# Linear Algebra

## Lecture Notes

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# Lecture 1. Systems of linear equations

# Outline

1. Systems of linear equations
   - Examples. Solutions. Substitution and elimination
   - Solutions in dimension 2
   - Solutions in dimension 3
2. Gaussian and Gauss–Jordan elimination
   - Reduction to an echelon form
   - Pivot and free variables
   - A reduced echelon form
3. Existence and uniqueness of solutions
   - Homogeneous and non-homogeneous systems
   - Existence and uniqueness of solutions
   - Nonsingular matrices. Rank of a matrix

# Systems of linear equations

Many applied models postulate linear dependence on some variables (e.g., economic factors) $x, y, z, \ldots$:

$$ax + by + cz + \cdots = \alpha$$

Here $a, b, c, \ldots, \alpha$ are given numbers, or scalars[a]

[a] Can be real ($\mathbb{R}$), complex ($\mathbb{C}$), quaternionic ($\mathbb{H}$), binary ($\mathbb{Z}_2$), in $\mathbb{Z}_p$ ...

- For many ($n$) parameters, use notations $x_1, x_2, \ldots, x_n$
- system of linear equations, or linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

- usually $x_1, \ldots, x_n$ are unknown; to be determined from (1)

**Definition (Solution to a system)**

A sequence $(x_1, x_2, \ldots, x_n)$ of numbers turning each equation in (1) into equality is a solution of the linear system (1).

# Lavrentief model (Nobel prize in economics in 1973):

- in 1949, divided the American economy in $N = 500$ sectors
- analyzed distribution of the outcomes
- got the $N \times N$ linear system ($N \leq 32$ feasible in 1949)

## Production matrix for a simple economy model (N=3):

| | Income Required per Dollar Output in | | |
|---|---|---|---|
| | Manufacturing | Agriculture | Utilities |
| Manufacturing | \$ 0.50 | \$ 0.10 | \$ 0.10 |
| Agriculture | \$ 0.20 | \$ 0.50 | \$ 0.30 |
| Utilities | \$ 0.10 | \$ 0.30 | \$ 0.40 |

$x_1, x_2, x_3$: produced output; $d_1, d_2, d_3$: demand

$$x_1 - (0.5x_1 + 0.1x_2 + 0.1x_3) = d_1$$
$$x_2 - (0.2x_1 + 0.5x_2 + 0.3x_3) = d_2$$
$$x_3 - (0.1x_1 + 0.3x_2 + 0.4x_3) = d_3$$

# Balancing chemical reactions

Combustion of propane produces carbon dioxide and water:

| $C_3H_8$ | + | $O_2$ | $\rightarrow$ | $CO_2$ | + | $H_2O$ |
|---|---|---|---|---|---|---|
| propane | | oxygen | | carbon dioxide | | water |

To balance the equation, take $x$ molecules of propane, $y$ of oxygen, $z$ of carbon dioxide and $w$ of water; now equate the number of atoms on both sides to get

$$\begin{aligned} 3x &= z && \text{(Carbon)} \\ 8x &= 2w && \text{(Hydrogen)} \\ 2y &= 2z + w && \text{(Oxygen)} \end{aligned}$$

Many solutions, e.g. $x = 1$, $y = 5$, $z = 3$, and $w = 4$:

$$C_3H_8 + 5O_2 = 3CO_2 + 4H_2O$$

# Polynomial approximation

**Theorem**

*For any n given pairs $(x_1, y_1), \dots, (x_n, y_n)$ with pairwise distinct $x_k$ there exists a unique polynomial p(x) of degree at most $n-1$ such that $p(x_k) = y_k$ for every $k = 1, \dots, n$.*

Search for $p$ of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1};$$

then we get

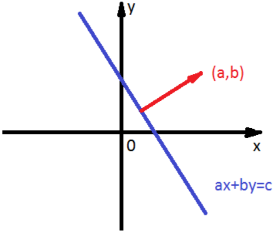
$$\begin{array}{ll} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} & = y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} & = y_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \dots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} & = y_n \end{array}$$

This is a linear system for the coefficients $a_0, a_1, \dots, a_{n-1}$!

