Masters Programme in Computer and Information Engineering

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Contents

Chapter 1

Basic Course in Mathematics

1.1 Basic

1.1.1 Mängder

$$\text{Naturliga tal: } \mathbb{N} = \{1, 2, 3...\}$$
$$\text{Heltal: } \mathbb{Z} = \{... - 2, 1, 0, 1, 2...\}$$
$$\text{Rationella tal: } \mathbb{Q} = \{\frac{a}{b}|a, b \in \mathbb{Z}, b$$
$$\text{Irrationella tal: } \mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \{x|x \in \mathbb{R}, x \notin \mathbb{Q}\}$$
$$\text{Reella tal: } \mathbb{R} = \mathbb{P} \cup \mathbb{Q}$$

1.1.2 Potenslagar

$$(\frac{a}{b})^{-3} = (\frac{b}{a})^3$$
$$\sqrt{a} = a^{\frac{1}{2}}$$

1.1.3 Logaritmer

Logaritmlagar:

Va vaksam över när a<0 då är logarytmen odefinerad.

$$\text{(1): } b = a^x \Leftrightarrow \log_a(b) = x \text{ för: } a > 0, b > 0, a \neq 1$$
$$\text{(2): } \log_a(\frac{b}{c}) = \log_a(b) - \log_a(c)$$
$$\text{(3): } \log_a(b * c) = \log_a(b) + \log_a(c)$$
$$\text{(4): } \log_a(b^d) = d \log_a(b)$$
$$\text{(5): } \log_a(b) = \frac{\log_f(b)}{\log_f(a)}$$
$$\text{(6): } \log_a(a) = 1$$
$$\text{(7): } \log_a(1) = 0$$
$$\text{(8): } a^{\log_a(x)} = x$$
$$\text{(9): } \log_{a^c}(b) = \frac{1}{c} \log_a(b)$$

1.1.4 Tillvägagångs sätt

$$\log_3(x) + \log_x(\frac{1}{27}) = 2 \tag{1.1}$$

Lösning

$$\log_3(x) - 3\log_x(3) = 2$$
$$\log_3(x) - 3\frac{\log_3(3)}{\log_3(x)} = 2$$
$$\log_3(x) - 3\frac{1}{\log_3(x)} = 2$$
$$\log_3(x)^2 - 3 = 2\log_3(x)$$
$$y = \log_3(x)^2$$
$$y^2 - 2y - 3 = 0$$
$$\text{pq-formeln: } y = 1 \pm \sqrt{4} = 1 \pm 2$$
$$\log_3(x) = 3 \Leftrightarrow x = 3^3 = 27$$
$$\log_3(x) = -1 \Leftrightarrow x = 3^{-1} = \frac{1}{3}$$

1.1.5 Intervall

$$[\,a, b] = \{x | a \leq x \leq b\} \tag{1.2}$$

$$[\,a, \infty[\tag{1.3}$$
$$]-\infty, \infty[\tag{1.4}$$

1.1.6 Tillvägagångs sätt

$$\frac{2}{x-3} < \frac{5}{x} \tag{1.5}$$

Lösning

$$\frac{2}{x-3} - \frac{5}{x} < 0$$
$$\frac{(x)2}{x(x-3)} - \frac{5(x-3)}{x(x-3)} < 0$$
$$\frac{2x - 5x - 15}{x(x-3)} < 0$$
$$\frac{-3x - 15}{x(x-3)} < 0$$
$$\frac{-3(x-5)}{x(x-3)} < 0$$
$$x \neq 0, x \neq 3$$

Värde tabell:

	x<0	0<x<3	3<x<5	5<x
x-5	-	-	-	+
-3	-	-	-	-
x	-	+	+	+
x-3	-	-	+	+
hela	+	-	+	-

1.2 Komplexa tal

$$
\begin{aligned}
&z = a + bi \\
&\text{Re}(z) = a \\
&\text{Im}(z) = b \\
&\overline{\text{z}} = a - bi \\
&|z| = \sqrt{a^2 + b^2} \text{ Där b(Im(z)) är för y-axeln och a(Re(z)) är för x-axeln}
\end{aligned}
$$

Samma räkneregler för reella tal gäller för komplexa tal

1.2.1 Polärform

$$
\begin{aligned}
&\arg(z) = \alpha + 2\pi * n \\
&\text{arg(z) är vinkeln mellan a (x-axeln) linjen |z|. n är heltal} \\
&\text{Arg}(z) \in]-\pi, \pi[\\
&\text{Arg}(z) \in \arg(z) \\
&a = |z| \cos \alpha \\
&b = |z| \sin \alpha \\
&z = |z| \cos \alpha + i * |z| \sin \alpha
\end{aligned}
$$

Polärform:

$$z = |z|(\cos \alpha + i * \sin \alpha) \tag{1.6}$$

Eulers formel:

$$z = |z|e^{i\alpha} \tag{1.7}$$

Sats:

$$
\begin{aligned}
&|z * w| = |z| * |w| \\
&\arg(z * w) = \arg(z) + \arg(w) \\
&\arg(\frac{z}{w}) = \arg(z) - \arg(w)
\end{aligned}
$$

De Movre's formel:

$$(\cos(\theta) + i * \sin(\theta))^n = \cos(n * \theta) + i * \sin(n * \theta)$$

Binomisk ekvation

$$(\cos(\theta) + i * \sin(\theta))^n = \cos(n * \theta) + i * \sin(n * \theta)$$

Tillvägagångs sätt

$$\text{Visa lösningarna i det komplexa tal planet } z^5 = \sqrt{3} + i \tag{1.8}$$

$$|\sqrt{3} + i| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$$
$$\begin{cases} \sqrt{3} = 2\cos(\alpha) \\ 1 = 2\sin(\alpha) \end{cases}$$
$$\alpha = \frac{\pi}{6}$$
$$|z|^5(\cos(5 * \theta) + i * \sin(5 * \theta)) = 2(\cos(\frac{\pi}{6}) + i * \sin(\frac{\pi}{6}))$$

Vilket ger:

$$|z|^5 = 2 \Leftrightarrow |z| = 2^{\frac{1}{5}}$$
$$5 * \theta = \frac{\pi}{6} + 2\pi n \Leftrightarrow \theta = \frac{\pi}{30} + \frac{2\pi n}{5} n \in \mathbb{Z}$$

I polärform blir det då:

$$z = 2^{\frac{1}{5}}(\cos\frac{\pi}{30} + \frac{2\pi n}{5} + i * \sin\frac{\pi}{30} + \frac{2\pi n}{5})$$

Eulers formel: $z = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{2\pi n}{5})}$

$$n = 0, 1, 2, 3, 4$$
$$z_1 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30})}$$
$$z_2 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{2\pi}{5})}$$
$$z_3 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{4\pi}{5})}$$
$$z_4 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{6\pi}{5})}$$
$$z_5 = 2^{\frac{1}{5}} e^{i(\frac{\pi}{30} + \frac{8\pi}{5})}$$

1.3 Absolut Belopp

1.3.1 Tillvägagångs sätt

$$|2x-8|+|1-x|-2|x-3|=8+3x \tag{1.9}$$

lösning

$$\begin{pmatrix} 2x-8=0 \\ x=4 \end{pmatrix}\begin{pmatrix} 1-x=0 \\ x=1 \end{pmatrix}\begin{pmatrix} x-3=0 \\ x=3 \end{pmatrix}$$

$$\mathrm{I}\begin{cases} x\leq 1 \\ -(2x-8)+(1-x)+2(x-3)=8+3x \end{cases}$$

$$\mathrm{I}\begin{cases} x\leq 1 \\ -4x=5 \end{cases}\quad\begin{cases} x\leq 1 \\ x=-\frac{5}{4}\text{ lösning} \end{cases}$$

$$\mathrm{II}\begin{cases} 1<x\leq 1 \\ -(2x-8)-(1-x)+2(x-3)=8+3x \end{cases}$$

$$\mathrm{II}\begin{cases} 1<x\leq 3 \\ -2x=7 \end{cases}\quad\begin{cases} 1<x<3 \\ x=-\frac{7}{2}\text{ Ej lösning} \end{cases}$$

$$\mathrm{III}\begin{cases} 3\leq x<4 \\ -(2x-8)-(1-x)-2(x-3)=8+3x \end{cases}$$

$$\mathrm{III}\begin{cases} 3\leq x\leq 4 \\ -(2x-8)-(4x-3)=8+3x \end{cases}\quad\begin{cases} 3x<4 \\ x=\frac{5}{6}\text{ Ej lösning} \end{cases}$$

$$\mathrm{IV}\begin{cases} x\geq 4 \\ (2x-8)-(1-x)-2(x-3)=8+3x \end{cases}$$

$$\mathrm{IV}\begin{cases} x\geq 4 \\ x=-\frac{11}{2}\text{ Ej lösning} \end{cases}$$

1.4 Summor

1.4.1 Aritmetiska summor

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n\,(a_1 + a_n)}{2} \tag{1.10}$$

1.4.2 Geometriska summor

Börjar altid med exponenten 0 och gör om summan så att den passar i följade talföljd:

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\,(k^n - 1)}{k - 1} \tag{1.11}$$

1.4.3 Tillvägagångs sätt

$$\sum_{k=n}^{2n} (2^k - k) \tag{1.12}$$

Lösning

$$\begin{aligned}
&\text{sätter f = 0 = k -n} \\
&\sum_{f=0}^{n} (2^{f+n} - (f+n)) = 2^n * \sum_{f=0}^{n} (2^f) - \sum_{f=0}^{n} (f+n) \\
&\frac{2^n(2^{n+1} - 1)}{2 - 1} - \frac{3n(n+1)}{2} = 2^{2n+1} - 2^n - \frac{3n(n+1)}{2}
\end{aligned}$$

1.5 Kombinatorik

1.5.1 Multiplikations principen

permetationer p(a,b) då ordningen spelar roll.
Antal sätt: (exemplet: antal sätt av måltider 7 företer 5 varmrätter 4 efterätter (7*5*4)

$$n_1 \cdot n_2 \cdot \ldots \cdot n_m \tag{1.13}$$

1.5.2 Kombinationer

Kombination c(a,b) då ordningen inte spelar roll.
Antal sätt: (exemplet: antal del-mängder två element (n = element k = antal element som kan väljas))

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \tag{1.14}$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n \cdot (n-1)(n-2) \cdot \ldots \cdot (n-1(k-1))}{k!} \tag{1.15}$$

Pascal triangel

n							
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1
	0	1	2	3	4	5	6
				k			

1.5.3 Binomial satsen

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{n-k} \tag{1.16}$$

1.6 Funktioner och kordinatsystem

Kom ihåg att när det stor (x-a) förflytas den i x-axeln a steg till höger –> medans (x+a) förflytas den a steg till vänster <–

1.6.1 Avståndsformeln

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.17}$$

1.6.2 Elipser

Förenkla till denna formeln:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1.18}$$

Där a är avståndet på x-axeln och b är avståndet på y-axeln

Hyperbol

Ser väldigt anerlunda ut från elipser

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{1.19}$$

Circkel

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \tag{1.20}$$

Där a = r

1.7 Polynom division

1.7.1 Tillvägagångs sätt

Man vet att ekvationen$z^4-2z^3-7z^2+26z-20=0$ har roten $z=2+i$. Lös ekvationen fullständigt. (1.21)

$z = 2 + i$ Är en läsning är också konjugatet en lösning enligt faktorsatsen $\overline{\mathrm{z}} = a - bi$

$z = 2 \pm i$

Vilket betyder att följande går att factorisera ut polynomet

$(z-(2+i))(z-(2-i)) = z^2 - 4z + 5$

Långdivition (liggande stolen):

$$
\begin{array}{r|rrrrr}
 & & & x^2 & +2x & -4 \\
\hline
x^2-4x+5) & x^4 & -2x^3 & -7x^2 & +26x & -20 \\
 & -x^4 & +4x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -12x^2 & +26x & \\
 & & -2x^3 & +8x^2 & -10x & \\
\cline{3-5}
 & & & -4x^2 & +16x & -20 \\
 & & & 4x^2 & -16x & +20 \\
\cline{4-6}
 & & & & & 0
\end{array}
$$

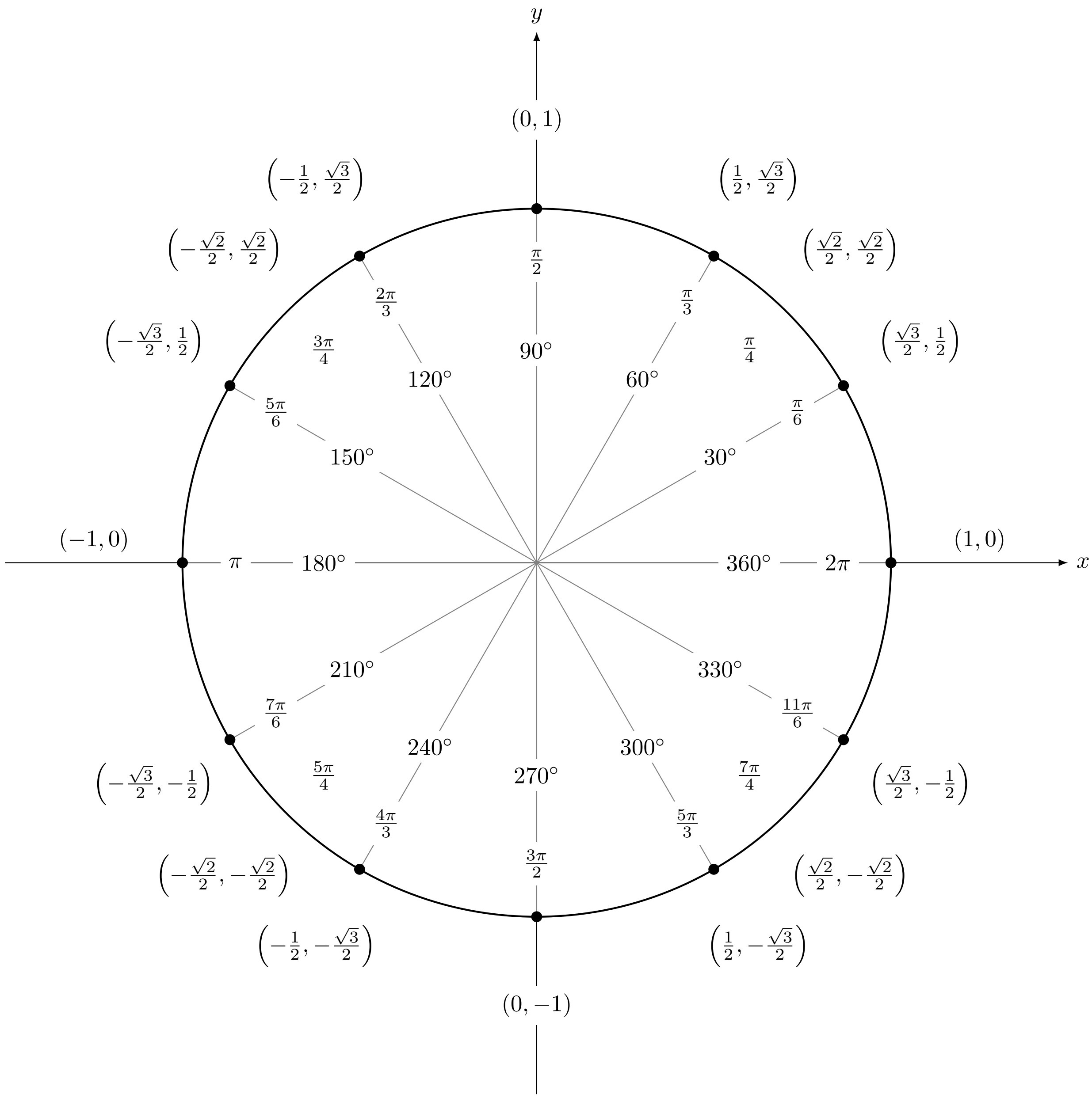
$z^2 + 2z - 4 = 0$ Är också en läsning som tilslut ger följande

$z = -1 \pm \sqrt{5}$

$z = 2 \pm i$

Varge n grads polynom har altid n stycken komplexa lösningar

1.8 Trigonometri



Sats:

$$
\begin{aligned}
&360^\circ = 2\pi rad \\
&v_g = v_r * \frac{180^\circ}{\pi} \\
&v_r = v_g * \frac{\pi}{180^\circ}
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&-1 \leq \sin t \leq 1 \\
&-1 \leq \cos t \leq 1
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&\cos(-t) = \cos(t) \\
&\sin(-t) = -\sin(t) \\
&\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)}
\end{aligned}
$$

Additionsformlerna:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha)
\end{aligned}
$$

Trigonometriska ettan:

$$
\begin{aligned}
&(\sin t)^2 + (\cos t)^2 = 1 \\
&\sin^2 t + \cos^2 t = 1
\end{aligned}
$$

1.8.1 Tillvägagångs sätt

$$
\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

Chapter 2

Program Design and Data Structures

2.1 Coding convension

VARIANT
A (recursive) function terminates if it has a variant. The variant need to follow all of the flowing rules

- Needs to decrease every recursive call
- All ways positive

Side effects
All IO functions have side effects in order to separate pure Haskell function with impure functions that changes the state with is commonly the case with imperative and object oriented programming. Every IO function has a side effects.
INVARIANT
A data types invariant is what value are allowed for the data type to work. Similar to preconditions for a function. An example is integer data type that can only use positive integers therefor the invariant is positive integers.

2.2 Design approach

- top-down design (Cheating): Is to break down a complex system in to subsystems to solve the problem. Most often is to write everything by scratch.
- bottom-up design (Stacking): Is to piece existing system together to create a more complex system. little is programmed, most is copied.
- dodging: Get some code working more quickly, make progress with some part of the system and back-paddle to the dodged part later. The reason is to develop insight that will help solve the larger problem.

2.2.1 Process

1. Data Description
2. Data Examples
3. Function Description
4. Function Examples
5. Function Template
6. Code
7. Tests
8. Review and Refactor

Programming to an Interface More dynamic, can change models, less code to write and a layer of abstraction. ADT

2.3 Recursion

2.3.1 Recursion types

1. Simple recursion: There is at most one recursive call (in each branch) and the argument is decremented by one.
2. Complete recursion: Some argument becomes smaller in the recursive call, but not necessarily.
3. Multiple recursion: There are multiple recursive calls (in the same branch).
4. Mutual recursion: Two or more functions are defined in terms of each other.
5. Nested recursion: An argument to a recursive call is computed by a recursive call.
6. Recursion on a generalized problem: Sometimes, no suitable recursion scheme is obvious.

2.4 Complexity

2.4.1 Growth

1. Big θ Notation: estimate of growth in intervals determine by constants Definition For non-negative functions f and g, $f(n) = \theta(g(n))$ if and only if there exist $n_0 \geq 0 and c_1, c_2 > 0$ such that for all $n > n_0$ $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$. $\theta(g(n))$ is the set of all functions $f(n)$ that are bounded below and above
2. Big Ω Notation: estimate of growth Lower bound
3. Big O: Notation: estimate of growth upper bound

Relation

$O(g(n)) \cap \omega(g(n)) = \theta(g(n))$

2.5 Recurrences

Example:
sumList [] = 0
sumList (x:xs) = x + sumList xs

1. pattern matching [] takes t_0 time
2. pattern matching $(x : xs)$ takes t_1 time
3. Adding x with recursive call takes t_{add}
4. Then the recursive call takes $T(n-1)$

$$T(n) = \begin{cases} t_0 & \text{if } n = 0 \\ T(n-1) + t_{add} + t_1 & \text{if } n > 0 \end{cases}$$

2.5.1 Closed Form

1. Use the substitution method to obtain a closed form for the following recurrence:

$$f(0) = 5$$
$$f(n) = f(n-1) + n + 2, n > 0$$

Hint: $1 + 2 + 3 + \; + n = \frac{n(n+1)}{2}$ **— you do not need to prove this fact.**

Expansion Method

$$
\begin{aligned}
&f(0) = 5\\
&f(1) = f(0) + n + 2 = n + 5 + 2\\
&f(2) = f(1) + n + 2 = 2n + 5 + 2 + 2\\
&f(3) = f(2) + n + 2 = 3n + 5 + 2 + 2 + 2\\
&f(n) = +n^2 + 5 + n \cdot 2
\end{aligned}
$$

Induction proof

Step1: test with base case sense the base case is predefine for 0 we do $n = 1$
$f(1)_{VL} = 1^2 + 5 + 1 \cdot 2 = 8,$
$f(1)_{HL} = f(0) + 1 + 2 = 5 + 3 = 8$
$f(1)_{VL} = f(1)_{HL}$

Step2: assumption for p $f(p) = p^2 + 5 + p \cdot 2$
$f(p+1)_{VL} = f(p) + (p+1) + 2 = (p^2 + 5 + p \cdot 2) + (p+1) + 2 =$
$= (p^2 + (p+1)) + 5 + (p+1) \cdot 2 = (p^2 + 2p + 1) + 5 + (p+1) \cdot 2$
$f(p+1)_{HL} = (p+1)^2 + 5 + (p+1) \cdot 2 = (p^2 + 2p + 1) + 5 + (p+1) \cdot 2$
$f(p+1)_{VL} = f(p+1)_{HL}$

Conclusion: according to induction hypothesis the recurrence of the function is equal to
$2n + 5 + \dfrac{n(1+n)}{2}$

Substitution Method

$$\begin{aligned}
f(n) &= f(n-1) + n + 2 \\
&= (f(n-2) + (n-1) + 2) + 1n + 2 \\
&= f(n-2) + (n-1) + n + 2 \cdot 2 \\
&= (f(n-3) + (n-2) + 2)(n-1) + n + 2 \cdot 2 \\
&= f(n-3) + (n-2) + (n-1) + n + 3 \cdot 2 \\
&. \\
&= f(n-k) + (n-(k-1)) + (n-(k-2)) + (n-(k-3)) + \ldots + n + k \cdot 2 \\
&. \\
&= f(n-n) + (n-(n-1)) + (n-(n-2)) + (n-(n-3)) + \ldots + n + n \cdot 2 \\
&= f(0) + 1 + 2 + 3 + \ldots + n + n \cdot 2 = 5 + 1 + 2 + 3 + \ldots + n + n \cdot 2
\end{aligned}$$

We can see the following patterns

$$2n + 5 + \sum_{k=1}^{n}(k) = 2n + 5 + \frac{n(1+n)}{2}$$

Induction proof

Step1: test with base case since the base case is predefine for 0 we do $n = 1$

$f(1)_{VL} = 2 \cdot 1 + 5 + \frac{1(1+1)}{2} = 2 + 5 + 1 = 8,$

$f(1)_{HL} = f(0) + 1 + 2 = 5 + 3 = 8$

$f(1)_{VL} = f(1)_{HL}$

Step2: assumption for p $f(p) = 2p + 5 + \frac{p(1+p)}{2}$

$f(p+1)_{HL} = f(p) + (p+1) + 2 = (2p + 5 + \frac{p(1+p)}{2}) + (p+1) + 2 =$

$= 2(p+1) + 5 + \frac{p(1+p)}{2} + p = 2(p+1) + 5 + \frac{2(p+1) + p(1+p)}{2} =$

$= 2(p+1) + 5 + \frac{p^2 + 2p + p + 2}{2} =$

$f(p+1)_{HL} = 2(p+1) + 5 + \frac{(p+1)(p+2)}{2} = 2(p+1) + 5 + \frac{p^2 + 2p + p + 2}{2}$

$f(p+1)_{VL} = f(p+1)_{HL}$

Conclusion: according to induction hypothesis the recurrence of the function is equal to $2n + 5 + \frac{n(1+n)}{2}$

2.6 Higher-Order Function

2.6.1 Higher-Order Functions on Lists

```
map :: (a -> b) -> [a] -> [b]
filter :: (a -> Bool) -> [a] -> [a]
foldl :: (a -> b -> a) -> a -> [b] -> a
foldr :: (a -> b -> b) -> b -> [a] -> b
```

map

maps a function to each element in list.

filter

filters elements with a condision

```
filter :: (a -> Bool) -> [a] -> [a]
filter (<6) [6,3,0,1,8,5,9,3] == [3,0,1,5,3]
```

foldl

foldl recurses over a list “from the left,” i.e., it initially applies the given operation to the first list element and the given start value. starts from the left (first element) and apply the function to with each element. Similar to an accumulator. No one uses it since it is some what useless.

```
foldl :: (a -> b -> a) -> a -> [b] -> a
foldl (*) 1 [1,2,3,4] == 24
```

foldr

foldr recurses over a list “from the right,” i.e., it initially applies the given operation to the last list element and the given start value. starts from the right (last element) and apply the function to with each element. Similar to an accumulator.

```
foldr :: (a -> b -> b) -> b -> [a] -> b
foldr (*) 1 [1,2,3,4] == 24
```

2.7 Data types

2.7.1 Basic

```
String :: ['char'] --list of charecters
List :: []         --undefine element types and elements
Tuple :: ()        --Predefine elements
Char :: ''         --single carecter
Int :: 1           --Hole number with define size
Integear :: 1      --Hole number with undefine size
Float :: 1.1       --Real number with double-precision
Double :: 1.1      --Real number with single-precision
```

2.7.2 Maybe Type

If the return is maybe "nothing" then the "Maybe type" is used, since it dose not have to return the a specific value. It is not polymorphic since you have to specify the type, however "Just" is at of it self polymorphic function. If a operation that requires a specific type one needs to remove "Just", for instance by a let function.

