Formula Tables

Signals and Transforms

October 19, 2021

1 Continuous Time Fourier Series

Trigonometric Continuous Time Fourier Series

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

with

$$a_0 = \frac{1}{T_0} \int_{(T_0)} x(t) \, \mathrm{d}t$$
$$a_n = \frac{2}{T_0} \int_{(T_0)} x(t) \cos(n\omega_0 t) \, \mathrm{d}t$$
$$b_n = \frac{2}{T_0} \int_{(T_0)} x(t) \sin(n\omega_0 t) \, \mathrm{d}t$$

Exponential Continuous Time Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{j}\, n\omega_0 t} \quad \text{with} \quad c_n = \frac{1}{T_0} \int_{(T_0)} x(t) e^{-\mathrm{j}\, n\omega_0 t} \, \mathrm{d}t$$

2 Continuous Time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}\,\omega t} \, \mathrm{d}t \quad \text{and} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{\mathrm{j}\,\omega t} \, \mathrm{d}\omega$$

Table 1. Properties of the continuous time Fourier transform.

Property	Time Domain	Frequency Domain
Symmetry	Real-valued	$X(-\omega) = X^*(\omega)$
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(\omega) + \beta X_2(\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\omega}{a}\right)$
Time shift	$x(t - \Delta t)$	$e^{-\mathrm{j}\,\omega \Delta t} X(\omega)$
Frequency shift	$x(t) e^{\mathrm{j}\,\omega_0 t}$	$X(\omega - \omega_0)$
Differentiation	$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$	$(\mathrm{j}\,\omega)^n X(\omega)$
Integration	$\int_{-\infty}^{t} x(\tau) \, \mathrm{d}\tau$	$\frac{1}{\mathrm{j}\,\omega} X(\omega) + \pi X(0) \delta(\omega)$
Convolution	$x_1(t) * x_2(t)$	$X_1(\omega) X_2(\omega)$
Multiplication	$x_1(t) x_2(t)$	$\frac{1}{2\pi} X_1(\omega) * X_2(\omega)$
Parseval's theorem	$E = \int_{-\infty}^{\infty} \lvert x(t) \rvert^2 \, \mathrm{d}t$	$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 \, \mathrm{d}\omega$

Table 2. Continuous time Fourier transforms of elementary signals.

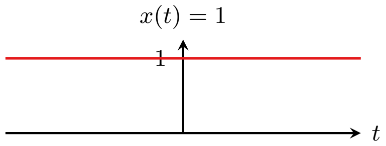
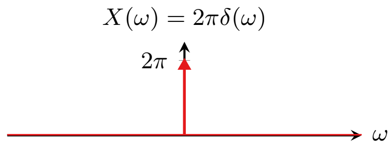
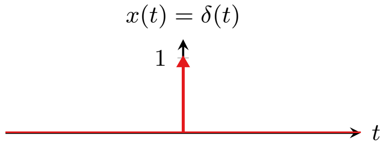
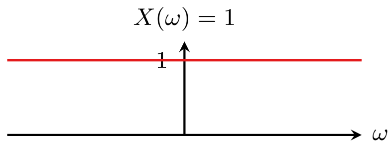
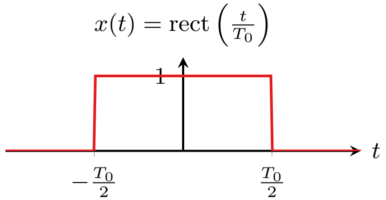
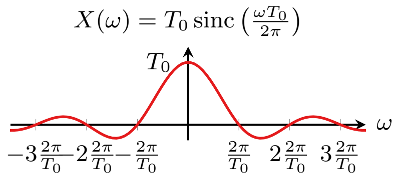
Time Domain $x(t)$	Fourier Transform $X(\omega)$
Periodic signals	
T_0-periodic ($\omega_0 = \frac{2\pi}{T_0}$)	$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_0)$ with $c_n = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-\mathrm{j} n\omega_0 t}\,\mathrm{d}t$
Constant	
$x(t) = 1$	$X(\omega) = 2\pi\delta(\omega)$
	
Dirac delta function	
$x(t) = \delta(t)$	$X(\omega) = 1$
	
Rectangular pulse	
$x(t) = \mathrm{rect}\left(\frac{t}{T_0}\right)$	$X(\omega) = T_0 \,\mathrm{sinc}\left(\frac{\omega T_0}{2\pi}\right)$
	

Table 2. Continuous time Fourier transforms of elementary signals (cont'd.).

