A Brief Introduction to Geometric Group Theory via Hyperbolic Groups

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January 2021

Abstract

Dehn [Deh11] proposed three fundamental problems in finitely presented groups which have since been shown to be undecidable in general. In this project, we introduce some elementary concepts of geometric group theory, and utilise them to show that at least two of Dehn's problems are solvable in hyperbolic groups as defined by Gromov [Gro87] .

Contents

Introduction

Max Dehn, in a 1911 paper [Deh11], identified that the theory of what he called "infinite discontinuous groups" was lacking. He went on to define three fundamental problems for these groups, which we now know as finitely presented groups. Dehn's problems were the following:

(1) The word problem: given two words of the generators, determine whether or not they represent the same element in the group.
(2) The conjugacy problem: given two words of the generators, determine whether or not they represent conjugate elements in the group.
(3) The isomorphism problem: determine whether or not two finitely presented groups are isomorphic.

Dehn did not provide a solution to these problems in general, but did describe an algorithm, now known as Dehn's algorithm, to decide the word problem for 'surface groups': groups arising as the fundamental group of certain 2-manifolds[1]. In a later paper ([Deh12]), Dehn also found a solution to the conjugacy problem in surface groups.

In his 1911 paper, Dehn noted that "the solution of [these problems] is very important and probably not possible without a thorough study of the matter." Dehn was correct. It was first shown by Novikov in 1955 [Nov55] and reaffirmed by Boone just three years later [Boo58] that there exist groups whose word problem is undecidable (in the sense of Turing) – that is to say that there are finitely presented groups for which no algorithm can solve their word problem.

In fact, all three of Dehn's problems are undecidable for finitely presented groups in general[2]. Dehn passed away in 1952, just three years before this was shown to be true.

This result is quite daunting. If we endeavour to understand finitely presented groups, surely the solution of these problems – or indeed even just one of them – is a fundamental obstacle. Are finitely presented groups simply beyond our understanding? These problems' undecidability only shows that there are *some* groups for which this is not possible, but hyperbolic groups provide a ray of hope.

Mikhail Gromov, in his 1987 paper [Gro87], introduced the concept of a hyperbolic group, and showed that they have many interesting properties. For example, following the work of Dehn, it has been shown the word problem and conjugacy problem are decidable in all hyperbolic groups. Recent work ([DG11]) has shown that the isomorphism problem is decidable when restricted to hyperbolic groups.

A particularly astounding property of hyperbolic groups is that, in a certain statistical sense, *almost all* finitely groups are hyperbolic. This was first formulated by Gromov [Gro93], and complete proofs have since been provided, for instance by Olshanskii [Ols11], though this is sensitive to the choice of

[1] Specifically, surface groups arise from compact, boundary-less 2-manifolds of genus at least 2. We do not mention this explicitly, but the example given in section 3.2 is a surface group.

[2] As Dehn [BH99] noted, the word problem is a specific instance of the conjugacy problem, so we conclude that the conjugacy problem must also be undecidable. The isomorphism problem is more difficult, and is a consequence of the Adian–Rabin theorem, first proved by Adian in 1955 [Adi55]. The theorem also shows that hyperbolicity is an undecidable property.

distribution.

Hyperbolic groups are therefore a beacon of hope in the study of finitely presented groups. Not only is there an interesting family of groups which are amenable to study, but this family is *vast!* What's more surprising, is that hyperbolicity is formulated in a purely geometric way, and yet it has bearing on the properties of groups.

In this project, we introduce some elementary tools of geometric group theory, and use them to prove some core (principally algorithmic) properties of hyperbolic groups. The author hopes that the project is a suitable introduction to some elementary ideas in geometric group theory for an undergraduate reader.

In section 1, we motivate the Cayley graph as a geometric interpretation of a group via quasi-isometries. We go on to prove geometric group theory's most fundamental observation: the Švarc–Milnor lemma.

In section 2, we introduce the hyperbolicity condition and hyperbolic groups, firstly introducing the requisite understanding of geodesic spaces. We go on to demonstrate that the word and conjugacy problems are solvable in hyperbolic groups, the former via an adaptation of Dehn's algorithm, and use the same techniques to show that they are also finitely presented.

In section 3, we briefly explore what hyperbolic groups look like. We classify all Abelian hyperbolic groups, and give a more tangible example of a hyperbolic group arising from isometries of the hyperbolic plane. We then go on to provide an explicit algorithm that solves the word problem in this group, using purely combinatorial methods.

Hyperbolic groups are the subject of ongoing research, and this project's scope includes only a very small portion of what is known. It is still an open question, for example, whether or not every hyperbolic group is *residually finite* – that is, given any non-identity element of a hyperbolic group, does there exist a finite-index normal subgroup that does not contain that element? We hope that interested readers will be sufficiently equipped by this project to approach further topics.

The author would like to thank his supervisor, Dr. Richard Webb, for his excellent advice every step of the way, which has been indispensable.

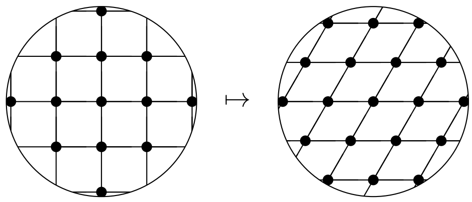

Figure 1.1: An example of a quasi-isometric embedding of a discrete subset of $\mathbb{R}^2$ into another. Left: the grid of integer-coordinate points. Right: the image of the grid under a certain map.

1 Groups as Geometric Objects

1.1 Quasi-Isometries

Quasi-isometries can be thought of as a way to bridge the gap between discrete and non-discrete metric spaces, allowing us to use properties of the latter to investigate the former. We will use the concept of a quasi-isometry to motivate an understanding of groups via a *coarse* rather than *fine-grained* perspective.

This approach allows us to focus on the very large-scale properties that a space admits. This is evidenced by the fact that, as we will see shortly, finite-diameter spaces are a degenerate case in this perspective.

Definition 1.1. A (λ, ε)-*quasi-isometric embedding* is a (not necessarily continuous) function $f\colon X \to Y$ on metric spaces (X, d_X) and (Y, d_Y) for which there exist real constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that the following inequality holds for all $a, b \in X$:

$$\frac{1}{\lambda} d_X(a, b) - \varepsilon \leq d_Y\big(f(a), f(b)\big) \leq \lambda d_X(a, b) + \varepsilon.$$

We say that f is a quasi-isometric embedding to mean implicitly that there exist constants λ and ε for which f is a (λ, ε)-quasi-isometric embedding.

This definition should be contrasted with that of a bi-Lipschitz map. In particular, this does not require the map to be continuous. The extra lenience that an added constant term provides is what allows this discontinuity.

Example. The map $f\colon \mathbb{Z} \to \mathbb{R}$; $z \mapsto \sin z + z$, where $\mathbb{Z}$ is seen as a subspace of $\mathbb{R}$, is a $(1, 2)$-quasi-isometric embedding. This is because $|f(a) - f(b)| \leq |a - b| + |\sin a| + |\sin b| \leq |a-b| + 2$, and $|f(a) - f(b)| \geq \big||a-b| - |\sin b - \sin a|\big| \geq |a-b| - 2$.

Remark. The constants λ and ε may not be unique. For instance, take f from the previous example, which was shown to be a $(1, 2)$-quasi-isometric embedding. Taking $\lambda = 3$ and $\varepsilon = 4/3$ also satisfies the definition, by the following argument: if $a = b$ then any constants work, so let us only consider the case where $a \neq b$. Then $|a - b| \geq 1$, so $|f(a) - f(b)| \leq |a - b| + 2 \leq 3|a - b|$, and in the opposite direction, $|f(a) - f(b)| \geq |a - b| - 2 = |a - b|/3 + 2|a - b|/3 - 2 \geq |a - b|/3 - 4/3$.

Example. Figure 1.1 illustrates a simple quasi-isometric embedding. Let X be the grid of integer-coordinate points in $\mathbb{R}^2$, and let Y be its image under the map $(x, y) \mapsto (x, x/2 + \sqrt{3}x/2)$, where both are considered under the Euclidean metric. This map can be shown to be a $(\lambda, 0)$-quasi-isometric embedding, where $\lambda = 2/\sqrt{3}$.

Proposition 1.2. *The composition of two quasi-isometric embeddings is itself also a quasi-isometric embedding.*

Proof. Let $f\colon X \to Y$ be a $(\lambda_1, \varepsilon_1)$-quasi-isometric embedding, and let $g\colon Y \to Z$ be a $(\lambda_2, \varepsilon_2)$-quasi-isometric embedding. By applying the definition twice, we obtain the following:

(1) $\frac{1}{\lambda_1\lambda_2} d_X(a,b) - \left(\frac{\varepsilon_1}{\lambda_2} + \varepsilon_2\right) \leq d_Z(gf(a), gf(b))$, and

(2) $d_Z(gf(a), gf(b)) \leq \lambda_1\lambda_2 d_X(a,b) + (\lambda_2\varepsilon_1 + \varepsilon_2)$.

The real constants $\lambda = \lambda_1\lambda_2$ and $\varepsilon = \lambda_2\varepsilon_1 + \varepsilon_2 \geq \varepsilon_1/\lambda_2 + \varepsilon_2$ can easily be shown to be $\lambda \geq 1$ and $\varepsilon \geq 0$, and by the above inequalities we may infer that the composition of f and g is a (λ, ε)-quasi-isometric embedding. $\square$

This terminology is very suggestive, and the reader may expect to soon see a definition of a quasi-inverse. The reader would be correct, but the possibly naïve notion of inverse (wherein the maps are mutually inverse) is a little too restrictive for our purposes. We use a slightly more relaxed notion of inverse, for which we first introduce an equivalence relation on quasi-isometric embeddings.

Definition 1.3. The relation $\sim$ on quasi-isometric embeddings $f, g\colon X \to X$ is defined as follows: $f \sim g$ if and only if there exists a real constant $K > 0$ such that $d_X(f(x), g(x)) \leq K$ for all $x \in X$. This is an equivalence relation.

