one-fifth of those occurring at Brockenhurst, Colwell Bay, and White Cliff are found at Barton." It would be certainly anomalous if the Venus-bed had more Barton forms than the Brockenhurst one, seeing that the former occupies a higher zone in the Middle Headon series.

An analysis is made of Prof. Judd's lists from which it appears that in the one list are nine species said to pass into Barton beds, while in the other list, these identical species have that range denied to them. Again, in his Brockenhurst and Colwell Bay list are 22 species of which the range into Barton beds is not recognized, while an examination of the Edward's collection in the British Museum shows that they pass up from Barton or Bracklesham beds.

From the authors' lists it appears, on the other hand, that the percentage of Barton forms in the Whitley Ridge bed is about 42 per cent., a lower proportion than at White Cliff, because of the number of corals special to the locality. At White Cliff Bay this bed has 52 per cent. The proportion of Barton forms from all the Brockenhurst localities, including the Roydon Zone, is 48 per cent. If for comparison the percentage of Barton forms in the Middle Headon of Headon Hill is calculated, it is found to be 29 per cent. Fossil evidence leads therefore to the conclusion that the Headon Hill marine bed is later in age than the Brockenhurst, the proportion of Barton forms in the latter being not one fifth, but nearly one half—a result in accordance with their stratigraphical position.

Similarly, to test by fossil evidence whether the Colwell Bay bed is nearer related to the Brockenhurst than is the Headon Hill one, the percentage of Barton forms is observed in each; in Colwell Bay, they were 29 per cent., in the Headon Hill bed also 29 per cent., while in the Brockenhurst bed they were 48 per cent.

In order to complete the proof from fossils, if any such were needed, it is noted that there are only two species in each case common to either Colwell Bay, or Headon Hill, and Brockenhurst and not occurring at the other locality, while there are 26 species common to Colwell Bay and Headon Hill, and not occurring at Brockenhurst.

These results are in perfect accordance with the stratigraphical succession, and show that Prof. Judd has misconceived the position of the Brockenhurst bed. The authors therefore reject his proposed term of "Brockenhurst series," and revert to the classification and nomenclature of the Geological Survey.

PROCEEDINGS

OF THE

Cambridge Philosophical Society.

February 7, 1881.

PROFESSOR NEWTON, PRESIDENT, IN THE CHAIR.

Horace Darwin, M.A., Trinity College, was balloted for and duly elected a Fellow of the Society.

The following communication was made to the Society:

Determination of the greatest height consistent with stability that a vertical pole or mast can be made, and of the greatest height to which a tree of given proportions can grow. By A. G. GREENHILL, M.A.

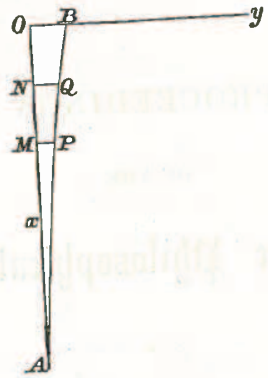
I. Suppose a uniform cylindrical pole or wire fixed in a vertical direction at its lowest point, and carried to such a height that the vertical position becomes unstable and flexure begins; it is required to determine this height.

Let $2a$ be the diameter in inches, and A the sectional area of the pole in square inches: and E be Young's modulus of elasticity of the substance, expressed in gravitation measure of ℔ to the square inch.

Then, if ρ be the radius of curvature of the central fibre of the pole, the bending moment of resilience (the unit being the inch-℔.)

$$= EI\frac{1}{\rho} = EAk^2\frac{1}{\rho}.$$

Take the origin O at the top of the pole in its vertical position and the axis Ox directed vertically downwards:



then if APB be the central line of the pole, supposed to be slightly curved under its own weight, we may put $\frac{1}{\rho}=\frac{dp}{dx}$, where $p=\frac{dy}{dx}$, and therefore the bending moment at P

$$=EAk^2\frac{dp}{dx}.$$

This must be equated to the moment of the weight of PB about P, which

$$=wA\int_0^x (y'-y)\,dx',$$

x', y' denoting the co-ordinates of any point Q between B and P, and w the density of the substance in pounds to the cubic inch.