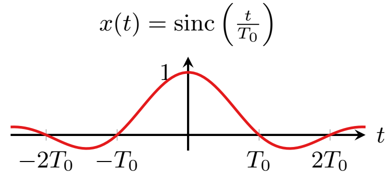
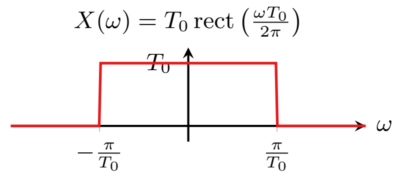
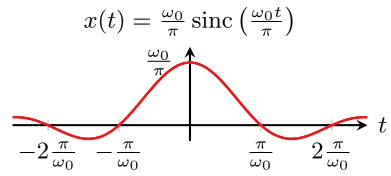
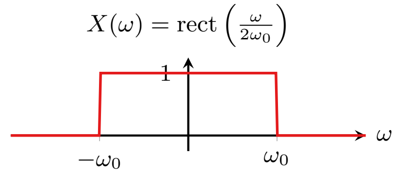
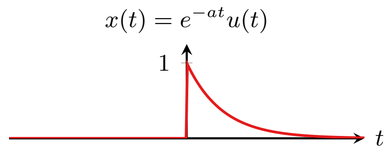
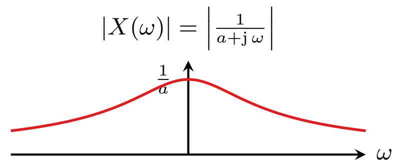
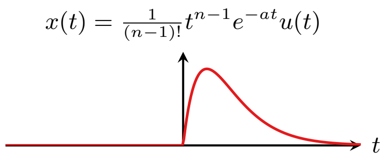
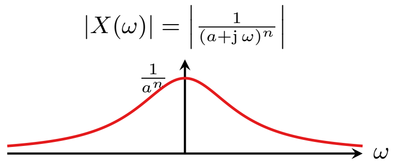
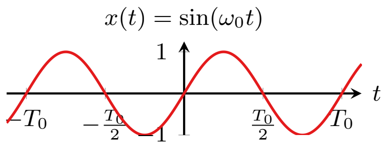
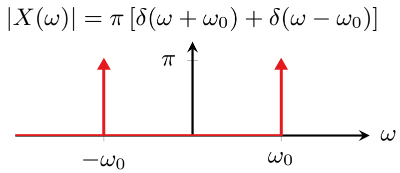
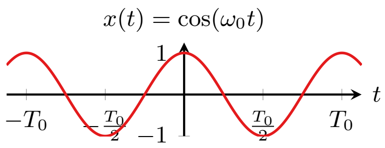
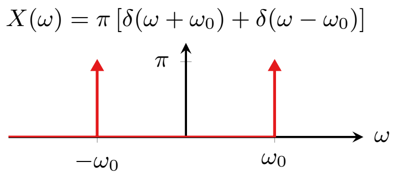
Time Domain $x(t)$	Fourier Transform $X(\omega)$
Sinc function	
$x(t) = \mathrm{sinc}\left(\frac{t}{T_0}\right)$	$X(\omega) = T_0 \,\mathrm{rect}\left(\frac{\omega T_0}{2\pi}\right)$
	
Sinc function (alternative parametrization)	
$x(t) = \frac{\omega_0}{\pi}\,\mathrm{sinc}\left(\frac{\omega_0 t}{\pi}\right)$	$X(\omega) = \mathrm{rect}\left(\frac{\omega}{2\omega_0}\right)$
	
Exponential	
$x(t) = e^{-at}u(t)$ $(a > 0)$	$X(\omega) = \frac{1}{a + \mathrm{j}\,\omega}$
	

Table 2. Continuous time Fourier transforms of elementary signals (cont'd.).

