Satisfiability Of Connected Contact Logics

Anton Dudov

2019-9-28

Contents

1 Tableau Method For Classic Propositional Logic

1.1 What Is A Tableau?

A tableau method is a formal proof procedure. First, it could be used as a refutation procedure: to show a formula X is valid we begin with some syntactical expression intended to assert it is not. This expression is broken down syntactically, generally splitting things into several cases. This part of a tableau procedure - the tableau expansion stage - can be thought of as a generalization of disjunctive normal form expansion. Generally, it moves from formulas to subformulas. Finally, there are rules for closing cases: impossibility conditions based on syntax. If each case closes, the tableau itself is said to be closed. A closed tableau beginning with an expression asserting that X is not valid is a tableau proof of X.

There is a second way of thinking about the tableau method: as a search procedure for models meeting certain conditions. Each branch of a tableau can be considered to be a partial description of a model. In automated theorem-proving, tableaus can be used to generate counter-examples.

The connection between the two roles for tableaus - as a proof procedure and as a model search procedure - is simple. If we use tableaus to search for a model in which X is false, and we produce a closed tableau, no such model exists, so X must be valid.

1.2 Classical Propositional Tableaus

We will look into the signed tableau system for classical propositional logic.

First, we need syntactical machinery for asserting the invalidity of a formula, and for doing case analysis. For this purpose two signs are introduced: T and F, where these are simply two new symbols, not part of the language of formulas. *Signed formulas* are expressions of the form FX and TX, where X is a formula. The intuitive meaning of FX is that X is *false* (in some model). Similarly, TX intuitively asserts that X is *true*. Then FX is the syntactical device for (informally) asserting the invalidity of X. A tableau proof of X begins with FX.

Next, we need machinery (rules) for breaking signed formulas down and doing a case division. We will define rules for each logical operator ($\neg \wedge \vee \Rightarrow \Leftrightarrow$).

The treatment of **negation** is straightforward: from $T\neg X$ we get FX and from $F\neg X$ we get TX. These rules can be conveniently presented as follows.

$$\frac{T\neg X}{FX} \qquad\qquad \frac{F\neg X}{TX}$$

The rules for **conjunction** are somewhat more complex. From truth tables we know that if $X \wedge Y$ is *true*, X must be *true* and Y must be *true*. Likewise, if $X \wedge Y$ is *false*, eigher X is *false* or Y is *false*. This involves a split into two cases. Corresponding syntactic rules are as follows.

$$\frac{TX \wedge Y}{\begin{array}{c} TX \\ TY \end{array}} \qquad\qquad \frac{FX \wedge Y}{FX \mid FY}$$

The rules for **disjunction** are similar. From truth tables we know that if $X \lor Y$ is *true*, eigher X is *true* or Y is *true*. This involves a split into two cases. Likewise, if $X \lor Y$ is *false*, X must be *false* and Y must be *false*. Corresponding syntactic rules are as follows.

$$\frac{T\,X \lor Y}{T\,X \mid T\,Y} \qquad\qquad \frac{F\,X \lor Y}{\begin{array}{c} F\,X \\ F\,Y \end{array}}$$

The rules for **implication**. From truth tables we know that if $X \Rightarrow Y$ is *true*, eigher X is *false* or Y is *true*. Likewise, if $X \Rightarrow Y$ is *false*, X must be *true* and Y must be *false*. Corresponding syntactic rules are as follows.

$$\frac{T\,X \Rightarrow Y}{F\,X \mid T\,Y} \qquad\qquad \frac{F\,X \Rightarrow Y}{\begin{array}{c} T\,X \\ F\,Y \end{array}}$$

The rules for **equivalence**. From truth tables we know that if $X \Leftrightarrow Y$ is *true*, eigher X is *true* and Y is *true* or X is *false* and Y is *false*. Likewise, if $X \Rightarrow Y$ is *false*, eigher X is *true* and Y is *false* or X is *false* and Y is *true*. Corresponding syntactic rules are as follows.

$$\frac{T\,X \Leftrightarrow Y}{\begin{array}{c|c} T\,X & F\,X \\ T\,Y & F\,Y \end{array}} \qquad\qquad \frac{F\,X \Leftrightarrow Y}{\begin{array}{c|c} T\,X & F\,X \\ F\,Y & T\,Y \end{array}}$$
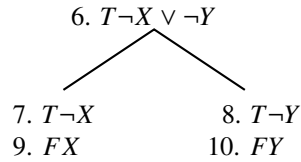
The standard way of displaying tableaus is as downward branching trees with signed formulas as node labels. Indeed, the tableau method is often referred to as the tree method. Think of a tree as representing the disjunction of its branches, and a branch as representing the conjunction of the signed formulas on it.