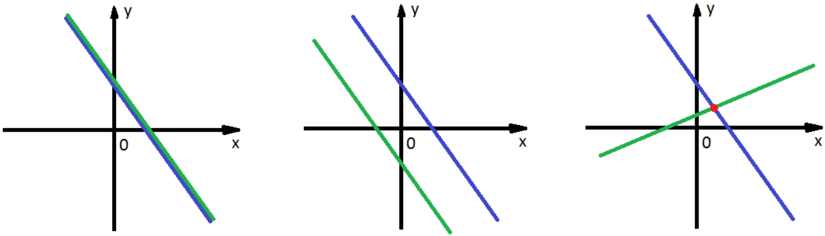
# Linear systems in dimension 2:

## Lines in $\mathbb{R}^2$

- In the $xy$-plane, equation $y = kx + m$ gives a line through $(0, m)$ with the slope $k$
- If $|a| + |b| > 0$, then
$$ax + by = c$$
describes the line with slope $k = -a/b$ through $(0, c/b)$ if $b \neq 0$, or the line $x \equiv c/a$ otherwise
- this line is orthogonal to the vector $(a, b)$



# Possible situations:




1. Lines coincide $\Longrightarrow$ infinitely many solutions
2. Lines parallel $\Longrightarrow$ no solutions
3. Lines intersect $\Longrightarrow$ single solution

# Solving linear systems in dimension 2:

## By substitution:

$$\begin{array}{rrl} 2x & +\ y & = 3 \\ 3x & -2y & = 8 \end{array} \implies \begin{array}{ll} x = \frac{1}{2}(3-y) & \times 3 \\ \frac{3}{2}(3-y) - 2y = 8 & \end{array}$$

From the second equation $y = -1$ and then the first gives $x = 2$

## By elimination:

$$\begin{array}{rrrl} 2x & +\ y & = & 3 \\ 3x & -2y & = & 8 \end{array} \quad -\frac{3}{2} \times (1) \implies \begin{array}{rrrl} 2x & +\ y & = & 3 \\ & -\frac{7}{2}y & = & \frac{7}{2} \end{array} \implies$$

$$\begin{array}{rrrl} 2x & +\ y & = & 3 \\ & y & = & -1 \end{array} \quad -1 \times (2) \implies \begin{array}{rrrl} 2x & & = & 4 \\ & y & = & -1 \end{array}$$

The two methods are only formally different

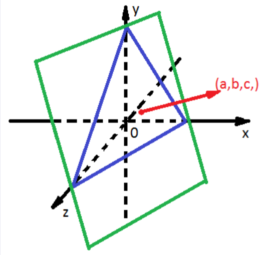
# Linear systems in dimension 3:

## Planes in $\mathbb{R}^3$

If $|a| + |b| + |c| > 0$, then the equation

$$ax + by + cz = d$$

describes in the $xyz$-space the plane orthogonal to the direction vector $(a, b, c)$