It is not difficult to see that this is an equivalence relation. Every function f is related to itself with $K = 0$, so the relation is reflexive. The relation is evidently symmetric as well, and transitivity follows from the triangle inequality.

Definition 1.4. A map $g\colon Y \to X$ is a *quasi-inverse* of a quasi-isometric embedding $f\colon X \to Y$ if g is a quasi-isometric embedding, and $gf \sim \mathrm{id}_X$ and $fg \sim \mathrm{id}_Y$, where id_A denotes the identity map on the set A.

Definition 1.5. A quasi-isometric embedding with a quasi-inverse is called a *quasi-isometry*, and in particular a (λ, ε)*-quasi-isometry* for specific constants.

The additional lenience that forgoing the naïve requirement allows us to formulate a very useful condition for being a quasi-isometry, which we work towards now.

Definition 1.6. A subset S of a metric space (X, d) is C*-quasi-dense* in X for some real constant $C > 0$ if for every point $x \in X$ there is some $s \in S$ such that $d_X(x, s) \leq C$.

Proposition 1.7. *A quasi-isometric embedding $f\colon X \to Y$ is a quasi-isometry if and only if $f(X)$ is quasi-dense in Y.*

Proof. If we assume f is a quasi-isometry, by definition the quasi-inverse g satisfies $fg \sim \mathrm{id}_Y$. This means that there is a $K > 0$ such that $d_Y(fg(y), y) \leq K$ for all $y \in Y$. Since $fg(y) \in f(X)$, we conclude $f(X)$ is K-quasi-dense in Y.

The real substance of this proposition is in the converse, which we now prove. Suppose that $f(X)$ is C-quasi-dense in Y, so we may produce map $g\colon Y \to X$ by choosing some point of $f(X)$ of distance at most C from the input. In particular that means that $d_Y(fg(y), y) \leq C$ for all y, which in particular means that

$fg \sim \mathrm{id}_Y$. As this may suggest, we will show that g is a quasi-inverse of f. First, we note two facts about g. Firstly:

$$\begin{aligned}(1) \qquad d_Y(fg(a), fg(b)) &\leq d_Y(fg(a), a) + d_Y(a, b) + d_Y(b, fg(b)) \\ &\leq d_Y(a, b) + 2C,\end{aligned}$$

And secondly:

$$\begin{aligned}(2) \qquad d_Y(a, b) &\leq d_Y(a, fg(a)) + d_Y(fg(a), fg(b)) + d_Y(fg(b), b) \\ &\leq d_Y(fg(a), fg(b)) + 2C.\end{aligned}$$

Now suppose f is a (λ, ε)-quasi-isometric embedding. By rearranging the lower bound in the definition, we see that:

$$d_X(g(a), g(b)) \leq \lambda d_Y(fg(a), fg(b)) + \lambda\varepsilon,$$

so applying the result in (1) we obtain:

$$d_X(g(a), g(b)) \leq \lambda d_Y(a, b) + \lambda(2C + \varepsilon).$$

Similarly, we see that the upper bound in the definition gives us

$$d_X(g(a), g(b)) \geq d_Y(fg(a), fg(b))/\lambda - \varepsilon/\lambda,$$

then by the result in (2), it follows that

$$d_X(g(a), g(b)) \geq d_Y(a, b)/\lambda - (2C + \varepsilon)/\lambda \geq d_Y(a, b)/\lambda - \lambda(2C + \varepsilon).$$

These previous two results show that g is a $(\lambda, \lambda(2C + \varepsilon))$-quasi-isometric embedding, so we need only show now that $gf \sim \mathrm{id}_X$.

Note first that $d_Y(fg(f(x)), f(x)) \leq C$, so the lower bound in the definition gives us $d_X(gf(x), x) \leq \lambda(C + \varepsilon)$ for any $x \in X$, so by definition $gf \sim \mathrm{id}_X$. □

Example. We noted earlier that there is a quasi-isometric embedding illustrated in figure 1.1. Proposition 1.7 allows us to conclude that this map is in fact a quasi-isometry, since the triangular grid is the image of the integer-coordinate grid under the quasi-isometry, and hence the image 0-quasi-dense.

Corollary 1.8. *The composition of two quasi-isometries is a quasi-isometry.*

Proof. Let $f\colon X \to Y$, $g\colon Y \to Z$ be quasi-isometries. By proposition 1.2 we know that gf is a quasi-isometric embedding, so by the previous proposition we need only show that $gf(X)$ is quasi-dense in Z.

By assumption we know that $f(X)$ is C-quasi-dense in Y, and $g(Y)$ is K-quasi-dense in Z. so, for any $z \in Z$ there is some $y \in Y$ for which $d_Z(z, g(y)) \leq K$. Similarly, there exists a $x \in X$ for which $d_Y(y, f(x)) \leq C$. If g is a (λ, ε)-isometric embedding, then recalling the upper bound, we see $d_Z(z, gf(x)) \leq d_Z(z, g(y)) + d_Z(g(y), gf(x)) \leq K + \lambda C + \varepsilon$.

Therefore $gf(X)$ is quasi-dense in Z, hence gf is a quasi-isometry. □

This corollary finally allows us to state a strong fact about the nature of quasi-isometries – a consequence of several of the previous propositions.

Corollary 1.9. *Let $X \approx Y$ denote that there is a quasi-isometry $f\colon X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces. This relation is an equivalence relation.*

Proof. The relation is reflexive since id_X is always a quasi-isometry, symmetric by definition of a quasi-isometry, and transitive due to corollary 1.8. $\square$

Example. We noted earlier that the grids in figure 1.1 are quasi-isometric. However, there is a simple proof of this that we now have access too. Both grids are under the Euclidean metric and are 1-quasi-dense in $\mathbb{R}^2$, hence are quasi-isometric to Euclidean 2-space. Ergo they are quasi-isometric to each other.

This provides us a helpful notion of equivalence of spaces: being quasi-isometric. We will (in section 1.3) use it to understand the coarse geometric structure of groups via their Cayley graphs.

We will explore some more properties of quasi-isometries in later sections – specifically concerning hyperbolicity – but there are many other interesting properties they have which will not be explored here.

We will end this section with a useful result regarding quasi-isometries.

Proposition 1.10. *Let d_1 and d_2 be Lipschitz equivalent metrics on the same set X. Then the metric spaces (X, d_1) and (X, d_2) are quasi-isometric.*

Proof. Given that the metrics are Lipschitz equivalent, there exists some $\lambda > 1$ for which $\frac{1}{\lambda} d_1(a, b) \leq d_2(a, b) \leq \lambda d_1(a, b)$. However $d_1(a, b) = d_2(\mathrm{id}_X(a), \mathrm{id}_X(b))$, so by definition id_X is a $(\lambda, 0)$-quasi-isometric embedding.

Furthermore the image of id_X is X itself, which is evidently quasi-dense in itself, so by proposition 1.7 id_X is a quasi-isometry. $\square$

1.2 Group Presentations

Group presentations are an abstract way of defining groups in terms of certain properties they have. We describe their construction here so that we may use them in later sections.

Definition 1.11. Let $\mathcal{A}$ be a set of symbols. A *group word* (or simply *word*) $\mathbf{w}$ in $\mathcal{A}$ is a finite string of symbols in $\mathcal{A} \cup \mathcal{A}^-$, where $\mathcal{A}^-$ is the set of symbols $\{a^{-1} \mid a \in \mathcal{A}\}$. The *empty word*, denoted by ϵ, is also a word in $\mathcal{A}$.

A *reduced word* $\mathbf{w}$ in $\mathcal{A}$ is a word in $\mathcal{A}$ within which no symbols a and a^{-1} are adjacent.

The *length* $|\mathbf{w}|$ of a word $\mathbf{w}$ in $\mathcal{A}$ is defined as the number of symbols within the word. Given two words $\mathbf{v}$ and $\mathbf{w}$ in $\mathcal{A}$, we may define their concatenation $\mathbf{v} +\!\!+\, \mathbf{w}$, which is also a word in $\mathcal{A}$.

Example. Given $\mathcal{A} = \{\mathrm{a}, \mathrm{b}\}$, $\mathbf{v} = \mathrm{aba}^{-1}$ and $\mathbf{w} = \mathrm{ab}^{-1}\mathrm{b}^{-1}$ are reduced words, but their concatenation $\mathbf{v} +\!\!+\, \mathbf{w} = \mathrm{aba}^{-1}\mathrm{ab}^{-1}\mathrm{b}^{-1}$ is not a reduced word.

Remark. Given any words $\mathbf{w}$ and $\mathbf{v}$, we have $|\mathbf{w} +\!\!+\, \mathbf{v}| = |\mathbf{w}| + |\mathbf{v}|$.

Given a word $\mathbf{w}$ in $\mathcal{A}$, we may take a *one-step reduction* by 'cancelling' from the word any adjacent inverses – for instance $\mathrm{a}^{-1}\mathrm{bb}^{-1}\mathrm{ab} \to \mathrm{a}^{-1}\mathrm{ab}$ is a one-step reduction. Similarly we may take a *one-step promotion* by inserting an adjacent symbol and its inverse – for instance $\mathrm{a}^{-1}\mathrm{bb}^{-1}\mathrm{ab} \to \mathrm{a}^{-1}\mathrm{bb}^{-1}\mathrm{abb}^{-1}\mathrm{b}$ is a one-step promotion.

We will call two words in $\mathcal{A}$ *equivalent* if there is a (possibly empty) sequence of one-step reductions and promotions that transforms one into the other. This is evidently an equivalence relation.

Definition 1.12. The *free group* $F(\mathcal{A})$ on a set of symbols $\mathcal{A}$ is the set of equivalence classes of words of $\mathcal{A}$ under equivalence of words, and whose operation is concatenation of a representative element.

Proof that the free group is a group. We must first show that the operation is well-defined. If words $\mathbf{w}$ and $\mathbf{v}$ have equivalent words $\mathbf{w}'$ and $\mathbf{v}'$, then we may apply the same reductions and promotions to the concatenation $\mathbf{w} \mathbin{+\!\!+} \mathbf{v}$ to firstly achieve equivalence with $\mathbf{w}' \mathbin{+\!\!+} \mathbf{v}$ and then with $\mathbf{w}' \mathbin{+\!\!+} \mathbf{v}'$.