When using a tree display, a tableau expansion is thought of temporally, and one talks about the stages of constructing a tableau, meaning the stages of growing a tree. The rules given above are thought of as branch-lengthening rules. Thus, a branch containing $T \neg X$ can be lengthened by adding a new node to its end, with $F\,X$ as a label. Likewise a branch containing $F\,X \lor Y$ can be lengthened with two new nodes, labelled $F\,X$ and $F\,Y$ (take the node with $F\,Y$ as the child of the one labelled $F\,X$). A branch containing $T\,X \lor Y$ can be split - its leaf is given a new left and a new right child, with one labelled $T\,X$, the other $T\,Y$. This is how the schematic rules above are applied to trees.

An important point to note is that the tableau rules are non-deterministic. They say what can be done, not what must be done. At each stage, we choose a signed formula occurrence on a branch and apply a rule to it. Since the order of choice is arbitrary, there can be many tableaus for a single signed formula.

Here is the final stage of a tableau expansion beginning with the signed formula $F(X \land Y) \Rightarrow (\neg X \land \neg Y)$.
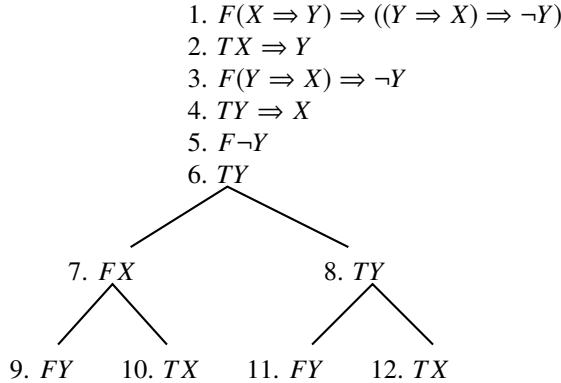
1. $F(X \land Y) \Rightarrow \neg(\neg X \lor \neg Y)$
2. $T\,X \land Y$
3. $F\neg(\neg X \lor \neg Y)$
4. $T\,X$
5. $T\,Y$

$$6.\ T\neg X \vee \neg Y$$

$$7.\ T\neg X \qquad\qquad 8.\ T\neg Y$$
$$9.\ FX \qquad\qquad 10.\ FY$$

In this we have added numbers for reference purposes. Items 2 and 3 are from 1 by $F \Rightarrow$. 4 and 5 are from 2 by $T\wedge$. 6 is from 3 by $F\neg$. 7 and 8 are from 6 by $T\vee$. 9 is from 7 by $T\neg$. 10 is from 8 by $T\neg$.

Finaly, the conditions for closing off a case (declaring a branch closed) are simple. A **branch is closed** if it contains a contradiction, i.e. TA and FA for some formula A. A **branch is opened** if it does not contain any contradiction. If each branch is closed, then the **tableau is closed**. A closed tableau for FX is a tableau proof of X, meaning that A is a tautology. The tableau displayed above is closed, so the formula $(X \wedge Y) \Rightarrow (\neg X \wedge \neg Y)$ has a tableau proof.

It may happen that no tableau proof is forthcoming, and we can think of the tableau construction as providing us with contraexamples. Consider the following attempt to prove $(X \Rightarrow Y) \Rightarrow ((Y \Rightarrow X) \Rightarrow \neg Y)$

$$1.\ F(X \Rightarrow Y) \Rightarrow ((Y \Rightarrow X) \Rightarrow \neg Y)$$
$$2.\ TX \Rightarrow Y$$
$$3.\ F(Y \Rightarrow X) \Rightarrow \neg Y$$
$$4.\ TY \Rightarrow X$$
$$5.\ F\neg Y$$
$$6.\ TY$$

$$7.\ FX \qquad\qquad 8.\ TY$$

$$9.\ FY \quad 10.\ TX \qquad 11.\ FY \quad 12.\ TX$$

Item 2 and 3 are from 1 by $F \Rightarrow$, as are 4 and 5 from 3. Item 6 is from 5 by $F\neg$. Items 7 and 8 are from 2 by $T \Rightarrow$, as are 9 and 10 from 4. Items 11 and 12 are also from 4 by $T \Rightarrow$. The leftmost branch is closed because of 6 and 9. The left-right branch is closed because of 7 and 10. The right-left branch is closed because of 8 and 11. But the rightmost branch is not closed. Notice that every non-atomic signed formula has had a rule applied to it on this branch and there is nothing left to do. (For clasical propositional logic it is sufficient to apply a rule to a formula on a branch only once.) In fact the branch yields a counterexample, as follows. Let v be a propositional valuation that maps X to *true* and Y to *true* in accordance to 8 and 12. Now, we work our way back up the branch. Since $v(Y) = true$, $v(\neg Y) = false$, item 5. From $v(X) = true$ follows that $v(Y \Rightarrow X) = true$, item 4. From $v(Y) = true$ follows that $v(X \Rightarrow Y) = true$, item 2. Since $v(Y \Rightarrow X) = true$ and $v(\neg Y) = false$ we have $v((Y \Rightarrow X) \Rightarrow \neg Y) = false$, item 3. Finally, $v((X \Rightarrow Y) \Rightarrow ((Y \Rightarrow X) \Rightarrow \neg Y)) = false$, item 1.