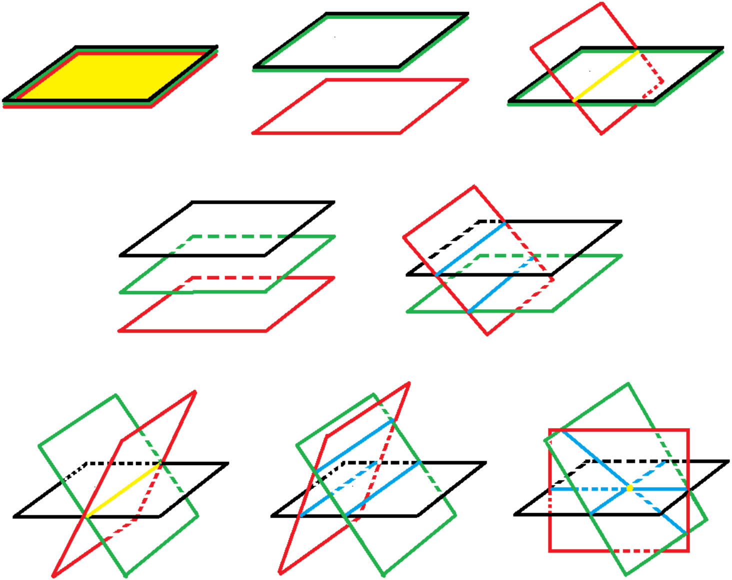



## Systems in $\mathbb{R}^3$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

describes the common (intersection) points of three planes in $x_1x_2x_3$-space

# Possible situations



# Solving linear systems in dimension 3:

## By elimination:

$$\begin{array}{rrrrl} 2x & +\ y & & = & 3 \\ 4x & -2y & +3z & = & 4 \qquad -2\times(r_1) \\ & 4y & -2z & = & 0 \end{array}$$

$$\begin{array}{rrrrl} 2x & +\ y & & = & 3 \\ & -4y & +3z & = & -2 \\ & 4y & -2z & = & 0 \qquad +1\times(r_2) \end{array}$$

$$\begin{array}{rrrrl} 2x & +\ y & & = & 3 \\ & -4y & +3z & = & -2 \qquad -3\times(r_3); \quad /(-4) \\ & & z & = & -2 \end{array}$$

$$\begin{array}{rrrrl} 2x & +\ y & & = & 3 \qquad -1\times(r_2); \quad /2 \\ & y & & = & -1 \\ & & z & = & -2 \end{array}$$

Solution: $(2,-1,-2)$

# How to solve linear systems for large $n, m$?

In real-world applications $m, n > 1000$

## Main questions:

1. Is there any solution to such a linear system?
2. How many?
3. How to find all/some special?

## Idea:

Transform a system to simpler form without changing the set of solutions

## Definition

- A system is called consistent if it possesses at least one solution
- Two linear systems are called equivalent if they possess the same set of solutions

# Shorthand matrix notations for linear systems:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \quad \rightsquigarrow \quad A\mathbf{x} = \mathbf{b} \qquad (2)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

is an $m \times n$ coefficient matrix

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = (x_1, x_2, \dots, x_n)^\top, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = (b_1, b_2, \dots, b_m)^\top$$

are the vector of unknowns and the (given) RHS vector resp.

# Augmented coefficient matrix

Often it is more convenient to combine the matrix notation for the coefficient matrix $A$ and the RHS $\mathbf{b}$ and to encode system (2) via the augmented coefficient matrix

$$(A \mid \mathbf{b}) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

# Elementary row operations that simplify linear systems

1. Multiply an equation/a row through by a nonzero constant
2. Add a constant times one equation/row to another
3. Interchange two equations/rows

**Definition**

Matrices $A$ and $B$ are row equivalent if $B$ can be obtained from $A$ via elementary row operations

**Properties**

1. Elementary row operations are reversible
2. Lead to equivalent systems/augmented matrices

# Example: elementary row operations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + y - z &= 1 \quad -2 \times (r_1) \\ x + y - z &= 0 \quad -1 \times (r_1) \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{aligned} x + 2y - z &= 2 \\ -3y + z &= -3 \quad (r_2) \leftrightarrow (r_3) \\ -y &= -2 \quad (r_2) \leftrightarrow (r_3) \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -3 & 1 & -3 \\ 0 & -1 & 0 & -2 \end{array}\right)$$

$$\begin{aligned} x + 2y - z &= 2 \\ -y &= -2 \\ -3y + z &= -3 \quad +3 \times (r_2) \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -3 & 1 & -3 \end{array}\right)$$

$$\begin{aligned} x + 2y - z &= 2 \quad -2 \times (r_2) + (r_3) \\ y &= 2 \\ z &= 3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

# An echelon form

## Definition

A rectangular matrix is in a row echelon form if it has the following three properties:

1. All nonzero rows are above any zero row
2. Each leading entry of a row (i.e., first nonzero entry in that row) is in the column to the right of of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

## Example ($\bullet$ stands for a nonzero number, $*$ for any number)

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

The first matrix is in echelon form, the second is not unless. . .