2.7.3 New types and enumeration types

New types: One create more relevant names and format of existing enumeration types. Overload is a problem with the use of the same operations that can not be used for the same data types. One needs to create new operations if it is not from the same type class.

Enumeration types: Instead of new types *type* enumeration types uses the operation call *data*. The difference is that enumeration types is independent from existing types therefor one becomes more flexible and precise.

example

```
data newTypeOfDataDerection = North | South | East | West
    deriving (Show) -- inorder to print

:t North
North :: newTypeOfDataDerection

-- We can use new types in pattern matching
oposit :: newTypeOfDataDerection -> newTypeOfDataDerection
oposit North = South
```

Type classes

A type class is a set of types that support certain related operations.

No function is applied by default to the new type, therefor you can write "deriving" the following functions are good to have

type classes to deriving for new data types

```
deriving (Show) -- in order to print in ghci and print normal
deriving (Eq) -- To test equality
deriving (Ord) -- the first one has the smallest value, order matter for comparison
```

New type classes

```
class Eq a where
    (==) :: a -> a -> Bool
```

New type classes Instances

```
instance Eq Colour where
  (==) = eqColour
```

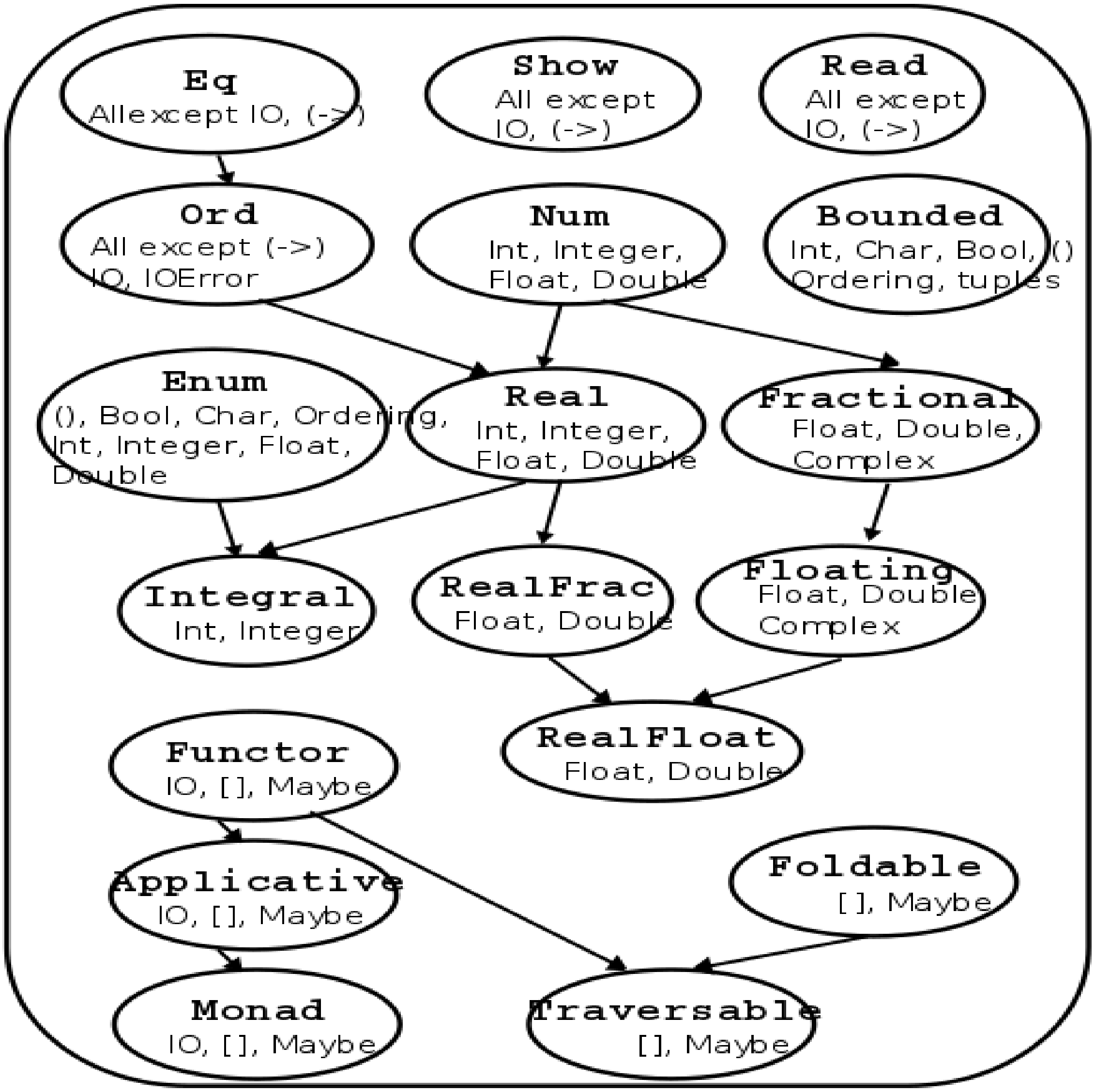


Figure 2.1: Type class

2.7.4 Inductive Data Types

Uses a base case then the inductive step as cases of. Multiple arguments. Type constructure

```
data AExp = Atom Int
    | Plus AExp AExp
    | Times AExp AExp

eval (Atom i)     = i
eval (Plus a b)   = eval a + eval b
eval (Times a b) = eval a * eval b
```

2.7.5 Trees

Terminology

1. Search tree: All nodes on the left side is less then the parent and opposite on the right side
2. Out-degree: is how many children it has. Binary trees has out-degree $0-2$
3. root Node: is a parent to any number of children at the top of the hierarchy
4. sub Node: is a parent to any number of children
5. Leaf: is a nod that has no children
6. Node: every element
7. Height: most steps from the root node to a leaf

Representation

1. Inorder: (Left, Root, Right)
2. Pre-order: (Root, Left, Right)
3. Post-order: (Left, Right, Root)

```
data FBTree a = Leaf a
   | Node (FBTree a) a (FBTree a)
   deriving (Show)

Node (Leaf 1) 5 (Leaf 21)
rootValue :: FBTree a -> a
rootValue (Node _ a _) = a
rootValue (Node x) = x

height :: FBTree a -> Int
height (Leaf _) = 0
height (Node b a c) = 1 + max (height b) (height c)
```

Binary tree

Insertion: $O(1)$ ($O(h)$ if search tree)
Delition: $O(n)$ ($O(h)$ if search tree)
Search: $O(n)$ ($O(h)$ if search tree)
Height: $O(n)$ (n nodes)

1. Full binary tree: has node out-degree either 2 or 0 worst-case complexity of $O(\log n)$.
2. Binary tree: each node has up to an out-degree of 2

Binary search tree

Insertion: $O(h)$ ($O(n)$ if search tree)
Delition: $O(h)$
Search: $O(n)$
Height: $O(n)$ (n nodes)

Red and black trees

Insertion: $O(\log n)$
Deletion: $O(\log n)$
Search: $O(\log n)$
Element: $O(2^r)$ (rank r)
Height: $O(2 \cdot \log_2(n+1))$ (n nodes)

Better then a normal binary tree since it balance the tree, therefor it becomes smaller and more efficient to search in. One should use red and black tree when there is a large number of nodes, say 50.

Definition: A red-black tree is a binary search tree where every node is colored either red or black, with the following balancing invariants:

1. No red node has a red parent.
2. Every path from the root to an empty subtree contains the same.
3. A red-black tree with n nodes has height at most $2 \cdot \log 2(n+1)$.
4. there are 4 cases to rebalance (1;4) is similar so is (2;3).

Algorithm

1. Perform a standard binary-search-tree insertion.
2. Color the new node red.
3. Rebalance the tree, if there is a red node with a red parent.

Binomial Trees and heaps

Insertion: $O(\log n)$
Search: $O(\log n)$ (number of trees n)
Element: 2^r (rank r, n=1 then r=0)

A heap can be used to implement a priority queue, where elements are added to a pool and assigned a priority. In a min-priority queue, extraction of an element yields an element with minimum priority. The smallest node is the root and every child is equal or larger then its parent.

Binomial Trees is a data structure by linking trees of rank $r-1$ together. A binomial heap (Vuillemin, 1978) is a list of binomial trees such that each tree satisfies the min-heap property (hence the root of each tree contains its minimum key); and the trees have strictly increasing ranks.

Terminology

1. Link: putting together two trees.
2. Merge: putting together two heaps
3. Binomial Heap: a list (forest!) of Binomial Trees
4. Binomial Trees have the largest subtree to the left, while Binomial

Binomial heaps

1. Heaps, extracting minimum element in worst case $O(\log |h|)$
2. Binomial Trees, The height (here: number of edges on the longest branch) is r.
3. Binomial Trees, There are 2^r nodes in the tree.
4. Binomial Trees, There are $\begin{pmatrix} r \\ k \end{pmatrix}$ nodes at level k. (Hence its name!)
5. Binomial Trees, The root has r subtrees of ranks $r-1, r-2, \ldots, 1, 0$.
6. A binomial heap h has at most $[\lg |h|] + 1$ binomial trees.
7. Inserting a binomial tree into a binomial heap is like addition with base 2.
8. merging is made with too cases either (case 1) when one is smaller or (case 2) when they are equal

```
BinoTree = Node Int Int [BinoTree] -- Node rank key (subtrees with decreasing rank)
```

2.7.6 Other data types

Tables, Stacking and queuing

1. Table: a list of key-value pairs
2. Stacks: elements accessed in Last-In First-Out (LIFO) order
3. Queues: elements accessed in First-In First-Out (FIFO) order

Table operations

```
empty :: Table k v
insert :: Eq k => Table k v -> k -> v -> Table k v
exists :: Eq k => Table k v -> k -> Bool
lookup :: Eq k => Table k v -> k -> Maybe v  -- value from key
delete :: Eq k => Table k v -> k -> Table k v
iterate :: Table k v -> (b -> (k, v) -> b) -> b -> b  -- Foldr
keys :: Table k v -> (b -> k -> b) -> b -> b  -- all keys
values :: Table k v -> (b -> v -> b) -> b -> b  -- all values
```

Stack operations

```
-- interface
newtype Stack a = StackImpl [a] -- opaque!

empty :: Stack a
isEmpty :: Stack a -> Bool
push :: a -> Stack a -> Stack a  -- insert
top :: Stack a -> a   -- the first value
pop :: Stack a -> (a,Stack a)  -- take out
```

Queue operations

```haskell
-- interface
newtype Queue a = Q [a] -- opaque

empty :: Queue a
isEmpty :: Queue a -> Bool
head :: Queue a -> a
enqueue :: Queue a -> a -> Queue a  -- take out element
dequeue :: Queue a -> Queue a  -- insert element
toList :: Queue a -> [a]
```

Hastables

- Key value lookup (an index)
- Array is only define for small index, hash has no limit on avalible keys.
- Typically we have n possible keys from set U for a hashtable (which is an array) with m slots, where n $\geq$ m.
- since there is infinitely many elements and limited amount of key there will be element with the same key, therefor a coalition is created.
- Worst-Case Retrieval: time complexity of retrieving a element.
- Load Factor: How much data is in the table $\frac{elements}{slots}$
- Rehashing: make the hastable more balanced.

Collision Resolution by Chaining

- Most commonly used collision resolution
- Let each array slot (also called a bin) hold a list of elements (called a chain).
- In other words, When collision then add it to a list in that element

Collision Resolution by Open Addressing

- Start with a table with each element is $\perp$ previously used Δ.
- Probing: is a function to insert items in a hastable therefore resolves collations
- Types of probing:
  Linear probing: $f(i) = i$.
  Quadratic probing: $f(i) = c_2 \cdot i^2 + c_1 \cdot i,\ where\ c_2 \neq 0$.
  Double hashing: $f(i) = i \cdot h''(i)$, where h'' is another hash function.

- Inserting with Linear Probing: Insert it to the next avalible key
- Deleting with Linear Probing: Ignores $\perp$ and Δ idex will change.

2.7.7 Graphs

Types of graphs

- list
- tree
- forest
- Directed Acyclic Graph (DAG)

Treminolage

- Node, vertex (plural: vertices)
- Edge connects two nodes.
- Self-loop edge from node to itself
- Adjacent nodes connected by an edge
- Degree number of edges from or to a node
- In-degree number of edges to a node
- Out-degree number of edges from a node

Representation

- **Adjacency Matrix** — a 2-dimensional array A of 0/1 values, with $A[i, j]$ containing the number of edges between nodes i and j (undirected graph), or from node i to node j
  + edge existence testing in $\theta(1)$ time
  − finding next outgoing edge in $O|V|$ time
  + compact representation for dense graphs (when $|E|$ is close to $|V|^2$)
- **Adjacency List** — a 1-dimensional array Adj of adjacency lists, with Adj[i] containing a list of the nodes adjacent to node i.
  + finding next outgoing edge in $\theta(1)$ time
  − edge existence testing in $O(|V|)$ time
  + compact representation for sparse graphs (when $|E|$ is much smaller than $|V|^2$)
- **Edge List** — a list of tuples, (i, j), for each edge (i, j) (plus a list of the nodes).
  − edge existence test in $\theta(|E|)$.
  − finding next outgoing edge in $O(|E|)$ (unless appropriately sorted).
  + compact representation for sparse graphs (when $|E|$ is much smaller than $|V|^2$)

topological sort

A topological sort is a linear ordering of all the nodes in a directed acyclic.

Algorithm

1. Select a node with in-degree 0.
2. Output it.
3. Remove it.
4. Repeat (from 1) until no nodes are left.

Total running time is $\theta(|V| + |E|)$.

Graph Traversals

- Breadth-first search (BFS). Uses a queue for eatch grey node.
  Time complexity: $\theta(|V| + |E|)$ — linear in the size of the graph.
- Depth-first search (DFS). Uses a stack for eatch (grey) node and has a rest of nodes (white).
  Time complexity: $\theta(|V| + |E|)$ — linear in the size of the graph.

Breadth-First Search: Algorithm
Input: Some node A.

1. Paint A gray. Paint other nodes white. Add A to an initially empty FIFO queue of gray nodes. All grey nodes is in the queue. It is a queue not a stack so first in first out.
2. Dequeue head node, X. Paint its undiscovered (white) adjacent nodes gray and enqueue them. Paint X black. Repeat until queue is empty. For every black node add it to BFS order (black)

Depth-First Search: Algorithm
Input: Some node A.

DFS(G)

1. Paint all nodes white.
2. For each node v in G: if v is (still) white, DFS-Visit(G,v). Each subsequent call to DFS-Visit in line 2 is called a restart.

DFS(G)

1. Colour v gray.
2. For each node u adjacent to v: if u is white, DFS-Visit(G,u).

Strongly-Connected Components

- Strongly connected component (SCC): maximal set of nodes where there is a path from each node to each other node.
- Many algorithms first divide a digraph into its SCCs, then process these SCCs separately, and finally combine the sub-solutions. (This is not divide & conquer, since a different algorithm is run on each SCC!)
- An undirected graph can be decomposed into its connected

1. Enumerate the nodes of G in DFS finish order, starting from any node
2. Compute the transpose G T (that is, reverse all edges)
3. Make a DFS in G T, considering nodes in reverse finish order from original DFS
4. Each tree in this depth-first forest is a strongly connected component

2.8 Important syntax

2.8.1 Let

```
  let x = 1 in x * 2 == 2
  case x of
  1 -> "Hello"
  2 -> "H"
  3 -> "Hel"

  input: 3 == "Hel"

  # other examples
  let f x y = x + 3 >= y + 3.1 in f 1 1 == False

  f x = let g z = z+1 in g (g x)
  f 1 == 3

:t div
div :: Integral a => a -> a -> a

:t (/)
(/) :: Fractional a => a -> a -> a
```

2.8.2 IO

monads Is used to return an IO and uses a operation *(»=)* that replaces *do-notation*

```
class Monad m where
  (>>=) :: m a -> (a -> m b) -> m b
  return :: a -> m a
```

2.9 Sorting Algorithms

1. Insertion Sort: One at a time
2. Bubble Sort: sort in order two a time starting from left and work to the right side.
3. Merge Sort: Divide and Conquer, split into even pies and sort them and then merge/sort again.
4. Quicksort: Divide and Conquer, takes a pivot to spit smaller and larger.

Chapter 3

Algebra 1

3.1 logik

Utsaga: ett påstående som erhåller antigen värderna sann(S) eller falsk(F) vilket ge en stluten utsaga eller ej ett sannings värde vilket kallas för öpna utsagor
Kombinerade utsagor:
Kunjuktion utsagor: består av två utsagor som vi kallar A och B
där and: $\wedge$ $(A \wedge B)$
Dissjuntion utsagor: består av två utsagor som vi kallar A eller B
or: $\vee$ $(A \vee B)$
icke utsaga A (motsatsen): $\neg A$
Alla: $\forall$
Minst en: $\exists$
$\neg(\forall x : A) \Leftrightarrow \exists x : \neg A$
$\neg(\exists x : A) \Leftrightarrow \forall x : \neg A$

Exempel:

A: Alla reala tal x gäller att $(x+1)^2 = 0$
$\forall x : (x+1)^2 = 0$
$\neg(\forall x : (x+1)^2 = 0) = \exists x : (x+1)^2 \neq 0$
$\neg A$: Det finns reala tal x gäller att $(x+1)^2 \not0$

3.1.1 värde tabeller

Kunjuktions värdetabel:

A	B	$A \wedge B$
S	S	S
S	F	F
F	S	F
F	F	F

Dissjuktions värdetabel:

A	B	$A \vee B$
S	S	S
S	F	S
F	S	S
F	F	F

3.1.2 Implikationer

A medför B: $A \Rightarrow B$ (implikaiton)
A och B medför varandra: $A \Leftrightarrow B$ (ekvivalens)

Implication värdetabell:

A	B	$A \Rightarrow B$
S	S	S
S	F	F
F	S	S
F	F	S

Ekvivalens värdetabel:

A	B	$A \Leftrightarrow B$
S	S	S
S	F	F
F	S	F
F	F	S

Exempel1:

$$x = \sqrt{6 - x} \tag{3.1}$$

$x = \sqrt{6 - x} \Rightarrow x^2 = 6 - x \Leftrightarrow x^2 + x - 6 = 0$

pq-formeln: $x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{24}{4}} \Leftrightarrow (x = -3) \vee (x = 2)$

Eftersom det är en implikation år höger och inte ekvivalens så behöver inte rötterna vara sanna

Testar för falska rötter:

$2 = \sqrt{6 - 2}$ Sann

$2 = \sqrt{6 - (-3)}$ Falsk $2 \neq \sqrt{6 - (-3)}$

Exempel2:

$$(x + 2)(x + 1) = 2x(x + 1) \tag{3.2}$$

$(x + 2)(x + 1) = 2x(x + 1)$ is not $\Rightarrow (x + 2) = 2x$ (x=0 is not allowed)

Insted do ass following:

$(x + 2)(x + 1) = 2x(x + 1) \Leftrightarrow (x + 2)(x + 1) - 2x(x + 1) = 0 \Leftrightarrow (x + 1)(x + 2 - 2x) = 0 \Leftrightarrow$

$\Leftrightarrow (x + 1 = 0) \vee (2 - x = 0)$

svar ej: x=1 och x=2

svar: x=1 eller x=2

svar: $x = 1 \vee x = 2$

3.2 Mängder

$$
\begin{aligned}
&\text{Naturliga tal: } \mathbb{N} = \{0, 1, 2, 3..\} \\
&\text{Heltal: } \mathbb{Z} = \{.. -2, 1, 0, 1, 2..\} \\
&\text{Rationella tal: } \mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b \\
&\text{Irrationella tal: } \mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \{x | x \in \mathbb{R}, x \notin \mathbb{Q}\} \\
&\text{Reella tal: } \mathbb{R} = \mathbb{P} \cup \mathbb{Q}
\end{aligned}
$$

$$
\begin{aligned}
&A \cup B = \{x : (x \in A) \vee (x \in B)\} \\
&A \cap B = \{x : (x \in A) \wedge (x \in B)\} \\
&A \setminus B = \{x : (x \in A) \wedge (x \notin B)\} \\
&A^{\#} = \{x : (x \in X) \wedge (x \notin A)\}
\end{aligned}
$$

Exempel:

$$\text{Bevisa: } X \setminus (A \cup B) = (X \setminus A) \wedge (X \setminus B) \tag{3.3}$$

$$
\begin{aligned}
&x \in (X \setminus (A \cup B)) \Rightarrow (x \in X) \wedge (x \notin (A \cup B)) \Rightarrow \\
&\Rightarrow (x \in X) \wedge (x \notin A) \wedge (x \notin B) \Rightarrow \\
&\Rightarrow (x \in X \setminus A) \wedge (x \in X \setminus B) \Rightarrow \\
&\Rightarrow x \in (X \setminus A) \wedge (X \setminus B) \Rightarrow \\
&\Rightarrow X \setminus (A \cup B) \subseteq (X \setminus A) \wedge (X \setminus B)
\end{aligned}
$$

3.3 Bevis

3.3.1 Induktions bevis

steg 1: bevisa att det gäller för basfallet

Steg 2: bevvisar att p => p a. Antar att det stämmer för pp b. Vissar att med att stopa in antagandet i pp+1 så blir hl = vl Tips: Förenkla hl först och sedan vl på klad papper fram och tillbacka **Exempel Recursion:**

$a_1 = 2, a_{n+1} = \frac{7a_n}{7-a_n}, n \in \mathbb{N}$

Vissa med induktion att $a_{n+1} = \frac{14}{7-2n}$

Bevis med induktions

steg 1: visar att påståendet som vi kallar p gäller för basfallet (n=1)

$VL_1 : a_{1+1} = a_2 = \frac{7 \cdot 2}{7-2} = \frac{14}{5}, HL_1 : a_{1+1} = a_2 = \frac{14}{7-2} = \frac{14}{5}$

steg 2: visar att $p_m \Rightarrow p_{m+1}$

steg 2a: antar att p_m gäller

$a_{m+1} = \frac{14}{7-2m}$

steg 2b: bevisar attt $p_m \Rightarrow p_{m+1}$ genom att andvända antagandet

$VL_{m+1} a_{m+2} = \frac{7a_{m+1}}{7-a_{m+1}} = \frac{7\frac{14}{7-2m}}{7-\frac{14}{7-2m}} = \frac{\frac{7 \cdot 14}{7-2m}}{\frac{7(7-2m)-14}{7-2m}} =$

$= \frac{7 \cdot 14}{7(7-2m+2)} = \frac{14}{7-2(m+1)}$

$HL_{m+1} : \frac{14}{7-2(m+1)}$

Enlight induktionsprincipen är p_m sann för alla $n = 1, 2, 3 \ldots$ VSB

3.3.2 Motsägelse bevis

Steg 1: Formulera utsagan och icke utsagan Steg 2: Hitta en motsägelse med utsagan Tips: Förenka båda led, tänk på teorin vi har, bättre att gå vaga moteveringar en inga alls

Bevis med motsägelse

Antar att motsatsen är sann

3.4 Delbarhet

$$\text{a är delbar med b, altså kvoten ger ingen rest. Vi följand: } a \mid b \tag{3.4}$$

Divitions algoritmen:

$$
\begin{aligned}
&a, b \in \mathbb{Z} \\
&a \geq 0 \wedge b \geq 0 \\
&a \mid b \Rightarrow (q \in \mathbb{Z} : q \geq 0) \wedge (r \in \mathbb{Z} : 0 \leq r \leq a) \text{ Sådant att} \\
&b = qa + r \\
&q = \text{ kvoten}, r = \text{ resten}
\end{aligned}
$$

3.4.1 Största Gemensama Delaren (SGD)

$$
\begin{aligned}
&SGD(a, b) \\
&a = bq + r \\
&0 \leq r \leq b
\end{aligned}
$$

Euklides algoritm:

$$
\begin{aligned}
&SGD(a, b) \\
&a = bq_1 + r_1 \\
&b = r_1 q_2 + r_2 \\
&r_1 = r_2 q_3 + r_3 \\
&r_2 = r_3 q_4 + r_4 \\
&. \\
&. \\
&. \\
&r_{k-3} = r_{k-2} q_{k-1} + r_k \\
&r_{k-2} = r_{k-1} q_k + 0 \\
&SGD(a, b) = r_k \\
&\text{Om } r_k = 1 \Rightarrow a \in \text{primtal} \vee b \in \text{primtal}
\end{aligned}
$$

Exempel:

$$\text{Förenkla } \frac{114}{96}$$

$$\begin{aligned}
&SGD(114, 96) : \\
&114 = 1 * 96 + 18 \\
&96 = 5 * 18 + 6 \\
&18 = 3 * 6 + 0 \\
&\frac{114}{6} = 19 \\
&\frac{96}{6} = 16 \\
&\frac{19}{16}
\end{aligned}$$