Later, we are going to use the tableau method to produce a model in which the initial formula is valid.

From a different point of view, we can think of a classical tableau simply as a set

of sets of signed formulas: a tableau is the set of its all branches, and a branch is the set of signed formulas that occur on it. Semantically, we think of the outer set as the disjunction of its members, and these members, the inner sets, as conjunctions of signed formulas they contain. Considered this way, a tableau is a generalization of disjunctive normal form (a generalization because of formulas more complex than literals can occur). Now, the tableau construction process can be thought of as a variation of the process of converting a formula into a disjunctive normal form.

2 Contact Logics

2.1 Syntax

The language of contact logic consist of:

- *Boolean variables* (a denumeratable set $\mathcal{V}$)

- *Boolean constants*: 0 and 1

- *Boolean operations*:

    - $\sqcap$ boolean meat
    - $\sqcup$ boolean join
    - * boolean complement

- *Boolean terms* (or simply *terms*)

- *Propositional connectivies*: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$

- *Propositional constants*: $\top$ and $\bot$

- *Modal connectives*: $\leq$(part-of) and C(contact)

- *Complex formulas* (or simply *formulas*)

Terms are defined in the following inductive process:

- Each Boolean variable is a term

- Each Boolean constant is a term

- If a is a term then a^* is a term

- If a and b are terms then $a \sqcap b$ and $a \sqcup b$ are terms

Atomic formulas are of the form $a \leq b$ and aCb, where a and b are terms.
Formulas are defined in the following inductive process:

- Each propositional constant is a formula

- Each atomic formula is a formula

- If ϕ is a formula then $\neg\phi$ is a formula

- If ϕ and ψ are formulas then $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \Rightarrow \psi)$ and $(\phi \Leftrightarrow \psi)$ are formulas

Abbreviations: $a = b \overset{\text{def}}{=\!=} (a \leq b) \wedge (b \leq a)$, $a \nleq b \overset{\text{def}}{=\!=} \neg(a \leq b)$, $a \neq b \overset{\text{def}}{=\!=} \neg(a = b)$

7

2.2 Relational semantics

Let $\mathcal{F} = (W, R)$ be a relational system with $W \neq \emptyset$ and $R \subseteq W^2$. We call such systems *frames*. Following Galton[TODO] we may give a spatial meaning of frames naming the elements of W, *cells* and the relation R, *adjacency relation*. Then $\mathcal{F}$ is called *adjacency space*. An example of adjacency space is the chess-board table, the cells are the squares, and two squares are adjacent if they have a common point.

Originally Galton assumed R to be a reflexive and symmetric relation but it is more natural for R to be an arbitrary relation. *Regions* in an adjacency space are arbitrary subsets of W and two sets a and b are in *contact* (aC_Rb) if for some $x \in a$ and $y \in b$ we have xRy. Another way to define this relation is the following. For a subset $a \subseteq W$ define as in modal logic $\langle R \rangle a = \{x \in W : (\exists y \in W)(xRy \text{ and } y \in a)\}$. Then aC_Rb iff $a \cap \langle R \rangle b \neq \emptyset$ iff $\langle R^{-1} \rangle a \cap b \neq \emptyset$, where R^{-1} is the converse relation of R. Note that if R is a symmetric relation then aCb iff $a \cap \langle R \rangle b \neq \emptyset$ iff $\langle R \rangle a \cap b \neq \emptyset$.

Definition 1. *By a **valuation** of the Boolean variables in $\mathcal{F}$ we mean any function $\upsilon : \mathcal{V} \rightarrow \mathcal{P}(W)$ assigning to each Boolean variable b a subset $\upsilon(b) \subseteq W$. The valuation υ is then extended inductively to all Boolean terms as follows:*

- $\upsilon(0) = \emptyset$

- $\upsilon(1) = W$

- $\upsilon(a \sqcap b) = \upsilon(a) \cap \upsilon(b)$

- $\upsilon(a \sqcup b) = \upsilon(a) \cup \upsilon(b)$

- $\upsilon(a^*) = W \setminus \upsilon(a) = -\upsilon(a)$

Deffinition 2. *The pair $\mathcal{M} = (\mathcal{F}, \upsilon)$ is called **model**. The truth of a formula ϕ in $\mathcal{M}$ ($\mathcal{M} \models \phi$ or $\mathcal{F}, \upsilon \models \phi$) is extended inductively to all Boolean terms as follows:*