The operation is associative due to concatenation already being so.

The identity exists as the equivalence class of the empty word ϵ.

Inverses exist: if $\mathbf{w} = s_1 \cdots s_n$ then $\mathbf{w}^{-1} = s_n^{-1} \cdots s_1^{-1}$, which is readily seen to be suitable. $\square$

Remark. Every equivalence class of words is represented by a unique reduced word, but we will not include the proof of this fact. For this reason we will treat the elements of the free group as reduced words of $\mathcal{A}$.

Definition 1.13. Let $\mathcal{A}$ be a set of symbols, and let $\mathcal{R}$ (the set of 'relators') be a subset of $F(\mathcal{A})$. The *group presentation* $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is given by the quotient group $F(\mathcal{A})/N$, where N is the smallest normal subgroup[3]of $F(\mathcal{A})$ which subsumes $\mathcal{R}$.

Using this quotient, reduced words in $F(\mathcal{A})$ may be used to represent elements in the group presentation, by considering what their image is under the natural homomorphism from $F(\mathcal{A})$ to $\langle \mathcal{A} \mid \mathcal{R} \rangle$. However, the word that represents a given element may not be unique.

The relators $\mathcal{R}$ can be thought of as the 'basis' for all elements that are sent to the identity by the natural homomorphism. This allows us to express some complicated groups in very simple ways. The following examples are all of finitely presented groups, which is to say that $\mathcal{A}$ and $\mathcal{R}$ are both finite.

Examples. The cyclic group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\langle \mathrm{a} \mid \mathrm{a}^n \rangle$ (that is, where $\mathcal{A} = \{\mathrm{a}\}$ and $\mathcal{R} = \{\mathrm{a}^n\}$). The dihedral group D_{2n} is isomorphic to $\langle \mathrm{r}, \mathrm{s} \mid \mathrm{r}^n, \mathrm{s}^2, \mathrm{srsr} \rangle$.

We contrast group presentations with the following definition.

Definition 1.14. Let G be a group. A *generating set* $\mathcal{A}$ is a subset (not necessarily a subgroup) of G with the property that $\langle \mathcal{A} \rangle = G$, and $1 \notin \mathcal{A}$. In other words every non-identity element of G may be expressed as a finite product of elements in $\mathcal{A}$ and their inverses.

Our use of the same symbol for generating sets and the set of symbols is suggestive and deliberate. Not only is $\mathcal{A}$ a generating set for any presentation which uses it as its set of symbols, but there is also a natural homomorphism from $F(\mathcal{A})$ to G achieved by multiplying each of the symbols in a reduced word.

It is necessary to distinguish equality as words and equality within a group. We will write $\mathbf{w} = \mathbf{v}$ to say that the words consist of the same symbols or when

[3] This is also known as the normal closure of $\mathcal{R}$, and may be constructed as the intersection of all normal subgroups of $F(\mathcal{A})$ which contain $\mathcal{R}$.

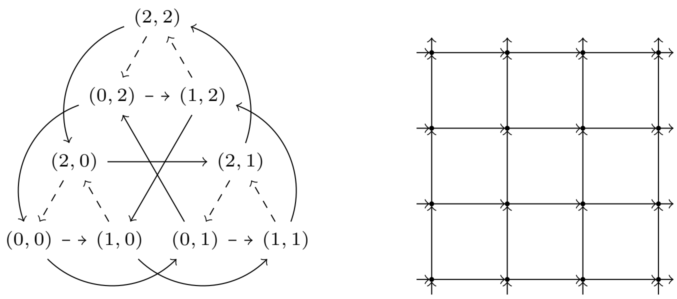

Figure 1.2: Two examples of combinatorial Cayley graphs. Left: the graph of $\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$ with generators $(1,0)$ and $(0,1)$ – this graph is not planar. Right: a small part of the graph of $\mathbb{Z}\times\mathbb{Z}$ with the generators $(1,0)$ and $(0,1)$.

they represent the same element in the group, but if confusion arises, we will write $\mathbf{w} =_G \mathbf{v}$ to say specifically that the words represent the same element of the group G.

1.3 Finitely Generated Groups as Metric Spaces

In order to study groups as geometric objects, we must first find a suitable geometric manifestation of groups. In this section we present Cayley graphs as such a manifestation. Not only will we define Cayley graphs, but we will also demonstrate that they are a natural way of thinking about groups with our understanding of quasi-isometries, as developed in section 1.1.

We make a distinction between *combinatorial* graphs and *metric* graphs. The exact notion used here is modelled after that in Bridson-Haefliger [BH99], but we will not consider the concept of a metric graph in the generality presented there, since we wish to avoid the various pathologies that can arise in them.

Definition 1.15. A *combinatorial Cayley graph* of a group G with respect to a generating set $\mathcal{A}$ is the directed graph whose vertices V are the elements of the group, and whose edges E are the ordered pairs $\{(g, ga) \mid g \in G, a \in \mathcal{A}\}$.

This is itself a metric space, whose points are the elements of G, and whose metric is d_G: the word metric, given by the length of the shortest path between to vertices. However, this space is topologically discrete: all of its points are isolated. Since we will later be reasoning about paths within spaces, this isn't quite good enough for our purposes. We introduce an alternative metric space now.

Definition 1.16. Let the G be a group, $\mathcal{A}$ be a generating set of G, and let E be the set of edges of the combinatorial Cayley graph of G with respect to $\mathcal{A}$. If $e \in E$, let $v_0(e)$ be the source and $v_1(e)$ be the target of the edge e.

A *metric Cayley graph* of a group G with respect to a generating set $\mathcal{A}$ is the set $\mathcal{C}_{\mathcal{A}}(G)$ of equivalence classes of $E \times [0,1]$ under the relation $(e_1, i) \sim (e_2, j)$ which holds when the pairs are equal, or $i, j \in \{0,1\}$ and $v_i(e_1) = v_j(e_2)$.

We will use the function $f_e(t)$ to denote the equivalence class of (e,t), so that $f_e : [0,1] \to \mathcal{C}_{\mathcal{A}}(G)$.

A metric Cayley graph may be thought of as an enrichment of the combinatorial Cayley graph by treating the edges between vertices as isometric to the interval $[0,1] \subseteq \mathbb{R}$.

To obtain a metric structure on the metric Cayley graph that is compatible with that of the combinatorial graph, we must define the notion of a path within these spaces.

Definition 1.17. A *piecewise linear path* is a function $\gamma : [0,1] \to \mathcal{C}_{\mathcal{A}}(G)$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each restriction $\gamma|_{[t_i,t_{i+1}]}$ is of the form $f_{e_i} \circ \gamma_i$, where $\gamma_i : [t_i, t_{i+1}] \to [0,1]$ is a linear polynomial.

The length of a piecewise linear path γ is defined to be the following:

$$l(\gamma) = \sum_{i=0}^{n-1} |\gamma_i(t_i) - \gamma_i(t_{i+1})|.$$

Proposition 1.18. *Let $\mathcal{C}_{\mathcal{A}}(G)$ be the metric Cayley graph of a group, and let $d(a,b)$ be the infimum of the lengths of piecewise linear paths γ with $\gamma(0) = a$ and $\gamma(1) = b$ Then $(\mathcal{C}_{\mathcal{A}}(G), d)$ is a metric space.*

We will not include the detailed proof of this fact, but we shall note that this is true for three main reasons: (1) the graph is connected, so distance is defined, (2) each edge's length is unity, so the distance between distinct points is strictly positive, and (3) the triangle inequality holds as a consequence of the triangle inequality holding for the distance between vertices already, and then between arbitrary points since $[0,1]$ also obeys the inequality.

These two notions, of course, are related. Suppose that G is not the trivial group, and therefore that $\mathcal{A}$ is non-empty and contains some element a. We may define a distance-preserving map p from the combinatorial Cayley graph into the metric one, defined by $g \mapsto p(g) = f_{(g,ga)}(0)$, which can be seen to be independent of the choice of a. This leads us to the following important fact.

Proposition 1.19. *Let G be a non-trivial group with generating set $\mathcal{A}$. Then the combinatorial and metric Cayley graphs of G with respect to $\mathcal{A}$ are quasi-isometric.*

Proof. The map p above is distance-preserving, so it is a $(1,0)$-quasi-isometric embedding by definition. Therefore by proposition 1.7 we need only show that the image of the combinatorial graph is quasi-dense in the metric graph. Indeed the image is 1-quasi-dense, since every point in the metric graph is distance less than 1 from an vertex, and the vertices are exactly those points in the image. $\square$

This proposition gives us insight into why quasi-isometries are important. By considering only the coarse geometry of these spaces, we obtain a very intuitive equivalence.

It may seem that a Cayley graph is very dependent on the choice of generating set, which as we noted is not unique. Indeed in general the choice of generating set does affect the Cayley graph's coarse geometry. For the special case in which the group is finitely generated, the choice does *not* matter significantly, as we observe now.

Proposition 1.20. *Let G be a group and let $\mathcal{A}$ and $\mathcal{B}$ be distinct finite generating sets of G. Then the Cayley graphs of G with respect to $\mathcal{A}$ and with respect to $\mathcal{B}$ are quasi-isometric.*

Proof. We need only show this holds for combinatorial Cayley graphs, since proposition 1.19 handles the metric case. Showing that the graphs' respective metrics $d_{\mathcal{A}}$ and $d_{\mathcal{B}}$ are Lipschitz equivalent will suffice, via an invocation of proposition 1.10.

Let $k = \max_{a \in \mathcal{A}} d_{\mathcal{B}}(1, a) \geq 0$. Suppose that $l = d_{\mathcal{A}}(g, h)$, in other words there exist $a_1, \ldots, a_l \in \mathcal{A} \cup \mathcal{A}^-$ such that $h = ga_1 \cdots a_l$. Since $\mathcal{B}$ generates $\mathcal{A}$, there exist words $\mathbf{B}_i$ of $\mathcal{B}$ such that $B_i = a_i$ for all $1 \leq i \leq l$. The length of each word $\mathbf{B}_i$ is less than k by definition, so we have found a path of length at most kl between g and h in $\mathcal{C}_{\mathcal{B}}(G)$. Therefore by definition $d_{\mathcal{B}}(g, h) \leq kd_{\mathcal{A}}(g, h)$.