Lemma:

$$\begin{aligned}
&a, b \in \mathbb{Z} \\
&x, y \in \mathbb{Z} \\
&SGD(a, b) = ax + by
\end{aligned}$$

Aritmetiska fundamentalsatsen:

$$a \in \mathbb{Z} \wedge a \geq 2 \Rightarrow$$
$$\Rightarrow a \text{ kan endast primtalsfaktoriseras på ETT SÄTT}$$

Lemma 2.7:

$$a \geq 2 \wedge a \notin \text{ Primatal} \Rightarrow$$
$$\Rightarrow q \in \text{ Primatal} \wedge q \mid a \wedge \text{ a går att primtals faktoriera}$$

3.4.2 Primtal

Sats:
För att bestäma om tal a är ett primtal

$$a \geq 1 \land a \in \text{ Primatal}$$
om ett tall p finns som dellar a gäller följande
$$2 \leq p \leq \sqrt{a} \leq a$$

Exemple:

$$\text{Bestäm om 211 är ett primtal} \tag{3.5}$$

Om 211 är ett primtal så finns det inte en äkta delare a
$$2 \leq a \leq \sqrt{211}$$
$$\sqrt{211} \approx 16$$
$$a : \{\cancel{2}, \cancel{3}, \cancel{5}, \cancel{7}, \cancel{1}1, \cancel{1}3\}$$
a kan inte vara en äktadelare

Euklides algoritm:

Det finns oändligt många primtal

Bevis:
Motsägelsebevis

Antar att det finns ändligt många primtal
$$p_1, p_2, P_3, .., p_n$$
$$M = \prod_{k=1} n + 1 \Rightarrow M > p_j, j = 1, 2, 3, .., n$$
Vissar att M är ett primtal
$1 < b < M \land b \mid M$ Där b är minsta äkta delaren av M $\Rightarrow$
$$\Rightarrow M = p_1 * b \Rightarrow p_1 \mid 1 \land p \geq 2$$
Detta är falskt eftersom båda utsågorna kan inte vara samtidigt
Eftersom motsatsen inte fungerar resulterar det i att satsen är sann

3.5 Diofantiska ekvationer

Sats:

$$
\begin{aligned}
&ax + by = c \wedge a, b, c \in \mathbb{Z} a \neq 0, b \neq b \\
&\Rightarrow \\
&ax + by = c \text{ Där } SGD(a, b) = 1 \\
&\text{Har den anmäla lösningen:} \\
&x = Cx_0 - nb \wedge y = Cy_0 + na
\end{aligned}
$$

Exempel:

En lastbil lastas med 12kg packet och 20kg paket. Totalt väger lasten 296, hur många av varge packet? (3.6)

$$
\begin{aligned}
&ax + by = C \Leftrightarrow 12x + 20y = 296 \\
&\text{steg1: Testar om } SGD(a, b) \mid c \\
&SGD(20, 12) = 4 \Rightarrow 4 \mid 269 \\
&\text{steg2: delar SGD med HL och VL} \\
&\frac{12x - 20y}{4} = \frac{269}{4} \Leftrightarrow 3x - 5y = 74 \\
&SGD(5, 3): \\
&5 = 1 \cdot 3 + 2 \\
&3 = 1 \cdot 2 + 1 \\
&2 = 2 \cdot 1 + 0 \\
&\text{steg3: Hjälp ekvation för att hitta} x_0, y_0 \\
&3x_0 - 5y_0 = 1 \\
&1 = 3 - 1 * 2 \\
&1 = 3 - (5 - 3) \\
&1 = 2 \cdot 3 - 1 \cdot 5 \\
&x_0 = 2, y_0 = -1 \\
&\text{steg4: almäna lösningen } x = Cx_0 - bn, y = Cy_0 + an \\
&x = 74 \cdot 2 - 5n = 148 - 5n \\
&y = 74 \cdot (-1) + 3n = -74 + 3n \\
&\text{steg5: hittar godtyckliga lösningar} \\
&\text{Intervallet som n ligger i för x-termen:} \\
&n = \not{2}9, 28, 27, \ldots, x = 148 - 5 \cdot 28 = 148 - 140 = 8 \\
&\text{Intervallet som n ligger i för y-termen:} \\
&n = \not{2}7, 28, 29, \ldots, x = -74 + 3 \cdot 28 = 74 - 84 = 10 \\
&n = 28, x = 8, y = 10 \\
&12 \cdot 8 + 20 \cdot 10 = 296
\end{aligned}
$$

3.6 Talbaser

3.6.1 konvertera från decimal bass till annan bas

$$175_8 = 1 \cdot 8^2 + 7 \cdot 8^1 + 5 \cdot 8^0$$

3.6.2 konvertera från annan bas till decimal bas

$$1609_{10} = (3 \cdot 8^3 + 1 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0) = 3111_8$$

eller så kan man andvända euklides algoritm

$$\begin{aligned}
&\text{skriv 517 i talbas 3} \\
&517 = 172 \cdot 3 + 1 \\
&172 = 57 \cdot 3 + 1 \\
&57 = 20 \cdot 3 + 0 \\
&20 = 6 \cdot 3 + 2 \\
&6 = 2 \cdot 3 + 0 \\
&2 = 0 \cdot 3 + 2 \\
&\text{Svar: } 517_{tio} = 202011_{tre}
\end{aligned}$$

3.6.3 Andra exempel

Exempel: Skriv 137_{nio} i bass tre

$$\begin{aligned}
137_{nio} &= 1 \cdot 9^2 + 3 \cdot 9^1 + 7 \cdot 9^0 = 1 \cdot 3^4 + 3 \cdot 3^2 + 7 \cdot 3^0 = \\
&= 1 \cdot 3^4 + 1 \cdot 3^3 + 0 \cdot 3^2 + 2 \cdot 3^1 + 1 \cdot 3^0 = 11021_{tre}
\end{aligned}$$

3.7 Functioner

Typer av funktioner:

Injektion: alla element x har olika värden y $f : A \to B, \{\forall x \in A : x_1 \neq x_2, f(x_1) \neq f(x_2)\}$
Surjektion: mängd D är definitions mängden $\{g : C \to D, g(x) = y, (\forall y \in D \wedge \exists x \in C)\}$
Bijektion: Injektion $\wedge$ Surjektion

Kareskaprodukten:

$$A \times B = \{(a, b) : a \in A \wedge b \in B\}$$
Låt $A = \{1, 2, 3\} \wedge B = \{x, y, z, w\}$
$$AxB :$$
$$\{(1, x), (1, y), (1, z), (1, w)$$
$$(2, x), (2, y), (2, z), (2, w)$$
$$(3, x), (3, y), (3, z), (3, w)\}$$

3.7.1 Inversen

En funktions invers kan enda

$f : A \to B \wedge Bijektiv \Rightarrow f^{-1}(x)$ Finns, där
(1) $x = f^{-1}(y) \Leftrightarrow y = f(x)$
(2) $D_{f^{-1}} = V_f \Leftrightarrow D_f = V_{f^{-1}}$
(3) $x = f^{-1}(f(x)), x \in D_f = V_{f^{-1}}$
(3) $y = f^{-1}(f(y)), y \in D_f = V_{f^{-1}}$

3.7.2 Relatioiner

Relation: xRy
Reflexiv: $\forall x \in X : xRx$
Symetrisk: $xRy \Rightarrow yRx, x \in X \wedge y \in X$
Transitiv: $(xRy) \wedge (yRz) \Rightarrow xRz, \forall x, y, z \in X$
Ekvivalnsrelation: Reflexiv och Symetrisk Transitiv

3.8 Summor

3.8.1 Aritmetiska summor

$$s_n = a_1 + a_2 + a_3 + \ldots + a_n = \frac{n\left(a_1 + a_n\right)}{2} \tag{3.7}$$

3.8.2 Geometriska summor

Börjar altid med exponenten 0 och gör om summan så att den passar i följade talföljd:

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\left(k^n - 1\right)}{k - 1} \tag{3.8}$$

Exempel:

$$\sum_{k=n}^{2n} (2^k - k) \tag{3.9}$$

sätter f = 0 = k -n

$$\sum_{f=0}^{n} (2^{f+n} - (f+n)) = 2^n * \sum_{f=0}^{n} (2^f) - \sum_{f=0}^{n} (f+n)$$

$$\frac{2^n(2^{n+1} - 1)}{2 - 1} - \frac{3n(n+1)}{2} = 2^{2n+1} - 2^n - \frac{3n(n+1)}{2}$$

3.9 Kongruensräkning

Räkneregler:

$$
\begin{aligned}
&a+b \pmod n \equiv a \pmod n + b \pmod n\\
&a\cdot b \pmod n \equiv a \pmod n \cdot b (modn)\\
&a^b \pmod n \equiv (a \pmod n)^b
\end{aligned}
$$

Exempel: Vilket är det minsta positiva rest som kan erhållas vid division av 19^{18} med 17?

$19^{18} \equiv 2^{18} \pmod{17} \equiv 2^4 \cdot 2^4 \cdot 2^4 \cdot 2^4 \cdot 2^2 \pmod{17} \equiv (-1)^4 \cdot (-1)^4 \cdot (-1)^4 \cdot (-1)^4 \cdot 2^2 \pmod{17}$
svar: resten blir 4

3.10 Kardinalitet

Kardinalitet eller “mäktighet” är ett sett att räkna med mängders sorlek och alla oändliga mängder har samma kardinalitet fast det är en delmängd. Naturliga tall har samma kardinalitet som reala tal trots att naturliga tal är en del mängd av reala talen

Låt A och B vara mängder. Vi sägger att A och B har samma kardinalitet då det finns en bijektion
$\exists f : A \to B, A \sim B$
Vi säger att A står i relation med B omm A och B har samma kardinalitetARB

3.10.1 Uppräkneligamängder

En mängd X sägs vara upräknerlig omm X har samma kardinalitet som $\mathbb{N}$
$\exists g : \mathbb{N} \to X$ där g är bijektiv
Exempel på upräknerliga mängder är $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \{1, 5, 78\}$
Exempel på ej upräknerliga mängder $\mathbb{R}, (0, 1)$

3.11 Polynom

3.11.1 Polynom division

Triviala delare: ej heltals kvot med delaren, har en kostant sådant $\lambda \cdot f, \lambda \notin \mathbb{Z}$

Äkta delare: heltals kvot med delaren $f(x), \exists a \in polynom : a \mid f(x)$

Irreducible: om polynomet endast har triviala delare det finns lösningar heltaslösningar $f(x) = 0$

Reducible: om polynomet har äkta delare

Multiplisitet: vilken grad polynomet har

Exempel:

Man vet att ekvationen $z^4-2z^3-7z^2+26z-20 = 0$ har roten $z = 2+i$. Lös ekvationen fullständigt. (3.10)

$z = 2 + i$ Är en lösning är också konjugatet en lösning enligt faktorsatsen $\overline{z} = a - bi$

$z = 2 \pm i$

Vilket betyder att följande går att factorisera ut polynomet

$(z - (2 + i))(z - (2 - i)) = z^2 - 4z + 5$

Långdivition (liggande stolen):

$$
\begin{array}{r|l}
 & x^2 + 2x - 4 \\
\hline
x^2 - 4x + 5 & x^4 - 2x^3 - 7x^2 + 26x - 20 \\
 & -x^4 + 4x^3 - 5x^2 \\
\hline
 & 2x^3 - 12x^2 + 26x \\
 & -2x^3 + 8x^2 - 10x \\
\hline
 & -4x^2 + 16x - 20 \\
 & 4x^2 - 16x + 20 \\
\hline
 & 0
\end{array}
$$

$z^2 + 2z - 4 = 0$ Är också en läsning som tilslut ger följande

$z = -1 \pm \sqrt{5}$

$z = 2 \pm i$

Varge n grads polynom har altid n stycken komplexa lösningar

3.11.2 Faktorsatsen

$$f(x) = (x - \alpha)p(x)$$

Exempel, Gemensam root hos två polynom:

$p(x) = x^4 - x^3 + x^2 + 2 = 0,\ g(x) = x^3 + 4x^2 + 4x + 3 = 0$ har en Gemensam root
Eftersom polynomen har en Gemensam root vet vi att det finns en gemensam$(x - \alpha)$

$f(x) = (x - \alpha)p_1(x)$
$g(x) = (x - \alpha)p_2(x)$
euklides algoritm:
$f(x) = (x - 5)g(x) + 17(x^2 + x + 1)$
$g(x) = (\frac{1}{17}(x + 3))(17(x^2 + x + 1)) + 17(x^2 + x + 1) + 0$

$h(x) = x^2 + x + 1$
$f(x) = (x - 5)(x + 3)h(x) + 17h(x) = h(x)((x - 5)(x + 3) + 17) = h(x)(x^2 - 2x + 2)$
$g(x) = (x + 3)h(x)$

$f(x) = 0 \Leftrightarrow (h(x) = 0 \vee x^2 - 2x + 2 = 0)$
$h(x) = 0 \Rightarrow x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
$x^2 + 2x + 2 = 0 \Rightarrow x = 1 \pm \sqrt{1 - 2} = 1 \pm i$

Exempel, Heltalslösning med okänd konstant:

$x^3 + bx^2 - 7x - 7 = 0$ har en heltals lösning
Eftersom ekvationen har en heltals lösning vet vi att $\alpha \mid 7$
Kandidaterna är $\{1, -1, 7, -7\}$
$I(x = 1)\ 1 + b \cdot 1 - 7 - 7 = 0 \Rightarrow b = 13 \in \mathbb{Z}$
$II(x = -1)\ -1 + b \cdot -1 + 7 - 7 = 0 \Rightarrow b = 1 \in \mathbb{Z}$
$III(x = 7)\ 7 + b \cdot 7 - 49 - 7 = 0 \Rightarrow b = \_ \notin \mathbb{Z}$
$IV(x = -7)\ 7 + b \cdot -7 + 49 - 7 = 0 \Rightarrow b = \_ \notin \mathbb{Z}$
svar: b=13, b=1

Exempel, Rationell root:

$g(x) = 2x^3 + 2x^2 + 2x + 3 = 0$ har en rationell root
Eftersom ploynomet har en rattionel root vet vi att:
$\frac{p}{q}\ p, q \in \mathbb{Z},\ SGD(p, q) = 1,\ p \mid 3 \wedge q \mid 2\ (a_0)$
Det möjliga delarna är $p = \pm 1, \pm 3,\ q = \pm 1, \pm 2$
$x = -\frac{3}{2}$ är den enda av kandidaterna som ger en sann root
Vilket vi löser genom polynom divition

Exempel, Renimaginär root:

$z^4 + 4z^3 + 8z^2 + 12z + 15 = 0$ har en renimachinär root
Eftersom ploynomet har en rentimaghinär root vet vi att $z = bi, b \in \mathbb{R}, b \neq 0$
$0 = b^4i^4 + 4b^3i^3 + 8b^2i^2 + 12bi + 15 = b^4 - 4b^3i - 8b^2 + 12bi + 15 = (b^4 - 8b^2 + 15) + (12b - 4b^3)i =$
$0 = b^4 - 8b^2 + 15$
$0 = 12b - 4b^3 = 4b(3 - b^2) \Rightarrow b = 0, b = \sqrt{3}, b = -\sqrt{3}$
$b = 0$ är ej en giltig lösning. Enlight faktorsatsen får vi följande
$p(z) = (z - \sqrt{3}i)(z + \sqrt{3}i)Q(x) = (z^2 + 3)Q(z)$
Vilket vi löser genom polynom divition

Exempel, Dubel root:

$p(t) = t^3 - 5t^2 + 3t + 9 = 0$ har en dubbel root
Eftersom ploynomet har en dubbel root vet vi att derivatan ger os en root$p'(t) = 0$
Att räkna ut derivatan gör man genm formeln $ax^b \Rightarrow b \cdot ax^{b-1}$
$p'(t) = 3t^2 - 10t + 3 = 0 \Rightarrow t_1 = 3, t_2 = \frac{1}{3}$ dock är sista en falsk root
$p(t) = x - 3^2Q(t)$
Vilket vi löser genom polynom divition

Exempel, SGD polynom:

Förenkla$\dfrac{x^4+x^3+2x-4}{x^4-x^3-2x-4}$

$SGD(x^4+x^3+2x-4, x^4-x^3-2x-4):$

$x^4+x^3+2x-4 = 1\cdot x^4-x^3-2x-4+(2x^3+4x)$

$x^4-x^3-2x-4 = \dfrac{x}{2}-\dfrac{1}{2}\cdot(2x^3+4x)+(-2x^2-4)$ med polynom divition

$2x^3+4x = -x\cdot(-2x^2-4)+0$

Med SGD så kan vi ta ett polynom som är hjämt delbart ex: $-2(x^2+x)$

polynom divition: $\dfrac{x^4+x^3+2x-4}{x^2+x} = x^2+x-2$

polynom divition: $\dfrac{x^4-x^3-2x-4}{x^2+x} = x^2-x-2$

$\dfrac{x^4+x^3+2x-4}{x^4-x^3-2x-4} = \dfrac{x^2+x-2}{x^2-x-2}$

Chapter 4

Single Variable Calculus

4.1 Basic

4.1.1 Mängder

Naturliga tal: $\mathbb{N} = \{0, 1, 2, 3..\}$ or $\{1, 2, 3..\}$
Heltal: $\mathbb{Z} = \{.. - 2, 1, 0, 1, 2..\}$
Rationella tal: $\mathbb{Q} = \{\frac{a}{b} | a, b \in \mathbb{Z}, b$
Irrationella tal: $\mathbb{P} = \frac{\mathbb{R}}{\mathbb{Q}} eller \{x | x \in \mathbb{R}, x \notin \mathbb{Q}\}$
Reella tal: $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$

$$A \cup B = \{x : (x \in A) \vee (x \in B)\}$$
$$A \cap B = \{x : (x \in A) \wedge (x \in B)\}$$
$$A \setminus B = \{x : (x \in A) \wedge (x \notin B)\}$$
$$A^{\#} = \{x : (x \in X) \wedge (x \notin A)\}$$

4.1.2 Intervall

$$\text{Open intervall} = (1, 4)$$
$$\text{Closed intervall} = [\,1, 4]$$

$$[\,a, b] \ = \{x | a \leq x \leq b\} \tag{4.1}$$

$$[\,a, \infty[\tag{4.2}$$

$$]\,-\infty, \infty[\tag{4.3}$$

Exempel: Olikheter och intervall

$$\frac{2}{x-3} < \frac{5}{x} \tag{4.4}$$

$$\begin{aligned}
&\frac{2}{x-3} - \frac{5}{x} < 0\\
&\frac{(x)2}{x(x-3)} - \frac{5(x-3)}{x(x-3)} < 0\\
&\frac{2x-5x-15}{x(x-3)} < 0\\
&\frac{-3x-15}{x(x-3)} < 0\\
&\frac{-3(x-5)}{x(x-3)} < 0\\
&x \neq 0, x \neq 3
\end{aligned}$$

Värde tabell:

	x<0	0<x<3	3<x<5	5<x
x-5	-	-	-	+
-3	-	-	-	-
x	-	+	+	+
x-3	-	-	+	+
hela	+	-	+	-

4.1.3 Funktion

$f : A \to B$

A: domain/definitionsmängd, B: Målmängd/kodomängd/värdemängd

Injektion: alla element x har olika värden y $f : A \to B, \{\forall x \in A : x_1 \neq x_2, f(x_1) \neq f(x_2)\}$

Surjektion: mängd D är definitions mängden $\{g : C \to D, g(x) = y, (\forall y \in D \wedge \exists x \in C)\}$

Bijektion: Injektion $\wedge$ Surjektion

$f \circ g = f(g(x))$

Begränsad: functionen är inom ett intervall, altså nedåt och uppåt

Begränsat nedåt: functionenn är endast begrensad nedåt

Begränsat uppåt: functionenn är endast begrensad uppåt

Jämn function: altså är den symetrisk, ex: $x^2,\ f(-x) = f(x)$

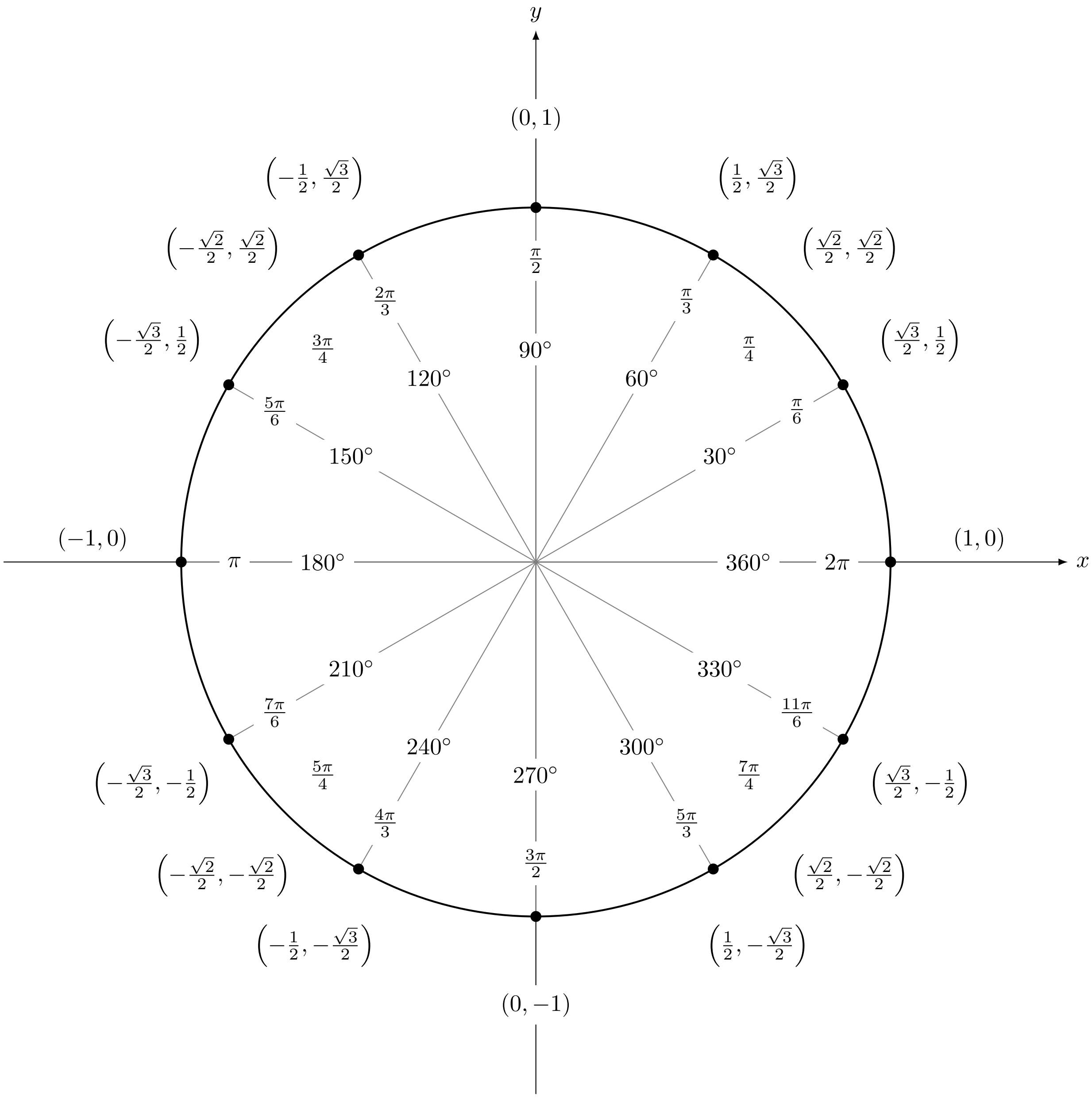
Udda function: altså är den spefelvänt symetrisk, ex: $x^3,\ f(-x) = -f(x)$

Polynom $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{n} (a_i x^i)$

Rationell funktion:$R(x) = \frac{P(x)}{Q(x)}$, ex: $x^{-1} = \frac{1}{x}$ ej polynom men rationell funktion

4.1.4 Trigonometri

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
cos	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$



Sats:

$$
\begin{aligned}
&360^\circ = 2\pi rad \\
&v_g = v_r * \frac{180^\circ}{\pi} \\
&v_r = v_g * \frac{\pi}{180^\circ}
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&-1 \leq \sin t \leq 1 \\
&-1 \leq \cos t \leq 1
\end{aligned}
$$

Sats:

$$
\begin{aligned}
&\cos(-t) = \cos(t) \\
&\sin(-t) = -\sin(t) \\
&\tan(-t) = \frac{\sin(-t)}{\cos(-t)} = \frac{-\sin(t)}{\cos(t)}
\end{aligned}
$$

Additionsformlerna:

$$
\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha) \\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) \\
\cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\beta)\sin(\alpha) \\
\cos(\alpha-\beta) &= \cos(\alpha)\cos(\beta) + \sin(\beta)\sin(\alpha)
\end{aligned}
$$