- *For atomic formulas:*

    - $\mathcal{M} \models \top$
    - $\mathcal{M} \not\models \bot$
    - $\mathcal{M} \models a \leq b \iff \upsilon(a) \subseteq \upsilon(b)$
    - $\mathcal{M} \models aCb \iff \upsilon(a) \, C_R \, \upsilon(b \iff (\exists x \in \upsilon(a))(\exists y \in \upsilon(b))(xRy)$

- *For complex formulas:*

    - $\mathcal{M} \models \neg\phi \iff \mathcal{M} \not\models \phi$
    - $\mathcal{M} \models \phi \wedge \psi \iff \mathcal{M} \models \phi \text{ and } \mathcal{M} \models \psi$
    - $\mathcal{M} \models \phi \vee \psi \iff \mathcal{M} \models \phi \text{ or } \mathcal{M} \models \psi$
    - $\mathcal{M} \models \phi \Rightarrow \psi \iff \mathcal{M} \not\models \phi \text{ or } \mathcal{M} \models \psi$
    - $\mathcal{M} \models \phi \Leftrightarrow \psi \iff (\mathcal{M} \models \phi \text{ and } \mathcal{M} \models \psi) \text{ or } (\mathcal{M} \not\models \phi \text{ and } \mathcal{M} \not\models \psi)$

Let us note that in the above semantics we evaluate formulas not locally at points[TODO ??], as it is in the standard modal semantics, but globally in the whole model and this is one of the main differences of the present modal approach with the standard Kripke approach.

Deffinition 3. *A model $\mathcal{M}$ is a **model of a formula** ϕ if ϕ is true in $\mathcal{M}$.*

Deffinition 4. *If ϕ has a model $\mathcal{M}$, then ϕ is **satisfiable**.*

Deffinition 5. *$\mathcal{M}$ is a **model of a set of formulas** A if $\mathcal{M}$ is a model of all formulas from A.*

Deffinition 6. *A formula ϕ **is true** (or **valid**) in a frame $\mathcal{F}$ ($\mathcal{F} \models \phi$), if $\mathcal{M} \models \phi$ for all models $\mathcal{M}$ based on $\mathcal{F}$, i.e. for all valuations υ we have $\mathcal{F}, \upsilon \models \phi$.*

Lemma 7. *(Equality of formulas) Let f and g are formulas. Then*
$$f = g \implies \upsilon(f) = \upsilon(g).$$

Lemma 8. *(Equality of terms) Let a and b are terms. Then*
$$a = b \implies \upsilon(a) = \upsilon(b).$$

Lemma 9. *(Zero term) Let a and b are terms. Then*
$$a \leq b \implies a \sqcap b^* = 0$$

Lemma 10. *(Non-zero term) Let a and b are terms. Then*
$$\neg(a \leq b) \implies a \sqcap b^* \neq 0$$

TODO: axiom or lemma or neither??

Axiom 11. *(Reflexivity) Let a be a term. Then*
$$a \neq 0 \implies aCa.$$

Axiom 12. *(Symmetry) Let a and b are terms. Then*
$$aCb \iff bCa.$$

Lemma 13. *(Monotonicity) Let a and b are terms. Then*
$$aCb \wedge a \leq a' \wedge b \leq b' \implies a'Cb'.$$

Lemma 14. *(Distributivity) Let a and b are terms. Then*
$$aC(b \sqcup c) \iff aCb \vee aCc, \quad \text{TODO what is the other one?} \quad aC(b \sqcup c) \iff aCb \vee aCc.$$

Lemma 15. *Let a, b, c are terms and f, g are formulas. The following formulas are true:*

- $f \wedge T \implies f, T \wedge f \implies f$

- $f \wedge F \implies F, F \wedge f \implies F$

- $f \vee T \implies T, T \vee f \implies T$

- $f \vee F \implies f, F \vee f \implies f$

- $a \sqcap 0 \implies 0, 0 \sqcap a \implies 0$

- $a \sqcup 0 \implies a, 0 \sqcup a \implies a$

- $a \sqcap 1 \implies a, 1 \sqcap a \implies a$

- $a \sqcup 1 \implies 1, 1 \sqcup a \implies 1$

- $(a \sqcup b)Cc \iff aCc \lor bCc$

- $(a \sqcup b) \leq c \iff a \leq c \land b \leq c$

- $aCb \implies a \neq 0 \land b \neq 0$

- $a \sqcap b \neq 0 \implies aCb$

- $a = 0 \lor b = 0 \implies \neg(aCb)$

- $0 \leq a \implies T$

- $a \leq 1 \implies T$

- $0C0 \implies F$

- $aC0 \implies F$

- $1C1 \implies T$

- $aC1 \implies a \neq 0$

- $a \neq 0 \implies aCa$

2.3 Formula satsfiability

Let ψ be a propositional formula. Let us build a tableau beginning with ψ. If the tableau has an opened branch then ψ is satisfiable. Unfortunately, for the contact logic this is not enough because we need to verify the modal connectives ($\leq$ and C).