Therefore the differential equation of the central line APB is

$$EAk^2\frac{dp}{dx}=wA\int_0^x (y'-y)\,dx',$$

or, differentiating with respect to x,

$$EAk^2\frac{d^2p}{dx^2}=-wA\int_0^x p\,dx'$$

$$=-wAxp,$$

or

$$x^2\frac{d^2p}{dx^2}+\frac{w}{Ek^2}x^3p=0 \qquad\qquad (1).$$

To solve this differential equation, first put $p=x^{\frac{1}{2}}z$, then

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}+\left(\frac{w}{Ek^2}x^3-\tfrac{1}{4}\right)z=0.$$

Again, put $x^3=r^2$, and then

$$x^2\frac{d^2z}{dx^2}+x\frac{dz}{dx}=\tfrac{9}{4}\left(r^2\frac{d^2z}{dr^2}+r\frac{dz}{dr}\right),$$

and therefore

$$r^2\frac{d^2z}{dr^2}+r\frac{dz}{dr}+\left(\frac{4w}{9Ek^2}r^2-\tfrac{1}{9}\right)z=0 \qquad\qquad (2).$$

This is of the form of Bessel's differential equation

$$r^2\frac{d^2z}{dr^2}+r\frac{dz}{dr}+(\kappa^2r^2-n^2)\,z=0 \qquad\qquad (3),$$

where $\kappa^2=\frac{4w}{9Ek^2}$, $n^2=\frac{1}{9}$.

The solution of (3) is

$$z=AJ_n(\kappa r)+BJ_{-n}(\kappa r),$$

where $J_n(x)=\dfrac{x^n}{\sqrt{\pi}\,2^n\Gamma(n+\frac{1}{2})}\displaystyle\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi\,d\phi.$

(Todhunter, *Functions of Laplace, Lamé, and Bessel*, p. 285.)

Consequently the solution of (2) is

$$z=AJ_{\frac{1}{3}}(\kappa r)+BJ_{-\frac{1}{3}}(\kappa r),$$

and the solution of (1) is

$$p=\sqrt{x}\,\{AJ_{\frac{1}{3}}(\kappa x^{\frac{3}{2}})+BJ_{-\frac{1}{3}}(\kappa x^{\frac{3}{2}})\}.$$

The condition that $\frac{dp}{dx}=0$ when $x=0$, makes $A=0$, and then

$$p=B\sqrt{x}J_{-\frac{1}{3}}(\kappa x^{\frac{3}{2}}) \qquad\qquad (4).$$

At A, the lowest point, we must have $p=0$; and therefore, supposing the height OA to be h,

$$J_{-\frac{1}{3}}(\kappa h^{\frac{3}{2}})=0.$$

If c be the least root of the equation $J_{-\frac{1}{3}}(c)=0$,

then

$$c=\kappa h^{\frac{3}{2}},$$

$$h=\left(\frac{c}{\kappa}\right)^{\frac{2}{3}}=\left(\frac{9Ek^2c^2}{4w}\right)^{\frac{1}{3}},$$

the greatest height to which the pole can reach for the vertical position to be stable; if carried up to a greater height, the pole will curve under its own weight if slightly displaced.

From the expansion of $J_n(x)$ in a series of ascending powers of x,

$$J_{-\frac{1}{3}}(x) = \frac{x^{-\frac{1}{3}}}{2^{-\frac{1}{3}}\Gamma(\frac{2}{3})}\left(1 - \frac{3x^2}{2.4} + \frac{3^2x^4}{2.4.4.10} - \frac{3^3x^6}{2.4.6.4.10.16} + \dots\right),$$

and we find by trial that $c = 1{\cdot}88$, $c^{\frac{2}{3}} = 1{\cdot}52$, and

$$h = 1{\cdot}52\left(\frac{9Ea^2}{16w}\right)^{\frac{1}{3}} = 1{\cdot}26\left(\frac{Ea^2}{w}\right)^{\frac{1}{3}}.$$

For instance, for a solid cylinder of pine,

$$E = 1500000, \text{ about};$$

$$w = \frac{{\cdot}6 \times 62{\cdot}5}{12^3} = \frac{37{\cdot}5}{12^3};$$

and if the diameter of the pole be six inches, $a = 3$, $k^2 = \frac{1}{4}a^2$, and therefore $h = 89{\cdot}45 \times 12$, and the height in feet is $89{\cdot}45$.

For a steel wire

$$E = 31000000,$$

$$w = \frac{7{\cdot}8 \times 62{\cdot}5}{12^3} = \frac{487{\cdot}5}{12^3};$$

and if the diameter of the wire be one-tenth of an inch, $a = \frac{1}{20}$,

and
$$h = 1{\cdot}26 \times 12\left(\frac{31000000}{487{\cdot}5 \times 400}\right)^{\frac{1}{3}}$$
$$= 6{\cdot}81 \times 12,$$

and the height in feet is $6{\cdot}81$.

If a load be concentrated at the top B, of weight equal to that of a length l of the shaft, then the differential equation of the central line becomes

$$EAk^2\frac{dp}{dx} = wA\int_0^x (y' - y)\,dx' + wAl\,(OB - y),$$

and differentiating

$$Ek^2\frac{d^2p}{dx^2} = -w(x + l)\,p,$$

the same differential equation as before, with $x + l$ for x.