Trigonometriska ettan:

$$
\begin{aligned}
&(\sin t)^2 + (\cos t)^2 = 1 \\
&\sin^2 t + \cos^2 t = 1
\end{aligned}
$$

4.1.5 Exempel: Trigonometri

$$
\begin{aligned}
\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}
\end{aligned}
$$

4.2 Gränsvärden

$$\lim_{x\to x_0} f = L \wedge \lim_{x\to x_0} g = M \Rightarrow \lim_{x\to x_0}(f+g) = L+M$$

Squeeze therum: $h(x) \leq f(x) \leq g(x),\ \lim_{x\to x_0} h(x) = \lim_{x\to x_0} g(x_0) = L \Rightarrow \lim_{x\to x_0} f(x) = L$

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

$\lim_{x\to\infty}\sin(x)$ är ej definerad

Example: Limit för sin

$$\text{Lös: } \lim_{x\to 0}\frac{\sin(2x^2)}{x^2}$$

$$\lim_{x\to 0}\frac{2\cdot\sin(2x^2)}{2x^2} = 2\cdot\lim_{x\to 0}\frac{\sin(2x^2)}{2x^2} = 2$$

Example: 0/0

$$\text{Lös: } \lim_{x\to 0}\frac{x\cdot\cos(x)+\frac{\sin(x)}{x}}{x+x^2}$$

$$\lim_{x\to 0}\frac{x\cdot\cos(x)+\frac{\sin(x)}{x}}{x(1+x)} = \frac{1+1}{1} = 2$$

Example: roten ur i nämnaren

$$\text{Lös: } \lim_{x\to 0}\frac{x}{\sqrt{x+9}-3}$$

$$\lim_{x\to 0}\frac{x(\sqrt{x+9}+3)}{(\sqrt{x+9}-3)(\sqrt{x+9}+3)} = \lim_{x\to 0}\frac{x(\sqrt{x+9}+3)}{x+9-9} = \lim_{x\to 0}\sqrt{x+9}+3 = 6$$

Example: roten ur i nämnaren

$$\text{Lös: } \lim_{x\to-\infty}\frac{2x-1}{\sqrt{3x^2+x+1}}$$

$$\lim_{x\to-\infty}\frac{x(2-\frac{1}{x})}{\sqrt{x^2(3+\frac{1}{x}+\frac{1}{x^2})}} = \lim_{x\to-\infty}\frac{x(2-\frac{1}{x})}{x\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}} = \frac{-2}{\sqrt{3}}$$

Example: infinity sin and squeeze

$$\text{Lös: } \lim_{x\to\infty}\frac{\sin(x)}{x}$$

$$\left|\frac{\sin(x)}{x}\right| = \frac{|\sin(x)|}{|x|} \leq \frac{1}{|x|} \Rightarrow \text{ squeeze } \lim_{x\to\infty}\left|\frac{\sin(x)}{x}\right| = 0$$

$$\Rightarrow \lim_{x\to\infty}\frac{\sin(x)}{x} = 0$$

4.2.1 Kontinuitet

en funktion är kontnuelig i punkt x_0, $\forall\varepsilon \wedge \exists\delta : |x - x_0| < \delta \Leftrightarrow |f(x) - f(c_0)| < \varepsilon$
Altså så finns det inga hopp i funktionen. Man kan dra penan på grafen utan att släppa
om functionen bestor av flera utryck är funktionen kontenuerlig om alla utrycken är det

4.3 Derivator

Tangent: linjen som har samma lutining i en punkt som functionens derivata
$y - f(x_0) = f'(x_0)(x - x_0) \Rightarrow y = ax + b$
Samband: *Deriverbar* $\Rightarrow$ *Kontinuerlig*

Räkneregler

f	f'
c	0
x	1
x^n	nx^{n-1}
a^x	$a^x \ln a$
e^{kx}	ke^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$1 + \tan^2 x \frac{1}{\cos^2 x}$
$\ln x$	$\frac{1}{x}$

$(f + g)' = f' + g'$
$(cf)' = cf'$
$(fg)' = f'g + fg'$ Produktregeln
$(\dfrac{f}{g})' = \dfrac{f'g - fg'}{g^2}$ Kvotregeln, $g \neq 0$

4.3.1 Kjedje regeln

om functionen g är deriverbar i x och functionen är deriverbar i $g(x)$
$\Rightarrow h(x) = f \circ g = f(g(x))$deriverbar i x
$h'(x) = (f(g(x)))' = f'(g(x))g'(x)$

Example: Kjedje regeln

Lös: $(x^x)'$

$$(x^x)' = (e^{x \ln x})' = e^(x \ln x)(\ln x + x\frac{1}{x}) = e^{x \ln x}(1 + \ln x) = x^x(1 + \ln x)$$

4.3.2 L'Hôpital's rule

$$f, g : \mathbb{R} \to \mathbb{R} \wedge (\lim_{x \to x_0} \frac{f}{g} = \frac{0}{0} \vee \text{``} \lim_{x \to x_0} \frac{f}{g} = \frac{\pm\infty}{\pm\infty}'') \Rightarrow \lim_{x \to x_0} \frac{f}{g} = \lim_{x \to x_0} \frac{f'}{g'}$$

Example: L'Hôpital's rule

Lös: $\lim_{x \to 1} \frac{x^4 - 6x^3 + 5x^2}{x^3 - 8x^2 + 7x}$

$\lim_{x \to 1} \frac{x^4 - 6x^3 + 5x^2}{x^3 - 8x^2 + 7x} = \text{``} \frac{0}{0} \text{``}$

L'Hôpital's rule $\Rightarrow \lim_{x \to 1} \frac{4x^3 - 18x^2 + 10x}{3x^2 - 16x + 7}$

$= \frac{-4}{-6} = \frac{2}{3}$

4.3.3 Medelvärdessatsen

Anta att f är kontinuerlig på $[a, b]$ och deriverbar på $(a, b) \Rightarrow \exists c \in (a, b)$

så att $\frac{f(b) - f(a)}{b - a} = f'(c)$

Example: Medelvärdessatsen

Bevisa att för varge $a > b$ gäller följande$|\cos a - \cos b| \leq |a - b|$

$f(x) = \cos x$är kont och deriverbar i $\mathbb{R} \wedge [a, b]$

MVS $\Rightarrow \exists c \in (a, b) : f'(c) = \frac{f(b) - (a)}{b - a} = \frac{\cos(b) - \cos(a)}{b - a} = -\sin(c)$

$\Rightarrow |\frac{\cos(b) - \cos(a)}{b - a}| = |-\sin(c)| \Rightarrow \frac{|\cos(b) - \cos(a)|}{|b - a|} = |-\sin(c)| \leq 1$

$\frac{|\cos(b) - \cos(a)|}{|b - a|} \leq 1 \Rightarrow |\cos(b) - \cos(a)| \leq |a - b|$

4.3.4 Rolle

Anta att f är kontinuerlig på $[a, b]$ och deriverbar på (a, b). Om $f(a) = f(b) \Rightarrow \exists c \in (a, b) : f'(c) = 0$

4.3.5 växande funktioner

växande: $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
stänkt växande: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
avtagande: $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$
stänkt avtagande: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

4.3.6 Högreordnings devivator

andragrads derivator: $(f')' = f'' = \dfrac{d^2 f}{dx^2}$
tredjegrads derivator: $(f'')' = f''' = \dfrac{d^3 f}{dx^3}$
ntegrads derivator: $f^{(n)} = f^n = \dfrac{d^n f}{dx^n}$

4.3.7 Impericit derivering

deriverar båda sidorna om modifierat

Example: Impericit derivering

Bestäm tangenten till $x^3 + y^3 = 6xy\ i\ (3,3)$
$(3^3 + 3^3 = 6 \cdot 3 \cdot 3)$ -sant
$y - y_0 = f'(x_0)(x - x_0) \Rightarrow y - 3 = f'(3)(x - 3)$
Implicit derivering: $3x^2 + 3y^2y' = 6y + 6xy'$
$\Rightarrow 3x^2y' - 6xy' = 6y - 6x^2 \Rightarrow y'(3x^2 - 6x) = 6y - 3x^2 \Rightarrow y' = \dfrac{6y - 3x^2}{3x^2 - 6x}$
$\Rightarrow y'(3) = \dfrac{63 - 3 \cdot 3^2}{3 \cdot 3^2 - 6 \cdot 3} = -1$
$\Rightarrow y - 3 = -(x - 3) \Rightarrow x + y = 6 \vee y = -x + 6$

4.3.8 invers funktioner

$f : A \to B \wedge Bijektiv \Rightarrow f^{-1}(x)$ Finns, där
(1) $x = f^{-1}(y) \Leftrightarrow y = f(x)$
(2) $D_{f^{-1}} = V_f \Leftrightarrow D_f = V_{f^{-1}}$
(3) $x = f^{-1}(f(x)), x \in D_f = V_{f^{-1}}$
(3) $y = f^{-1}(f(y)), y \in D_f = V_{f^{-1}}$

4.3.9 exponetial och logaritm

$$\frac{a}{b}^{-3} = \frac{b}{a}^{3}$$
$$\sqrt{a} = a^{\frac{1}{2}}$$
$$a^{x+y} = a^x a^y$$
$$(a^x)^y = a^x a^y$$
$$a^{-x} = \frac{1}{a^x}$$
$$a^{\frac{m}{n}} = \sqrt{a^m}^n$$

(1): $b = a^x \Leftrightarrow \log_a(b) = x$ för: $a > 0, b > 0, a \neq 1$

(2): $\log_a(\frac{b}{c}) = \log_a(b) - \log_a(c)$

(3): $\log_a(b * c) = \log_a(b) + \log_a(c)$

(4): $\log_a(b^d) = d \log_a(b)$

(5): $\log_a(b) = \frac{\log_f(b)}{\log_f(a)}$

(6): $\log_a(a) = 1$

(7): $\log_a(1) = 0$

(8): $a^{\log_a(x)} = x$

(9): $\log_{a^c}(b) = \frac{1}{c} \log_a(b)$

4.3.10 odefinerad form

$$\frac{0}{0}, \infty \cdot 0, 1^{\infty}, \infty^0, 0^0$$

4.3.11 inversa trigometriska funktioner

arcsin

$$
\begin{aligned}
&\cos : [\frac{-\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \\
&\arcsin 1 = \frac{\pi}{2} \\
&\arcsin 0 = 0 \\
&\arcsin \pi = \text{odefinerad} \\
&\sin(\arcsin(x)) = x \\
&\arcsin(\sin(x)) = x \\
&(\arcsin(x))' = \frac{1}{\sin'(\arcsin(x))} = \frac{1}{\cos(\arcsin(x))} = \\
&\frac{1}{\sqrt{\cos^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

arccos

$$
\begin{aligned}
&\cos : [0, \pi] \to [-1, 1] \\
&\arccos 1 = 0 \\
&\arccos 0 = \frac{\pi}{2} \\
&\arccos \pi = \text{odefinerad} \\
&\cos(\arccos(x)) = x \\
&\arccos(\cos(x)) = x \\
&(\arccos(x))' = \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

arctan

$$
\begin{aligned}
&\cos : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R} \\
&\tan(\arctan(x)) = x \\
&\arctan(\tan(x)) = x \\
&(\arctan(x))' = \frac{1}{1 + x^2}
\end{aligned}
$$

exempel

$$
\begin{aligned}
\tan(\arccos x) &= \frac{\sin(\arccos x)}{\cos\arccos x} = \frac{\sqrt{1-x^2}}{x} \\
\cos(\arctan x) &= \cos\frac{\arcsin x}{\arccos x} = \frac{1}{1+x^2} \\
\sin(\arccos x) &= \sqrt{1-x^2} \\
\cos(\arcsin x) &= \sqrt{1-x^2}
\end{aligned}
$$

4.4 Grafritning

Ta reda på extrem punkterna och rita utifrån det

Extremvärden

global minimum punkt: $x_0,\ \forall x : f(x) \geq f(x_0)$
lokal minimum punkt: $x_0,\ \forall x$ nära $x_0 : f(x) \geq f(x_0)$
global maximum punkt: $x_0,\ \forall x : f(x) \leq f(x_0)$
lokal maximum punkt: $x_0,\ \forall x$ nära $x_0 : f(x) \leq f(x_0)$
Kritisk punkt: $f'(x) = 0$

Komplexity

konvex: Tangent liger altid under funktionen $y'' > 0$
konkav: Tangent liger altid över funktionen $y'' < 0$
Inflektionspunct: då funktionen byter från konvex till konkav eller tvärtom

Exemple: Datan som behövs beräknas vid grafritning

Rita functionen $y = (x^2 - 1)^3$

(1) Kollar om functionen är Konternuelig:

Eftersom funkrionen består av polynom är den konternuerlig

(2) Extrem punkter:

(I) Derivatan: functionen är deriverbar$y' = 3(x^2 - 1)^2 \cdot 2x = 0$

$\Rightarrow x = 0$

(II) Singulär punkter:

eftersom funktionen är deriverbar i alla punkter i definitions mängden

Så funns det inga singulär punkter

(III) End värden: Eftersom funktionen är ej definerad i ett intervall så finns det inga

(3) Komplexitet:

Andra derivatan avgör om funktionen är knvex eller konkav

$y'' = (6x(x^2 - 1)^2)' = (6x(x^4 - 2x^2 + 1))' = (6x^5 - 12x^3 + 6x)' = 30x^4 - 36x^2 + 6 = 0$

$\Rightarrow t^2 - \frac{6}{5}t + \frac{1}{5} = 0 \Rightarrow t = \frac{6}{10} \pm \sqrt{\frac{36}{100} - \frac{20}{100}} = \frac{6}{10} \pm \frac{4}{10}$

$t = 1 \vee t = \frac{1}{5} \Rightarrow x = \pm 1 \vee x = \pm\frac{1}{\sqrt{5}}$

(4) Asymptoter:

(I) lodräta asymptoter:

$\lim_{x->+\infty} y = +\infty$

$\lim_{x->-\infty} y = +\infty$

(II) vågräta asymptoter:

Eftersom funktionen inte är en kvot finns det inga sådana asymptoter

	$x < -1$	$x = -1$	$-1 < x < -\frac{1}{\sqrt{5}}$	$x = -\frac{1}{\sqrt{5}}$	...
f'	$-$	$-$	$-$	$-$	...
f''	$-$	0	$+$	0	...
f	avtagande	avtagande	avtagande	avtagande	...
	kokav	inflektions punkt	konvex	inflektions punk	...

4.5 Optemering

Sats: om funktionen $f(x)$ är knternuerlig på det slutna intervalet$[a, b]$
så antar det sitt största värde och minsta värde där$\exists x_1, x_2 \in [a, b]$
så att$f(x_1) \leq f(x) \leq f(x_2)$

Exempel: Max/Min värde

Låt $f(x) : [-1, 2] \to \mathbb{R}, f(x) = x^3 - 3|x|$
(1) Konternuelig:
f är konternuerlig i intervallet

(2) Extrem punkter:
(I) Derivatan: functionen är deriverbar i intervalet förutom då $x = 0$
låt f bestå av $f_1(x) = x^3 - 3x, x \geq 0 \wedge f_1(x) = x^3 + 3x, x < 0$
$\Rightarrow f_1'(x) = 3x^2 - 3, x \geq 0 \wedge f_2'(x) = 3x^2 + 3, x < 0$
$f_1'(x) = 0 \Rightarrow x = 1 \in [1, 2], f(1) = -2$
$f_2'(x) = 0 \Rightarrow x =$ Odefinerad
(II) Singulär punkter:
f är inte deriverbar då $x = 0, f(0) = 0$
(III) End punkterna: Eftersom funktionen är ej definerad i ett intervall så finns det inga
$f(-1) = -4$
$f(2) = 2$

(3) Största och minsta värdet:
$f(-1) < f(1) < f(0) < f(2)$
Svar: Max är 2, min är -4

4.6 Talfölder och serier

Def: Talföljder

En talföljd är en funktion $a : \mathbb{N} \to \mathbb{N}$
Vi skriver a_n istället för $a(n)$, $a_2 = a(2)$
Vi säger att $a_n \to a \in \mathbb{N}$ så är den **konvergent**
Vi säger att $a_n \to \infty$ så är den **divergent** eller ej existerande
Konvergent kan vara begränsad uppåt eller nedåt
Talföljed kan vara växande eller avtagande

$a_n \to a \vee b_n \to b$ Omm a och b existerar $a_n + b_n \to a + b$

Exempel: Derivatan av serier

$$\text{Ange } f'(2),\ f(x) = \sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 2^{2n}}$$
$$\text{Anger den första termen } (\frac{x-2}{4})' = \frac{1}{4}$$
$$f'(x) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{n(x-2)^{n-1}}{n^2 2^{2n}} \text{ Inre och yttre derivatan}$$
$$f'(2) = \frac{1}{4} + \sum_{n=2}^{\infty} \frac{n(2-2)^{n-1}}{n^2 2^{2n}} = \frac{1}{4} + 0 = \frac{1}{4}$$

Sats:

$$\text{Om } (a_n)^{\infty} \text{ är Konvergent } \Rightarrow (a_n),\ n = 1 \text{ är begränsad}$$

Sats:

Låt $a > 0$
(I) $a^n \to 0$ om $a < 1$
(II) $a^n \to +\infty$ om $a > 1$
$$(a_n)_{n=1}{}^{\infty} = \sum_{k=0}^{n} a_n = a_1 + a_2 + a_3 + \dots$$

Sats: Geometrisk serie

$$s_n = a + ak + ak^2 + \ldots + ak^{n-1} = \frac{a\,(k^n - 1)}{k-1}$$
$$\sum_{n=0}^{\infty} r^n \Rightarrow$$
Konvergent om$|r| < 1$
divergent om$|r| < 1$

Sats:

$$\sum_{n=0}^{\infty} a_n \text{ konvergent } \Rightarrow a_n \to 0$$

Sats: p-serie

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{1}{n^p} \Rightarrow \\ &\text{konvergent om } \Rightarrow p > 1 \\ &\text{divergent om } \Rightarrow p \leq 1 \end{aligned}$$

Sats:

$$\begin{aligned} &\text{Antag att } 0 \leq a_n \leq b_n \text{ för varge } n \in \mathbf{N} \\ &\sum_{n=1}^{\infty} a_n \text{ konvergent } \Rightarrow a_n \to 0 \\ &(I) \sum_{n=1}^{\infty} b_n \text{ konvergerar} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ konvergerar} \\ &(II) \sum_{n=1}^{\infty} a_n \text{ derigerar} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ derigerar} \end{aligned}$$

Sats:

$$\begin{aligned} &\text{Antag att } a_n > 0,\ b_n > 0 \text{ för varje } n \in \mathbf{N} \text{ och antag att } \frac{a_n}{b_n} \to L \neq 0\ \wedge L < \pm\infty \\ &\text{Då gäller att serien } \sum_{n=1}^{\infty} a_n \text{ och } \sum_{n=1}^{\infty} b_n \\ &\text{är båda divergenta} \end{aligned}$$

Sats: kvotkriterium

$$\begin{aligned} &\text{Låt } a_n > 0 \wedge \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \to L \Rightarrow \\ &(I)\ 0 \leq L \leq 1 \Rightarrow \text{ Konvergerar } \sum_{n=1}^{\infty} a_n (a_n \to 0) \\ &(II)\ L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ derigerar } (a_n \to +\infty) \end{aligned}$$

Exemple: Måste kunna

$$\sum_{n=1}^{\infty} \frac{5^n n!}{n^n}$$

$$\text{Låt } a_n = \frac{5^n n!}{n^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{5^{n+1}(n+1)!/n^n}{5^n n!/n^n} = 5\frac{(n+1)n^n}{(n+1)^{n+1}} = 5(\frac{n}{n+1})^n =$$

$$= 5\frac{1}{(1+1/n)^n} \to \frac{5}{e} > 1 \Rightarrow \text{ derigerar}$$

Sats: Rotkriteriet (Ovanlig)

$$\text{Låt } a_n > 0 \wedge \lim_{n\to\infty} \sqrt[n]{a_n} \to L \Rightarrow$$

$$(I)\ 0 \le L \le 1 \Rightarrow \text{ Konvergerar } \sum_{n=1}^{\infty} a_n (a_n \to 0)$$

$$(II)\ L > 1 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ derigerar } (a_n \to +\infty)$$

4.6.1 Serier med varierande tecken

Sats: leibinz

Antag att $\sum_{n=1}^{\infty} a_n$ är en alternerande serie

och att om $|a_n| \to 0 \wedge |a_n| \ge |a_{n+1}| \Rightarrow konvergent$

Def: absolut konvergent

Endast om $\sum_{n=1}^{\infty} |a_n|$ konvagerar $\Rightarrow$ då är den absolutkonvergent

om seriern är absolutkonvergent så är den också konvergent behöver inte vara tvärtom

4.6.2 Potensserier

Def: Potensserier

En potensserier kring x_0 är en serie på formen $\sum_{n=1}^{\infty} a_n (x - x_0)^n,\ a_n \in \mathbb{R}$ (koefficienten)

x är en variabel

Def: Konvergens radie

$T(x) = \sum_{n=1}^{\infty} \frac{f(x_0)^{(n)}}{n!}(x - x_0)^n$

vid exempelvis grad 2 så blir det andra derivatan man räknar ut

$R = L^{-1} = \frac{1}{L}$ (L är där serien konvergerar, R är konvergens radie) $\sum_{n=1}^{\infty} a_n(x - x_0)^n,\ a_n \in \mathbb{R}$ (koefficienten)

x är en variabel

Om konvergens radien är noll så finns det inte några intervar för konvergens eller divergens

Sats: Potensserier

En potensserie kring x_0 är en serie på formen $\sum_{n=1}^{\infty} a_n(x - x_0)^n a_n \in \mathbb{R}$ a_n koficienten x är en variabel

När det står ex $(3x + 2)^n$ så måste det stå med i $a_n, 3^n$

Exempel: Potensserier

$\sum_{n=1}^{\infty} \frac{(x-7)^n}{n(n+1)}$ är en potensserie där $x_0 = 7, a_n = \frac{1}{n(n+1)}$

$|\frac{a_{n+1}}{a_n}| = |\frac{1/((n+1)(n+2))}{1/(n(n+1))}| = |\frac{n}{n+2}| = |\frac{n}{n(1+2/n)}|$

$\Rightarrow |\frac{1}{1}| = 1,$ då $n \to \infty$

$\Rightarrow R = \frac{1}{1} = 1 \Rightarrow$ vi får följande intervall av potenserien

Absolutkonverget för $|x - 7| < 1 \Leftrightarrow 6 < x < 8$

Divergent för $|x - 7| > 1 \Leftrightarrow 8 < x \vee x < 6$

Testar edge fallen $x = 6, x = 8$

$\sum_{n=1}^{\infty} \frac{(8-7)^n}{n(n+1)} \Rightarrow |\sum_{n=1}^{\infty} \frac{(1)^n}{n(n+1)}|,\ |\frac{(1)^n}{n(n+1)}| \to 0 \Rightarrow$ absolutkonvergerar

$\sum_{n=1}^{\infty} \frac{(6-7)^n}{n(n+1)} \Rightarrow |\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}|,\ |\frac{(-1)^n}{n(n+1)}| \to 0 \Rightarrow$ absolutkonvergerar

Svar:

Absolutkonverget för $6 \leq x \leq 8$

Divergent för $8 < x \vee x < 6$

4.6.3 Taylor serier

Def: Talorpolynom

Taylorpolynom av grad $n \in \mathbb{N}$, *kring* $x = x_0$

$$f(x) = P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k, \; 0! = 1, f^{(0)} = f$$

När$x \approx x_0 \Rightarrow f(x) \approx p_n(x)$

Sats: Talor sats

Antag att $f \in c^{n+1}$.