Let ϕ be a formula. Let us build a tableau beginning with ϕ. Let the tableau has an opened branch B. The branch B is a set of signed atomic formulas of the following type:

- $TC(a, b)$

- $FC(e, f)$

- $Ta \leq b$

- $Fa \leq b$

where a and b are terms.
B is an opened branch, so there are no contradicting formulas in it, i.e. $(\neg \exists X)(TX \in B \land FX \in B)$. The signed atomic formulas could be written as atomic formulas as follows:

- $TC(a, b) \rightarrow C(a, b)$

- $FC(a, b) \rightarrow \neg C(a, b)$

- $Ta \le b \rightarrow a \le b \rightarrow a \sqcap b^* = 0 \rightarrow g = 0$ (zero term)

- $Fa \le b \rightarrow \neg(a \le b) \rightarrow a \sqcap b^* \ne 0 \rightarrow d \ne 0$ (non-zero term)

where a,b,d and g are terms.

All atomic formulas in the branch should be satisfied, so we can think of it as a conjunction of them. Let's call it a **branch conjunction**. It is sufficient to build a satisfiable model for the branch conjuction to verify that ϕ is satisfiable. Building such a model could be done a lot more effective than building a model for an arbitrary formula because it is just a conjuction.

Deffinition 16. *Let ϕ be a formula. Let T be a tableau beginning with ϕ. Let B be a set of all atomic signed formulas in a branch of T, as follows:*

$$B = \{TC(a_i, b_i) \mid i \in \{1, \ldots, I\}\} \cup \{Fd_j = 0 \mid j \in \{1, \ldots, J\}\} \cup \tag{1}$$
$$\{FC(e_k, f_k) \mid k \in \{1, \ldots, K\}\} \cup \{Tg_l = 0 \mid l \in \{1, \ldots, L\}\}$$

A **branch cojnuction** β is the following formula:

$$\bigwedge_{i=1}^{I} C(a_i, b_i) \ \wedge \ \bigwedge_{j=1}^{J} d_j \ne 0 \ \wedge \ \bigwedge_{k=1}^{K} \neg C(e_k, f_k) \ \wedge \ \bigwedge_{l=1}^{L} g_l = 0 \tag{2}$$

2.4   Branch conjunction model building

The branch conjunction formula β is satisfiable if β has a model $\mathcal{M}$. We have to construct such a model $\mathcal{M} = (\mathcal{F}, \upsilon) = ((W, R), \upsilon)$.

Deffinition 17. *A denumeratable set of n boolean variables $\mathcal{V}_n$ is defined as follows:*

$$\mathcal{V}_n = \{x_1, x_2, \ldots, x_n\} \tag{3}$$

Let n be the number of the unique boolean variables in β. Let $\mathcal{V}_n$ be the set of them. Out of the four types of atomic formulas in β only two require existance of modal points. These are the contacts and non-zero terms. The valuation υ should assign a set of modal points to each boolean variable $x \in \mathcal{V}_n$. One way to define υ and W is the following:

- Assign an empty set of modal points for each boolean variable $x \in \mathcal{V}_n$.

  $\upsilon = \{ \ <\!x, \emptyset\!> \mid x \in \mathcal{V}_n \ \}$

- Add a modal point for each term in the contacts and non-zero terms of β.

  $W = \{ \ p_i \mid C(a_i, b_i) \in \beta \ \} \cup \{ \ p_{I+i} \mid C(a_i, b_i) \in \beta \ \} \cup \{ \ p_{I*2+j} \mid d_j \ne 0 \in \beta \ \}$

- Extend υ such as:

- $p_i \in \upsilon(a_i)$ for each $C(a_i, b_i) \in \beta$
- $p_{I+i} \in \upsilon(b_i)$ for each $C(a_i, b_i) \in \beta$
- $p_{I*2+j} \in \upsilon(d_j)$ for each $d_j \neq 0 \in \beta$

Definition 18. *Let $\mathcal{V}_n$ be a denumeratable set of n boolean variables. Let W be a set of modal points. Let υ be a valuation over $\mathcal{V}_n$ and W. Then $X_i \subseteq W$ is the evaluation set of the boolean variable x_i in $\mathcal{V}_n$.*

$$X_i = \upsilon(x_i) \; for \, x_i \in \mathcal{V}_n \tag{4}$$

Definition 19. *Let $\mathcal{V}_n$ be a denumeratable set of n boolean variables. Let W be a set of modal points. Let υ be a valuation over $\mathcal{V}_n$ and W. Then X is the union of all evaluations of boolean variables in $\mathcal{V}_n$.*

$$X = \{\upsilon(x_i) \mid x_i \in \mathcal{V}_n\} = \{X_i \mid x_i \in \mathcal{V}_n\} \tag{5}$$

Let t be an arbitrary term. Let $p \notin W$ be a new modal point. Let us extend υ such as $p \in \upsilon(t)$. By the valuation definition $\upsilon(t)$ is a composition of interesections, unions and compliments of some $X_i \subseteq W$. Therefore, the modal point p should be added to zero or more X_i depending on the boolean operations in the term t. With n boolean variables there are 2^n ways of adding the point p.