A similar argument allows us to obtain a constant $k' \geq 0$ such that $\frac{1}{k'} d_{\mathcal{A}}(g, h) \leq d_{\mathcal{B}}(g, h)$, so the metrics are Lipschitz equivalent, and we are done. ☐

Corollary 1.21. *The Cayley graph of a finitely generated group is unique up to quasi-isometry.*

This wonderful fact provides ample evidence that Cayley graphs are a suitable geometric interpretation of finitely generated groups. For this reason we may refer to a finitely generated group G as a metric space – specifically the combinatorial Cayley graph (G, d_G) – without ambiguity, so long as we work up to quasi-isometry.

Remark. The same cannot be said for groups that are not finitely generated. For instance, consider the group of functions $\mathbb{N} \to \mathbb{Z}$ under addition. Define the generating set $\mathcal{A} = \{f_i \,|\, i \in \mathbb{N}\}$ where $f_i(x) = 1$ if $i = x$ and $f_i(x) = 0$ otherwise. Similarly define $\mathcal{B} = \{g_i \,|\, i \in \mathbb{N}\}$ where $g_i = f_0 + \cdots + f_i$. Then $d_G(1, g_n) = n$ and $d_{\mathcal{B}}(1, g_n) = 1$, meaning we may exceed the bounds required for any constants λ and ε.

1.4 The Švarc–Milnor Lemma

It is a natural question to ask to what extent the nature of a group of isometries of a space reflects the nature of the space itself. Given the material covered up to this point, we may be inclined to consider groups that act on a space in a discrete way – after all, we have been considering finitely generated groups. Remarkably, provided a few subtle conditions, a group acting upon a space is finitely generated and quasi-isometric to that space.

This observation is known as the Švarc–Milnor lemma (sometimes Schwarz–Milnor) which was discovered independently by Albert Schwarz and John Milnor. It is often known as the fundamental observation in geometric group theory, because it strongly motivates the study of groups as geometric objects by linking them to pre-existing spaces. It will later help us identify hyperbolic groups.

We begin by pinning down the key concept of a group action by isometries.

Definition 1.22. Let Γ be a group, and let $\mathrm{Isom}(X)$ denote the group of isometries of a metric space X. An *action by isometries* on X is a homomorphism $\phi\colon \Gamma \to \mathrm{Isom}(X)$. We write $\gamma \cdot x$ to denote $\phi(\gamma)(x)$.

Equivalently, we may define an action as satisfying the following properties for all $\gamma_1, \gamma_2 \in \Gamma$ and $x, y \in X$:

(1) $1 \cdot x = x$, where $1 \in \Gamma$ is the identity,
(2) $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1\gamma_2) \cdot x$, and
(3) $d_X(x, y) = d_X(\gamma_1 \cdot x, \gamma_1 \cdot y)$.

Remark. Every group acts by isometries on all of its own Cayley graphs. If g is a vertex (i.e., an element of Γ) then we define $\gamma \cdot g = \gamma g$, and if v is a point on the edge (g, ga) then $\gamma \cdot v$ is the corresponding point on the edge $(\gamma g, \gamma ga)$. It may be readily verified that this is an action by isometries. What's more, one may find an group element that sends any vertex to any other given vertex.

Readers familiar with ordinary group actions may find it instructive to contrast actions by isometry to ordinary group actions, wherein $\mathrm{Isom}(X)$ is substituted for $\mathrm{Sym}(X)$, which is the group of bijections on X.

Definition 1.23. We write $\Gamma \cdot x$ to denote the *orbit* $\{\gamma \cdot x \mid \gamma \in \Gamma\}$, where $x \in X$.
We also write $\gamma \cdot S$ where $S \subseteq X$ to denote $\{\gamma \cdot s \mid s \in S\}$.
Finally, for some $S \subseteq X$, we write $\Gamma \cdot S$ to denote $\{\gamma \cdot s \mid \gamma \in \Gamma, s \in S\}$. (Equivalently we could define $\Gamma \cdot S = \bigcup_{\gamma \in \Gamma} \gamma \cdot S$).

We now introduce a complicated definition that is key to the main result of this section.

Definition 1.24. Let Γ act by isometries on X. We will say that the action satisfies the *Švarc–Milnor condition* if all the following hold:
(1) The space X is a geodesic metric space;
(2) The space X is *proper*, meaning that all closed balls in X are compact;
(3) Γ acts *cocompactly* on X, meaning that there exists some compact set $D \subseteq X$ such that $\Gamma \cdot D = X$;
(4) Γ acts *properly discontinuously* on X, meaning that for all compact sets $K \subseteq X$, the set $\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \varnothing\}$ is finite.

Example. Let $X = \mathbb{R}^2$ and let $\Gamma = \mathbb{Z} \times \mathbb{Z}$, with the action defined by $(n, m) \cdot (x, y) = (x + n, y + m)$. This action satisfies the Švarc–Milnor condition.
Firstly, it is evident that X is a geodesic metric space, and by the Heine–Borel theorem it is also proper.
Letting D be the square $[0, 1]^2$ we attain $\Gamma \cdot D = X$, and D is compact by the Heine–Borel theorem – this proves that Γ acts cocompactly on X.
If $K \subseteq X$ is a compact set, then it has a finite diameter. We may therefore choose some open ball $B_r(x)$ for some $r > 0$ and $x \in K$ that envelops K. However we may readily see that there are only finitely many $\gamma \in \Gamma$ for which $d(x, \gamma \cdot x) < 2r$, and hence the action is properly discontinuous.

The Švarc–Milnor condition is indeed subtle. It may be conceptualised as requiring the action to resemble a tessellation of the metric space as so: the cocompactness corresponds to a repeated shape covering the space, and the proper discontinuity encodes the 'discreteness' of the repetition. In fact, we will later use tessellations to show that certain groups are quasi-isometric to particular spaces.
The following proofs are adapted from the proof given by Bridson and Haefliger [BH99] and that given by Farb and Margalit [FM11].

Lemma 1.25 (The Švarc–Milnor lemma, first part)**.** *If* Γ *acts by isometries on* X *in a way that satisfies the Švarc–Milnor condition, then* Γ *is finitely generated.*

Proof. Choose any point $x_0 \in X$. By the cocompactness of the action, there is a compact set D for which $\Gamma \cdot D = X$. Therefore there is a $\gamma \in \Gamma$ for which $\gamma \cdot D \ni x_0$. The set D must have finite diameter, so there is some $R > 0$ for which $\gamma \cdot D$ is enveloped by the open ball $B = B_{R/3}(x_0)$ of radius $R/3$ around x_0. Since $\Gamma \cdot (\gamma \cdot D) = X$, we conclude that $\Gamma \cdot B = X$. Choosing the ball to be radius $R/3$ seems arbitrary, but it will be useful later on.

Let $\mathcal{A} = \{g \in \Gamma \mid g \cdot B_R(x_0) \cap B_R(x_0) \neq \varnothing\}$. Since X is proper and the action is properly discontinuous, the set is finite. We shall now show that this set generates Γ.

Suppose $\gamma \in \Gamma$, and consider the geodesic $[x_0, \gamma \cdot x_0]$. Since this curve has finite length $d_X(x_0, \gamma \cdot x_0)$, we may partition it by points $x_0 = p_1, p_2, \ldots, p_n = \gamma \cdot x_0$ along the geodesic, with $d_X(p_i, p_{i+1}) \leq R/3$. Then since $\Gamma \cdot B = X$, there are group elements $1 = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma$ such that $d_X(p_i, \gamma_i \cdot x_0) \leq R/3$. Setting $s_i = \gamma_i^{-1}\gamma_{i+1}$, The triangle inequality now tells us $d_X(x_0, s_i \cdot x_0) = d_X(\gamma_i \cdot x_0, \gamma_{i+1} \cdot x_0) \leq R$, so $s_i \in \mathcal{A}$. Furthermore, $\gamma = s_1 s_2 \cdots s_{n-1}$. Since γ was chosen arbitrarily, we have shown that $\mathcal{A}$ generates Γ. □

Remark. As noted by Bridson and Haefliger [BH99] theorem I.8.8.10, the set $\mathcal{A}$ defined above can be shown to generate the group Γ with only the requirements that X be connected and the action be cocompact. However, the finiteness of $\mathcal{A}$ is a consequence of the properness of X and the proper discontinuity of the action.

Theorem 1.26 (The Švarc–Milnor lemma, second part)**.** *If Γ acts by isometries on X in a way that satisfies the Švarc–Milnor condition, then Γ is finitely generated and quasi-isometric to X.*

Proof. We continue from the proof of lemma 1.25.

We will now proceed to show that the map $\gamma \mapsto \gamma \cdot x_0$ is a quasi-isometry. To do so, we would ordinarily need to show that $d_\Gamma(\gamma', \gamma)$ is bounded above and below by expressions involving $d_X(\gamma' \cdot x_0, \gamma \cdot x_0)$, but since Γ acts by isometries on both spaces, it is sufficient to consider the case when γ' is the identity.

To see the upper bound, recall the geodesic $[x_0, \gamma \cdot x_0]$ as in the lemma, and consider the partition $p_1, \cdots, p_n$ with the widest possible intervals. That is, the partition with $d_X(p_i, p_{i+1}) = R/3$ for all appropriate i, except $i = n - 1$. In this case, $n \leq (3/R) d_X(x_0, \gamma \cdot x_0) + 1$. In the previous lemma we showed that $\gamma = s_1 \cdots s_{n-1}$, so $d_\Gamma(1, \gamma) \leq n \leq (3/R) d_X(x_0, \gamma \cdot x_0) + 1$.