$f(x) = p_n(x) + R_{n+1}(x)$ Där felet är $R_{n+1}(x)$

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \; x \vee x_0 \leq c \leq x_0 \vee x$$

exempel: felet

Upskata felet hos $f(x) = \sin x$, med grad 5

Sista termen är $\frac{x^5}{5!} \Rightarrow$ Felet $\frac{f^{(6)}(c)}{6!}x^6, (0 < c < 1)$

$$|\frac{f^{(6)}(c)}{6!}x^6| \leq \frac{|f^{(6)}(c)|}{6!} = \frac{|\sin c|}{6!} \leq \frac{1}{6!} = \frac{1}{720}$$

Exempel: grad n vid specificerad punkt

Lös: Hitta Taylorpolynomet, ording $2n - 1$, för $\sin 2x$ vid $x = \frac{\pi}{2}$

$f(x) = \sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f'(x) = -2\cos(2x)$ $\qquad f(\frac{\pi}{2}) = -2$

$f''(x) = -2^2 \sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f^{(3)}(x) = -2\cos(2x)$ $\qquad f(\frac{\pi}{2}) = 2^3$

$f^{(4)}(x) = -2^2 \sin(2x)$ $\qquad f(\frac{\pi}{2}) = 0$

$f^{(5)}(x) = -2^4 f'(x)$ $\qquad f(\frac{\pi}{2}) = -2^5$

$$p_{2n-1}(x) = -2(x - \frac{\pi}{2}) + \frac{2^3}{3!}(x - \frac{\pi}{2})^3 - \frac{2^5}{5!}(x - \frac{\pi}{2})^5 + \ldots + \frac{(-1)^n \cdot 2^{2n-1}}{(2n-1)!}(x - \frac{\pi}{2})^{(2n-1)}$$

Def: maclaurin polynomial

Taylorpolynom av grad $n \in \mathbb{N}$, då $x_0 = 0$

Big-O notation: Vi säger att $g(x) = O(f(x))$ när $x \approx x_0$, värsta fall

Man kan också säga istället för Big-O notation en rest $R_{n+1}(x) = x^{n+1}H(x)$ där $H(x)$ begränsad i intervalet

$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + O(x^{n+1})$

$\sin x = \sum_{k=0}^{n} \frac{(-1)^{k-1}}{(2k-1)!} x^{2k-1} + O(x^{2n})$

$\cos x = \sum_{k=0}^{n} \frac{(-1)^{k}}{(2k)!} x^{2k} + O(x^{2n+19})$

$\ln(1+x) = \sum_{k=0}^{n} \frac{x^k}{k} (x)^x + O(x^{n+1})$

4.7 Integraler

Regler no.1

$$\int_a^a f(x)dx = 0$$

Regler no.2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Regler no.3 (Linjearitet)

$A, B \in \mathbb{R}$

$$\int_a^b (A \cdot f(x)dx + B \cdot g(x)) = A(\int_a^b f(x)dx) + B(\int_a^b g(x)dx)$$

Fungerar på subtraction men INTE multiplicatione eller divition!

Regler no.4

$c \in [a, b]$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Regler no.5

$f(-x) = -f(x)$ är udda

$$\int_{-a}^a f(x)dx = 0$$

Regler no.6

$f(-x) = f(x)$ är jämn

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Regler no.7

$$|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$$

Regler no.8

$$f(x) \leq g(x) \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Sats: Medelvärdessatsen för integraler

$$f:[a,b]\to\mathbb{R} \text{ konternuelig } \wedge \exists c\in(a,b) \Rightarrow \int_a^b f(x)dx = f(c)(b-a)$$

Sats: Fanalysens huvudsats, Fundemetal theorem of calculus
Satsen är den vi andvänder för att lösa integraler utan geometrisk tolkning

$$f:[a,b]\to\mathbb{R} \text{ konternuelig } \Rightarrow F(x) = \int_a^x f(t)dt,\ a\leq x\leq b$$

Sats:

$$f:[a,b]\to\mathbb{R} \text{ konternuelig } \Rightarrow F \text{ är e primitiv funktion till } f(F'(x)=f(x))$$
$$\int_a^b f(x)dx = F(b)-F(a)$$
$$\text{Vi skriver } \int_a^b f(x)dx = [F(x)]_a{}^b = F(b)-F(a)$$

Exempel: arean mellan två grafer

$$\int_a^b (f(x)-g(x))dx \text{ där f är översta funnktionen och g är understa}$$
$$\int_0^1 (x-x^2)dx = [x^2/2-x^3/3]_0{}^1 = 1/2-1/3 = 1/6$$

Primitiva funktioner (Obestämda integralen)

$\int f dx$	F
$\int 1dx$	$x+c$
$\int n^n dx$	$x^{n+1}/(n+1)+c$
$\int 1/xdx$	$\ln\lvert x\rvert+c$
$\int \sin xdx$	$-\cos x+c$
$\int \cos xdx$	$\sin x+c$
$\int e^x dx$	e^x+c
$\int 1/\sqrt{1-x^2}dx$	$\arcsin x+c$
$\int 1/(x^2+1)dx$	$\arctan x+c$
$\int 1/\cos^2 xdx$	$\tan x+c$

Exempel:

$$\int_a^b e^x dx = [e^x]_a{}^b = e^b - e^a$$
$$\int_e^x dx = e^x + c,\ c\in\mathbb{R}$$

Def: medelvärdet

$$Avgf = \frac{1}{b-a}\int_a^b f dx$$

4.7.1 variabelsubstition

$$\int f'(g(x))g'(x)dx = f(g(x)) + c$$
$$\int_e^x dx = e^x + c,\ c \in \mathbb{R}$$

Exempel:

$$\int e^{x^2} 2x dx$$
$$\text{Låt } u = x^2 \Rightarrow du = 2xdx \Rightarrow \int e^{x^2} 2xdx = \int e^u du =$$
$$= e^u + c = e^{x^2} + c,\ c \in \mathbb{R}$$

Integraller som följer mönster

Integralles som inehåller $\sqrt{x^2 + a^2},\ a > 0$
$x = a \tan u \Rightarrow u = \arctan x,\ -\pi/2 < u < \pi/2$
$$\Rightarrow \sqrt{x^2 + a^2} = \sqrt{a^2 + a^2 \tan^2 u} = \sqrt{a^2(1 + \tan^2 u)} = a\sqrt{\frac{\sin^2 u + \cos^2 u}{\cos^2 u}} = a\frac{1}{\cos u}$$

Exempel: $\displaystyle\int \frac{1}{(1 + 25x^2)^{3/2}} dx$
$$\int \frac{dx}{\sqrt{1 + 25x^2}^3} = \int \frac{dx}{\sqrt{25(1/25 + x^2)}^3} = \frac{1}{5^3} \int \frac{dx}{\sqrt{1/205x^2}^3}$$
Låt $x = 1/5 \tan u \Leftrightarrow u = \arctan 5x$
$$\Rightarrow dx = \frac{1}{5\cos^2 u} du$$
$$\Rightarrow \frac{1}{5^3} \int \frac{dx}{\sqrt{1/205x^2}^3} = \frac{1}{5^3} \int \frac{\cos^3 u}{1/5^3} \frac{1}{5\cos^2 u} du = \frac{1}{5} \int \cos u du =$$
$= \frac{1}{5} \sin u + c = \frac{1}{5} \sin \arctan 5x + c$ Kan nu förenkla med trigo regler
och få: $\displaystyle\frac{x}{\sqrt{1 + 25x^2}} + c$

Integralles som inehåller $\sqrt{x^2 - a^2},\ a > 0$
$$x = \frac{a}{\cos u} \Rightarrow u = \arccos \frac{a}{x}$$
$$\Rightarrow \sqrt{x^2 - a^2} = \sqrt{\frac{a^2}{\cos^2 u} - a^2} = \sqrt{\frac{a^2 - a^2\cos^2 u}{\cos^2 u}} = \sqrt{\frac{a^2 \sin^2 u - a^2 \cos^2 u}{a^2 \cos^2 u}} = a|\tan u|$$

Integralles som inehåller $\sqrt{ax+b}$
$ax+b=u^2$

Exempel: $\int \frac{dx}{2+\sqrt{x}}$
Låt: $u^2 = x \Rightarrow 2udu = dx$
$\Rightarrow \int \frac{2udu}{2+u} = 2\int \frac{u+2-2}{2+u} du = 2(\int du - 2\int \frac{du}{2+u}) = 2u - 4(\ln 2 + u) + c$

4.7.2 Integration av rationella funktioner

Om det står beräkna generaliserade integrallen så ska man beräkna med detta
Rationella funktioner: $R(x) = \frac{P(x)}{Q(x)}$, P, Q är polynom
Algoritm
1. Polynom divition om grad av täljare är stören än grad av nämnare
2. Faktorisera nämnaren i realla faktorer
3. Partiella bråk och sedan integrera dem, (dela up i mindre lösbara integraller)

Partiella bråk

$x-a$	$\frac{A}{x-a}$
$(x-a)^k$	$\frac{A_1}{x-a} + \frac{A_2}{x-a^2} + \ldots + \frac{A_k}{x-a^k}$
$x^2+bx+c,\ (b^2-4c<0)$	$\frac{Ax+B}{x^2+bx+c}$
$x^2+bx+c,\ (b^2-4c<0)$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \ldots + \frac{A_1x+B_1}{(x^2+bx+c)^k}$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c$$
$$\int \frac{x}{x^2+a^2}dx = \frac{1}{2}\ln|x^2+a^2| + c$$
$$\int \frac{x}{x^2-a^2}dx = \frac{1}{2}\ln|x^2-a^2| + c$$
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^- 1\frac{x}{a} + c$$
$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + c$$

$$\int \frac{x^3+2}{x^2-5x+4}dx$$

1. Polynomdivition

$P(x) = x^3 + 2,\ Q(x) = x^2 - 5x + 4$

Med polynom divition så får vi kvot $x + 5$ och rest $21x - 18$

$x^3 + 2 = (x + 5)(x^2 - 5x + 4) + 21x - 18$

2. Faktorisera nämnaren i realla faktorer

$$\Rightarrow \int \frac{x^3+2}{x^2-5x+4}dx = \int \frac{(x+5)(x^2-5x+4)+21x-18}{x^2-5x+4}dx \int (x+5)dx + \int \frac{21x-18}{x^2-5x+4}$$

3. Integrerar partiella bråk

$$\frac{21x-18}{x^2-5x+4} = \frac{21x-18}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} \Rightarrow \frac{21x-18}{(x-4)(x-1)} = \frac{Ax-A+Bx-4B}{(x-4)(x-1)}$$

$\Rightarrow A + B = 21 \wedge -A - 4B = -18 \Rightarrow A = 22 \wedge B = -1$

$$\int (x+5)dx + \int \frac{22}{x-4}dx - \int \frac{1}{x-1}dx$$

Svar: $x^2/2 + 5x + 22\ln|x-4| - \ln|x-1| + c,\ c \in \mathbb{R}$

4.7.3 Partiell integration

Om det är integraler med två funktioner så hjälper oftast partiell integration

Formel

$$\int_a^b f'gdx = [fg]_a^b - \int_a^b fg'dx$$
$$\int f'gdx = fg - \int fg'dx$$

Bevis

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \Rightarrow$$
$$\int_a^b (fg)'dx = \int_a^b f'gdx + \int_a^b fg'dx \Rightarrow \int_a^b f'gdx = \int_a^b (fg)'dx - \int_a^b fg'dx$$
$$\int_a^b f'gdx = [fg]_a^b - \int_a^b fg'dx$$

Exempel: polynom * trignomotrik

$$\int_0^{\pi/2} x \sin x dx$$
$$\int_0^{\pi/2} (-\cos x)' x dx = [-x\cos x]_0^{\pi/2} + \int_0^{\pi/2} \cos x (x)' dx =$$
$$[-x\cos x]_0^{\pi/2} + [\sin x]_0^{\pi/2} = 1$$

Exempel: polynom * exponensiel

$$\int e^x x^2 dx$$
$$\int (e^x)' x^2 dx = [e^x x^2] - \int e^x 2x dx = e^x x^2 - 2\int (e^x)' x dx = e^x x^2 - 2e^x x + 2\int (e^x)' dx = e^x x^2 - 2e^x x + 2e^x + c$$

Exempel: exponensiel * trignomotrik

$$\int e^x \cos x dx$$
Låt $I = \int e^x \cos x dx = \int (e^x)' \cos x = e^x \cos x - \int e^x \sin x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx \Rightarrow$
$$I = e^x \cos x + e^x \sin x - I \Rightarrow I = \frac{e^x \cos x + e^x \sin x}{2} + c$$

Exempel: bonus ln

$$\int_1^2 \ln x dx$$
$$\int_1^2 x' \ln x dx = \ldots = 2\ln 2 - 1$$

4.7.4 Generaliserande integraler

Def:

Antag att f är kontinuerlig i a, b och att $\lim_{x \to a^+} f(x) = \infty$

Vi definerar den generaliserade integralen $\int_a^b f(x)dx = \lim_{\varepsilon \to 0^+} \int_{a+c} f(x)dx, , , , konstingst$

Om gränsvärdet existerar och är ändlig säger vi att integralen är konvergentn, anart är den divergent

Betekning

ε Andvänds för små tal $\lim_{\varepsilon \to 0^+}$

M Andvänds för stora tal $\lim_{M \to +\infty}$

c Andvänds för konstanter $+c$

Sats:

$$\int_a^{+\infty} \frac{dp}{x^p}, \ a > 0$$

$p > 1 \Rightarrow$ Konvergerar

$p \leq 1 \Rightarrow$ Divergerar

Sats: Jämförelsesatsen

Anta att f och g är konternuerliga och $0 \leq f(x) \leq g(x), (a \in [-\infty, +\infty), b \in (-\infty, \infty])$

(I) om integralen är konvergent $\int_a^b g(x)dx \Rightarrow \int_a^b f(x)dx$ är också konvergent

(II) om integralen är divergent $\int_a^b f(x)dx \Rightarrow \int_a^b g(x)dx$ är också divergent

Sats:

Om $f(x)$ är positiv konternuerlig och avtagande i intervallet $x \geq \mathbb{N}$,

så är serien $\sum_{n=N}^{\infty} f(n)$ konvergent

precis när $\int_N^{\infty} f(x)dx$ är konvergent

Exempel:

Betämm om serien konvergerar eller divergerar $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$

$f(x) = \dfrac{1}{x \ln x (\ln(\ln x))^2}$

är positiv, konternuerlig och avtagande då nämnaren är stängt positivt ökande för $x \geq 10$

$\displaystyle\int_{10}^{+\infty} f(x)dx = \lim_{M \to +\infty} \int_{10}^{M} f(x)dx, \ (u = \ln \ln x \Rightarrow du = 1/\ln x \cdot 1/x dx) \Rightarrow \lim_{M \to +\infty} \int_{\ln \ln 10}^{\ln \ln M} \frac{1}{u^2}$

$\displaystyle\lim_{M \to +\infty} [-1/u]_{\ln \ln 10}^{\ln \ln M} = \lim_{M \to +\infty} (-1/\ln \ln M + 1/\ln \ln M) = 1/\ln \ln M$

$\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^2}$ konverger

Sats:

Antag att $a_n > 0, \ b_n > 0$ för varje $n \in \mathbf{N}$ och antag att $\dfrac{a_n}{b_n} \to L \neq 0 \ \wedge L < \pm\infty$

Då gäller att serien $\displaystyle\int_{n=1}^{\infty} a_n$ och $\displaystyle\int_{n=1}^{\infty} b_n$ är båda divergenta

4.7.5 Volymberäkningar

$V = \int_a^b A(x)dx$ Rotationsvolymer runt x-axeln $\Leftrightarrow \pi \int_a^b ((g(x))^2 - (f(x))^2)$ g är övre f är undre

$V = 2\pi \int_a^b xf(x)dx$ Rotationsvolymer runt y-axeln $\Leftrightarrow 2\pi \int_a^b x(g(x) - f(x))dx$ g är övre f är undre

exempel

Beräkna volymen av den kropp som uppstår när området som begränsas av kurnvan $y = 4x - x^2 - 3$ och x-axeln roteras kring y-axeln

$$y = 0 \Leftrightarrow 4x - 3 - x^2 = 0 \Leftrightarrow x = 1 \vee x = 3$$
$$\wedge\, y > 0,\ x \in [1, 3]$$
$$\Rightarrow V = 2n \int_1^3 x(4x - 3 - x^2)dx = 2n \int_1^3 (4x^2 - 3x - x3)dx$$
$$= 2n[4/3x^3 - 3/2x^2 - x^4/4]_1^3 = 16\pi/3$$

Kurvlängd

$$L = ||\vec{x_0} - \vec{x_0}|| = \sqrt{(a-c)^2 + (b-d)^2}$$
$$\Rightarrow L = \int_a^b \sqrt{1 + (f'(x))^2}$$

exempel

Find the lenght of curve $y = x^3/12 + 1/x$ from $x = 1$ to $x = 4$

$$f(x) = x^3/12 + 1/x$$
$$L = \int_1^4 \sqrt{1 + (f'(x))^2}dx = \int_1^4 \sqrt{1 + (x^2/4 - 1/x^2)^2}dx$$
$$= \int_1^4 \sqrt{1 + x^4/16 - 1/2 + 1/x^4}dx = \int_1^4 \sqrt{x^4/16 + 1/x^4 + 1/2}dx$$
$$= \int_1^4 \sqrt{(x^2/4 + 1/x^2)^2}dx = \int_1^4 (x^2/4 + 1/x^2)dx$$
$$= [x^2/4 + 1/x^2]_1^4 = 6$$

Rotationsarea

$$A = \int_a^b 2\pi|f(x)|\sqrt{1+(f'(x))^2}dx$$

exempel: Gabriel's Horn/Torricell's trumpet

Bestäm volymen och arean av den kropp som uppstår när området som begränsas av kurvan $y = 1/x, x \geq 1$ och x-axeln roteras kring x-axeln

$$V = \pi \lim_{M\to+\infty} \int_1^M (\frac{1}{x})^2 dx = \pi \lim_{M\to+\infty} [-1/x]_1^M = \pi \cdot 1 = \pi$$

$$A = \lim_{M\to+\infty} \int_1^M 2\pi|1/x|\sqrt{1+(-1/x^2)^2}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{1+1/x^4}}{x}dx$$
$$= 2\pi \lim_{M\to+\infty} \int_1^M \frac{x^2\sqrt{1+1/x^4}}{x^3}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{x^4+1}}{x^3}dx$$
$$> 2\pi \lim_{M\to+\infty} \int_1^M \frac{\sqrt{x^4}}{x^3}dx = 2\pi \lim_{M\to+\infty} \int_1^M \frac{x^2}{x^3}dx$$
$= 2\pi \lim_{M\to+\infty} \int_1^M \frac{1}{x}dx$ Diveigerar enlight p-satsen till ∞

Eftersom integrallen har en undre begränsing som divergerar divegerar också integrallen till $+\infty$

4.8 Differential ekvationer

Tangent plan (Slope field): Andvänds för att se hur kurvan ser ut utefrån olika start värden.

grad: högsta graden på derivatan $ty'''(t) - 4y'(t) + 5t^2y(t) = e^t$ har grad 3

Linjär diffirential ekvation: diff funktionerna har ingen uphöjning
$y'' - 4t^2t' + e^ty = 0$ är linjär
$t^2y''' + 5ty' - 4y^2 = 5$ är inte linjär

Homogen: $om h(t) = 0 \Rightarrow$ ODE är homogen
Inhomogen: $om h(t) \neq 0 \Rightarrow$ ODE är inhomogen
$y''' - \sin^2(t)y' = 5y$ är homogen
$e^{t^2}y^{(5)} - y'' + 4ty(t) = e^t + t^3$ är inhomogen

Sats: superposition princple

Låt $a_ny^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0$
Om $y_1(x), y_2(x)$ är lösningar till differantials ekvanationen
$\Rightarrow Ay_1(x) + By_2(x),\ A, b \in \mathbb{R}$ är en lösning till differantials ekvanationen

4.8.1 superabla ekvationer

Def: superabla ekvationer

En differentialekvation är seperabel om den kan skrivas på formeln

$$\frac{dy}{dx} = f(x)g(y)$$

Exempel: Vilka är seperabel

$(I)\ y' = x + y$ är inte seperabel

$(II)\ \dfrac{dy}{dx} = 1 + e^y$ är separabel

Exempel: beräkning

$$\text{Lös} y'(x) = (1 + e^{-x})(y^2 - 1)$$

$$y = \pm 1$$

$$\frac{dy}{y^2 - 1} = (1 + e^-1)dx \Rightarrow \int \frac{dy}{y^2 - 1} = \int (1 + e^{-1})dx(*)$$

$$\int \frac{1}{(y+1)(y-1)}dy = \int \frac{((y+1) - (y-1))}{2(y+1)(y-1)}dy = \int \frac{1}{2(y-1)}dy - \int \frac{1}{2(y+1)}dy$$

$$(*) \Rightarrow \int \frac{1}{2(y-1)}dy - \int \frac{1}{2(y+1)}dy = \int (1 + e^{-1})dx$$

$$\Rightarrow 1/2 \ln|y - 1| - 1/2 \ln|y + 1| = x - e^{-x} + c \Rightarrow \ln|\frac{y-1}{y+1}| = 2x - 2e^{-x} + 2c$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}|\frac{y-1}{y+1}|} = e^{2x - 2e^{-x} + 2c}$$

$$\Rightarrow y = \frac{1 + e^{2x - 2e^{-x}} + 2c}{1 - e^{2x - 2e^{-x}} + 2c} \vee y = \frac{1 - e^{2x - 2e^{-x}} + 2c}{1 + e^{2x - 2e^{-x}} + 2c} \vee y = \pm 1$$

4.8.2 Linjära differentialekvationer av ordning 1

Methodology

$y' + p(x)y = q(x)$
Om$g(x) = 0 \Rightarrow$ homogen och därmed seperable
Om$g(x) \equiv 0$ Multiplisera med $e^{M(x)}$ För att kuna andvända produktregeln så vi kan slå ihop y, y'
$M(x) = \int p(x)dx$ är en primitiv dunktion till $p(x) \Rightarrow c = 0$

$\Rightarrow e^{M(x)}y' + e^{M(x)}p(x)y(x) = e^{M(x)}q(x)$ Antifunktionen slår ut $p(x)$
$\Rightarrow e^{M(x)}y' + (e^{M(x)})'y(x) = e^{M(x)}q(x)$ VL kan vi andvända produktregeln bakvänt
$\Rightarrow (e^{M(x)}y(x))' = e^{M(x)}q(x)$ Integrerar båda led
$\Rightarrow \int (e^{M(x)}y(x))'dx = \int (e^{M(x)}q(x))dx$ Antiderivatan slår ut derivatan
$\Rightarrow e^{M(x)}y(x) = \int (e^{M(x)}q(x))dx$ Får y ensamt
$\Rightarrow y(x) = e^{-M(x)} \int (e^{M(x)}q(x))dx$

Exempel

$$\text{Lös}(1+t^2)y' + ty = \frac{1}{\sqrt{1+t^2}}$$

$$y' + \frac{t}{(1+t^2)y} = \frac{1}{(1+t^2)\sqrt{1+t^2}} \text{ Linjär ode, ordning 1}$$
$$p(t) = \frac{t}{(1+t^2)y},\ q(t) = \frac{1}{(1+t^2)\sqrt{1+t^2}}$$
$$M(t) = \int \frac{t}{(1+t^2)y}dt,\ \text{Låt } u = 1+t^2 \Rightarrow dt = du/2t$$
$$\Rightarrow M(t) = \int \frac{1}{2u}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln t^2 + 1$$
$$e^{M(t)} = e^{\frac{1}{2}\ln t^2+1} = \sqrt{t^2+1}$$
$$\Rightarrow \sqrt{t^2+1}y' + \frac{t\sqrt{t^2+1}}{t^2+1}y = \frac{1}{1+t^2}$$
$$\Rightarrow \sqrt{t^2+1}y' + \frac{t}{\sqrt{t^2+1}}y = \frac{1}{1+t^2}$$
$$\Rightarrow (\sqrt{t^2+1}y)' = \frac{1}{1+t^2} \Rightarrow \sqrt{t^2+1}y = \int \frac{1}{1+t^2}$$
$$\Rightarrow \sqrt{t^2+1}y = \arctan(t) + c$$
$$\Rightarrow y = \frac{\arctan(t)}{\sqrt{t^2+1}} + \frac{c}{\sqrt{t^2+1}},\ c \in \mathbb{R}$$

4.8.3 Linjära differentialekvationer av ordning 2

Methodology

Det finns 3 metoder för att lösa Linjära differentialekvationer av ordning 2
$ay'' + by' + cy = 0$ vilket är den generella formeln. Antag att lösningen är på formen
$y = e^{rx},\ y' = re^{rx},\ y'' = r^2e^{rx} \Rightarrow ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$
$(ar^2 + br + c)e^{rx} = 0 \Rightarrow ar^2 + br + c = 0$ andvänder pq-formeln och får följande
$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(i)Om $b^2 - 4ac > 0 \Rightarrow r_1, r_2 \in \mathbb{R} \wedge r_1 \neq r_2 \Rightarrow y_1 = e^{r_1x},\ y_2 = e^{r_2x}$
är lösningen till $ay'' + by' + cy = 0 \Rightarrow y_k = c_1e^{r_1x} + c_2e^{r_2x},\ c_1, c_2 \in \mathbb{R}$ superposition
$(ii)b^2 - 4ac = 0 \Rightarrow r_1 = r_2 \in \mathbb{R}$ dubbelrot $y_1 = e^{r_1x},\ y_2 = xe^{r_1x}$

(iii)Om $b^2 - 4ac < 0 \Rightarrow r_1 \neq r_2 \in \mathbb{C}$
$r_1 = k + li,\ r_2 = k - li$ Koplexatals konjugat enlight euler formel, alla lösningar är
$y = c_1e^{r_1x} + c_2e^{r_2x}, c_1, c_2 \in \mathbb{C}$
$y = c_1e^{r_1x} + c_2e^{r_2x} = c_1e^{(k+li)x} + c_2e^{(k-li)x} = c_1e^{kx}e^{lix} + c_2e^{kx}e^{-lix}$
$c_1e^{kx}(\cos lx + i\sin lx) + c_2e^{kx}(\cos -lx + i\sin -lx)$
$c_1e^{kx}(\cos lx + i\sin lx) + c_2e^{kx}(\cos lx - i\sin lx) = e^{kx}((c_1 + c_2)\cos lx + (c_1 - c_2)i\sin lx)$
$e^{kx}(\vec{c_1}\cos lx + \vec{c_2}\sin lx)$
$y = e^{kx}(c_1\cos lx + c_2\sin lx),\ c_1, c_2 \in \mathbb{R}$
Vi för ett svar som är realt, men vi andvänder genväg med komplexa tal