# Pivot and free variables

## Question

*Is an echelon form of a matrix unique?*

- No, because it depends on what elementary row operations and in what order were applied
- However, the positions of the leading entries are fixed!

## Definition

Let $A$ be a matrix of a linear system. A pivot column of $A$ is a column with a leading entry in an echelon form of $A$. The corresponding variable is called a pivot variable. Otherwise a variable is called free.

## Example (Columns corresponding to free variables are in red )

$$\begin{pmatrix} \bullet & * & * & * & * \\ 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}$$

# Why free variables are free?

$$\left(\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right) \iff \begin{array}{rcl} x_1 + 2x_2 - 2x_3 + x_4 &=& 0 \\ x_3 - x_4 &=& 2 \end{array}$$

Therefore

1. $x_3 = 2 + x_4$
2. $x_1 = -2x_2 + 2x_3 - x_4 = 4 - 2x_2 + x_4$
3. by introducing parameters $s$ and $t$ via $x_2 := s$ and $x_4 := t$, we get the following parametric representation of general solutions of the system for every values of $s$ and $t$:

$$\begin{aligned} x_1 &= 4 - 2s + t \\ x_2 &= s \\ x_3 &= 2 + t \\ x_4 &= t \end{aligned}$$

# A reduced echelon form

## Definition

A rectangular matrix in an echelon form is in a reduced row echelon form if in addition:

1. The leading entry in each nonzero row is 1
2. Each leading 1 is the only nonzero element in its column

## Example ($*$ stands for any number)

$$\begin{pmatrix} 1 & 0 & * & * & 0 \\ 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & * & 0 & * & 0 \\ 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1. Matrix in echelon form can be transformed to a reduced echelon form using elementary row operations
2. A reduced echelon form of a matrix is unique
3. If the augmented matrix is in the reduced echelon form, solutions are easy to write in terms of free variables

# Gaussian and Gauss–Jordan elimination

Forward substitution $\rightsquigarrow$ an echelon form

**Example (Gaussian elimination $\iff$ substitution)**

$$\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= 1 \end{aligned} \iff \begin{aligned} x + 2y &= 1 \\ 3(1 - 2y) + 4y &= 1 \end{aligned} \iff \begin{aligned} x + 2y &= 1 \\ -2y &= -2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -2 & -2 \end{array}\right)$$

Backward substitution $\rightsquigarrow$ reduced echelon form

**Example (Gauss–Jordan elimination $\iff$ back substitution)**

$$\begin{aligned} x + 2y &= 1 \\ -2y &= -2 \end{aligned} \iff \begin{aligned} x + 2 \cdot 1 &= 1 \\ y &= 1 \end{aligned} \iff \begin{aligned} x \qquad &= -1 \\ y &= 1 \end{aligned}$$

# Solution set of homogeneous systems

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ always has a trivial solution $\mathbf{x} = \mathbf{0}$ (is always consistent)
- If $\mathbf{x} = (x_1, \ldots, x_n)^\top$ is a solutions, then for every scalar $k$, $k\mathbf{x} = (kx_1, \ldots, kx_n)^\top$ is a solution as well:
$$a_{i1}(kx_1) + \cdots + a_{in}(kx_n) = k(a_{i1}x_1 + \cdots + a_{in}x_n) = 0$$
- If $\mathbf{x} = (x_1, \ldots, x_n)^\top$ and $\mathbf{y} = (y_1, \ldots, y_n)^\top$ are solutions, then so is $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)^\top$:
$$\begin{aligned} &a_{i1}(x_1 + y_1) + \cdots + a_{in}(x_n + y_n) \\ &\qquad = (a_{i1}x_1 + \cdots + a_{in}x_n) + (a_{i1}y_1 + \cdots + a_{in}y_n) = 0 \end{aligned}$$
- Thus the solution set of a homogeneous linear system is a linear space

**Theorem (Solutions to a homogeneous system)**

*A homogeneous system $A\mathbf{x} = \mathbf{0}$ may have one solution (the trivial solution $\mathbf{x} = \mathbf{0}$) or infinitely many solutions; moreover, its solution set is linear.*