We now show the lower bound. Let $\mu = \max_{a \in \mathcal{A} \cup \mathcal{A}^-} d(x_0, a \cdot x_0)$. Let $\gamma \in \Gamma$, so that if $m = d_\Gamma(1, \gamma)$ then $\gamma = a_1 a_2 \cdots a_m$, where $a_i \in \mathcal{A} \cup \mathcal{A}^-$ since $\mathcal{A}$ generates Γ. Then by the triangle inequality, we see the following:

$$
\begin{aligned}
d_X(x_0, \gamma \cdot x_0) &\leq d_X(x_0, a_1 \cdot x_0) + d_X(a_1 \cdot x_0, a_1 a_2 \cdot x_0) \\
&\quad + \cdots + d_X(a_1 \cdots a_{m-1} \cdot x_0, \gamma \cdot x_0) \\
&= d_X(x_0, a_1 \cdot x_0) + d_X(x_0, a_2 \cdot x_0) + \cdots + d_X(x_0, a_m \cdot x_0) \\
&\leq \mu m = \mu d_\Gamma(1, \gamma),
\end{aligned}
$$

Therefore $\mu^{-1} d_X(x_0, \gamma \cdot x_0) \leq d_\Gamma(1, \gamma)$.

We have now shown that $\gamma \mapsto \gamma \cdot x_0$ is a quasi-isometric embedding. Since $\Gamma \cdot B = X$ is the image of Γ under the map, it is evidently $R/3$-quasi-dense in X. By proposition 1.7, the map is therefore a quasi-isometry. □

2 Hyperbolic Groups

2.1 The Hyperbolicity Condition

Before we turn our attention to hyperbolic groups, we must define hyperbolicity more generally. Hyperbolicity may be thought of as describing a kind of curvature in a metric space. However there are other ways of describing curvature, and for our purposes hyperbolicity might be better though of as describing a certain 'closeness' within a space.

Before we begin with hyperbolicity, we must do some work to describe the tools we will use to define it.

Definition 2.1. A *geodesic* in a metric space X is a map $\gamma\colon [a,b] \to X$ such that given $t, t' \in [a,b]$, we have $d(\gamma(t), \gamma(t')) = |t - t'|$. Note that geodesics are always continuous. A *geodesic space* is a metric space such that given any two points, there is a geodesic whose endpoints are those points.

Examples. Euclidean n-space is a geodesic space, whose geodesics are unique up to their image. Any metric Cayley graph is a geodesic space, but a combinatorial Cayley graph is a geodesic space if and only if the group is trivial. There may be geodesics in a Cayley graph between the same points but with different paths.

Remark. Every geodesic space is connected, but the converse is not true. Consider the subset of Euclidean 2-space obtained by removing a disc of radius 1. This space is easily shown to be connected, but it is not geodesic, since for example there is no geodesic from $(-2,0)$ to $(2,0)$

Certain well-behaved paths have an associated length, which we define now.

Definition 2.2. Let X be a metric space. The *length* of a continuous path $g\colon [a,b] \to X$ is defined as:

$$\operatorname{len}(g) = \sup_{(t_i)_{0 \le i \le n}} \sum_{i=0}^{n-1} d(g(t_i), g(t_{i+1})),$$

where the supremum ranges over all $(t_i)_{0 \le i \le n}$ such that $a = t_0 \le \cdots \le t_n = b$ and n is a positive integer.

Remark. Not all paths have finite length, but all paths' lengths are bounded below by the distance between their endpoints.

An example of a path of infinite length is the path from $[0,1]$ to $\mathbb{R}^2$ defined by $t \mapsto (t\cos(\pi/t), t\sin(\pi/t))$ for $0 < t \le 1$ and $t \mapsto (0,0)$ for $t = 0$. This path has finite length along many of its segments, but the entire path has infinite length.

A monstrously pathological example of a path with infinite length could be constructed via a Wierstrass function, or some other fractal curve such as the Koch snowflake – an example of this kind would have infinite length for almost all of its segments.

The notion of length corresponds in a natural way to the notion of a geodesic, as the following proposition shows.

Proposition 2.3. *Let X be a metric space, and let $g\colon [a,b] \to X$ be a path of finite length. Then $\operatorname{len}(g) = d(g(a), g(b))$ if and only if g can be re-parametrised to be a geodesic.*

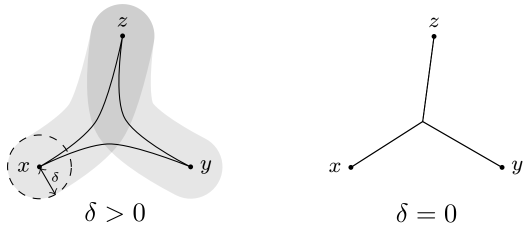

Figure 2.1: Two examples of δ-slim triangles. Right: As δ approaches 0, the triangle increasingly resembles a tree.

Proof. If g is a geodesic, then

$$\sum_{i=1}^{n-1} d(g(t_i), g(t_{i+1})) = \sum_{i=1}^{n-1} |t_{i+1} - t_i| = |b - a| = d(g(a), g(b)),$$

and so one implication is done.

For the converse, we suppose $l = \text{len}(g) = d(g(a), g(b))$, and suppose by re-parametrisation that it is locally non-constant. Define $\lambda\colon [a, b] \to [0, l]; t \mapsto \text{len}(g|_{[a,t]})$. This is a strictly increasing continuous function with $\lambda(a) = 0$ and $\lambda(b) = 1$, so it is bijective and has a continuous inverse $\lambda^{-1}\colon [0, l] \to [a, b]$. We claim that the re-parametrisation $\gamma\colon [0, l] \to X$ given by $\gamma = g \circ \lambda^{-1}$ is a geodesic, which we now show.

Let $t \geq t'$ both be in $[0, l]$. The definition of λ tells us $d(\gamma(0), \gamma(x)) = x$ for all $x \in [0, l]$, so we see that $|t - t'| = t' - t = d(\gamma(0), \gamma(t')) - d(\gamma(0), \gamma(t)) \leq d(\gamma(t), \gamma(t')) \leq \text{len}(\gamma|_{[t,t']}) = |t - t'|$, and we are done. $\square$

These definitions are stated mostly to introduce the reader to relevant concepts. There are some more evident properties – for instance that the sum of lengths of sub-paths is the length of the whole path – that we will not prove.

We can now turn our attention to hyperbolicity. In the following definition, we use the notation $[x, y]$ not to refer to intervals of the real line, but to refer to some geodesic in a metric space with endpoints x and y. This may not be unique or totally descriptive, but it is useful in denoting a particular 'side'.

Definition 2.4 (Rips' hyperbolicity condition[4])**.** A geodesic metric space X is δ-*hyperbolic* for some real $\delta \geq 0$ if all geodesic triangles in X are δ-*slim.* That is, given points $x, y, z \in X$ and geodesics $[x, y], [x, z], [y, z]$, every point on $[x, y]$ is distance less than δ from some point on $[x, z]$ or $[y, z]$.

We say that a space is hyperbolic to say that there exists some $\delta \geq 0$ for which it is δ-hyperbolic.

As we see in figure 2.1, as δ approaches 0, the triangle described approaches a tree-like figure. It may be useful to think of δ as how close a particular triangle is to a tree for this reason.

[4]Gromov [Gro87] attributes this definition to Eliyahu Rips, and presents an equivalent (on geodesic spaces) definition using his now eponymous product, which remarkably does not require X to be a geodesic space. However it is not always preserved under quasi-isometries in non-geodesic spaces, and so is not appropriate for our purposes.

Example. The hyperbolic plane is, unsurprisingly, hyperbolic. One way to show this is to place a hyperbolic semicircle within a triangle, centred on the midpoint of a given side. By the Gauss-Bonnet theorem, the area of the triangle is less than π. However, the area of the semicircle is unbounded as its radius increases. Ergo there is some radius at which the semicircle's area exceeds π and hence the area of triangle. Every point on the chosen side is therefore a distance at most the radius from a point on the other two sides – this shows that the radius is a δ for which the hyperbolic plane is δ-hyperbolic.

The area of a hyperbolic semicircle with radius δ is given by $2\pi \sinh^2(\delta/2)$. Solving for δ when we reach an area of π gives us $\delta = 2\sinh^{-1}(1/\sqrt{2})$, or approximately 1.3170. This is a suitable δ, but may not be the lowest possible.

Proposition 2.5. *In a δ-hyperbolic space, a given geodesic lies in the δ-neighbourhood of any geodesic with the same endpoints.*

Proof. We need only show that any point on one geodesic is distance at most δ from some point on the other. If x is a common endpoint of the geodesics, then $\gamma\colon [0,0] \to X$ given by $\gamma(t) = x$ is a geodesic. Together with the two other geodesics this forms a geodesic triangle, and the definition of hyperbolicity gives us the result immediately. $\square$

Proving that a given space is hyperbolic is not necessarily an easy task. Fortunately, the following fact gives us an alternative way of showing that a space is hyperbolic.

Theorem 2.6. *Let X and Y be quasi-isometric geodesic spaces. Then X is hyperbolic if and only if Y is hyperbolic.*

The proof of this fact lies on a different path to the one we follow in this project. Interested readers will find the proof of this fact in Bridson–Haefliger's corollary III.H.1.8 [BH99], among other texts.

2.2 Hyperbolic Groups and the Word Problem

Given a group presentation $G = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$, it is a very natural question to ask if two given words $\mathbf{w}_1$ and $\mathbf{w}_2$ of the generators represent the same element in the group. Clearly this is a fundamental hurdle in understanding the group! It is not hard to see that an equivalent condition is finding a method of determining if a given word $\mathbf{s}$ is the identity, since setting $\mathbf{s} = \mathbf{w}_1\mathbf{w}_2^{-1}$ determines if $\mathbf{w}_1 =_G \mathbf{w}_2$, and setting $\mathbf{w}_1 = \epsilon$ and $\mathbf{w}_2 = \mathbf{s}$ delivers the converse.

This is the first of Dehn's three fundamental problems, which he outlined in [Deh11]. As we noted, there are finitely presented groups for which this problem is unsolvable, computationally speaking. In this section, we will identify hyperbolic groups, and show that they have the remarkable property that their word problems are solvable.