Exempel Positiv kvot

Lös $y''(x) - 4y'(x) + 3y(x) = 0$
$r^2(x) - 4r(x) + 3 = 0 \Rightarrow r_1 = 1,\ r_2 = 3 \Rightarrow y(x) = c_1e^x + c_2e^{3x},\ c_1, c_2 \in \mathbb{R}$

Exempel noll kvot

Lös $y''(t) - 4y'(t) + 4$
$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ dubbel rot
$y(t) = c_1e^{2t} + c_2te^{2t},\ c_1, c_2 \in \mathbb{R}$

Exempel koplex kvot

Lös $y''(t) + 25y(t) = 0$
$r^2 + 25 = 0 \Rightarrow r^2 = -25 \Rightarrow r = \pm 5i$
$y(t) = c_1\cos(5t) + c_2\sin 5t,\ c_1, c_2 \in \mathbb{R}$

Partikulär lösning

Methodology

Partikulär lösning är för att gissa lösningen till det ike homogena lösningarna
$ay'' + by' + cy = h(t),\ y(t) = y_h(t) + y_p(t)$

If $f(x) = P_n(x),$ try $y_p = x^m A_n(x)$
If $f(x) = P_n(x)e^{rx},$ try $y_p = x^m A_n(x)e^{rx}$
If $f(x) = P_n(x)e^{rx}\cos(kx),$ try $y_p = x^m e^{rx}(A_n(x)\cos(kx) + B_n(x)\sin(kx))$
If $f(x) = P_n(x)e^{rx}\sin(kx),$ try $y_p = x^m e^{rx}(A_n(x)\cos(kx) + B_n(x)\sin(kx))$

För att ta reda på t eller k så kan tänka att den homogena lösningen är ej en lösning så börja med k=0 sen gå up tills det är sant

Exempel trigonometrisk

Lös $y'' + 9y = \sin 3x$
Sats: diff ekv kan skrivas på följande formel $y = y_h + y_p$

Homogena lösningen
$r^2 + 9 = 0 \Rightarrow r = \pm 3i \Rightarrow y_h = A\sin 3x + B\cos 3x$
Particulöra lösningen
Sats $y_p = x^m(A_1 \sin 3x + A_2 \cos 3x), m = 1$
är godtaglig då det är första ekv som inte kan skrivas på homogen förmel
$y_p = A_1 x\sin 3x + A_2 x\cos 3x$
$y_p' = A_1(\sin 3x + 3x\cos 3x) + A_2(\cos 3x - 3x\sin 3x)$
$y_p'' = A_1(3\cos 3x + 3\cos 3x - 9x\sin 3x) + A_2(-3\sin 3x - 3\sin 3x - 9x\cos 3x)$
$\Rightarrow y_p'' = A_1(6\cos 3x - 9x\sin 3x) + A_2(-6\sin 3x - 9x\cos 3x) + 9(A_1 x\sin 3x + A_2 x\cos 3x) = \sin 3x$
$6A_1\cos 3x - 6A_2\sin 3x = \sin 3x \Rightarrow A_1 = 0, A_2 = -1/6$
$\Rightarrow y_p = -x/6\cos 3x$
Den almäna lösningen blir på följande
$y(x) = A\sin 3x + B\cos 3x - x/6\cos 3x$

Exempel polynom och Exponensiel

Lös $y'' + y' - 2y = 4e^2x + x^2$
Sats: diff ekv kan skrivas på följande formel $y = y_h + y_p$

Homogena lösningen
$r^2 + r - 2 = 0 \Rightarrow r_1 = 1,\ r_2 = -2$
$y_h = c_1e^x + c_2e^{-2x},\ c_1, c_2 \in \mathbb{R}$
Particulöra lösningen
Sats $y_p = y_{p1} + y_{p2}$
$y'' + y' - 2y = 4e^{2x}$
$y_{p1} = x^m(Ae^{2x}), m = 0 \Rightarrow y'_{p1} = 2Ae^{2x} \Rightarrow y''_{p1} = 4Ae^{2x}$
$\Rightarrow 4Ae^2x + 2Ae^2x - 2Ae^2x = 4e^2x \Rightarrow A = 1 \Rightarrow y_{p1} = e^{2x}$
$y_{p2} = Ax^2 + Bx + C \Rightarrow y'_{p2} = 2Ax + B \Rightarrow y''_{p2}2A$
$\Rightarrow (2A) + (2Ax + B) - (2(Ax^2 + Bx + C)) = x^2$
$\Rightarrow -2A = 1,\ 2A - 2B = 0,\ 2A + B - 2C = 0 \Rightarrow A = -1/2,\ B = -1/2,\ C = -3/4$
$\Rightarrow y_{p2} = -1/2x^2 - 1/2x - 3/4$

$\Rightarrow y(x) = c_1e^x + c_2e^{-2x} + e^{2x} - 1/2x^2 - 1/2x - 3/4$

Resonans

Exempel Resonans

$$
\begin{aligned}
&\text{Lös} y'' + y = \sin 2t \\
&r^2 + 1 = 0 \Rightarrow r = \pm i \\
&y_h = c_1 \cos t + c_2 \sin t \\
&y_p = A \sin t + B \cos t \\
&y_p' = 2A \cos 2t - 2B \sin 2t \\
&y_p'' = -4A \sin 2t - 4B \cos 2t \\
&y_p'' + y_p = \sin 2t \Rightarrow -4A \sin 2t - 4B \cos 2t + A \sin t + B \cos t = \sin 2t \\
&\Rightarrow A = -1/3,\ B = 0 \Rightarrow y_p = -1/3 \sin 2t \Rightarrow y(t) = c_1 \cos t + c_2 \cos t - 1/2 \sin 2t
\end{aligned}
$$

Serielösningar

Antag att serien $\sum_{n=0}^{\infty} c_n (x - x_0)^n$ konvergerar för alla $n = 0$

$x \in x_0 - R, x_0 + R$ Då är funktionen $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ deriverbar i

$(x_0 - R, x_0 + R) \wedge f'(x) \sum_{n=0}^{\infty} n c_n (x - x_0)^{n-1}$ Den sista serien konvergerar också i intervallet $(x_0 - R, x_0 + R)$

Chapter 5

Computer Architecture

5.1 ISA1

5.1.1 MIPS instructions

The instructions that a assembly languge use for the MIPS proses.
Terminology

- Register file: 32 registers from R0-R31, each is 32 bits.
- Memory: Large each adders stores a word (4 x Registers). Address is all ways increments of 4. Memory is segmented into 4 8-bits of data to create a word.
- PC (Project counter): Increase with 4 each time to go to the next memory address.
- Instruction Registers: Retrieve the current instructions.
- Control: Inserts data to the register file and add the operation to ALU (Compute).
- ALU (Compute): Calculates the value.
- Memory Address: Call the value from a specific address. From ex “lw”.
- Data Register: Retries and directs value from memory to register file.

The operations most used

Function	Instruction	Effect
add	add R1, R2, R3	R1 = R2 + R3
sub	sub R1, R2, R3	R1 = R2 – R3
add immediate	addi R1, R2, 145	R1 = R2 + 145
multiply	mult R2, R3	hi, lo = R2 * R3
divide	div R2, R3	low = R2/R3, hi = remainder
and	and R1, R2, R3	R1 = R2 & R3
or	or R1, R2, R3	R1 = R2 | R3
and immediate	andi R1, R2, 145	R1 = R2 & 14
or immediate	ori R1, R2, 145	R1 = R2 | 145
shift left logical	sll R1, R2, 7	R1 = R2 << 7
shift right logical	srl R1, R2, 7	R1 = R2 >> 7
load word	lw R1, 145(R2)	R1 = memory[R2 + 145]
store word	sw R1, 145(R2)	memory[R2 + 145] = R1
load upper immediate	lui R1, 145	R1 = 145 << 16
branch on equal	beq R1, R2, 145	if (R1 == R2) go to PC + 4 + 145*4
branch on not equal	bne R1, R2, 145	if (R1 != R2) go to PC + 4 + 145*4
set on less than	slt R1, R2, R3	if (R2 < R3) R1 = 1, else R1 = 0
set less than immediate	slti R1, R2, 145	if (R2 < 145) R1 = 1, else R1 = 0
jump	j 145	go to 145
jump register	jr R31	go to R31
jump and link	jal 145	R31 = PC + 4; go to 145

Figure 5.1: MIPS oporation tabel

note: memory operations like (sw R1, 0(R2)) stores the word in poison R2+0 where 0 is the increment since if it is a loop it is needed. The value at that position is R1.

5.1.2 Sequencing

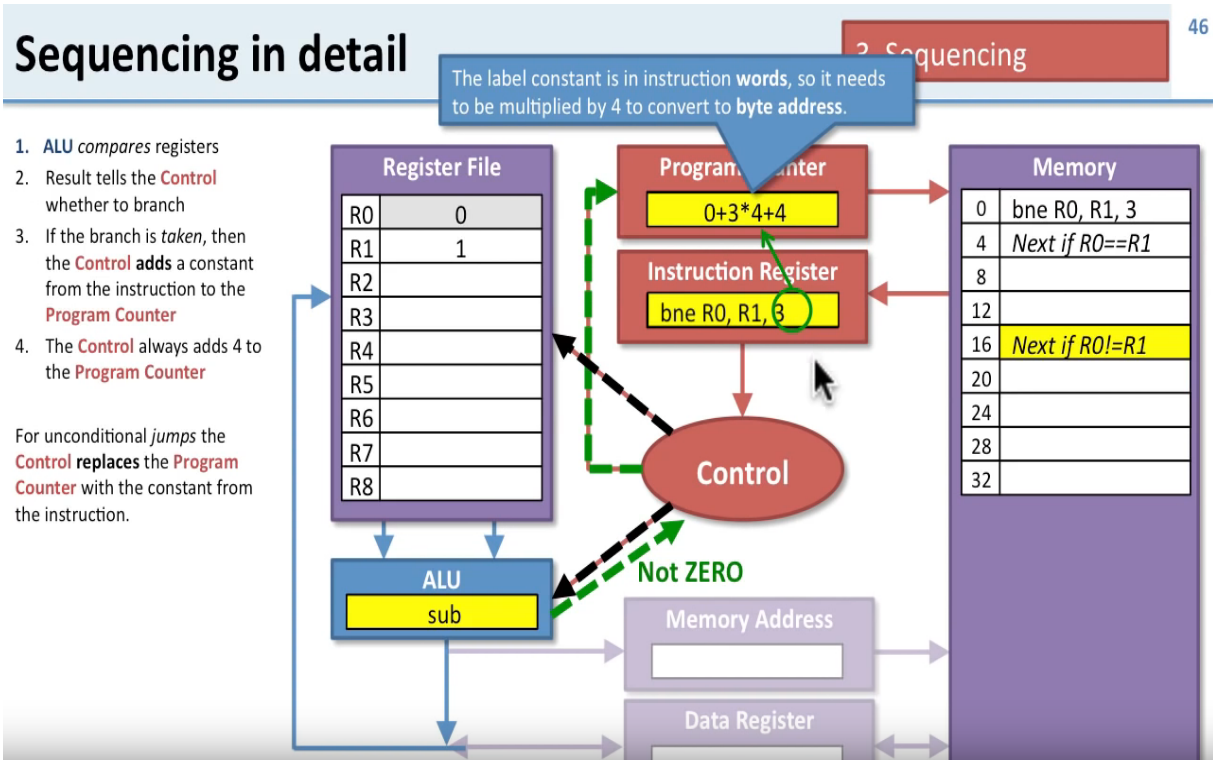


Figure 5.2: Sequencing

5.1.3 Converge to Assambly code

If Else example

```
if (i==j)
  h = i+j;

else
  h = i-j;
```

```
R1 = i; R2 = j; R3 = h

 bne R1, R2, DoElse
  add R3, R1, R2
  j SkipElse
DoElse:
  sub R3, R1, R2
SkipElse:
```

Figure 5.3: Assambly exampel

5.2 ISA2

5.2.1 MIPS type format

- R-format, Aretmetic and lodgical (add)
- I-format, Load/store, branch and intemidiats (addi, beq)
- J-format, Jump (j skipelse)

Name	Bit Fields						Notes
	6 bits	5 bits	5 bits	5 bits	5 bits	6 bits	(32 bits total)
R-Format	op	rs	rt	rd	shmt	funct	Arithmetic, logic
I-format	op	rs	rt	address/immediate (16)			Load/store, branch, immediate
J-format	op	target address (26)					Jump

Figure 5.4: MIPS Types

A instruction allwas ends with 00 it means that when a instruction is done we update the program couner to by 4 or more it changes the posistion of the 16bit intermidjet for i-format and can. (bne, beq)

5.2.2 Procedure

Procedure is how a function call is handeld. It is devided into two parts Caller and Callee, the caller call the procedure (callee). The oporation for contolling procedure is called jump-and-link (jal). It stores the return address (PC+4) in $ra (R31). Where (ra = register address (updates automatically)) (PC = project counter) (R31 = value). jr $ra returns the value.

Stacking

To avoid conflicts with writing over the callers register files, one uses staking in predefined ranges to allow store data independent from the callers. (R29 = store stack data $sp) When a register file is added the stock pointer is moved and when it is done it returns with (jr) and then reset the previously used register files. when it has used what it needs it removes them when leaded one value at a time

```
integer_array_sum:
        addi $sp, $sp, -4   # increase stack-pointer by one word
        sw $s0, 0($sp)      # save array index i to stack
        addi $sp, $sp, -4   # increase stack-pointer by one word
        sw $s1, 0($sp)      # save element n to stack

        addi $v0, $zero, 0      # Initialize Sum to zero.
        add $s0, $zero, $zero   # Initialize i to zero.

ia_loop:
        beq $s0, $a1, ia_done   # Done if i == N
        lw $s1, 0($a0)          # n = A[i]
        add $v0, $v0, $s1       # Sum = Sum + n
        addi $a0, $a0, 4        # address = ARRAY + 4*i
        addi $s0, $s0, 1        # i++
        j ia_loop               # next element



ia_done:
        lw $s1, 0($sp)     # loads i from stack-pointer
        lw $s0, 4($sp)     # loads n from stack-pointer
        addi $sp, $sp, 8   # restore stack-pointer

        jr   $ra           # return to caller
```

Figure 5.5: code ex

Notice that the register files for the callee save is $sx and for the caller $tx.

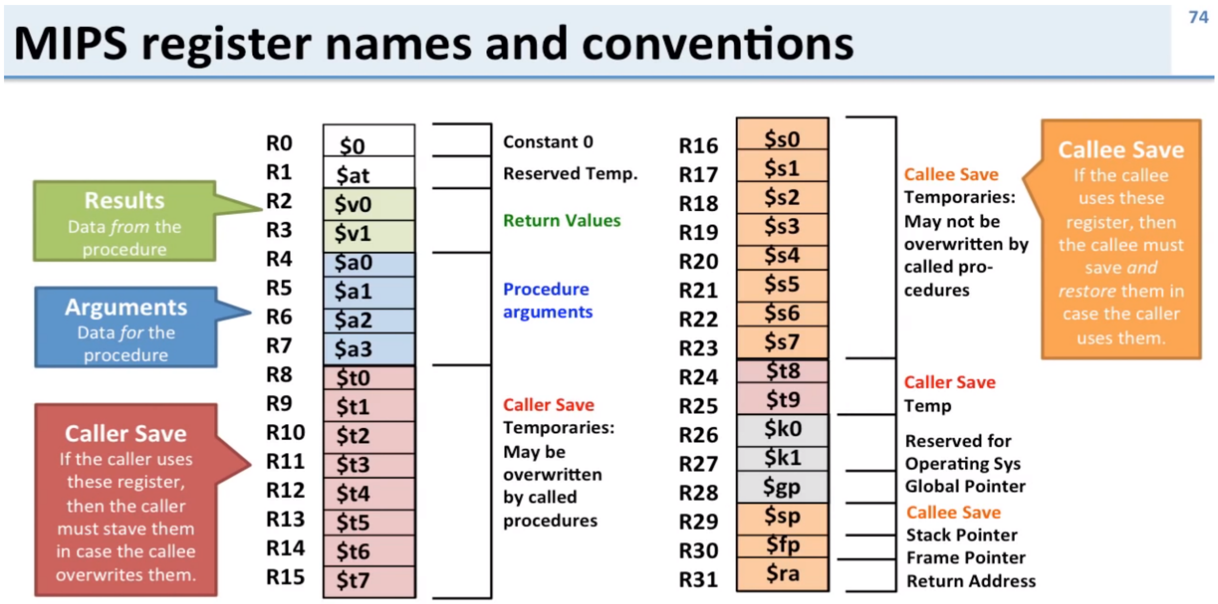


Figure 5.6: MIPS register

5.2.3 Other ISA

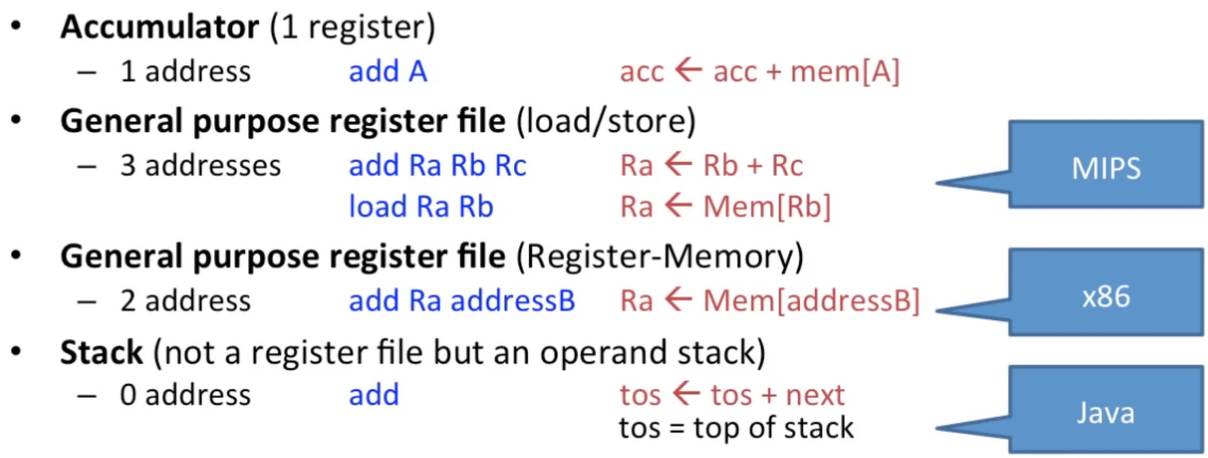


Figure 5.7: Other ISA

Stack	Accumulator	Register (Register-memory)	Register (Load-store)
Push A	Load A	Load R1, A	Load R1, A
Push B	Add B	Add R1, B	Load R2, B
Add	Store C	Store C, R1	Add R3, R2, R1
Pop C			Store C, R3
JVM	PDP-8, 8008, 8051	x86	MIPS, PPC, ARM, SPARC

Figure 5.8: Other ISA instructions

5.3 Arithetic

5.3.1 Binary numbers

- Binary numbers reduses noice and disregard the exact voltitge to rusult in on off mode.
- In comuter sience octal and hexadecimals are also often used becose of it properties. every octo didgets represent 3 in decimal every hex didgets represent 4 in decimal
- msb=most significant bit
- lsb=lest significant bit
- Optemazation: $0010 * 0101$ 2 oporations $\rightarrow 0101 * 0010$ 1 oporations
- Multiplication: (fixed point need to shift desimal point) $0010*0101 = 0010+(0010 shifting[00]) = 1010$ $0011.010 * 0001.110 =$
- Addition: (Same for fixed points and integers) $1110+1000 = carry[1]0110$ Overflow $0101+0001 = 0110$ Not overflow

5.3.2 Negative integes

- *Signed magnetude:* msb is the sign $1011_2 = -(2+1)_{10} = -3_{10}$
- *tow's complement:* if msb is 1 then negative number every other digiet after is positiv $1011_2 = -8+2+1_{10} = -5_{10}$

5.3.3 Oporations

5.3.4 Non integer numbers (Floating and fixed point)

- Converting binary to decimal: $0.111 = 1/2 + 1/4 + 1/8 = 0.875$
- Fixed point: for defined ranges 1.001 and 0.100 one can chose large and small representations
- Floating point: IEEE standard, se following image
- Standard floting point is in binary so FFFF is at most 1111 and with two's complement 0111

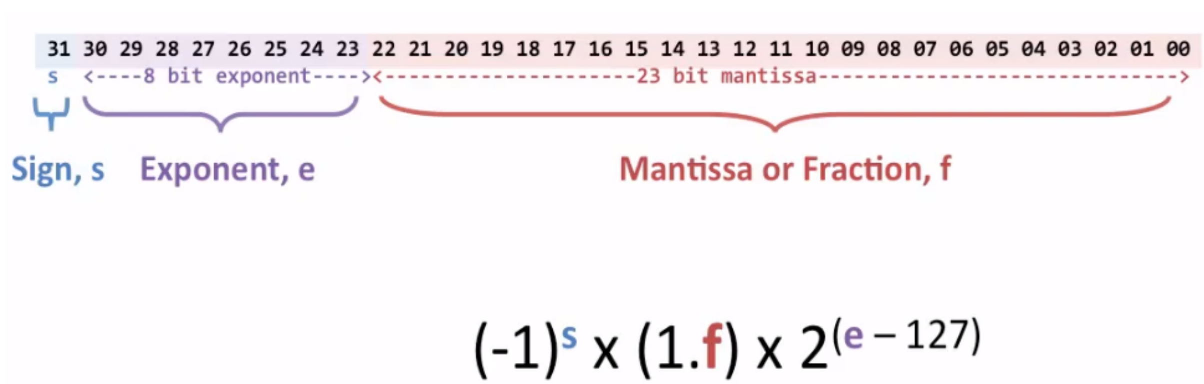
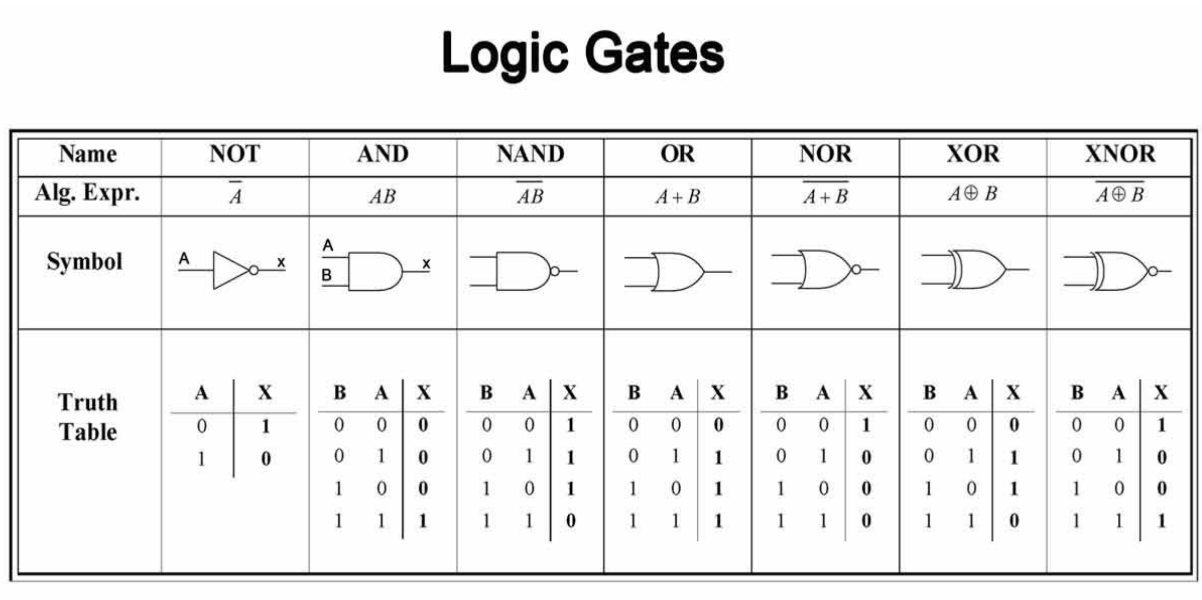


Figure 5.9: Floating Point

5.3.5 Overflow

- input: msb = 1 for boath numbers and output: msb = 0 (overflow)
- input: msb = 0 for boath numbers and output: msb = 1 (overflow)
- input: msb = 0 for boath numbers and output: msb = 0 (Not overflow)
- input: msb = 1 for boath numbers and output: msb = 1 (Not overflow)
- input: msb = 1 and msb = 0 (Maby overflow)

5.4 Logic



Logic Gates

Name	NOT	AND	NAND	OR	NOR	XOR	XNOR
Alg. Expr.	$\overline{A}$	AB	$\overline{AB}$	$A+B$	$\overline{A+B}$	$A \oplus B$	$\overline{A \oplus B}$
Symbol	A x	A B x					
Truth Table	A X 0 1 1 0	B A X 0 0 0 0 1 0 1 0 0 1 1 1	B A X 0 0 1 0 1 1 1 0 1 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 1	B A X 0 0 1 0 1 0 1 0 0 1 1 0	B A X 0 0 0 0 1 1 1 0 1 1 1 0	B A X 0 0 1 0 1 0 1 0 0 1 1 1

Figure 5.10: Floating Point

- truth table, is used to represent all possible inputs and then the corresponding output.
- To determined the possible schematics one can write for the input A and B "!A and B" not A and B. One often get a unnecessary complexity, one can often simplify the schematics. It is only done with the inputs that have an output of one.
- De Morgan's Law $!(A+B) = !A \cdot !B$ + and $\cdot$ works the same way but are different.
- karno map is used to easily determine the schematics. It is a matrix. It is possible to use more inputs then two. In the matrix one locks at when output is one.
- A cabal with 4 wires can take 4 bits. Overflow needs one extra wire on output

Building blocks

- Building blocks are the first layer of abstraction since one can not se the logical gates constructed of. For example add is a building block.
- Two types of logic, combinational and sequential. combinational: output just depends on input sequential: output deepens on the state. State is stored in memory update on clock.
- MUXes (Mulitplexrs): choose input from a buss/ routing signal. 2-bit decision for input 4-bit selection. Starts from position 0.
- DEMUXes (Demulitplexrs): oposit, take on signal and decides where it want to be sent to Buss is a multi-bit signal "wire"
- Decoder: binary to hot, can choose position in array.
- Encoder: hot to binary.