Definition 20. *A **variable evaluation** $\mathcal{E}_n$ for n boolean variables is a sequence of 1s and 0s, as follows:*

$$\mathcal{E}_n = \, < e_1, e_2, \ldots, e_n >, \; where \; e_1, \ldots, e_n \in \{0, 1\} \tag{6}$$

For n boolean variables there are 2^n unique variable evaluations. We will define the modal points as variable evaluations. By the definition of the valuation, it is not possible to distinguish two or more different modal points in some subsets. For example, the $W \setminus X$ subset. The modal point representation also have this limitaion. The modal connectives require existence of a point in those sets. It is sufficient to work with only one point from them.

Definition 21. *Let $\mathcal{E}_n$ be a variable evaluation for n boolean variables. Then $(\mathcal{E}_n)^i$ is the i-th element in the sequence $\mathcal{E}_n$.*

Definition 22. *Let $\mathcal{V}_n$ be a denumeratable set of n boolean variables. The set of **all unique variable evaluations** W_n over n variables is defined as follows:*

$$W_n = \{< e_1, e_2, \ldots, e_n >\mid e_1, \ldots, e_n \in \{0, 1\}\} \tag{7}$$

Definition 23. *Let $\mathcal{V}_n$ be a denumeratable set of n boolean variables. Let W_n be the set of all unique points over n variables. Let $W \subseteq W_n$. Then the **valuation** $\upsilon_n : \mathcal{V}_n \rightarrow \mathcal{P}(W)$ is inductively defined as follows:*

- *Boolean constants*

  - $\upsilon_n(0) = \emptyset$

- $v_n(1) = W$

- *Boolean variables*

  - $v_n(x_i) = \{\mathcal{E}_n \mid \mathcal{E}_n \in W \wedge (\mathcal{E}_n)^i = 1\}, \text{ for } i \in \{1, \ldots, n\}$

- *Boolean terms*

  - $v_n(a \sqcap b) = v_n(a) \cap v_n(b)$
  - $v_n(a \sqcup b) = v_n(a) \cup v_n(b)$
  - $v_n(a^*) = W \setminus v_n(a) = -v_n(a)$

TODO: move to each definition which uses this things somehow.. Let β be a branch conjunction as in 16 . Let W be a set of modal points. Let $\mathcal{V}_n$ be the set of boolean variables in β. Let v_n be a valuation as in 23 . Let R $\subseteq W^2$ be a relation.

Definition 24. *The zero terms in β are satisfied iff each zero term's evaluation is the empty set.*

$$(\forall g_l \in \beta)(v_n(g_l) = \emptyset) \tag{8}$$

Definition 25. *The non contacts in β are satisfied iff:*

$$(\forall_{\neg C(e_k, f_k) \in \beta}) \neg (\exists x \in v_n(e_k))(\exists y \in v_n(f_k))(xRy) \tag{9}$$

Lemma 26. *The valuations of the non contact terms in β does not have a common modal point. Follows from the reflexivity of R.*

$$(\forall_{\neg C(e_k, f_k) \in \beta})(v_n(e_k) \cap v_n(f_k) = \emptyset) \tag{10}$$

Definition 27. *Let t be an arbitrary boolean term. Then $\mathcal{V}_t$ is the set of the boolean variables used in t.*

Definition 28. *Let $\mathcal{V}_n$ be a denumeratable set of boolean variables. Then $\mathcal{T}_n$ is the set of all boolean terms with variables of $\mathcal{V}_n$.*

$$\mathcal{T}_n = \{t \mid t \text{ is a boolean term and } \mathcal{V}_t \subseteq \mathcal{V}_n\} \tag{11}$$

Definition 29. *Let $\mathcal{V}_n$ be a denumeratable set of boolean variables. Let $\mathcal{T}_n$ be the set of all boolean terms with variables of $\mathcal{V}_n$. Let W_n be the set of all unique variable evaluations for n varialbes. By a **boolean valuation** of a boolean term in $\mathcal{T}_n$ we mean the function η. It assigns a boolean constant for each pair of term $t \in \mathcal{T}_n$ and variable evaluation $\mathcal{E}_n \in W_n$.*

$$\eta(t, \mathcal{E}_n) \in \{0, 1\} \tag{12}$$

Let $t \in \mathcal{T}_n$. Let $\mathcal{E}_n \in W_n$. The inductively definition of η on the structure of the term t is as follows:

- $\eta(0, E_n) = 0$

- $\eta(1, E_n) = 1$

- $\eta(x_i, E_n) = (E_n)^i$

- $\eta(a \sqcap b, E_n) = \eta(a, E_n) \wedge \upsilon(b, E_n)$

- $\eta(a \sqcup b, E_n) = \upsilon(a, E_n) \vee \upsilon(b, E_n)$

- $\eta(a^*, E_n) = \neg\eta(a, E_n)$

TODO: define the set of all variables evaluations which evaluates term t to 1

2.5 Model building

Let β be a branch conjunction as in 16 :

$$\bigwedge_{i=1}^{I} C(a_i, b_i) \ \wedge \ \bigwedge_{j=1}^{J} d_j \neq 0 \ \wedge \ \bigwedge_{k=1}^{K} \neg C(e_k, f_k) \ \wedge \ \bigwedge_{l=1}^{L} g_l = 0$$

Let $\mathcal{V}_n$ be the set of variables in β. Let $\mathcal{F}$ = (W, R) be a relational system with $W \subseteq W_n$ and $R \subseteq W^2$. Let $\mathcal{M} = (\mathcal{F}, \upsilon)$ be a model, where $\upsilon : \mathcal{V}_n \to \mathcal{P}(W)$ is a valuation defined as in 23 over W. There are four types of atomic formulas: contacts C, $\neg C$, zero terms and non-zero terms. Only the contacts and non-zero terms requires point existence. We will create points only for them. The pseudocode bellow implements the main idea for building $\mathcal{M}$ as a model of β. Note that there are some more details and optimizations which will be shown in the implementation section.

```
1  Function CheckNegContact(p):
2      for k ← 1 to K by 1 do
           /* Checking ¬C(e_k, f_k)                                     */
3          if (p ∈ υ_n(e_k) ∧ p ∈ υ_n(f_k)) then
4              return false
5      end
6      return true
7  Function CheckNegContact(p_1, p_2):
8      for k ← 1 to K by 1 do
           /* Checking ¬C(e_k, f_k)                                     */
9          if (p_1 ∈ υ_n(e_k) ∧ p_1 ∈ υ_n(f_k)) ∨ (p_2 ∈ υ_n(e_k) ∧ p_2 ∈ υ_n(f_k)) ∨ (p_1 ∈
           υ_n(e_k) ∧ p_2 ∈ υ_n(f_k)) ∨ (p_2 ∈ υ_n(e_k) ∧ p_1 ∈ υ_n(f_k)))) then
10             return false
11     end
12     return true
13 Function CheckZeroTerms(p):
14     for l ← 1 to L by 1 do
           /* Checking g_l = 0                                          */
15         if (x ∈ υ_n(g_l) then
16             return false
17     end
18     return true
19
20 R ← ∅
21 W ← ∅
   /* Create points for the contacts                                   */
22 for i ← 1 to I by 1 do
       /* Processing C(a_i, b_i)                                        */
23     for x ∈ υ_n(a_i) do
24         for y ∈ υ_n(b_i) do
               /* Verify that the pair <x, y> does not breaks any ¬C and zero
                  term                                                  */
25             if CheckNegContact(x, y) ∧ CheckZeroTerms(x) ∧
               CheckZeroTerms(y) then
26                 W ← W ∪ {x, y}
27                 R ← R ∪ {< x, x >, < y, y >, < x, y >, < y, x >}
28                 go to 22
29         end
30     end
31     Model building fails.
32 end
   /* Create points for the non-zero terms                             */
33 for j ← 1 to J by 1 do
       /* Processing d_j ≠ 0                                            */
34     for x ∈ υ_n(d_j) do
           /* Verify that x does not breaks any ¬C and zero term        */
35         if CheckNegContact(x) ∧ CheckZeroTerms(x) then
36             W ← W ∪ {x}
37             R ← R ∪ {< x, x >}         15
38             go to 33
39     end
40     Model building fails.
41 end
```

3 Connected Contact Logics

3.1 Connectivity

In topology and related branches of mathematics, a connected space is a topological space that cannot be represented as the union of two or more disjoint non-empty open subsets. Connectedness is one of the principal topological properties that are used to distinguish topological spaces.

Theorem 30. *(Connectivity) Let b is a term. Then*

$$b \neq 0 \land b \neq 1 \implies bCb^* \tag{12}$$

Let $\mathcal{F}$ = (W, R) be a relational system with W $\neq \emptyset$, R $\subseteq W^2$ and a, b are terms. Let us recall the deficition of C:

$$aCb \iff (\exists x \in \upsilon(a))(\exists y \in \upsilon(b))(xRy) \tag{12}$$

The connectivity theorem can be written as follows:

$$\upsilon(b) \neq \emptyset \land \upsilon(b) \neq W \implies (\exists x \in \upsilon(b))(\exists y \in W \setminus \upsilon(b))(xRy) \tag{12}$$

Deffinition 31. *Let G = (W, R) be a graph. W is the set of vertexes and R the set of edges. A **path** $\pi(v_1, v_n)$ is a sequence of vertexes $(v_1, v_2, \ldots, v_n)$ such that $v_1, \ldots, v_n \in V$ and $v_i R v_{i+1}$, for $i \in \{1, \ldots, n-1\}$.*