Equipped with the understanding of the previous sections, we have a strong motivation for the following definition, which is the one introduced by Mikhail Gromov.

Definition 2.7. A *hyperbolic group* is a finitely generated group whose metric Cayley graph is a hyperbolic space.

The ideas that allow this to be well-defined are twofold: firstly we recall corollary 1.21 which shows that the metric Cayley graph is independent of choice of generators up to quasi-isometry, and secondly we recall theorem 2.6 which lets us conclude that hyperbolicity is independent of the same choice.

Remark. All finite groups are hyperbolic groups, choosing δ to be the diameter of the metric Cayley graph. For this reason, we are more concerned with *infinite* groups, since finite groups are indistinguishable both in terms of hyperbolicity and quasi-isometry.

We will briefly demonstrate that the free groups discussed in section 1.2 are hyperbolic. As figure 2.1 might lead us to believe, free groups are in fact hyperbolic with $\delta = 0$, which is to say that they are trees in the graph-theoretic sense.

Proposition 2.8. *The free group $F(\mathcal{A})$ on finitely many generators is hyperbolic.*

Proof. Suppose $a, b, c \in F(\mathcal{A})$ are reduced words. Since $F(\mathcal{A})$ acts on its own Cayley graph by isometries, it suffices to consider the case where c is the empty word ϵ.

Reduced words are unique, so the geodesic $[c, a]$ can be described by the reduced word representing $ac^{-1} = a$. That is, vertices p along $[c, a]$ are initial subword of a – which is to say that $p = s_1 \cdots s_k$, where $a = s_1 \cdots s_n$ and $k \leq n$. The same argument characterises the vertices of $[c, b]$.

Let P be the largest common initial subword of a and b. That is, given $a = s_1 \cdots s_n$ and $b = r_1 \cdots r_m$, let $P = s_1 \cdots s_i = r_1 \cdots r_i$ where i is the greatest index such that $s_i = r_i$ (note that i may be 0, in which case $P = \epsilon$.

The path from a to P and then to b given by $s_n^{-1} \cdots s_{i+1}^{-1} r_{i+1} \cdots r_m$ is a reduced word, since otherwise $s_{i+1} = r_{i+1}$, contradiction the maximality of i. Hence this word describes the unique geodesic $[a, b]$.

Consider any point along the geodesic $[a, b]$. This is either a vertex or is on the edge between two vertices. In either case, the path above allows us to conclude that this point falls on either $[c, a]$ or $[c, b]$, and hence is distance 0 from a point on those two geodesics, namely itself. □

It is not difficult to see that this is merely a consequence of the uniqueness of geodesics between points, which is the defining characteristic of a graph-theoretic tree.

We now introduce a few concepts that will be useful in solving the word problem.

Definition 2.9. A *vertex path* is a continuous path $g \colon [a, b] \to \mathcal{C}_{\mathcal{A}}(G)$ in a metric Cayley graph such that:

(1) The endpoints of g are vertices, and
(2) The path is locally injective, meaning that given $x \in [a, b]$ there exists some $\varepsilon > 0$ for which $g|_{[x-\varepsilon, x+\varepsilon]}$ is injective.

The significance of vertex paths is that they correspond to reduced words of $\mathcal{A}$, and in particular are always of integer length.

If we have a word, we may start at the vertex corresponding to the identity, and follow the edge corresponding to each letter of that word. If the word is the identity in the group, then the vertex path that we would obtain would in fact be a loop. We will show that we can always reduce such a loop below a certain threshold, reducing the infinite search space to a finite one.

To prove that we can always find a shorter path, we need to find a way to identify vertex paths that are not at their shortest possible. The following definition realises this.

Definition 2.10. Let $k > 0$, and let X be a metric space. A k-local geodesic is a path $g\colon [a,b] \to X$ such that, for all $t, t' \in [a,b]$, if $|t - t'| \leq k$ then $d(g(t), g(t')) = |t - t'|$.

Remark. Any vertex path can be re-parametrised to be a 1-local geodesic.

The following proof is quoted from Bridson and Haefliger [BH99] – we include it here for completeness.

Theorem 2.11. *Let X be a δ-hyperbolic space, and suppose $g\colon [a,b] \to X$ is a k-local geodesic, and $k > 8\delta$. Then every point of g is contained in the 2δ-neighbourhood of any geodesic connecting its endpoints.*

Proof. Since $[a,b]$ is compact, there must be a point $t \in [a,b]$ such that $x = g(t)$ is a maximal distance from $[g(a), g(b)]$. We consider the first case, that both $|t-a|$ and $|t-b|$ are greater than 4δ, in which case we have a subarc g' whose midpoint is at x, and whose length is greater than 8δ but strictly less than k, so by definition it is a geodesic. Let y and z be the endpoints of this path, and let y' and z' be their closest cousins on $[g(a), g(b)]$. The geodesics g, $[z, z']$, $[y, y']$, and the segment of $[g(a), g(b)]$ which we call $[y', z']$ form a quadrilateral, and adding a diagonal $[y', z]$ shows us that there exists some w on one of the sides (except g) such that $d(x, w) \leq 2\delta$.

We now need only prove that w lies on $[y', z']$. Suppose for contradiction that w lies on $[y, y']$. Then:

$$\begin{aligned}
d(x, y') - d(y, y') &\leq \big(d(x,w) + d(w,y')\big) - \big(d(y,w) + d(w,y')\big) \\
&= d(x,w) + d(w,y') \\
&\leq d(x,w) - \big(d(y,x) - d(x,w)\big) \\
&= 2d(x,w) - d(x,y) \\
&\leq 4\delta - 4\delta = 0.
\end{aligned}$$

This is to say that x is not a maximal distance from $[g(a), g(b)]$, since y is evidently further – a contradiction.

A dual argument shows that w does not lie on $[z, z']$. In the case that at least one of $|t-a|$ and $|t-b|$ are less than or equal to 4δ, we may draw a geodesic triangle instead of a quadrilateral, and by a similar argument x must be within the δ-neighbourhood of $[g(a), g(b)]$. ☐

Proposition 2.12. *Let X be a δ hyperbolic metric space, and suppose g is a k-local geodesic with $k > 8\delta$. Then either g is constant, or its endpoints are not equal.*

Proof. Suppose there is some k-local geodesic $g\colon [a,b] \to X$ with $g(a) = g(b)$. Then every point of g must lie in the 2δ-neighbourhood of the trivial geodesic connecting its endpoints. In the case that $a < b$ and $|a - b| > k$, then $2\delta \geq$

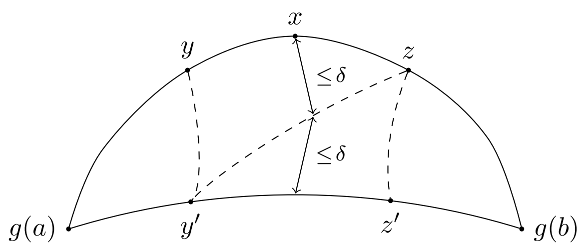

Figure 2.2: An illustration of the proof of theorem 2.11, in the case where both $|t-a|$ and $|t-b|$ are greater than 4δ.

$d(g(a), g(a+k)) = |a-(a+k)| = k > 8\delta$, which cannot be true. In the case that $a < b$ and $|a-b| \leq k$, then $0 = d(g(a), g(b)) = |a-b| \neq 0$, which also cannot be true. Therefore the only remaining case is when $a = b$, which is to say that g is constant. □

The following theorem is the crux of this section, and will be used to show that the word problem is solvable.

Theorem 2.13. *Let $\mathcal{C}_{\mathcal{A}}(G)$ be the metric Cayley graph of a δ-hyperbolic group, where $\delta > 0$ is an integer. Suppose there is a vertex path $\gamma\colon [0,1] \to \mathcal{C}_{\mathcal{A}}(G)$ such that $\gamma(0) = \gamma(1)$ and $\operatorname{len}(\gamma) > 16\delta$. Then there exist $s, t \in [0,1]$ such that:*
(1) $\gamma(s)$ and $\gamma(t)$ are vertices,
(2) $d(\gamma(s), \gamma(t)) \leq \operatorname{len}(\gamma|_{[s,t]}) - 1$, and
(3) $d(\gamma(s), \gamma(t)) + \operatorname{len}(c_{[s,t]}) \leq 16\delta$.

Proof. Let $k = 8\delta + \frac{1}{2}$. Since γ is a loop of length 16δ, it is not constant, and so by proposition 2.12, it cannot be a k-local geodesic, since $k > 8\delta$. This means that there is some sub-path γ' of γ of length less than $8\delta + \frac{1}{2}$ which cannot be re-parametrised to be a geodesic. Let $s, t \in [0,1]$ be the least and greatest numbers for which $\gamma(s)$ and $\gamma(t)$ are vertices that lie on γ'.

We must have $1 < \operatorname{len}(\gamma|_{[s,t]})$ otherwise the segment is a geodesic, and similarly $d(\gamma(s), \gamma(t)) < \operatorname{len}(\gamma|_{[s,t]})$, so since they are integers, $d(\gamma(s), \gamma(t)) \leq \operatorname{len}(\gamma|_{[s,t]}) - 1$. Furthermore, $\operatorname{len}(\gamma|_{[s,t]}) \leq 8\delta$, so $d(\gamma(s), \gamma(t)) + \operatorname{len}(\gamma|_{[s,t]}) \leq 8\delta + 8\delta = 16\delta$, and we are done. □

This is a remarkable result and has important consequences. First, we restate this in the language of finitely generated groups – this is achieved by corresponding vertex paths to words, and length of those paths with length of their words.

Corollary 2.14. *Let $\mathcal{A}$ be a generating set for a δ-hyperbolic group G. Let the reduced word $\mathbf{w} = w_1 \cdots w_n$ of $\mathcal{A}$ represent the identity in G, and suppose $n = |\mathbf{w}| > 16\delta$. There exists a sub-word $\mathbf{w}' = w_i \cdots w_j$ and a word $\mathbf{v}$ such that:*
(1) $\mathbf{w}' =_G \mathbf{v}$ (in other words $\mathbf{w}'\mathbf{v}^{-1}$ represents the identity),
(2) $|\mathbf{w}'| < |\mathbf{v}|$, and
(3) $|\mathbf{w}'| + |\mathbf{v}| \leq 16\delta$.