This state of things will be represented in nature by an exogenous tree like the Areca Palm, growing in a cylindrical shaft, and having a cluster of leaves at the top.

II. Suppose the rod to taper uniformly up to a point, being a right circular cone of semi-vertical angle α; then at the point P the bending moment, supposing r the radius of the cross section of the rod at P,

$$\tfrac{1}{4}\pi Er^4\frac{dp}{dx} = \pi w\int_0^x (y' - y)\,r'^2\,dx',$$

and differentiating with respect to x,

$$\tfrac{1}{4}\pi E\frac{d}{dx}\left(r^4\frac{dp}{dx}\right) = -\pi wp\int_0^x r'^2\,dx'.$$

Now $r = x\tan\alpha$, and therefore

$$\tfrac{1}{4}\pi E\tan^4\alpha\frac{d}{dx}\left(x^4\frac{dp}{dx}\right) = -\pi wp\tan^2\alpha\int_0^x x'^2\,dx'$$
$$= -\tfrac{1}{3}\pi w\tan^2\alpha\, x^3 p,$$

or
$$x^2\frac{d^2p}{dx^2} + 4x\frac{dp}{dx} + \frac{4w}{3E\tan^2\alpha}\,xp = 0 \ldots\ldots\ldots\ldots(1),$$

the differential equation of the central line of the conical rod, the flexure under its own weight being small.

First put $p = x^{-\frac{3}{2}}z$, then

$$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(\frac{4w}{3E\tan^2\alpha}\,x - \tfrac{9}{4}\right)z = 0;$$

and again, putting $x = r^2$,

$$r^2\frac{d^2z}{dr^2} + r\frac{dz}{dr} + \left(\frac{16w}{3E\tan^2\alpha}\,r^2 - 9\right)z = 0 \ldots\ldots\ldots\ldots(2),$$

Bessel's differential equation, in which

$$\kappa^2 = \frac{16w}{3E\tan^2\alpha}, \qquad n^2 = 9.$$

Therefore the solution of (1), subject to the condition that p is finite when $x = 0$, is

$$p = Ax^{-\frac{3}{2}}J_3(\kappa x^{\frac{1}{2}}),$$

and supposing h the height OA, and c the least root of the equation $J_3(x) = 0$, then

$$c = \kappa h^{\frac{1}{2}},$$

or

$$h = \left(\frac{c}{\kappa}\right)^2,$$

$$= \frac{3E \tan^2 \alpha c^2}{16w}.$$

The value of c for $n = 3$ is 6·379 (Table B. p. 274, Rayleigh, *Theory of Sound*, Vol. I.), and therefore

$$h = 7{\cdot}63 \frac{E}{w} \tan^2 \alpha.$$

For instance with pine, where

$$E = 1500000, \text{ and } w = \frac{37{\cdot}5}{12^3},$$

$$h = 527380000 \tan^2 \alpha;$$

and if b be the radius of the base in inches, $\tan \alpha = \dfrac{b}{h}$,

$$h^3 = 527380000\, b^2,$$

or

$$h = 807{\cdot}9 b^{\frac{2}{3}},$$

h and b being given in inches.

III. If the rod be in the form of a paraboloid of revolution, then the differential equation of the central line

$$\tfrac{1}{4}\pi E \frac{d}{dx}\left(r^4 \frac{dp}{dx}\right) = -\pi w p \int_0^x r'^2 dx',$$

becomes, since $r^2 = 4mx$, where $4m$ is the latus rectum of the generating parabola,

$$4\pi m^2 E \frac{d}{dx}\left(x^2 \frac{dp}{dx}\right) = -4\pi m w p \int_0^x x' dx'$$

$$= -2\pi m w x^2 p,$$

or

$$x^2 \frac{d^2 p}{d^2 x} + 2x \frac{dp}{dx} + \frac{w}{2Em} x^2 p = 0,$$

or

$$x \frac{d^2 p}{dx^2} + 2 \frac{dp}{dx} + \frac{w}{2Em} xp = 0,$$

which may be written

$$\frac{d^2}{dx^2}(xp) + \frac{w}{2Em}(xp) = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

the solution of which is, subject to the conditions that p is finite and $\dfrac{dp}{dx} = 0$ when $x = 0$,

$$xp = A \sin \sqrt{\left(\frac{w}{2Em}\right)} x \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

and therefore the height h is given by

$$\sin \sqrt{\left(\frac{w}{2Em}\right)} h = 0,$$

or

$$\sqrt{\left(\frac{w}{2Em}\right)} h = \pi,$$

or

$$h = \pi \sqrt{\left(\frac{2Em}{w}\right)}.$$

If b be the radius of the base, then $b^2 = 4mh$, or

$$h = \pi \sqrt{\left(\frac{E}{2wh}\right)} b,$$

or

$$h^{\frac{3}{2}} = \pi \sqrt{\left(\frac{E}{2w}\right)} b.$$

If k denote the elasticity of volume, and n the elasticity of figure, or as it is called, the rigidity of the substance, then

$$E = \frac{9nk}{3k + n}$$

(Thomson and Tait, *Natural Philosophy*, p. 521); and for a substance like jelly, n is small compared with k, and we may put $E = 3n$.