Let us think of W and R as an undirected graph $G = (W, R)$. W is the set of vertexes and R the set of edges. The connectivity theorem implies that the graph G should be connected, i.e. for each vertex $x \in W$ there should be a path to each vertex in W.

$$(\forall x \in W)(\forall y \in W)(\exists \pi(x, y))(\pi(x, y) \text{ is a path in } G) \tag{12}$$

Deffinition 32. *A **connected model** is a model which satisfies the connectivity theorem(30).*

Let $\mathcal{F}$ = (W, R) be a relational system with W $\neq \emptyset$, R $\subseteq W^2$. Let $\mathcal{M} = (\mathcal{F}, \upsilon)$ be a model, where υ is an valuation. In order $\mathcal{M}$ to be a *connected model* the graph $G = (W, R)$ should be connected.

3.2 Connected model building

Let ϕ be a branch conjunction as in 16 :

$$\bigwedge_{i=1}^{I} C(a_i, b_i) \land \bigwedge_{j=1}^{J} d_j \neq 0 \land \bigwedge_{k=1}^{K} \neg C(e_k, f_k) \land \bigwedge_{l=1}^{L} g_l = 0$$

Let $\mathcal{V}_n$ be the set of variables in ϕ. Let $\mathcal{F} = (W, R)$ be a relational system with $W \subseteq W_n$ and $R \subseteq W^2$. Let $\mathcal{M} = (\mathcal{F}, \upsilon)$ be a model, where $\upsilon : \mathcal{V}_n \to \mathcal{P}(W)$ is a valuation defined as in 23 over W. In order $\mathcal{M}$ to be a connected model for ϕ, W should be non-empty and the graph $G = (W, R)$ should be connected (from 3.1). The main idea for building a connected model is as follows:

- Create the set W from all points in W_n which does not interfere with the $\neg C$ and zero terms in ϕ:

$$W = \{p \mid p \in W_n \wedge (\forall_l)(p \notin \upsilon_n(g_l)) \wedge (\forall_k)(p \notin \upsilon_n(e_k) \vee p \notin \upsilon_n(f_k))\} \quad (12)$$

- Validate that all contact terms are non-zero:

$$(\forall_i)(\exists_{p_1 \in W})(\exists_{p_2 \in W})(p_1 \in \upsilon(a_i) \wedge p_2 \in \upsilon(b_i)) \quad (12)$$

- Create relations between all points but skip those which interfere with $\neg C$ actomic formulas:

$$R = \{< x, y > \mid x, y \in W \wedge \neg(\exists_j)((x \in \upsilon(e_j) \wedge y \in \upsilon(f_j)) \vee$$
$$(x \in \upsilon(f_j) \wedge y \in \upsilon(e_j)) \vee$$
$$(x \in \upsilon(e_j) \wedge y \in \upsilon(e_j)) \vee$$
$$(x \in \upsilon(f_j) \wedge y \in \upsilon(f_j)))\}$$

- Validate that all atomic contact formulas in ϕ are satisfied:

$$(\forall_i)(\exists_{x \in \upsilon(a_i)})(\exists_{y \in \upsilon(b_i)})(xRy \vee yRx) \quad (12)$$

Note that now $\mathcal{M}$ is a model of ϕ.

- Get all connected components of G in the set $Comp$:

$$Comp = \{G' = (W', R') \mid W' \subseteq W \wedge R' \subseteq R \wedge G' \text{ is connected}\} \quad (12)$$

- If Comp has a graph G' which defines a model of ϕ $\mathcal{M}' = (W', R', \upsilon')$, then $\mathcal{M}'$ is a connected model. $\upsilon' : \mathcal{V}_N \to \mathcal{P}(W')$ is defained as υ but over W'. Note that we only have to check the non-zero terms and the contact atomic formulas because $\mathcal{M}'$ does not introduce new points or relations:

$$(\exists_{(W', R') \in Comp})((\forall_j)(\exists_{p \in W'})(p \in \upsilon(d_j))) \wedge (\forall_i)((\exists_{x \in \upsilon'(a_i)})(\exists_{y \in \upsilon'(b_i)})(xR'y \wedge yR'x))$$

4 Implementation Introduction

5 Tableaux Implementation

6   Model Implementation

7 Connected Contact Logics Implementation

TODO: separate line for the authors

References

[1] Handbook of Tableau Methods M. DâĂŹAgostino, D. M. Gabbay, R. HÂÍahnle
and J. Posegga, eds.

[2] Modal Logics for Region-based Theories of Space Philippe Balbiani, Thinko
Tinchev, Dimiter Vakarelov

[3] Connected Space or go to the next url: `https://en.wikipedia.org/wiki/Connected_space`