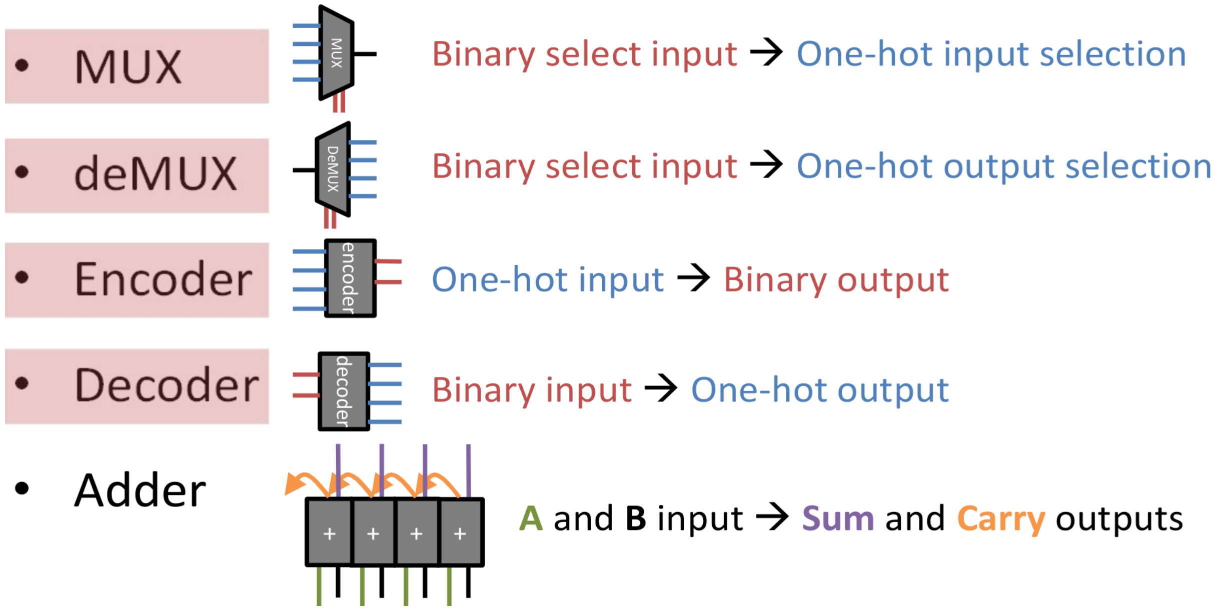


Figure 5.11: Floating Point

- Adders: adds carrys on extra bit to handel overflow

Latches

- One use is for counters with a clock statest trigers the count. So when input is one the output is 0 and when the clock clicks it is one then when the input is 0 the output is still 1.
- flip flop: is a edge trigerd latche has a clock trigger to open or close lathce and then can save it to SRAM cell

Memory

- SRAM (Static Random Access Memory) Big, fast and expensive. Loops thou value in order to store it. To write to memory one uses a switch to update value.
- DRAM (Dynamic Random Access Memory) Smaller, Slower and cheaper. Uses a transistor to store value and a capacitor to store the charge so the data dose not disappear. That is why it is slower become the capacitor needed to be recharge

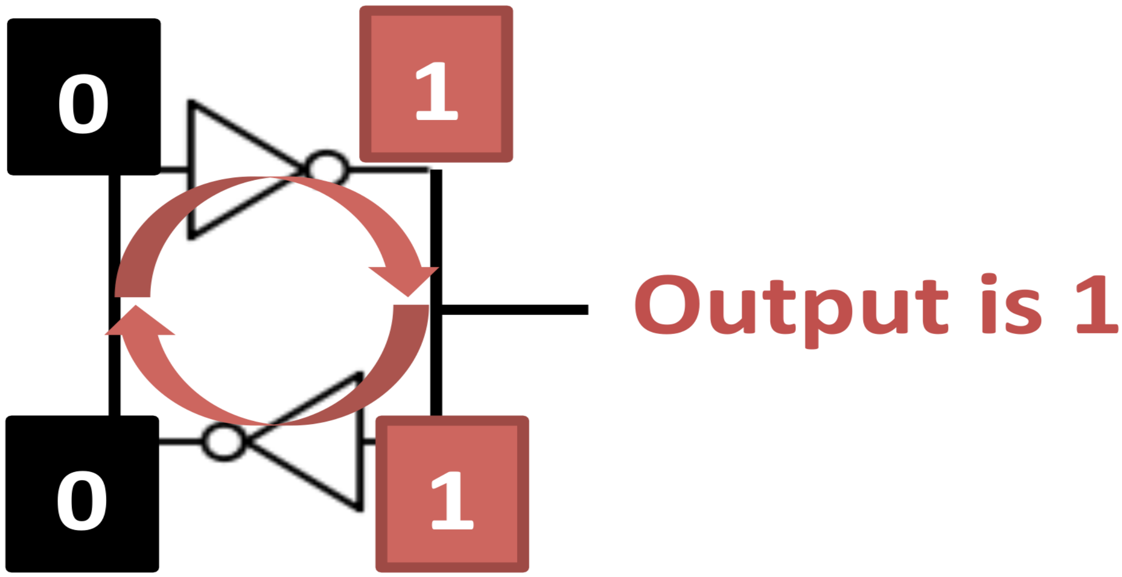


Figure 5.12: latche

5.5 processor control and datapath

There is 3 main part of the mips datapath as seen in the following image:

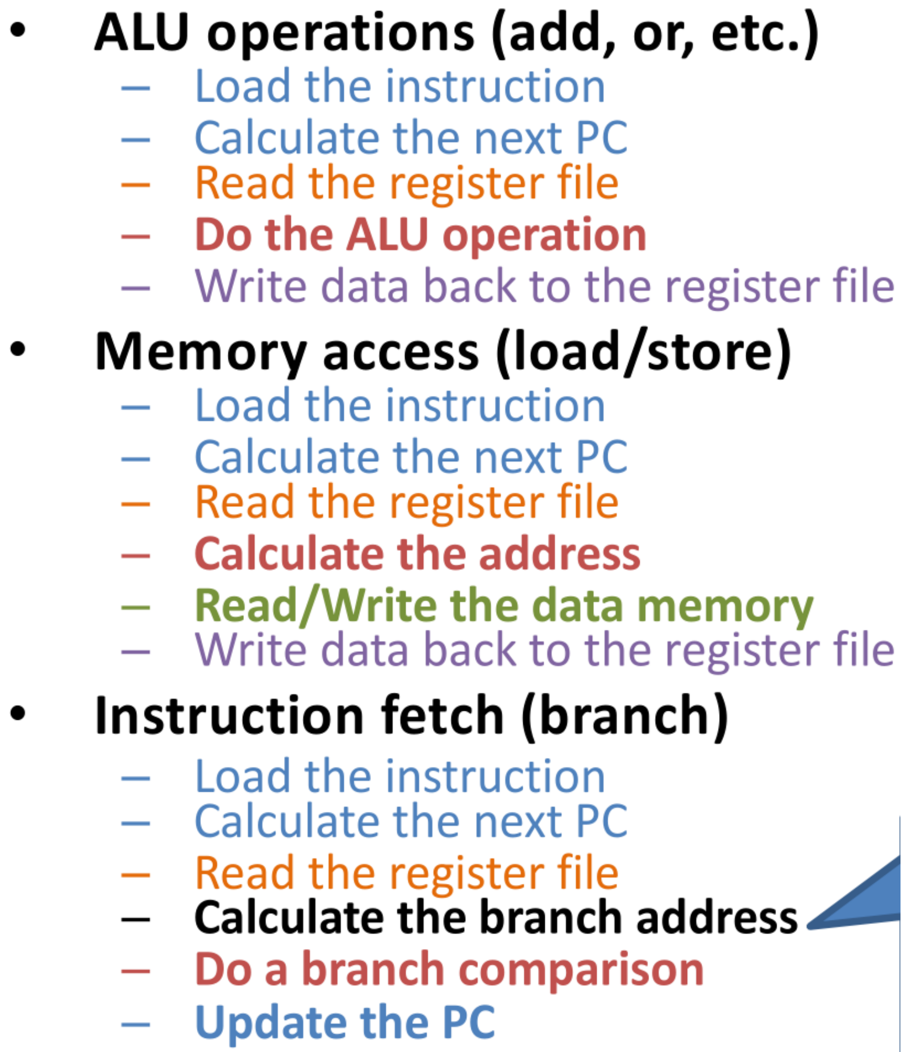


Figure 5.13: mips datapath 3 parts

An overview of the mips datapath with j-format instruction:
Think about the controller like a decoder that decodes the opt code from the different formats.

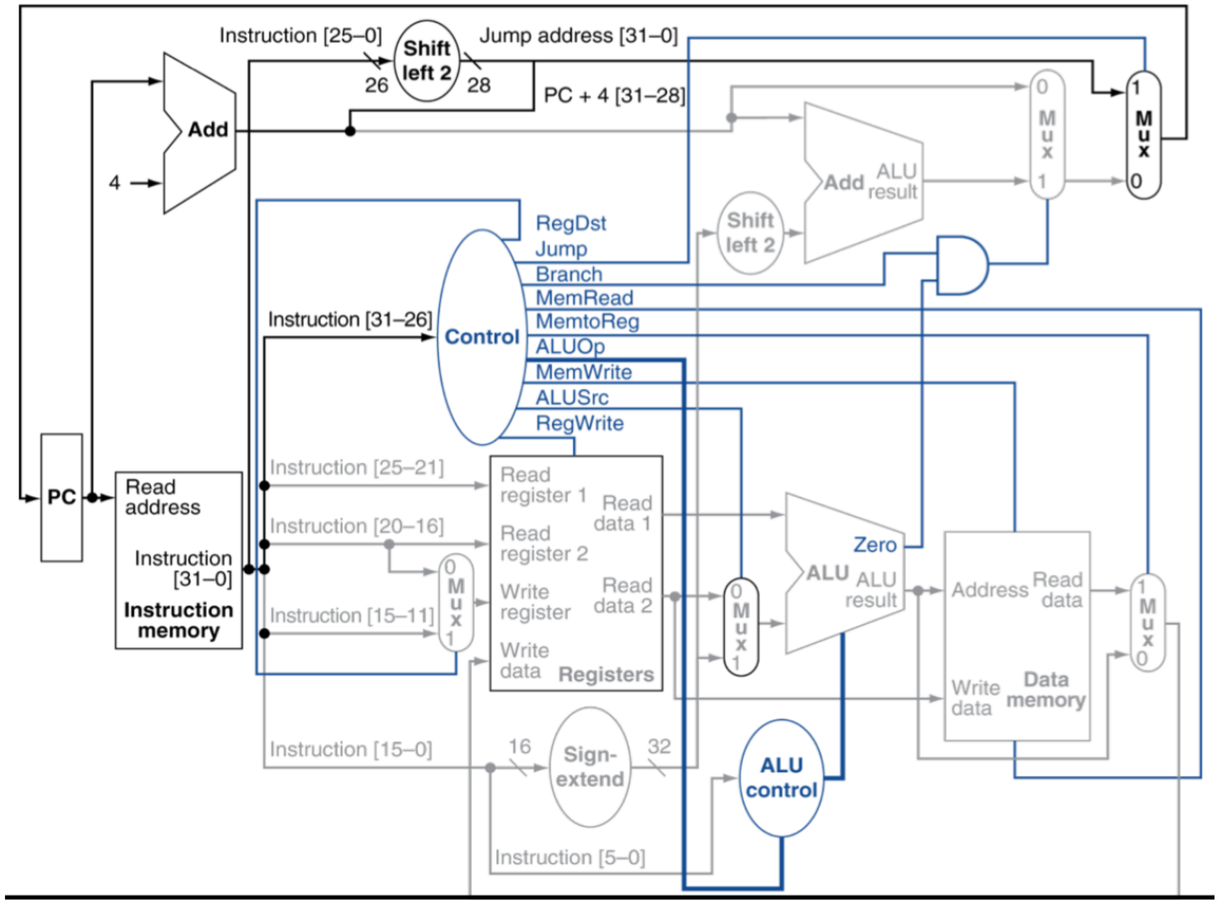


Figure 5.14: mips datapath

5.5.1 Clock

A clock is used to update the state and continue with the other instructions. Every state element uses a clock. In mips prossesor clock goes to memory, pc, rf and dm. The clock unit is in (MHz) converting from (ns) is simple. Ex 10ns: 1/10ns=0.1MHz.

5.5.2 Critical path

The longest path of the datapath is the critical path. Often PC is the fastest so one then calculates the amount of time it takes for the instruction has travel the data path and reternd to know what the critical path is. The longest part is data memory size it is big and slow. One often say read and write takes constant time regardless of the amount of different read and write.

5.6 pipline

The purpose is to split a instruction cycle into multiple stages to run instructions in peralel to improve preformans. Itch stage has its own pipline register file for storing the nessasary registers. Pipline registers has a preferment reduction since it takes time, therfore more pipeline stages dose not equal greater preformans over a certain amount. One issue of preformans is balance the stages inorder to get a good clock frequency. Not all stages are needed for eatch operation or instruction

Mips stages

- IF= Instruction fetch
- ID= Decode and RF Read
- Ex= ALU Execute
- MEM= Memory
- WB= RF Write back

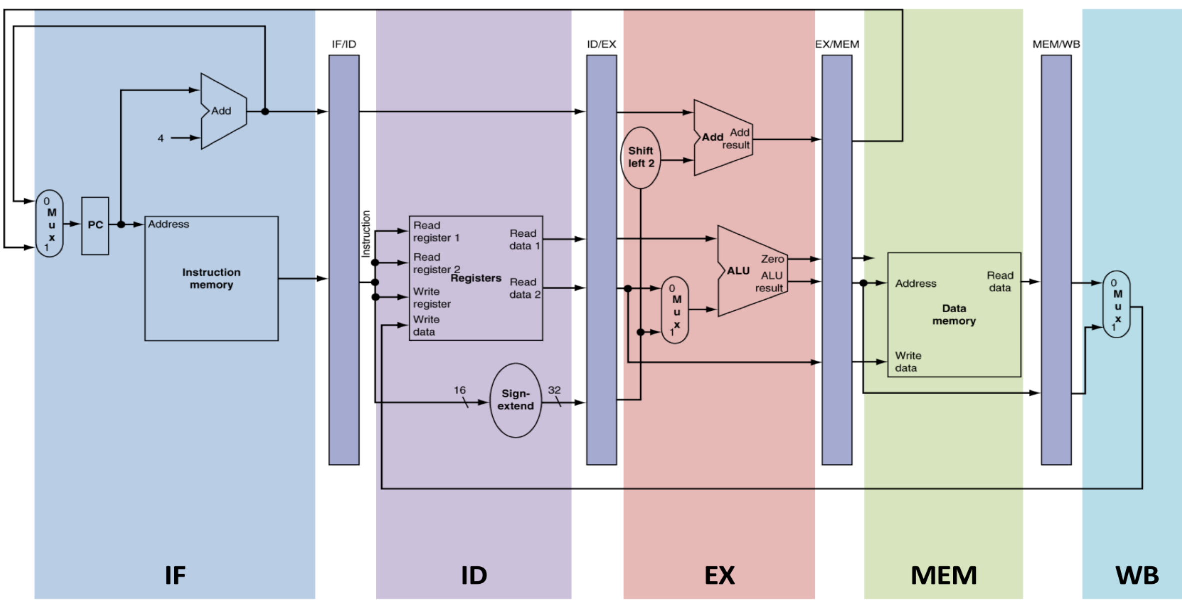


Figure 5.15: Pipline

Terminology

- Bubbles= Is detected by the hardware where there is no instuctions
- Nop= Is a sudo instruction for a stall type instruction (do not do anything)
- Delay slots= a stall type for branches. Try to fill in those slots with usful instructions.
- Interference= Can not read and write at the same time.
- Double pumping= Spiting write to first clock cycle and the second cycle is read.
- Forwarding= Getting data from a different pipeline stage from a register file.

Calculating time coplexity

- Latency: (stages*(penalty per stage))
  overhead: (instructions time (ex 100ns)) / (stages (ex 1000)) + (register overhead (1ns)) = 1.1ns Total time: 1.1*1000

5.7 Pipline hazards

5.7.1 Data hazards

The issue

- Data is not available where we need it (later in the pipeline; not written back yet).
- Data is not available when we need it (need to read memory first).

Fixing the issue

- Forward the data to where we need it.
- Stall if data is not ready yet (NOPs or Bubbles).

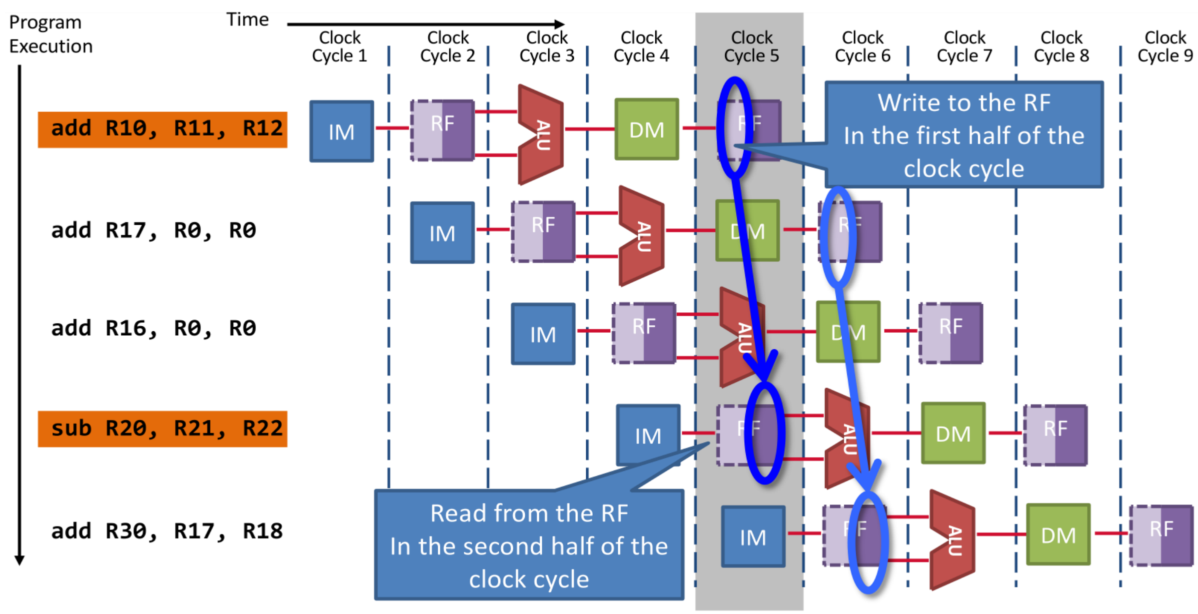


Figure 5.16: Bubble pump

5.7.2 Control hazards

The issue

- Don't know which instruction is next when we need to fetch it.

Fixing the issue

- Calculate branch as early as possible.
- Stall with a branch delay slot.

5.7.3 Structural hazards

The issue

- Can't do the instruction because the hardware is busy.

Fixing the issue

- Build more hardware (double-pumped register file). Double-pumped write at the first half cycle and read at the other half

Calculating time coplexity

- Branch delay
  (How many branches (ex 20%)) / (How many usful instuctions can be filled in (ex 50%)) = 20%/0.5 = 10%

- Instructions per cycle

5.8 Predicting Branches and Exceptions

When the prediction is wrong, clean up is needed. – “Kill” or “Squash” so they don’t execute (they were wrong) – Prevent them from writing: disable RegWrite, MemWrite in the pipeline – Turn them into NOPs: change opcode to add R0, R0, R0 in the pipeline

5.8.1 Static pridictors

- Predict alwas not taken
- Predict alwas taken
- Backwards-Taken, Forward-Not-Taken (BTFNT)

5.8.2 Dynamic pridictors

The implementation of branch predictors look something like this:

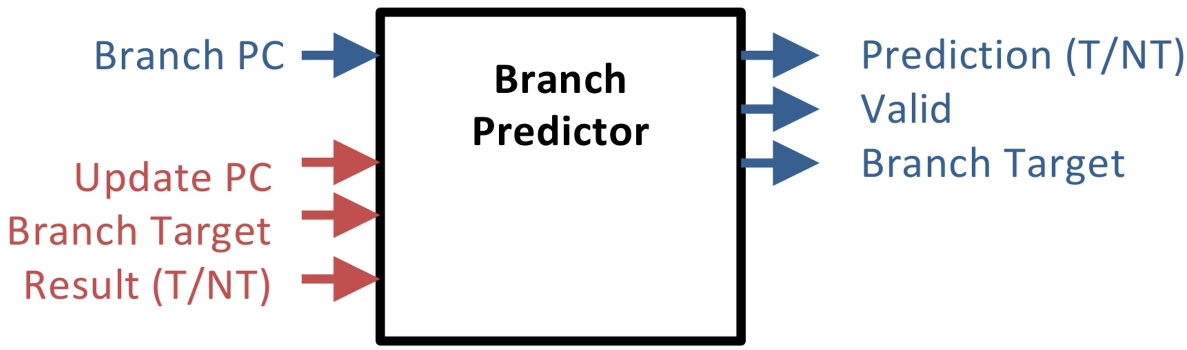


Figure 5.17: branch-predictor

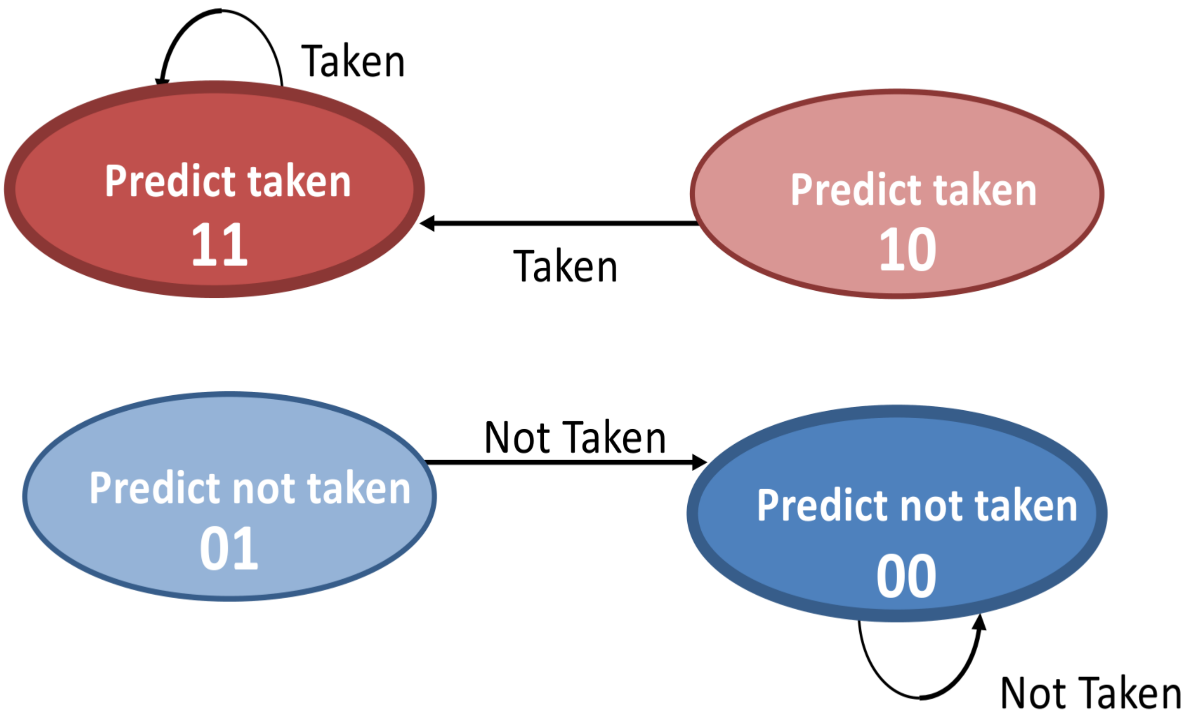


Figure 5.18: 2-bit predict

BTB

- Branch Target Buffer (BTB)
- Save a table with PC in order to have history that we can predict on