Now E and n being expressed in gravitation measure, the rate of propagation v of transversal vibrations is given by

$$v^2 = g\frac{n}{w} = \frac{Eg}{3w},$$

and therefore

$$h^{\frac{3}{2}} = \pi \sqrt{\frac{3}{2g}} \cdot vb,$$

from which the greatest height a jelly in the form of a paraboloid may be made can be inferred.

IV. Generally, if we have a solid of revolution like a tree, of which the radius of the section of the trunk at the depth x below the top is r, and if W be the weight in pounds of the part of the tree above this section, then if the tree be of such a height as to be slightly bent under its own weight, the differential equation of the central line of the tree is, as before,

$$\tfrac{1}{4}\pi E r^4 \frac{dp}{dx} = \int_0^x (y' - y)\frac{dW'}{dx'}\, dx',$$

and differentiating with respect to x,

$$\tfrac{1}{4}\pi E \frac{d}{dx}\left(r^4 \frac{dp}{dx}\right) = -p\int_0^x \frac{dW'}{dx'}\, dx' = -Wp \text{ .... (1).}$$

This differential equation is equivalent to Bessel's differential equation if r and W are proportional to powers of x.

For suppose $r = \lambda x^m$, $W = \mu x^n$; then

$$\tfrac{1}{4}\pi E\lambda^4 \frac{d}{dx}\left(x^{4m}\frac{dp}{dx}\right) + \mu x^n p = 0,$$

or

$$x^{4m}\frac{d^2p}{dx^2} + 4mx^{4m-1}\frac{dp}{dx} + \frac{4\mu}{\pi E\lambda^4}x^n p = 0,$$

or

$$x^2\frac{d^2p}{dx^2} + 4mx\frac{dp}{dx} + \frac{4\mu}{\pi E\lambda^4}x^{n-4m+2}p = 0 \text{ ....(2),}$$

which is reduced to Bessel's differential equation by putting

$$p = x^{\frac{1-4m}{2}} z, \text{ and } x^{n-4m+2} = r^2.$$

For instance, in I. $m = 0$, $n = 1$; in II. $m = 1$, $n = 3$; in III. $m = \tfrac{1}{2}$, $n = 2$.

The solution of (2), subject to the condition that p is finite or $\frac{dp}{dx} = 0$ when $x = 0$, is as before

$$p = Ax^{\frac{1-4m}{2}} J_{\frac{4m-1}{n-4m+2}}\left(\kappa x^{\frac{n-4m+2}{2}}\right),$$

where

$$\kappa^2 = \frac{16\mu}{\pi E\lambda^4 (n-4m+2)^2};$$

and the greatest height h to which the tree can grow without flexure is given by

$$c = \kappa h^{\frac{n-4m+2}{2}},$$

or

$$h = \left(\frac{c}{\kappa}\right)^{\frac{2}{n-4m+2}},$$

where c is the least positive root of the equation

$$J_{\frac{4m-1}{n-4m+2}}(x) = 0.$$

By assigning different values to m and n according to the growth of the tree, and to E, μ, and λ according to the elasticity and density of the wood, the greatest straight vertical growth of a tree can be inferred. The application of these formulæ to the case of the large trees of California would be interesting, but in the absence of the numerical data required, I am unable to carry this out.

This paper was written for Dr Asa Gray, Professor of Botany in Harvard University, Cambridge, Mass., and was to have been read at the meeting of the American Association last year, but arrived too late.

A pine tree, as described in Sproat's *British Columbia* (1875), is said to have grown in one straight tapering stem to a height of 221 feet, and to have measured only 20 inches in diameter at the base.

Considered as an example of Article II., and neglecting the weight of the branches, a diameter of 20 inches at the base would allow of a vertical stable growth of about 300 feet.

Perhaps the best assumptions to make for our purpose as to the growth of a tree are, (i) to assume a uniformly tapering trunk as a central column, and (ii) to adopt Ruskin's assumption (*Modern Painters*), that the sectional area of the branches of a tree, made by any horizontal plane, is constant.

This is equivalent to putting $m = 1$ and $n = 1$ in equation (2), and then the solution depends on the least root of the associated Bessel's function of the order -3.

Generally, for a homogeneous body, $n = 2m + 1$, and the diameter at the base must increase as the $\tfrac{3}{2}$ power of the height, which accounts for the slender proportions of young trees, compared with the stunted appearance of very large trees.