Time Domain $x(t)$	Fourier Transform $X(\omega)$
Multiple real-valued poles	
$x(t) = \dfrac{1}{(n-1)!} t^{n-1} e^{-at} u(t)$ $(a > 0,\ n = 1, 2, \ldots)$	$X(\omega) = \dfrac{1}{(a + \mathrm{j}\,\omega)^n}$
	
Sine	
$x(t) = \sin(\omega_0 t)$	$X(\omega) = \mathrm{j}\,\pi\,[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$
	
Cosine	
$x(t) = \cos(\omega_0 t)$	$X(\omega) = \pi\,[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$
	

3 Laplace Transform

$$X(s) = \int_0^\infty x(t) e^{-st}\,\mathrm{d}t \quad \text{and} \quad x(t) = \frac{1}{2\pi\,\mathrm{j}} \int_{\sigma-\mathrm{j}\,\omega}^{\sigma+\mathrm{j}\,\omega} X(s) e^{st}\,\mathrm{d}s$$

Table 3. Properties of the Laplace transform.

Property	Time Domain	Laplace Domain
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$
Scaling $(a > 0)$	$x(at)$	$\dfrac{1}{a} X\left(\dfrac{s}{a}\right)$
Time shift	$x(t - \Delta t)$	$e^{-s\Delta t} X(s)$
s-domain shift	$x(t) e^{at}$	$X(s - a)$
Differentiation (first order)	$\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s) - x(0)$
Differentiation (nth order)	$\dfrac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$	$s^n X(s) - s^{n-1} x(0) - s^{n-2} x^{(1)}(0) - \cdots - x^{(n-1)}(0)$
Integration	$\displaystyle\int_0^t x(\tau)\,\mathrm{d}\tau$	$\dfrac{1}{s} X(s)$
Convolution	$x_1(t) * x_2(t)$	$X_1(s) X_2(s)$
Multiplication	$x_1(t) x_2(t)$	$\dfrac{1}{2\pi\,\mathrm{j}} X_1(s) * X_2(s)$
Initial value theorem	$\lim\limits_{t\to 0} x(t)$	$\lim\limits_{s\to\infty} sX(s)$
Final value theorem	$\lim\limits_{t\to\infty} x(t)$	$\lim\limits_{s\to 0} sX(s)$

Table 4. Table of unilateral Laplace transform pairs for causal signals $x(t)$ ($x(t) = 0$ for $t < 0$).

Time domain $x(t)$	Laplace transform $X(s)$	ROC
Dirac delta function		
$x(t) = \delta(t)$	$X(s) = 1$	All s
Unit step		
$x(t) = u(t)$	$X(s) = \frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
Exponential		
$x(t) = e^{-at}u(t)$	$X(s) = \frac{1}{a+s}$	$\mathrm{Re}\{s\} > -a$
Ramp		
$x(t) = tu(t)$	$X(s) = \frac{1}{s^2}$	$\mathrm{Re}\{s\} > 0$
Higher order ramp		
$x(t) = t^n u(t)$	$X(s) = \frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
Cosine		
$x(t) = \cos(\omega_0 t)u(t)$	$X(s) = \frac{s}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Sine		
$x(t) = \sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{\omega_0^2 + s^2}$	$\mathrm{Re}\{s\} > 0$
Decaying cosine		
$x(t) = e^{-at}\cos(\omega_0 t)u(t)$	$X(s) = \frac{a+s}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$
Decaying sine		
$x(t) = e^{-at}\sin(\omega_0 t)u(t)$	$X(s) = \frac{\omega_0}{(a+s)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$

4 Discrete Time Fourier Series

$$x[k] = \sum_{n=0}^{K_0-1} c_n e^{\mathrm{j}\, n\Omega_0 k} \quad \text{with} \quad c_n = \frac{1}{K_0}\sum_{k=0}^{K_0-1} x[k]e^{-\mathrm{j}\, n\Omega_0 k}$$

5 Discrete Time Fourier Transform

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x[k]e^{-\mathrm{j}\,\Omega k} \quad \text{and} \quad x[k] = \frac{1}{2\pi}\int_0^{2\pi} X(\Omega)e^{\mathrm{j}\,\Omega k}\,\mathrm{d}\Omega$$

Table 5. Properties of the discrete time Fourier transform.