# Solution set of non-homogeneous systems

- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ need not be consistent (may not have any solutions)
- If $\mathbf{x} = (x_1, \ldots, x_n)^\top$ and $\mathbf{y} = (y_1, \ldots, y_n)^\top$ are solutions, then $\mathbf{x} - \mathbf{y} = (x_1 - y_1, \ldots, x_n - y_n)^\top$ solves the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\begin{aligned} a_{i1}(x_1 - y_1) + \cdots + a_{in}(x_n - y_n) \\ = (a_{i1}x_1 + \cdots + a_{in}x_n) - (a_{i1}y_1 + \cdots + a_{in}y_n) \\ = b_i - b_i = 0 \end{aligned}$$

- The solution set of a non-homogeneous linear system is an affine set (shifted linear set)

**Theorem (Solutions to a non-homogeneous system)**

*A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ may have no solutions, one solution, or infinitely many solutions. Any solution has the form $\mathbf{x} = \mathbf{x}_{\text{par}} + \mathbf{x}_{\text{hom}}$, where $\mathbf{x}_{\text{par}}$ is a particular solution, and $\mathbf{x}_{\text{hom}}$ is any solution to a homogeneous system $A\mathbf{x} = \mathbf{0}$*

# Existence of solutions

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ is always consistent
- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent

  $\iff$ an echelon form of its augmented matrix $(A \,|\, \mathbf{b})$ has no row $(0 \; \ldots \; 0 \,|\, \bullet)$ with a non-zero $\bullet$
- A system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$

  $\iff$ an echelon form of $A$ has no zero rows

  $\iff$ $A$ has a pivot position in every its row
- For every $A$, the system $A\mathbf{x} = \mathbf{b}$ is consistent for some choice of $\mathbf{b}$.

# Uniqueness of solutions

- A homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution
  $\iff$ there is no free variables
- A non-homogeneous system $A\mathbf{x} = \mathbf{b}$ has a unique solution
  $\iff$ no rows $(0 \ldots 0 \,|\, \bullet)$ in echelon form of $(A \,|\, \mathbf{b})$ and no free variables
- A system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$
  $\iff$ no zero rows in an echelon form of $A$ and no free variables
  $\implies$ $A$ must be a square matrix $(m = n)$

# Nonsingular matrices

## Definition

A coefficient matrix $A$ is called nonsingular if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for *every* choice of **b**.

## Theorem

*An $m \times n$ matrix $A$ is nonsingular if and only if $m = n$ and $A$ has $m$ pivot columns*

## Proof:

- should be no free variables (otherwise nonuniqueness)
- an echelon form of $A$ should contain no zero rows (otherwise nonexistence) $\square$

## Example

$$\begin{pmatrix} 1 & 2 & a \\ 3 & b & 0 \\ c & 0 & 0 \end{pmatrix} \overset{c\neq 0}{\sim} \begin{pmatrix} 1 & 2 & a \\ 0 & b & 0 \\ 1 & 0 & 0 \end{pmatrix} \overset{b\neq 0}{\sim} \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & a \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}$$

Thus $A$ is nonsingular $\iff abc \neq 0$

# Rank of a matrix

## Definition

The number of nonzero rows in an echelon form of a matrix $A$ is called its rank and denoted $\operatorname{rank} A$.

## Theorem (Some properties of the rank)

*Assume that $A$ is the $m \times n$ coefficient matrix of a linear system and $\hat{A}$ is the corresponding augmented matrix. Then*

1. $\operatorname{rank} A \le \min\{m, n\}$ *and* $\operatorname{rank} A \le \operatorname{rank} \hat{A}$
2. *a solution of* $A\mathbf{x} = \mathbf{b}$ *exists* $\iff \operatorname{rank} A = \operatorname{rank} \hat{A}$
3. $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $\mathbf{b} \iff \operatorname{rank} A = m$
4. $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b} \iff \operatorname{rank} A = n$
5. $A$ *is nonsingular* $\iff \operatorname{rank} A = m = n$.