This demonstrates that the word problem may be reduced to a finite problem. If we have computed all words of length at most 16δ representing the identity, then we may repeatedly invoke corollary 2.14 to reduce a word until it is length

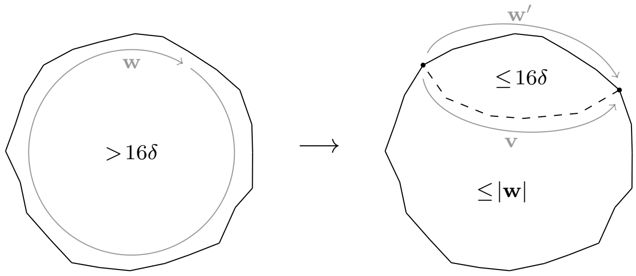

Figure 2.3: A visualisation of the reduction performed in corollary 2.14. A particular bound of the length of a loop in the images is indicated within the loop.

at most 16δ, at which point we may simply inspect it. This method is essentially the one introduced by Dehn [Deh11], and might be called Dehn's algorithm.

For this reason we may non-constructively say that there exists an algorithm to determine all words of length at most 16δ which represent the identity. Hence there must be an algorithm that determines whether or not an arbitrary word represents the identity. In other words, the following is true, though non-constructively.

Corollary 2.15. *The word problem is solvable in hyperbolic groups.*

Corollary 2.14 also indicates that every word representing the identity is within the normal closure of the set of words of length at most 16δ which themselves represent the identity. This tells us that the smallest normal subgroup of $F(\mathcal{A})$ containing all such words of length at most 16δ is the kernel of the obvious homomorphism from $F(\mathcal{A})$ to G. Applying the first isomorphism theorem, this gives us a group presentation for G. In other words, the following holds.

Corollary 2.16. *Every hyperbolic group is finitely presented.*

Remark. We take a moment contrast these tidy results with the case of monoids (groups without inverses) and semigroups (monoids without identity). It may be tempting to use similar arguments to the above to show that a hyperbolicity condition solves the word problem, however the same arguments do not apply. For instance, inverses do not necessarily exist, and so vertex paths may not correspond to reduced words as they do in groups.

2.3 The Conjugacy Problem

We now turn our attention to Dehn's second problem. Given two elements g and h of a group G, does there exist a third element k for which $g = k^{-1}hk$? This is known as the *conjugacy problem.* More precisely, we will ask this for finitely generated groups, where g and h are represented by words.

It may not be evident why this is a fundamental obstacle in understanding a group. The conjugacy classes of a group can reveal its structure in important ways. For example, we might observe that rotations in the Euclidean plane are conjugate by a translation when their angles of rotation are equal – a key step in classifying all isometries of Euclidean space. Similar observations in other groups allow us to reason about them.

Dehn notes in [Deh11] that the word problem is actually a specific instance of the conjugacy problem, namely when g is the identity and h is any word. The conjugacy problem is therefore not solvable in general, since its solubility would imply the solubility of the word problem. We have already shown that hyperbolic groups have a solvable word problem, and remarkably they also have solvable conjugacy problem. We will now show that this is the case.

Definition 2.17. A *geodesic word* $\mathbf{w}$ of $\mathcal{A}$, a generating set of a group G, is a word such that $|\mathbf{w}| = d(1, \mathbf{w})$. That is to say, it represents an element of the group with the smallest possible length.

It is clear that every element of the group can be represented by a geodesic word, by definition of the word metric. Since we know that the word problem is solvable, it is therefore possible to take a given word and find a geodesic word that represents the same element in the group. This can be done simply by iterating through every word of length less than a given word, and choosing the shortest one that represents the same element.

We may observe that we can cyclically permute the symbols of a word via conjugation, like so:

$$s_1 s_2 \cdots s_n \mapsto s_1^{-1}(s_1 s_2 \cdots s_n)s_1 = s_2 \cdots s_n s_1.$$

In the case that $s_1 = s_n^{-1}$, cancellation occurs. For this reason we would like to consider *cyclically reduced* words, which are reduced words with $s_1 \neq s_n^{-1}$ – it can be shown that a cyclically reduced word is unique up to cyclic permutation, and it is evidently conjugate to the word which we reduce. Note also that if a cyclically reduced word is a geodesic word, then so are all of its cyclic permutations.

Theorem 2.18. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be geodesic, cyclically reduced words that represent conjugate elements in a hyperbolic group* G*. Then one of the following holds:*

(1) The lengths $|\mathbf{v}|, |\mathbf{w}| \leq 8\delta + 1$*, or*

(2) There exist cyclic permutations $\mathbf{v}'$ *and* $\mathbf{w}'$ *of* $\mathbf{v}$ *and* $\mathbf{w}$ *respectively such that there exists a word* $\mathbf{u}$ *with length* $|\mathbf{u}| \leq 2\delta + 1$ *such that* $\mathbf{u}\mathbf{v}'\mathbf{u}^{-1} =_G \mathbf{w}'$*.*

This theorem shows that the conjugacy problem can be solved, since we may find cyclically reduced geodesic words corresponding to any word in the generators, then apply the lemma, allowing us to exhaustively search every word of length at most $2\delta + 1$ until we find that some cyclic permutation is conjugate to another.

Since this applies to all words of length greater than $8\delta + 1$, there remains only finitely many words whose conjugacy must be determined. This problem is finite, so is solvable, though we have not explicitly constructed an algorithm to solve it.

The proof of this theorem is adapted from Bridson and Haefliger [BH99]. We include it here for completeness.

Proof of theorem 2.18. We will work in the metric Cayley graph of G. Considering a quadrilateral representing the conjugacy of $\mathbf{v}$ and $\mathbf{w}$, we may assume without loss of generality via cyclic permutations that if $\mathbf{u}\mathbf{v}\mathbf{u}^{-1} = \mathbf{w}$, then the distance between any two points along $\mathbf{v}$ and $\mathbf{w}$ is at least $|\mathbf{u}|$. This is illustrated in figure 2.4: if $\mathbf{w}$ and $\mathbf{v}$ are conjugate but there are points along the two with

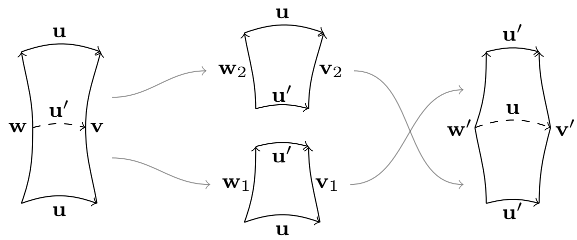

Figure 2.4: An illustration of the minimisation argument in the proof of theorem 2.18. The leftmost quadrilateral encodes that $\mathbf{w}$ is conjugate to $\mathbf{v}$, and the rightmost that $\mathbf{w}' = \mathbf{w}_2\mathbf{w}_1$ is conjugate to $\mathbf{v}' = \mathbf{v}_2\mathbf{v}_1$.

distance less than $|\mathbf{u}|$, then we can cyclically permute the words to achieve the shortest distance in such a way that preserves the conjugacy of the words.

Following figure 2.4, we will call the path described by $\mathbf{w}$ the left side, and $\mathbf{v}$ the right. Consider the midpoint p of the left side. Adding a diagonal, we see that p is distance at most 2δ from a point q on the right side, or on the top and bottom. We now consider these two cases.

Suppose that q is on the right geodesic side – that is, along the path of $\mathbf{v}$. In that case, the vertices of the Cayley graph that are closest to p and q are a distance at most $2\delta + 1$ apart from each other. In this case $|\mathbf{u}| \leq 2\delta + 1$, since otherwise this contradicts the minimality of $|\mathbf{u}|$.

Now suppose that q is on the top or bottom sides. Let x and y be the vertices on the left and right corners of the quadrilateral on the side to which q belongs. By the minimality of $|\mathbf{u}|$, we have $|\mathbf{u}| - (1/2) \leq d(p, y)$, and by the triangle inequality, $d(p, y) \leq 2\delta + d(q, y)$. Since $d(x, q) + d(q, y) = |\mathbf{u}|$, we obtain $d(x, q) \leq 2\delta + (1/2)$. Finally, we see that $|\mathbf{w}| = 2d(p, x) \leq 2d(p, q) + 2d(q, x) \leq (4\delta) + (4\delta + 1) = 8\delta + 1$.

A dual argument shows the result for $\mathbf{v}$ as well. □

3 Examples of Hyperbolic Groups

3.1 Abelian Hyperbolic Groups

We would like to characterise all Abelian hyperbolic groups. It is a well-known fact of finitely generated Abelian groups that they are a product of a finite number of cyclic groups. In particular, we say that such a group has rank n if it is isomorphic to $\mathbb{Z}^n \times G$, where G is some finite group. We will show that Abelian hyperbolic groups are exactly those of rank 1 or rank 0

Firstly, we will prove the following useful result.

Proposition 3.1. *Let G be a finitely generated group with a subgroup H of finite index. That is, H has finitely many right (or equivalently, left) cosets in G. Then H is finitely generated, and furthermore the two groups are quasi-isometric.*

We defer proof momentarily to inspect this marvellous fact more closely.

Why is it that we require H to be of finite index in G? If the reader is familiar with free groups, they will know that there is a subgroup of the free group on two symbols which is isomorphic to the free group on countably many symbols, however it is not a finite-index subgroup.

Furthermore many infinite groups have finite subgroups. If these groups were quasi-isometric, then the infinite group would necessarily have finite radius, which is not possible if it is finitely generated. The finite-index condition is therefore necessary for both purely algebraic and purely geometric reasons!

Proof of proposition 3.1. The first half of this proposition – that H is also finitely generated – is a lemma of Schreier. We present a geometric argument here, which is in the same vein as the proof of the Švarc–Milnor lemma.