Property	Time Domain	Frequency Domain
Periodicity	-	$X(\Omega) = X(\Omega + 2\pi)$
Symmetry	Real-valued	$X(-\Omega) = X^*(\Omega)$
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(\Omega) + \beta X_2(\Omega)$
Time shifting (k_0: integer)	$x[k - k_0]$	$e^{-\mathrm{j}\,\Omega k_0} X(\Omega)$
Time differencing	$x[k] - x[k-1]$	$\left(1 - e^{-\mathrm{j}\,\Omega}\right) X(\Omega)$
Frequency domain differentiation	$-\mathrm{j}\, kx[k]$	$\frac{\mathrm{d}X(\Omega)}{\mathrm{d}\Omega}$
Time summation	$\sum_{n=-\infty}^{k} x[n]$	$\frac{X(\Omega)}{1 - e^{-\mathrm{j}\,\Omega}} + \pi X(0)\sum_{m=-\infty}^{\infty}\delta(\Omega - 2\pi m)$
Convolution	$x_1[k] * x_2[k]$	$X_1(\Omega)X_2(\Omega)$
Multiplication†	$x_1[k]x_2[k]$	$\frac{1}{2\pi}X_1(\Omega) * X_2(\Omega)$

Table 5. Properties of the discrete time Fourier transform (cont'd.).

Property	Time Domain	Frequency Domain
Parseval's theorem	$E = \sum_{k=-\infty}^{\infty} \lvert x[k]\rvert^2$	$E = \frac{1}{2\pi}\int_0^{2\pi} \lvert X(\Omega)\rvert^2 \, \mathrm{d}\Omega$

†The convolution is on the interval $0 \ldots 2\pi$.

Table 6. Discrete time Fourier transforms of elementary signals.

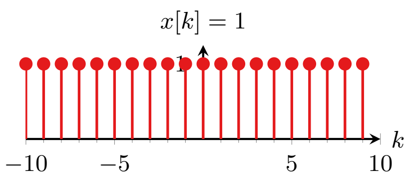
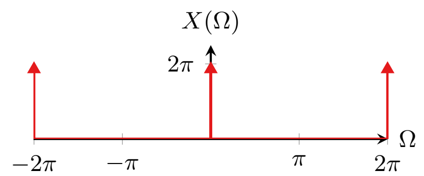
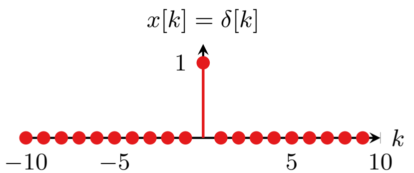
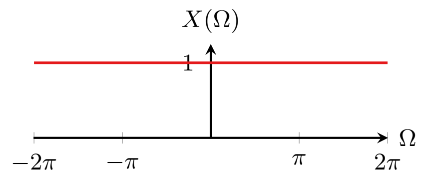
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Periodic signals	
K_0-periodic ($\Omega_0 = \frac{2\pi}{K_0}$)	$X(\Omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\Omega - n\Omega_0)$ with $c_n = \frac{1}{K_0}\sum_{k=0}^{K_0-1} x[k] e^{-\mathrm{j}\, n\Omega_0 k}$
Constant	
$x[k] = 1$	$X(\Omega) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$
	
