning

Resorcess

- ...

19.1 Design thinking

19.2 Litreture search

Tips

- If you tink about it, chanses are that somone else has as well. Therefore it is very important that there is a stable foundation on projects that has not already beed done. If it has been done but the project differs from the existing solutions, that is fine. But empfasise what the diffrence is.
- A reaseach project often draws knolage from several fealds. An example would be a project that looked at the application of artificial intelligence methods to predict breast cancer rates in patients – thus drawing together the two fields of medical science and artificial intelligence.
- First define what are you looking for.
- *relevance tree*, *spider diagram* or *research territory map* are good ways to find the realevance in the litreture you find useful in our prodject.

- vi moste höja vår nivå, det är högre nivå i denna kursen åsikter ska inte varamed Kolla författaren Bra referenser. Ha flera omberoende referenser kolla regler system booken ta insperation för hur man skriver bra texter som man läser vi kommer läsa mer än 30 totalt

IEEE should be use. BUt go for the mall. Dont just have number but try to understand it. Figure 3 is best to describe if it is on diffrent page. Dont have clickle text.

Vi har relaterat arbete. Ett par sidor relaterat arbete.

Man år andvända vi, blir lättare att läsa. Lätt läst är viktigt ej talspråk ej förtärknings språk objektiv (käller, ej åskikter) konsekvent stil svårt att ha flera,

19.3 g

usa (privacy) att ingen ska ta din data

eu (integretet) att datan ska haterat på ett bra och godkänt sätt

gdpr if personlig data if behandling (insamling också, digital) if registerande inom eu then gdpr

if not lagrig grund eller not princisp if ansvarige

behöver vi data balansera andvändaren och företaget

anymisering du behandlar data därmef gdpr. sen behöver du inte tänka på det

måste kunna bevisa att du har gjort det

DPbD data protektion by design

spara aldrig lösenord i klar text

19.4 Pattent

självstudie kurs kolla studium.

Namnet får inte referera till produktern

25år design kyd geographismkt

upphovsrätt R namn år

håll hemligt innan patent annars får man inte det

teknat, avtal

19.5 df

små steg energi intense

dont have alwae running

we should use mål as a refrence, this is realy imortant even fn says it is

make disition based on mål
färg blind
kolla på vad som tar energi
development hålbarhat readability

19.6 c

backrund läg fokus

innan man har fåt granskad kan man publicera på denna sidan https://arxiv.org/

Kolla på wikipedia för primär källor

`https://libguides.ub.uu.se/` får bra resurcer kola tidskrifter för att kolla peer review

Bibliography

[BS] D. Black-Schaffer. Isa 1.8 instruction sequencing example. Youtube. [Online]. Available: https://www.youtube.com/watch?v=dPa4T_EZ7Gs

[PH11] D. A. Patterson and J. L. Hennessy, Computer Organization and Design: The Hardware/Software Interface 4th Edition. Morgan Kaufmann, 2011.

[Wik] Wikipedia. Haskell. 'Accessed 29 Mar. 2022'. [Online]. Available: https://en.wikipedia.org/wiki/Haskell