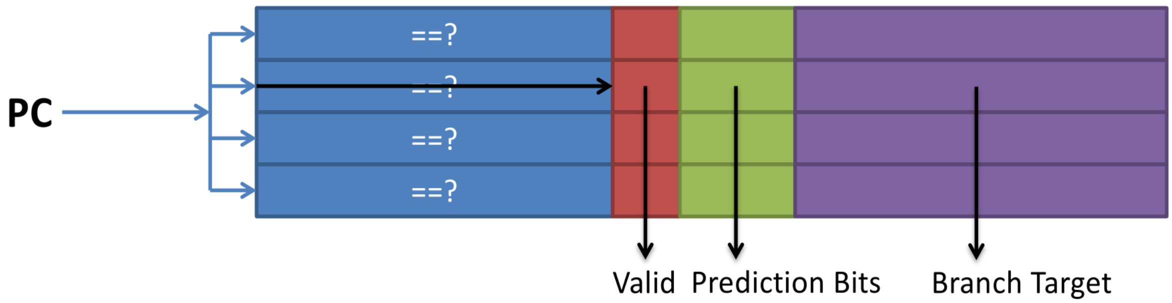


Figure 5.19: btb

5.8.3 Exceptions

Exceptions are non-normal events that interrupt the normal flow of instructions

- Divide by zero
- Misaligned memory access
- Page fault
- Memory protection violation

Interrupts are external events that interrupt the normal flow of

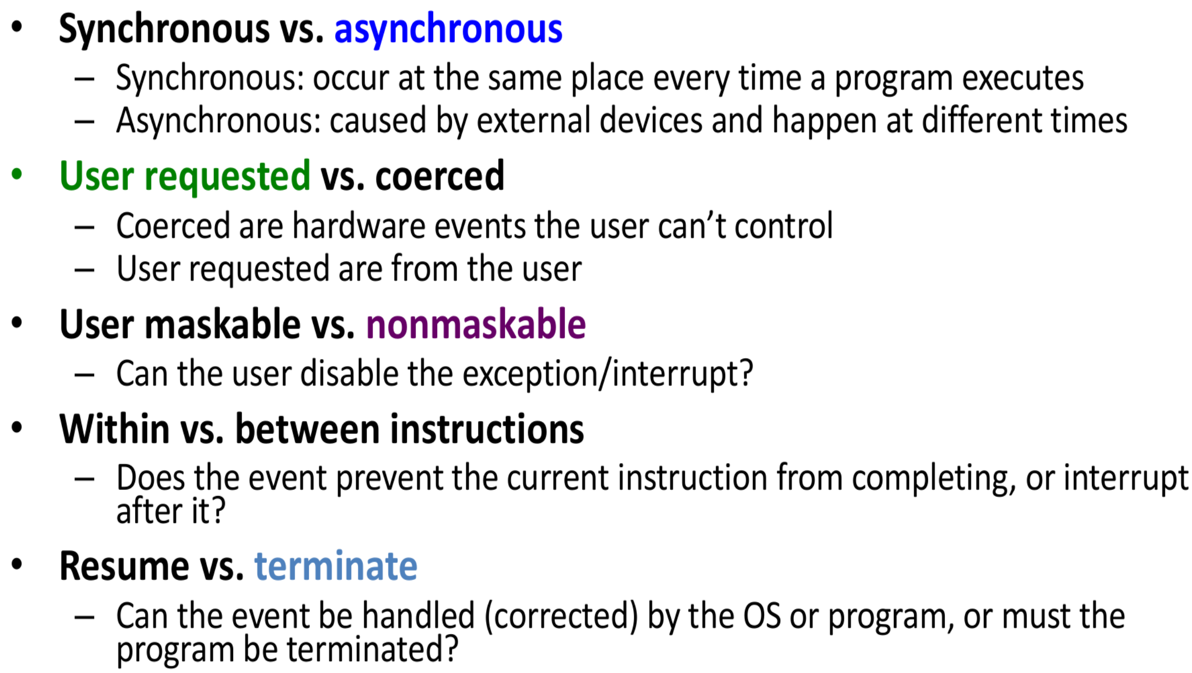


Figure 5.20: exceptions

5.9 Input/Output

Terminology

- nvram (flash): Similar to ram but it saves data when there is no power
- Busses: Parallel: many bits at once (e.g., 32 bits together in one clock) Used to be used everywhere Still used inside chips
- Serial links: Serial: one bit at a time (e.g., 32 clock cycles to send 32 bits) Used to be used only where distances were long (e.g., networks) Now used for most off-chip communications
- memory-mapped I/O: Manual handling of I/O devices Map portions of the address space to I/O devices Read and write to those addresses to access the
- direct memory access (DMA): Hardware who manage I/O devices (dynamically)
- Polling: The device puts its status somewhere The OS repeatedly checks for it to change
- Interrupt: When the device is ready, it gets the processor's attention by signaling an interrupt The OS then jumps to an interrupt handler to handle the event
- Throughput: x/s The read and write speed
- Latency: s/x Accessing time.
- Overhead: any combination of excess or indirect computation time, memory, bandwidth, or other resources that are required to perform a specific task

Data transfers: manual copy

```
add    R2, R0, R0       // Counter starts at 0
loop:
  lw R4, 0x12f0(R0)   // Read the I/O device
  sw R4, 0xfe00(R2)   // Store the results
  addi R2, R2, 4        // Next location
  bne R3, R2, loop
done:
```

Data transfers: DMA

```
addi R2, R0, 0xfe00 // destination
addi R3, R0, 230400  // number of words to copy
sw R1, 0x100b(R0) // set the device address
sw R2, 0x1008(R0) // set the destination address
sw R3, 0x1004(R0) // set the length and start
wait:
  lw R2, R0(0x1000) // Wait for it to be done
  bne R2, R0, wait
done:
```

5.10 Cache

5.10.1 Memory hierarchy

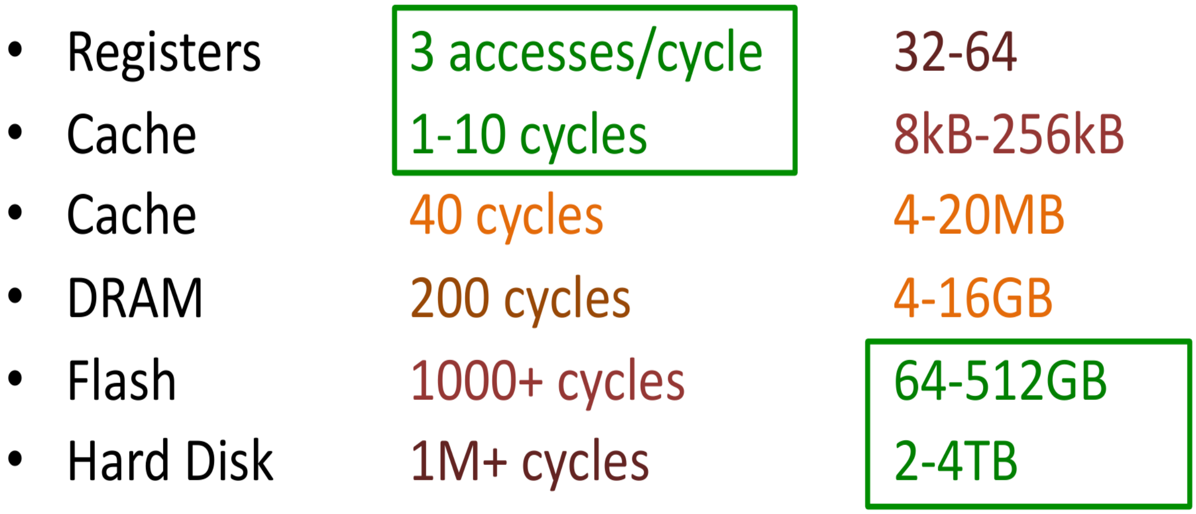


Figure 5.21: memory-hierarchy

3-types of cache types

- Fully-associative: Have to search all blocks, but very flexible
- Direct-mapped: Only one place for each block, no flexibility
- Set-associative: Only have to search one set for each block, flexible because each set can store multiple blocks in its ways

Write policies

- Write-through: slow, simple
- Write-back: fast (keeps the data just in the cache), more complex

Terminology

- Data: What is being stored
- Tag: What address the data has
- Index: What we use to determen where we want to put the data
- Cache line (block) size: How many tag's there are
- Valid bit: If the data is correct
- Dirty bit: The data has been change and we need to write it to memory
- Type of cache: Fully-associative (FA), Set-associative (SA), Direct-mapped (DM)
- Replacement policy: Write-through, Write-back

Cache blocks

- tag has 30bits and the 2 other bits is to determen what data 63 bits for eatch entry 32bit data 30bit tag 1bit valid-bit
- larger block of data -> fewer tags -> more efficient storage
- 1 word cache block we have 1 32bit data
- 2 word cache block we have 2 32bit data
- 4 word cache block we have 4 32bit data
- we load more data when we have space for it we will then have spatial locality
- Last 2: determine the byte within the word
- Next N: determine which word in the cache block
- Remaining 32-N-2: tag

Calculate overhead

- Data= Cache-Lines * Byte-Lines
- Overhead= Cache-Lines * (Tags-per-line + valid-bit)
- % data= Overhead / Data

Calculate Address (Tag Index Offset)

- Direct maped:
  Offest= log (number of data blocks) (+ 2-bit (4-bite cache block size if 64 then log 64))
  Index= log (number of cache lines)
  Tags= 32(bit processor)- (Index+Offset)
- x-set associative:
  Offest= log (number of data blocks) (+ 2-bit determens how the address looks like)
  Index= log ((number of cache lines)/x)
  Tags= 32(bit processor)- (Index+Offset)
- fully associative:
  Offest= log (number of data blocks) (+ 2-bit determens how the address looks like)
  Index= 0
  Tags= 32(bit processor)- (Offset)

Calculate Avrage memory access time

- Avrage-cycles= CacheL1*Cycles + CacheL2*Cycles + Dram*Cycles

5.10.2 Cache misses $3C's$

- cold/compulsory miss: When the program first begin it doesn't have anything in the cache
- conflicts miss: to small to store the necessary data
- capacity miss: map's at the same block an then overwrite unnecessarily

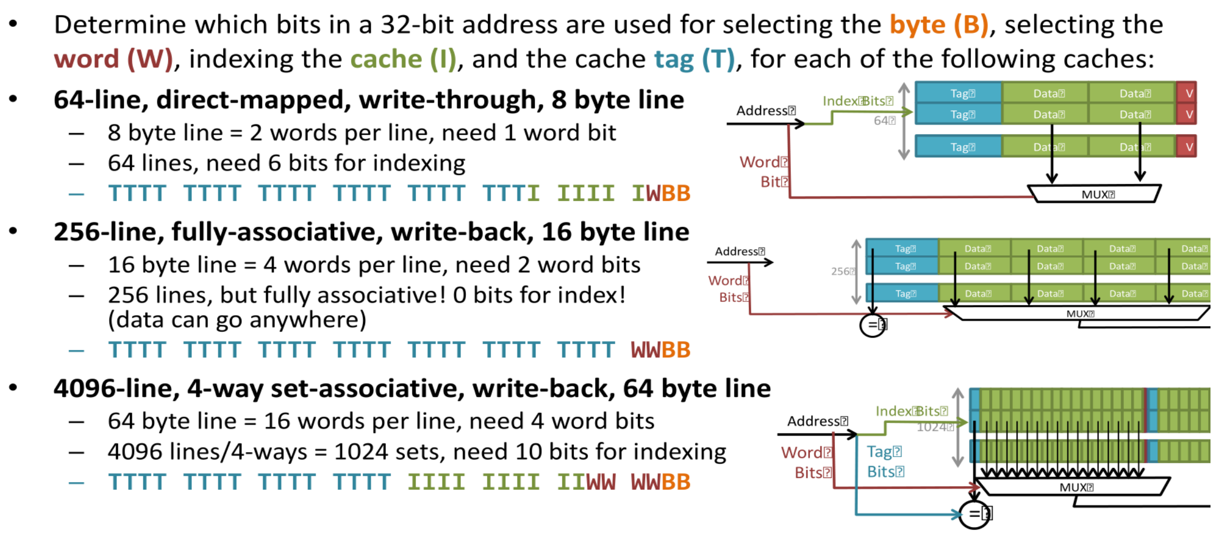


Figure 5.22: cache-format

5.11 Virtual Memory

Why use VM

- Map memory to disk (“unlimited” memory)
- Keep programs from accessing each other’s memory (security)
- Fill holes in the RAM address space (efficiency)

Terminology

- Translation= map address
- Page tables= for each program keep track of all translations
- Fine grain= page table with spesific address
- Coarse grain= page table with address mapped ranges
- Page Table Entries (PTEs)= number of translation
- Translation Lookaside Buffer TLB= All of the pages, fast translation via hardware
- VA= Virtual program addresses
- PA= Physical RAM addresses
- Page offset= point to a range and then use the offset to determen where
- Translation Lookaside Buffer (TLB)= page table cache (Faster)
- Multilevel page table translation= page table point to other page tables (inseption)

Combining TLB and cache

- Physical caches: slow and needs the translation first and then save get the PA also known as Pysical-Index, Physical-Tagged (PIPT)
- Virtual caches: fast uses only virtual addresses, no translation, difficult for protection also known as Virtual-Index, Virtual-Tagged (VIVT):
- Virtual-Index, Phisically-Tagged (**VIPT**): VA for index PA for tags, fast dose translation and fetch from cache at the same time, most commonly used. We need a comperitor to se if the PA tags from cache matches the TLB PA only then we can say if we had a hit or a miss. The VA offset is what the PA tag is selected and the VA tag (number of virtual pages) is what the TLB uses. Mostly used for L1 cache not L2.

TLB

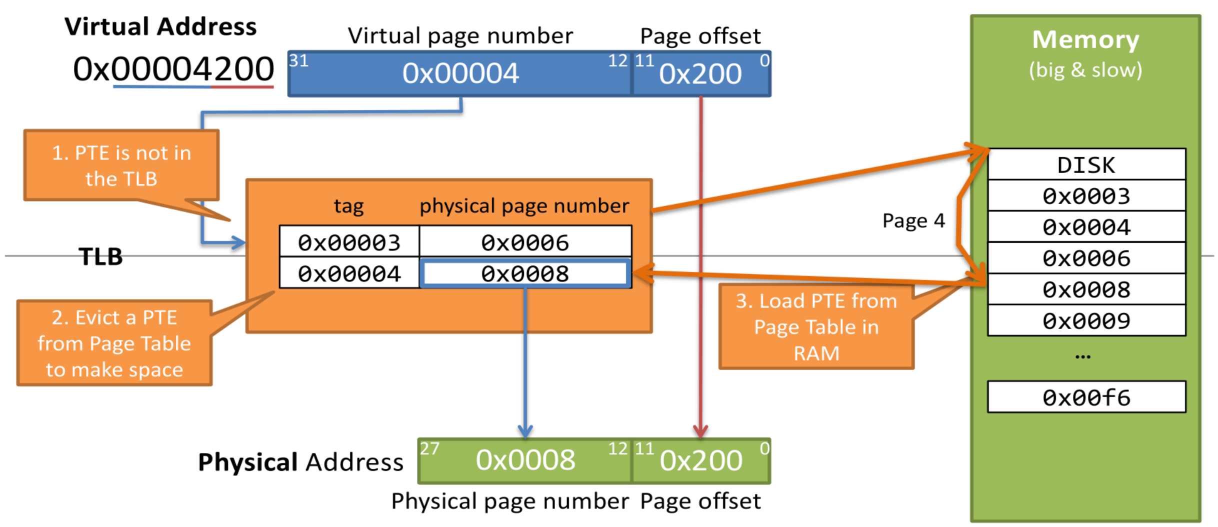


Figure 5.23: tlb

usful conversion: $2^n = xB$

0		10	1kB	20	1MB	30	1GB
1	2B	11	2kB	21	2MB	31	2GB
2	4B	12	4kB	22	4MB	32	4GB
3	8B	13	8kB	23	8MB		
4	16B	14	16kB	24	16MB		
5	32B	15	32kB	25	32MB		
6	64B	16	64kB	26	64MB		
7	128B	17	128kB	27	128MB		
8	256B	18	256kB	28	256MB		
9	512B	19	512kB	29	512MB		

Figure 5.24: conversion

Calculating Page sizing

- Number-of-Virtual-Pages= 2^{32}/pages
- Bits-used-for-Page-Offset= log(pages)
- Bits-used-for-VPN= 32(bit prossessor)- Bits-used-for-Page-Offset

Calculating TLB size

- TLB-size=Entries*Pages

5.12 Parallism

5.12.1 Multicore

powerwall

$$\begin{aligned}
&P = CV^2 f \\
&C = \text{ capasiter, smaller transistors smaller capasiters} \\
&V = \text{ voltage, decreasing makes it slower} \\
&f = \text{ frequency, reducing clock speed}
\end{aligned}$$

5.12.2 Paralel programming

Avrage processors

Calculate how many cores are used in avrage we need to know
how chunks (work) there is in total
how many time units (cycles think of a reverse pyramid)

there is 16 input data and we have 8-cores, we want to calculate the total sum
what is the avrage prossesor being used
$15chunks,\ 4timeunits \Rightarrow 15/4 = 3.75$

parallel issues

- Most programs can not utilize parallelism, need to devid the program to diffrent executions.
- We also need to share cache and therefore have performance issue since we cant use the entire cache.

How much faster

$75\%parallel,\ 25\%nonparallel$ we have hundred thousand cores
$\Rightarrow$ Parallaism takes $(0.75/100000 + 0.25 * 1)$
Singular takes $(0.75 * 1 + 0.25 * 1) \Rightarrow 4*$ faster

Speedup with Amdahl's law $Speedup = \dfrac{1}{(1-P) + P/S}$
$P =$ Parallel aktion
$S =$ Speed up of the parrallel part

$$\Rightarrow \frac{1}{(1-0.75) + 0.75/100000} = \frac{1}{0.25+0} = 4$$

5.12.3 Synchronization

- Fix the issue with 2 processors accessing the same value at the same time.
- We can use atomic swap to first set lock to 1 then check the data.

locks

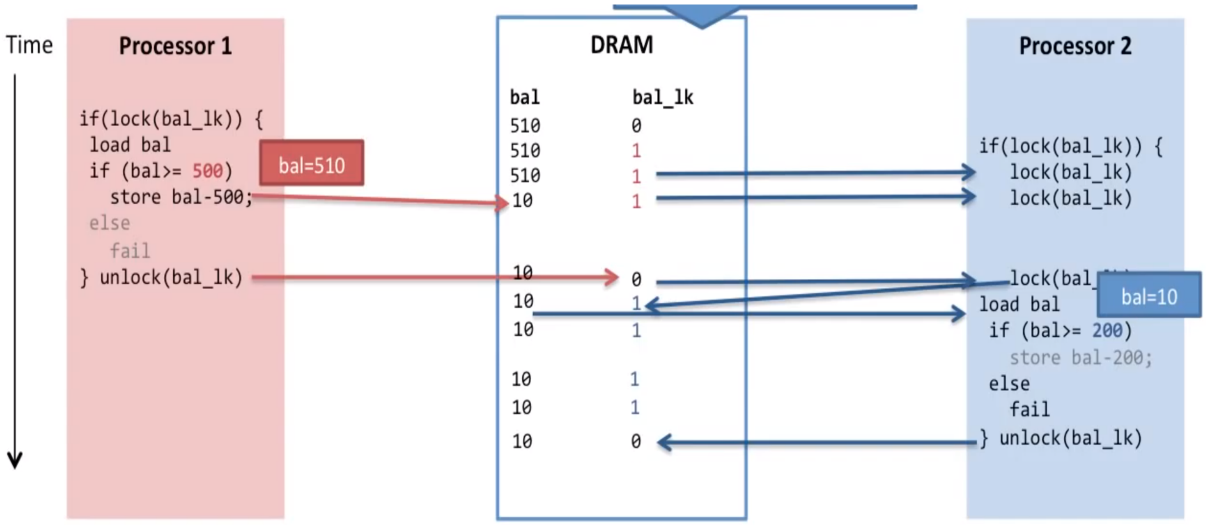


Figure 5.25: locks

5.12.4 Cache coherency

- How we use caches to save memory
- Locks: if the data has been acceded. (not a guaranty of protection)
- Snooping: Look what the other processors are storing in there private cache

snooping

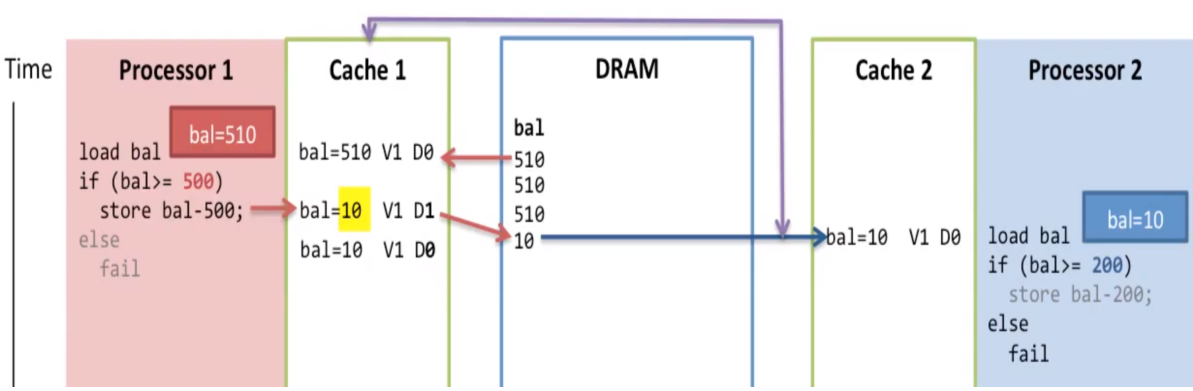


Figure 5.26: locks

5.12.5 ILP

- Instruction level parallelism (ILP)
- better to have out-of-oder execution since we can find independent instructions
- We use Dual-issue pipline so we can have one for memory instructions and one for non memory instruction
- It makes ISA promise with in order execution
- Issue with data hazards, more complexity

dual issue pipeline

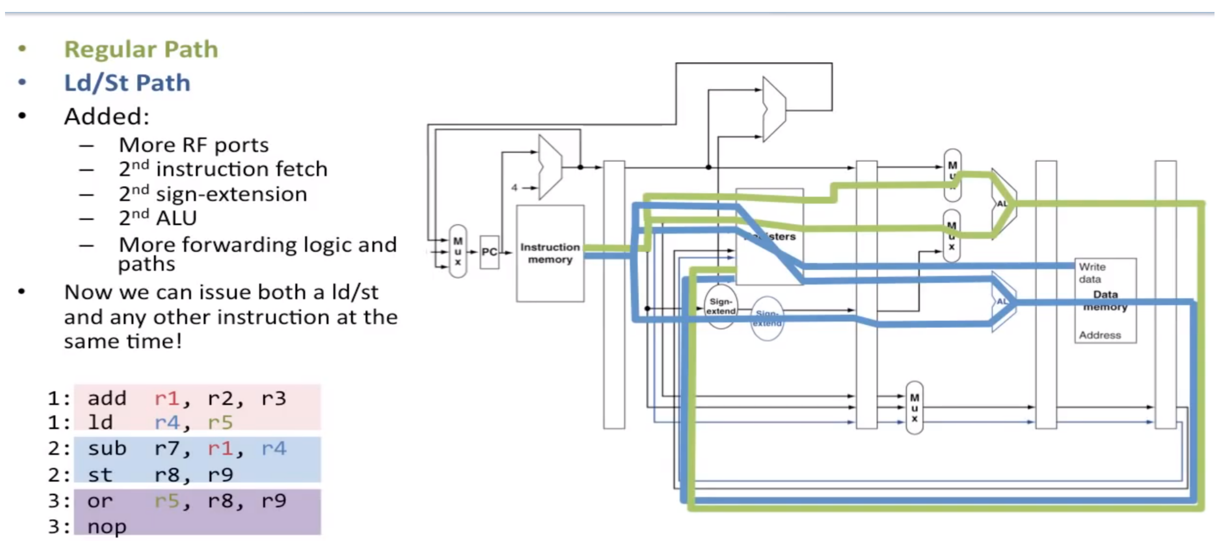


Figure 5.27: dual-issue-pipline