Unit impulse	
$x[k] = \delta[k]$	$X(\Omega) = 1$
	

Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

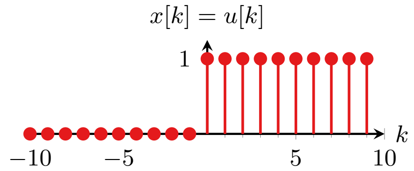
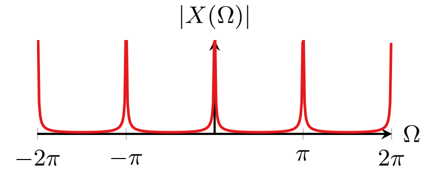
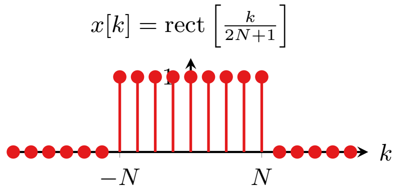
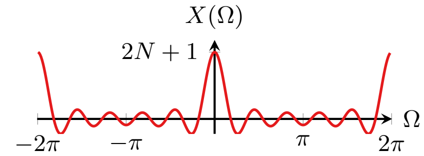
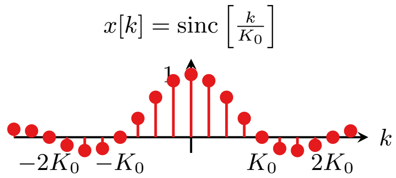
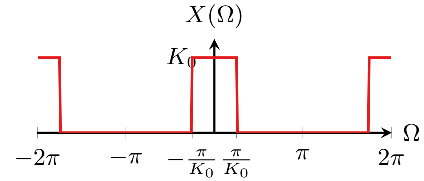
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Unit step	
$x[k] = u[k]$	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) + \frac{1}{1 - e^{-\mathrm{j}\Omega}}$
	
Rectangular pulse	
$x[k] = \mathrm{rect}\left[\frac{k}{2N+1}\right]$	$X(\Omega) = \frac{\sin\left(\frac{(2N+1)\Omega}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$
	
Sinc function	
$x[k] = \mathrm{sinc}\left[\frac{k}{K_0}\right]$	$X(\Omega) = K_0 \sum_m \mathrm{rect}\left(\frac{\Omega - 2\pi m}{\frac{2\pi}{K_0}}\right)$
	

Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

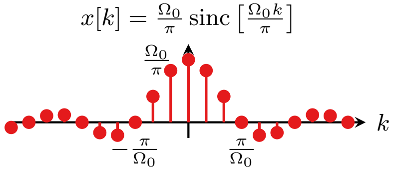
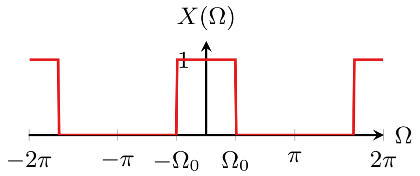
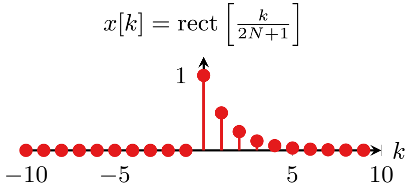
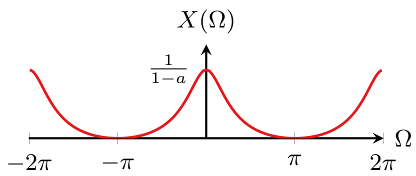
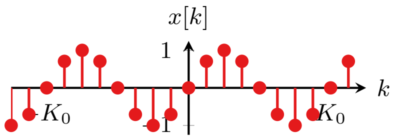
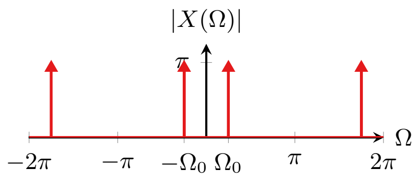
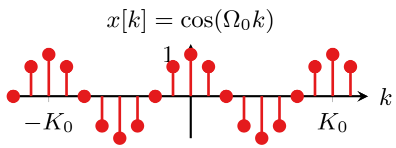
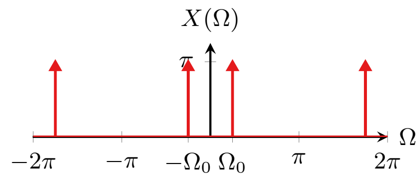
Time Domain $x[k]$	Fourier Transform $X(\Omega)$
Sinc function (alternative parametrization)	
$x[k] = \frac{\Omega_0}{\pi}\,\mathrm{sinc}\left[\frac{\Omega_0 k}{\pi}\right]$	$X(\Omega) = \sum_m \mathrm{rect}\left(\frac{\Omega - 2\pi m}{2\Omega_0}\right)$
	