Consider the combinatorial Cayley graph of G. Since H is of finite index in G, there are finitely many representatives $g_1, \ldots, g_n$ of its right cosets in G. Letting $R = \max\{d_G(1, g_1), \ldots, d_G(1, g_n)\}$, we note that H is R-quasi-dense in G. Let $f\colon G \to H$ be the closest element of H to some element of G – the closest element may not be unique, so we may use the axiom of choice. This allows us to see that $d_G(g, f(g)) \leq R$.

There is a path in the Cayley graph of G between any elements h and h' of H, so that $h' = ha_1 \cdots a_m$. Let $g_i = ha_1 \cdots a_i$. The triangle inequality tells us that $d_G(f(g_i), f(g_{i+1})) \leq d_G(f(g_i), g_i) + d_G(g_i, g_{i+1}) + d_G(g_{i+1}, f(g_{i+1})) \leq 2R + 1$.

We may now see that $h, f(g_1), f(g_2), \ldots, f(g_n)$ is a sequence of elements of H with distance $2R + 1$ between elements. This shows that H is generated by the set of elements of H with distance at most $2R + 1$ from the origin, which is a finite set. Hence H is finitely generated.

The above argument also shows that, in the generating set we have chosen for H, that $d_H(1, h) \leq d_G(1, h) \leq (2R + 1)d_H(1, h)$, so that the inclusion map from H to G is a quasi-isometric embedding. Since we've already seen that H is quasi-dense in G, so by proposition 1.7 G and H are quasi-isometric. □

Since $\mathbb{Z}$ is isomorphic to the free group on one generator, by proposition 2.8 it is hyperbolic. However, this is not true for $\mathbb{Z}^n$ if $n > 1$, which we see now.

Proposition 3.2. *The group $\mathbb{Z}^n$ is hyperbolic if and only if $n = 1$ or $n = 0$.*

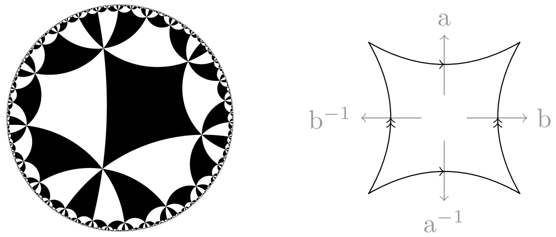

Figure 3.1: A visualisation of the tessellation of the hyperbolic plane in question. Left: the tessellation of the hyperbolic plane in question, reproduced from the public domain work of Sherwood [She20]. Right: The sides of the quadrilateral are paired by isometries, and these isometries generate the Fuchsian group.

Proof. As we have seen, $\mathbb{Z}$ is hyperbolic, so it remains to show that $\mathbb{Z}^n$ is not hyperbolic for $n > 1$.

We may see that $\mathbb{Z}^n$ is quasi-isometric to $\mathbb{R}^n$, by invoking the Švarc–Milnor lemma (theorem 1.26) under the action of $\mathbb{Z}^n$ given by $(z_1, \ldots, z_n) \cdot (x_1, \ldots, x_n) = (z_1 + x_1, \cdots, z_n + x_n)$. The n-cube $[0, 1]^n$ allows us to show cocompactness, and the other conditions are easily seen to hold.

Evidently $\mathbb{R}^2$ is not hyperbolic, and it may be embedded in all $\mathbb{R}^n$ for $n > 1$ to show that the higher-dimensional spaces are also not hyperbolic. Since hyperbolicity is preserved under quasi-isometry, we conclude that the group $\mathbb{Z}^n$ is not hyperbolic for $n > 1$. $\square$

This now allows us to characterise Abelian hyperbolic groups. Since a finitely generated Abelian group of rank n has $\mathbb{Z}^n$ as a subgroup of finite index, by proposition 3.1 it is quasi-isometric to $\mathbb{Z}^n$. Then by proposition 3.2, in order to be hyperbolic it must have $n = 1$ or $n = 0$, and in the latter case it is finite.

Corollary 3.3. *A hyperbolic group is Abelian if and only if it has rank* 1 *or is finite.*

Put simply, Abelian hyperbolic groups are uninteresting. This is fundamentally a product of their relationship with Euclidean space, as we saw in proposition 3.2.

3.2 An Example of a Fuchsian Group

Many examples of hyperbolic groups may be found in Fuchsian groups: discrete subgroups of the isometries of the hyperbolic plane. In particular, Fuchsian groups arising from tessellations of the hyperbolic plane satisfy the Švarc–Milnor condition automatically. What's more, these tessellations strongly resemble the structure of the Cayley graph, which we will use to our benefit.

In this section, we will use results in the study of the hyperbolic plane that vastly exceed the scope of this project. These results are quoted from a course given by Charles Walkden [Wal20], who advises the author that the notes may become a book in the future.

Proposition 3.4. *The group* $G = \langle \mathrm{a}, \mathrm{b} \mid (\mathrm{a}^{-1}\mathrm{b}^{-1}\mathrm{ab})^2 \rangle$ *is hyperbolic.*

Proof. Via Poincaré's theorem (theorem 19.1.1 in [Wal20]), the presentation may be seen to correspond to a tessellation of the hyperbolic plane by regular

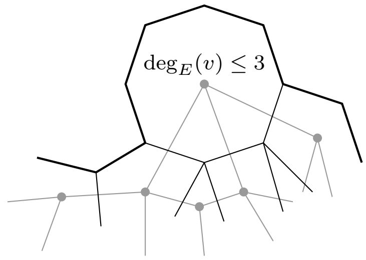

Figure 3.2: The situation we aim for in the proof of proposition 3.5. The thick line is the boundary: the word which represents the identity. The large octagon has the property that more than half of its edges are on the boundary, reflected in the degree of the dual vertex (in grey).

quadrilaterals with internal angle $\pi/4$, meaning that G is isomophic to a Fuchsian group.

This Fuchsian group acts on the hyperbolic plane in an obvious way, since it is a group of isometries. What's more, this action is properly discontinuous and cocompact, by virtue of corresponding to a tessellation. Applying theorem 1.26, we see that G is quasi-isometric to the hyperbolic plane, and by theorem 2.6, G is therefore hyperbolic. □

This proposition may suggest a way to solve the word problem for G: first determine the isometries that the elements a and b correspond to, then simply compute whatever word we wish to determine the identity of. This is in theory possible since isometries of the hyperbolic plane (which are Möbius transformations) may be handled entirely numerically.

However, the quantities involved are not very amenable to these calculations and might cause headaches involving precision. Instead we will present an algorithm – similar to the one described in theorem 2.13 – which solves the word problem in a combinatorial way.

We first consider the dual graph of the tessellation presented. This is constructed by connecting the midpoints of the quadrilaterals if they are adjacent. In this way, we obtain an octagonal tessellation of the hyperbolic plane, with four octagons at each corner.

Since the edges of this graph correspond to the generators (see figure 3.1), a word representing the identity is a loop in the graph, and if it is cyclically reduced then it is a cycle in the graph-theoretic sense. This leads us to the solution.

Proposition 3.5. *Every cyclically reduced word representing the identity in G has a subword of length at least* 5 *which is also a subword of a cyclic permutation of the relator or its inverse.*

Proof. Consider such a cycle of the octagonal tessellation, and note that it may be tiled by octagonal faces. We are looking for an octagon with at least 5 edges on the boundary, or in other words with at most 3 edges adjacent to another octagon. If we work within the dual graph, we are looking for a vertex with degree at most 3, which we illustrate in figure 3.2.

The dual graph here is planar, so it follows that $V - E + F = 1$ after a theorem of Euler[5], where V, E, and F are the number of vertices, edges, and faces respectively.

The nature of the octagonal tessellation means that every edge of the dual graph has 2 vertices, and its every face has 4 vertices. If $\deg_E(v)$ is the degree of a vertex v and $\deg_F(v)$ is the number of faces surrounding it, we then obtain the following, via double-counting.

$$1 = V - E + F = \sum_{v \text{ vertex}} \left(1 - \frac{\deg_E(v)}{2} + \frac{\deg_F(v)}{4}\right)$$

We divide $\deg_E(v)$ by 2 since we count each edge twice, and similarly divide $\deg_F(v)$ by 4 since we count each face four times.

This sum is positive, so there exists some vertex v for which $1 - \deg_E(v)/2 + \deg_F(v)/4 > 0$. Noting that $\deg_E(v) \geq \deg_F(v)$, we may find that $\deg_E(v) < 4$, so there is a vertex v of degree at most 3 as required. □

This solves the word problem. We have shown that in a word representing the identity, we may always find an application of the relator that reduces the length of the word. Simply repeating this process until we either cannot reduce any more or reach the empty word, we can decide if a given cyclically reduced word is the identity.

We briefly note that this result is easily generalisable. Supposing that we have a tessellation of $3 \leq n$-sided polygons, meeting m at a corner, under what circumstances does the same result hold? We have already seen that this works for $n = 4$ and $m = 8$

Via the same argument in the proof of the above proposition, we may obtain that there is always a vertex v in the analogous dual graph with $\deg_E(v) < 2n/(n-2)$. If we want the same argument to work, we require $2n/(n-2) \leq m/2$ – note that equality held in the example. After expansion, we may find the following equivalent condition:

$$\frac{1}{n} + \frac{2}{m} \leq \frac{1}{2}.$$

In particular, that means that $1/n + 1/m < 1/2$: this exactly the condition for the existence of a tessellation of the hyperbolic plane consisting of n-gons meeting m at a corner (theorem 7.3.1 in [Wal20]) – though this is of course a necessary but insufficient condition for the same argument to apply.

This fact shows a very direct connection between the condition for Dehn's algorithm and resemblance to the hyperbolic plane. This is a hint of a deeper result – that Dehn's algorithm in fact *only* applies to hyperbolic groups – but we hope at least that this offers a fresh perspective on why exactly these groups are called 'hyperbolic'.

[5] The reader may be more familiar with the formula $V - E + F = 2$. In this case, we are not considering the region outside the cycle to be a face, and so we reduce the number of faces by 1, so that $V - E + F = 1$.

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