Exponential	
$x[k] = a^k u[n]$ $(\lvert a\rvert < 1)$	$X(\Omega) = \frac{1}{1 - a e^{-\mathrm{j}\Omega}}$
	
Sine	
$x[k] = \sin(\Omega_0 k)$	$X(\Omega) = \mathrm{j}\,\pi \sum_m [\delta(\Omega + \Omega_0 - 2\pi m) - \delta(\Omega - \Omega_0 - 2\pi m)]$
	
Cosine	
$x[k] = \cos(\Omega_0 k)$	$X(\Omega) = \pi \sum_{m=-\infty}^{\infty} [\delta(\Omega + \Omega_0 - 2\pi m) + \delta(\Omega - \Omega_0 - 2\pi m)]$

Table 6. Discrete time Fourier transforms of elementary signals (cont'd.).

Time Domain $x[k]$	Fourier Transform $X(\Omega)$
	

6 Discrete Fourier Transform

$$X[l] = \sum_{k=0}^{K-1} x[k] e^{-\mathrm{j}\,lk\frac{2\pi}{K}} \quad \text{and} \quad x[k] = \frac{1}{K}\sum_{l=0}^{K-1} X[l] e^{\mathrm{j}\,lk\frac{2\pi}{K}}$$

7 z-Transform

$$X(z) = \sum_{k=0}^{\infty} x[k] z^{-k} \quad \text{and} \quad x[k] = \frac{1}{2\pi\,\mathrm{j}} \oint X(z) z^{k-1}\,\mathrm{d}z$$

Table 7. Properties of the z-transform.

Property	Time Domain	z-Domain
Linearity	$\alpha x_1[k] + \beta x_2[k]$	$\alpha X_1(z) + \beta X_2(z)$
Time shifting	$x[k-m]$	$z^{-m} X(z)$
Convolution	$x_1[k] * x_2[k]$	$X_1(z) X_2(z)$
Scaling	$a^k x[k]$	$X\left(\frac{z}{a}\right)$
Time difference	$x[k] - x[k-1]$	$(1 - z^{-1}) X(z)$
Accumulation	$\sum_{m=0}^{k} x[m]$	$\frac{1}{1 - z^{-1}} X(z)$
Initial value theorem	-	$x[0] = \lim_{z\to\infty} X(z)$

Table 7. Properties of the z-transform (cont'd.).

Property	Time Domain	z-Domain
Final value theorem	-	$\lim_{k\to\infty} x[k] = \lim_{z\to 1}(z-1)X(z)$

Table 8. Table of unilateral z-transform pairs for causal signals $x[k]$ ($x[k] = 0$ for $k < 0$).

Time domain $x[k]$	z-Transform $X(z)$	ROC
Unit impulse		
$x[k] = \delta[k]$	$X(z) = 1$	all z
Unit step		
$x[k] = u[k]$	$X(z) = \dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
Exponential		
$x[k] = a^k u[k]$	$X(z) = \dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Ramp		
$x[k] = ku[k]$	$x(z) = \dfrac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$
Cosine		
$x[k] = \cos(\Omega_0 k)u[k]$	$X(z) = \dfrac{1-z^{-1}\cos(\Omega_0)}{1-2z^{-1}\cos(\Omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
Sine		
$x[k] = \sin(\Omega_0 k)u[k]$	$X(z) = \dfrac{z^{-1}\sin(\Omega_0)}{1-2z^{-1}\cos(\Omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
Decaying cosine		
$x[k] = a^k\cos(\Omega_0 k)u[k]$	$X(z) = \dfrac{1-az^{-1}\cos(\Omega_0)}{1-2az^{-1}\cos(\Omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
Decaying sine		
$x[k] = a^k\sin(\Omega_0 k)u[k]$	$X(z) = \dfrac{az^{-1}\sin(\Omega_0)}{1-2az^{-1}\cos(\Omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$